In this note we will construct (cf. formulas (1)) an example of a non-negative, strictly increasing, continuously differentiable function $d\colon [0,\infty) \to [0,\infty)$, bounded from above by a convex function $f\colon [0,\infty) \to [0,\infty)$, $f(0) = 0$, such that the function d is not convex on any interval $I \subseteq [0,\infty)$. Such considerations are interesting from the point of view of [2, p. 167]. It is (implicitly) stated there, that if a decreasing function $\widetilde{d} \geqslant 0$ is bounded from above by a convex function $\widetilde{f}$ vanishing at some point $b > 0$, then the function $\widetilde{d}$ is convex on some interval arbitrary close to b. After considering the mappings

$$d(x) = \widetilde{d}(b-x)\,,\ f(x) = \widetilde{f}(b-x) \quad (x \in [0,b])\,,$$

one sees that instead of decreasing function $\widetilde{d}$, we can consider increasing function d.

We will use the following results

Theorem 1. *There exists a continuous, bounded and nowhere differentiable function $w\colon \mathbb{R} \to \mathbb{R}$.*

Example of such function can be found in [1, Example 8, p. 38].

Theorem 2. *If the function g is convex in an interval (a,b), then g is continuous in (a,b). Moreover, g' exists except at most in a countable set and is monotone increasing.*

The proof of above Theorem can be found in [3, Theorem 7.40, p. 120].

Theorem 3. *Let $g\colon (a,b) \to \mathbb{R}$ be monotone increasing. Then the function g has a measurable, non-negative derivative g' almost everywhere in (a,b).*

For the proof of the above we refer to [3, Theorem 7.21, page 111].

Consider $\alpha \geqslant 0$ and let $w\colon \mathbb{R} \to [0,1]$ be a continuous, nowhere differentiable function. Define the functions $d, f\colon [0,\infty) \to \mathbb{R}$ as follows

$$\begin{aligned} d(x) &= \int_0^x s^\alpha w(s)\ \mathrm{d}s \quad (x \geqslant 0)\,, \\ f(x) &= \int_0^x s^\alpha\ \mathrm{d}s = \frac{x^{\alpha+1}}{\alpha+1} \quad (x \geqslant 0)\,. \end{aligned} \tag{1}$$

Remark 4. The function f is convex and $f(0) = f'(0) = 0$.

Remark 5. The following inequality holds true

$$0 \leqslant d(x) \leqslant f(x) \quad (x \geqslant 0)\,.$$

Proof. Observe that

$$0 \leqslant d'(s) = s^\alpha w(s) \leqslant s^\alpha = f'(s) \quad (s > 0)\,.$$

Integrating above inequality over the set $[0,x]$ gives the claim. □

Proposition 6. *The function d is continuously differentiable and if $\alpha > 1$ then $d''(0) = 0$ ([1]).*

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[1]here we consider $d''(0)$ as the limit $\lim_{h\to 0^+} \frac{d'(h)-d'(0)}{h}$.

Proof. Observe that $d'(x) = x^\alpha w(x)\,(x \in [0,\infty))$. Moreover, we have

$$\frac{d'(h) - d'(0)}{h} = \frac{h^\alpha w(\alpha)}{h} \xrightarrow[h\to 0^+]{} 0,$$

for all $\alpha > 1$. ☐

Proposition 7. *The function d' is not differentiable for $x > 0$.*

Proof. Suppose the contrary and consider the quotient

$$\begin{aligned}
\frac{d'(x+h) - d'(x)}{h} &= \frac{(x+h)^\alpha\, w(x+h) - x^\alpha w(x)}{h} \\
&= \frac{(x+h)^\alpha\, w(x+h) - (x+h)^\alpha\, w(x) + (x+h)^\alpha\, w(x) - x^\alpha w(x)}{h} \\
&= (x+h)^\alpha\, \frac{w(x+h) - w(x)}{h} + w(x)\, \frac{(x+h)^\alpha - x^\alpha}{h}
\end{aligned} \tag{2}$$

By our assumption the quotient of the left hand side of (2) converges to $d''(x)$ as $h \to 0$. By the continuity of the function w and and differentiability of the function $(0,\infty) \ni s \longmapsto s^\alpha \in (0,\infty)$, we get that the limit $\lim_{h\to 0} \frac{w(x+h)-w(x)}{h}$ exists. Therefore the function w is differentiable at point x. Contradiction. ☐

Proposition 8. *The function d is strictly increasing.*

Proof. Clearly $d'(x) = x^\alpha w(x) \geqslant 0\,(x \geqslant 0)$, therefore

$$d(a) - d(b) = \int_b^a d'(s)\ \mathrm{d}s \geqslant 0 \quad (a \geqslant b \geqslant 0)\,.$$

Suppose that there exist some points $a > b \geqslant 0$, such that

$$0 = d(a) - d(b) = \int_b^a d'(s)\ \mathrm{d}s.$$

Since $d' \geqslant 0$, we have that $d'(s) = 0$ for almost all $s \in [a,b]$. By the continuity of the function d', we have that $d'(s) = 0\ (s \in [a,b])$. Hence, the function d' is differentiable on the interval (a,b), which gives a contradiction (cf. Proposition 7). ☐

Proposition 9. *The function d is not convex on any of the intervals $I \subseteq [0,\infty)$.*

Proof. Suppose that there exists an interval $I \subseteq [0,\infty)$ such that the function $d_{|I}$ is convex. Theorem 2 implies that the function d' is monotone increasing. Therefore, by Theorem 2 the function d' is differentiable almost everywhere (with respect to Leagues measure) on I. This gives the contradiction together with Proposition 7. ☐

Remark 10. Let the functions d and f be as above (cf. formula (1)) and let $b > 0$. Consider functions $\widetilde{d}, \widetilde{f} \colon [0,b] \to [0,\infty)$ defined as follows:

$$\begin{aligned}
\widetilde{d}(x) &= d(b-x) \quad (x \in [0,b])\,, \\
\widetilde{f}(x) &= f(b-x) \quad (x \in [0,b])\,.
\end{aligned}$$

Then

- $0 \leqslant \widetilde{d}(x) \leqslant \widetilde{f}(x)$ for all $x \in [0,b]$ (cf. Remark 5),
- the function $\widetilde{f}$ is convex on the interval $[0,b]$ and $\widetilde{f}(0) = \widetilde{f}'(b) = 0$ (cf. Remark 4),
- the function $\widetilde{d}$ is of the class $\mathcal{C}^1$ on $[0,b]$ (cf. Proposition 6),
- the function $\widetilde{d}$ is strictly decreasing on the interval $[0,b]$ (cf. Proposition 8),
- the function $\widetilde{d}$ is not convex on any interval $I \subseteq [0,b]$ (cf. Proposition 9).

References

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