PHY 511

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Equation of motion

- 2nd Newton's law: $\frac{dp_r}{dt} = F_r$ (1)
- changing variables: $\xi = ct - z$ (2)

$$\Downarrow$$

$$\frac{d}{dt} = (c - v_z)\frac{d}{d\xi} \quad (3)$$

where v_z - velocity of electrons along z-axis

- relativistic case: $\vec{p} = \gamma m \vec{v}$ (4)

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (5)$$

- notation: $\frac{dr}{d\xi} = r'$

- (1), (3) $\Longrightarrow$

$$\frac{dp_r}{dt} = (c - v_z)\frac{dp_r}{d\xi} \quad (6)$$

(3), (4) $\Rightarrow$

$$p_r = \gamma m \frac{dr}{dt} = \gamma m (c - v_z)\frac{dr}{d\xi} \quad (7)$$

$$\frac{dp_r}{dt} = (c - v_z)\frac{d}{d\xi}\left(\gamma m (c - v_z)\frac{dr}{d\xi}\right) \quad (8)$$

$$\frac{dp_r}{dt} = \frac{c - v_z}{c} \frac{d}{d\xi}\left(\gamma m c (c - v_z) r'\right) \quad (9)$$

- Integral of motion

$$\vec{P} = \vec{p} + q\frac{\vec{A}}{c} \quad (10) \text{ — canonical momentum}$$

$$H = \gamma m c^2 + q\Phi \quad (11) \text{ — Hamiltonian}$$

↖ scalar potential

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = c\frac{\partial H}{\partial \xi} = -c\frac{\partial H}{\partial z} = c\frac{dP_z}{dt}$$

Heisenberg equation

quasistatic approx. $\partial_t = c\partial_\xi$, $\partial_z = -\partial_\xi$

Hamilton's equations

$$\left\{ \begin{array}{l} \text{Hamilton's equations} \\ \dfrac{\partial P_z}{\partial t} = -\dfrac{\partial H}{\partial z} \\ v_z = \dfrac{\partial H}{\partial P_z} \end{array} \right\}$$

$$\frac{dH}{dt} = c\frac{dP_z}{dt} \quad (12)$$

$$\Rightarrow \frac{d}{dt}(H - cP_z) = 0 \Rightarrow H - cP_z = \text{const}$$

Initially $v_z = 0$, $\Phi = 0 \Rightarrow H|_{t=0} = mc^2$,

$$P_z|_{t=0} = 0 \Rightarrow H - cP_z = mc^2 \quad (13)$$

(10), (11), (13) $\Rightarrow$

$$\gamma mc^2 + q\Phi - cp_z - qA_z = mc^2 \qquad (14)$$

- Pseudo potential

$$\Psi = \Phi - A_z \qquad (15)$$

$$p_z = \gamma m v_z$$

$$\gamma mc^2 + q\Psi - \gamma m v_z c = mc^2 \qquad (14')$$

$$\gamma mc(c - v_z) + q\Psi = mc^2 \qquad (14'')$$

$$\frac{mc^2 - q\Psi}{mc(c - v_z)} = \gamma$$

- (9), (14')

$$F_r = \frac{dp_r}{dt} = \frac{c - v_z}{c} \frac{d}{d\xi}\left([mc^2 - q\Psi]\, r'\right)$$

$$\frac{d}{d\xi}\left[(mc^2 - q\Psi)\frac{dr}{d\xi}\right] = \frac{c}{c - v_z} F_r \qquad (16)$$

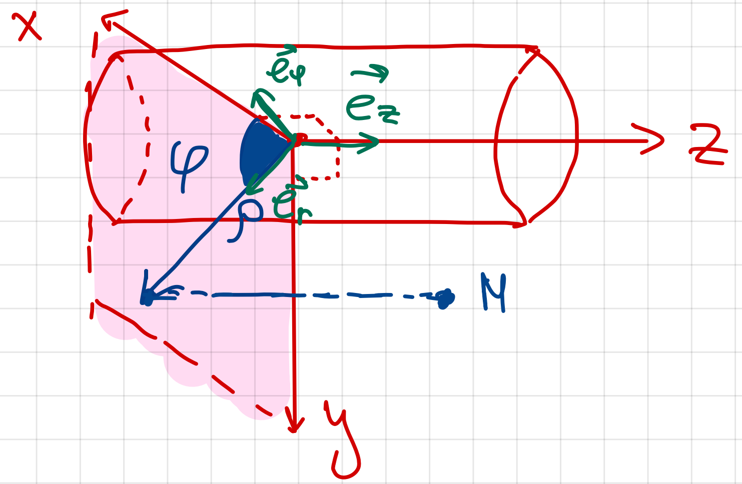
should find them

Radial force

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]$$

$$\vec{B} = \mathrm{rot}\vec{A} = \vec{\nabla}\times\vec{A} =$$

$$\vec{v}\times\vec{B} = \vec{v}\times\vec{\nabla}\times\vec{A}$$



$$\vec{B} = \mathrm{rot}\vec{A} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\varphi & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix} =$$

$$= \vec{e}_r \underbrace{\left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)}_{B_r} - \vec{e}_\varphi\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) +$$

$$+ \vec{e}_z \underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}\right)}_{B_z}$$

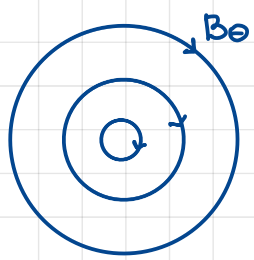
$$\vec{v}\times\vec{B} = \begin{vmatrix} \frac{1}{r}\vec{e}_r & \vec{e}_\varphi & \frac{1}{r}\vec{e}_z \\ v_r & rv_\varphi & v_z \\ B_r & rB_\varphi & B_z \end{vmatrix} =$$

$$= \vec{e}_r\left(v_\varphi B_z - v_z B_\varphi\right) - \vec{e}_\varphi\left(v_r B_z - v_z B_r\right) +$$

$$+ \vec{e}_z\left(v_r B_\varphi - v_\varphi B_r\right)$$

Because of symmetry $B_r = B_z = 0$:

$$\vec{v}\times\vec{B} = \vec{e}_r\left(-v_z B_\varphi\right) + \vec{e}_z\left(v_r B_\varphi\right)$$



Also, $E_\theta = 0$

$$\vec{F} = q\left(E_r - \frac{v_z B_\varphi}{c}\right)\vec{e}_r + q\left(E_z + \frac{v_r B_\varphi}{c}\right)\vec{e}_z \quad (17)$$

$$F_r = q\left(E_r - \frac{v_z}{c} B_\varphi\right) \qquad (17')$$

$$B_\varphi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow E_r = -\frac{\partial \Phi}{\partial r} - \frac{1}{c}\frac{\partial A_r}{\partial t}$$

$$\Rightarrow F_r = q\left(-\frac{\partial \Phi}{\partial r} - \frac{1}{c}\frac{\partial A_r}{\partial t} - \frac{v_z}{c}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\right)$$

$\left\{ \begin{array}{l} \text{quasistatic approx.} \\ \partial_t = c\,\partial_\xi \;,\; \partial_z = -\partial_\xi \end{array} \right\}$

$$F_r = q\left(-\frac{\partial \Phi}{\partial r} - \frac{\partial A_r}{\partial \xi} - \frac{v_z}{c}\left(-\frac{\partial A_r}{\partial \xi} - \frac{\partial A_z}{\partial r}\right)\right)$$

$\left\{ \psi = \Phi - A_z \;,\; q = -e \right\}$

$$F_r = e\left(\frac{\partial \psi}{\partial r} + \frac{\partial A_z}{\partial r} + \frac{\partial A_r}{\partial \xi} - \frac{v_z}{c}\left(\frac{\partial A_r}{\partial \xi} + \frac{\partial A_r}{\partial r}\right)\right)$$

$$F_r = e\left[\frac{\partial \psi}{\partial r} + \left(1 - \frac{V_z}{c}\right)\frac{\partial A_z}{\partial r} + \underbrace{\left(1 - \frac{V_z}{c}\right)\frac{\partial A_r}{\partial \xi}}_{\text{we will neglect}}\right] \qquad (18)$$

Derivation of V_z

2nd method

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad ; \quad \gamma^2 \left(1 - \frac{V_z^2 + V_\perp^2}{c^2}\right) = 1$$

$$\gamma^2 - \frac{\gamma^2}{c^2} V_z^2 - \frac{\gamma^2}{c^2} V_\perp^2 = 1$$

$$\gamma^2 c^2 - \gamma^2 V_z^2 - \gamma^2 V_\perp^2 = c^2$$

$$\gamma^2 (c^2 - V_z^2) = c^2 + \gamma^2 V_\perp^2 \qquad (*)$$

From 14" we have

$$\gamma mc(c - v_z) = mc^2 + e\Psi \qquad (**)$$

$$(*) : (**)^2$$

$$\frac{c + v_z}{m^2 c^2 (c - v_z)} = \frac{c^2 + \gamma^2 V_\perp^2}{(mc^2 + e\Psi)^2}$$

$$\frac{c + v_z}{c - v_z} = \underbrace{m^2 c^2 \left[\frac{c^2 + \frac{p_\perp^2}{m^2}}{(mc^2 + e\Psi)^2}\right]}_{T}$$

$$\frac{c + v_z}{c - v_z} = T \quad , \quad c + v_z = Tc - Tv_z$$

$$v_z (1 + T) = c(T - 1) \Rightarrow \boxed{v_z = \frac{c(T-1)}{T+1}}$$

$$\gamma mc\,(c - v_z) = mc^2 + e\Psi$$

$$\gamma mc\left(c - \frac{c(T-1)}{T+1}\right) = mc^2 + e\Psi$$

$$\gamma mc^2\left(\frac{T+1-(T-1)}{T+1}\right) = mc^2 + e\Psi$$

$$\gamma mc^2\left(\frac{2}{T+1}\right) = mc^2 + e\Psi$$

$$\gamma = \frac{\left(1 + \frac{e\Psi}{mc^2}\right)(T+1)}{2} =$$

$$\boxed{\gamma = \frac{(mc^2 + e\Psi)(T+1)}{2mc^2}}$$

$$T = T(\Psi, p_\perp)$$

Potentials

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = 4\pi\frac{\vec{J}}{c}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = 4\pi\rho$$

Because of quazist. appr:

$$\frac{\partial^2}{\partial \varsigma^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial z^2} \Rightarrow$$

$$\begin{cases} -\nabla_\perp^2\Phi = 4\pi\rho \\ -\nabla_\perp^2\vec{A} = 4\pi\frac{\vec{J}}{c} \end{cases} \Rightarrow -\nabla_\perp^2 A_z = \frac{4\pi}{c} j_z$$

$$\Psi = \Phi - A_z \Rightarrow -\nabla_\perp^2\Psi = 4\pi\left(\rho - \frac{j_z}{c}\right)$$

$$j_z = \rho v_z \Rightarrow -\nabla_\perp^2\Psi = 4\pi\rho\left(1 - \frac{v_z}{c}\right)$$

- $$-\nabla_\perp^2\Psi = 4\pi\rho_e\left(1 - \frac{v_z}{c}\right) + 4\pi\rho_i\left(1 - \frac{0}{c}\right) + 4\pi\rho_b\left(1 - \underbrace{\frac{c}{c}}_{\to 0}\right) = 4\pi\rho_e\left(1 - \frac{v_z}{c}\right) + 4\pi\rho_i$$

- $$-\nabla_\perp^2 A_z = \frac{4\pi}{c}\rho_e v_z + \frac{4\pi}{c}\rho_i\cdot 0 + \frac{4\pi}{c}\rho_b\cdot c = 4\pi\rho_e\frac{v_z}{c} + 4\pi\rho_b$$

$$\begin{cases} -\nabla_\perp^2\Psi = 4\pi\rho_e\left(1 - \frac{v_z}{c}\right) + 4\pi\rho_i \\ -\nabla_\perp^2 A_z = 4\pi\rho_e\frac{v_z}{c} + 4\pi\rho_b \end{cases}$$

Summary

$$\frac{d}{d\xi}\left[(mc^2 - q\psi) \frac{dr}{d\xi} \right] = \frac{c}{c - v_z} F_r$$

[1] eq. of mot.

$$F_r = e\left[\frac{\partial \psi}{\partial r} + \left(1 - \frac{v_z}{c}\right)\frac{\partial A_z}{\partial r} + \left(1 - \frac{v_z}{c}\right)\frac{\partial A_r}{\partial \xi} \right]$$

[2] rad. force

$$\begin{cases} -\nabla_\perp^2 \psi = 4\pi \rho_e \left(1 - \frac{v_z}{c}\right) + 4\pi \rho_i \\ -\nabla_\perp^2 A_z = 4\pi \rho_e \frac{v_z}{c} + 4\pi \rho_b \end{cases}$$

[3] potent.

$$\boxed{v_z = \frac{c\,(T-1)}{T+1}} \qquad \boxed{\gamma = \frac{(mc^2 + e\psi)(T+1)}{2mc^2}}$$

[4] v_z, γ ,

where $T = m^2c^2 \left[\dfrac{c^2 + \frac{p_\perp^2}{m^2}}{(mc^2 + e\psi)^2} \right]$

$$p_\perp = \gamma m v_\perp = \gamma m \frac{dr}{dt} = \gamma m (c - v_z) \frac{dr}{d\xi} =$$

$$= \left(mc + \frac{e}{c}\psi\right) \frac{dr}{d\xi}$$

Let's make everything dimensionless

$v \to v/c$

$p \to p/mc$

$\psi, A \to \dfrac{e\psi}{mc^2}, \dfrac{eA}{mc^2}$

$r, \xi \to \dfrac{r\omega_p}{c}, \dfrac{\xi\omega_p}{c}$

$F \to F/(mc\omega_p)$

$n \to n/n_0$, n_0 - background plasma density

(Transformations)

$$\frac{d}{d\xi}\left[(mc^2 - q\psi)\frac{dr}{d\xi}\right] = \frac{c}{c - v_z} F_r$$

$$\Rightarrow \frac{d}{d\xi}\left[1 + \frac{e\psi}{mc^2}\right]\frac{dr}{d\xi} = \frac{1}{1 - v_z/c}\frac{F_r}{mc^2}$$

$$\frac{d}{d\xi}\left[1 + \hat{\psi}\right]\frac{dr}{d\xi} = \frac{1}{1 - \hat{v}_z}\frac{F_r}{mc^2}$$

$$F_r = e\left[\frac{\partial\psi}{\partial r} + \left(1 - \frac{v_z}{c}\right)\frac{\partial A_z}{\partial r} + \left(1 - \frac{v_z}{c}\right)\frac{\partial A_r}{\partial\xi}\right]$$

$$F_r = e\left[\frac{mc^2}{e}\frac{\partial\tilde{\psi}}{\partial r} \pm (1 - \hat{v}_z)\frac{mc^2}{e}\frac{\partial\tilde{A}_z}{\partial r} + (1 - \hat{v}_z)\frac{mc^2}{e}\frac{\partial\tilde{A}_r}{\partial\xi}\right]$$

$$F_r = mc^2\frac{\omega_p}{c}\left[\frac{\partial\tilde{\psi}}{\partial\tilde{r}} + (1 - \hat{v}_z)\frac{\partial\tilde{A}_z}{\partial\tilde{r}} + (1 - \hat{v}_z)\frac{\partial\tilde{A}_r}{\partial\tilde{\xi}}\right]$$

$$\tilde{F}_r = \frac{F_r}{mc\omega_p} = \frac{\partial\tilde{\psi}}{\partial\tilde{r}} + (1 - \hat{v}_z)\frac{\partial\hat{A}_z}{\partial\tilde{r}} + (1 - \hat{v}_z)\frac{\partial\hat{A}_r}{\partial\tilde{\xi}}$$

$$\frac{\omega_p}{c}\frac{d}{d\tilde{\xi}}\left[1 + \hat{\psi}\right]\frac{d\tilde{r}}{d\tilde{\xi}} = \frac{1}{1 - \tilde{v}_z} mc\omega_p \hat{F}_r \frac{1}{mc^2}$$

$$\frac{d}{d\tilde{\xi}}\left[1 + \tilde{\psi}\right]\frac{d\tilde{r}}{d\tilde{\xi}} = \frac{1}{1 - \tilde{v}_z}\tilde{F}_r$$

$$\boxed{v_z = \frac{c(T-1)}{T+1}} \qquad \boxed{\gamma = \frac{(mc^2 + e\psi)(T+1)}{2mc^2}} \qquad T = m^2c^2\left[\frac{c^2 + \frac{P_\perp^2}{m^2}}{(mc^2 + e\psi)^2}\right]$$

$$\hat{v}_z = \frac{T-1}{T+1}, \quad \gamma = \frac{(1 + \hat{\psi})(T+1)}{2},$$

$$T = \frac{(mc^2)^2}{(mc^2 + e\psi)^2}\left[1 + \hat{P}_\perp^2\right] = \frac{1 + \tilde{P}_\perp^2}{(1 + \hat{\psi})^2}$$

$$p_\perp = \left(mc + \frac{e}{c}\Psi\right)\frac{dr}{d\xi} = mc\left(1+\tilde{\Psi}\right)\frac{d\tilde{r}}{d\tilde{\xi}};$$ (Transformations)

$$\tilde{p}_\perp = (1+\tilde{\Psi})\frac{d\tilde{r}}{d\tilde{\xi}}$$

$$\begin{cases} -\nabla_\perp^2 \Psi = 4\pi\rho_e\left(1-\frac{v_z}{c}\right) + 4\pi\rho_i \\ -\nabla_\perp^2 A_z = 4\pi\rho_e \frac{v_z}{c} + 4\pi\rho_b \end{cases}$$

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$-\left(\frac{\omega_p}{c}\right)^2 \tilde{\nabla}_\perp^2 \frac{mc^2}{e}\tilde{\Psi} = 4\pi(-en_e)(1-\tilde{v}_z) + 4\pi e n_0$$

$$-\frac{\omega_p^2 m}{4\pi e^2 n_0}\tilde{\nabla}_\perp^2\tilde{\Psi} = -(1-\tilde{v}_z)\tilde{n}_e + 1$$

$$\frac{\omega_p^2 m}{4\pi e^2 n_0} = 1$$

↖ ions

$$-\tilde{\nabla}_\perp^2\tilde{\Psi} = -(1-\tilde{v}_z)\tilde{n}_e + 1$$

$$-\tilde{\nabla}_\perp^2\tilde{A}_z = -\tilde{v}_z\tilde{n}_e - \tilde{n}_b$$

From this point we will omit ~ and consider everything as dimensionless.

Full dimensionless set of equations

1) $$\frac{d}{d\xi}\left[1+\psi\right]\frac{dr}{d\xi} = \frac{1}{1-v_z} F_r$$

2) $$F_r = \frac{\partial \psi}{\partial r} + (1-v_z)\frac{\partial A_z}{\partial r} + (1-v_z)\frac{\partial A_r}{\partial \xi}$$

3) $$-\nabla_\perp^2 \psi = -\frac{(1-v_z)}{1-v_z} n_e + 1 \quad , \quad n_e, n_i, n_b > 0$$
$$-\nabla_\perp^2 A_z = -\frac{v_z n_e}{1-v_z} - n_b$$

4) $$v_z = \frac{T-1}{T+1}$$

$$\gamma = \frac{(1+\psi)(T+1)}{2}$$

$$T = \frac{1+p_\perp^2}{(1+\psi)^2}$$

5) $$p_\perp = (1+\psi)\frac{dr}{d\xi}$$

Continuity equation for 1D fluid along z:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} n v_z = 0$$

QSA: $$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} = -c\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \xi}(c-v_z)n = 0$$

$$\Rightarrow (c-v_z)n = c n_0$$

$$\Rightarrow n = \frac{n_0}{1-v_z/c}$$

$$\oint_L \frac{\partial A_z}{\partial r}\, dl = \frac{\partial A_z}{\partial r} \cdot 2\pi r = \iint_S \left(\frac{v_z\, n_e}{1-v_z} + n_B \right) dS$$

$$\frac{\partial A_z}{\partial r} = \frac{1}{2\pi r} \left[\int_0^r \frac{v_z}{1-v_z} \overbrace{n_e \cdot 2\pi r' dr'}^{r_0 \Delta r_0} + \int_0^r n_B \cdot 2\pi r' dr' \right] =$$

$$= \frac{\Delta r_0}{r} \sum_{j : r_j < r} \frac{v_{zj}}{1-v_{zj}} r_{0j} + \frac{1}{r} \int_0^r n_B(r') \cdot r' dr'$$

Pseudo potential integration

$$\nabla_\perp^2 \psi = n_e - 1 \Rightarrow \oint_L \frac{\partial \psi}{\partial r}\, dl = 2\pi r \frac{\partial \psi}{\partial r} = \iint_S (n_e - 1)\, dS$$

$$\frac{\partial \psi}{\partial r} = \frac{1}{2\pi r}(-\pi r^2) + \frac{1}{r}\int_0^r n_e(r') \cdot r' dr'$$

$$\frac{\partial \psi}{\partial r'} = -\frac{r'}{2} + \frac{1}{r'}\int_0^{r'} dr'' r'' n_e(r'')$$

charge of jth ring

$$\psi = \int_r^\infty \frac{1}{2} r' dr' - \int_r^\infty dr' \frac{1}{2\pi r'} \sum_j Q_j\, \Theta(r' - r_j) =$$

$$= \frac{1}{4}(r_\infty^2 - r^2) - \sum_j \frac{Q_j}{2\pi} \int_r^\infty \frac{dr' \cdot \Theta(r' - r_j)}{r'}$$

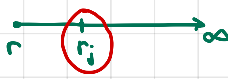
Let's take the integral by parts

$$\left\{ \int u\, dv = uv - \int v\, du \; ; \quad dv \to d(\ln r'), \quad u \to \Theta(r' - r_j), \quad du \to \delta(r' - r_j) \right\}$$

$$\int_r^\infty \frac{dr'}{r'} \Theta(r' - r_j) = \ln r' \cdot \Theta(r' - r_j) \Big|_r^\infty - \int_r^\infty \ln r' \cdot \delta(r' - r_j)\, dr' =$$

$$= \ln r_\infty \cdot 1 - \ln r \cdot \Theta(r - r_j) - \ln r_j \cdot \underbrace{\Theta(r_j - r)} =$$

We need it to make sure we have a point where $r' = r_j$



$$= \ln r_\infty - \ln r\, \Theta(r - r_j) -$$
$$- \ln r_j \left[1 - \Theta(r - r_j) \right] =$$

$$= \ln r_\infty - \Theta(r - r_j)\left[\ln r - \ln r_j \right] - \ln r_j =$$

$$= \ln r_\infty - \ln r_j - \Theta(r - r_j) \cdot \ln \frac{r}{r_j}$$

$$\psi(r) = \frac{1}{4}(r_\infty^2 - r^2) - \sum_j \frac{Q_j}{2\pi}\Big(\underset{(*)}{\ln r_\infty} - \underset{(**)}{\ln r_j} -$$
$$- \underset{(***)}{\Theta(r - r_j)} \ln \frac{r}{r_j} \Big)$$

$$(*):\quad \ln r_\infty \cdot \sum_j \frac{Q_j}{2\pi} = \frac{\ln r_\infty}{2\pi}\cdot \pi r_\infty^2 = \frac{1}{2} r_\infty^2 \cdot \ln r_\infty$$

$$(**):\quad \sum_j \frac{Q_j}{2\pi} \ln r_j = \sum_j \frac{Q_j}{2\pi}\left(\ln \frac{r_j}{r_{j0}} + \ln r_{j0}\right) =$$

$$= \frac{1}{2\pi}\sum_j Q_j \ln \frac{r_j}{r_{j0}} + \underbrace{\sum_j r_{j0}\,\Delta r_{j0} \ln r_{j0}}$$

$$\int_0^\infty r \ln r\, dr = \frac{r^2 \ln r}{2}\Big|_0^\infty - \frac{r^2}{4}\Big|_0^\infty = \frac{r_\infty^2 \ln r_\infty}{2} - \frac{r_\infty^2}{4}$$

$$\Rightarrow \quad \Psi(r) = \cancel{\frac{1}{4} r_\infty^2} - \frac{1}{4} r^2 - \cancel{\frac{1}{2} r_\infty^2 \ln r_\infty} +$$

$$+ \frac{1}{2\pi} \sum_j Q_j \ln \frac{r_j}{r_{j0}} + \cancel{\frac{1}{2} r_\infty^2 \ln r_\infty} - \cancel{\frac{1}{4} r_\infty^2} +$$

$$+ \sum_j \frac{Q_j}{2\pi} \cdot \Theta(r - r_j) \ln \frac{r}{r_j} =$$

$$= -\frac{1}{4} r^2 + \frac{1}{2\pi} \sum_j Q_j \left[\ln \frac{r_j}{r_{j0}} + \Theta(r - r_j) \ln \frac{r}{r_j}\right]$$

$$\Psi(r) = -\frac{1}{4} r^2 + \frac{1}{2\pi} \sum_j Q_j \left[\ln \frac{r_j}{r_{j0}} + \Theta(r - r_j) \ln \frac{r}{r_j}\right]$$

$$= -\frac{1}{4} r^2 + \Delta r_0 \sum_j r_{0j} \left[\ln \frac{r_j}{r_{j0}} + \Theta(r - r_j) \ln \frac{r}{r_j}\right]$$

1) $$\frac{d}{d\xi}\left[1+\psi\right]\frac{dr}{d\xi} = \frac{1}{1-v_z} F_r \rightarrow \frac{d}{d\xi} P_\perp = \frac{1}{1-v_z} F_r$$

2) $$F_r = \frac{\partial \psi}{\partial r} + (1-v_z)\frac{\partial A_z}{\partial r}$$

3) $$-\nabla_\perp^2 \psi = -n_e + 1$$

$$-\nabla_\perp^2 A_z = -\frac{v_z n_e}{1-v_z} - n_b$$

4) $$v_z = \frac{T-1}{T+1}$$

$$\gamma = \frac{(1+\psi)(T+1)}{2}$$

$$T = \frac{1+p_\perp^2}{(1+\psi)^2}$$

5) $$p_\perp = (1+\psi)\frac{dr}{d\xi}$$

$$\psi(r) = -\frac{1}{4}r^2 + \Delta r_0 \sum r_{0j}\left[\ln\frac{r_j}{r_{j0}} + \Theta(r-r_j)\ln\frac{r}{r_j}\right]$$

$$\frac{\partial \psi}{\partial r} = -\frac{r}{2} + \frac{1}{r}\int_0^r dr'\, r'\, n_e(r') = \frac{-r}{2} + \frac{\Delta r_0}{r}\sum_{j:r_j<r} r_{0j}$$

$$\frac{\partial A_z}{\partial r} = \frac{\Delta r_0}{r}\sum_{j:r_j<r}\frac{v_{zj}}{1-v_{zj}} r_{0j} + \frac{1}{r}\int_0^r n_b(r')\cdot r'\,dr'$$