Paper solutions

Exam TDDE07 2020 06/04

1 a)

1. a)

$$X \mid N, \theta \sim Bin(N, \theta) \quad N = 50$$

$$\theta \sim U(0,1) \Rightarrow p(\theta) = 1$$

$$p(\theta \mid x) \propto p(x \mid \theta) p(\theta) \propto \binom{N}{x} \theta^x (1-\theta)^{N-x}$$

$$\propto \theta^x (1-\theta)^{50-x} \propto \frac{\theta^x (1-\theta)^{50-x} \Gamma(52)}{\Gamma(x+1)\Gamma(51-x)} \sim Beta(x+1, 51-x)$$

2 d)

2 d) Method 1: Normal approximation

One can approximate the posteror distrubution $N(\bar{\theta}, J^{-1}(\hat{\theta}))$ where $\hat{\theta}$: Posterior mode and J = negative $\frac{d^2 \log(p(\theta|x))}{d^2\theta}(\hat{\theta})$ (seond derivate of log posterior likelihood) at $\theta = \hat{\theta}$.

The posterior mode can be found ~~by~~ optimizing $\max_\theta \log(p(\theta|x))$, using optim in R for instance; same with J.

Method 2: Hamiltonian monte carlo, which adds momentum to the sampling distrubution, akin to a physics simulation, using expressions for kinetic and potential energy of the posterion.

1. Sample from momentum to begin
2. Simulate new theta proposal and momentum (using leapfrog algorithm)
3. compute acceptance probability

$$\alpha = \min\left(1, \frac{p(\bar{y}|\theta_p)p(\theta_p)}{p(\bar{y}|\theta^{(i-1)})p(\theta^{(i-1)})} \cdot \frac{p(\phi_p)}{p(\phi_s)}\right)$$

(θ_p: theta proposal, ϕ_p: momentum proposal)

4. with probability α, set $\theta^{(i)} = \theta_p$, else $\theta^{(i)} = \theta^{(i-1)}$.

Repeat N times from step 2.

4 a)

4.a) $X \overset{i.i.d}{\sim} Bern(\theta)$ $\quad \theta \sim Beta(2,2)$

θ: Prob for female

$$p(x=1) = \int p(x=1|\theta)\,p(\theta|\bar{x})\,d\theta$$

$$p(\theta|\bar{x}) \propto p(\bar{x}|\theta)\,p(\theta) \propto \theta^{16}(1-\theta)^{4}\,\theta^{1}(1-\theta)^{1} \propto$$

$$Beta(18, 6) = Beta(18, 6)$$

$$\int p(x=1|\theta)\,p(\theta|\bar{x})\,d\theta = \int \frac{\theta \cdot \theta^{17}(1-\theta)^{5}\,d\theta}{\frac{\Gamma(18)\Gamma(6)}{\Gamma(24)}} =$$

$$\frac{\Gamma(24)}{\Gamma(18)\Gamma(6)} \int \theta^{18}(1-\theta)^{5}\,d\theta = \frac{\Gamma(24)\cdot\Gamma(19)\Gamma(6)}{\Gamma(18)\Gamma(6)\Gamma(25)}$$

$$p(x=1) = \frac{\Gamma(24)\Gamma(18)\cdot 18}{\Gamma(18)\Gamma(24)\cdot 24}$$

// using $\Gamma(x+1) = \Gamma(x)x$ //

$$\Leftrightarrow p(x=1) = \frac{18}{24} = \frac{3}{4}$$

4 b)

4. b) $length_M = x \sim N(\mu_{ML}, 2^2)$ $\sigma_{ML} = 2$

$$p(\mu_{ML} \mid \bar{x}) \propto p(\bar{x} \mid \mu_{ML}) \underbrace{p(\mu_{ML})}_{1;\ \text{Uniform prior}} \propto$$

$$\propto \exp\left(\frac{-(\mu_{ML} - \bar{x})^2}{2(\sigma_{ML}^2/n)} \right) \sim N\left(\bar{x}, \frac{2^2}{4}\right)$$

//n=4//

$\mu_{ML} = \bar{x} + \varepsilon$ // $\varepsilon \sim N\left(0, \frac{2^2}{4}\right)$ =>

$y_{pred} = \mu_{ML} + v$ // $v \sim N(0, 2^2)$

$y_{pred} = \bar{x} + \varepsilon + v \sim N\left(\bar{x}, 2^2\left(1 + \frac{1}{4}\right)\right) =$

$y_{pred} = N(12, 5)$

predictive variance = population variance + posterior variance