

In this note we will construct (cf. formulas (1)) an example of a non-negative, strictly increasing, continuously differentiable function  $d: [0, \infty) \rightarrow [0, \infty)$ , bounded from above by a convex function  $f: [0, \infty) \rightarrow [0, \infty)$ ,  $f(0) = 0$ , such that the function  $d$  is not convex on any interval  $I \subseteq [0, \infty)$ . Such considerations are interesting from the point of view of [2, p. 167]. It is (implicitly) stated there, that if a decreasing function  $\tilde{d} \geq 0$  is bounded from above by a convex function  $\tilde{f}$  vanishing at some point  $b > 0$ , then the function  $\tilde{d}$  is convex on some interval arbitrary close to  $b$ . After considering the mappings

$$d(x) = \tilde{d}(b - x), \quad f(x) = \tilde{f}(b - x) \quad (x \in [0, b]),$$

one sees that instead of decreasing function  $\tilde{d}$ , we can consider increasing function  $d$ .

We will use the following results

**Theorem 1.** *There exists a continuous, bounded and nowhere differentiable function  $w: \mathbb{R} \rightarrow \mathbb{R}$ .*

Example of such function can be found in [1, Example 8, p. 38].

**Theorem 2.** *If the function  $g$  is convex in an interval  $(a, b)$ , then  $g$  is continuous in  $(a, b)$ . Moreover,  $g'$  exists except at most in a countable set and is monotone increasing.*

The proof of above Theorem can be found in [3, Theorem 7.40, p. 120].

**Theorem 3.** *Let  $g: (a, b) \rightarrow \mathbb{R}$  be monotone increasing. Then the function  $g$  has a measurable, non-negative derivative  $g'$  almost everywhere in  $(a, b)$ .*

For the proof of the above we refer to [3, Theorem 7.21, page 111].

Consider  $\alpha \geq 0$  and let  $w: \mathbb{R} \rightarrow [0, 1]$  be a continuous, nowhere differentiable function. Define the functions  $d, f: [0, \infty) \rightarrow \mathbb{R}$  as follows

$$\begin{aligned} d(x) &= \int_0^x s^\alpha w(s) \, ds \quad (x \geq 0), \\ f(x) &= \int_0^x s^\alpha \, ds = \frac{x^{\alpha+1}}{\alpha+1} \quad (x \geq 0). \end{aligned} \tag{1}$$

*Remark 4.* The function  $f$  is convex and  $f(0) = f'(0) = 0$ .

*Remark 5.* The following inequality holds true

$$0 \leq d(x) \leq f(x) \quad (x \geq 0).$$

*Proof.* Observe that

$$0 \leq d'(s) = s^\alpha w(s) \leq s^\alpha = f'(s) \quad (s > 0).$$

Integrating above inequality over the set  $[0, x]$  gives the claim.  $\square$

**Proposition 6.** *The function  $d$  is continuously differentiable and if  $\alpha > 1$  then  $d''(0) = 0$  <sup>(1)</sup>.*

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<sup>1</sup>here we consider  $d''(0)$  as the limit  $\lim_{h \rightarrow 0+} \frac{d'(h) - d'(0)}{h}$ .

*Proof.* Observe that  $d'(x) = x^\alpha w(x)$  ( $x \in [0, \infty)$ ). Moreover, we have

$$\frac{d'(h) - d'(0)}{h} = \frac{h^\alpha w(\alpha)}{h} \xrightarrow{h \rightarrow 0^+} 0,$$

for all  $\alpha > 1$ . □

**Proposition 7.** *The function  $d'$  is not differentiable for  $x > 0$ .*

*Proof.* Suppose the contrary and consider the quotient

$$\begin{aligned} \frac{d'(x+h) - d'(x)}{h} &= \frac{(x+h)^\alpha w(x+h) - x^\alpha w(x)}{h} \\ &= \frac{(x+h)^\alpha w(x+h) - (x+h)^\alpha w(x) + (x+h)^\alpha w(x) - x^\alpha w(x)}{h} \\ &= (x+h)^\alpha \frac{w(x+h) - w(x)}{h} + w(x) \frac{(x+h)^\alpha - x^\alpha}{h} \end{aligned} \quad (2)$$

By our assumption the quotient of the left hand side of (2) converges to  $d''(x)$  as  $h \rightarrow 0$ . By the continuity of the function  $w$  and differentiability of the function  $(0, \infty) \ni s \mapsto s^\alpha \in (0, \infty)$ , we get that the limit  $\lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h}$  exists. Therefore the function  $w$  is differentiable at point  $x$ . Contradiction. □

**Proposition 8.** *The function  $d$  is strictly increasing.*

*Proof.* Clearly  $d'(x) = x^\alpha w(x) \geq 0$  ( $x \geq 0$ ), therefore

$$d(a) - d(b) = \int_b^a d'(s) \, ds \geq 0 \quad (a \geq b \geq 0).$$

Suppose that there exist some points  $a > b \geq 0$ , such that

$$0 = d(a) - d(b) = \int_b^a d'(s) \, ds.$$

Since  $d' \geq 0$ , we have that  $d'(s) = 0$  for almost all  $s \in [a, b]$ . By the continuity of the function  $d'$ , we have that  $d'(s) = 0$  ( $s \in [a, b]$ ). Hence, the function  $d'$  is differentiable on the interval  $(a, b)$ , which gives a contradiction (cf. Proposition 7). □

**Proposition 9.** *The function  $d$  is not convex on any of the intervals  $I \subseteq [0, \infty)$ .*

*Proof.* Suppose that there exists an interval  $I \subseteq [0, \infty)$  such that the function  $d|_I$  is convex. Theorem 2 implies that the function  $d'$  is monotone increasing. Therefore, by Theorem 2 the function  $d'$  is differentiable almost everywhere (with respect to Leagues measure) on  $I$ . This gives the contradiction together with Proposition 7. □

*Remark 10.* Let the functions  $d$  and  $f$  be as above (cf. formula (1)) and let  $b > 0$ . Consider functions  $\tilde{d}, \tilde{f}: [0, b] \rightarrow [0, \infty)$  defined as follows:

$$\begin{aligned} \tilde{d}(x) &= d(b-x) \quad (x \in [0, b]), \\ \tilde{f}(x) &= f(b-x) \quad (x \in [0, b]). \end{aligned}$$

Then

- $0 \leq \tilde{d}(x) \leq \tilde{f}(x)$  for all  $x \in [0, b]$  (cf. Remark 5),
- the function  $\tilde{f}$  is convex on the interval  $[0, b]$  and  $\tilde{f}(0) = \tilde{f}(b) = 0$  (cf. Remark 4),
- the function  $\tilde{d}$  is of the class  $\mathcal{C}^1$  on  $[0, b]$  (cf. Proposition 6),
- the function  $\tilde{d}$  is strictly decreasing on the interval  $[0, b]$  (cf. Proposition 8),
- the function  $\tilde{d}$  is not convex on any interval  $I \subseteq [0, b]$  (cf. Proposition 9).

## References

- [1] John M. H. Olmsted Bernard R. Gelbaum. *Counterexamples in Analysis*. Mathesis series. Holden-Day, 1964.
- [2] Steven Levandosky. Stability and instability of fourth-order solitary waves. *Journal of Dynamics and Differential Equations*, 10(1):151–188, Jan 1998.
- [3] Antoni Zygmund Richard L. Wheeden. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.