Commutative Diagrams for Algebras

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An *algebra* (U, a) over a functor $T\colon \mathcal{C} \to \mathcal{C}$ consists of a carrier object U and a structure morphism $a\colon T(U) \to U$.

A morphism of T-algebras $f\colon (U, a) \to (V, b)$ is a morphism $f\colon U \to V$ in $\mathcal{C}$ such that the following square commutes:

$$\begin{tikzcd} T(U) \arrow[r, "T(f)"] \arrow[d, "a"'] & T(V) \arrow[d, "b"] \\ U \arrow[r, "f"'] & V \end{tikzcd} \tag{1}$$

That is, $f \circ a = b \circ T(f)$, meaning f respects the algebra structures. Such morphisms are called *algebra homomorphisms* or simply *natural*.

These morphisms compose, giving rise to sequences in the category of T-algebras:

$$\left(T(U) \xrightarrow{a} U\right) \xrightarrow{f} \left(T(V) \xrightarrow{b} V\right) \xrightarrow{g} \left(T(W) \xrightarrow{c} W\right) \tag{2}$$

The category formed by T-algebras and their morphisms is called the *Eilenberg–Moore category* $\mathcal{C}^T$.

The following diagram shows that functorial composition is compatible with algebra structure:

$$\begin{tikzcd} T(U) \arrow[r, "T(f)"] \arrow[d, "a"'] \arrow[rr, bend left, "T(g\circ f)"] & T(V) \arrow[r, "T(g)"] \arrow[d, "b"'] & T(W) \arrow[d, "c"] \\ U \arrow[r, "f"'] \arrow[rr, bend right, "g\circ f"'] & V \arrow[r, "g"'] & W \end{tikzcd} \tag{3}$$

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This diagram ensures that composition in $\mathcal{C}^T$ is well-defined: morphisms of algebras preserve structure not just individually, but also under composition, reflecting the functoriality of T and the coherence of the algebra morphisms.