Regression methods on the Runge function
FYS-STK3155 - Project 1*

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This project investigates polynomial regression methods for approximating Runge's function $f(x) = 1/(1 + 25x^2)$ on the interval $[-1, 1]$. We implement and compare Ordinary Least Squares (OLS), Ridge regression, and Lasso regression using different polynomial basis functions and sampling strategies. The main focus is on model performance and numerical stability issues, particularly the Runge phenomenon that occurs when using high-degree polynomials with equispaced points.

Our analysis includes gradient descent implementations with various optimization algorithms , bootstrap resampling for bias-variance analysis, and cross-validation techniques. We achieve target accuracy of $RMSE < 10^{-2}$ using Vandermonde polynomials with Chebyshev node sampling, demonstrating the importance of proper basis function selection for numerical stability

I. INTRODUCTION

Polynomial regression is a fundamental technique in machine learning and numerical analysis [1], yet it presents significant challenges when applied to certain functions. Runge's function, $f(x) = 1/(1 + 25x^2)$, serves as a classic example of these difficulties due to the Runge phenomenon - the tendency for high-degree polynomial interpolation to produce large oscillations near the boundaries of the interpolation interval.

This project explores various regression techniques to approximate Runge's function while investigating the interplay between polynomial degree, sampling strategy, and numerical stability. The work is motivated by the need to understand when and why certain regression methods fail, and how alternative approaches can mitigate these issues.

II. METHODS

A. Method 1/X

B. Implementation

C. Use of AI tools

ChatGPT was used as a coding assistant for debugging, improving code readability and greating poilerplate code. It also helped with writing, such as improving grammar or structure.

III. RESULTS AND DISCUSSION

A. Bias–Variance Tradeoff

Figure 1 shows the training and test mean squared error for polynomial regression on the Runge function, plotted against model complexity. The figure illustrates the bias–variance tradeoff: low-degree models underfit (high bias), while high-degree models overfit (high variance), with an intermediate degree achieving the best test performance.

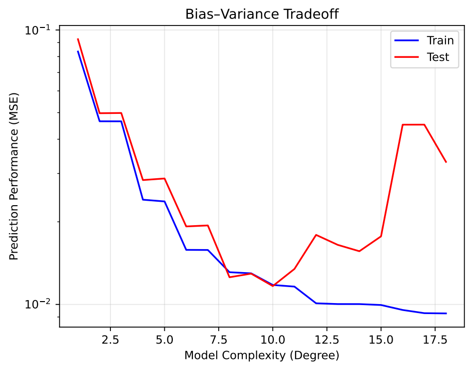

Figure 1: Training and test mean squared error (MSE) for polynomial regression on the Runge function with $n = 100$ equispaced points and Gaussian noise of standard deviation 0.1. The horizontal axis shows the polynomial degree, the vertical axis the prediction error on a logarithmic scale. The plot highlights the bias–variance tradeoff: test error decreases initially with model complexity but increases again for very high degrees.

*https://github.com/viktorbgulbrandsen/fysstk3155/tree/main/project-1

B. Ridge Heatmap

Figure 2 shows the test mean squared error for ridge regression across polynomial degrees and regularization parameters. The plot highlights how prediction error varies with model complexity and the choice of λ. The regression was done utilizing the `SkRidge` function made by Sklearn [2]

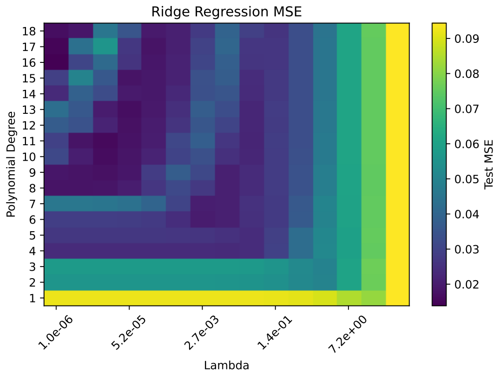

Figure 2: Test mean squared error (MSE) for ridge regression on the Runge function. The vertical axis shows polynomial degree, the horizontal axis the regularization parameter λ on a logarithmic scale. Lower values correspond to better predictive accuracy.

IV. CONCLUSION

[1] T. Hastie, R.Tibshirani, and J.Friedman, *The Elements of Statistical Learning: Data Mining, Inference, and Prediction, Second Edition. Springer Series in Statistics* (Springer, New York, 2009), URL `https://link.springer.com/book/10.1007%2F978-0-387-84858-7`.

[2] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, et al., Journal of Machine Learning Research **12**, 2825 (2011).