4.3 Jacobian for positional and angular displacements

Assume again that we are at a joint configuration $\mathbf{q}$ with associated forward kinematics $\mathbf{T}_{base}^{tool}(\mathbf{q})$ and that there is only a differential displacement between $\mathbf{T}_{base}^{tool}(\mathbf{q})$ and $[\mathbf{T}_{base}^{tool}]_{desired}$, which we now write as $[\mathbf{du}]_{desired}$. We now wish to derive the Jacobian so that we can find $\mathbf{dq}$ so that $\mathbf{T}_{base}^{tool}(\mathbf{q}+\mathbf{dq}) = [\mathbf{T}_{base}^{tool}]_{desired}$ by solving $\mathbf{J}(\mathbf{q})\mathbf{dq} = [\mathbf{du}]_{desired}$.

In the following, we assume for simplicity that the reference frame is the "base" frame. Thus, when we refer to e.g. an origin $\mathbf{o}^i$, we refer to the coordinates of the origin of frame i computed in the base frame. Assume that we have placed frames $\mathbf{F}^i \quad i = 1,\ldots,n$ so that $\mathbf{o}^i$ is located on joint axis i and $\mathbf{z}^i$ along the positive direction of joint i. Assume that the robot is at a joint configuration $\mathbf{q} = (q_1,\ldots,q_n)^T$. The transformations $\mathbf{T}_{base}^{tool}(\mathbf{q})$ and $\mathbf{T}_{base}^{i}(q_1,\ldots,q_i)$ for any i are then given. Although we thus realize that $\mathbf{T}_{base}^{i}$ only depends on the first i joint angles, we for the sake of brevity write it as $\mathbf{T}_{base}^{i}(\mathbf{q})$.

Assume now that we move joint i by a signed infinitesimal amount dq_i. We then obtain a new transformation $\mathbf{T}_{base}^{tool}(\mathbf{q}+\mathbf{dq}_i)$ where $\mathbf{dq}_i = (0,\ldots,0,dq_i,0,\ldots,0)^T$. If joint i is rotational, we immediately obtain

$$\mathbf{p}_{base}^{tool}(\mathbf{q}+\mathbf{dq}_i) = \mathbf{p}_{base}^{tool}(\mathbf{q}) + dq_i\,[\mathbf{z}_{base}^{i}(\mathbf{q})] \times [\mathbf{p}_{base}^{tool}(\mathbf{q}) - \mathbf{p}_{base}^{i}(\mathbf{q})]$$

where in general $\mathbf{p}_i^j(\mathbf{q})$ is the positional part of $\mathbf{T}_i^j(\mathbf{q})$ and $\mathbf{z}_i^j(\mathbf{q})$ is the z-axis of $\mathbf{T}_i^j(\mathbf{q})$ (third column in the rotation matrix).

If joint i is prismatic, we get

$$\mathbf{p}_{base}^{tool}(\mathbf{q}+\mathbf{dq}_i) = \mathbf{p}_{base}^{tool}(\mathbf{q}) + dq_i\mathbf{z}_{base}^{i}(\mathbf{q})$$

In order to proceed, we define a joint type function

$$\xi_i = \begin{cases} 1 \text{ if joint } i \text{ is revolute} \\ 0 \text{ if joint } i \text{ is prismatic} \end{cases} \tag{4.6}$$

We may then write the positional displacement as

$$\begin{aligned} \mathbf{dp}_{base}^{tool}(\mathbf{q},\mathbf{dq}_i) &\equiv \mathbf{p}_{base}^{tool}(\mathbf{q}+\mathbf{dq}_i) - \mathbf{p}_{base}^{tool}(\mathbf{q}) \\ &= \xi_i dq_i\,[\mathbf{z}_{base}^{i}(\mathbf{q})] \times [\mathbf{p}_{base}^{tool}(\mathbf{q}) - \mathbf{p}_{base}^{i}(\mathbf{q})] + (1-\xi_i)dq_i\mathbf{z}_{base}^{i}(\mathbf{q}) \end{aligned}$$

Consider now the angular displacement between two rotations $\mathbf{R}_{base}^{tool}(\mathbf{q})$ and $\mathbf{R}_{base}^{tool}(\mathbf{q}+\mathbf{dq})$. This displacement can be uniquely determined as a vector $\mathbf{dw}_{base}^{tool}(\mathbf{q},\mathbf{dq})$ given in the base frame so that $\mathbf{R}_{base}^{tool}(\mathbf{q}+\mathbf{dq}) = \mathbf{R}(\mathbf{dw}_{base}^{tool}(\mathbf{q},\mathbf{dq}))\mathbf{R}_{base}^{tool}(\mathbf{q})$ where $\mathbf{R}(\mathbf{dw}_{base}^{tool}(\mathbf{q},\mathbf{dq}))$ is the matrix corresponding to a rotation around $\mathbf{dw}_{base}^{tool}(\mathbf{q},\mathbf{dq})$ with the size of the rotation equal to the size of the vector. For small rotations, we may simplify this to $\mathbf{R}_{base}^{tool}(\mathbf{q}+\mathbf{dq}) = RPY(d\theta_x,d\theta_y,d\theta_z)\mathbf{R}_{base}^{tool}(\mathbf{q})$, so that we see that $\mathbf{dw}_{base}^{tool}(\mathbf{q},\mathbf{dq}) = (d\theta_x,d\theta_y,d\theta_z)$.

Consider now again a displacement dq_i of joint i. Clearly, if joint i is prismatic, we obtain no angular displacement. If the joint is revolute, we may write the angular displacement as the unique infinitesimal rotation vector $\mathbf{dw}^{tool}_{base}(\mathbf{q}, \mathbf{dq}_i) \equiv dq_i \mathbf{z}^i_{base}(\mathbf{q})$. Notice thus in general that an angular displacement is represented as a vector in the direction around which the rotation takes place and with length equal to the size of the rotation. We can formally write the angular displacement of the tool frame as the vector

$$\mathbf{dw}^{tool}_{base}(\mathbf{q}, \mathbf{dq}) = \xi_i dq_i \mathbf{z}^i_{base}(\mathbf{q})$$

We now consider a general displacement $\mathbf{dq} = (dq_1, \ldots, dq_n)$. We now use the result of the previous section, that the total displacement may then be found as a sum of the individual displacements. That is, we obtain a positional displacement

$$\begin{aligned}
\mathbf{dp}^{tool}_{base}(\mathbf{q}, \mathbf{dq}) &= \mathbf{p}^{tool}_{base}(\mathbf{q} + \mathbf{dq}) - \mathbf{p}^{tool}_{base}(\mathbf{q}) \\
&= \sum_{i=1}^{n} dq_i \Big\{ \xi_i \left[\mathbf{z}^i_{base}(\mathbf{q})\right] \times \left[\mathbf{p}^{tool}_{base}(\mathbf{q}) - \mathbf{p}^i_{base}(\mathbf{q})\right] \\
&\quad + (1 - \xi_i)\mathbf{z}^i_{base}(\mathbf{q}) \} \\
&\equiv \mathbf{A}(\mathbf{q})\mathbf{dq}
\end{aligned} \tag{4.7}$$

where $\mathbf{A}(\mathbf{q})$ is a $3 \times n$ matrix with ji'th element

$$[\mathbf{A}(\mathbf{q})]_{ji} = \Big\{ \xi_i \left[\mathbf{z}^i_{base}(\mathbf{q})\right] \times \left[\mathbf{p}^{tool}_{base}(\mathbf{q}) - \mathbf{p}^i_{base}(\mathbf{q})\right] + (1 - \xi_i)\mathbf{z}^i_{base}(\mathbf{q}) \Big\}_j \tag{4.8}$$

The total angular displacement of the tool frame is given by

$$\begin{aligned}
\mathbf{dw}^{tool}_{base}(\mathbf{q}, \mathbf{dq}) &= \sum_{i=1}^{n} dq_i \left\{ \xi_i \mathbf{z}^i_{base}(\mathbf{q}) \right\} \\
&\equiv \mathbf{B}(\mathbf{q})\mathbf{dq}
\end{aligned} \tag{4.9}$$

where $\mathbf{B}(\mathbf{q})$ is a $3 \times n$ matrix with ji'th element

$$[\mathbf{B}(\mathbf{q})]_{ji} = \left\{ \xi_i \mathbf{z}^i_{base}(\mathbf{q}) \right\}_j \tag{4.10}$$

We thus have the relations

$$\begin{aligned}
\mathbf{A}(\mathbf{q})\mathbf{dq} &= \mathbf{p}^{tool}_{base}(\mathbf{q} + \mathbf{dq}) - \mathbf{p}^{tool}_{base}(\mathbf{q}) \\
\mathbf{B}(\mathbf{q})\mathbf{dq} &= \mathbf{dw}(\mathbf{R}^{tool}_{base}(\mathbf{q}), \mathbf{R}^{tool}_{base}(\mathbf{q} + \mathbf{dq}))
\end{aligned} \tag{4.11}$$

where $\mathbf{dw}(\mathbf{R}^{tool}_{base}(\mathbf{q}), \mathbf{R}^{tool}_{base}(\mathbf{q} + \mathbf{dq}))$ denotes the unique small rotation vector for rotating from $\mathbf{R}^{tool}_{base}(\mathbf{q})$ to $\mathbf{R}^{tool}_{base}(\mathbf{q} + \mathbf{dq})$. We now formally define the Jacobian

$$\mathbf{J}(\mathbf{q}) \equiv \begin{bmatrix} \mathbf{A}(\mathbf{q}) \\ - - - - - - - - \\ \mathbf{B}(\mathbf{q}) \end{bmatrix} \tag{4.12}$$

where $\mathbf{A}(\mathbf{q}), \mathbf{B}(\mathbf{q})$ are given by Eq.4.8 and Eq.4.10 respectively. The $6 \times n$ matrix $\mathbf{J}(\mathbf{q})$ is called the *Manipulator Jacobian* (or sometimes just the *Jacobian*) of the robot.

We can now gather our derivations of the desired displacement and the Jacobian into 6 linear relations in n unknowns

$$\mathbf{A}(\mathbf{q})\mathbf{dq} = [\mathbf{dp}_{base}^{tool}]_{desired}$$
$$\mathbf{B}(\mathbf{q})\mathbf{dq} = [\mathbf{dw}_{base}^{tool}]_{desired}$$

that should be solved for the desired $\mathbf{dq}$. Again, we may formally write these as

$$\mathbf{J}(\mathbf{q})\mathbf{dq} = [\mathbf{du}]_{desired} \tag{4.13}$$

In some textbooks such as [*], an analytical formula for the Jacobian based on the Denavit-Hartenberg formulation is presented. It should be pointed out that these formulae are not necessary and give only very little additional insight.

[*] R.J.Schilling, Fundamentals of Robotics: Analysis and Control

4.3.1 Example: $\mathbf{J}(\mathbf{q})$ for a 3-axis robot

Consider the three axis robot from the previous section also illustrated in Figure 4.1. The frames $\mathbf{F}^{base}, \mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3, \mathbf{F}^{tool}$ are shown on the figure. All joints are revolute. We have $L = 3$, $a_2 = a_3 = 2$ and $\mathbf{q} = (q_1, q_2, q_3) = (0, -\frac{\pi}{6}, \frac{\pi}{6})$ where q_i is chosen as the angle between $\mathbf{x}^{i-1}$ and $\mathbf{x}^i$ around $\mathbf{z}^i$. A numerical forward kinematics routine yields

$$\mathbf{p}^1_{base}(\mathbf{q}) = \mathbf{p}^2_{base}(\mathbf{q}) = (0,0,3) \ ; \ \mathbf{p}^3_{base}(\mathbf{q}) = (1,0,3+\sqrt{3}) \ ; \ \mathbf{p}^{tool}_{base}(\mathbf{q}) = (3,0,3+\sqrt{3})$$
$$\mathbf{z}^1_{base}(\mathbf{q}) = (0,0,1) \ ; \ \mathbf{z}^2_{base}(\mathbf{q}) = (0,-1,0) \ ; \ \mathbf{z}^3_{base}(\mathbf{q}) = (0,-1,0)$$

Thus, we now obtain

$$\begin{aligned}
[\mathbf{A}(\mathbf{q})]_{*1} &= \mathbf{z}^1_{base}(\mathbf{q}) \times (\mathbf{p}^{tool}_{base}(\mathbf{q}) - \mathbf{p}^1_{base}(\mathbf{q})) = (0,0,1) \times (3,0,\sqrt{3}) = (0,3,0) \\
[\mathbf{A}(\mathbf{q})]_{*2} &= \mathbf{z}^2_{base}(\mathbf{q}) \times (\mathbf{p}^{tool}_{base}(\mathbf{q}) - \mathbf{p}^2_{base}(\mathbf{q})) = (0,-1,0) \times (3,0,\sqrt{3}) = (-\sqrt{3},0,3) \\
[\mathbf{A}(\mathbf{q})]_{*3} &= \mathbf{z}^3_{base}(\mathbf{q}) \times (\mathbf{p}^{tool}_{base}(\mathbf{q}) - \mathbf{p}^3_{base}(\mathbf{q})) = (0,-1,0) \times (2,0,0) = (0,0,2) \\
[\mathbf{B}(\mathbf{q})]_{*1} &= \mathbf{z}^1_{base}(\mathbf{q}) = (0,0,1) \\
[\mathbf{B}(\mathbf{q})]_{*2} &= \mathbf{z}^2_{base}(\mathbf{q}) = (0,-1,0) \\
[\mathbf{B}(\mathbf{q})]_{*3} &= \mathbf{z}^3_{base}(\mathbf{q}) = (0,-1,0)
\end{aligned}$$

where e.g. $[\mathbf{A}(\mathbf{q})]_{*1}$ denotes the first column of $\mathbf{A}$. We now obtain

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

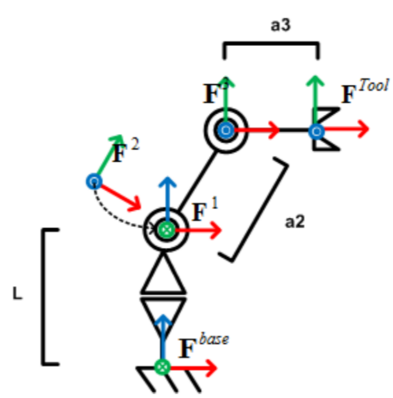

Figure 4.1: 3-axis robot with coordinate frames