

In this note we will construct (cf. formulas (1)) an example of a non-negative, strictly increasing, continuously differentiable function $d: [0, \infty) \rightarrow [0, \infty)$, bounded from above by a convex function $f: [0, \infty) \rightarrow [0, \infty)$, $f(0) = 0$, such that the function d is not convex on any interval $I \subseteq [0, \infty)$. Such considerations are interesting from the point of view of [2, p. 167]. It is (implicitly) stated there, that if a decreasing function $\tilde{d} \geq 0$ is bounded from above by a convex function \tilde{f} vanishing at some point $b > 0$, then the function \tilde{d} is convex on some interval arbitrary close to b . After considering the mappings

$$d(x) = \tilde{d}(b - x), \quad f(x) = \tilde{f}(b - x) \quad (x \in [0, b]),$$

one sees that instead of decreasing function \tilde{d} , we can consider increasing function d .

We will use the following results

Theorem 1. *There exists a continuous, bounded and nowhere differentiable function $w: \mathbb{R} \rightarrow \mathbb{R}$.*

Example of such function can be found in [1, Example 8, p. 38].

Theorem 2. *If the function g is convex in an interval (a, b) , then g is continuous in (a, b) . Moreover, g' exists except at most in a countable set and is monotone increasing.*

The proof of above Theorem can be found in [3, Theorem 7.40, p. 120].

Theorem 3. *Let $g: (a, b) \rightarrow \mathbb{R}$ be monotone increasing. Then the function g has a measurable, non-negative derivative g' almost everywhere in (a, b) .*

For the proof of the above we refer to [3, Theorem 7.21, page 111].

Consider $\alpha \geq 0$ and let $w: \mathbb{R} \rightarrow [0, 1]$ be a continuous, nowhere differentiable function. Define the functions $d, f: [0, \infty) \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} d(x) &= \int_0^x s^\alpha w(s) \, ds \quad (x \geq 0), \\ f(x) &= \int_0^x s^\alpha \, ds = \frac{x^{\alpha+1}}{\alpha+1} \quad (x \geq 0). \end{aligned} \tag{1}$$

Remark 4. The function f is convex and $f(0) = f'(0) = 0$.

Remark 5. The following inequality holds true

$$0 \leq d(x) \leq f(x) \quad (x \geq 0).$$

Proof. Observe that

$$0 \leq d'(s) = s^\alpha w(s) \leq s^\alpha = f'(s) \quad (s > 0).$$

Integrating above inequality over the set $[0, x]$ gives the claim. \square

Proposition 6. *The function d is continuously differentiable and if $\alpha > 1$ then $d''(0) = 0$ ⁽¹⁾.*

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¹here we consider $d''(0)$ as the limit $\lim_{h \rightarrow 0+} \frac{d'(h) - d'(0)}{h}$.

Proof. Observe that $d'(x) = x^\alpha w(x)$ ($x \in [0, \infty)$). Moreover, we have

$$\frac{d'(h) - d'(0)}{h} = \frac{h^\alpha w(\alpha)}{h} \xrightarrow{h \rightarrow 0^+} 0,$$

for all $\alpha > 1$. □

Proposition 7. *The function d' is not differentiable for $x > 0$.*

Proof. Suppose the contrary and consider the quotient

$$\begin{aligned} \frac{d'(x+h) - d'(x)}{h} &= \frac{(x+h)^\alpha w(x+h) - x^\alpha w(x)}{h} \\ &= \frac{(x+h)^\alpha w(x+h) - (x+h)^\alpha w(x) + (x+h)^\alpha w(x) - x^\alpha w(x)}{h} \\ &= (x+h)^\alpha \frac{w(x+h) - w(x)}{h} + w(x) \frac{(x+h)^\alpha - x^\alpha}{h} \end{aligned} \quad (2)$$

By our assumption the quotient of the left hand side of (2) converges to $d''(x)$ as $h \rightarrow 0$. By the continuity of the function w and differentiability of the function $(0, \infty) \ni s \mapsto s^\alpha \in (0, \infty)$, we get that the limit $\lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h}$ exists. Therefore the function w is differentiable at point x . Contradiction. □

Proposition 8. *The function d is strictly increasing.*

Proof. Clearly $d'(x) = x^\alpha w(x) \geq 0$ ($x \geq 0$), therefore

$$d(a) - d(b) = \int_b^a d'(s) \, ds \geq 0 \quad (a \geq b \geq 0).$$

Suppose that there exist some points $a > b \geq 0$, such that

$$0 = d(a) - d(b) = \int_b^a d'(s) \, ds.$$

Since $d' \geq 0$, we have that $d'(s) = 0$ for almost all $s \in [a, b]$. By the continuity of the function d' , we have that $d'(s) = 0$ ($s \in [a, b]$). Hence, the function d' is differentiable on the interval (a, b) , which gives a contradiction (cf. Proposition 7). □

Proposition 9. *The function d is not convex on any of the intervals $I \subseteq [0, \infty)$.*

Proof. Suppose that there exists an interval $I \subseteq [0, \infty)$ such that the function $d|_I$ is convex. Theorem 2 implies that the function d' is monotone increasing. Therefore, by Theorem 2 the function d' is differentiable almost everywhere (with respect to Lebesgue measure) on I . This gives the contradiction together with Proposition 7. □

Remark 10. Let the functions d and f be as above (cf. formula (1)) and let $b > 0$. Consider functions $\tilde{d}, \tilde{f}: [0, b] \rightarrow [0, \infty)$ defined as follows:

$$\begin{aligned} \tilde{d}(x) &= d(b-x) \quad (x \in [0, b]), \\ \tilde{f}(x) &= f(b-x) \quad (x \in [0, b]). \end{aligned}$$

Then

- $0 \leq \tilde{d}(x) \leq \tilde{f}(x)$ for all $x \in [0, b]$ (cf. Remark 5),
- the function \tilde{f} is convex on the interval $[0, b]$ and $\tilde{f}(0) = \tilde{f}(b) = 0$ (cf. Remark 4),
- the function \tilde{d} is of the class \mathcal{C}^1 on $[0, b]$ (cf. Proposition 6),
- the function \tilde{d} is strictly decreasing on the interval $[0, b]$ (cf. Proposition 8),
- the function \tilde{d} is not convex on any interval $I \subseteq [0, b]$ (cf. Proposition 9).

References

- [1] John M. H. Olmsted Bernard R. Gelbaum. *Counterexamples in Analysis*. Mathesis series. Holden-Day, 1964.
- [2] Steven Levandosky. Stability and instability of fourth-order solitary waves. *Journal of Dynamics and Differential Equations*, 10(1):151–188, Jan 1998.
- [3] Antoni Zygmund Richard L. Wheeden. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.