In this note we will construct (cf. formulas (1)) an example of a positive, strictly increasing, continuously differentiable function $d\colon [0,\infty) \to [0,\infty)$, bounded from above by a convex function $f\colon [0,\infty) \to [0,\infty)$, $f(0) = 0$, such that the function d is not convex on any interval $I \subseteq [0,\infty)$. Such considerations are interesting from the point of view of [2, p. 167]. It is (implicitly) stated there, that if a decreasing function $\widetilde{d} \geqslant 0$ is bounded from above by a convex function $\widetilde{f}$ vanishing at some point $b > 0$, then the function $\widetilde{d}$ is convex on some interval arbitrary close to b. After considering the mappings

$$d(x) = \widetilde{d}(b - x), \ f(x) = \widetilde{f}(b - x) \quad (x \in [0, b]),$$

one sees that instead of decreasing function $\widetilde{d}$, we can consider increasing function d.

We will use the following results

Theorem 1. *There exists a continuous, bounded and nowhere differentiable function $w\colon \mathbb{R} \to \mathbb{R}$.*

Example of such function can be found in [1, Example 8, p. 38].

Theorem 2. *If the function g is convex in an interval (a, b), then g is continuous in (a, b). Moreover, g' exists except at most in a countable set and is monotone increasing.*

The proof of above Theorem can be found in [3, Theorem 7.40, p. 120].

Theorem 3. *Let $g\colon (a, b) \to \mathbb{R}$ be monotone increasing. Then the function g has a measurable, non-negative derivative g' almost everywhere in (a, b).*

For the proof of the above we refer to [3, Theorem 7.21, page 111].

Consider $\alpha \geqslant 0$ and let $w\colon \mathbb{R} \to [0, 1]$ be a continuous, nowhere differentiable function. Define the functions $d, f\colon [0, \infty) \to \mathbb{R}$ as follows

$$d(x) = \int_0^x s^\alpha w(s) \, \mathrm{d}s \quad (x \geqslant 0),$$

$$f(x) = \int_0^x s^\alpha \, \mathrm{d}s = \frac{x^{\alpha+1}}{\alpha + 1} \quad (x \geqslant 0).$$

(1)

Remark 4. The function f is convex and $f(0) = f'(0) = 0$.

Remark 5. The following inequality holds true

$$0 \leqslant d(x) \leqslant f(x) \quad (x \geqslant 0).$$

Proof. Observe that

$$0 \leqslant d'(s) = s^\alpha w(s) \leqslant s^\alpha = f'(s) \quad (s > 0).$$

Integrating above inequality over the set $[0, x]$ gives the claim. $\square$

Proposition 6. *The function d is continuously differentiable and if $\alpha > 1$ then $d''(0) = 0$ ([1]).*

Proof. Observe that $d'(x) = x^\alpha w(x) \, (x \in [0, \infty))$. Moreover, we have

$$\frac{d'(h) - d'(0)}{h} = \frac{h^\alpha w(\alpha)}{h} \xrightarrow[h \to 0^+]{} 0,$$

for all $\alpha > 1$. $\square$

---

[1] here we consider $d''(0)$ as the limit $\lim_{h \to 0^+} \frac{d'(h) - d'(0)}{h}$.

Proposition 7. *The function d' is not differentiable for $x > 0$.*

Proof. Suppose the contrary and consider the quotient

$$
\begin{aligned}
\frac{d'(x+h) - d'(x)}{h} &= \frac{(x+h)^\alpha w(x+h) - x^\alpha w(x)}{h} \\
&= \frac{(x+h)^\alpha w(x+h) - (x+h)^\alpha w(x) + (x+h)^\alpha w(x) - x^\alpha w(x)}{h} \qquad (2) \\
&= (x+h)^\alpha \frac{w(x+h) - w(x)}{h} + w(x) \frac{(x+h)^\alpha - x^\alpha}{h}
\end{aligned}
$$

By our assumption the quotient of the left hand side of (2) converges to $d''(x)$ as $h \to 0$. By the continuity of the function w and and differentiability of the function $(0,\infty) \ni s \longmapsto s^\alpha \in (0,\infty)$, we get that the limit $\lim_{h\to 0} \frac{w(x+h)-w(x)}{h}$ exists. Therefore the function w is differentiable at point x. Contradiction. $\qquad\square$

Proposition 8. *The function d is strictly increasing.*

Proof. Clearly $d'(x) = x^\alpha w(x) \geqslant 0 \, (x \geqslant 0)$, therefore

$$
d(a) - d(b) = \int_b^a d'(s) \, \mathrm{d}s \geqslant 0 \quad (a \geqslant b \geqslant 0).
$$

Suppose that there exist some points $a > b \geqslant 0$, such that

$$
0 = d(a) - d(b) = \int_b^a d'(s) \, \mathrm{d}s.
$$

Since $d' \geqslant 0$, we have that $d'(s) = 0$ for almost all $s \in [a,b]$. By the continuity of the function d', we have that $d'(s) = 0 \, (s \in [a,b])$. Hence, the function d' is differentiable on the interval (a,b), which gives a contradiction (cf. Proposition 7). $\qquad\square$

Proposition 9. *The function d is not convex on any of the intervals $I \subseteq [0,\infty)$.*

Proof. Suppose that there exists an interval $I \subseteq [0,\infty)$ such that the function $d_{|I}$ is convex. Theorem 2 implies that the function d' is monotone increasing. Therefore, by Theorem 2 the function d' is differentiable almost everywhere (with respect to Leagues measure) on I. This gives the contradiction together with Proposition 7. $\qquad\square$

Remark 10. Let the functions d and f be as above (cf. formula (1)) and let $b > 0$. Consider functions $\widetilde{d}, \widetilde{f} \colon [0,b] \to [0,\infty)$ defined as follows:

$$
\begin{aligned}
\widetilde{d}(x) &= d(b-x) \quad (x \in [0,b]), \\
\widetilde{f}(x) &= f(b-x) \quad (x \in [0,b]).
\end{aligned}
$$

Then

- $0 \leqslant \widetilde{d}(x) \leqslant \widetilde{f}(x)$ for all $x \in [0,b]$ (cf. Remark 5),

- the function $\widetilde{f}$ is convex on the interval $[0,b]$ and $\widetilde{f}(0) = \widetilde{f}'(b) = 0$ (cf. Remark 4),

- the function $\widetilde{d}$ is of the class $\mathcal{C}^1$ on $[0,b]$ (cf. Proposition 6),

- the function $\widetilde{d}$ is strictly decreasing on the interval $[0,b]$ (cf. Proposition 8),

- the function $\widetilde{d}$ is not convex on any interval $I \subseteq [0,b]$ (cf. Proposition 9).

References

[1] John M. H. Olmsted Bernard R. Gelbaum. *Counterexamples in Analysis*. Mathesis series. Holden-Day, 1964.

[2] Steven Levandosky. Stability and instability of fourth-order solitary waves. *Journal of Dynamics and Differential Equations*, 10(1):151–188, Jan 1998.

[3] Antoni Zygmund Richard L. Wheeden. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.