In this note we will construct (cf. formulas (1)) an example of a non-negative, strictly increasing, continuously differentiable function $d: [0, \infty) \to [0, \infty)$, bounded from above by a convex function $f: [0, \infty) \to [0, \infty)$, f(0) = 0, such that the function d is not convex on any interval $I \subseteq [0, \infty)$. Such considerations are interesting from the point of view of [2, p. 167]. It is (implicitly) stated there, that if a decreasing function $\tilde{d} \geqslant 0$ is bounded from above by a convex function \tilde{f} vanishing at some point b > 0, then the function \tilde{d} is convex on some interval arbitrary close to b. After considering the mappings

$$d(x) = \widetilde{d}(b-x), \ f(x) = \widetilde{f}(b-x) \quad (x \in [0,b]),$$

one sees that instead of decreasing function \widetilde{d} , we can consider increasing function d. We will use the following results

Theorem 1. There exists a continuous, bounded and nowhere differentiable function $w : \mathbb{R} \to \mathbb{R}$.

Example of such function can be found in [1, Example 8, p. 38].

Theorem 2. If the function g is convex in an interval (a,b), then g is continuous in (a,b). Moreover, g' exists except at most in a countable set and is monotone increasing.

The proof of above Theorem can be found in [3, Theorem 7.40, p. 120].

Theorem 3. Let $g:(a,b) \to \mathbb{R}$ be monotone increasing. Then the function g has a measurable, non-negative derivative g' almost everywhere in (a,b).

For the proof of the above we refer to [3, Theorem 7.21, page 111].

Consider $\alpha \geqslant 0$ and let $w \colon \mathbb{R} \to [0,1]$ be a continuous, nowhere differentiable function. Define the functions $d, f \colon [0, \infty) \to \mathbb{R}$ as follows

$$d(x) = \int_0^x s^{\alpha} w(s) \, ds \quad (x \ge 0),$$

$$f(x) = \int_0^x s^{\alpha} \, ds = \frac{x^{\alpha+1}}{\alpha+1} \quad (x \ge 0).$$
(1)

Remark 4. The function f is convex and f(0) = f'(0) = 0.

Remark 5. The following inequality holds true

$$0 \leqslant d(x) \leqslant f(x) \quad (x \geqslant 0).$$

Proof. Observe that

$$0 \leqslant d'(s) = s^{\alpha} w(s) \leqslant s^{\alpha} = f'(s) \quad (s > 0).$$

Integrating above inequality over the set [0, x] gives the claim.

Proposition 6. The function d is continuously differentiable and if $\alpha > 1$ then d''(0) = 0 (1).

Repository with the most recent version of this document: https://github.com/vil02/nowhere_convex ¹here we consider d''(0) as the limit $\lim_{h\to 0^+} \frac{d'(h)-d'(0)}{h}$.

Proof. Observe that $d'(x) = x^{\alpha}w(x)$ $(x \in [0, \infty))$. Moreover, we have

$$\frac{d'(h) - d'(0)}{h} = \frac{h^{\alpha}w(\alpha)}{h} \xrightarrow[h \to 0^{+}]{} 0,$$

for all $\alpha > 1$.

Proposition 7. The function d' is not differentiable for x > 0.

Proof. Suppose the contrary and consider the quotient

$$\frac{d'(x+h) - d'(x)}{h} = \frac{(x+h)^{\alpha} w(x+h) - x^{\alpha} w(x)}{h}
= \frac{(x+h)^{\alpha} w(x+h) - (x+h)^{\alpha} w(x) + (x+h)^{\alpha} w(x) - x^{\alpha} w(x)}{h}
= (x+h)^{\alpha} \frac{w(x+h) - w(x)}{h} + w(x) \frac{(x+h)^{\alpha} - x^{\alpha}}{h}$$
(2)

By our assumption the quotient of the left hand side of (2) converges to d''(x) as $h \to 0$. By the continuity of the function w and and differentiability of the function $(0, \infty) \ni s \longmapsto s^{\alpha} \in (0, \infty)$, we get that the limit $\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$ exists. Therefore the function w is differentiable at point x. Contradiction.

Proposition 8. The function d is strictly increasing.

Proof. Clearly $d'(x) = x^{\alpha}w(x) \ge 0 \ (x \ge 0)$, therefore

$$d(a) - d(b) = \int_{b}^{a} d'(s) \, \mathrm{d}s \geqslant 0 \quad (a \geqslant b \geqslant 0) \, .$$

Suppose that there exist some points $a > b \ge 0$, such that

$$0 = d(a) - d(b) = \int_{b}^{a} d'(s) ds.$$

Since $d' \ge 0$, we have that d'(s) = 0 for almost all $s \in [a, b]$. By the continuity of the function d', we have that d'(s) = 0 ($s \in [a, b]$). Hence, the function d' is differentiable on the interval (a, b), which gives a contradiction (cf. Proposition 7).

Proposition 9. The function d is not convex on any of the intervals $I \subseteq [0, \infty)$.

Proof. Suppose that there exists an interval $I \subseteq [0, \infty)$ such that the function $d_{|I|}$ is convex. Theorem 2 implies that the function d' is monotone increasing. Therefore, by Theorem 2 the function d' is differentiable almost everywhere (with respect to Leagues measure) on I. This gives the contradiction together with Proposition 7.

Remark 10. Let the functions d and f be as above (cf. formula (1)) and let b > 0. Consider functions $\widetilde{d}, \widetilde{f} : [0, b] \to [0, \infty)$ defined as follows:

$$\widetilde{d}(x) = d(b - x) \quad (x \in [0, b]),$$

$$\widetilde{f}(x) = f(b - x) \quad (x \in [0, b]).$$

Then

- $0 \leqslant \widetilde{d}(x) \leqslant \widetilde{f}(x)$ for all $x \in [0, b]$ (cf. Remark 5),
- the function \widetilde{f} is convex on the interval [0,b] and $\widetilde{f}(0)=\widetilde{f}'(b)=0$ (cf. Remark 4),
- the function \widetilde{d} is of the class C^1 on [0,b] (cf. Proposition 6),
- the function \widetilde{d} is strictly decreasing on the interval [0, b] (cf. Proposition 8),
- the function \widetilde{d} is not convex on any interval $I \subseteq [0,b]$ (cf. Proposition 9).

References

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