Thèse de Doctorat

Université de Limoges
ED 653 – Sciences et Ingénierie (SI)
Faculté des Sciences et Techniques – Institut de Recherche XLIM

Thèse pour obtenir le grade de
Docteur de l'Université de Limoges
Section 25 - Mathématiques

Présentée et soutenue par

Lucas LEGRAND

Le 16/09/2025

Gröbner bases over polyhedral algebras

Thèse dirigée par Moulay BARKATOU et Tristan VACCON

JURY :

Président du jury
M. Frédéric CHYZAK – Directeur de Recherche – MATHEXP, Centre Inria de Saclay

Rapporteurs
M. Xavier CARUSO – Directeur de Recherche – CNRS-IMB, Université de Bordeaux
M. Frédéric CHYZAK – Directeur de Recherche – MATHEXP, Centre Inria de Saclay

Examinateurs
Mme Mercedes HAIECH – Maîtresse de Conférences – XLIM, Université de Limoges
M. Bernard LE STUM – Maitre de Conférences – IRMAR, Université de Rennes
Mme Fatemeh MOHAMMADI – Professeure – Department of Mathematics, KU Leuven

Remerciements

Il y aurait tant de personnes à remercier, mais, à l'image de ce manuscrit, je souhaite aller à l'essentiel.

Je tiens à exprimer ma profonde gratitude à mes directeurs de thèse, M. Moulay Barkatou et M. Tristan Vaccon, pour la confiance qu'ils m'ont accordée dès le début, ainsi que pour leur accompagnement et leur soutien tout au long de cette aventure, qui a abouti à la publication de deux articles de recherche et à la rédaction de ce manuscrit.

Je remercie chaleureusement M. Xavier Caruso et M. Frédéric Chyzak d'avoir accepté de rapporter cette thèse et de faire partie du jury. Leurs remarques, conseils et rapports ont été d'une grande valeur.

Je remercie également Mme Mercedes Haiech, Mme Fatemeh Mohammadi et M. Bernard Le Stum d'avoir accepté de faire partie du jury.

Enfin, je remercie mes parents pour leur soutien inconditionnel, tout au long de la préparation de cette thèse et bien avant.

Table of Contents

0 Introduction

In this thesis, we present a Gröbner bases theory for polyhedral algebras. These are algebras of power series satisfying convergence constraints given by a polyhedron. They are central objects in the field of tropical analytic geometry. This work contributes to a sequence of modifications and extensions towards tropical and rigid geometry of the original Gröbner theory for polynomial rings introduced by Buchberger in his 1965 thesis ([Buc06]).

Recall the basic principle behind Gröbner theory in polynomial rings. We choose a term order that allows us to define the leading monomial of a polynomial, and likewise, the leading ideal of an ideal. Not every finite generating set of an ideal has the property that the leading monomials of its generators generate the leading ideal. But some do — these are called Gröbner bases. Given any generating set, the Buchberger algorithm transforms the set into a Gröbner basis by adding new generators until the leading ideals coincide. When we have a Gröbner basis, we can perform many effective computations: membership tests, elimination of variables, and more. The reader can consult the excellent book [CLO15] for a primer on this topic.

The core result of this thesis is the realization for polyhedral algebras of the same foundational program as presented above: define a way to order terms, specify what a Gröbner basis is, and provide a Buchberger algorithm that transforms any generating set of an ideal into a Gröbner basis as previously defined. We show that our theory is effective and present an explicit computation of a Gröbner basis using SageMath. Of course, there are challenges behind such a program, because polyhedral algebras are algebras of power series with convergence constraints rather than polynomial rings.

Before stating our results and how they are organized in this manuscript, let us clarify the prior advancements in Gröbner theory upon which our contributions are built, introduce in more detail polyhedral algebras, and explain why it is interesting to develop a Gröbner theory for them.

Recall from Gröbner theory in polynomial rings that for an ideal $I \subseteq K[x_1, \ldots, x_n]$ and a weight vector $w \in \mathbb{R}^n$, the initial ideal of I with respect to w is defined as

$$\mathrm{in}_w(I) := \{\mathrm{in}_w(f) \mid f \in I\}$$

where $\mathrm{in}_w(f)$ is the w-initial part of the polynomial f. For K a valued field, this definition is adjusted to incorporate the valuation of the coefficient. For a polynomial $f = \sum_{u \in \mathbb{N}^n} c_u x^u$, it is defined as:

$$\mathrm{in}_w(f) := \sum_{\mathrm{val}(c_u) + w \cdot u = W} c_u x^u,$$

where $W = \min(\mathrm{val}(c_u) + w \cdot u : \ c_u \neq 0)$ [1]. This construction connects Gröbner bases with the zero locus of a polynomial or ideal in the torus $(\overline{K}^*)^n$, while recovering the previous definition when the valuation is trivial. It yields a simple one-way criterion: if $\mathrm{in}_w(f)$ is a monomial, then f has no zero

[1] See [MS15, Def. 2.4.2], where c_u is sent to the residue field $\Bbbk$ using a splitting of the valuation map, so $\mathrm{in}_w(f) \in \Bbbk[x_1, \ldots, x_n]$. If K is discretely valued, such a splitting always exists.

$(a_1, \ldots, a_n) \in (\overline{K}^*)^n$ satisfying $\mathrm{val}(a_i) = w_i$. Proving the converse and generalizing it to ideals is the focus of the early chapters of the introductory book on tropical geometry [MS15]. Topics include the existence of tropical bases, the fundamental theorem of tropical geometry, and the rich structure of tropicalizations as weighted polyhedral complexes. Part of these results relies on the prior construction of the Gröbner complex of an ideal for this modified initial ideal definition ([MS15, Theorem 2.5.3]). The authors stress the need for an effective Gröbner theory with Buchberger's algorithm in this context, especially for computational purposes.

This is done in the paper [CM18]. Given a weight vector, terms are compared by weight-valuation first ([CM18, Definition 2.3]), taking into account the coefficient (the smaller the valuation, the larger the term). A term order is used to break ties if needed. The main issue is that the standard normal form algorithm may not terminate under this ordering. A sequence that is strictly decreasing in the valuation-first order may fail to be strictly decreasing with respect to the tie-breaking term order. Thus, we cannot use well-ordering of term orders to guarantee termination as usual (see [CM18, Example 2.2] for an example). They address this issue by using a variant of Mora's weak normal form algorithm. Another strategy is to use linear algebra as in [Vac15] and [VVY21].

A strictly decreasing sequence for the val-first order also has an interesting property when the valuation image is a discrete subset of $\mathbb{R}$: its valuation must tend to $+\infty$, ensuring that the associated series converges in an appropriate metric space. For example, when the weight vector is zero, such series define exactly the elements of a Tate algebra. This naturally leads into rigid analytic geometry. This connection starts a series of papers beginning in 2019 with [CVV19], developing a Gröbner theory in Tate algebras [CVV20; CVV21; CVV22; VV23]. The authors systematically extend known results from polynomial Gröbner theory: FGLM, F4, signature-based algorithms, and more. Finally, in [VV23], they sketch an alternative to [MR20] for computing tropical varieties of polynomial ideals via their various completions. This paper also opens the door in [VV23, Section 6] to broader generalizations through polyhedral algebras (they restrict to polyhedral algebras with positive exponents, which we will study in Chapter 2).

Polyhedral algebras belong to rigid geometry as they are affinoid algebras (quotients of Tate algebras). They can be realized as power series algebras with convergence conditions coming from a given pointed polyhedron. This was first proven for polytopal algebras in [EKL06] (the defining polyhedron is a polytope), and later extended to the general case in [Rab12]. As affinoid algebras, they arise as an admissible open in the rigidification of the toric variety defined by the recession cone of the polyhedron. As noted in [Rab12], one could adopt the Berkovich space viewpoint, but Tate algebras are sufficient for our goals, especially since we focus on computational aspects. We work exclusively in the affine setting and do not consider global constructions.

They also belong to tropical geometry, since the zero locus of an ideal in a polyhedral algebra admits a tropicalization map whose image lies in the defining polyhedron (more precisely in some partial compactification of the polyhedron, see [Rab12, Section 5]). Most results of tropical geometry in polynomial rings extend to polyhedral algebras ([Rab12, Theorem 7.9], [Rab12, Lemma 8.4]), though not all. The existence of a tropical basis for an ideal has not yet been proved and was raised as an interesting open problem in [Rab12, Remark 8.8]. Precisely, quoting the author:

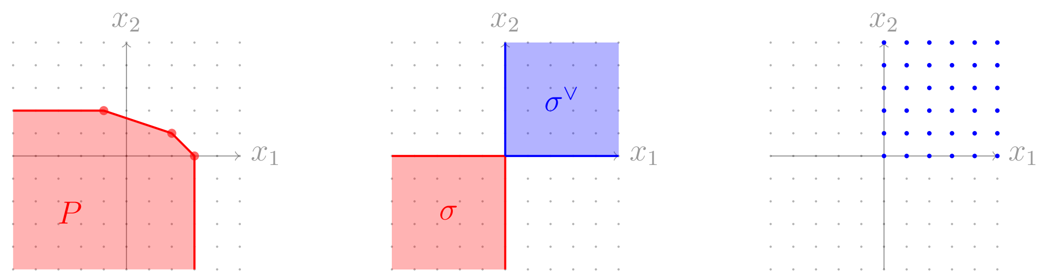
> The author would guess that the tropical basis theorem is true for closed subspaces of polyhedral domains. The proof in the algebraic case uses Gröbner theory, hence does not translate in an evident way to the analytic setting. This issue is certainly deserving of further study as such a theorem would form an important part of the foundations of a theory of tropical analytic geometry.

In fact, the existence of tropical bases for ideals in Laurent polynomial rings is established using Gröbner theory ([MS15, Theorem 2.6.6]), so it is natural to attempt to develop a similar framework in polyhedral algebras. The natural extension to polyhedral algebras of the recent Gröbner theory for Tate algebras developed in [CVV19; CVV20; CVV21; CVV22; VV23], together with this earlier open question, forms the main motivation behind our work. We do not resolve the problem, but only set up the basic foundations, which already pose significant challenges. We hope these results will contribute to further progress towards a positive resolution of the question.

Now from a very practical point of view, we will work in algebras of power series of the following form:

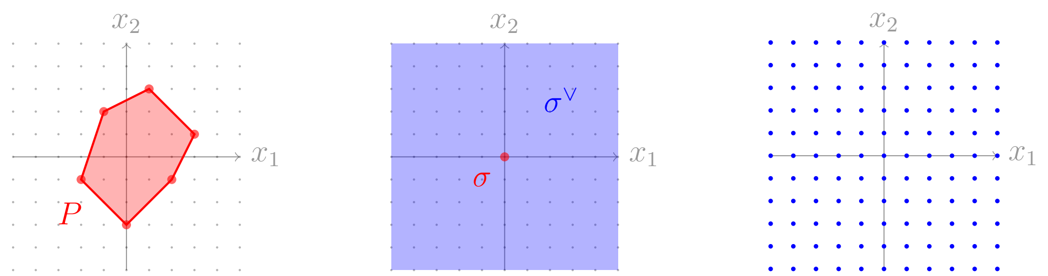
$$\sum_{\nu \in S_\sigma} a_\nu x^\nu \mid \forall r \in P,\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to +\infty$$

Here, P is a pointed polyhedron, and S_σ is a submonoid of $\mathbb{Z}^n$, defined as the integer monoid of the dual cone of the recession cone of P (definitions of dual and recession cones are recalled in Section 1.1.7). Graphically, this looks like:



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Figure 0.1: Example of polyhedral algebra for a non-bounded P



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Figure 0.2: Example of polyhedral algebra for P a polytope

The reader will often encounter this type of figure, where the left shows the defining polyhedron, the middle its recession cone and dual, and the right the corresponding exponent monoid. It summarizes in one go all the essential structural information about a particular type of polyhedral algebra.

If we compare this power series with those in Tate algebras, we have:

$\sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid \mathrm{val}(a_\nu) \to \infty$	$\sum_{\nu \in S_\sigma} a_\nu x^\nu \mid \forall r \in P,\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty$
Tate algebras	Polyhedral algebras

Notice from the above comparison that taking the exponent monoid $S_\sigma = \mathbb{N}^n$ and the polyhedron $P = \{(0, \ldots, 0)\}$ on the right-hand side recovers exactly the definition of series in Tate algebras on the left. In the other direction, moving from Tate to polyhedral algebras introduces two new difficulties.

The first, highlighted in blue, is that the exponent monoid is not generally $\mathbb{N}^n$. It may be non-pointed, such as $\mathbb{Z}^n$ in Figure 0.2, meaning it contains nontrivial units. This precludes the existence of standard term orders. This obstruction has already been studied for Laurent polynomial ideals in [PZ96] and [PU99]. We adapt their idea to the polyhedral setting, replacing term orders with generalized orders when it is necessary to break ties between monomials. We refer to this issue as the exponent problem. Notice that when working with Laurent polynomials, one can still perform positive homogenization. With power series, such homogenization is not possible because the exponents in a series can be arbitrarily large in norm.

The second, highlighted in red, is that the polyhedron may have more than one vertex, leading to multiple independent convergence conditions. As a result, the infimum valuation is not multiplicative, but only submultiplicative, which is a general property of affinoid algebras. This prevents the use of naive valuation-first orderings as in Tate algebras. We refer to this issue as the valuation problem. As far as we know, this is the first instance in the literature where terms are ordered using a submultiplicative valuation.

With that in mind, our strategy to extend Gröbner theory to polyhedral algebras is to tackle one problem at a time by working on specific types of algebras where only one of the two issues is active, the other being neutralized. We then combine the different results to handle the general case. Notice that in one case, we need to find solutions for a new problem (the valuation problem), while in the other, we need to incorporate an existing solution for our problem (the exponent problem) into a different framework. The two main diverging points from usual Gröbner theory that we will reach are the following. First, the monomials reached by series in an ideal must be split into good parts, each requiring a definition of its own leading monomial. Second, within such a part, the least common multiple of two leading monomials may not be unique. All of this will manifest in the final form of the Buchberger algorithm.

We will now give a more detailed summary of our results.

Contributions and organization of the thesis.

Chapter 1 - Settings. We fix some general notations and recall common mathematical knowledge. In particular, we briefly present the basic notions of polyhedral geometry needed later. We then present established material drawn from two different areas:

1. rigid geometry *à la* Tate and a particular type of affinoid algebra called polyhedral algebras
2. Gröbner theory in Tate algebras and in Laurent polynomial rings

In each topic, the exposition is minimal: we present only what is practically needed and do not develop the full general theory. The reader will find relevant references for further study in the chapter's introduction and in each section. The main references are:

1. Chapters 1 to 6 of Rabinoff's paper *Tropical analytic geometry, Newton polygons, and tropical intersections* ([Rab12]), the paper introducing polyhedral algebras
2. the first paper ([CVV19]) in the series on Gröbner bases in Tate algebras (at least Section 2)

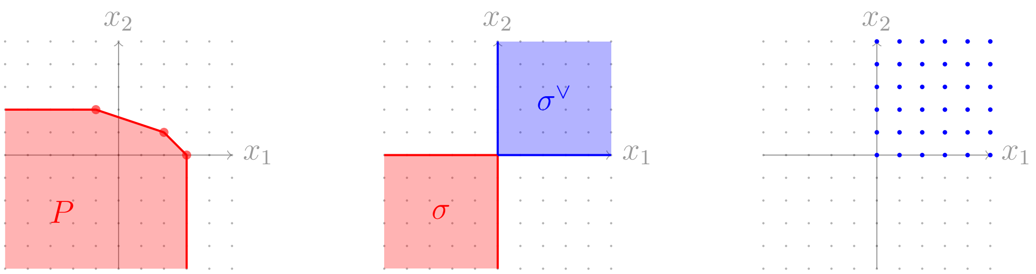
3. the second of the two papers ([PZ96] and [PU99]) on Gröbner bases in Laurent polynomial rings

The minimal practical prerequisites presented in this chapter and needed for the following ones are:

1. the definitions of Tate and polyhedral algebras as power series algebras with valuation maps
2. the basics of Gröbner theory in Tate algebras: the orderings used, the definition of Gröbner bases, and the corresponding Buchberger algorithm
3. the notion of generalized order on a monoid

The structure of polyhedral algebras given in Examples 1.2.2 to 1.2.7 is important and should be well understood.

Chapter 2 - Positive exponents. As the name suggests, we develop a Gröbner theory for polyhedral algebras whose defining polyhedron P is such that its monoid of exponents is $\mathbb{N}^n$. Graphically, we are in the following situation:



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Figure 0.3: Positive exponents

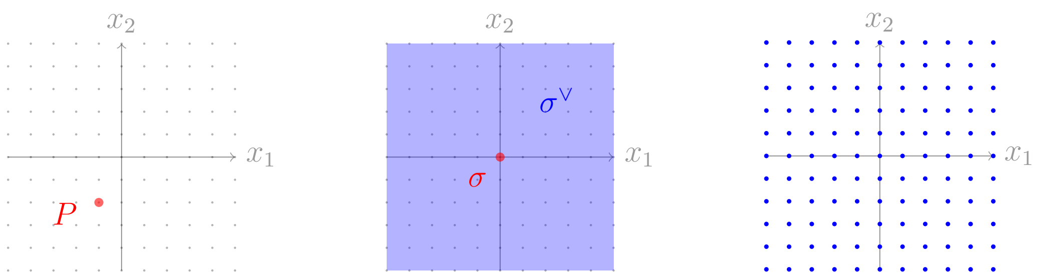
In this setting, the exponent problem disappears, and the only remaining issue is the submultiplicativity of the valuation. As a consequence of submultiplicativity, the natural definition of the leading monomial of a series f is not compatible with multiplication by monomials: we may find monomials m such that $\mathrm{lm}(m \times f) \neq m \times \mathrm{lm}(f)$.

We overcome this difficulty by introducing, in Definition 2.1.7, a new way of ordering terms using an intermediate breaking step involving what we call a *vertex ordering*. It is just an arbitrary fixed ordering of the (finitely many) vertices of the defining polyhedron P, used to break ties when the valuation of a term — which is the minimum over the valuations defined at each vertex (see Section 1.2) — is reached at more than one vertex. This extension requires decomposing the monoid of monomials into separate regions, one for each vertex. In each region, the global submultiplicative valuation acts like the local multiplicative valuation associated with the corresponding vertex. This calls for a careful redefinition of the leading monomial of a series and of an ideal, along with new objects tailored to this setting: there is one leading monomial and one leading ideal for each vertex. This is done in Definition 2.1.10, and the properties derived in Proposition 2.1.12 are important consequences for what follows. We can then define what a Gröbner basis of an ideal is in Definition 2.2.1, and present a reduction algorithm adapted to this ordering (Algorithm 6).

A major addition is the following possibly strange fact for readers familiar with Gröbner theory in polynomial rings: the least common multiple of the leading monomials of two series f, g is not unique. Its definition depends on the chosen vertex, and for each vertex there are generally several (always finitely many) possibilities. This is important because the crucial notion of the critical pair of two series (Definition 2.3.1) relies on these LCMs. Consequently, for a fixed pair of series, there will be as many critical pairs as LCMs, and this will be reflected in the final Buchberger algorithm. This marks a significant divergence from the theory developed for Tate algebras.

Finally, after several technical lemmas that require careful handling (Lemmas 2.3.2, 2.3.3, and 2.3.4), we prove the final Buchberger criterion for polyhedral algebras with positive exponents (Proposition 2.4.1), and derive the corresponding Buchberger algorithm (Algorithm 7).

Chapter 3 - Torus. This chapter is the counterpart of the preceding one: we restrict to polyhedral algebras for which the valuation problem is neutralized (we have a multiplicative valuation), but the exponent monoid is the whole of $\mathbb{Z}^n$. This corresponds to polyhedral algebras whose defining polyhedron is just a single vertex. Graphically, this gives:



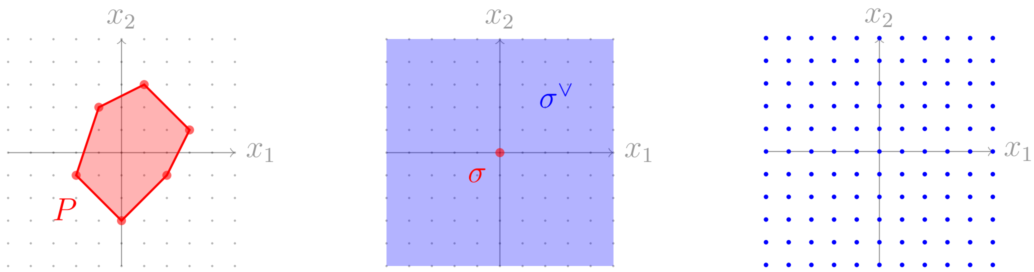
(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Figure 0.4: Torus

To overcome the exponent problem, we use generalized order. This is a tool introduced to allow a notion of Gröbner basis on Laurent polynomial ideals. The core idea is to cover the monoid $\mathbb{Z}^n$ by smaller monoids, each of which admits a term order. This requires defining one leading monomial per covering monoid and adapting the definition of Gröbner basis accordingly. As in Chapter 2, there will generally be, for a pair of series f, g, multiple LCMs per covering monoid for their leading monomials.

This chapter is shorter and simpler than the previous one, because we rely heavily on the work already done for Laurent polynomial ideals, presented in Section 1.4. Our main task is to show that the theory on Laurent polynomial ideals extends to series with convergence conditions and a valuation-first ordering. We present a reduction algorithm (Proposition 3.2.2) and prove a Buchberger criterion in this context (Proposition 3.4.1). The final result is a Buchberger algorithm (Algorithm 10) for this kind of polyhedral algebras. The content of this chapter has been published in the paper [BLV24].

Chapter 4 - Polytopal algebras. We bring together the results of Chapters 2 and 3 to develop a Gröbner theory in polytopal algebras, that is, polyhedral algebras whose defining polyhedron is a polytope. Graphically:



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Figure 0.5: Polytopal algebra

As one can see, we have at the same time a submultiplicative valuation (multiple vertices) and an exponent monoid equal to $\mathbb{Z}^n$. The main point of this chapter is that the two different decompositions of the monoid of monomials, introduced in the previous chapters for distinct reasons, cannot be combined arbitrarily. Indeed, when ordering terms, we first compare them using the valuation, and ultimately break ties using a generalized order. For the latter to be consistent with the former, the decomposition of monomials arising from the generalized order (Chapter 3) must *refine* the decomposition coming from the vertices of the polytope (Chapter 2). Equipped with such a decomposition, we redefine what leading monomials and leading ideals are in this context (Definition 4.1.12), and develop our Gröbner theory up to the final Buchberger criterion in Proposition 4.4.1.

In this chapter, we work with polytopal algebras to emphasize the combination of Chapters 2 and 3, but the content is in fact independent of the exponent monoid, and thus applies to any polyhedral algebra. In particular, we recover exactly the results of Chapters 2 and 3, as explained in Section 4.5. The results of Chapter 3 were discovered first (and published in [BLV24]), followed by those of Chapter 2 and 4 published in [BLV25]. They are presented here in reverse order, as it is more logical due to the dependence of one on the others in the definition of orderings and monoid decompositions.

We could have presented the theory for general polyhedral algebras—mixing everything from Chapters 2 to 4 in a single chapter. However, we found it more intuitive and instructive to develop the results step by step. The downside is that there is inevitably some redundancy, especially in the proofs of the Buchberger criteria across these three chapters, which are *essentially* the same. As a general guideline, we have taken care to provide the most detailed proofs possible for all results, even when they may appear trivial.

Chapter 5 - Effective computations. In this chapter, we investigate how to perform effective computations on a computer for the Gröbner theory in polyhedral algebras developed in Chapters 2 to 4. Since our objects are algebras of power series, we introduce a precision model to work with finite data: each series is represented by a polynomial obtained by truncating it at a chosen precision. Given such truncated inputs, we ask whether the final Buchberger algorithm, Algorithm 12, can be executed effectively in practice. The next task to address is the computation of the newly introduced least common multiples of a pair of series (f, g).

We prove two results:

1. In Section 5.1.1, we show that if the defining polyhedron is full-dimensional, a conic decomposition can be chosen such that computing the LCMs of (f, g) reduces to a classical problem in convex geometry: computing the Hilbert basis of a rational cone.

2. In Theorem 5.1.5, we show that if the defining polyhedron reduces to a single point (as in Chapter 3), generalized orders exist for which each pair (f, g) admits a unique LCM per cone in the decomposition, which can be computed directly.

We illustrate the first result by running Algorithm 12 on a simple example using a preliminary SageMath implementation.

1

Setting

Summary

Introduction

This chapter introduces preliminary material needed in subsequent chapters. We start with notation and key definitions. Section 1.2 presents Tate and polyhedral algebras. Section 1.3 focuses on Gröbner bases over Tate algebras. Section 1.4 covers the theory of generalized orders.
Each section is concise, including only what is needed in later chapters. Below is a list of the primary references on which each section is based for further reading.

1. Section 1.2. For Tate algebras, we use [Bos14] and [FP04]. For polyhedral algebras, the reference is [Rab12] by J. Rabinoff.

2. Section 1.3. For Gröbner bases over Tate algebras, we refer to the series of works [CVV19], [CVV20], [CVV21], [CVV22], and [VV23], where the theory has been developed since 2019.

3. Section 1.4. For generalized orders, we rely on [PZ96] and [PU99], the only existing references on the subject.

This chapter is purely expository and contains only established material without any new contributions.

1.1 Notation, definitions

We introduce notation and recall basic definitions and theorems that will be frequently used. Let R be a commutative ring.

1.1.1 Monoids and modules over a monoid

A (multiplicative) monoid M is a set together with an associative binary operation and a distinguished identity element. It is called commutative if $ab = ba$ for all $a, b \in M$. The monoid M is finitely generated if there exists a finite subset $S \subseteq M$ such that every element of M can be written as a finite product of elements of S. An ideal of M is a subset $I \subseteq M$ such that for all $m \in M$ and $i \in I$, one has $mi \in I$.

An M-module E is a set equipped with an action

$$M \times E \to E, \quad (m, e) \mapsto m \cdot e,$$

satisfying $1 \cdot e = e$ and $(mn) \cdot e = m \cdot (n \cdot e)$ for all $m, n \in M$, $e \in E$. An M-module E is finitely generated if there exists a finite subset $F \subseteq E$ such that

$$E = \bigcup_{f \in F} M \cdot f.$$

Any ideal I of M naturally becomes an M-module under the restricted action. We say that I is finitely generated if it is finitely generated as an M-module under the restricted action.

Proposition 1.1.1 ([GB09]). *Let M be a finitely generated monoid and N a finitely generated M-module. Then every M-submodule of N is finitely generated.*

In particular, if M is a finitely generated monoid, then every ideal of M is finitely generated.

Proposition 1.1.2. *Let M be a finitely generated monoid. Then M satisfies the ascending chain condition on ideals: every ascending chain of ideals stabilizes.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in M, and set $I = \bigcup_{k \geq 1} I_k$. Then I is an ideal of M. Since M is finitely generated, I is finitely generated by Proposition 1.1.1. Let $x_1, \ldots, x_m$ be generators of I. Each x_j lies in some I_{N_j}. Let $N = \max_j N_j$. Then $x_j \in I_N$ for all j, so $I \subseteq I_N$. As the chain is ascending, we have $I = I_N$, and hence $I_k = I_N$ for all $k \geq N$. Thus the chain stabilizes. □

1.1.2 Monomials, terms

We use the notation x to denote an n-tuple of variables $(x_1, \ldots, x_n)$, $n > 0$. For $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we write x^u for the product $x_1^{u_1} \ldots x_n^{u_n}$. We define:

- $\mathcal{M} := \{x^u, u \in \mathbb{Z}^n\}$

- $\mathcal{M}_+ := \{x^u, u \in \mathbb{N}^n\}$
- $\mathcal{T}(R) := \{ax^u, a \in R, u \in \mathbb{Z}^n\}$
- $\mathcal{T}_+(R) := \{ax^u, a \in R, u \in \mathbb{N}^n\}$

If the coefficient ring R is understood, we simply write $\mathcal{T}$ and $\mathcal{T}_+$. All these sets are commutative monoids with neutral element 1. $\mathcal{M}$ is a group, and if R is a field, $\mathcal{T}(R)$ is also a group. The letters $\mathcal{M}$ and $\mathcal{T}$ are chosen to help the reader throughout the manuscript: $\mathcal{M}$ stands for "Monomial", while $\mathcal{T}$ stands for "Term".

1.1.3 Seminorm, norm

A (multiplicative) seminorm on R is a map $|\cdot| : R \to \mathbb{R}_{\geq 0}$ satisfying, for all $a, b \in R$:

1. $|0| = 0$.
2. $|ab| = |a||b|$.
3. $|a + b| \leq |a| + |b|$.

We say that $|\cdot|$ is:

- a *norm* if condition (1) is strengthened to $|a| = 0 \iff a = 0$.
- *non-Archimedean* if condition (3) is replaced by $|a + b| \leq \max(|a|, |b|)$.
- *submultiplicative* if condition (2) is relaxed to $|ab| \leq |a||b|$.
- *power-multiplicative* if $|a^n| = |a|^n$ for all positive integers n.

1.1.4 Semi-valuation, valuation

A (multiplicative) non-Archimedean semi-valuation is a map $\mathrm{val} : R \to \mathbb{R} \cup \{\infty\}$ satisfying, for all $a, b \in R$:

1. $\mathrm{val}(0) = \infty$.

2. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$.

3. $\mathrm{val}(a + b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$.

We say that val is:

1. a *valuation* if condition (1) is strengthened to $\mathrm{val}(a) = \infty \iff a = 0$.

2. *submultiplicative* if condition (2) is replaced by $\mathrm{val}(ab) \geq \mathrm{val}(a) + \mathrm{val}(b)$.

3. *power-multiplicative* if $\mathrm{val}(a^n) = n \times \mathrm{val}(a)$ for all positive integers n.

4. discrete if $\mathrm{val}(R \setminus \{0\})$ is a discrete subset of $\mathbb{R}$.

A valuation induces a norm by setting $| \cdot | = e^{-\mathrm{val}(\cdot)}$.

1.1.5 Valued field

A valued field is a field endowed with a valuation val. It is complete if it is complete for the distance defined by the norm associated to val, and discrete if its valuation is discrete. An example of a complete, discrete valued field (CDVF) is the field of p-adic numbers $\mathbb{Q}_p$.

1.1.6 Term order

A term order (or monomial order) on a monoid M is a total order $\preceq$ satisfying:

1. $1 \preceq u$ for all $u \in M$

2. $a \preceq b \implies au \preceq bu$ for all $a, b, u \in M$

If a monoid contains a non-trivial unit, no term order can exist on it. Indeed, let t be invertible with $t \neq 1$. Then by (1), we have $1 \prec t$. By (2), multiplying by t^{-1} gives $t^{-1} \prec 1$, contradicting (1). For example, the additive monoid $\mathbb{Z}^n = \mathcal{M}$ admits no term order.

1.1.7 Polyhedral geometry

The purpose of this section is to recall several basic notions from polyhedral geometry that will be used later: polyhedra, polyhedral cones and their duals, as well as the recession cone and the normal fan associated with a polyhedron. These concepts are classical and well-known, so we omit detailed references. For background the reader may consult [Zie95], or the presentation in Chapter 2 of [Rab12], which covers all the material presented here.

We write $a \cdot b$ for the standard scalar product in $\mathbb{R}^n$.

Polyhedra

A half-space is a set of the form

$$H := \{u \in \mathbb{R}^n,\ u \cdot v \geq a\}$$

for some $v \in \mathbb{R}^n \setminus \{0\}$ and $a \in \mathbb{R}$. It is rational if v can be chosen in $\mathbb{Q}^n$ and a in $\mathbb{Q}$. If $a = 0$, it is called linear; otherwise, it is affine.

A nonempty set $P \subseteq \mathbb{R}^n$ is called a polyhedron if there exist finitely many half-spaces $H_1, H_2, \ldots, H_m$ such that

$$P = \bigcap_{i=1}^{m} H_i.$$

It is rational if there exists such a description of P with each H_i a rational half-space. A polytope is a bounded polyhedron.

For $S \subseteq \mathbb{R}^n$, the convex hull $\mathrm{Conv}(S)$ is the intersection of all half-spaces containing S.

Given a polyhedron P and $v \in \mathbb{R}^n$, define

$$\mathrm{face}_v(P) := \{u \in P,\ \forall t \in P,\ u \cdot v \geq t \cdot v\}.$$

A face F of P is any set of the form $\mathrm{face}_v(P)$. We write $F \prec P$.

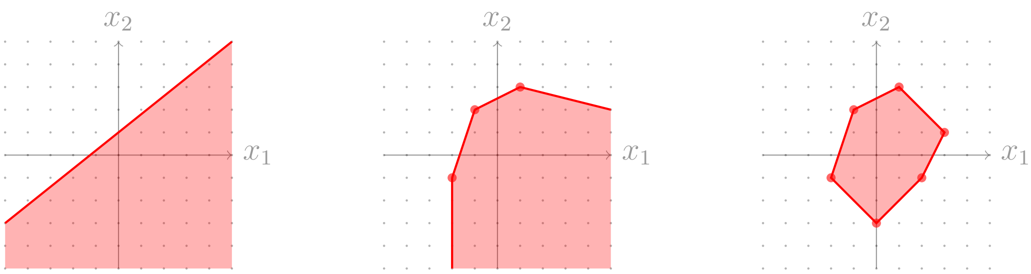
A vertex is a face consisting of a single point. The set of all vertices is $\mathrm{Vert}(P)$.

The span of P, denoted $\mathrm{span}(P)$, is the smallest linear subspace containing P. Its dimension is defined as $\dim(P) := \dim(\mathrm{span}(P))$.

Faces of polyhedra satisfy the following:

1. Each face F of a polyhedron is itself a polyhedron, and it satisfies $\dim(F) < \dim(P)$ if $F \neq P$.
2. A polyhedron has finitely many faces.
3. Faces inherit rationality.
4. The intersection of two faces is a face of each.
5. If P is a polytope, then $P = \mathrm{Conv}(\mathrm{Vert}(P))$.

Several polyhedra are illustrated in Figure 1.1.

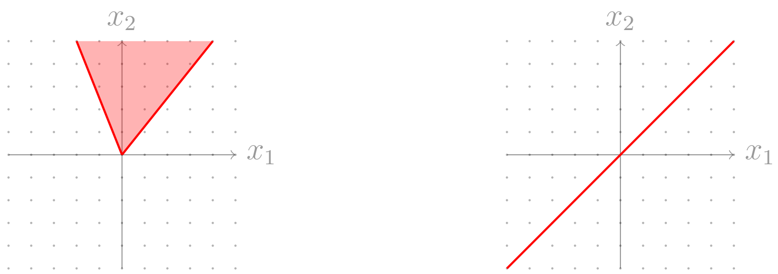


(a) Half-space **(b)** Polyhedron **(c)** Polytope

Figure 1.1: Polyhedra

Cones and their duals

A polyhedral cone C is a polyhedron which is the intersection of finitely many *linear* half-spaces. A cone is pointed if $0 \in \mathrm{Vert}(C)$, or equivalently, if it contains no non-zero linear subspace. Any face of a cone is itself a cone.



(a) Pointed cone **(b)** Non-pointed cone

Figure 1.2: Pointed and non-pointed cone

Given vectors $v_1, \dots, v_r \in \mathbb{R}^n$, the set

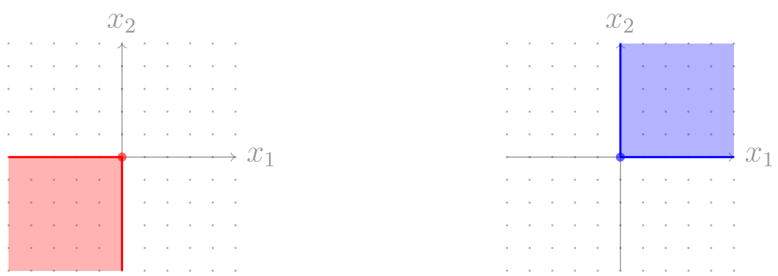
$$\sum_{i=1}^{r} \mathbb{R}_{\geq 0}\, v_i$$

is a cone. If moreover $v_1, \dots, v_r \in \mathbb{Q}^n$, then the cone is rational.
Conversely, every cone (resp. every rational cone) can be written in the form

$$\sum_{i=1}^{r} \mathbb{R}_{\geq 0}\, v_i$$

for suitable vectors $v_1, \dots, v_r \in \mathbb{R}^n$ (resp. $\in \mathbb{Q}^n$).
The (polar) dual cone of C is defined as

$$C^\vee := \{u \in \mathbb{R}^n,\ u \cdot v \leq 0,\ \forall v \in C\}.$$



(a) Cone C **(b)** Its (polar) dual $C^\vee$

Figure 1.3: A cone and its dual

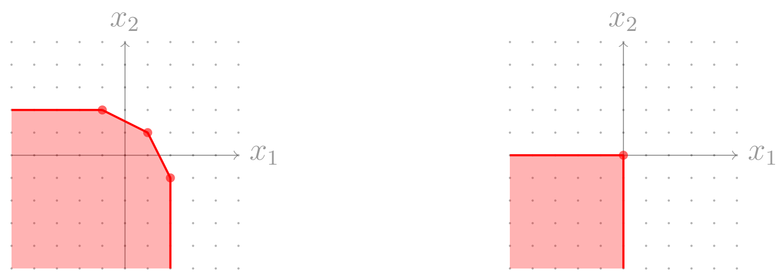
We have $(C^\vee)^\vee = C$, and rationality is preserved by duality.

The recession cone of a polyhedron

The recession cone of a polyhedron P, denoted $\mathrm{rec}(P)$, is defined by:

$$\mathrm{rec}(P) := \{u \in \mathbb{R}^n : \forall x \in P,\ x + u \in P\}.$$

It is the largest cone C (with respect to inclusion) such that $P + C \subseteq P$. We say that P is pointed if its recession cone is pointed. This condition holds if and only if P does not contain an affine subspace of positive dimension.



(a) P **(b)** $\mathrm{rec}(P)$

Figure 1.4: A polyhedron and its recession cone

The normal fan of a polyhedron

A polyhedral complex is a finite collection D of polyhedra in $\mathbb{R}^n$, called cells, satisfying:

1. If $P_1, P_2 \in D$ and $P_1 \cap P_2 \neq \emptyset$, then $P_1 \cap P_2$ is a face of both.
2. If $P \in D$ and $F \prec P$, then $F \in D$.

The support of D is $\mathrm{supp}(D) := \bigcup_{P \in D} P$. Its dimension is the maximum dimension among its cells. It is rational if all cells are.

(a) Polyhedral complex. One color per cell.

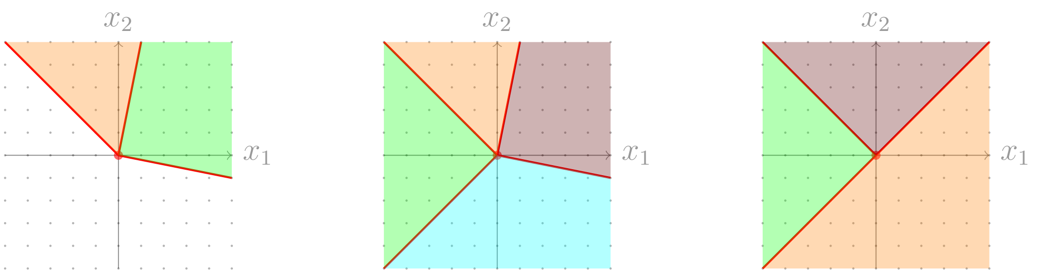
(b) Support

Figure 1.5: Polyhedral complex of dimension 2 and its support

A fan is a polyhedral complex whose cells are cones. A fan is:

1. Pointed if all cones are pointed.

2. Complete if $\operatorname{supp}(D) = \mathbb{R}^n$.



(a) Non-complete, pointed

(b) Complete, pointed

(c) Complete, non-pointed

Figure 1.6: Fans, one color per maximal cell

Given a polyhedron P, the normal fan $\mathcal{N}(P)$ of P is the fan whose cones are defined for each face F of P by:

$$C(P, F) := \{u \in \mathbb{R}^n,\ F \subseteq \text{face}_u(P)\}.$$

By definition, there is a canonical bijection between the faces of P and the cones of $\mathcal{N}(P)$. Under this bijection, the vertices of P correspond to the maximal cones of $\mathcal{N}(P)$, and we have:

$$\mathrm{Supp}(\mathcal{N}(P)) = \bigcup_{C \in \mathrm{MaxCones}(\mathcal{N}(P))} C,$$

where $\mathrm{MaxCones}(\mathcal{N}(P))$ is the set of maximal cones of $\mathcal{N}(P)$.
If P is rational, then so is $\mathcal{N}(P)$. Also:

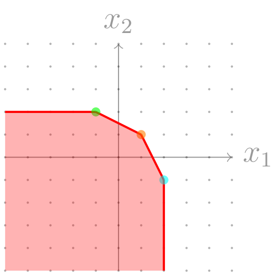
1. P is a polytope if and only if $\mathcal{N}(P)$ is complete.

2. $\mathcal{N}(P)$ is pointed if and only if $\dim(P) = n$.

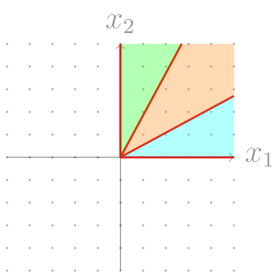
An alternative description of $\mathrm{rec}(P)$ can be given using $\mathcal{N}(P)$. Specifically, the recession cone of P is exactly the dual of the support of $\mathcal{N}(P)$. In summary, we have:

$$\mathrm{Supp}(\mathcal{N}(P)) = \mathrm{rec}(P)^\vee = \bigcup_{C \in \mathrm{MaxCones}(\mathcal{N}(P))} C \tag{1.1}$$

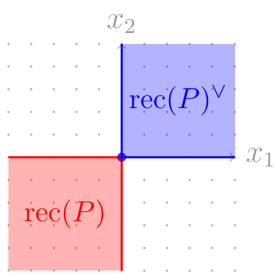
The notions introduced in this section are illustrated in Figures 1.7 to 1.10 below.



(a) P

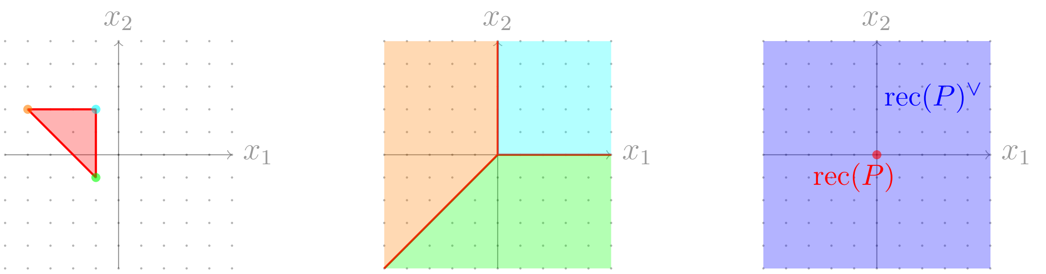


(b) $\mathcal{N}(P)$



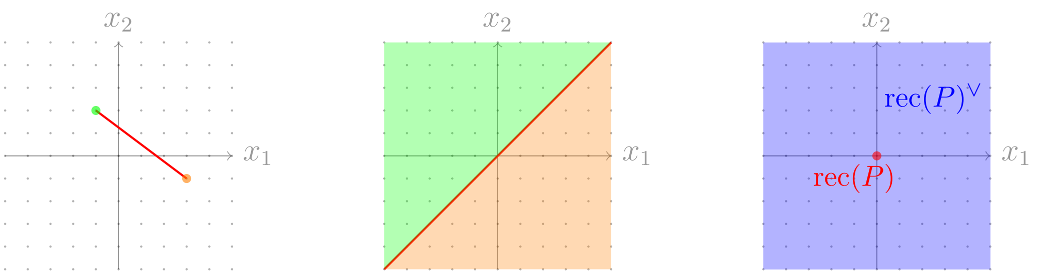
(c) $\mathrm{rec}(P)$ and $\mathrm{rec}(P)^\vee = \mathrm{Supp}(\mathcal{N}(P))$

Figure 1.7: A polyhedron, its normal fan, its recession cone and dual



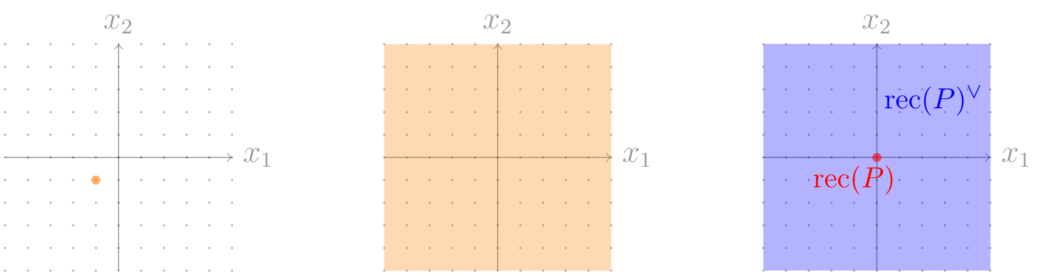
(a) P **(b)** $\mathcal{N}(P)$ **(c)** $\mathrm{rec}(P)$ and $\mathrm{rec}(P)^\vee$

Figure 1.8: A polytope, its normal fan and recession cone



(a) P **(b)** $\mathcal{N}(P)$ **(c)** $\mathrm{rec}(P)$ and $\mathrm{rec}(P)^\vee$

Figure 1.9: A non full dimensional polytope, its normal fan and recession cone



(a) P **(b)** $\mathcal{N}(P)$ **(c)** $\mathrm{rec}(P)$ and $\mathrm{rec}(P)^\vee$

Figure 1.10: A polytope consisting of a single vertex, its normal fan and its recession cone

1.1.8 Gordon's lemma and Hilbert bases

Let $C \subseteq \mathbb{R}^n$ be a rational cone Then the set of integer points in C,

$$C \cap \mathbb{Z}^n,$$

forms a finitely generated monoid under addition.
In other words, there exists a finite set

$$\{x_1, x_2, \dots, x_k\} \subset C \cap \mathbb{Z}^n,$$

such that every element $z \in C \cap \mathbb{Z}^n$ can be written as

$$z = m_1 x_1 + m_2 x_2 + \cdots + m_k x_k,$$

with coefficients $m_i \in \mathbb{Z}_{\geq 0}$ for all i. Such a representation is not necessarily unique.

If C is pointed, there exists a unique minimal generating set of $C \cap \mathbb{Z}^n$, called the *Hilbert basis* of C.

1.2 Tate and polyhedral algebras

In this section, we briefly introduce Tate algebras and affinoid spaces, then focus on polyhedral affinoids, a specific type of affinoid space central to tropical analytic geometry.
We present this object from a computational perspective. Globalization is not discussed, as we work only within affinoid settings defined by power series rings.

For a broader and more theoretical introduction to Tate algebras and rigid geometry, see Tate's original article [Tat71], [FP04], and [Bos14].
For polyhedral affinoids, the main reference is [Rab12]. For general concepts and historical background of analytic tropical geometry, see [EKL06], [Gub07a], and [Gub07b].

1.2.1 Tate algebras and affinoids

Rigid geometry is based on affinoid algebras, which are analytic analogues of coordinate rings in algebraic geometry. These algebras generalize polynomial rings while enforcing convergence conditions suitable for non-Archimedean analysis. Let K be a complete non-Archimedean field with a nontrivial valuation

$$\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}.$$

Denote by $\overline{K}$ the algebraic closure of K.

We consistently use the valuation rather than the norm, to align with the conventions in [CVV19]-[CVV22].

Define the closed unit ball in $\overline{K}^n$ as

$$\mathbb{B}^n(\overline{K}) = \{(x_1, \ldots, x_n) \in \overline{K}^n \mid \mathrm{val}(x_i) \geq 0\}.$$

The Tate algebra in n variables is

$$K\langle x \rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid a_\nu \in K,\ \mathrm{val}(a_\nu) \to \infty \text{ as } |\nu| \to \infty \right\}.$$

Its valuation,

$$\mathrm{val}\left(\sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \right) = \min_\nu (\mathrm{val}(a_\nu))$$

is called the Gauss valuation. Elements of $K\langle x \rangle$ are precisely power series converging on $\mathbb{B}^n(\overline{K})$.

Key properties of $K\langle x \rangle$ include (see [FP04], [Bos14]):

1. $K\langle x \rangle$ is complete for the Gauss norm, hence a K-Banach algebra.
2. It is noetherian, an integral domain, factorial, Jacobson, and has Krull dimension n.
3. It satisfies the maximum principle: $\mathrm{val}(f(a)) \geq \mathrm{val}(f)$ for all $a \in \mathbb{B}^n(\overline{K})$, with equality for some $a \in \mathbb{B}^n(\overline{K})$.

More generally, fix a log-radii $r \in \mathbb{Q}^n$. The Tate algebra with log-radii r is

$$K_r\langle x\rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid a_\nu \in K,\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty \right\},$$

with valuation

$$\mathrm{val}_r\left(\sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \right) = \min_\nu (\mathrm{val}(a_\nu) - r \cdot \nu).$$

$K_r\langle x\rangle$ has the same properties as $K\langle x\rangle$, but its elements converge on the poly-disk with log-radii r:

$$\mathbb{B}^n_r(\overline{K}) := \{(x_1, \ldots, x_n) \in \overline{K}^n \mid \mathrm{val}(x_i) \geq r_i\}.$$

The algebra $K\langle x\rangle$ corresponds to the special case $K_r\langle x\rangle$ with $r = (0, \ldots, 0)$.

A K-affinoid algebra is a K-algebra A isomorphic to a Tate algebra quotient, that is $A \simeq K\langle x\rangle/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Since $K\langle x\rangle$ is noetherian, $\mathfrak{a} = (f_1, \ldots, f_s)$ for some series $f_i \in K\langle x\rangle$.
Elements of $K\langle x\rangle/\mathfrak{a}$ can be interpreted as $\overline{K}$-valued functions on the zero set $V(\mathfrak{a}) \subseteq \mathbb{B}^n(\overline{K})$.
The infimum valuation of $f \in K\langle x\rangle/\mathfrak{a}$ is defined by

$$\mathrm{val}_{\inf}(f) := \inf_{a \in V(\mathfrak{a})} (\mathrm{val}(f(a))).$$

The norm corresponding to this valuation is the supremum norm, denoted by $|\cdot|_{\sup}$.
The main properties of the infimum valuation are:

1. It is power-multiplicative.
2. It satisfies the maximum principle: there exists $a \in V(\mathfrak{a})$ such that $\mathrm{val}_{\inf}(f) = \mathrm{val}(f(a))$.
3. It coincides with the Gauss valuation when the affinoid algebra is $K\langle x\rangle$ itself.

Importantly, unlike the Gauss valuation, the infimum valuation is not multiplicative. For all $f, g \in K\langle x\rangle/\mathfrak{a}$, we only have the inequality

$$\mathrm{val}_{\mathrm{inf}}(fg) \geq \mathrm{val}_{\mathrm{inf}}(f) + \mathrm{val}_{\mathrm{inf}}(g).$$

This fact will be important later on.

1.2.2 Polyhedral algebras

We now introduce the notion of polyhedral algebras. Our exposition follows a streamlined adaptation of [Rab12, Chap. 6]. These algebras correspond to admissible affinoids in the analytification of toric varieties over K. In practice, they are defined as power series rings whose convergence domain is determined by a polyhedron P satisfying specific conditions. Such algebras can be interpreted as function algebras converging on the inverse image of P under the valuation map. It is important to understand the six examples from Examples 1.2.2 to 1.2.7.

Convention 1.2.1. *Starting from now on and* ***for the rest of the thesis****,* K *will denote a complete discretely valued field equipped with a nontrivial valuation map*

$$\mathrm{val} : K \to \mathbb{Q} \cup \{\infty\},$$

normalized so that $\mathrm{val}(K^\times) = \mathbb{Z}$*. An example is* $K = \mathbb{Q}_p$ *with its normalized* p*-adic valuation.*

Let $P \subseteq \mathbb{R}^n$ be a pointed rational polyhedron with recession cone σ.
Define S_σ as the monoid $S_\sigma := \sigma^\vee \cap \mathbb{Z}^n$. It is finitely generated by Gordon's Lemma (see Section 1.1.8).
The polyhedral algebra associated with P is

$$K_P\{x\} := \left\{ \sum_{\nu \in S_\sigma} a_\nu x^\nu \mid \forall r \in P,\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty \right\}$$

Such power series represent functions converging on:

$$U_P := \{(x_1, \dots, x_n) \in \overline{K}^n,\ -(\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) \in P\}.$$

Each $r \in P$ defines a valuation val_r on $K_P\{x\}$ via

$$\mathrm{val}_r\left(\sum_{\nu \in S_\sigma} a_\nu x^\nu\right) = \min_{\nu \in S_\sigma} \left(\mathrm{val}(a_\nu) - r \cdot \nu\right).$$

For $f = \sum_{\nu \in S_\sigma} a_\nu x^\nu$ in $K_P\{x\}$, the infimum

$$\inf_{\nu \in S_\sigma,\ r \in P} \left(\mathrm{val}(a_\nu) - r \cdot \nu\right) = \inf_{r \in P} \left(\mathrm{val}_r(f)\right)$$

is finite and attained at some vertex of P.
Thus, we have the equalities

$$\inf_{\nu \in S_\sigma,\ r \in P} \left(\mathrm{val}(a_\nu) - r \cdot \nu\right) = \min_{\nu \in S_\sigma,\ r \in \mathrm{Vert}(P)} \left(\mathrm{val}(a_\nu) - r \cdot \nu\right) = \min_{r \in \mathrm{Vert}(P)} \left(\mathrm{val}_r(f)\right)$$

Since P is pointed, $\mathrm{Vert}(P)$ is finite and nonempty. Therefore, this infimum is the minimum of finitely many valuations, one for each vertex of P.
The main result of [Rab12, Chapter 3] is that $K_P\{x\}$ is affinoid with infimum valuation

$$\mathrm{val}_P(f) := \min_{r \in \mathrm{Vert}(P)} \mathrm{val}_r(f).$$

Moreover, it suffices to impose convergence conditions at the vertices of P only:

$$K_P\{x\} = \left\{\sum_{\nu \in S_\sigma} a_\nu x^\nu \mid \forall r \in \mathrm{Vert}(P),\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty\right\}$$

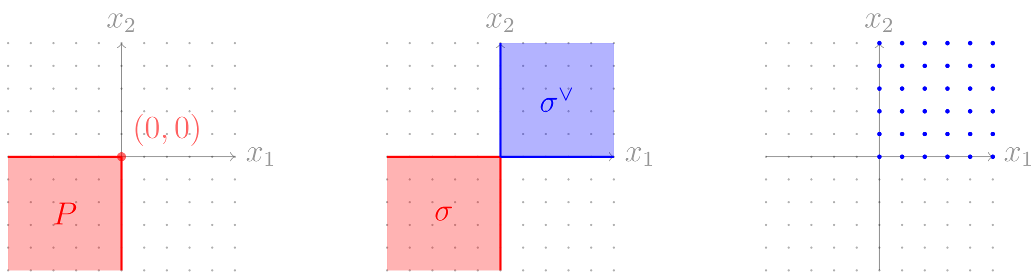
Example 1.2.2 (Tate Algebra)**.** *Take* $P = (-\infty, 0]^n$. *Then:*

- $\mathrm{Vert}(P) = \{(0, \dots, 0)\}$
- $\sigma = \mathbb{R}^n_{\leq 0}$
- $\sigma^\vee = \mathbb{R}^n_{\geq 0}$

- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Thus, for this P, we recover the Tate algebra:

$$K_P\{x\} = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid \mathrm{val}(a_\nu) \to \infty \text{ as } |\nu| \to \infty \right\} = K\langle x \rangle$$



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

Figure 1.11: Tate algebra

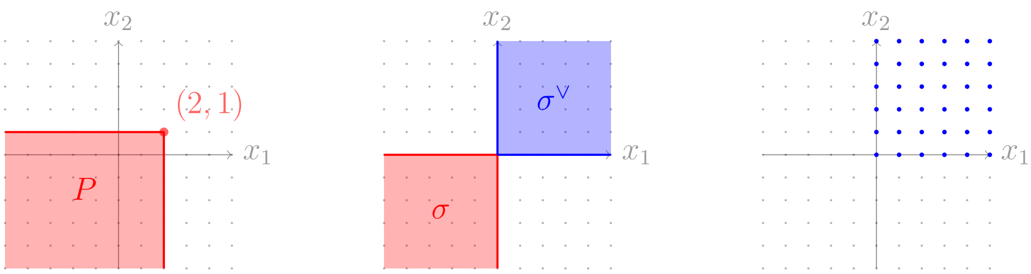
Example 1.2.3 (Tate algebra, general log-radii)**.** *Let $r \in \mathbb{Q}^n$ and set $P = r + (-\infty, 0]^n$. Then:*

- $\mathrm{Vert}(P) = \{r\}$
- $\sigma = \mathbb{R}^n_{\leq 0}$
- $\sigma^\vee = \mathbb{R}^n_{\geq 0}$
- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Hence, for this P, we obtain the Tate algebra with log-radii r:

$$K_P\{x\} = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty \right\} = K_r\langle x \rangle$$

Example 1.2.2 is the special case $r = (0, \ldots, 0)$.



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

Figure 1.12: Tate algebra with log-radii $r = (2, 1)$

Example 1.2.4 (Positive exponents)**.** *Let* $r_1, \ldots, r_s \in \mathbb{Q}^n$*. Define*

$$P = \mathrm{Conv}\left(\bigcup_i (r_i + (-\infty, 0]^n)\right).$$

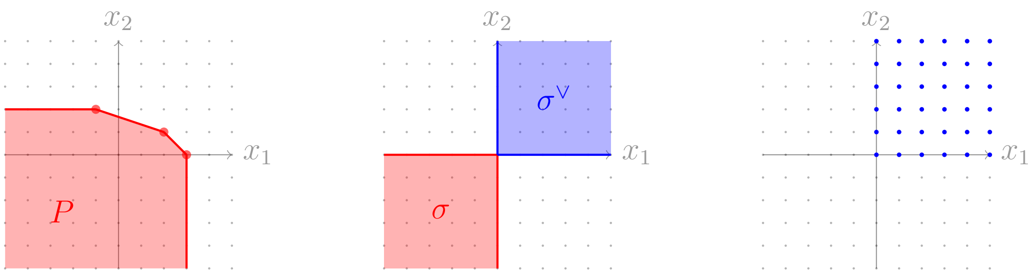
Clearly, $\mathrm{Vert}(P) \subseteq \{r_1, \ldots, r_s\}$*. We may assume, by removing some* r_i *if needed, that* $\mathrm{Vert}(P) = \{r_1, \ldots, r_s\}$*.*
We have:

- $\mathrm{Vert}(P) = \{r_1, \ldots, r_s\}$
- $\sigma = \mathbb{R}^n_{\leq 0}$
- $\sigma^\vee = \mathbb{R}^n_{\geq 0}$
- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Therefore, the associated polyhedral algebra is

$$K_P\{x\} = \left\{\sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \mid \forall r \in \{r_1, \ldots, r_s\},\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty\right\}.$$

We call this a polyhedral algebra with positive exponents, since all exponents are nonnegative.
These algebras are studied in Chapter 2. Any polyhedral algebra with positive exponents arises from a polyhedron of this form.
Examples 1.2.2 and 1.2.3 correspond to the special case $s = 1$*.*



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

Figure 1.13: Positive exponents

Example 1.2.5 (Poly circles)**.** *Let $r \in \mathbb{Q}^n$ and define $P = \{r\}$. Then:*

- $\mathrm{Vert}(P) = \{r\}$
- $\sigma = \{(0, \ldots, 0)\}$
- $\sigma^\vee = \mathbb{R}^n$
- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

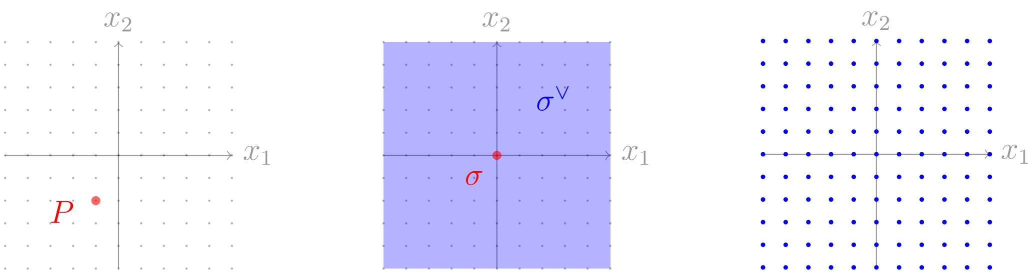
Thus, the corresponding polyhedral algebra is

$$K_P\{x\} = \left\{ \sum_{\nu \in \mathbb{Z}^n} a_\nu x^\nu \mid \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \ \textit{as}\ |\nu| \to \infty \right\}.$$

The only difference with $K_r\langle x\rangle$ is that exponents now range over $\mathbb{Z}^n$ instead of being restricted to $\mathbb{N}^n$.
Series in this algebra converge on the poly circle

$$C_r(\overline{K}) := \{a \in \overline{K}^n \mid \forall i \in \{1, \ldots, n\},\ \mathrm{val}(a_i) = -r_i\}.$$

In this case, we use the notation $K_r\{x\}$.



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

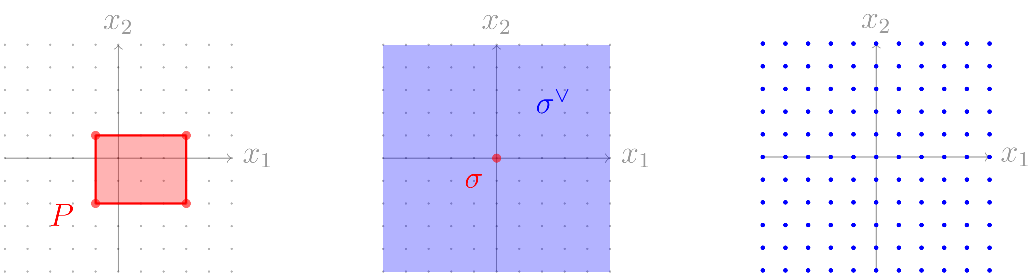
Figure 1.14: Poly circle

Example 1.2.6 (Poly annulus)**.** *Let* $P = \prod_{i=1}^{n}[r_i; s_i]$ *for some* $r_i < s_i \in \mathbb{Q}$. *Then:*

- $|\mathrm{Vert}(P)| = 2^n$
- $\sigma = \{(0, \dots, 0)\}$
- $\sigma^\vee = \mathbb{R}^n$
- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Series in the corresponding polyhedral algebra represent functions converging on the poly annulus

$$A_{r,s} := \{a \in \overline{K}^n \mid \forall i \in \{1, \dots, n\},\ -\mathrm{val}(a_i) \in [r_i; s_i]\}.$$



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

Figure 1.15: Poly annulus

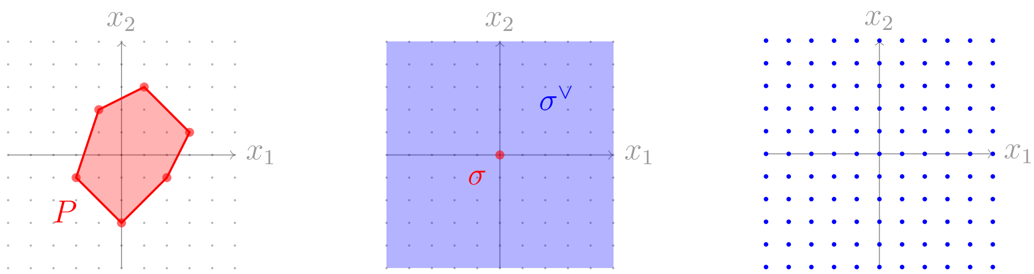
Example 1.2.7 (Polytopal algebra)**.** *Let* $P = \mathrm{Conv}(\{r_1, \ldots, r_s\})$ *for some* $r_1, \ldots, r_s \in \mathbb{Q}^n$*. Clearly,* $\mathrm{Vert}(P) \subseteq \{r_1, \ldots, r_s\}$*. We may assume, by removing some* r_i *if needed, that* $\mathrm{Vert}(P) = \{r_1, \ldots, r_s\}$*. Then:*

- $\mathrm{Vert}(P) = \{r_1, \ldots, r_s\}$
- $\sigma = \{(0, \ldots, 0)\}$
- $\sigma^\vee = \mathbb{R}^n$
- $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Thus, the corresponding polyhedral algebra is

$$K_P\{x\} = \left\{ \sum_{\nu \in \mathbb{Z}^n} a_\nu x^\nu \mid \forall r \in \{r_1, \ldots, r_s\},\ \mathrm{val}(a_\nu) - r \cdot \nu \to \infty \text{ as } |\nu| \to \infty \right\}.$$

Such a polyhedral algebra where P *is a polytope (i.e., a bounded polyhedron) is called a polytopal algebra.*
Examples 1.2.5 and 1.2.6 are special cases of this construction.



(a) P **(b)** σ and $\sigma^\vee$ **(c)** S_σ

Figure 1.16: Polytopal algebra

Remark 1.2.8. *In [Rab12], the assumptions on* K *are weaker than those in Convention 1.2.1:* K *need not be discretely valued, and its value group* $\Gamma := \mathrm{val}(K^\times) \subseteq \mathbb{R}$ *is not required to embed in* $\mathbb{Q}$*. Defining* $K_P\{x\}$ *for such fields would require introducing rational* Γ*-affine polyhedra in Section 1.1.7 as in [Rab12, Chapter 2], which would be needed only in the present section* *because Convention 1.2.1 is required from the next section onward. To simplify, we preferred not to introduce* Γ *and stick to rational polyhedral geometry in Section 1.1.7, which explains why Convention 1.2.1 is stated before introducing* $K_P\{x\}$*, even though* $K_P\{x\}$ *can in fact be defined for more general fields* K*.*

1.3 Gröbner theory in Tate algebras

In this section, we present a partial survey of the series of works [CVV19], [CVV20], [CVV21], [CVV22], [VV23].

We focus on the results that we extend to the polyhedral setting in the following chapters. These include the definition of Gröbner bases over $K_r\langle x\rangle$ using a valuation-first ordering with a term order as a tie-breaker, and the corresponding Buchberger algorithm, all of which are covered in [CVV19].

In the works cited above, Gröbner theory is developed for ideals in $K_r\langle x\rangle$ as well as its valuation ring:

$$K_r^\circ\langle x\rangle := \{f \in K_r\langle x\rangle \mid \mathrm{val}_r(f) \geq 0\}.$$

In this thesis, for simplicity, we restrict our attention to the case of $K_P\{x\}$. Possible extensions to the valuation ring

$$K_P^\circ\{x\} := \{f \in K_P\{x\} \mid \mathrm{val}_P(f) \geq 0\}.$$

are briefly discussed in Section 4.6.

Recall that Convention 1.2.1 is in force. The fact that K must be discretely valued is a crucial condition, appearing already in the proof of Lemma 1.3.2 below. It is essential for the established Gröbner theory in Tate algebras, and for the extension to polyhedral algebras that we will develop.

To order terms in $K_r\langle x\rangle$, we use a valuation-first ordering with a term order as a tie-breaker, as in [CM18].

Definition 1.3.1 ([CVV19, Def. 2.11])**.** *Let* $\leq_t$ *be a term order on* $\mathcal{M}_+$*. Define* *a preorder* $\leq_r$ *on* $\mathcal{T}_+$ *by:*

$$ax^u \leq_r bx^v \iff \begin{cases} \mathrm{val}_r(bx^v) < \mathrm{val}_r(ax^u) & (1) \\ \textbf{or} \\ \text{equality in (1) } \textbf{and } x^v \geq_t x^u \end{cases}$$

The leading term, monomial, and coefficient of $f \in K_r\langle x\rangle$*, denoted* $\mathrm{lt}(f)$, $\mathrm{lm}(f)$, *and* $\mathrm{lc}(f)$ *respectively, are defined using* $\leq_r$*.*

With this ordering, due to the valuation-first rule, strictly decreasing sequences are not necessarily finite. Nevertheless, since the valuation increases and is assumed discrete, we obtain topological convergence:

Lemma 1.3.2 ([CVV19, Lem. 2.14])**.** *Let* $(t_j)_{j\geq 0}$ *be a strictly decreasing sequence of terms for* $\leq_r$*. Then*

$$\lim_{j\to+\infty} \mathrm{val}_r(t_j) = +\infty.$$

With $\leq_r$ as the ordering, the definition of Gröbner basis in $K_r\langle x\rangle$ is the same as in the polynomial case:

Definition 1.3.3 ([CVV19], Def. 2.17)**.** *Let* J *be an ideal in* $K_r\langle x\rangle$*. We say that* $(g_1, \ldots, g_s) \in K_r\langle x\rangle$ *is a Gröbner basis of* J *if:*

$$\mathrm{lm}(J) = \langle \mathrm{lm}(g_1), \ldots, \mathrm{lm}(g_s)\rangle_{K_r\langle x\rangle}$$

As in the classical polynomial case, any ideal admits a Gröbner basis ([CVV19, Prop. 2.19]), and any Gröbner basis generates the ideal ([CVV19, Prop. 2.18]).

Algorithm 1: ReductionTateAlgebra ([CVV19, Algorithm 1])

input : $f \in K_r\langle x\rangle$, $(g_1, \ldots, g_s) \in K_r\langle x\rangle^s$
output : $(q_1, \ldots, q_s) \in K_r\langle x\rangle^s$ and $b \in K_r\langle x\rangle$ satisfying Prop 1.3.4

1 $q_1, \ldots, q_s, b \leftarrow 0$;
2 **while**$_\infty$ $f \neq 0$
3   **while** $\exists i \in \{1, \ldots, m\}$ *s.t.* $\mathrm{lm}(g_i) \mid \mathrm{lm}(f)$ **do**
4         $q_i \leftarrow q_i + \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_i)}$;
5         $f \leftarrow f - \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_i)} g_i$;
6     $b \leftarrow b + \mathrm{lt}(f)$;
7     $f \leftarrow f - \mathrm{lt}(f)$;
8 **return**$_\infty$ $(q_1, \ldots, q_s), b$

In Algorithm 1, the classical multivariate division algorithm is adapted to the setting of $K_r\langle x\rangle$. The difference is that the procedure may not terminate in the usual sense: the while-loop in line 2 can run indefinitely, producing infinitely many terms in the quotients $q_1, \ldots, q_s$ and in the remainder b. However, by construction these sequences converge in $K_r\langle x\rangle$. To formalize this, we introduce an *infinite while-loop*, denoted **while**$_\infty$. The notation **return**$_\infty$ indicates that the outputs $(q_1, \ldots, q_s)$ and b are the limits in $K_r\langle x\rangle$ of the sequences constructed inside the **while**$_\infty$ loop.

Thus Algorithm 1 is not a proper algorithm in the usual sense (e.g., [Knu97, Section 1.1]). However, when the inputs are finitely represented as an approximate polynomial with a precision cap on val_r, the **while**$_\infty$ loop terminates after finitely many iterations: by construction $\mathrm{val}_r(f) \to \infty$, hence $\mathrm{val}_r(f)$ eventually exceeds the cap and the update becomes null at the given precision. In such case, Algorithm 1 behaves like a classical algorithm. See also the discussion just below [CVV19, Algorithm 2].

The notation **while**$_\infty$/**return**$_\infty$ allows us to reason formally about convergent infinite processes, while still preserving a classical algorithmic interpretation for inputs given at finite precision. The reader interested in this topic can also consult [CCM25, Section 4.1] and its discussion on topological rewriting theory.

This notation will reappear in Algorithms 6, 9, and 12, which extend Algorithm 1 to polyhedral algebras. The only difference is that the sequences produced inside the **while**$_\infty$ loop now converge in $K_P\{x\}$.

Proposition 1.3.4 ([CVV19, Prop. 3.1]). *Let $f, g_1, \ldots, g_s \in K_r\langle x\rangle$. Algorithm 1 called on input $(f, (g_1, \ldots, g_s))$ outputs $(q_1, \ldots, q_s) \in K_r\langle x\rangle^s$ and $b \in K_r\langle x\rangle$ such that:*

1. *$f = q_1 g_1 + \cdots + q_s g_s + b$*
2. *For all i and all monomials m of b, $\mathrm{lm}(g_i)$ does not divide m*
3. *For all i and all terms t of q_i, $\mathrm{lt}(tg_i) \leq_r \mathrm{lt}(f)$*

We now come to the notion of critical pair in $K_r\langle x\rangle$. It is defined in the usual way:

Definition 1.3.5 ([CVV19, Def. 3.5]). *The critical pair of $f, g \in K_r\langle x\rangle$ is*

$$S(f,g) := \mathrm{lc}(g)\frac{\mathrm{lcm}(\mathrm{lm}(f),\mathrm{lm}(g))}{\mathrm{lm}(f)}f - \mathrm{lc}(f)\frac{\mathrm{lcm}(\mathrm{lm}(f),\mathrm{lm}(g))}{\mathrm{lm}(g)}g.$$

We have the following Buchberger criterion:

Theorem 1.3.6 ([CVV19, Th. 3.7]). *Let $J = \langle g_1, \ldots, g_s\rangle$ be an ideal in $K_r\langle x\rangle$. Then $G := (g_1, \ldots, g_s)$ is a Gröbner basis of J if and only if all critical pairs $S(g_i, g_j)$ with $i \neq j$ reduce to zero via Algorithm 1 called on input $(S(g_i, g_j), G)$.*

Finally, the Buchberger algorithm in this context is identical to the classical version in polynomial rings. We simply replace the multivariate division algorithm with Algorithm 1 at line 6.

Algorithm 2: BuchbergerTateAlgebra ([CVV19, Algorithm 2])

input : $g_1, \ldots, g_s \in K_r\langle x \rangle$

output : G a Gröbner basis of the ideal generated by the g_i's

1 $G \leftarrow g_1, \ldots, g_s$;

2 $B \leftarrow \{(f_i, f_j),\ 1 \leq i < j \leq s\}$;

3 **while** $B \neq \emptyset$ **do**

4     $(f, g) \leftarrow$ an element of B;

5     $B \leftarrow B \setminus \{(f, g)\}$;

6   _, $b =$ ReductionTateAlgebra$(S(f, g), G)$;

7   **if** $a \neq 0$ **then**

8         $B \leftarrow B \cup \{(g, b),\ g \in G\}$;

9         $G \leftarrow G \cup \{b\}$;

10 **return** G

The notation "_" in Algorithm 2 line 6 indicates that the corresponding output is discarded.

To conclude, readers familiar with classical Gröbner theory in polynomial rings will notice that the theory over Tate algebras follows it closely, with no major difficulties. In Chapters 2 to 4, we progressively extend this theory to the polyhedral algebra setting. This transition introduces more significant challenges. The definition of Gröbner basis must be modified, and both the notion of least common multiple in $K_P\{x\}$ and the concept of critical pair for $f, g \in K_P\{x\}$ require substantial revision.

1.4 Generalized orders

In subsequent chapters, we will need to order terms of power series $\sum_{a \in M} c_a x^a$, where M is a submonoid of $\mathbb{Z}^n$. For $M = \mathbb{N}^n$, a term order is typically used. Recall from 1.1.6 that a term order is a total order where 1 is the smallest monomial and multiplication is order-compatible. However, for a general M containing non-trivial invertible elements, such orders cannot exist (see 1.1.6).

Instead, we use a more general concept, namely that of generalized orders, introduced in [PZ96; PU99] to study systems of difference equations.

Let M be a submonoid of $\mathcal{M}$, and let F be a field.

1.4.1 Conic decomposition

The core idea of generalized orders is to decompose the monoid M into smaller monoids on each of which the neutral element is the only invertible element. This process is known as a conic decomposition of M.

Definition 1.4.1 ([PU99, Definition 2.1])**.** *A conic decomposition of M is a finite family $(M_i)_{i\in I}$ of finitely generated submonoids of M such that*

1. $\forall t \in M_i,\ (t^{-1} \in M_i \implies t = 1)$
2. *The group generated by M_i, denoted ${}_{\mathrm{gr}}\langle M_i\rangle$, is equal to $\mathcal{M}$.*
3. $\cup_{i\in I} M_i = M$.

Example 1.4.2. *Let $M = \mathcal{M}_+$. Then M is a conic decomposition of itself.*

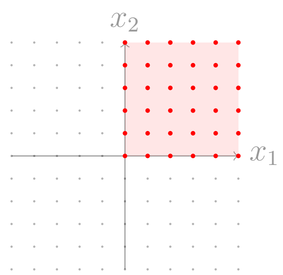
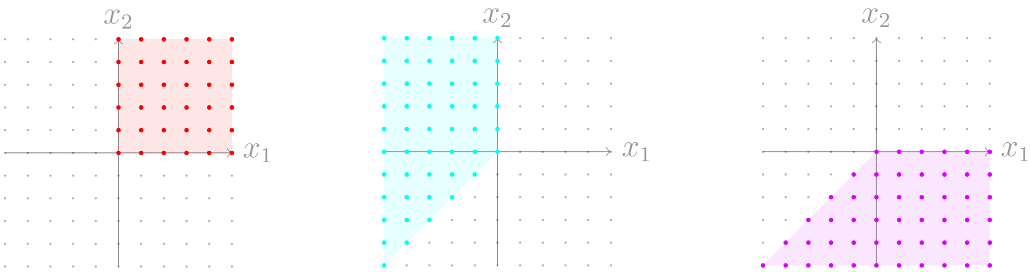


Figure 1.17: The trivial conic decomposition of $\mathcal{M}_+$ from Example 1.4.2

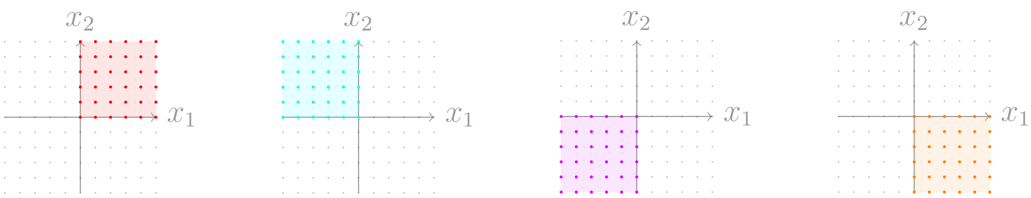
Example 1.4.3. *Let $M = \mathcal{M}$, and define $M_0 := \{x^k,\ k \in \mathbb{N}^n\}$. For $1 \le j \le n$, let M_j be generated by $\{x_1^{-1} \dots x_n^{-1}\} \cup \{x_1, \dots, \hat{x_j}, \dots, x_n\}$, where the hat means the corresponding variable is omitted. That is, M_j contains monomials where the exponent of x_j is non-positive and less than any other exponent. Then $(M_j)_{0\le j\le n}$ is a conic decomposition of $\mathcal{M}$ with $n+1$ cones.*



(a) M_0 **(b)** M_1 **(c)** M_2

Figure 1.18: Conic decomposition for Example 1.4.3 for $n = 2$

Example 1.4.4. *Let $M = \mathcal{M}$ and let D_n be the set of all maps from $\{1, \ldots, n\}$ to $\{-1, 1\}$. For d in D_n, define M_d as the monoid generated by $\{x_1^{d(1)}, \ldots, x_n^{d(n)}\}$. Then $(M_d)_{d \in D_n}$ is a conic decomposition of $\mathcal{M}$ with 2^n cones.*



(a) $M_{(1,2)\to(1,1)}$ **(b)** $M_{(1,2)\to(-1,1)}$ **(c)** $M_{(1,2)\to(-1,-1)}$ **(d)** $M_{(1,2)\to(1,-1)}$

Figure 1.19: Conic decomposition for Example 1.4.4 for $n = 2$

Example 1.4.5. *Let $M = \{x^s,\ s \in S\}$, where S is a full-dimensional rational cone in $\mathbb{R}^n$. Let $(P_i)_{i \in I}$ be the maximal cones of a pointed rational polyhedral fan whose support contains S. Define $M_i := M \cap \{x^s,\ s \in P_i\}$. Each M_i is finitely generated by Gordon's Lemma. Then $(M_i)_{i \in I}$ is a conic decomposition of M. Examples 1.4.2 to 1.4.4 are special cases of Example 1.4.5.*

1.4.2 Leading terms

The definition of generalized order depends on a prior conic decomposition of M. Different orders on M may share the same decomposition.

Definition 1.4.6 ([PU99, Definition 2.2])**.** *A generalized monomial order (or g.m.o) on M for the decomposition $(M_i)_{i \in I}$ is a total order $\leq$ such that*

1. $\forall t \in M, 1 \leq t$

2. $\forall r \in M, \forall i \in I$, $(s, t \in M_i \text{ and } r < s) \implies rt < st$

Remark 1.4.7. *For each $i \in I$, the restriction of $\leq$ to M_i is a monomial order (take r in M_i in 2. of Definition 1.4.6).*

Remark 1.4.8. *In [PU99], the notion of g.m.o is defined in a more general setting, where $K[M]$ is replaced by V, a finite-dimensional free $R[M]$-module with basis B, with R a Noetherian ring. We do not need this level of generality and restrict to the polynomial case, which corresponds to $R = K$, $V = K[M]$, and $B = \{1\}$, as stated just before [PU99, Definition 2.2].*

A generalized order is not compatible with multiplication, but it remains a well-order:

Lemma 1.4.9 ([PU99, Lemma 2.2])**.** *Every strictly decreasing sequence in M is finite.*

Given a conic decomposition, Lemma 1.4.10 provides a method to construct a g.m.o using an auxiliary function $\phi : M \to \mathbb{Q}_{\geq 0}$, which behaves well with respect to the decomposition. Examples of such ϕ are given in Examples 1.4.12 to 1.4.14.

Lemma 1.4.10 ([PU99, Lemma 2.1])**.** *Let $(M_i)_{i\in I}$ be a conic decomposition of M. Let E be either $\{1\}$ or some M_i. Let $<_G$ be a total order on M compatible with the monoid structure. Let $\phi : M \to \mathbb{Q}_{\geq 0}$ satisfy:*

1. $\forall t \in M \setminus E$, $\phi(t) > 0$.

2. $\forall s, t \in M$, $\phi(st) \leq \phi(s) + \phi(t)$.

3. $\forall i \in I$, $\phi|_{M_i}$ *is a monoid homomorphism.*

The order $<$ defined by

$$r < s \iff \phi(r) < \phi(s) \text{ or } (\phi(r) = \phi(s) \text{ and } r <_G s)$$

is a g.m.o on M for the decomposition $(M_i)_{i\in I}$.

The notions of leading monomial, leading coefficient, and leading term for a g.m.o are defined in $F[M]$ as in the case of term orders.

Definition 1.4.11. *Fix a g.m.o on M and let $f \in F[M]$. The leading monomial* $\mathrm{lm}(f)$ *of f is defined as* $\max(x^j,\ j \in \mathrm{supp}(f))$. *The leading coefficient* $\mathrm{lc}(f)$ *of f is the coefficient of* $\mathrm{lm}(f)$ *in f. The leading term* $\mathrm{lt}(f)$ *of f is the product* $\mathrm{lc}(f)\mathrm{lm}(f)$.

Examples 1.4.12 to 1.4.14 illustrate Lemma 1.4.10 and Definition 1.4.11. In each case, the order on $\mathcal{M}$ is the lexicographical order (with $x_1 > \cdots > x_n$), and we show for $n = 2$ how the monomials of

$$f = x_1x_2^{-2} + x_1^{-2}x_2^{-2} + x_1^{-1}x_2^{-2} + x_2^2 \in F[x_1^{\pm 1}, x_2^{\pm 1}]$$

are ordered.

Example 1.4.12. *Take the decomposition $(M_0, M_1, \ldots, M_n)$ of Example 1.4.3. Let $E = \{1\}$ and define $\phi : M \to \mathbb{Q}_{\geq 0}$ by*

$$\phi(i_1, \ldots, i_n) = i_1 + \cdots + i_n - (n+1)\min(0, i_1, \ldots, i_n).$$

Then

$$x_1x_2^{-2} > x_1^{-1}x_2^{-2} > x_2^2 > x_1^{-2}x_2^{-2} \text{ and so } \mathrm{lm}(f) = x_1x_2^{-2}.$$

Example 1.4.13. *Take the decomposition $(M_0, M_1, \ldots, M_n)$ of Example 1.4.3. Let $E = M_0$ and define $\phi : M \to \mathbb{Q}_{\geq 0}$ by*

$$\phi(i_1, \ldots, i_n) = -\min(0, i_1, \ldots, i_n).$$

Then

$$x_1x_2^{-2} > x_1^{-1}x_2^{-2} > x_1^{-2}x_2^{-2} > x_2^2 \text{ and so } \mathrm{lm}(f) = x_1x_2^{-2}.$$

Example 1.4.14. *Take the conic decomposition $(M_d)_{d \in D_n}$ of Example 1.4.4. Let $E = \{1\}$ and define $\phi : M \to \mathbb{Q}_{\geq 0}$ by*

$$\phi(i_1, \ldots, i_n) = |i_1| + \cdots + |i_n|.$$

Then

$$x_1^{-2}x_2^{-2} > x_1x_2^{-2} > x_1^{-1}x_2^{-2} > x_2^2 \text{ and so } \mathrm{lm}(f) = x_1^{-2}x_2^{-2}.$$

Since a g.m.o is not fully compatible with multiplication, the leading monomial is not necessarily stable under multiplication.

Example 1.4.15. *Take the g.m.o of Example 1.4.12 in the case $n = 2$ and*

$$f = x_1x_2 + x_2^{-1} \in K[x_1^{\pm 1}, x_2^{\pm 1}].$$

Then

$$\mathrm{lm}(x_2f) = x_1x_2^2, \quad \textit{but} \quad x_2\mathrm{lm}(f) = 1 \neq x_1x_2^2.$$

However, by the compatibility condition (2) in Definition 1.4.6, there is exactly one "leading monomial per cone." Before proving this in Lemma 1.4.18, a definition is introduced:

Definition 1.4.16. *For $i \in I$ and $f \in F[M]$, define*

$$M_i(f) := \{t \in M \mid \mathrm{lm}(tf) \in M_i\}.$$

Proposition 1.4.17. *Let $i \in I$ and $f \in F[M]$. We have $M_i(f) \neq \emptyset$.*

Proof. Since M_i is finitely generated and generates $\mathcal{M}$ as a group, each monomial s of f can be written as $s = u_sv_s^{-1}$ for some $u_s, v_s \in M_i$. Define t as the product of the monomials v_s for all monomials s of f. This ensures $\mathrm{supp}(tf) \subseteq M_i$, and thus $\mathrm{lm}(tf) \in M_i$. This shows $t \in M_i(f)$. □

Lemma 1.4.18 ([PU99, Lemma 2.3])**.** *Let $i \in I$, $f \in F[M]$, and $u, v \in M_i(f)$. Write $\mathrm{lm}(uf) = ut_u \in M_i$ and $\mathrm{lm}(vf) = vt_v \in M_i$ for some monomials t_u, t_v of f. Then $t_u = t_v$.*

Definition 1.4.19. *Let $i \in I$, $f \in F[M]$ and $t \in M_i(f)$. We define $\mathrm{lm}_i(f) := \mathrm{lm}(tf)t^{-1}$, which is a monomial of f. It is well defined by Lemma 1.4.18 (i.e it does not depend on a particular $t \in M_i(f)$). By construction, we have:*

$$t \in M_i(f) \implies \mathrm{lm}(tf) = t\mathrm{lm}_i(f)$$

We also define $\mathrm{lc}_i(f)$ as the coefficient of $\mathrm{lm}_i(f)$ in f, and $\mathrm{lt}_i(f) = \mathrm{lc}_i(f)\mathrm{lm}_i(f)$.

To compute $\mathrm{lm}_i(f)$, it suffices to find an element $t \in M_i(f)$. Then, by Lemma 1.4.18, we obtain

$$\mathrm{lm}_i(f) = \mathrm{lm}(tf)\, t^{-1}.$$

An element $t \in M_i(f)$ can be constructed as in the proof of Proposition 1.4.17. This leads to Algorithm 3 to compute such a t and Algorithm 4 to compute $\mathrm{lm}_i(f)$.

Algorithm 3: TranslatorForCone

---

input : $f \in F[M]$, $(g_1, \ldots, g_k)$ generators of M_i
output : A monomial t such that $\mathrm{Supp}(tf) \subseteq M_i$

$t \leftarrow 1$;
for *each monomial s of f* **do**
  find some $n_1, \ldots, n_k \in \mathbb{Z}^n$ such that $s = g_1^{n_1} \ldots g_k^{n_k}$;
  $v \leftarrow \prod g_j^{-n_j}$ for all $n_j \leq 0$;
  $t \leftarrow t \times v$;
return t

Algorithm 4: LeadingMonomialForCone

---

input : $f \in F[M]$, $(g_1, \ldots, g_k)$ generators of M_i
output : The monomial $\mathrm{lm}_i(f)$

$t \leftarrow$ TranslatorForCone$(f, (g_1, \ldots, g_k))$;
return $\frac{\mathrm{lm}(tf)}{t}$

Example 1.4.20 (Example 1.4.12 continued.)**.** *We have* $\mathrm{lm}_0(f) = \mathrm{lm}_1(f) = y^2$ *and* $\mathrm{lm}_2(f) = \mathrm{lm}(f) = xy^{-2}$.

Example 1.4.21 (Example 1.4.13 continued.)**.** *We have* $\mathrm{lm}_0(f) = \mathrm{lm}_2(f) = \mathrm{lm}(f) = xy^{-2}$ *and* $\mathrm{lm}_1(f) = x^{-2}y^{-2}$.

Remark 1.4.22. *We have* $\mathrm{lm}(f) = \mathrm{lm}_i(f)$ *for at least one index* $i \in I$*. For a fixed* f*, the* $\mathrm{lm}_i(f)$*'s are not necessarily distinct, and* $\mathrm{lm}_i(f)$ *is generally not an element of* M_i*.*

Lemma 1.4.23. *For all $f, g \in F[M]$ and $i \in I$ the sets:*

1. $\mathrm{lm}_i(f)M_i(f) \subseteq M_i$
2. $\mathrm{lm}_i(f)M_i(f) \cap \mathrm{lm}_i(g)M_i(g) \subseteq M_i$

are finitely generated M_i-module.

Proof. Let us first prove that $M_i(f)$ is an M_i-module. Take $s \in M_i(f)$, so that $\mathrm{lm}(sf) \in M_i$, and let $t \in M_i$. If u is a term of sf, then $u \leq \mathrm{lm}(sf)$. So we have

$$t,\ \mathrm{lm}(sf) \in M_i \quad \text{and} \quad u \leq \mathrm{lm}(sf).$$

By item 2 of Definition 1.4.6, we obtain $tu \leq t\mathrm{lm}(sf)$. Thus, every term of tsf, which can be written as tu for some term u of sf, is smaller than the term $t\mathrm{lm}(sf)$, showing that

$$\mathrm{lm}(tsf) = t\mathrm{lm}(sf) \in M_i.$$

Hence $ts \in M_i(f)$, proving that $M_i(f)$ is indeed an M_i-module.

Consequently, $\mathrm{lm}_i(f)M_i(f)$ is also an M_i-module, isomorphic to $M_i(f)$ via the bijection

$$M_i(f) \ \to\ \mathrm{lm}_i(f)M_i(f), \quad a \ \mapsto\ \mathrm{lm}_i(f)a.$$

Moreover, since $\mathrm{lm}_i(f)M_i(f) \subseteq M_i$ by definition, both $\mathrm{lm}_i(f)M_i(f)$ and $\mathrm{lm}_i(f)M_i(f) \cap \mathrm{lm}_i(g)M_i(g)$ are M_i-ideals. Finally, as M_i is finitely generated, these ideals are finitely generated M_i-modules by Proposition 1.1.1.

□

The generators of $\mathrm{lm}_i(f)M_i(f) \cap \mathrm{lm}_i(g)M_i(g) \subseteq M_i$ can be interpreted as the least common multiples of f and g with respect to the cone M_i. Later, in Theorem 5.1.5, we will show that for a g.m.o with underlying conic decomposition given by Example 1.4.3, the module $M_i(f)$ is cyclic, i.e. generated by a single element.

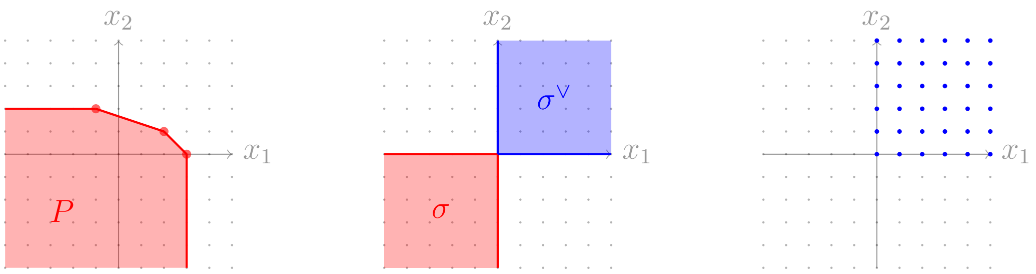
2

Positive exponents

Summary

Introduction

In this chapter, we develop a Gröbner bases theory for polyhedral algebras with positive exponents, where the monoid of exponents S_σ is $\mathbb{N}^n$. Recall from Example 1.2.4 that these algebras are defined by a polyhedron of the following form:



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

Figure 2.1: Positive exponents

and that it contains the case of $K_r\langle x\rangle$.
We use the following notation in this chapter: if $r_1, \dots, r_s \in \mathbb{Q}^n$, we define

$$\mathrm{Conv}^+(r_1, \dots, r_s) := \mathrm{Conv}\left(\bigcup_i (r_i + (-\infty, 0]^n)\right).$$

Fix a polyhedron P such that $P = \mathrm{Conv}^+(r_1, \dots, r_s)$ for some $r_1, \dots, r_s \in \mathbb{Q}^n$. We always assume that $r_1, \dots, r_s$ are exactly the vertices of P, that is, no r_i is contained in the interior of $\mathrm{Conv}^+(r_1, \dots, r_s)$. For a polyhedral algebra defined by such a polyhedron, we use the notation $K_P^+\{x\}$ to emphasize that exponents are positive.

When working in $K_P^+\{x\}$, the advantage is that we can use a standard term order (lexicographic, degrevlex, etc.) to order monomials in $K_P^+\{x\}$ in a way compatible with multiplication. That way, compared to the known case of $K_r\langle x\rangle$, we only need to handle the fact that val_P is not multiplicative, but only submultiplicative:

$$\mathrm{val}_P(fg) \geq \mathrm{val}_P(f) + \mathrm{val}_P(g).$$

This creates difficulties in properly defining the leading part of an ideal in $K_P^+\{x\}$. Indeed, if we choose to order the terms of a series $f \in K_P^+\{x\}$ first by val_P, and then by a term order as a tie-breaker it may happen that $\mathrm{in}_P(mf) \neq m \times \mathrm{in}_P(f)$. Here is an example:

Example 2.0.1. *Take* $r_1 = (1,2)$*,* $r_2 = (2,1)$*,* $P = \mathrm{Conv}^+(r_1, r_2)$*,* $K = \mathbb{Q}_2$*. Let* $f = x_1^2 + \frac{1}{2}x_2 \in K_P^+\{x_1, x_2\}$ *and* $m = x_2^2 \in \mathcal{M}_+$*. We have* $\mathrm{in}_P(f) = x_1^2$*, but:*

$$\mathrm{in}_P(mf) = m \times \frac{1}{2}x_2 \neq m \times \mathrm{in}_P(f). \tag{2.1}$$

A direct consequence is that, with such an ordering, in general for $m \in \mathcal{M}_+$:

$$\mathrm{lm}(m \times f) \neq m \times \mathrm{lm}(f). \tag{2.2}$$

The issue in Example 2.0.1 is that $\mathrm{val}_P(f)$ is attained at val_{r_2}, while $\mathrm{val}_P(m)$ is attained at val_{r_1}, which allows a compensation to occur.
On the other hand, if both $\mathrm{val}_P(f)$ and $\mathrm{val}_P(m)$ are attained at the same unique vertex r_i, then it is easy to check that $\mathrm{in}_P(mf) = m \times \mathrm{in}_P(f)$, and more precisely:

$$\mathrm{in}_P(mf) = \mathrm{in}_{r_i}(mf) = m \times \mathrm{in}_{r_i}(f) = m \times \mathrm{in}_P(f) \tag{2.3}$$

As one can see in this case, val_P behaves *locally*, for this choice of f and m, like val_{r_i}, which is a multiplicative valuation.

From this, the strategy becomes clear: we partition $\mathcal{M}_+$ into regions, each associated to a vertex of P, on which the favorable behavior of Equation 2.3 holds. However, difficulties arise in the ambiguous case where $\mathrm{val}_P(f)$ is attained at more than one vertex. To resolve this, we first fix an arbitrary order on the vertices of P to break ties among the r_i's attaining val_P, before ultimately selecting the leading term using a term order (see Definition 2.1.7).

The consequence of the above strategy is that we need to define one leading monomial for each vertex r_i, and likewise for the least common multiple of the leading monomials of two series f, g: there will be multiple lcm's per vertex. Apart from these modifications, the proof strategy for establishing the final

Buchberger criterion (Proposition 2.4.1) follows the same pattern as in the classical polynomial case.

Recall that Convention 1.2.1 is in force: K is a complete discretely valued field equipped with a nontrivial valuation $\mathrm{val} : K \to \mathbb{Q} \cup \{\infty\}$ normalized so that $\mathrm{val}(K^\times) = \mathbb{Z}$.

2.1 Setting

We assign an (arbitrary) indexing to the vertices of P: $\mathrm{vert}(P) = \{r_1, \ldots, r_s\}$. Let I_P denote the set of indices: $I_P := \{1, \ldots, s\}$. This indexing will be used to resolve ties in Definition 2.1.7, when $\mathrm{val}_P(f) = \mathrm{val}_{r_i}(f)$ for more than one r_i.

Definition 2.1.1. *For $f \in K_P^+\{x\}$, we define*

$$I_P(f) := \{i \in I_P \mid \mathrm{val}_P(f) = \mathrm{val}_{r_i}(f)\} \subseteq I_P.$$

Thus, $I_P(f) \subseteq I_P$ is the subset of indices at which $\mathrm{val}_P(f)$ is attained. We give two examples to illustrate Definition 2.1.1.

Example 2.1.2. *Set $r_1 = (1, 2)$, $r_2 = (2, 1)$. Take $P = \mathrm{Conv}^+(r_1, r_2)$, $K = \mathbb{Q}_2$, and*

$$f = x_1^2 + x_2^2 + \frac{1}{2}x_1x_2 + x_1 + x_2 \in K_P^+\{x_1, x_2\}.$$

We have

$$\mathrm{val}_P(f) = \min_{i \in I_P} \mathrm{val}_{r_i}(f) = \mathrm{val}_{r_1}(f) = \mathrm{val}_{r_2}(f) = -4,$$

so $I_P(f) = \{1, 2\}$. This common minimum is reached at:

- x_1^2 *and* $\frac{1}{2}x_1x_2$ *for* val_{r_1}
- x_2^2 *and* $\frac{1}{2}x_1x_2$ *for* val_{r_2}

Notice that the term $\frac{1}{2}x_1x_2$ appears in both $\mathrm{in}_{r_1}(f) = x_1^2 + \frac{1}{2}x_1x_2$ and $\mathrm{in}_{r_2}(f) = x_2^2 + \frac{1}{2}x_1x_2$.

We now take as illustration the case of f being a single term:

Example 2.1.3. *Set* $r_1 = (3,0)$, $r_2 = (2,2)$, $r_3 = (0,3)$. *Take* $P = \mathrm{Conv}^+(r_1, r_2, r_3)$, $K = \mathbb{Q}_2$, *and*

$$f = x_1^2 x_2 \in K_P^+\{x_1, x_2\}.$$

We have

$$\mathrm{val}_P(f) = \min_{i \in I_P} \mathrm{val}_{r_i}(f) = \mathrm{val}_{r_1}(f) = \mathrm{val}_{r_2}(f) = -6,$$

so $I_P(f) = \{1, 2\}$. *Since* f *is a single term* $x_1^2 x_2$, *this common minimum is necessarily reached at* $x_1^2 x_2$.

For each $i \in I_P$, we consider monomials $x^\alpha \in \mathcal{M}_+$ satisfying

$$\mathrm{val}_P(x^\alpha) = \mathrm{val}_{r_i}(x^\alpha),$$

and those for which i is the smallest index (in the vertex ordering) achieving this equality. These sets, introduced in Definition 2.1.4, are denoted V_i and $V_{i,<}$, respectively. Note that $V_{i,<}$ is a subset of V_i, and depends on the chosen vertex ordering.

Definition 2.1.4. *For each* $i \in I_P$, *we define:*

- *The set* $V_i \subseteq \mathcal{M}_+$ *by:*

$$\begin{aligned} V_i &= \{x^\alpha \in \mathcal{M}_+ \mid i \in I_P(x^\alpha)\} \\ &= \{x^\alpha \in \mathcal{M}_+ \mid \mathrm{val}_P(x^\alpha) = \min_{j \in I_P} \mathrm{val}_{r_j}(x^\alpha) = \mathrm{val}_{r_i}(x^\alpha)\} \\ &= \{x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P,\ \mathrm{val}_{r_i}(x^\alpha) \leq \mathrm{val}_{r_j}(x^\alpha)\} \\ &= \{x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P,\ r_i \cdot \alpha \geq r_j \cdot \alpha\} \\ &= \{x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P,\ (r_i - r_j) \cdot \alpha \geq 0\}. \end{aligned}$$

- *The subset $V_{i,<} \subseteq V_i$ by:*

$$\begin{aligned} V_{i,<} &= \{x^\alpha \in \mathcal{M}_+ \mid \min(I_P(x^\alpha)) = i\} \\ &= \left\{ x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P \quad \begin{cases} \mathrm{val}_{r_i}(x^\alpha) < \mathrm{val}_{r_j}(x^\alpha) & \textit{if } j < i \\ \mathrm{val}_{r_i}(x^\alpha) \leq \mathrm{val}_{r_j}(x^\alpha) & \textit{if } j \geq i \end{cases} \right\} \\ &= \left\{ x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P \quad \begin{cases} r_i \cdot \alpha > r_j \cdot \alpha & \textit{if } j < i \\ r_i \cdot \alpha \geq r_j \cdot \alpha & \textit{if } j \geq i \end{cases} \right\} \\ &= \left\{ x^\alpha \in \mathcal{M}_+ \mid \forall j \in I_P \quad \begin{cases} (r_i - r_j) \cdot \alpha > 0 & \textit{if } j < i \\ (r_i - r_j) \cdot \alpha \geq 0 & \textit{if } j \geq i \end{cases} \right\}. \end{aligned}$$

We introduce C_i and $C_{i,<}$, the rational counterparts of V_i and $V_{i,<}$. They are defined by the same equations, replacing each monomial with its exponent, and working in $\mathbb{R}_+^n$ instead of $\mathbb{N}^n$.

Definition 2.1.5. *For each $i \in I_P$, we define:*

- *The rational cone $C_i \subseteq \mathbb{R}_+^n$ by:*

$$\begin{aligned} C_i &= \{\alpha \in \mathbb{R}_+^n \mid i \in I_P(x^\alpha)\} \\ &= \left\{\alpha \in \mathbb{R}_+^n \mid \forall j \in I_P,\ (r_i - r_j) \cdot \alpha \geq 0\right\} \\ &= \left\{ \alpha \in \mathbb{R}^n \mid \begin{cases} \forall j \in I_P,\ (r_i - r_j) \cdot \alpha \geq 0 \\ \forall k \in \{1, \ldots, n\},\ \alpha_k \geq 0 \end{cases} \right\} \end{aligned}$$

- *The subset $C_{i,<} \subseteq C_i$ by:*

$$\begin{aligned} C_{i,<} &= \{\alpha \in \mathbb{R}_+^n \mid i = \min(I_P(x^\alpha))\} \\ &= \left\{ \alpha \in \mathbb{R}^n \mid \forall j \in I_P,\ \forall k \in \{1, \ldots, n\}, \ \begin{cases} (r_i - r_j) \cdot \alpha > 0 & \text{if } j < i \\ (r_i - r_j) \cdot \alpha \geq 0 & \text{if } j \geq i \\ \alpha_k \geq 0 \end{cases} \right\} \end{aligned}$$

The following is immediate from Definitions 2.1.4, 2.1.5 and Section 1.1.7. See Example 2.1.6 for illustrations.

1. The cones $(C_i)_{i\in I_P}$ are the maximal cones of $\mathcal{N}(P)$, and C_i corresponds to the vertex r_i under the canonical bijection between the cones of $\mathcal{N}(\mathcal{P})$ and the faces of P.

2. $\bigcup_{i\in I_P} C_i = \mathrm{Supp}(\mathcal{N}(P)) = \mathrm{rec}(P)^\vee = \mathbb{R}^n_+$.

3. $V_i = \{x^\alpha \mid \alpha \in C_i \cap \mathbb{N}^n\}$, that is, the monoid V_i can be identified with the set of integer points in C_i.

4. $\bigcup_{i\in I_P} V_i = \mathcal{M}_+$

5. $V_{i,<}$ is a V_i-module.

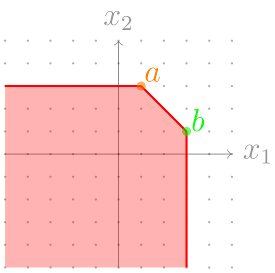
6. $V_{i,<}$ (resp. $C_{i,<}$) is obtained from V_i (resp. C_i) by replacing each inequality involving $j < i$ with a strict inequality, hence the sign $<$ in the notation.

7. $V_{i,<} = \{x^\alpha \mid \alpha \in C_{i,<} \cap \mathbb{N}^n\}$, that is, the subset $V_{i,<} \subseteq V_i$ corresponds to the integer points in $C_{i,<}$.

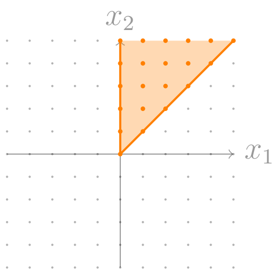
8. The sets $V_{i,<}$ are pairwise disjoint, and $\coprod_{i\in I_P} V_{i,<} = \mathcal{M}_+$.

9. The sets $C_{i,<}$ are pairwise disjoint, and $\coprod_{i\in I_P} C_{i,<} = \mathbb{R}^n_+$.

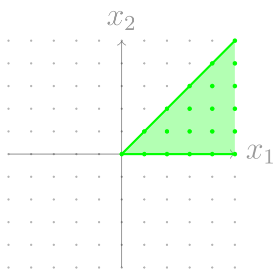
Example 2.1.6. *We illustrate Definitions 2.1.4 and 2.1.5. Let $a = (1,3)$, $b = (3,1)$, and $P = \mathrm{Conv}^+(a,b)$. For now, the vertices a and b are not ordered. We denote by C_a and C_b the cones of $\mathcal{N}(P)$ corresponding to a and b, respectively.*



(a) P



(b) C_a and V_a



(c) C_b and V_b

Now we illustrate $C_{i,<}$ and $V_{i,<}$, which require a choice of ordering between a and b. First, set $r_1 = a$ and $r_2 = b$. For this choice, we have:

(d) $C_{1,<}$ and $V_{1,<}$ **(e)** $C_{2,<}$ and $V_{2,<}$

Figure 2.2: $C_{i,<}$ and $V_{i,<}$ for $r_1 = a$, $r_2 = b$

Now choose the other possible ordering, that is $r_1 = b$ and $r_2 = a$. We have:

(a) $C_{2,<}$ and $V_{2,<}$ **(b)** $C_{1,<}$ and $V_{1,<}$

Figure 2.3: $C_{i,<}$ and $V_{i,<}$ for $r_1 = b$, $r_2 = a$

Notice that whichever ordering is chosen, we always have $C_1 = C_{1,<}$ and $V_1 = V_{1,<}$. This is not specific to this example: it is always the case, since the condition $j < 1$ in the definitions of $V_{1,<}$ and $C_{1,<}$ is vacuous, so we recover the definitions of V_1 and C_1. On the other hand, for any index $i > 1$, the inclusions $V_{i,<} \subsetneq V_i$ and $C_{i,<} \subsetneq C_i$ are always strict.

To order terms in $K_P^+\{x\}$, we use a val_P-first ordering with a term order as tie-breaker. But between these two, we insert an extra rule: when two terms have the same val_P, we prioritize the term for which val_P is reached at r_i with the smallest $i \in I_P$. If equality still holds, we finally use a term order to definitively break ties.

Definition 2.1.7. *Let $\leq_t$ be a term order on $\mathcal{M}_+$. We define a preorder $\leq_{P,t}$ (or simply $\leq_P$ if $\leq_t$ is understood) on $\mathcal{T}_+$ by:*

$$ax^u \leq_P bx^v \iff \begin{cases} \mathrm{val}_P(bx^v) < \mathrm{val}_P(ax^u) & (1) \\ \textbf{or} & \\ \text{equality in (1) } \textbf{and } \min(I_P(bx^v)) < \min(I_P(ax^u)) & (2) \\ \textbf{or} & \\ \text{equality in (2) } \textbf{and } x^v \geq_t x^u & \end{cases}$$

Note that in Definition 2.1.7, when $\mathrm{Vert}(P) = \{r\}$ (the Tate algebra case), val_P coincides with val_r, and condition (2) becomes vacuous and can be omitted. In this situation, the preorder $\leq_P$ reduces to $\leq_r$ as defined in 1.3.1 for Tate algebra.

Example 2.1.8. *Take $r_1 = (-1, -2)$, $r_2 = (-2, -1)$, $P = \mathrm{Conv}^+(r_1, r_2)$, $K = \mathbb{Q}_2$, and let $\geq_t$ be the lexicographic term order.*
Let us consider the order of the terms

$$\left\{x_1^3, \frac{1}{2}x_1, 2x_1, \frac{1}{8}x_1x_2, 2x_2, \frac{1}{2}x_2, 4x_1^2\right\}$$

using $\geq_P$ for this data.
We have three val_P-blocks:

1. *The block $\{\frac{1}{2}x_1, \frac{1}{8}x_1x_2, \frac{1}{2}x_2\}$ with valuation 0. The valuation is attained at r_1 for $\frac{1}{2}x_1$, at both r_1 and r_2 for $\frac{1}{8}x_1x_2$, and at r_2 for $\frac{1}{2}x_2$.*

   By the vertex ordering rule, we have:

   $$\left\{\frac{1}{8}x_1x_2, \frac{1}{2}x_1\right\} \geq_P \frac{1}{2}x_2$$

   Next, we break the tie between $\frac{1}{2}x_1$ and $\frac{1}{8}x_1x_2$ using the lexicographic order:

   $$\frac{1}{8}x_1x_2 \geq_P \frac{1}{2}x_1 \geq_P \frac{1}{2}x_2.$$

2. *The block $\{2x_1, 2x_2\}$ with valuation 2. The valuation is attained solely at r_1 for $2x_1$ and solely at r_2 for $2x_2$.*

So we do not need the lexicographic order, and directly obtain, via the vertex ordering rule:

$$2x_1 \ \geq_P \ 2x_2.$$

3. *The block* $\{4x_1^2, x_1^3\}$ *with valuation* 6. *The valuation is attained only at* r_2 *for both terms, so we do not need the vertex ordering rule, and use the lexicographic order directly:*

$$x_1^3 \ \geq_P \ 4x_1^2.$$

Altogether, we obtain the following ordering:

$$\frac{1}{8}x_1x_2 \ \geq_P \ \frac{1}{2}x_1 \ \geq_P \ \frac{1}{2}x_2 \ \geq_P \ 2x_1 \ \geq_P \ 2x_2 \ \geq_P \ x_1^3 \ \geq_P \ 4x_1^2$$

This example illustrates that a val_P*-block may consist of terms that attain the same* val_P *through different vertices: with overlap (first block), without overlap (second block), or from a single vertex (third block).*

Lemma 2.1.9. *Let* $(t_j)_{j\geq 0}$ *be a strictly decreasing sequence of terms for* $\leq_P$*. Then*

$$\lim_{j\to+\infty} \mathrm{val}_P(t_j) = +\infty.$$

Proof. Since the sequence $(t_j)_{j\geq 0}$ is strictly decreasing for $\leq_P$, the sequence $(\mathrm{val}_P(t_j))_{j\geq 0}$ is nondecreasing, that is, for all j we have $\mathrm{val}_P(t_{j+1}) \geq \mathrm{val}_P(t_j)$. Because val is assumed to be discrete, val_P takes values in $\frac{1}{D}\mathbb{Z}$ for some integer D (for example, the least common multiple of the denominators of the coefficients of the vertices of P). Now, since $\geq_t$ is a well-order and there is only a finite number of vertices, for each v in the discrete set $\frac{1}{D}\mathbb{Z}$, there can only be finitely many indices j such that $\mathrm{val}_P(t_j) = v$. Therefore, the sequence $(\mathrm{val}_P(t_j))_{j\geq 0}$ must tend to $+\infty$. □

Definition 2.1.10. *For* $f \in K_P\{x\}$*,* $i \in I_P$*, define:*

1. $\mathrm{lm}(f), \mathrm{lc}(f)$ *and* $\mathrm{lt}(f)$*, the leading monomial, coefficient and term of* f *for the preorder* $\leq_P$ *of Definition 2.1.7*

2. $\mathrm{in}_{r_i}(f) := \sum_{\{t \in \mathrm{terms}(f) \mid \mathrm{val}_{r_i}(t) = \mathrm{val}_{r_i}(f)\}} t$

3. $\mathrm{in}_{P,<}(f) := \mathrm{in}_{r_i}(f)$ *where* $i = \min(I_P(\mathrm{lm}(f)))$.

4. $\mathrm{lm}^i(f), \mathrm{lc}^i(f)$ *and* $\mathrm{lt}^i(f)$ *the leading monomial, coefficient and term of* $\mathrm{in}_{r_i}(f)$ *for the term order* $\geq_t$.

5. $V_{i,<}(f) := \{m \in \mathcal{M}_+,\ \mathrm{lm}(mf) \in V_{i,<}\}$

6. $\mathrm{LM}^i(f) := \{\mathrm{lm}(mf),\ m \in \mathcal{M}_+\} \cap V_{i,<}$.

We will need the following lemma in the proof of item 3 in Proposition 2.1.12 below.

Lemma 2.1.11. *Suppose* $f \in K_P^+\{x\}$ *satisfies* $\mathrm{lm}(f) \in V_{i,<}$ *and let* $t \in V_i$. *Then* $\mathrm{lm}(tf) \in V_{i,<}$.

Proof. We have:

$$\mathrm{val}_P(tf) = \min_{j \in I_P}(\mathrm{val}_{r_j}(tf)) = \min_{j \in I_P}(\mathrm{val}_{r_j}(t) + \mathrm{val}_{r_j}(f)). \tag{2.4}$$

Since $t \in V_i$, we have $i \in I_P(t)$, that is $\min_{j \in I_P}(\mathrm{val}_{r_j}(t)) = \mathrm{val}_{r_i}(t)$. Similarly, $\mathrm{lm}(f) \in V_{i,<} \subseteq V_i$, so $i \in I_P(f)$ and $\min_{j \in I_p}(\mathrm{val}_{r_j}(f)) = \mathrm{val}_{r_i}(f)$. Thus for any $(j,k) \in I_P^2$, we have $\mathrm{val}_{r_i}(t) + \mathrm{val}_{r_i}(f) \leq \mathrm{val}_{r_j}(t) + \mathrm{val}_{r_k}(f)$. This is true in particular when $j = k$, so the minimum in (2.4) is reached at i, that is $i \in I_P(tf)$.
Now we show that if $j \in I_P(tf)$, then $i \leq j$. Suppose that $j \in I_P(tf)$. Then $\mathrm{val}_{r_j}(t) + \mathrm{val}_{r_j}(f) = \mathrm{val}_{r_i}(t) + \mathrm{val}_{r_i}(f)$. Since $t \in V_i$, $\mathrm{val}_{r_j}(t) \geq \mathrm{val}_{r_i}(t)$, so $\mathrm{val}_{r_j}(f) \leq \mathrm{val}_{r_i}(f) = \mathrm{val}_P(f)$. Thus $\mathrm{val}_{r_j}(f) = \mathrm{val}_P(f)$ and $j \in I_P(f)$. In addition $\mathrm{lm}(f) \in V_{i,<}$, thus $i = \min(I_P(f))$, and then $i \leq j$.
Finally $i = \min(I_P(tf))$, that is $\mathrm{lm}(tf) \in V_{i,<}$. ☐

Proposition 2.1.12. *Let* $f, g \in K_P^+\{x\}$, $i \in I_P$ *and* $a \in V_{i,<}(f)$. *Then:*

1. $\mathrm{lm}(af) = a \times \mathrm{lm}^i(f)$.

2. $\mathrm{LM}^i(f) = \mathrm{lm}^i(f) V_{i,<}(f)$.

3. $\mathrm{LM}^i(f)$ *and* $\mathrm{LM}^i(f) \cap \mathrm{LM}^i(g)$ *are finitely generated* V_i*-modules.*

Proof. 1. By definition, $a \in V_{i,<}(f)$ means that $\operatorname{lm}(af) \in V_{i,<}$. The elements of $V_{i,<}$ are precisely the monomials t in $\mathcal{M}_+$ such that $\min(I_P(t)) = i$, so $\min(I_P(\operatorname{lm}(af))) = i$. By (3) of Definition 2.1.10, it follows that $\operatorname{in}_{P,<}(af) = \operatorname{in}_{r_i}(af)$. Now by Definition 2.1.7, $\operatorname{lm}(af)$ is the greatest term for $\geq_t$ of $\operatorname{in}_{P,<}(af) = \operatorname{in}_{r_i}(af)$. Because $\geq_t$ is a term order and val_{r_i} a multiplicative valuation, this greatest term is equal to a times the greatest term for $\geq_t$ of $\operatorname{in}_{r_i}(f)$, which by (4) of Definition 2.1.10 equals $\operatorname{lm}^i(f)$. Finally, $\operatorname{lm}(af) = a \times \operatorname{lm}^i(f)$.

2. By definition, $\operatorname{LM}^i(f) := \{\operatorname{lm}(mf) \mid m \in \mathcal{M}_+\} \cap V_{i,<}$, which can be rewritten as
$$\operatorname{LM}^i(f) = \{\operatorname{lm}(mf) \mid m \in V_{i,<}(f)\}.$$
Applying item (1), we get:
$$\operatorname{LM}^i(f) = \{m \times \operatorname{lm}^i(f) \mid m \in V_{i,<}(f)\} = \operatorname{lm}^i(f) V_{i,<}(f).$$

3. We first show that $\operatorname{LM}^i(f)$ (and therefore $\operatorname{LM}_i(g)$ and $\operatorname{LM}^i(f) \cap \operatorname{LM}^i(g)$) is a V_i-module. Since $\operatorname{LM}^i(f) = \operatorname{lm}^i(f) V_{i,<}(f)$ by item (2), it suffices to show that $V_{i,<}(f)$ is a V_i-module.

   Let $t \in V_{i,<}(f)$ and $s \in V_i$. By definition of $V_{i,<}(f)$, $\operatorname{lm}(tf) \in V_{i,<}$. By Lemma 2.1.11, $\operatorname{lm}(stf) \in V_{i,<}$. Hence $st \in V_{i,<}(f)$, so $V_{i,<}(f)$ is a V_i-module.

   Now, since $\operatorname{LM}^i(f) \subseteq V_i$ and $\operatorname{LM}^i(f) \cap \operatorname{LM}^i(g) \subseteq V_i$, both $\operatorname{LM}^i(f)$ and $\operatorname{LM}^i(f) \cap \operatorname{LM}^i(g)$ are V_i-ideals. As V_i is finitely generated, these ideals are finitely generated V_i-modules by proposition 1.1.1.

□

2.2 Gröbner bases and reduction

In this section, we define Gröbner bases for ideals in $K_P^+\{x\}$ and adapt the reduction algorithm for Tate algebras (Algorithm 1) to $K_P^+\{x\}$, resulting in Algorithm 6.

Definition 2.2.1. *Let J be an ideal in $K_P^+\{x\}$, and let G be a finite subset of $J \setminus \{0\}$. We say that G is a Gröbner basis of J with respect to $\geq_P$ if the following holds:*

$$\mathrm{lm}(J) := \{\mathrm{lm}(f) \mid f \in J\} = \bigcup_{i \in I_P,\ g \in G} \mathrm{LM}^i(g).$$

Definition 2.2.2. *Let $g, f \in K_P^+\{x\}$. We say that g is a reducer of f if there exists $m \in \mathcal{M}_+$ such that* $\mathrm{lm}(mg) = \mathrm{lm}(f)$.

Algorithm 5: FindReducerPositiveExponents

input : $f \in K_P^+\{x\}$ and $(g_1, \dots, g_s) \in K_P^+\{x\}^s$
output : $\emptyset$ if there is no reducer of f within $\{g_1, \dots, g_s\}$
Otherwise a triple (m, k, i) with $m \in \mathcal{M}_+$, $k \in \{1, \dots, s\}$ and $i \in I_P$ such that $\mathrm{lm}(mg_k) = \mathrm{lm}(f) \in V_{i,<}$

```
i ← the unique i ∈ I_P such that lm(f) ∈ V_{i,<};
for k ∈ {1, ..., s} do
    m ← lm(f) / lm^i(g_k);
    if m ∈ M_+ and lm(m g_k) = lm(f) then
        return (m, k, i)
return ∅
```

Algorithm 6: ReductionPositiveExponents

input : $f \in K_P^+\{x\}, (g_1, \dots, g_s) \in K_P^+\{x\}^s$

output : $(q_1, \dots, q_s) \in K_P^+\{x\}^s$ and $r \in K_P^+\{x\}$ satisfying Prop 2.2.3

$q_1, \dots, q_s, r \leftarrow 0$;
while$_\infty$ $f \neq 0$
  while FindReducerPositiveExponents$(f, (g_1, \dots, g_s))$ *returns a triple* (m, k, i) **do**
    $t \leftarrow \frac{\mathrm{lc}(f)}{\mathrm{lc}^i(g_k)} m$;
    $q_k \leftarrow q_k + t$;
    $f \leftarrow f - t g_k$;
  $r \leftarrow r + \mathrm{lt}(f)$;
  $f \leftarrow f - \mathrm{lt}(f)$;
return$_\infty$ $(q_1, \dots, q_s), r$

Proposition 2.2.3. *Let $f \in K_P^+\{x\}$ and let G be a finite subset of $K_P^+\{x\}$. Algorithm 6 called on input (f, G) outputs a family $(q_g)_{g \in G}$ and r in $K_P^+\{x\}$ such that:*

1. $f = \sum_{g \in G} q_g g + r$
2. *For all monomial m of r,*
$$m \notin \bigcup_{i \in I_P,\ g \in G} \mathrm{LM}^i(g). \tag{2.5}$$
3. *For all $g \in G$ and all term t of q_g, $\mathrm{lt}(tg) \leq_P \mathrm{lt}(f)$.*

Proof. We construct by induction sequences $(f_a)_{a \geq 0}$, $(q_{g,a})_{a \geq 0}$ for $g \in G$ and $(r_a)_{a \geq 0}$ such that for all $a \geq 0$:

$$f = f_a + \sum_{g \in G} q_{g,a} g + r_a,$$

and $\mathrm{lt}(f_a)_{a \geq 0}$ is strictly decreasing for $\leq_P$.

1. **Initialization**: Set $f_0 = f$, $r_0 = 0$ and $q_{g,0} = 0$ for all $g \in G$.

2. **Induction**: Let $i \in I_P$ be the (unique) index such that $\mathrm{lm}(f_a) \in V_{i,<}$.

   - If there exists $g \in G$ and $m \in V_{i,<}(g)$ such that
     $$\mathrm{lm}\,(mg) = \mathrm{lm}(f_a) \in V_{i,<},$$
     set $t = \frac{\mathrm{lc}(f_a)}{\mathrm{lc}^i(g)} m$, $f_{a+1} = f_a - tg$ and $q_{g,a+1} = q_{g,a} + tg$, and leave unchanged r_a and the other $q_{g,a}$'s.
   - Otherwise, set $f_{a+1} = f_a - \mathrm{lt}(f_a)$ and $r_{a+1} = r_a + \mathrm{lt}(f_a)$, leaving unchanged the $q_{g,a}$'s.

By construction, the sequence $(\mathrm{lt}(f_a))_{a\geq 0}$ is strictly decreasing for $\leq_P$. By Lemma 2.1.9, $\mathrm{val}_P(r_{a+1} - r_a)$ and the $\mathrm{val}_P(q_{g,a+1} - q_{g,a})$'s tend to $+\infty$ when $a \to +\infty$. Thus r_a and the $q_{g,a}$'s converge in $K_P^+\{x\}$. Their limits satisfy the requirements of the proposition. □

Let us comment on Algorithm 5, which is called in line 3 of Algorithm 6. Assuming that $\mathrm{lm}(f) \in V_{i,<}$, its purpose is to test whether there exists a divisor g_k and a monomial $m \in V_{i,<}(g_k)$ such that $\mathrm{lm}(mg_k) = \mathrm{lm}(f) \in V_{i,<}$.
If such g_k and m exist, then

$$\mathrm{lm}(mg_k) = m\mathrm{lm}^i(g_k) = \mathrm{lm}(f),$$

which implies

$$m = \frac{\mathrm{lm}(f)}{\mathrm{lm}^i(g_k)} \in \mathcal{M}_+.$$

Hence, we can test each g_k sequentially: first check if $m_k = \frac{\mathrm{lm}(f)}{\mathrm{lm}^i(g_k)} \in \mathcal{M}_+$, and, if this passes, verify $\mathrm{lm}(m_k g_k) = \mathrm{lm}(f)$. If any g_k passes both checks, a reducer is found.

2.3 Critical pairs

In this section, we introduce the concept of a critical pair for elements $f, g \in K_P^+\{x\}$. The definition is vertex-dependent, and for a given vertex, several distinct critical pairs may occur. We prove three lemmas needed for the Buchberger criterion in Section 2.4.

Definition 2.3.1 (Critical pair)**.** *Let $f, g \in K_P^+\{x\}$ and $i \in I_P$.*
Let $\mathrm{Lcm}(i, f, g)$ *be a minimal finite set of generators of the V_i-module*

$$\mathrm{LM}^i(f) \cap \mathrm{LM}^i(g)$$

(which exists by 2.1.12). For $v \in \mathrm{Lcm}(i, f, g)$, we define:

$$S(i, f, g, v) := \mathrm{lc}^i(g)\frac{v}{\mathrm{lm}^i(f)}f - \mathrm{lc}^i(f)\frac{v}{\mathrm{lm}^i(g)}g.$$

Notice that the quotients $\frac{v}{\mathrm{lm}^i(f)}$ and $\frac{v}{\mathrm{lm}^i(g)}$ are in $\mathcal{M}_+$ by definition of $\mathrm{LM}^i(f)$ and $\mathrm{LM}^i(g)$, so $S(i, f, g, v)$ is indeed in $K_P^+\{x\}$.

Lemma 2.3.2. *Let $h_1, \dots, h_m \in K_P^+\{x\}$ and $i \in I_P$.*
Suppose that there are terms $t_1, \dots, t_m \in \mathcal{T}_+$, a monomial $u \in V_{i,<}$ and $c \in$ $\mathrm{val}(K^\times)$ *such that*

- *for all $k \in \{1, \dots, m\}$,* $\mathrm{lt}(t_k h_k) = c_k u$ *with* $\mathrm{val}(c_k) = c$
- $\mathrm{lt}(\sum_{k=1}^m t_k h_k) <_P c_1 u$.

For $1 \le k \le m-1$, let $\mathrm{Lcm}(i, h_k, h_{k+1})$ *be a finite system of generators of the V_i-module* $\mathrm{LM}^i(h_k) \cap \mathrm{LM}^i(h_{k+1})$ *which exists by Proposition 2.1.12.*
Then there are elements $d_k \in K$, $v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$ for $1 \le k \le m-1$ and a term $t'_m \in \mathcal{T}_+$ such that:

1. $\sum_{j=k}^m t_k h_k = \sum_{k=1}^{m-1} d_k \frac{u}{v_k} S(i, h_k, h_{k+1}, v_k) + t'_m h_m$.
2. $\mathrm{val}_P(t'_m h_m) > \mathrm{val}_P(u c_1)$.
3. $\frac{u}{v_k} \in V_i$ *for all $k < m$.*
4. *For all $k < m$,* $\mathrm{val}\left(d_k \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})\right) \ge c$.

Proof. Write

$$p_k = \frac{t_k h_k}{c_k}, \quad e_k = \sum_{s=1}^k c_s, \quad t_k = \gamma_k \tilde{t}_k$$

for some $\gamma_k \in K$ and some monomial $\tilde{t}_k$.
By hypothesis, u is in $V_{i,<}$ and

$$u = \tilde{t}_k \mathrm{lm}^i(h_k) \in \mathrm{LM}^i(h_k) \quad \text{for all } k.$$

This implies that for all $k < m$, we have:

$$u \in \mathrm{LM}^i(h_k) \cap \mathrm{LM}^i(h_{k+1}) = V_i \cdot \mathrm{Lcm}(i, h_k, h_{k+1}).$$

We deduce that for all $k < m$, there exist

$$s_k \in V_i \quad \text{and} \quad v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$$

such that $u = s_k v_k$.
Now write

$$\sum_{k=1}^{m} t_k h_k = e_1(p_1 - p_2) + \cdots + e_{m-1}(p_{m-1} - p_m) + e_m p_m.$$

For all $k < m$, we have

$$\mathrm{lt}(t_k h_k) = c_k u = \gamma_k \mathrm{lc}^i(h_k) \tilde{t}_k \mathrm{lm}^i(h_k),$$

hence

$$\frac{t_k}{c_k \tilde{t}_k} = \frac{1}{\mathrm{lc}^i(h_k)}.$$

For any $k < m$, define $P_k = p_k - p_{k+1}$.
We can then write:

$$P_k = \frac{u}{v_k} \left(\frac{v_k}{u} p_k - \frac{v_k}{u} p_{k+1} \right).$$

Expanding,

$$P_k = \frac{u}{v_k} \left(\frac{t_k v_k h_k}{c_k \tilde{t}_k \mathrm{lm}^i(h_k)} - \frac{t_{k+1} v_k h_{k+1}}{c_{k+1} \tilde{t}_{k+1} \mathrm{lm}^i(h_{k+1})} \right).$$

This simplifies to:

$$P_k = \frac{u}{v_k} \left(\frac{1}{\mathrm{lc}^i(h_k)} \frac{v_k}{\mathrm{lm}^i(h_k)} h_k - \frac{1}{\mathrm{lc}^i(h_{k+1})} \frac{v_k}{\mathrm{lm}^i(h_{k+1})} h_{k+1} \right).$$

Factoring further,

$$P_k = \frac{1}{\mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})}\frac{u}{v_k}\left(\mathrm{lc}^i(h_{k+1})\frac{v_k}{\mathrm{lm}^i(h_k)}h_k - \mathrm{lc}^i(h_k)\frac{v_k}{\mathrm{lm}^i(h_{k+1})}h_{k+1}\right).$$

Thus,

$$P_k = \frac{1}{\mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})}\frac{u}{v_k}S(i, h_k, h_{k+1}).$$

Plugging this into the sum expression, we obtain:

$$d_k = \frac{e_k}{\mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})}, \quad t'_m = \frac{e_m}{c_m}t_m.$$

This satisfies 1.
Since the hypothesis forces

$$\mathrm{val}(e_m) > \mathrm{val}(c_m),$$

we get

$$\mathrm{val}_P(t'_m h_m) = \mathrm{val}(e_m) + \mathrm{val}_P(u) > \mathrm{val}(c_m) + \mathrm{val}_P(u) = \mathrm{val}_P(c_1 u),$$

which proves 2.
In addition, since

$$\frac{u}{v_k} = s_k \in V_i,$$

this proves 3.
Finally, using that $\mathrm{val}(e_k) \geq c$ and

$$d_k = \frac{e_k}{\mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})},$$

we obtain item 4. □

Lemma 2.3.3. *For $f, g \in K_P\{x\}$, $i \in I_P$ and $v \in \mathrm{Lcm}(i, f, g)$, we have*

$$\mathrm{lt}(S(i, f, g, v)) <_P \mathrm{lc}^i(f)\mathrm{lc}^i(g)v.$$

Proof. Since $v \in \mathrm{Lcm}(i, f, g) \subseteq \mathrm{LM}^i(g) \cap \mathrm{LM}^i(f)$, there exist $m_f \in V_{i,<}(f)$ and

$m_g \in V_{i,<}(g)$ such that

$$v = \mathrm{lm}(m_f f) = \mathrm{lm}(m_g g) = m_f \mathrm{lm}^i(f) = m_g \mathrm{lm}^i(g).$$

Then the leading terms of

$$\mathrm{lc}^i(g) \frac{v}{\mathrm{lm}^i(f)} f \quad \text{and} \quad \mathrm{lc}^i(f) \frac{v}{\mathrm{lm}^i(g)} g$$

are both equal to $\mathrm{lc}^i(f)\mathrm{lc}^i(g)v$. They cancel out, hence

$$\mathrm{lt}(S(i, f, g, v)) <_P \mathrm{lc}^i(f)\mathrm{lc}^i(g)v.$$

□

Lemma 2.3.4. *If $f \in K_P^+\{x\}$ and $i \in I_P$ are such that $\mathrm{lt}(f) <_P u$ for some term u satisfying $\mathrm{lm}(u) \in V_{i,<}$, then for any $v \in V_i$ we have $\mathrm{lt}(vf) <_P vu$.*

Proof. Take t a term of f. Then $t <_P u$. Since $\mathrm{lm}(u), v \in V_i$, we have

$$\mathrm{val}_P(uv) = \mathrm{val}_{r_i}(uv) = \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v) = \mathrm{val}_P(u) + \mathrm{val}_P(v).$$

We consider 3 cases, according to Definition 2.1.7:

1. Case $\mathrm{val}_P(u) < \mathrm{val}_P(t)$.

   Then $\mathrm{val}_P(uv) = \mathrm{val}_P(u) + \mathrm{val}_P(v) < \mathrm{val}_P(t) + \mathrm{val}_P(v) \leq \mathrm{val}_P(tv)$. Thus $tv <_P uv$.

2. Case $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $\min(I_P(u)) < \min(I_P(t))$.

   Then $\mathrm{val}_P(uv) = \mathrm{val}_P(u) + \mathrm{val}_P(v) = \mathrm{val}_P(t) + \mathrm{val}_P(v) \leq \mathrm{val}_P(tv)$. If $\mathrm{val}_P(uv) < \mathrm{val}_P(tv)$, then $tv <_p uv$ and we are done, so let's suppose $\mathrm{val}_P(uv) = \mathrm{val}_P(tv)$. Then we have

   $$\mathrm{val}_P(uv) = \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v) = \mathrm{val}_P(tv) = \min_{k \in I_P}(\mathrm{val}_{r_k}(t) + \mathrm{val}_{r_k}(v)) \quad (2.6)$$

   We have $\mathrm{lm}(u) \in V_{i,<}$, so $i = \min(I_P(u))$. Now since $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $i = \min(I_P(u)) < \min(I_P(t))$, we have $\mathrm{val}_{r_k}(t) > \mathrm{val}_{r_i}(u)$ for $k \leq i$.

Also, since $v \in V_i$, we have $\mathrm{val}_{r_k}(v) \geq \mathrm{val}_{r_i}(v)$ for any $k \in I_P$. Thus for $k \leq i$, we have $\mathrm{val}_{r_k}(t) + \mathrm{val}_{r_k}(v) > \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v)$. It follows that the minimum in (2.6) can be reached only for a $k > i$. This shows $\min(I_P(tv)) > \min(I_P(uv))$ and so $tv <_P vu$.

3. Case $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $\min(I_P(u)) = \min(I_P(t))$ and $t <_t u$.

   Reasoning as in case 2 with $\min(I_P(u)) = \min(I_P(t))$ this time, one gets that the minimum in (2.6) can only be reached for indices $k \geq i$, so $\min(I_P(tv)) \geq \min(I_P(uv))$. If the inequality is strict, we get $tv <_P uv$ and we are done, otherwise we use $\leq_t$ to break ties between uv and tv. Since by hypothesis $t <_t u$ and $\leq_t$ is a term order, we have $tv <_t uv$, and so $tv <_P uv$.

Now since the maximum on the vt's for $\leq_P$ equals $\mathrm{lt}(vf)$, we conclude $\mathrm{lt}(vf) <_P vu$. □

2.4 Buchberger criterion and algorithm

Based on our definition of Gröbner basis in $K_P^+\{x\}$ and the adapted notion of critical pair, we prove a Buchberger criterion and we introduce the corresponding Buchberger algorithm. The proof follows the same strategy as in Theorem 1.3.6 for Tate algebras.

Proposition 2.4.1 (Buchberger criterion)**.** *Let $H = (h_1, \ldots, h_m)$ be a family in $K_P^+\{x\}$ and J be the ideal generated by H. For each $i \in I_P$ and two distinct elements h_k and h_s from H, let $\mathrm{Lcm}(i, h_k, h_s)$ be given by Definition 2.3.1. The following are equivalent:*

1. *H is a Gröbner basis of J.*

2. *For all $i \in I_P$, all distinct h_k and h_s, and all $v \in \mathrm{Lcm}(i, h_s, h_k)$, we have:*

$$\mathrm{ReductionPositiveExponents}(S(i, h_s, h_k, v), H) = 0. \tag{2.7}$$

Proof. By contradiction, assume that (2) is true and that H is not a Gröbner basis of J. Then there exists $f \in J$ such that $\mathrm{lm}(f) \notin \bigcup_{i \in I_P,\ h \in H} \mathrm{LM}^i(h)$. Since, $f \in J = (h_1, \ldots, h_m)$, we can write $f = \sum_{k=1}^m q_k h_k$ for some q_k in $K_P^+\{x\}$. Write $\Delta(k)$ to be the set of terms of q_k. We can rewrite f as $\sum_{k=1}^m \sum_{\alpha \in \Delta(k)} t_{k,\alpha} h_k$. For such a writing of f, define

$$u = \max_{\geq_P} \{\mathrm{lt}(t_{k,\alpha} h_k), 1 \leq k \leq m, \alpha \in \Delta(k)\}$$

and write the term u as $u = c\tilde{u}$ for some $c \in K$ and some monomial $\tilde{u}$. We have $\mathrm{lt}(f) <_P u$ because $\mathrm{lm}(f) \notin \cup_{i \in I_P} \mathrm{LM}^i(h_k)$. Thus, $\mathrm{val}_P(u)$ is upper-bounded. Since val is discrete, there is a maximal $\mathrm{val}_P(u)$ among all possible expressions of $f = \sum_{k=1}^m q_k h_k$. Among the expressions reaching this valuation, Lemma 2.1.9 ensures there is one such that u is minimal. Let $i \in I_P$ be such that $u \in V_{i,<}$. Define

$$Z = \{(k, \alpha) \in \{1, \ldots, m\} \times \Delta(k), \text{ s.t. } \mathrm{lt}(t_{k,\alpha} h_k) =_P u\}$$

and

$$Z' = \{(k, \alpha) \in \{1, \ldots, m\} \times \Delta(k), \text{ s.t. } \mathrm{lt}(t_{k,\alpha} h_k) <_P u\}.$$

We can then write:

$$f = \sum_{(k,\alpha) \in Z} t_{k,\alpha} h_k + \sum_{(k,\alpha) \in Z'} t_{k,\alpha} h_k \tag{2.8}$$

Let $g := \sum_{(k,\alpha) \in Z} t_{k,\alpha} h_k$. We have

$$\mathrm{lt}(g) \leq_P \max(\mathrm{lt}(f), \mathrm{lt}(\sum_{(k,\alpha) \in Z'} t_{k,\alpha} h_k)) <_P u$$

and $\mathrm{lt}(t_{k,\alpha} h_k) = c_{k,\alpha} \tilde{u}$ for all $(k, \alpha) \in Z$, where the $c_{k,\alpha}$ all have the same valuation. So g satisfies the conditions of Lemma 2.3.2 and we can write

$$g = \sum_{k=1}^{m-1} d_k \frac{\tilde{u}}{v_k} S(i, h_k, h_{k+1}, v_k) + t'_m h_m \tag{2.9}$$

for some $d_k \in K$, $v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$, $\mathrm{val}\left(d_k \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})\right) \geq \mathrm{val}(c)$ and $\tilde{u}/v_k \in V_i$ for $k < m$, and with $\mathrm{lt}(t'_m h_m) <_P u$. Now we use the hypothesis that all the critical pairs of elements of H reduce to zero. For each $k < m$ we can write

$$S(i, h_k, h_{k+1}, v_k) = \sum_{l=1}^{m} q_l^{(k)} h_l,$$

for some $q_l^{(k)}$'s in $K_P^+\{x\}$ satisfying

$$\begin{aligned} \mathrm{lt}(q_l^{(k)} h_l) &\leq_P \mathrm{lt}\left(S(i, h_k, h_{k+1}, v_k)\right), \\ &<_P \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1}) v_k, \end{aligned}$$

where the last inequality comes from Lemma 2.3.3. Since $v_k \in V_{i,<}$ and $\tilde{u}/v_k \in V_i$, we can apply Lemma 2.3.4 to get:

$$\begin{aligned} \mathrm{lt}\left(\frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) &<_P \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1}) v_k \frac{\tilde{u}}{v_k}, \\ &= \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})\tilde{u}. \end{aligned}$$

Finally, using that $\mathrm{val}\left(d_k \mathrm{lc}^i(h_k)\mathrm{lc}^i(h_{k+1})\right) \geq \mathrm{val}(c)$, we deduce that for all $l \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m-1\}$;

$$\mathrm{lt}\left(d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) <_P u.$$

Inserting the expressions of $d_k \frac{\tilde{u}}{v_k} S(i, h_k, h_{k+1}, v_k)$ as $\sum_{l=1}^{m} d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l$ in Equations (2.9) and then (2.8), we get an expression of f in terms of the h_k's with strictly smaller u for $\leq_P$, contradicting its minimality. □

Algorithm 7: BuchbergerPositiveExponents

input : $J = (h_1, \dots, h_m)$ an ideal of $K_P\{x\}$
output : a Gröbner basis of J (Definition 2.4.1)

1 $H \leftarrow \{h_1, \dots, h_m\}$;
2 $B \leftarrow \{(h_s, h_k),\ 1 \leq s < k \leq m\}$
3 **while** $B \neq \emptyset$ **do**
4     $(f, g) \leftarrow$ element of B;
5     $B \leftarrow B \setminus \{(f, g)\}$;
6   **for** $i \in I_P$ **do**
7         $\mathrm{Lcm}(i, f, g) \leftarrow$ finite set of generators of $\mathrm{LM}^i(g) \cap \mathrm{LM}^i(f)$;
8   **for** $v \in \mathrm{Lcm}(i, f, g)$ **do**
9             $\_, r \leftarrow$ ReductionPositiveExponents$(S(i, f, g, v), H)$;
10   **if** $r \neq 0$ **then**
11                 $B \leftarrow B \cup \{(h, r),\ h \in H\}$;
12                 $H \leftarrow H \cup \{r\}$
13 **return** H

Remark 2.4.2. *We defer the computation of the finite set* $\mathrm{Lcm}(i, f, g)$ *in line 7 of Algorithm 7 to Section 5.1.1.*

Proposition 2.4.3. *Algorithm 7 is correct and terminates, in the sense that it calls Algorithm* 6 *a finite number of times.*

Proof. Correctness of the output is immediate from the Buchberger criterion of Proposition 2.4.1.

For the termination, we prove that the addition of a new r to H (on Line 12) can only happen a finite amount of times. Indeed, let us assume that there is some input J such that there is an infinite amount of non-zero r produced on Line 9. By the pigeonhole principle, there is $i \in I_P$ such that there is an infinite amount of $\mathrm{lm}(r)$ in $V_{i,<} \subseteq V_i$. Let H_j be an indexing of all the states of the set H throughout Algorithm 7. Using the second property of multivariate division in Proposition 2.2.3, we can extract from the nondecreasing sequence

$$\left(\bigcup_{g\in H_j} \mathrm{LM}^i(g)\right)_{j\in\mathbb{N}}$$

of V_i-ideals a strictly increasing subsequence. This is not possible by Proposition 1.1.2. $\square$

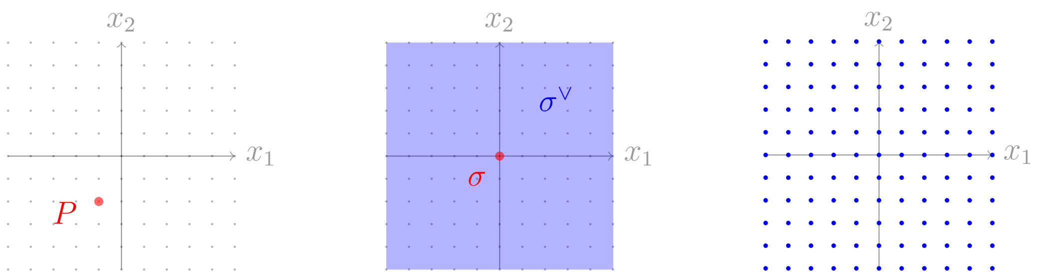
3

Torus

Summary

Introduction

In this chapter, we develop a Gröbner theory in $K_P\{x\}$ for the case where $P = \{r\}$ for some $r \in \mathbb{Q}^n$. In this situation, we use the notation $K_r\{x\}$. Recall that series in $K_r\{x\}$ represent functions converging on the poly-circle with log-radii r. In this case, val_P reduces to val_r, which is a multiplicative valuation. However, the difficulty now is that the monoid of exponents S_σ is the whole of $\mathbb{Z}^n$.



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Figure 3.1: Poly circle

To replace term orders, which cannot exist on $\mathbb{Z}^n$, we use generalized orders, as introduced in Section 1.4. The key point is that a generalized order is a well-order (Lemma 1.4.9), and being a well-order is the only property of a term order used in the proof of Lemma 1.3.2 for Tate algebras. Thus, we can replace a term order by a generalized order and establish the equivalent Lemma 3.1.2 for $K_r\{x\}$. We then inject the machinery of Gröbner theory for Laurent polynomial ideals into $K_r\{x\}$, and unfold the theory up to Buchberger's algorithm. In this way, we combine the Gröbner theory for Laurent polynomial rings [PU99] with the one for Tate algebras [CVV19].

One word about notation: in the previous chapter, we used superscript notation for leading monomials depending on a vertex r_i of P: $\mathrm{lm}^i(f), \mathrm{LM}^i(f), \ldots$. In this chapter, we use subscript notation to denote leading monomials depending on a cone of the conic decomposition underlying the generalized order used: $\mathrm{lm}_i(f), \mathrm{LM}_i(f)$. Both notations will appear in the next chapter. We will then use double indexing notation $\mathrm{lm}^i_j(f), \mathrm{LM}^i_j(f)$, since these objects will depend on both a vertex and a cone.

Recall that Convention 1.2.1 is in force: K is a complete discretely valued field equipped with a nontrivial valuation $\mathrm{val} : K \to \mathbb{Q} \cup \{\infty\}$ normalized so that $\mathrm{val}(K^\times) = \mathbb{Z}$.

3.1 Setting

Let $\leq_g$ be a **generalized** order for a conic decomposition $(M_i)_{i \in I}$ of $\mathcal{M}$ (Definition 1.4.6).

Definition 3.1.1. *We define a preorder $\leq_r$ on $K_r\{x\}$ by:*

$$ax^u \leq_r bx^v \iff \begin{cases} \mathrm{val}_r(bx^v) < \mathrm{val}_r(ax^u) & (1) \\ \textbf{\textit{or}} & \\ \text{equality in (1) and } x^v \geq_g x^u & \end{cases}$$

Definition 3.1.1 is the same as Definition 1.3.1 for Tate algebra, with the term order $\geq_t$ replaced by the generalized order $\geq_g$. We have:

Lemma 3.1.2. *Let $(t_j)_{j \geq 0}$ be a strictly decreasing sequence of terms for $\leq_r$. Then*

$$\lim_{j \to +\infty} \mathrm{val}_r(t_j) = +\infty.$$

Proof. Exactly the same as the proof in Lemma 1.3.2 for Tate algebra, because the only property used for the term order $\geq_t$ is that it is a well-order, a property which is shared with $\geq_g$.

We redo the proof for completeness. Since the sequence $(t_j)_{j \geq 0}$ is strictly decreasing for $\geq_r$, the sequence $(\mathrm{val}_r(t_j))_{j \geq 0}$ is nondecreasing, that is, for all j we have $\mathrm{val}_r(t_{j+1}) \geq \mathrm{val}_r(t_j)$. Because val is assumed to be discrete, val_r takes values in $\frac{1}{D}\mathbb{Z}$ for some integer D (for example, the least common multiple of the coefficients of r). Now, since $\geq_g$ is a well-order, for each v in the discrete set $\frac{1}{D}\mathbb{Z}$, there can only be finitely many indices j such that $\mathrm{val}_r(t_j) = v$. Therefore, the sequence $(\mathrm{val}_r(t_j))_{j \geq 0}$ must tend to $+\infty$. □

In the following definition, we use the map:

$$K_r\{x\} \to K[x^{\pm 1}] : f \mapsto \mathrm{in}_r(f)$$

to extend to series in $K_r\{x\}$ all the definitions introduced in Section 1.4 for Laurent polynomials.

Definition 3.1.3. *For $f \in K_r\{x\}$, $i \in I$, define:*

1. $\mathrm{in}_r(f) := \sum_{\{t \in \mathrm{terms}(f),\ \mathrm{val}_r(t)=\mathrm{val}_r(f)\}} t$.
2. $\mathrm{lm}(f), \mathrm{lc}(f)$ *and* $\mathrm{lt}(f)$*, the leading monomial, coefficient, and term of f for the preorder $\leq_r$. These are also the leading monomial, coefficient and term of* $\mathrm{in}_r(f)$ *for $\geq_g$ as defined in 1.4.11 for polynomials.*
3. $\mathrm{lm}_i(f), \mathrm{lc}_i(f)$ *and* $\mathrm{lt}_i(f)$*, the leading monomial, coefficient, and term of* $\mathrm{in}_r(f)$ *for the cone M_i (Definition 1.4.19).*
4. $M_i(f) := \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M_i\}$.
5. $\mathrm{LM}_i(f) := \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_i$.

The following proposition was already proved for Laurent polynomials in Section 1.4. Its extension to series in $K_r\{x\}$ is straightforward.

Proposition 3.1.4. *Let $f, g \in K_r\{x\}$, $i \in I$ and $a \in M_i(f)$. Then:*

1. $\mathrm{lm}(af) = a \times \mathrm{lm}_i(f)$.
2. $\mathrm{LM}_i(f) = \mathrm{lm}_i(f) M_i(f)$.
3. $\mathrm{LM}_i(f)$ *and* $\mathrm{LM}_i(f) \cap \mathrm{LM}_i(g)$ *are finitely generated M_i-modules.*

Proof. 1.

$$\begin{aligned}\mathrm{lm}(af) = \mathrm{lm}(\mathrm{in}_r(af)) = \mathrm{lm}(a \times \mathrm{in}_r(f)) &= a \times \mathrm{lm}_i(\mathrm{in}_r(f)) \\ &= a \times \mathrm{lm}_i(f).\end{aligned}$$

2. By definition, $\mathrm{LM}_i(f) := \{\mathrm{lm}(mf) \mid m \in \mathcal{M}\} \cap M_i$, which can be rewritten as
$$\mathrm{LM}_i(f) = \{\mathrm{lm}(mf) \mid m \in M_i(f)\}.$$
Applying item 1., we get:
$$\mathrm{LM}_i(f) = \{m \times \mathrm{lm}_i(f) \mid m \in M_i(f)\} = \mathrm{lm}_i(f) M_i(f).$$

3. Since $\mathrm{lm}(mf) = \mathrm{lm}(\mathrm{in}_r(mf)) = \mathrm{lm}(m \times \mathrm{in}_r(f))$, we have:

$$\begin{aligned}\mathrm{LM}_i(f) &= \{\mathrm{lm}(mf),\ m \in M\} \cap M_i \\ &= \{\mathrm{lm}(m \times \mathrm{in}_r(f)),\ m \in M\} \cap M_i \\ &= \mathrm{LM}_i(\mathrm{in}_r(f)),\end{aligned}$$

and the same for $\mathrm{LM}_i(g)$. Then just apply Lemma 1.4.23 to $\mathrm{LM}_i(\mathrm{in}_r(f))$ and $\mathrm{LM}_i(\mathrm{in}_r(f)) \cap \mathrm{LM}_i(\mathrm{in}_r(g))$.

□

3.2 Gröbner bases and reduction

The following definition of Gröbner bases in $K_r\{x\}$ is exactly the same as Definition 2.4 for Laurent polynomial ideals in the paper [PU99]. The fact that we work with series is encapsulated in the definition of $\mathrm{LM}_i(g)$. It is also the same as Definition 2.2.1 from the previous chapter for $K_P^+\{x\}$, replacing I_p by I (the indexing set of the conic decomposition underlying $\geq_g$), and $\mathrm{LM}^i(g)$ by $\mathrm{LM}_i(g)$.

Definition 3.2.1. *Let J be an ideal in $K_r\{x\}$ and G be a finite subset of $J \setminus \{0\}$. We say that G is a Gröbner basis of J (with respect to $\geq_r$) when:*

$$\mathrm{lm}(J) := \{\mathrm{lm}(f),\ f \in J\} = \bigcup_{i \in I,\ g \in G} \mathrm{LM}_i(g)$$

Algorithm 8: FindReducerPolyCircle

input : $f \in K_r\{x\}$, $(g_1, \ldots, g_s) \in K_r\{x\}^s$

output : $\emptyset$ if there is no reducer of f within $\{g_1, \ldots, g_s\}$

Otherwise a triple (m, k, i) with $m \in \mathcal{M}$, $k \in \{1, \ldots, s\}$ and $i \in I$ such that $\mathrm{lm}(mg_k) = \mathrm{lm}(f) \in M_i$

for $i \in I$ *such that* $\mathrm{lm}(f) \in M_i$ **do**
    for $k \in \{1, \ldots, s\}$ **do**
        $m \leftarrow \frac{\mathrm{lm}(f)}{\mathrm{lm}_i(g_k)}$;
        if $\mathrm{lm}(mg_k) = \mathrm{lm}(f)$ **then**
            return (m, k, i)
return $\emptyset$

Algorithm 9: ReductionPolyCircle

input : $f \in K_r\{x\}$, $(g_1, \ldots, g_s) \in K_r\{x\}^s$

output : $(q_1, \ldots, q_s) \in K_r\{x\}^s$ and $b \in K_r\{x\}$ satisfying Prop 3.2.2

$q_1, \ldots, q_k, b \leftarrow 0$;
while$_\infty$ $f \neq 0$
    while FindReducerPolyCircle$(f, (g_1, \ldots, g_s))$ *returns a triple* (m, k, i) **do**
        $t \leftarrow \frac{\mathrm{lc}(f)}{\mathrm{lc}_i(g_k)} m$;
        $q_k \leftarrow q_k + t$;
        $f \leftarrow f - tg_k$;
    $b \leftarrow b + \mathrm{lt}(f)$;
    $f \leftarrow f - \mathrm{lt}(f)$;
return$_\infty$ $(q_1, \ldots, q_k), b$

Proposition 3.2.2. *Let $f \in K_r\{x\}$ and let G be a finite subset of $K_r\{x\}$. Algorithm 9 called on input (f, G) outputs a family $(q_g)_{g \in G}$ and b in $K_r\{x\}$ such that:*

1. $f = \sum_{g \in G} q_g g + b$

2. *For all monomial* m *of* b,

$$m \notin \bigcup_{i \in I,\ g \in G} \mathrm{LM}_i(g). \tag{3.1}$$

3. *For all* $g \in G$ *and all term* t *of* q_g, $\mathrm{lt}(tg) \leq_r \mathrm{lt}(f)$.

Proof. We construct by induction sequences $(f_a)_{a \geq 0}$, $(q_{g,a})_{a \geq 0}$ for $g \in G$ and $(b_a)_{a \geq 0}$ such that for all $a \geq 0$:

$$f = f_a + \sum_{g \in G} q_{g,a} g + b_a,$$

and $\mathrm{lt}(f_a)_{a \geq 0}$ is strictly decreasing for $\leq_r$.

1. **Initialization**: Set $f_0 = f$, $b_0 = 0$ and $q_{g,0} = 0$ for all $g \in G$.
2. **Induction**: Let $i \in I$ be an index such that $\mathrm{lm}(f_a) \in M_i$ (there exists at least one).
   - If there exists $g \in G$ and $m \in M_i(g)$ such that

     $$\mathrm{lm}\,(mg) = \mathrm{lm}(f_a) \in M_i,$$

     set $f_{a+1} = f_a - mg$ and $q_{g,a+1} = q_{g,a} + mg$, and leave unchanged b_a and the other $q_{g,a}$'s. Otherwise,
   - Otherwise, set $f_{a+1} = f_a - \mathrm{lt}(f_a)$ and $b_{a+1} = b_a + \mathrm{lt}(f_a)$, leaving unchanged the $q_{g,a}$'s.

By construction, the sequence $(\mathrm{lt}(f_a))_{a \geq 0}$ is strictly decreasing for $\leq_r$. By Lemma 3.1.2, we deduce that $\mathrm{val}_r(b_{a+1} - b_a)$ and the $\mathrm{val}_r(q_{g,a+1} - q_{g,a})$'s tend to $+\infty$ when $a \to +\infty$. Thus b_a and the $q_{g,a}$'s converge in $K_r\{x\}$. Their limits satisfy the requirements of the proposition. □

Remark 3.2.3. *Algorithm 8 is similar to Algorithm 5. The difference is that, since we work in the whole of* $\mathcal{M}$*, the monomial* m *constructed in line 3 is automatically an element of* $K_r\{x\}$*. In Algorithm 5, we instead had to check whether* $m \in \mathcal{M}_+$ *to ensure that it lies in* $K_P^+\{x\}$*.*

3.3 Critical pairs

Definition 3.3.1 (Critical pair). *Let $f, g \in K_r\{x\}$ and $i \in I$.*
Let $\mathrm{Lcm}(i, f, g)$ *be a minimal finite set of generators of the M_i-module*

$$\mathrm{LM}_i(f) \cap \mathrm{LM}_i(g).$$

For $v \in \mathrm{Lcm}(i, f, g)$, we define:

$$S(i, f, g, v) := \mathrm{lc}_i(g)\frac{v}{\mathrm{lm}_i(f)}f - \mathrm{lc}_i(f)\frac{v}{\mathrm{lm}_i(g)}g.$$

Lemma 3.3.2. *Let $h_1, \dots, h_m \in K_r\{x\}$ and $i \in I$.*
Suppose that there are terms $t_1, \dots, t_m \in \mathcal{T}$, a monomial $u \in M_i$ and $c \in \mathrm{val}(K^\times)$ such that

- *for all $k \in \{1, \dots, m\}$, $\mathrm{lt}(t_k h_k) = c_k u$ with $\mathrm{val}(c_k) = c$*
- $\mathrm{lt}(\sum_{k=1}^m t_k h_k) <_r c_1 u$.

For $1 \le k \le m-1$, let $\mathrm{Lcm}(i, h_k, h_{k+1})$ be a finite system of generators of the M_i-module $\mathrm{LM}_i(g) \cap \mathrm{LM}_i(f)$ which exists by Proposition 3.1.4.

Then there are elements $d_k \in K$, $v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$ for $1 \le k \le m-1$ and a term $t'_m \in \mathcal{T}$ such that:

1. $\sum_{j=k}^m t_k h_k = \sum_{k=1}^{m-1} d_k \frac{u}{v_k} S(i, h_k, h_{k+1}, v_k) + t'_m h_m$.
2. $\mathrm{val}_r(t'_m h_m) > \mathrm{val}_r(u c_1)$.
3. $\frac{u}{v_k} \in M_i$ *for all $k < m$.*
4. *For all $k < m$,* $\mathrm{val}\left(d_k \mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})\right) \ge c$.

Proof. Write

$$p_k = \frac{t_k h_k}{c_k}, \quad e_k = \sum_{s=1}^{k} c_s, \quad t_k = \gamma_k \tilde{t}_k$$

for some $\gamma_k \in K$ and some monomial $\tilde{t}_k$.

By hypothesis, u is in M_i and

$$u = \tilde{t}_k \mathrm{lm}_i(h_k) \in \mathrm{LM}_i(h_k) \quad \text{for all } k.$$

This implies that for all $k < m$, we have:

$$u \in \mathrm{LM}_i(h_k) \cap \mathrm{LM}_i(h_{k+1}) = T_i \cdot \mathrm{Lcm}(i, h_k, h_{k+1}).$$

We deduce that for all $k < m$, there exist

$$s_k \in M_i \quad \text{and} \quad v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$$

such that $u = s_k v_k$.
Now write

$$\sum_{k=1}^{m} t_k h_k = e_1(p_1 - p_2) + \cdots + e_{m-1}(p_{m-1} - p_m) + e_m p_m.$$

For all $k < m$, we have

$$\mathrm{lt}(t_k h_k) = c_k u = \gamma_k \mathrm{lc}_i(h_k) \tilde{t}_k \mathrm{lm}_i(h_k),$$

hence

$$\frac{t_k}{c_k \tilde{t}_k} = \frac{1}{\mathrm{lc}_i(h_k)}.$$

For any $k < m$, define $P_k = p_k - p_{k+1}$.
We can then write:

$$P_k = \frac{u}{v_k} \left(\frac{v_k}{u} p_k - \frac{v_k}{u} p_{k+1} \right).$$

Expanding,

$$P_k = \frac{u}{v_k} \left(\frac{t_k v_k h_k}{c_k \tilde{t}_k \mathrm{lm}_i(h_k)} - \frac{t_{k+1} v_k h_{k+1}}{c_{k+1} \tilde{t}_{k+1} \mathrm{lm}_i(h_{k+1})} \right).$$

This simplifies to:

$$P_k = \frac{u}{v_k} \left(\frac{1}{\mathrm{lc}_i(h_k)} \frac{v_k}{\mathrm{lm}_i(h_k)} h_k - \frac{1}{\mathrm{lc}_i(h_{k+1})} \frac{v_k}{\mathrm{lm}_i(h_{k+1})} h_{k+1} \right).$$

Factoring further,

$$P_k = \frac{1}{\mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})}\frac{u}{v_k}\left(\mathrm{lc}_i(h_{k+1})\frac{v_k}{\mathrm{lm}_i(h_k)}h_k - \mathrm{lc}_i(h_k)\frac{v_k}{\mathrm{lm}_i(h_{k+1})}h_{k+1}\right).$$

Thus,

$$P_k = \frac{1}{\mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})}\frac{u}{v_k}S(i, h_k, h_{k+1}, v_k).$$

Plugging this into the sum expression, we obtain:

$$d_k = \frac{e_k}{\mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})}, \quad t'_m = \frac{e_m}{c_m}t_m.$$

This satisfies 1.
Since the hypothesis forces

$$\mathrm{val}(e_m) > \mathrm{val}(c_m),$$

we get

$$\mathrm{val}_r(t'_m h_m) = \mathrm{val}(e_m) + \mathrm{val}_r(u) > \mathrm{val}(c_m) + \mathrm{val}_r(u) = \mathrm{val}_r(c_1 u),$$

which proves 2.
In addition, since

$$\frac{u}{v_k} = s_k \in M_i,$$

this proves 3.
Finally, using that $\mathrm{val}(e_k) \geq c$ and

$$d_k = \frac{e_k}{\mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})},$$

we obtain 4. □

Lemma 3.3.3. *For $f, g \in K_r\{x\}$, $i \in I$ and $v \in \mathrm{Lcm}(i, f, g)$, we have:*

$$\mathrm{lt}(S(i, f, g, v)) <_r \mathrm{lc}_i(f)\mathrm{lc}_i(g)v.$$

Proof. Since $v \in \mathrm{LM}_i(g) \cap \mathrm{LM}_i(f)$, there exists $m_f \in M_i(f)$ and $m_g \in M_i(g)$

such that

$$v = \mathrm{lm}(m_f f) = \mathrm{lm}(m_g g) = m_f \mathrm{lm}_i(f) = m_g \mathrm{lm}_i(g).$$

Then the leading terms of $\mathrm{lc}_i(g)\frac{v}{\mathrm{lm}_i(f)} f$ and $\mathrm{lc}_i(f)\frac{v}{\mathrm{lm}_i(g)} g$ are both equal to $\mathrm{lc}_i(f)\mathrm{lc}_i(g)v$. They cancel out leaving

$$\mathrm{lt}(S(i, f, g, v)) <_r \mathrm{lc}_i(f)\mathrm{lc}_i(g)v.$$

□

Lemma 3.3.4. *If $f \in K_r\{x\}$ and $i \in I$ are such that $\mathrm{lt}(f) <_r u$ for some term u satisfying $\mathrm{lm}(u) \in M_i$, then for any $v \in M_i$ we have $\mathrm{lt}(vf) <_r vu$.*

Proof. We can replace f by the polynomial $\mathrm{in}_r(f)$. Then this is just an application of item 2. of Definition 1.4.6. □

3.4 Buchberger criterion and algorithm

Once again, the proof of the following Buchberger criterion follows the usual strategy.

Proposition 3.4.1 (Buchberger criterion)**.** *Let $H = (h_1, \ldots, h_m)$ be a family in $K_r\{x\}$ and J the ideal generated by H. For each $i \in I$ and distinct elements h_k and h_s in H, let $\mathrm{Lcm}(i, h_k, h_s)$ be given by Definition 3.3.1. The following are equivalent:*

1. *H is a Gröbner basis of J*
2. *For all $i \in I$, all distinct h_k and h_s, and all $v \in \mathrm{Lcm}(i, h_s, h_k)$:*

$$\mathrm{ReductionPolyCircle}(S(i, h_s, h_k, v), H) = 0. \tag{3.2}$$

Proof. By contradiction, assume that (2) is true and that H is not a Gröbner basis of J. Hence there exists $f \in J$ such that

$$\mathrm{lm}(f) \notin \bigcup_{i \in I,\ h \in H} \mathrm{LM}_i(h).$$

Since, $f \in J = (h_1, \ldots, h_m)$, we can write $f = \sum_{k=1}^m q_k h_k$ for some q_k in $K_r\{x\}$. Write $\Delta(k)$ to be the set of terms of q_k. We can rewrite f as

$$\sum_{k=1}^{m} \sum_{\alpha \in \Delta(k)} t_{k,\alpha} h_k.$$

For such a writing of f, define

$$u = \max\{\mathrm{lt}(t_{k,\alpha} h_k), 1 \leq k \leq m, \alpha \in \Delta(k)\}$$

and write the term u as $u = c\tilde{u}$ for some $c \in K$ and some monomial $\tilde{u}$. We have $\mathrm{lt}(f) <_r u$ because $\mathrm{lm}(f) \notin \cup_{i \in I, h \in H} \mathrm{LM}_i(h)$. Thus, $\mathrm{val}_r(u)$ is upper-bounded. Since val is discrete, there is a maximal $\mathrm{val}_r(u)$ among all possible expressions of $f = \sum_{k=1}^m q_k h_k$. Among the expressions reaching this valuation, Lemma 3.1.2 ensure there is one such that u is minimal.
Let $i \in I$ be such that $u \in M_i$. Define

$$Z = \{(k, \alpha) \in \{1, \ldots, m\} \times \Delta(k), \text{ s.t. } \mathrm{lt}(t_{k,\alpha} h_k) =_r u\}$$

and

$$Z' = \{(k, \alpha) \in \{1, \ldots, m\} \times \Delta(k), \text{ s.t. } \mathrm{lt}(t_{k,\alpha} h_k) <_r u\}.$$

We can then write:

$$f = \sum_{(k,\alpha) \in Z} t_{k,\alpha} h_k + \sum_{(k,\alpha) \in Z'} t_{k,\alpha} h_k \tag{3.3}$$

Let $g := \sum_{(k,\alpha) \in Z} t_{k,\alpha} h_k$. We have

$$\mathrm{lt}(g) \leq_r \max \left(\mathrm{lt}(f), \mathrm{lt}(\sum_{(k,\alpha) \in Z'} t_{k,\alpha} h_k) \right) <_r u$$

and $\mathrm{lt}(t_{k,\alpha} h_k) = c_{k,\alpha} \tilde{u}$ for all $(k, \alpha) \in Z$, where the $c_{k,\alpha}$ all have the same valuation. So g satisfies the conditions of Lemma 3.3.2 and we can write

$$g = \sum_{k=1}^{m-1} d_k \frac{\tilde{u}}{v_k} S(i, h_k, h_{k+1}, v_k) + t'_m h_m \tag{3.4}$$

for some $d_k \in K$, $v_k \in \mathrm{Lcm}(i, h_k, h_{k+1})$, $\mathrm{val}\left(d_k \mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})\right) \geq \mathrm{val}(c)$ and $\tilde{u}/v_k \in M_i$ for $k < m$, and with $\mathrm{lt}(t'_m h_m) <_P u$.
Now we use the hypothesis that all the critical pairs of elements of H reduce to zero. For each $k < m$ we can write

$$S(i, h_k, h_{k+1}, v_k) = \sum_{l=1}^{m} q_l^{(k)} h_l,$$

for some $q_l^{(k)}$'s in $K_r\{x\}$ satisfying

$$\begin{aligned} \mathrm{lt}(q_l^{(k)} h_l) &\leq_r \mathrm{lt}\left(S(i, h_k, h_{k+1}, v_k)\right), \\ &<_r \mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1}) v_k, \end{aligned}$$

where the last inequality comes from Lemma 3.3.3. Since $v_k \in M_i$ and $\tilde{u}/v_k \in M_i$, we can apply Lemma 3.3.4:

$$\begin{aligned} \mathrm{lt}\left(\frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) &<_r \mathrm{lc}_{i,j}(h_k)\mathrm{lc}_{i,j}(h_{k+1}) v_k \frac{\tilde{u}}{v_k}, \\ &= \mathrm{lc}_{i,j}(h_k)\mathrm{lc}_{i,j}(h_{k+1}) \tilde{u}. \end{aligned}$$

Finally, using that $\mathrm{val}\left(d_k \mathrm{lc}_i(h_k)\mathrm{lc}_i(h_{k+1})\right) \geq \mathrm{val}(c)$, we deduce that for all $l \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m-1\}$

$$\mathrm{lt}\left(d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) <_r u.$$

Inserting the expressions of $d_k \frac{\tilde{u}}{v_k} S(i, h_k, h_{k+1}, v_k)$ as $\sum_{l=1}^{m} d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l$ in Equations (3.4) and then (3.3), we get an expression of f in terms of the h_k's with strictly smaller u for $\leq_r$, contradicting its minimality. □

Algorithm 10: BuchbergerSinglePoint

input : $J = (h_1, \dots, h_m)$ an ideal of $K_r\{x\}$

output : a Gröbner basis of J (Definition 3.2.1)

1 $H \leftarrow \{h_1, \dots, h_m\}$;
2 $B \leftarrow \{(h_s, h_k),\ 1 \leq s < k \leq m\}$;
3 **while** $B \neq \emptyset$ **do**
4     $(f, g) \leftarrow$ element of B;
5     $B \leftarrow B \setminus \{(f, g)\}$;
6   **for** $i \in I$ **do**
7         $\mathrm{Lcm}(i, f, g) \leftarrow$ finite set of generators of $\mathrm{LM}_i(g) \cap \mathrm{LM}_i(f)$;
8   **for** $v \in \mathrm{Lcm}(i, f, g)$ **do**
9             $\_, b \leftarrow$ ReductionPolyCircle$(S(i, f, g, v), H)$;
10   **if** $b \neq 0$ **then**
11                $B \leftarrow B \cup \{(h, b),\ h \in H\}$;
12                $H \leftarrow H \cup \{b\}$
13 **return** H

Proposition 3.4.2. *Algorithm 10 is correct and terminates, in the sense that it calls Algorithm* 9 *a finite number of times.*

Proof. Correctness of the output is immediate from the Buchberger criterion of Proposition 3.4.1.

For the termination, we prove that the addition of a new b to H (on Line 12) can only happen a finite amount of times. Indeed, let us assume that there is some input J such that there is an infinite amount of non-zero b happening on Line 9. By the pigeonhole principle, there is a $i \in I$ such that there is an infinite amount of $\mathrm{lt}(b)$ in M_i. Let H_j be an indexing of all the states of the set H throughout Algorithm 10. Using the second property of multivariate division in Proposition 3.2.2, we can extract from the nondecreasing sequence

$$\left(\bigcup_{g \in H_j} \mathrm{LM}_i(g) \right)_{j \in \mathbb{N}}$$

of M_i-ideals a strictly increasing subsequence. This is not possible by Proposition 1.1.2. $\square$

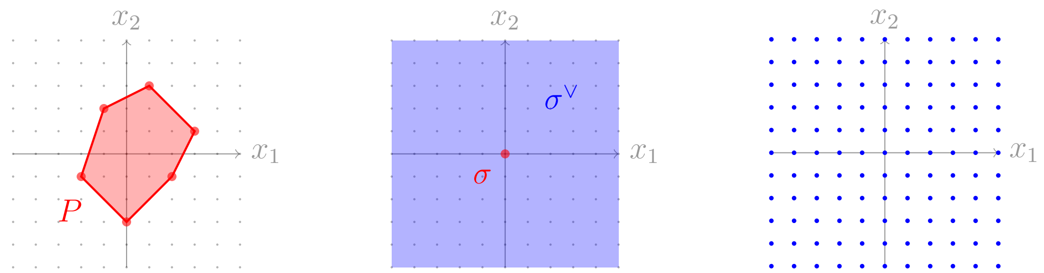
4

Polytopal algebras

Summary

Introduction

In this chapter, we develop a Gröbner bases theory for polytopal algebras. Recall from Example 1.2.7 that these are polyhedral algebras where P is a polytope. This implies that val_P is only submultiplicative and that the exponent monoid S_σ is equal to $\mathbb{Z}^n$.



(a) P **(b)** σ and $\sigma^\vee$ **(c)** $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n$

Figure 4.1: Polytopal algebra

We naturally wish to combine the results of Chapter 2, where we addressed the problem of val_P using an intermediate vertex ordering, and Chapter 3, where we handled the monoid $\mathbb{Z}^n$ through generalized orders. Following this idea, one can take two different approaches:

1. Replace in Definition 2.1.7 the term order $\geq_t$ by a chosen generalized order $\geq_g$

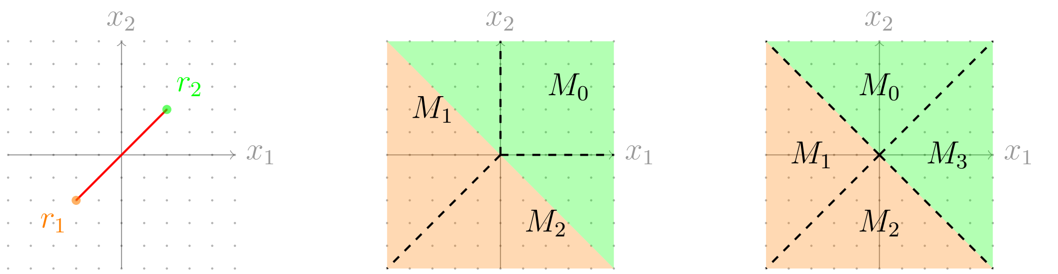
2. Or, modify Definition 3.1.1 by inserting an intermediate breaking step from a chosen vertex ordering

Whichever path is chosen, this leads to the same term ordering, introduced in Definition 4.1.10. Note that the V_i's are determined by the polytope P; we have no choice there. However, the generalized order, and thus the underlying conic decomposition, is entirely up to us. Yet, with an arbitrary conic decomposition, a consistent definition of leading monomial is not possible. Indeed, since the generalized order acts **after** the vertex ordering, the conic decomposition $(M_j)_{j\in J}$ underlying $\geq_g$ must **refine** the decomposition induced by the vertices of P in the following sense:

$$\forall M_j,\ \exists i \in I_P \text{ such that } M_j \subseteq V_i$$

In other words, a cone M_j from the chosen conic decomposition must be fully contained in a V_i, which is necessarily unique by Lemma 4.1.5.

We provide a quick visual summary of this idea. A more thorough explanation will appear later. The monoids V_i, determined by the vertices of P, are shown in color, while the cones M_j from the chosen conic decomposition are indicated by black dashed lines. In case (b), M_1 and M_2 are not fully contained in either V_1 or V_2, making this conic decomposition a bad decomposition for the given polytope P. In case (c), each M_j lies entirely within a single V_i.



(a) P

(b) Non-refining conic decomposition

(c) Conic decomposition refining V_1 and V_2

Figure 4.2: Non-refining and refining conic decompositions

Finally, in Section 4.5, we explain how, with a slight modification, the theory developed here applies to any polyhedral algebra. In particular, it recovers the theory of $K_P^+\{x\}$ from Chapter 2 and that of $K_r\{x\}$ from Chapter 3.

Recall that Convention 1.2.1 is in force: K is a complete discretely valued field equipped with a nontrivial valuation $\mathrm{val} : K \to \mathbb{Q} \cup \{\infty\}$ normalized so that $\mathrm{val}(K^\times) = \mathbb{Z}$.

4.1 Setting

As in Chapter 2, let us assign an indexing to the vertices of P: $\mathrm{vert}(P) = \{r_1, \ldots, r_t\}$, and let I_P represent the set of indices: $I_P := \{1, \ldots, t\}$.

Definition 4.1.1. *For $f \in K_P\{x\}$, we define*

$$I_P(f) := \{i \in I_P,\ \mathrm{val}_P(f) = \mathrm{val}_{r_i}(f)\} \subseteq I_P.$$

Thus, $I_P(f) \subseteq I_P$ is the subset of indices at which $\mathrm{val}_P(f)$ is attained.

Example 4.1.2. *Set $r_1 = (1, 2)$, $r_2 = (2, 1)$, $r_3 = (0, 0)$. Take $P = \mathrm{Conv}(\{r_1, r_2, r_3\})$, $K = \mathbb{Q}_2$ and*

$$f = \frac{1}{2}x_1 + x_2^{-2} + x_1x_2 \in K_P\{x_1^{\pm 1}, x_2^{\pm 1}\}.$$

We have $\mathrm{val}_P(f) = \min_{i \in I_P} \mathrm{val}_{r_i}(f) = \mathrm{val}_{r_1}(f) = \mathrm{val}_{r_2}(f) = -3$, so $I_P(f) = \{1, 2\}$. This common minimum is reached at x_1x_2 for val_{r_1} and at x_1x_2 and $\frac{1}{2}x_1$ for val_{r_2}.

Example 4.1.3. *Set $r_1 = (0, 0, 3), r_2 = (1, 0, 0), r_3 = (-1, 1, 2), r_4 = (1, 1, 1)$. Take $P = \mathrm{Conv}(\{r_1, r_2, r_3, r_4\})$, $K = \mathbb{Q}_2$ and*

$$t = 2x_1x_2x_3^{-1} \in K_P\{x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\}.$$

We have $\mathrm{val}_P(t) = \min_{i \in I_P} \mathrm{val}_{r_i}(t) = \mathrm{val}_{r_2}(t) = \mathrm{val}_{r_4}(t) = 0$, so $I_P(t) = \{2, 4\}$. Since t is a single term, this common minimum is necessarily reached at t.

The following definition of V_i, $V_{i,<}$, C_i, and $C_{i,<}$ is exactly the same as in the $K_P^+\{x\}$ case, except we replace the monoid $\mathcal{M}_+$ by $\mathcal{M}$, and thus $\mathbb{R}_+^n$ by $\mathbb{R}^n$, since exponents now lie in $\mathbb{Z}^n$. These definitions were given in detail in Definitions 2.1.4 and 2.1.5, so we slightly shorten them here:

Definition 4.1.4. *For each $i \in I_P$, we define:*

- *The monoid $V_i \subseteq \mathcal{M}$ by:*

$$\begin{aligned} V_i &= \{x^\alpha \in \mathcal{M},\ i \in I_P(x^\alpha)\} \\ &= \{x^\alpha \in \mathcal{M},\ (r_i - r_j) \cdot \alpha \geq 0,\ \forall j \in I_P\}. \end{aligned}$$

- *The subset $V_{i,<} \subseteq V_i$ by:*

$$\begin{aligned} V_{i,<} &= \{x^\alpha \in \mathcal{M},\ \min(I_P(x^\alpha)) = i\} \\ &= \left\{ x^\alpha \in \mathcal{M},\ \begin{cases} (r_i - r_j)\cdot\alpha > 0 & \text{if } j < i \\ (r_i - r_j)\cdot\alpha \geq 0 & \text{if } i \leq j \end{cases}, \right\}. \end{aligned}$$

- *The rational cone $C_i \subseteq \mathbb{R}^n$ by:*

$$\begin{aligned} C_i &= \{\alpha \in \mathbb{R}^n,\ i \in I_P(x^\alpha)\} \\ &= \{\alpha \in \mathbb{R}^n,\ (r_i - r_j)\cdot\alpha \geq 0,\ \forall j \in I_P\} \end{aligned}$$

- *The subset $C_{i,<} \subseteq C_i$ by:*

$$\begin{aligned} C_{i,<} &= \{\alpha \in \mathbb{R}^n,\ i = \min(I_P(x^\alpha))\} \\ &= \left\{ \alpha \in \mathbb{R}^n,\ \begin{cases} (r_i - r_j)\cdot\alpha > 0, & \text{if } j < i \\ (r_i - r_j)\cdot\alpha \geq 0, & \text{if } j \geq i \end{cases} \right\} \end{aligned}$$

Similarly to Section 2.1, we have the following properties:

1. The cones $(C_i)_{i\in I_P}$ are the maximal cones of $\mathcal{N}(P)$, and C_i corresponds to the vertex r_i under the canonical bijection between the cones of $\mathcal{N}(\mathcal{P})$ and the faces of P.

2. $\bigcup_{i\in I_P} C_i = \mathrm{Supp}(\mathcal{N}(P)) = \mathrm{rec}(P)^\vee = \mathbb{R}^n$.

3. $V_i = \{x^\alpha \mid \alpha \in C_i \cap \mathbb{Z}^n\}$, that is, the monoid V_i can be identified with the set of integer points in C_i.

4. $\bigcup_{i\in I_P} V_i = \mathcal{M}$

5. $V_{i,<}$ is a V_i-module.

6. $V_{i,<}$ (resp. $C_{i,<}$) is obtained from V_i (resp. C_i) by replacing each inequality involving $j < i$ with a strict inequality, hence the sign $<$ in the notation.

7. $V_{i,<} = \{x^\alpha \mid \alpha \in C_{i,<} \cap \mathbb{Z}^n\}$, that is, the subset $V_{i,<} \subseteq V_i$ corresponds to the integer points in $C_{i,<}$.

8. The sets $V_{i,<}$ are pairwise disjoint, and $\coprod_{i \in I_P} V_{i,<} = \mathcal{M}$.

9. The sets $C_{i,<}$ are pairwise disjoint, and $\coprod_{i \in I_P} C_{i,<} = \mathbb{R}^n$.

We now turn to the main point of this chapter. Let $\geq_g$ be a generalized order associated with a conic decomposition that refines the regions V_i. That is, for every cone M in the decomposition, there exists V_i such that $M \subseteq V_i$. Such a V_i is necessarily unique by Lemma 4.1.5 below.

Lemma 4.1.5. *Let M be a cone from a conic decomposition of $\mathcal{M}$. Suppose that there exists $i \in I_P$ such that $M \subseteq V_i$. Then we have*

$$M \subseteq V_k \implies V_k = V_i.$$

Proof. Assume, for the sake of contradiction, that $M \subseteq V_i$ and $M \subseteq V_k$ with $V_i \neq V_k$. Then the corresponding cones C_i and C_k in $\mathcal{N}(P)$ are distinct maximal cones, associated to different vertices of P under the canonical bijection between faces of P and cones of $\mathcal{N}(P)$ (see Section 1.1.7).
It follows that

$$\dim(C_i \cap C_k) < \dim(C_i) = \dim(C_k) = n,$$

and hence

$$_{\mathrm{gr}}\langle V_i \cap V_k \rangle \subsetneq \mathcal{M}.$$

On the other hand, by item 2 of Definition 1.4.1, we must have $_{\mathrm{gr}}\langle M \rangle = \mathcal{M}$. But since $M \subseteq V_i \cap V_k$ by assumption, we obtain

$$_{\mathrm{gr}}\langle M \rangle \subseteq {}_{\mathrm{gr}}\langle V_i \cap V_k \rangle,$$

which yields the contradiction

$$\mathcal{M} = {}_{\mathrm{gr}}\langle M \rangle \subseteq {}_{\mathrm{gr}}\langle V_i \cap V_k \rangle \subsetneq \mathcal{M}.$$

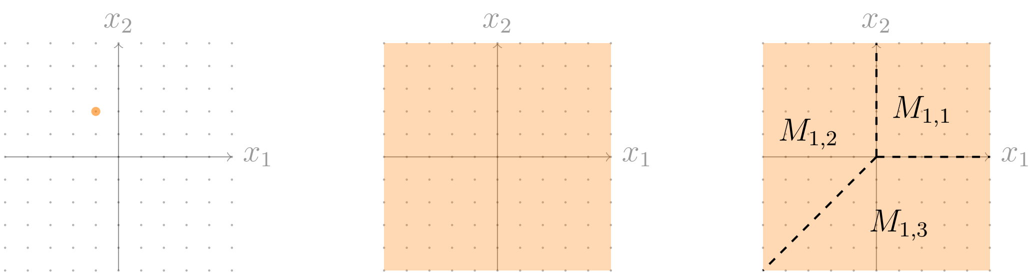
□

For each V_i, let A_i be a finite index set for the cones M satisfying $M \subseteq V_i$. The conic decomposition can then be written using double indices as follows:

$$(M_{i,j})_{(i \in I_p,\ j \in A_i)}.$$

We give many different example of this construction, together with comments.

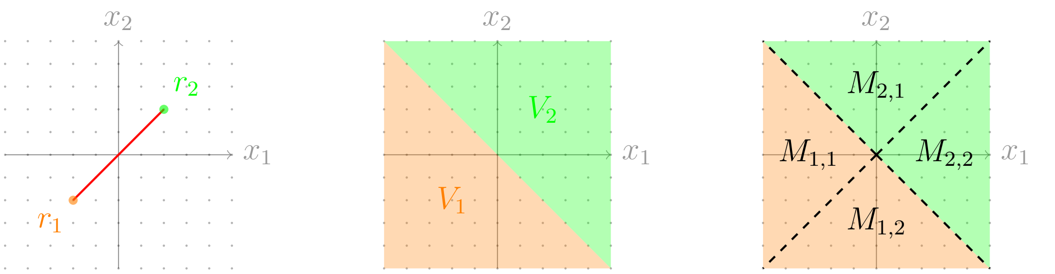
Example 4.1.6. *If P is a single vertex r, then there is only one monoid V_i, which we denote V_1, and it equals the whole $\mathcal{M}$. In this case, any conic decomposition of $\mathcal{M}$ refines $\mathcal{M}$, and the double indexing becomes unnecessary. We simply recover a conic decomposition of $\mathcal{M}$ as in Chapter 3.*



(a) P **(b)** V_1 **(c)** Any conic decomposition of $V_1 = \mathcal{M}$ is fine

Figure 4.3: A single vertex : we recover $K_r\{x\}$

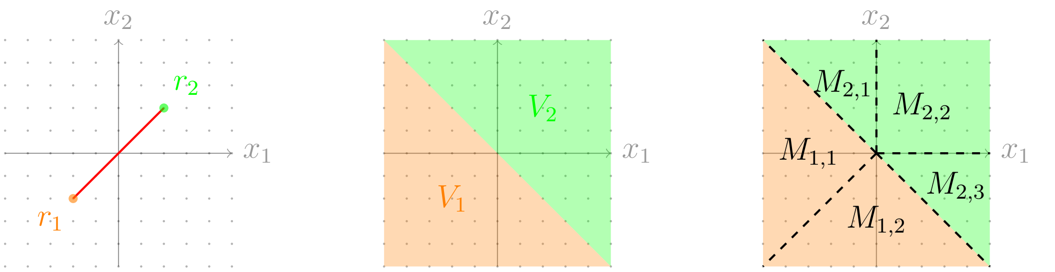
Example 4.1.7. *In this example, note that the monoids V_1 and V_2 contain non-trivial invertible elements, so a refining decomposition is necessary. The decomposition below splits V_1 and V_2 into the minimal number of cones (2 each), yielding a conic decomposition of $\mathcal{M}$ refining the V_i's with the minimal number of cones (4).*



(a) P **(b)** V_1 and V_2 **(c)** Each V_i is decomposed in two cones

Figure 4.4: Minimal refining conic decomposition

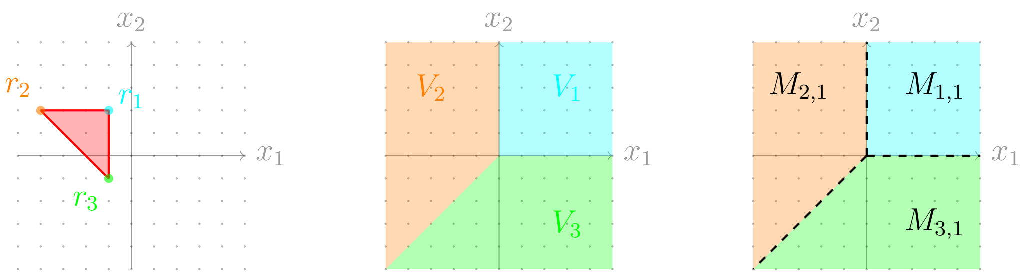
It is entirely possible to decompose, for example, V_2 into more than two cones, as shown below.



(a) P **(b)** V_1 and V_2 **(c)** V_2 is decomposed in three cones

Figure 4.5: Non-minimal refining conic decomposition

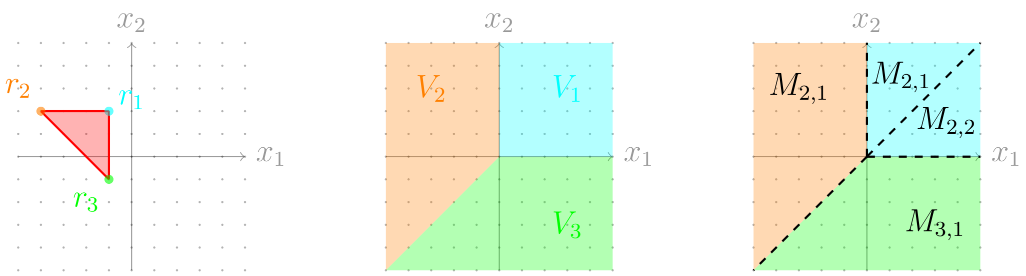
Example 4.1.8. *In this example, each V_i is already a finitely generated monoid containing no non-trivial invertible elements, so each can serve directly as a cone. As in Example 4.1.6, the double indexing becomes unnecessary.*



(a) P **(b)** V_1, V_2, V_3 **(c)** Each V_i is a cone

Figure 4.6: A full dimensional polytope

As in Example 4.1.7, the following decomposition is also possible:



(a) P **(b)** V_1, V_2, V_3 **(c)** V_1 is decomposed in two cones

Figure 4.7: Further decomposition of V_1 in two cones

Remark 4.1.9. *Simple computations show that when decomposing some V_i into more cones than necessary, as illustrated in Figures 4.5c and 4.7c, the size of the output of Algorithm 12 grows rapidly. Therefore, if one aims to compute a minimal Gröbner basis with as few generators as possible, we heuristically advise avoiding such "unnecessary" decompositions unless there is a specific reason to do so. In other words, when V_i is pointed, do not decompose it further; if it is not pointed, decompose it using the minimal number of cones. Note that even when V_i is decomposed minimally, there remain several possible ways to do so.*

Let us fix $\leq_g$, a generalized order on $\mathcal{M}$, whose underlying conic decomposition $(M_{i,j})_{(i \in I_P,\ j \in A_i)}$ refines the V_i's.

The following definition is the natural combination of Definition 2.1.7 for $K_P^+\{x\}$ and Definition 3.1.1 for $K_r\{x\}$.

Definition 4.1.10. *We define a preorder $\leq_P$ on $K_P\{x\}$ by:*

$$ax^u \leq_P bx^v \iff \begin{cases} \mathrm{val}_P(bx^v) < \mathrm{val}_P(ax^u) & (1) \\ \textbf{or} & \\ \text{equality in (1) } \textbf{and } \min(I_P(bx^v)) < \min(I_P(ax^u)) & (2) \\ \textbf{or} & \\ \text{equality in (2) } \textbf{and } x^v \geq_g x^u & \end{cases}$$

Combining the arguments from the proofs of Lemma 2.1.9 and Lemma 3.1.2, we obtain:

Lemma 4.1.11. *Let $(t_k)_{k\geq 0}$ be a strictly decreasing sequence of terms for $\leq_P$. Then*

$$\lim_{k\to+\infty} \mathrm{val}_P(t_k) = +\infty.$$

Proof. Since the sequence $(t_k)_{k\geq 0}$ is strictly decreasing for $\geq_P$, the sequence $(\mathrm{val}_P(t_k))_{k\geq 0}$ is increasing, that is, for all k we have $\mathrm{val}_P(t_{k+1}) \geq \mathrm{val}_P(t_k)$. Because val is assumed to be discrete, val_P takes values in $\frac{1}{D}\mathbb{Z}$ for some integer D (for example, the least common multiple of the denominators of the coefficients of the vertices of P). Now, since $\geq_g$ is a well-order and there is only a finite number of vertices, for each v in the discrete set $\frac{1}{D}\mathbb{Z}$, there can only be finitely many indices k such that $\mathrm{val}_P(t_k) = v$. Therefore, the sequence $(\mathrm{val}_P(t_k))_{k\geq 0}$ must tend to $+\infty$. □

Notice in the following definition, starting from item 4, the use of double indexing, which reflects our choice of conic decomposition.

Definition 4.1.12. *For $f \in K_P\{x\}$, $i \in I_P$, $j \in A_i$ define:*

1. $\mathrm{lm}(f), \mathrm{lc}(f)$ *and* $\mathrm{lt}(f)$*, the leading monomial, coefficient and term of f for the preorder $\geq_P$.*

2. $\mathrm{in}_{r_i}(f) := \sum_{\{t\in \mathrm{terms}(f)\ \mid\ \mathrm{val}_{r_i}(t)=\mathrm{val}_{r_i}(f)\}} t$

3. $\mathrm{in}_{P,<}(f) := \mathrm{in}_{r_i}(f)$ *where* $i = \min(I_P(\mathrm{lm}(f)))$.

4. $\mathrm{lm}^i_j(f), \mathrm{lc}^i_j(f)$ *and* $\mathrm{lt}^i_j(f)$ *the leading monomial, coefficient and term of* $\mathrm{in}_{r_i}(f)$ *for the cone* $M_{i,j}$ *for* $\geq_g$ *(Definition 1.4.19).*

5. $M^<_{i,j} = M_{i,j} \cap V_{i,<}$

6. $M^<_{i,j}(f) := \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M^<_{i,j}\}$

7. $\mathrm{LM}^i_j(f) := \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M^<_{i,j}$.

We need the following lemma, which is identical to Lemma 2.1.11. The proof is word-for-word the same, so we omit it.

Lemma 4.1.13. *Suppose* $f \in K_P\{x\}$ *satisfies* $\mathrm{lm}(f) \in V_{i,<}$ *and let* $t \in V_i$. *Then* $\mathrm{lm}(tf) \in V_{i,<}$.

Proof. See the proof of Lemma 2.1.11. □

Proposition 4.1.14. *Let* $f, g \in K_P\{x\}$ *and* $a \in M^<_{i,j}(f)$. *Then:*

1. $\mathrm{lm}(af) = a \times \mathrm{lm}^i_j(f)$.

2. $\mathrm{LM}^i_j(f) = \mathrm{lm}^i_j(f) M^<_{i,j}(f)$.

3. $\mathrm{LM}^i_j(f)$ *and* $\mathrm{LM}^i_j(f) \cap \mathrm{LM}^i_j(g)$ *are finitely generated* $M_{i,j}$*-modules.*

Proof. 1. By definition, $a \in M^<_{i,j}(f)$ means that $\mathrm{lm}(af) \in M^<_{i,j} \subseteq V_{i,<}$. The elements of $V_{i,<}$ are precisely the monomials m in $\mathcal{M}$ such that $\min(I_P(m)) = i$, so $\min(I_P(\mathrm{lm}(af))) = i$. By item (2) of Definition 4.1.12, it follows that $\mathrm{in}_{P,<}(af) = \mathrm{in}_{r_i}(af)$, and the latter is equal to $a \times \mathrm{in}_{r_i}(f)$, since val_{r_i}, unlike val_P, satisfies $\mathrm{val}_{r_i}(af) = \mathrm{val}_{r_i}(a) + \mathrm{val}_{r_i}(f)$.

   Now by Definition 4.1.10, $\mathrm{lm}(af)$ is the greatest term under the generalized order $\leq_g$ of $\mathrm{in}_{P,<}(af) = a \times \mathrm{in}_{r_i}(f)$. Since $a \in M_{i,j}(f)$, this greatest term is equal to $a \times \mathrm{lm}^i_j(f)$ by item (4) of Definition 4.1.12.

2. By definition, $\mathrm{LM}^i_j(f) := \{\mathrm{lm}(mf) \mid m \in \mathcal{M}_+\} \cap M^<_{i,j}$, which can be rewritten as
$$\mathrm{LM}^i_j(f) = \{\mathrm{lm}(mf) \mid m \in M^<_{i,j}(f)\}.$$

Applying item (1), we get:

$$\mathrm{LM}^i_j(f) = \{m \times \mathrm{lm}^i_j(f) \mid m \in M^<_{i,j}(f)\} = \mathrm{lm}^i_j(f) M^<_{i,j}(f).$$

3. We first show that $\mathrm{LM}^i_j(f)$ is a $M_{i,j}$-module. Since $\mathrm{LM}^i_j(f) = \mathrm{lm}^i_j(f) M^<_{i,j}(f)$ by item 2, it suffices to show that $M^<_{i,j}(f)$ is a $M_{i,j}$-module. Let $t \in M^<_{i,j}(f)$ and $s \in M_{i,j}$. By definition of $M^<_{i,j}(f)$, we have $\mathrm{lm}(tf) \in M^<_{i,j}$. Since $M_{i,j} \subseteq V_{i,<}$, Lemma 4.1.13 gives $\mathrm{lm}(stf) \in V_{i,<}$. Also, since $\mathrm{lm}(tf) \in M_{i,j}$ and $s \in M_{i,j}$, we may apply item 2 of Definition 1.4.6, as in the proof of Lemma 1.4.23, to get $\mathrm{lm}(stf) \in M_{i,j}$. Therefore, $\mathrm{lm}(stf) \in M_{i,j} \cap V_{i,<} = M^<_{i,j}$, which implies $sf \in M^<_{i,j}(f)$.

   This shows that $\mathrm{LM}^i_j(f)$ is an $M_{i,j}$-module, and likewise $\mathrm{LM}^i_j(g)$ is an $M_{i,j}$-module. It follows that $\mathrm{LM}^i_j(f) \cap \mathrm{LM}^i_j(g)$ is also an $M_{i,j}$-module.

   Now, since $\mathrm{LM}^i_j(f) \subseteq M_{i,j}$ and $\mathrm{LM}^i_j(f) \cap \mathrm{LM}^i_j(g) \subseteq M_{i,j}$, both $\mathrm{LM}^i_j(f)$ and $\mathrm{LM}^i_j(f) \cap \mathrm{LM}^i_j(g)$ are $M_{i,j}$-ideals. As $M_{i,j}$ is finitely generated, these ideals are finitely generated $M_{i,j}$-modules by proposition 1.1.1.

□

4.2 Gröbner bases and reduction

For all the proofs that follow, it is convenient to slightly shorten the double indexing of the conic decomposition $(M_{i,j})_{(i \in I_P,\ j \in A_i)}$ by setting

$$L := \{(i,j) \mid i \in I_P,\ j \in A_i\},$$

so that we can write

$$(M_{i,j})_{(i \in I_P,\ j \in A_i)} = (M_{i,j})_{(i,j) \in L}.$$

Definition 4.2.1. *Let J be an ideal in $K_P\{x\}$ and G be a finite subset of*

$J \setminus \{0\}$. We say that G is a Gröbner basis of J (with respect to $\geq_P$) when:

$$\mathrm{lm}(J) = \bigcup_{(i,j)\in L,\ g\in G} \mathrm{LM}^i_j(g)$$

Algorithm 11: ReductionPolytopal

input : $f \in K_P\{x\}, (g_1, \dots, g_m) \in K_P\{x\}^m$

output : $(q_1, \dots, q_m) \in K_P\{x\}^m$ and $r \in K_P\{x\}$ satisfying Prop 4.2.2

$q_1, \dots, q_m, r \leftarrow 0$;
while$_\infty$ $f \neq 0$
  while $\exists (i,j) \in L$ *and* $k \in \{1, \dots, m\}$ *such that* $\mathrm{lm}\left(\frac{\mathrm{lm}(f)}{\mathrm{lm}^i_j(g_k)} g_k\right) = \mathrm{lm}(f)$ **do**
    $t \leftarrow \frac{\mathrm{lt}(f)}{\mathrm{lt}^i_j(g_k)}$;
    $q_k \leftarrow q_k + t$;
    $f \leftarrow f - t g_k$;
  $r \leftarrow r + \mathrm{lt}(f)$
  $f \leftarrow f - \mathrm{lt}(f)$;
return$_\infty$ $(q_1, \dots, q_m), r$

Proposition 4.2.2. *Let $f \in K_P\{x\}$ and G be a finite subset of $K_P\{x\}$. Algorithm 11) called on input (f, G) outputs a family $(q_g)_{g\in G}$ and r in $K_P\{x\}$ such that:*

1. $f = \sum_{g\in G} q_g g + r$
2. *for all monomial m in r,*

$$m \notin \bigcup_{(i,j)\in L, g\in G} \mathrm{LM}^i_j(g) \tag{4.1}$$

3. *for all $g \in G$ and all term t in q_g, $\mathrm{lt}(tg) \leq_P \mathrm{lt}(f)$.*

Proof. We construct by induction sequences $(f_k)_{k\geq 0}$, $(q_{g,k})_{k\geq 0}$ for $g \in G$ and

$(r_k)_{k\geq 0}$ such that for all $k \geq 0$:

$$f = f_k + \sum_{g\in G} q_{g,k} g + r_k,$$

and $\mathrm{lt}(f_k)_{k\geq 0}$ is strictly decreasing for $\leq_P$.

1. **Initialization**: Set $f_0 = f$, $r_0 = 0$ and $q_{g,0} = 0$ for all $g \in G$.

2. **Induction**:

   - If there exists $(i, j) \in L$ and $g \in G$ such that

     $$\mathrm{lm}\left(\frac{\mathrm{lm}(f_k)}{\mathrm{lm}_j^i(g)} g\right) = \mathrm{lm}(f_k) \in M_{i,j}^{<},$$

     we set $f_{k+1} = f_k - tg$ and $q_{g,k+1} = q_{g,k} + tg$ where $t = \frac{\mathrm{lt}(f_k)}{\mathrm{lt}^i(g)}$, and leave unchanged r_k and the other $q_{g,k}$'s.

   - Otherwise, set $f_{k+1} = f_k - \mathrm{lt}(f_k)$ and $r_{k+1} = r_k + \mathrm{lt}(f_k)$, leaving unchanged the $q_{g,k}$'s.

By construction, the sequence $(\mathrm{lt}(f_k))_{k\geq 0}$ is strictly decreasing for $\leq_P$. By Lemma 4.1.11, $\mathrm{val}_P(r_{k+1} - r_k)$ and the $\mathrm{val}_P(q_{g,k+1} - q_{g,k})$'s tend to $+\infty$ when $k \to +\infty$. Thus r_k and the $q_{g,k}$'s converge in $K_P\{x\}$. Their limits satisfy the requirements of the proposition. $\square$

4.3 Critical pairs

Once again, we introduce the notion of a critical pair for polytopal algebras, and prove several technical lemmas needed for the proof of the Buchberger criterion in Section 4.4.

Definition 4.3.1 (S-pair). *Let $f, g \in K_P\{x\}$ and $(i, j) \in L$.*
Let $\mathrm{Lcm}(i, j, f, g)$ be a minimal finite set of generators of the $M_{i,j}$-module

$$\mathrm{LM}_j^i(f) \cap \mathrm{LM}_j^i(g)$$

(which exists by item (3) of proposition 4.1.14).
For $v \in \mathrm{Lcm}(i,j,f,g)$*, we define:*

$$S(i,j,f,g,v) := \mathrm{lc}_j^i(g)\frac{v}{\mathrm{lm}_j^i(f)}f - \mathrm{lc}_j^i(f)\frac{v}{\mathrm{lm}_j^i(g)}g.$$

Lemma 4.3.2. *Let* $h_1,\ldots,h_m \in K_P\{x\}$ *and* $(i,j) \in L$*. Suppose that there are* $t_1,\ldots,t_m \in \mathcal{M}$*,* $u \in M_{i,j}^{<}$ *and* $c \in \mathrm{val}(K^\times)$ *such that*

- *for all* $k \in \{1,\ldots,m\}$*,* $\mathrm{lt}(t_k h_k) = c_k u$ *with* $\mathrm{val}(c_k) = c$
- $\mathrm{lt}(\sum_{k=1}^m t_k h_k) <_P c_1 u$*.*

For $1 \leq k \leq m-1$*, let* $\mathrm{Lcm}(i,j,h_k,h_{k+1})$ *be a finite system of generators of the* $M_{i,j}$*-module*

$$\mathrm{LM}_j^i(f) \cap \mathrm{LM}_j^i(g)$$

which exists by Proposition 4.1.14.
Then there are elements $d_k \in K$*,* $v_k \in \mathrm{Lcm}(i,j,h_k,h_{k+1})$ *for* $1 \leq k \leq m-1$ *and* $t'_m \in K_P\{x\}$ *such that:*

1. $\sum_{j=k}^m t_k h_k = \sum_{k=1}^{m-1} d_k \frac{u}{v_k} S(i,j,h_k,h_{k+1},v_k) + t'_m h_m$*.*
2. $\mathrm{val}_P(t'_m h_m) > \mathrm{val}_P(uc_1)$*.*
3. $\frac{u}{v_k} \in M_{i,j}$ *for all* $k < m$*.*
4. *For all* $k < m$*,* $\mathrm{val}\left(d_k \mathrm{lc}_j^i(h_k)\mathrm{lc}_j^i(h_{k+1})\right) \geq c$*.*

Proof. Write

$$p_k = \frac{t_k h_k}{c_k}, \quad e_k = \sum_{s=1}^k c_s, \quad t_k = \gamma_k \tilde{t}_k$$

for some $\gamma_k \in K$ and some monomial $\tilde{t}_k$. By hypothesis u is in $M_{i,j}^{<}$ and

$$u = \tilde{t_k}\mathrm{lm}_j^i(h_k) \in M_{i,j}(h_k)\mathrm{lm}_j^i(h_k) \quad \text{for all } k.$$

This implies that for all $k < m$ we have:

$$u \in M_{i,j}(h_k)\mathrm{lm}_j^i(h_k) \cap M_{i,j}(h_{k+1})\mathrm{lm}_j^i(h_{k+1}) = M_{i,j} \cdot \mathrm{Lcm}(i,j,h_k,h_{k+1})$$

We deduce that for all $k < m$, there exist $s_k \in M_{i,j}$ and $v_k \in \mathrm{Lcm}(i, j, h_k, h_{k+1})$ such that $u = s_k v_k$. Now write

$$\sum_{k=1}^{m} t_k h_k = e_1(p_1 - p_2) + \cdots + e_{m-1}(p_{m-1} - p_m) + e_m p_m \tag{4.2}$$

For all $k < m$, we have

$$\mathrm{lt}(t_k h_k) = c_k u = \gamma_k \mathrm{lc}_j^i(h_k) \tilde{t}_k \mathrm{lm}_j^i(h_k),$$

hence

$$\frac{t_k}{c_k \tilde{t}_k} = \frac{1}{\mathrm{lc}_{i,j}(h_k)}.$$

For any $k < m$, put $P_k = p_k - p_{k+1}$. We can then write:

$$\begin{aligned}
P_k &= \frac{u}{v_k}\left(\frac{v_k}{u} p_k - \frac{v_k}{u} p_{k+1}\right) \\
&= \frac{u}{v_k}\left(\frac{t_k v_k h_k}{c_k \tilde{t}_k \mathrm{lm}_j^i(h_k)} - \frac{t_{k+1} v_k h_{k+1}}{c_{k+1} \tilde{t}_{k+1} \mathrm{lm}_j^i(h_{k+1})}\right) \\
&= \frac{u}{v_k}\left(\frac{1}{\mathrm{lc}_j^i(h_k)} \frac{v_k}{\mathrm{lm}_j^i(h_k)} h_k - \frac{1}{\mathrm{lc}_j^i(h_{k+1})} \frac{v_k}{\mathrm{lm}_j^i(h_{k+1})} h_{k+1}\right) \\
&= \frac{1}{\mathrm{lc}_j^i(h_k) \mathrm{lc}_j^i(h_{k+1})} \frac{u}{v_k}\left(\mathrm{lc}_j^i(h_{k+1}) \frac{v_k}{\mathrm{lm}_j^i(h_k)} h_k - \mathrm{lc}_j^i(h_k) \frac{v_k}{\mathrm{lm}_j^i(h_{k+1})} h_{k+1}\right) \\
&= \frac{1}{\mathrm{lc}_j^i(h_k) \mathrm{lc}_j^i(h_{k+1})} \frac{u}{v_k} S(i, j, h_k, h_{k+1}).
\end{aligned}$$

Plugging in the last expression back into equation (4.2) gives the desired equality with

$$d_k = \frac{e_k}{\mathrm{lc}_j^i(h_k) \mathrm{lc}_j^i(h_{k+1})}, \quad t'_m = \frac{e_m}{c_m} t_m.$$

t satisfies 1. The hypothesis forces $\mathrm{val}(e_m) > \mathrm{val}(c_m)$. Then we have

$$\mathrm{val}_P(t'_m h_m) = \mathrm{val}(e_m) + \mathrm{val}_P(u) > \mathrm{val}(c_m) + \mathrm{val}_P(u) = \mathrm{val}_P(c_1 u),$$

which proves 2. In addition,

$$\frac{u}{v_k} = s_k \in M_{i,j},$$

which proves 3.
Finally, using that $\mathrm{val}(e_k) \geq c$ and

$$d_k = \frac{e_k}{\mathrm{lc}_j^i(h_k)\mathrm{lc}_j^i(h_{k+1})},$$

one gets 4. □

Lemma 4.3.3. *For $f, g \in K_P\{x\}$, $(i,j) \in L$ and $v \in \mathrm{Lcm}(i,j,f,g)$, we have*

$$\mathrm{lt}(S(i,j,f,g,v)) <_P \mathrm{lc}_j^i(f)\mathrm{lc}_j^i(g)v.$$

Proof. Since $v \in \mathrm{Lcm}(i,j,f,g) \subseteq M_{i,j}^{<}(f)\mathrm{lm}_j^i(f) \cap M_{i,j}^{<}(g)\mathrm{lm}_j^i(g)$, there exists $m_f \in M_{i,j}^{<}(f)$ and $m_g \in M_{i,j}^{<}(g)$ such that

$$v = \mathrm{lm}(m_f f) = \mathrm{lm}(m_g g) = m_f \mathrm{lm}_j^i(f) = m_g \mathrm{lm}_j^i(g).$$

Then the leading terms of

$$\mathrm{lc}_j^i(g)\frac{v}{\mathrm{lm}_j^i(f)}f \quad \text{and} \quad \mathrm{lc}_j^i(f)\frac{v}{\mathrm{lm}_j^i(g)}g$$

are both equal to $\mathrm{lc}_j^i(f)\mathrm{lc}_j^i(g)v$. They cancel out leaving

$$\mathrm{lt}(S(i,j,f,g,v)) <_P \mathrm{lc}_j^i(f)\mathrm{lc}_j^i(g)v.$$

□

The first two items in the proof of the following lemma are identical to those in the corresponding lemma (Lemma 2.3.4) for $K_P^+\{x\}$ proved in Chapter 2. Only the third item, which involves the generalized order $\geq_g$, requires adaptation.

Lemma 4.3.4. *If $f \in K_P\{x\}$ and $(i,j) \in L$ are such that* $\mathrm{lt}(f) <_P u$ *for some term u satisfying* $\mathrm{lm}(u) \in M_{i,j}^{<}$*, then for any $v \in M_{i,j}$ we have* $\mathrm{lt}(vf) <_P vu$.

Proof. Take t a term of f. Then $t <_P u$. Since $\mathrm{lm}(u), v \in V_i$, we have $\mathrm{val}_P(uv) = \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v) = \mathrm{val}_P(u) + \mathrm{val}_P(v)$. We separate in 3 cases following Definition 4.1.10:

- Case $\mathrm{val}_P(u) < \mathrm{val}_P(t)$.

  Then $\mathrm{val}_P(uv) = \mathrm{val}_P(u) + \mathrm{val}_P(v) < \mathrm{val}_P(t) + \mathrm{val}_P(v) \leq \mathrm{val}_P(tv)$. Thus $tv <_P uv$.

- Case $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $\min(I_P(u)) < \min(I_P(t))$.

  Then $\mathrm{val}_P(uv) = \mathrm{val}_P(u) + \mathrm{val}_P(v) = \mathrm{val}_P(t) + \mathrm{val}_P(v) \leq \mathrm{val}_P(tv)$. If $\mathrm{val}_P(uv) < \mathrm{val}_P(tv)$, then $tv <_P uv$ and we are done, so let's suppose $\mathrm{val}_P(uv) = \mathrm{val}_P(tv)$. Then we have

  $$\mathrm{val}_P(uv) = \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v) = \mathrm{val}_P(tv) = \min_{k \in I_P}(\mathrm{val}_{r_k}(t) + \mathrm{val}_{r_k}(v)) \quad (4.3)$$

  We have $\mathrm{lm}(u) \in V_{i,<}$, so $i = \min(I_P(u))$. Now since $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $i = \min(I_P(u)) < \min(I_P(t))$, we have $\mathrm{val}_{r_k}(t) > \mathrm{val}_{r_i}(u)$ for $k \leq i$. Also, since $v \in V_i$, we have $\mathrm{val}_{r_k}(v) \geq \mathrm{val}_{r_i}(v)$ for any $k \in I_P$. Thus for $k \leq i$, we have $\mathrm{val}_{r_k}(t) + \mathrm{val}_{r_k}(v) > \mathrm{val}_{r_i}(u) + \mathrm{val}_{r_i}(v)$. It follows that the minimum in (4.3) can be reached only for a $k > i$. This shows $\min(I_P(tv)) > \min(I_P(uv))$ and so $tv <_P vu$.

- Case $\mathrm{val}_P(u) = \mathrm{val}_P(t)$ and $\min(I_P(u)) = \min(I_P(t))$ and $u >_\omega t$

  Reasoning like in case 2 with $\min(I_P(u)) = \min(I_P(t))$ this time, one gets that the minimum in (4.3) can only be reached for indices $k \geq i$, so $\min(I_P(tv)) \geq \min(I_P(uv))$. If the inequality is strict, we get $tv <_P uv$ and we are done, otherwise this is the generalized order $\leq_g$ who breaks ties between uv and tv. Now because u, v are in the same cone $M_{i,j}$, we can apply item 2 of Definition 1.4.6 to get $tv <_g uv$, and so $tv <_P uv$.

Now since the maximum on the vt's for $\leq_P$ equals $\mathrm{lt}(vf)$, we conclude $\mathrm{lt}(vf) <_P vu$. □

4.4 Buchberger algorithm

Proposition 4.4.1 (Buchberger criterion)**.** *Let $H = (h_1, \ldots, h_m)$ be a family in $K_P\{x\}$ and J the ideal generated by H. For all $(i, j) \in L$ and all distinct h_k and h_s in H, let $\mathrm{Lcm}(i, j, h_k, h_s)$ be given by Definition 4.3.1.*
The following are equivalent:

1. *H is a Gröbner basis of J*

2. *For all $(i, j) \in L$, all distinct h_k and h_s in H and all $v \in \mathrm{Lcm}(i, j, h_s, h_k)$:*

$$\mathrm{ReductionPolytopal}(S(i, j, h_s, h_k, v), H) = 0. \tag{4.4}$$

Proof. By contradiction, assume that (2) is true and that H is not a Gröbner basis of J. Then there exists $f \in J$ such that

$$\mathrm{lm}(f) \notin \bigcup_{(i,j)\in L,\ 1\leq k\leq m} \mathrm{LM}^i_j(h_k).$$

Since $f \in J = (h_1, \ldots, h_m)$, we can write $f = \sum_{k=1}^m q_k h_k$ for some q_k in $K_P\{x\}$. Write $\Delta(k)$ to be the set of terms of q_k. We can rewrite f as

$$\sum_{k=1}^{m} \sum_{\alpha\in\Delta(k)} t_{k,\alpha} h_k.$$

For such a writing of f, define

$$u = \max\{\mathrm{lt}(t_{k,\alpha} h_k), 1 \leq k \leq m, \alpha \in \Delta(k)\},$$

and write the term u as $u = c\tilde{u}$ for some $c \in K$ and some monomial $\tilde{u}$. We have $\mathrm{lt}(f) <_P u$ because $\mathrm{lm}(f) \notin \cup_{\{(i,j)\in L,\ 1\leq k\leq m\}} \mathrm{LM}^i_j(h_k)$. Thus, $\mathrm{val}_P(u)$ is upper-bounded. Since val is discrete, there is a maximal $\mathrm{val}_P(u)$ among all possible expressions of $f = \sum_{k=1}^m q_k h_k$. Among the expressions reaching this valuation, Lemma 4.1.11 ensures there is one such that u is minimal. Let's select this one and the corresponding u.

Now, let $(i,j) \in L$ be such that $u \in M_{i,j}^{<}$. Define

$$Z = \{(k,\alpha) \in \{1,\dots,m\} \times \Delta(k),\ \text{s.t. } \mathrm{lt}(t_{k,\alpha}h_k) =_P u\}$$

and

$$Z' = \{(k,\alpha) \in \{1,\dots,m\} \times \Delta(k),\ \text{s.t. } \mathrm{lt}(t_{k,\alpha}h_k) <_P u\}.$$

We can then write:

$$f = \sum_{(k,\alpha)\in Z} t_{k,\alpha}h_k + \sum_{(k,\alpha)\in Z'} t_{k,\alpha}h_k \tag{4.5}$$

Let $g := \sum_{(k,\alpha)\in Z} t_{k,\alpha}h_k$. We have

$$\mathrm{lt}(g) \leq_P \max\left(\mathrm{lt}(f), \mathrm{lt}\left(\sum_{(k,\alpha)\in Z'} t_{k,\alpha}h_k\right)\right) <_P u$$

and $\mathrm{lt}(t_{k,\alpha}h_k) = c_{k,\alpha}\tilde{u}$ for all $(k,\alpha) \in Z$, where the $c_{k,\alpha}$ all have the same valuation. So g satisfies the conditions of Lemma 4.3.2 and we can write

$$g = \sum_{k=1}^{m-1} d_k \frac{\tilde{u}}{v_k} S(i,j,h_k,h_{k+1},v_k) + t'_m h_m \tag{4.6}$$

for some $d_k \in K$, $v_k \in \mathrm{Lcm}(i,j,h_k,h_{k+1})$, $\mathrm{val}\left(d_k \mathrm{lc}_j^i(h_k)\mathrm{lc}_j^i(h_{k+1})\right) \geq \mathrm{val}(c)$ and $\tilde{u}/v_k \in M_{i,j}$ for $k < m$, and with $\mathrm{lt}(t'_m h_m) <_P u$.
Now we use the hypothesis that all the critical pairs of elements of H reduce to zero. For each $k < m$ we can write

$$S(i,j,h_k,h_{k+1},v_k) = \sum_{l=1}^{m} q_l^{(k)} h_l,$$

for some $q_l^{(k)}$'s in $K_P\{x\}$ satisfying

$$\begin{aligned} \mathrm{lt}(q_l^{(k)}h_l) &\leq_P \mathrm{lt}\left(S(i,j,h_k,h_{k+1},v_k)\right), \\ &<_P \mathrm{lc}_j^i(h_k)\mathrm{lc}_j^i(h_{k+1})v_k, \end{aligned}$$

where the last inequality comes from Lemma 4.3.3.
Since $v_k \in M^{<}_{i,j}$ and $\tilde{u}/v_k \in M_{i,j}$, we can apply Lemma 4.3.4:

$$\begin{aligned}\operatorname{lt}\left(\frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) &<_P \operatorname{lc}_j^i(h_k)\operatorname{lc}_j^i(h_{k+1}) v_k \frac{\tilde{u}}{v_k},\\ &= \operatorname{lc}_j^i(h_k)\operatorname{lc}_j^i(h_{k+1})\tilde{u}.\end{aligned}$$

Finally, using that $\operatorname{val}\left(d_k \operatorname{lc}_j^i(h_k)\operatorname{lc}_j^i(h_{k+1})\right) \geq \operatorname{val}(c)$, we deduce that for all $l \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m-1\}$,

$$\operatorname{lt}\left(d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l\right) <_P u.$$

Inserting the expressions of $d_k \frac{\tilde{u}}{v_k} S(i, j, h_k, h_{k+1}, v_k)$ as $\sum_{l=1}^m d_k \frac{\tilde{u}}{v_k} q_l^{(k)} h_l$ in Equations (4.6) and then (4.5), we get an expression of f in terms of the h_k's with strictly smaller u for $\leq_P$, contradicting its minimality. □

Algorithm 12: BuchbergerPolytopal

input : $J = (h_1, \ldots, h_m)$ an ideal of $K_P\{x\}$
output: a Gröbner basis of J (Definition 4.2.1)

$H \leftarrow \{h_1, \ldots, h_m\}$;
$B \leftarrow \{(h_s, h_k),\ 1 \leq s < k \leq m\}$;
while $B \neq \emptyset$ **do**
  $(f, g) \leftarrow$ element of B;
  $B \leftarrow B \setminus \{(f, g)\}$;
  for $(i, j) \in L$ **do**
    $\operatorname{Lcm}(i, j, f, g) \leftarrow$ finite set of generators of $\operatorname{LM}_j^i(f) \cap \operatorname{LM}_j^i(g)$;
    for $v \in \operatorname{Lcm}(i, j, f, g)$ **do**
      $\_, r \leftarrow$ ReductionPolytopal$(S(i, j, f, g, v), H)$; `// Algo 11`
      if $r \neq 0$ **then**
        $B \leftarrow B \cup \{(h, r),\ h \in H\}$;
        $H \leftarrow H \cup \{r\}$

return H

Proposition 4.4.2. *Algorithm 12 is correct and terminates, in the sense that it calls Algorithm* 11 *a finite number of times.*

Proof. Correctness of the output is immediate from the Buchberger criterion of Proposition 4.4.1.

For the termination, we prove that the addition of a new r to H (on Line 12) can only happen a finite amount of times. Indeed, let us assume that there is some input J such that there is an infinite amount of non-zero r happening on Line 9. By the pigeonhole principle, there a $(i, j) \in L$ such that there is an infinite amount of $\mathrm{lm}(r)$ in $M_{i,j}^{<} \subseteq M_{i,j}$. Let H_k be an indexation of all the states of the set H throughout Algorithm 12. Using the second property of multivariate division in Proposition 4.2.2, we can extract from the nondecreasing sequence

$$\left(\bigcup_{g \in H_k} \mathrm{LM}_j^i(g) \right)_{k \in \mathbb{N}}$$

of $M_{i,j}$-ideals a strictly increasing one. This is not possible by Proposition 1.1.2. □

4.5 Polyhedral algebras

In Sections 1–4 of the present chapter, where the exponent module S_σ is $\mathbb{Z}^n$, we did not use any property specific to $\mathbb{Z}^n$ as opposed to a general S_σ. Therefore, the theory developed so far generalizes immediately to any polyhedral algebra. In particular, it is interesting to see how the cases of $K_P^+\{x\}$ and $K_r\{x\}$ from Chapters 2 and 3 fit into this.

For Chapter 3 and the case of $K_r\{x\}$, this is simply a special case of a polytope reduced to a single vertex r. As explained in Example 4.1.6, in this case $I_P = \{1\}$ and $V_1 = V_{1,<} = \mathcal{M}$. The double indexing $(M_{i,j})_{(i \in I_P,\ j \in A_i)}$ thus becomes $(M_{1,j})_{(j \in A_1)}$, or simply $(M_j)_{j \in I}$ by setting $I = A_1$. We then recover a conic decomposition of $\mathcal{M}$. We have:

1. val_P becomes val_r and $\geq_P$ becomes $\geq_r$.

2. $\mathrm{in}_{P,<}(f)$ becomes $\mathrm{in}_r(f)$

3. $M_{i,j}^{<}$ becomes

$$M_{1,j} \cap V_{1,<} = M_{1,j} \cap V_1 = M_{1,j} \cap \mathcal{M} = M_{1,j} = M_j.$$

4. $\mathrm{lm}_j^i(f)$, $\mathrm{lc}_j^i(f)$, and $\mathrm{lt}_j^i(f)$ become $\mathrm{lm}_j^1(f)$, $\mathrm{lc}_j^1(f)$, and $\mathrm{lt}_j^1(f)$, or simply $\mathrm{lm}_j(f)$, $\mathrm{lc}_j(f)$, and $\mathrm{lt}_j(f)$. This recovers item (3) of Definition 3.1.3.

5. $M_{i,j}^{<}(f)$ becomes

$$\begin{aligned} M_{1,j}^{<}(f) &= \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M_{1,j}^{<}\} \\ &= \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M_j\} \\ &= M_j(f) \end{aligned}$$

which gives item (4) of Definition 3.1.3.

6. $\mathrm{LM}_j^i(f) = \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_{i,j}^{<}$ becomes

$$\begin{aligned} \mathrm{LM}_j^1(f) &= \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_{1,j}^{<} \\ &= \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_j \\ &= \mathrm{LM}_j(f), \end{aligned}$$

which corresponds to item (5) of Definition 3.1.3.

For Chapter 2 and the case of $K_P^+\{x\}$, the conic decomposition is directly given by the V_i's (as explained in Example 4.1.8). Thus, the double indexing $(M_{i,j})_{(i\in I_P,\ j\in A_i)}$ becomes $(M_{i,1})_{i\in I_P} = (V_i)_{i\in I_P}$. In this case, any term order $\geq_t$ is a generalized order for this decomposition. So we can take $\geq_g = \geq_t$. Then we have:

1. $M_{i,j}^{<}$ becomes

$$M_{i,1} \cap V_{i,<} = M_{i,1} \cap V_{i,<} = V_i \cap V_{i,<} = V_{i,<}.$$

2. $\mathrm{lm}^i_j(f)$, $\mathrm{lc}^i_j(f)$, and $\mathrm{lt}^i_j(f)$ become $\mathrm{lm}^i_1(f)$, $\mathrm{lc}^i_1(f)$, and $\mathrm{lt}^i_1(f)$, or simply $\mathrm{lm}^i(f)$, $\mathrm{lc}^i(f)$, and $\mathrm{lt}^i(f)$. This recovers item (4) of Definition 2.1.10.

3. $M^<_{i,j}(f)$ becomes

$$\begin{aligned} M^<_{i,1}(f) &= \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M^<_{i,1}\} \\ &= \{m \in \mathcal{M},\ \mathrm{lm}(mf) \in V_{i,<}\} \\ &= V_{i,<}(f) \end{aligned}$$

   which gives item (5) of Definition 2.1.10.

4. $\mathrm{LM}^i_j(f) = \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M^<_{i,j}$ becomes

$$\begin{aligned} \mathrm{LM}^i_1(f) &= \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M^<_{i,1} \\ &= \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap V_{i,<} \\ &= \mathrm{LM}^i(f), \end{aligned}$$

   which corresponds to item (6) of Definition 2.1.10.

In general, every time S_σ is a pointed monoid, we will take a term order for $\geq_g$. In other words, we use a generalized order that is not a term order only if we need to.

4.6 Gröbner bases in $K^\circ_P\{x\}$

A natural addition to the theory developed so far would be to adapt everything to the valuation ring $K^\circ_P\{x\}$ of $K_P\{x\}$, as done in [CVV19] for $K^\circ_r\langle x\rangle$. We recall the definition of $K^\circ_P\{x\}$:

$$K^\circ_P\{x\} := \{f \in K_P\{x\},\ \mathrm{val}_P(f) \geq 0\}.$$

The difference is that there are more units in $K_P\{x\}$ than in $K^\circ_P\{x\}$. For example, $J = (px_1, x_1x_2)$ is monogeneous in $\mathbb{Q}_p\langle x\rangle$, since p is invertible, so $x_1x_2 = (\frac{1}{p}x_2) \times px_1$. But this is not valid in $\mathbb{Q}^\circ_p\langle x\rangle$.

When working in $K_P^\circ\{x\}$, the order $\geq_P$ on terms remains unchanged, but we must replace the leading monomial of a series by its leading term modulo $(K^\circ)^\times$. Therefore, all definitions introduced earlier must be adapted accordingly. We believe our Gröbner theory should extend to $K_P^\circ\{x\}$. We also hope the precision theorem from [CVV19, Theorem 3.8] generalises well here.

5

Computations

Summary

Introduction

Through Chapters 2 to 4, we gradually built a Gröbner theory in $K_P\{x\}$. The final Buchberger algorithm (Algorithm 12) is effective in the following sense. Given an ideal $J = (f_1, \dots, f_s) \subseteq K_P\{x\}$ and approximate inputs $f'_1, \dots, f'_s \in K[S_\sigma]$ with a precision error on val_P:

$$f = f'_1 + O_P(b), \quad \cdots, \quad f_s = f'_s + O_P(b),$$

the algorithm terminates. Here $f_i = f'_i + O_P(b)$ means that $\mathrm{val}_P(f_i - f'_i) \geq b \in \frac{1}{D}\mathbb{Z}$ for an appropriate $D \in \mathbb{N}$ (for example an LCM of the denominators appearing in the coefficients of the vertices of P). To complete the construction of an effective Buchberger algorithm in $K_P\{x\}$, it remains to address the computation of the finite set $\mathrm{Lcm}(i, j, f, g)$ appearing in line 7 in Algorithm 12.

We show in Section 5.1.1 that, when the chosen conic decomposition is $(V_i)_{i\in I_P}$, computing $\mathrm{Lcm}(i, f, g)$ reduces to computing the Hilbert basis of a pointed rational cone. This applies to $K_P^+\{x\}$, and more generally to $K_P\{x\}$ when P is full-dimensional, in which case $(V_i)_{i\in I_P}$ can be chosen as the conic decomposition.

For $K_r\{x\}$, we show in Theorem 5.1.5 that with a specific conic decomposition (the one in Example 1.4.3), $\mathrm{Lcm}(i, f, g)$ always contains a unique element that can be computed using Algorithm 13.

Finally we provide an effective computation for an ideal in $K_P^+\{x\}$ in Section 5.3.

Recall that Convention 1.2.1 is in force: K is a complete discretely valued field equipped with a nontrivial valuation $\mathrm{val} : K \to \mathbb{Q} \cup \{\infty\}$ normalized so that $\mathrm{val}(K^\times) = \mathbb{Z}$.

5.1 Computing least common multiples

5.1.1 Case where the conic decomposition is $(V_i)_{i\in I_P}$

Define $M_\sigma := \{x^\alpha,\ \alpha \in S_\sigma\}$.

Suppose the conic decomposition satisfies: each cone $M_{i,j}$ equals a (necessarily unique) V_i. In other words, the decomposition is exactly $(V_i)_{i\in I_P}$. This includes the case of $K_P^+\{x\}$ (recall Section 4.6), and, more generally, polyhedral algebras with full-dimensional polyhedra (as in Figure 4.6 for example), for which all V_i are pointed monoids, so one may choose the conic decomposition $(V_i)_{i\in I_P}$. We show that in these cases, computing $\mathrm{Lcm}(i, f, g)$ reduces to a known problem in polyhedral geometry. We provide illustrated examples of the computation of $\mathrm{Lcm}(i, f, g)$ for all i in Examples 5.1.2 and 5.1.3.

For all $i \in I_P$, we have:

$$
\begin{aligned}
V_{i,<}(f) &= \{x^\alpha \in M_\sigma \mid \mathrm{lm}(x^\alpha f) \in V_{i,<}\} \\
&= \left\{ x^\alpha \in M_\sigma \mid \forall j \in I_P \ \begin{cases} \mathrm{val}_{r_i}(x^\alpha f) < \mathrm{val}_{r_j}(x^\alpha f) & \text{if } j < i \\ \mathrm{val}_{r_i}(x^\alpha f) \leq \mathrm{val}_{r_j}(x^\alpha f) & \text{if } j \geq i \end{cases} \right\} \\
&= \left\{ x^\alpha \in M_\sigma \mid \forall j \in I_P \ \begin{cases} r_i \cdot \alpha - \mathrm{val}_{r_i}(f) > r_j \cdot \alpha - \mathrm{val}_{r_j}(f) & \text{if } j < i \\ r_i \cdot \alpha - \mathrm{val}_{r_i}(f) \geq r_j \cdot \alpha - \mathrm{val}_{r_j}(f) & \text{if } j \geq i \end{cases} \right\} \\
&= \left\{ x^\alpha \in M_\sigma \mid \forall j \in I_P \ \begin{cases} (r_i - r_j) \cdot \alpha > \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) & \text{if } j < i \\ (r_i - r_j) \cdot \alpha \geq \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) & \text{if } j \geq i \end{cases} \right\}.
\end{aligned}
$$

Now using our assumption that val is discrete, we can write that val_P takes values in $\frac{1}{D}\mathbb{Z}$, where D is the least common multiple of the denominators of the coefficients of the vertices of P. As such, we can replace the strict inequality in the case $j < i$ by a non-strict inequality as follows ($\frac{1}{|D|}$ is added on the right side when $j < i$):

$$
V_{i,<}(f) = \left\{ x^\alpha \in M_\sigma \mid \forall j \in I_P \ \begin{cases} (r_i - r_j) \cdot \alpha \geq \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) + \frac{1}{|D|} & \text{if } j < i \\ (r_i - r_j) \cdot \alpha \geq \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) & \text{if } j \geq i \end{cases} \right\}.
$$

Identifying a monomial with its vector of exponents in $\mathbb{Z}^n$, the set $V_{i,<}(f)$ can thus be identified with the set of integer points inside the following rational polyhedron, which we denote $P_{i,<}(f)$:

$$(S_\sigma \otimes_{\mathbb{N}} \mathbb{R}) \cap \left\{ \alpha \in \mathbb{R}^n,\ \begin{cases} (r_i - r_j)\cdot \alpha \geq \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) + \frac{1}{|D|} & \text{if } j < i \\ (r_i - r_j)\cdot \alpha \geq \mathrm{val}_{r_i}(f) - \mathrm{val}_{r_j}(f) & \text{if } j \geq i \end{cases} \right\}.$$

Note that the recession cone of $P_{i,<}(f)$ is C_i, because the inequalities defining $P_{i,<}(f)$ are exactly the same as those of C_i but affine (with a nonzero constant term instead of zero in the inequalities).
Finally, noting $e_{i,f}$ and $e_{i,g}$ the exponent vectors in $\mathbb{Z}^n$ of $\mathrm{lm}^i(f)$ and $\mathrm{lm}^i(g)$ respectively, since

$$\mathrm{LM}^i(f) \cap \mathrm{LM}^i(g) = \mathrm{lm}^i(f)V_{i,<}(f) \cap \mathrm{lm}^i(g)V_{i,<}(g),$$

we can identify the monomials in $\mathrm{LM}^i(f) \cap \mathrm{LM}^i(g)$ with the integer points in the following rational polyhedron in $\mathbb{R}^n$:

$$P_{i,<}(f,g) := (e_{i,f} + P_{i,<}(f)) \cap (e_{i,g} + P_{i,<}(g)) \subseteq C_i.$$

So finally, computing $\mathrm{Lcm}(i,f,g)$ reduces to the following problem:

"Given a pointed rational polyhedron P with recession cone C, compute a minimal set of generators of $P \cap \mathbb{Z}^n$ as a $C \cap \mathbb{Z}^n$-module."

This problem is classical in polyhedral geometry, for which we can cite the standard reference [Sch98]. Such an affine problem involving a polyhedron can be reduced to a linear one involving a cone by way of homogenization. Specifically, define the homogenization of P, which is a pointed rational cone in $\mathbb{R}^{n+1}$:

$$\mathrm{Hom}(P) := \mathrm{Cone}\big(\{(a_1,\dots,a_n,1) \mid (a_1,\dots,a_n) \in P\}\big) \subseteq \mathbb{R}^{n+1}.$$

Then compute a Hilbert basis H of $\mathrm{Hom}(P)$ (see Section 1.1.8). Project all elements of H with last coordinate 1 onto the hyperplane $\{z = 0\}$, where z is the added coordinate. These projected elements form a minimal generating set of $P \cap \mathbb{Z}^n$ as a $C \cap \mathbb{Z}^n$-module. We illustrate this computation in Example 5.1.1. In practice, we use the reference implementation in the dedicated library Normaliz (or through its SageMath interface) to solve this problem.

Example 5.1.1. *Let P be defined by the following inequalities:*

$$P := \{x \in \mathbb{R}^2 \mid (-1,1)\cdot x \geq -4,\ (2,0)\cdot x \geq 3,\ (2,4)\cdot x \geq 11\}$$

Its recession cone C is $(0,1)\mathbb{R}_+ + (1,1)\mathbb{R}_+$. Let's compute a Hilbert basis H of the cone

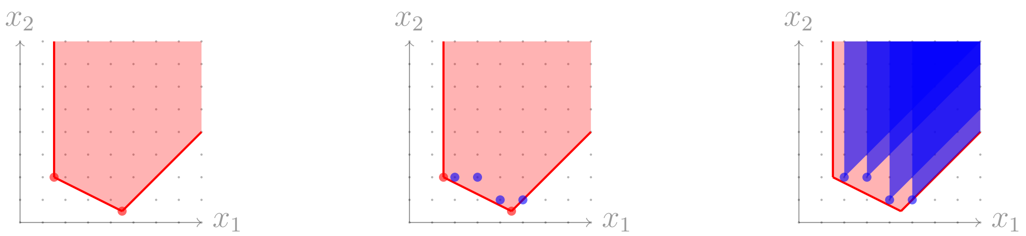
$$\mathrm{Hom}(P) := \{\mathrm{Cone}(\{(x,1),\quad x \in P \subseteq \mathbb{R}^3\}.$$

We obtain:

$$H = \{(0,1,0),(1,1,0),(2,2,1),(4,1,1),(5,1,1),(3,2,1)\}.$$

Retaining the first two coordinates of points in H whose last coordinate is 1, we obtain the following minimal generating set:

$$A = \{(2,2),(4,1),(5,1),(3,2)\}$$



(a) P **(b)** The minimal generating set A **(c)** $P \cap \mathbb{Z}^2 = A + C \cap \mathbb{Z}^2$

Figure 5.1

Now let's see an example of the computation of $\mathrm{Lcm}(i,f,g)$ in $K_P^+\{x\}$.

Example 5.1.2. *Take $K = \mathbb{Q}_2$, $r_1 = (1,5)$, $r_2 = (4,3)$, $r_3 = (5,1)$, $P = \mathrm{Conv}^+(r_1,r_2,r_3)$ and*

$$f = 4x_1x_2^3 + \frac{1}{4}x_1^3 + 2^{42}x_2 + 2^{121} \quad , \quad g = \frac{1}{2}x_2^4 + x_1^2x_2 + x_1^2 + 8x_2.$$

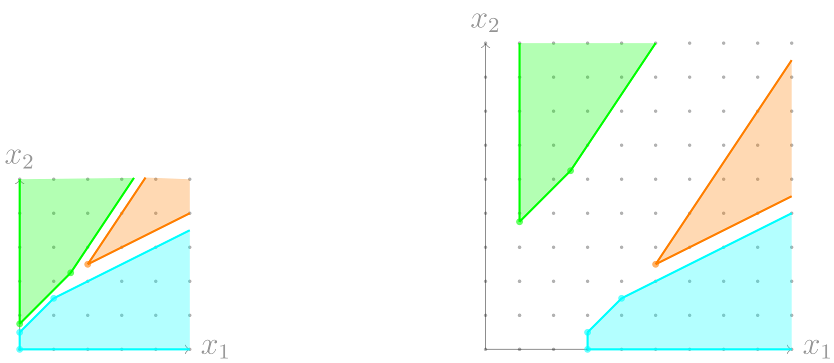
(a) P **(b)** C_1, C_2 and C_3

Figure 5.2

The term order used is the degree reverse lexicographical order. For $i \in \{1, 2, 3\}$*, let* $e_{i,f}$ *(resp.* $e_{i,g}$*) be the exponent vector in* $\mathbb{Z}^2$ *of* $\mathrm{lm}^i(f)$ *(resp.* $\mathrm{lm}^i(g)$*). A computation gives:*

1. $e_{1,f} = (1, 3)$ *and* $e_{1,g} = (0, 4)$
2. $e_{2,f} = (3, 0)$ *and* $e_{2,g} = (0, 4)$
3. $e_{3,f} = (3, 0)$ *and* $e_{3,g} = (2, 1)$

Now we represent the polyhedra $P_{i,<}(f)$ *(left graphic) and their translation* $e_{i,f} + P_{i,<}(f)$ *(right graphic). The integer points in* $P_{i,<}(f)$ *represent* $V_{i,<}(f)$*, while those in* $e_{i,f} + P_{i,<}(f)$ *represent* $\mathrm{LM}^i(f) = \mathrm{lm}^i(f)V_{i,<}(f)$*.*



(a) $P_{i,<}(f)$ **(b)** $e_{i,f} + P_{i,<}(f)$

Figure 5.3

Observe that

1. $\coprod_{i\in\{1,2,3\}} P_{i,<}(f) \cap \mathbb{Z}^2 = \mathbb{N}^2$.

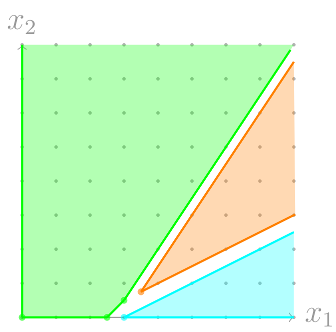
2. *Although* $P_{i,<}(f) \cap \mathbb{Z}^2 \subsetneq C_i \cap \mathbb{Z}^2$ *in general, after translation by* $e_{i,f}$*, we now have an inclusion*

$$(e_{i,f} + P_{i,<}(f)) \cap \mathbb{Z}^2 \subseteq C_i \cap \mathbb{Z}^2.$$

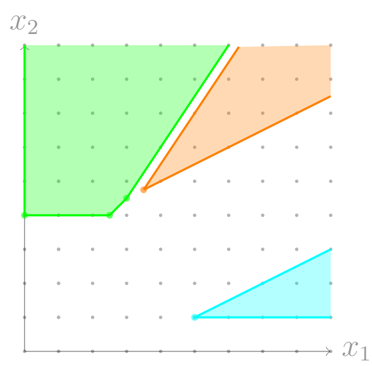
This is of course in conformity with the definition of $\mathrm{LM}^i(f)$ *(Definition 2.1.10):*

$$\mathrm{LM}^i(f) := \{\mathrm{lm}(mf),\ m \in \mathcal{M}_+\} \cap V_{i,<} \subseteq V_i.$$

We represent the same objects, for g this time:



(a) $P_{i,<}(g)$



(b) $e_{i,g} + P_{i,<}(g)$

Figure 5.4

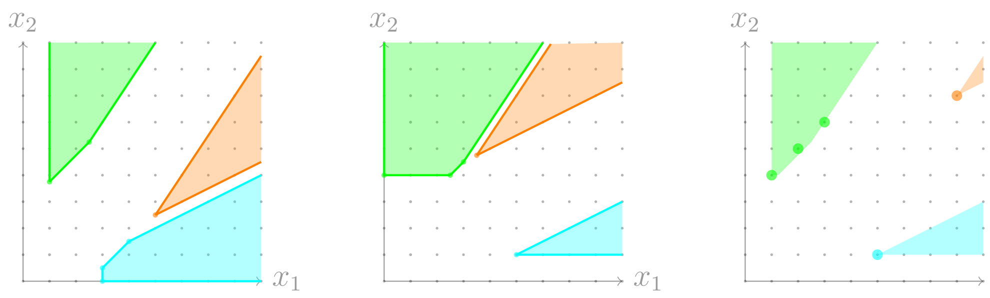
Finally, we represent together the polyhedra

1. $e_{i,f} + P_{i,<}(f)$

2. $e_{i,g} + P_{i,<}(g)$

3. $P_{i,<}(f,g) = (e_{i,f} + P_{i,<}(f)) \cap (e_{i,g} + P_{i,<}(g))$

and the minimal generating sets of $P_{i,<}(f,g) \cap \mathbb{Z}^2$ *as a* $C_i \cap \mathbb{Z}^2$*-module.*



(a) $e_{i,f} + P_{i,<}(f)$ **(b)** $e_{i,g} + P_{i,<}(g)$ **(c)** $P_{i,<}(f,g)$ and $\mathrm{Lcm}(i,f,g)$

Figure 5.5

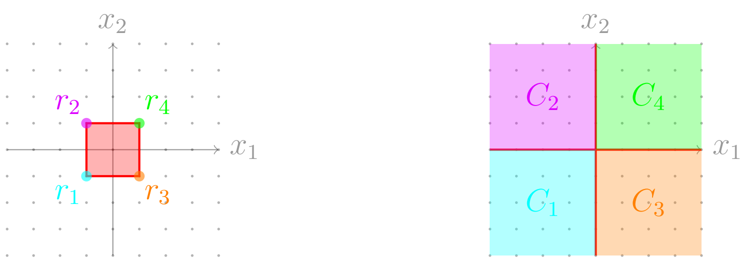
The integer points in $P_{i,<}(f,g)$ *represent* $\mathrm{LM}^i(f) \cap \mathrm{LM}^i(g)$, *while the minimal generating sets represent* $\mathrm{Lcm}(i,f,g)$. *We conclude that:*

$$\mathrm{Lcm}(1,f,g) = \{x_1x_2^4, x_1^2x_2^5, x_1^3x_2^6\} \quad , \quad \mathrm{Lcm}(2,f,g) = \{x_1^8x_2^7\}$$
$$\mathrm{Lcm}(3,f,g) = \{x_1^5x_2\}$$

Let's see an example for $f, g \in K_P\{x\}$ for P a full dimensional polytope. We will take P such that we are in the poly-annulus case (Example 1.2.6). We use as generalized order the order of Example 1.4.14.

Example 5.1.3. *Take* $r_1 = (-1,-1)$, $r_2 = (-1,1)$, $r_3 = (1,-1)$, $r_4 = (1,1)$, $P = \mathrm{Conv}(r_1, r_2, r_3, r_4)$, $K = \mathbb{Q}_2$ *and*

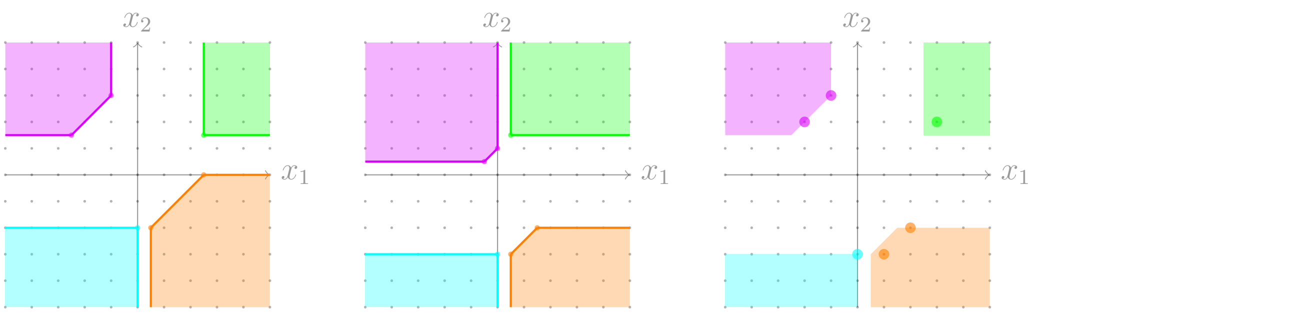
$$f = 2x_1^2x_2^{-2} - 4x_1^{-1} + x_1^2x_2^{-3} \quad , \quad g = 2x_2^2x_1 + 8x_1^2x_2^{-1}.$$



(a) P **(b)** C_1, C_2, C_3, C_4

Figure 5.6

We give directly the final figure combining all objects as in Figure 5.5.



(a) $e_{i,f} + P_{i,<}(f)$ **(b)** $e_{i,g} + P_{i,<}(g)$ **(c)** $P_{i,<}(f, g)$ and $\mathrm{Lcm}(i, f, g)$

Figure 5.7

We conclude that:

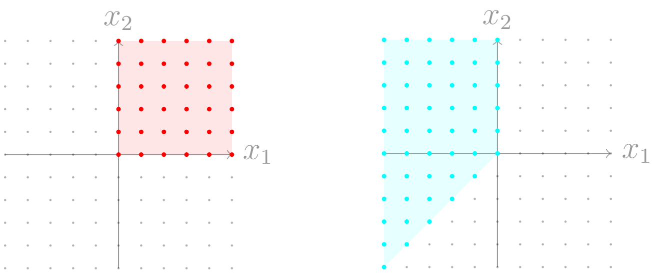
$$\mathrm{Lcm}(1, f, g) = \{x_2^{-3}\} \quad , \quad \mathrm{Lcm}(2, f, g) = \{x_1^{-2}x_2^2, x_1^{-1}x_2^3\}$$
$$\mathrm{Lcm}(3, f, g) = \{x_1x_2^{-3}, x_1^2x_2^{-2}\} \quad , \quad \mathrm{Lcm}(4, f, g) = \{x_1^3x_2^2\}$$

5.1.2 Case of $K_r\{x\}$ for a specific decomposition

Recall from Chapter 3 and Section 4.5, that in the case of $K_r\{x\}$, the conic decomposition is single-indexed: $(M_i)_{i\in I}$. We aim to compute the finite set of generators of the M_i-module $\mathrm{LM}_i(f) \cap \mathrm{LM}_i(g)$ for $f, g \in K_r\{x\}$, which is denoted by $\mathrm{Lcm}(i, f, g)$.
Our result is partial: we prove in Theorem 5.1.5 that if the conic decomposition is $(M_0, M_1, \dots, M_n)$ from Example 1.4.3, the module $M_i(f)$ is cyclic (i.e generated by a single element) over M_i for every index i. We then present a simple algorithm (Algorithm 13) to compute its generator. From this result, one easily derives that $\mathrm{LM}_i(f) \cap \mathrm{LM}_i(g)$ is also cyclic, and its generator, which is by definition $\mathrm{Lcm}(i, f, g)$, can be computed from the generators of $M_i(f)$ and $M_i(g)$.
Recall that the conic decomposition from Example 1.4.3 contains $n + 1$ cones and is defined by:

1. $M_0 := \{x^k \mid k \in \mathbb{N}^n\}$

2. $M_j := \{x_1^{-1} \dots x_n^{-1}\} \cup \{x_1, \dots, \hat{x_j}, \dots, x_n\}$, for $1 \leq j \leq n$



(a) M_0 **(b)** M_1 **(c)** M_2

Figure 5.8: Conic decomposition for Example 1.4.3 for $n = 2$

We will need this lemma:

Lemma 5.1.4. *Let $\geq_g$ be a generalized order on $\mathcal{M}$ such that the underlying conic decomposition $(M_i)_{i \in I}$ satisfies:*

$$\forall i, j \in I, \ {}_{\mathrm{gr}}\langle M_i \cap M_j \rangle \cap M_i = M_i \cap M_j, \tag{5.1}$$

where ${}_{\mathrm{gr}}\langle M_i \cap M_j \rangle$ is the group generated by the monoid $M_i \cap M_j$. For all $i, j \in I, f \in K_r\{x\}$, we have:

$$(s \in M_i \cap M_j, \ t \in M_i(f), \ st \in M_i(f) \cap M_j(f)) \implies t \in M_j(f).$$

Proof. Suppose that $s \in M_i \cap M_j$, $t \in M_i(f)$, and $st \in M_i(f) \cap M_j(f)$. From the condition $st \in M_i(f) \cap M_j(f)$, we deduce

$$\mathrm{lm}(stf) = st\mathrm{lm}_i(f) = st\mathrm{lm}_j(f),$$

which implies that the leading monomials agree, i.e.

$$l := \mathrm{lm}_i(f) = \mathrm{lm}_j(f).$$

Since $t \in M_i(f)$, it follows that

$$\mathrm{lm}(tf) = tl \in M_i,$$

while $st \in M_i(f) \cap M_j(f)$ yields

$$\mathrm{lm}(stf) = stl \in M_i \cap M_j.$$

Therefore,

$$\mathrm{lm}(tf) = t\mathrm{lm}_j(f) = tl = s^{-1}(stl) \in {}_{\mathrm{gr}}\langle M_i \cap M_j\rangle \cap M_i = M_i \cap M_j \subseteq M_j$$

by Equation 5.1. This shows precisely that $t \in M_j(f)$. □

Notice that the conic decomposition from Example 1.4.3 satisfies Lemma 5.1.4.

In the proof of Theorem 5.1.5, we work with exponents instead of monomials. That is, we use the isomorphism $\mathcal{M} \simeq \mathbb{Z}^n$ and represent monomials by vectors in $\mathbb{Z}^n$.

Theorem 5.1.5. *Let $\geq_g$ be a generalized order such that the underlying conic decomposition is $(M_0, M_1, \ldots, M_n)$ as defined in Example 1.4.3. For all $f \in K_r\{x\}$ and $0 \leq i \leq n$, $M_i(f)$ is a cyclic (i.e generated by a single element) M_i-module.*

Proof. Assume, for the sake of contradiction, that there exists an index i and $f \in K_r\{x\}$ such that the set of minimal generators of $M_i(f)$ contains at least two elements $a \neq b \in \mathbb{Z}^n$. We will focus on the case where $i = 0$; the proof for $i \neq 0$ is essentially the same. We have $a + M_0 = \bigcap_j \{x \in \mathbb{Z}^n \mid x_j \geq a_j\}$ and $b + M_0 = \bigcap_j \{x \in \mathbb{Z}^n \mid x_j \geq b_j\}$. Let $p \in \mathbb{Z}^n$ be the vector such that $p_j = \min(a_j, b_j)$ for $1 \leq j \leq n$, which is necessarily different from a and b. By definition of p, we have $(a + M_0) \cup (b + M_0) \subset p + M_0$. Therefore, $p \notin M_0(f)$; otherwise, $\{a, b\}$ wouldn't be contained in a minimal set of generators of $M_0(f)$. Since the $M_i(f)$'s cover $\mathbb{Z}^n$, there exists $k \neq 0$ such that $p \in M_k(f)$. We have $M_0 \cap M_k = \{x_k = 0\} \cap \left(\bigcap_{j\neq k} \{x_j \geq 0\}\right)$. Consequently, the set $p + (M_0 \cap M_k)$ contains either a or b, depending on whether $\min(a_k, b_k) = a_k$ or b_k. Without loss of generality, assume it contains a. From $a \in p + (M_0 \cap M_k)$, we deduce that $a - p \in M_0 \cap M_k$. Also, from $a \in p + (M_0 \cap M_k) \subset p + M_k$ and $p \in M_k(f)$, we get that $a \in M_k(f)$ and thus $a \in M_k(f) \cap M_0(f)$. Consequently we can write $(a - p) + p = a$ with $(a - p) \in M_k \cap M_0$, $p \in M_k(f)$ and

$a \in M_k(f) \cap M_0(f)$. According to Lemma 5.1.4, this implies $p \in M_0(f)$, contradicting $p \notin M_0(f)$. $\square$

Knowing from Theorem 5.1.5 that $M_0(f)$ is cyclic, we can find its generator by first locating an element in $M_0(f)$, and then performing a simple descent, coordinate by coordinate, until reaching the generator (Algorithm 13).
Then, from the generator s_f of $M_0(f)$, we get that $\mathrm{lm}_0(f)s_f$ is the generator of $\mathrm{LM}_0(f)$, that is:

$$\mathrm{LM}_0(f) = \mathrm{lm}_0(f)M_0(f) = \mathrm{lm}_0(f)s_f M_0,$$

and similarly for $\mathrm{LM}_0(g)$:

$$\mathrm{LM}_0(g) = \mathrm{lm}_0(g)M_0(g) = \mathrm{lm}_0(g)s_g M_0.$$

It follows that $\mathrm{LM}_0(f) \cap \mathrm{LM}_0(g)$ is also monogeneous over M_0, with generator $\max(\mathrm{lm}_0(f)s_f, \mathrm{lm}_0(g)s_g)$, where the max is taken coordinate by coordinate.
Finally:

$$\mathrm{LM}_0(f) \cap \mathrm{LM}_0(g) = \max(\mathrm{lm}_0(f)s_f, \mathrm{lm}_0(g)s_g)M_0,$$

that is,

$$\mathrm{Lcm}(0, f, g) = \{\max(\mathrm{lm}_0(f)s_f, \mathrm{lm}_0(g)s_g)\}.$$

The procedure is similar for all the $M_i(f)$'s.

Algorithm 13: Generator

input : an index i s.t $0 \leq i \leq n$, $f \in K_r\{x\}$, $H = \{h_1, \ldots, h_n\}$ a set of generators of M_i

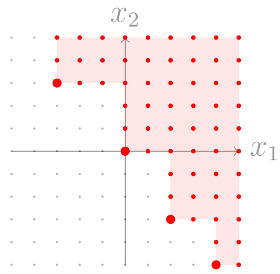
output: $g \in \mathcal{M}$ such that $M_i(f) = gM_i$

$m \leftarrow \text{TranslatorForCone}(f, H)$;
for $h \in H$ **do**
  while $m \in M_i(f)$ **do**
    $m \leftarrow m - h$;
  $m \leftarrow m + h$;
return m

5.2 Gröbner bases for Laurent polynomial ideals

Theorem 5.1.5 belongs to the theory of generalized orders and does not rely on any specific property of $K_r\{x\}$. Although stated for a series $f \in K_r\{x\}$, the identity $M_i(f) = M_i(\text{in}_r(f))$ implies it also applies to polynomials in Laurent polynomial rings. Thus, Theorem 5.1.5 and Algorithm 13 can be used to compute Gröbner bases *à la* Pauer-Unterkircher (see [PU99]) in $F[X^{\pm 1}]$, for any field F. We implemented this method in SageMath. A short demo is available in Appendix A.1.

Effectivity is not addressed in general in the original article [PU99]. Our approach allows one, when the choice of the generalized order and thus the conic decomposition is free, to compute Gröbner bases in $F[X^{\pm 1}]$ by choosing the decomposition of Example 1.4.3. But, in general, a formula to compute the generators of $M_i(f)$ remains to be found. For an arbitrary conic decomposition, $M_i(f)$ is generally neither cyclic nor equal to the set of integer points of a polyhedron. For example, the typical shape of $M_{(1,2)\to(1,1)}(f)$ under the decomposition of Example 1.4.4 in two variables is:



(a) Typical shape of $M_i(f)$ for the cone $M_{(1,2)\to(1,1)}$ of Example 1.4.4

In our attempts to find a general formula for the generators of $M_i(f)$, we derived a theoretical expression which we find interesting to present in Theorem 5.2.2. What makes it difficult to handle is the use of the set complement in item 3 of Definition 5.2.1 below:

Definition 5.2.1. *We define:*

1. $A_i(f) := \{t \in \mathcal{M},\ t\mathrm{lm}_i(f) \in M_i\}$.
2. $\Delta_{i,j}(f) := \{t \in A_i(f) \cap A_j(f),\ t\mathrm{lm}_i(f) >_g t\mathrm{lm}_j(f)\}$.
3. $U_i(f) := A_i(f) \cap \bigcap_{j\in I,\ \mathrm{lm}_i(f)\neq \mathrm{lm}_j(f)} \left(A_j(f)^c \cup \Delta_{i,j}(f)\right)$.

Notice in Definition 5.2.1 that $M_i(f) \subseteq A_i(f)$ because if $m \in M_i(f)$, then $\mathrm{lm}(mf) = m\mathrm{lm}_i(f) \in M_i$ by definition. The converse is not true.

Theorem 5.2.2. *For any $i \in I$,*

$$M_i(f) = U_i(f). \tag{5.2}$$

Proof. We first prove that $M_i(f) \subseteq U_i(f)$. Let $t \in M_i(f)$. and let $j \in I$ be such that $\mathrm{lm}_i(f) \neq \mathrm{lm}_j(f)$. We can write that $t\mathrm{lm}_i(f) = \mathrm{lm}(tf) \geq_g t\mathrm{lm}_j(f)$ and since $\mathrm{lm}_i(f) \neq \mathrm{lm}_j(f)$, this inequality becomes strict: $t\mathrm{lm}_i(f) >_g t\mathrm{lm}_j(f)$. Depending on whether $t\mathrm{lm}_j(f) \in M_j$ or not, t is then in $A_j^c(f)$ or $\Delta_{i,j}(f)$. In conclusion, $t \in U_i(f)$ and $M_i(f) \subseteq U_i(f)$.
We prove the converse inclusion. Let

$$t \in U_i(f) = A_i(f) \cap \bigcap_{j\in I,\ \mathrm{lm}_i(f)\neq \mathrm{lm}_j(f)} \left(A_j^c(f) \cup \Delta_{i,j}(f)\right).$$

From $t \in A_i(f)$, we get that $t\mathrm{lm}_i(f) \in M_i$. Let $j \in I$ be such that $\mathrm{lm}(tf) \in M_j$. Then $\mathrm{lm}(tf) = t\mathrm{lm}_j(f) \in M_j$ and $t \in A_j(f)$. If j is such that $\mathrm{lm}_i(f) = \mathrm{lm}_j(f)$ then $t\mathrm{lm}_i(f) = t\mathrm{lm}_j(f)$, hence $\mathrm{lm}(tf) = t\mathrm{lm}_i(f)$ and $\mathrm{lm}(tf) \in M_i$. Otherwise, our assumptions provide that $t \in U_i$ but not in $A_j(f)$, hence $t \in \Delta_{i,j}(f)$. Consequently, $t\mathrm{lm}_i(f) >_g t\mathrm{lm}_j(f) = \mathrm{lm}(tf)$ which is in contradiction with the definition of $\mathrm{lm}(tf)$. In conclusion, $\mathrm{lm}(tf) = t\mathrm{lm}_i(f) \in M_i$, hence $t \in M_i(f)$ and $M_i(f) = U_i(f)$. □

5.3 Implementation

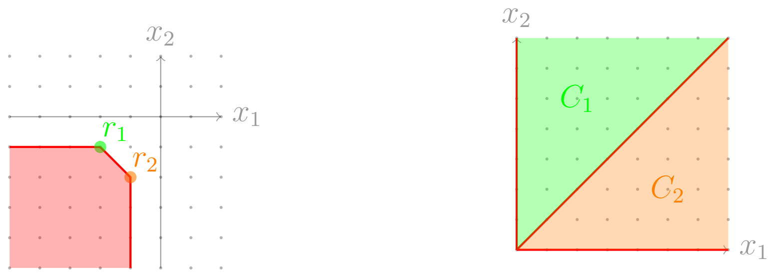
A prototype implementation[1] has been coded in SageMath. Three classes can be loaded into a Sage session:

1. `PositiveExponentsAlgebra` for computations in $K_P^+\{x\}$
2. `PolyCircleAlgebra` for computations in $K_r\{x\}$ using the conic decomposition of Example 1.4.3, where Theorem 5.1.5 applies
3. `PolyAnnulusAlgebra` for computations in $K_P\{x\}$ over a poly-annulus domain (Example 1.2.6) using the generalized order of Example 1.4.14

In this development, the main focus has been placed on $K_P^+\{x\}$ through the `PositiveExponentsAlgebra` class, for which all presented algorithms have been implemented. The two other classes are testing tools that allow only basic computations within $K_r\{x\}$ and poly-annuli.
We report below on an explicit Gröbner basis computation in $K_P^+\{x\}$. Note that the implementation relies on Sage's Normaliz backend for polyhedral computations, which must be available. The coefficient field is currently restricted to $\mathbb{Q}_p$, and the vertices of P must lie in $\mathbb{Z}^n$.

A computation in $K_P^+\{x\}$. Take $r_1 = (-2,-1)$, $r_2 = (-1,-2)$, $P = \mathrm{Conv}^+(r_1, r_2)$, and let $K = \mathbb{Q}_2$. We use the degree reversed lexicographical order ('degrevlex') as term order.

[1] https://gist.github.com/vilanele/1d0396e3d602be9ab84e31ec1354697f



(b) P **(c)** C_1 and C_2

Figure 5.9

Consider the following two series:

$$f = (1 + 2 + O(2^6))x_1x_2^2 + (2^2 + O(2^7))x_1^3 + O_P(10),$$
$$g = (1 + 2 + O(2^4))x_1^2x_2^2 + (2^3 + O(2^6))x_1^2x_2 + O_P(10).$$

The notation $O_P(10)$ hides a series l such that $\mathrm{val}_P(l) \geq 10$, while $O(2^e)$ refers to the usual 2-adic precision in the coefficient field $K = \mathbb{Q}_2$. Note that if a series is known up to $O_P(b)$, then for any known term $t = c \cdot m$ (with $c \in K$ and m a monomial), the 2-adic precision of c needs to be at most $\mathrm{val}_P(m) + b$. When the coefficient is known exactly at that precision, we omit the explicit 2-adic precision for readability. With these conventions, we may simply write:

$$f = (1 + 2)x_1x_2^2 + 2^2x_1^3 + O_P(10),$$
$$g = (1 + 2)x_1^2x_2^2 + 2^3x_1^2x_2 + O_P(10).$$

Now let us run Algorithm 7 with input $H = (f, g)$. We first compute the least common multiples of f and g with respect to r_1 and r_2:

$$\mathrm{Lcm}(1, f, g) = \{x_1^2x_2^2\}, \quad \text{and} \quad \mathrm{Lcm}(2, f, g) = \{x_1^7x_2^2\}.$$

From these, we compute the corresponding critical pairs:

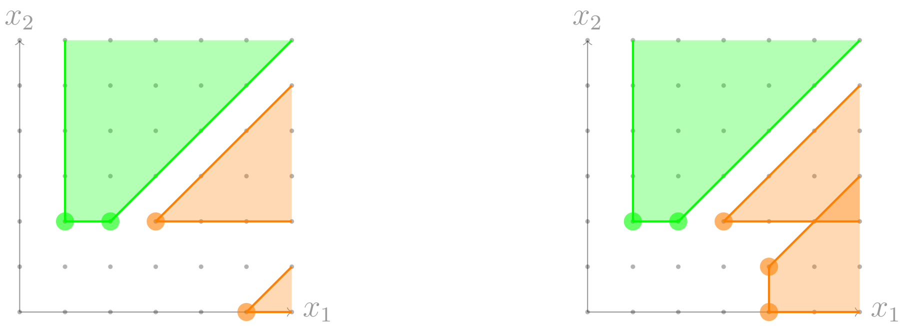
$$S(1, f, g, x_1^2x_2^2) = (2^2 + 2^3)x_1^4 + (2^3 + 2^5)x_1^2x_2 + O_P(10),$$
$$S(2, f, g, x_1^7x_2^2) = (1 + 2^3)x_1^5x_2^4 + (2^5 + 2^6 + 2^7)x_1^7x_2 + O_P(17).$$

Reducing $S(1, f, g, x_1^2x_2^2)$ by $H = (f, g)$ gives:

$$r = (2^2 + 2^4 + 2^5)x_1^4 + (2^3 + 2^4)x_1^2x_2 + O_P(10),$$

which we add to H. The reduction of $S(1, f, g, x_1^7x_2^2)$ by $H = (f, g, r)$ yields 0. Next, all critical pairs for the tuples (f, r) and (g, r) reduce to 0 under $H = (f, g, r)$, so the algorithm terminates and outputs $H = (f, g, r)$.

To conclude, we provide a side-by-side illustration of the leading monomials reached by the initial generating tuple (f, g) and the output (f, g, r):



(a) $\bigcup_{i=1,2} \mathrm{LM}^i(f) \cup \mathrm{LM}^i(g)$

(b) $\bigcup_{i=1,2} \mathrm{LM}^i(f) \cup \mathrm{LM}^i(g) \cup \mathrm{LM}^i(r)$

Figure 5.10: Leading monomials reached by (f, g) and the output of BuchbergerPositiveExponents$((f, g))$

6
Conclusion

The reader will have noticed throughout this thesis that once the right objects are properly defined, the strategy and proofs of the Gröbner theory developed here closely follow the classical one in polynomial rings. The general principle is the following: whenever a leading term is involved, select a cone $M_{i,j}$ from the chosen conic decomposition that contains this leading term. Then, locally for this term, everything behaves nicely. We then use the finiteness of the vertex set and the well-ordering property of generalized orders to conclude globally. This approach appears, for example, in the proof of topological convergence (Lemma 4.1.11) and in the proof of the sum of critical pairs (Lemma 4.3.3), amongst others.

Another point is that, since the given polytope P is a core datum of $K_P\{x\}$, it seems unavoidable to introduce the sets V_i, $V_{i,<}$, $V_{i,<}(f)$, and more generally all new definitions that depend on the vertices of P and the polyhedral convergence constraints. However, the use of generalized orders seems more artificial: it was added simply because we could not find another way to bypass the exponent problem. A natural question is: can we avoid introducing generalized orders at all?

Finally, we strongly believe that all results proved for Tate algebras in the series of papers [CVV19]–[CVV22] should extend to $K_P\{x\}$. This includes, in particular, all the algorithms designed to avoid useless reductions of critical pairs to zero during the course of Buchberger's algorithm, which we did not discuss in this thesis: the F5 criterion of [CVV19, Section 3.3], the signature-based algorithms introduced in [CVV20], and the FGLM adaptation to zero-dimensional ideals in [CVV21]. These tools are especially important, because our Buchberger algorithm in $K_P\{x\}$ generally involves reducing more than one critical pair per pair (f, g).

A
Appendix

Summary

A.1 Gröbner bases in Laurent polynomial ring

Short demo (https://github.com/sagemath/sage/pull/37241).
A new class constructor named `GeneralizedOrder` is introduced. The supported g.m.o's are those for which the underlying conic decomposition is the decomposition of Example 1.4.3 (so we can benefit of Theorem 5.1.5) and for which the order is defined by a group order and a score function as in Lemma 1.4.10. The group order and score function can be specified using the keywords `group_order` and `score_function` in the constructor. Defaults are the lexicographical order on $\mathbb{Z}^n$ and the `min` function of Example 1.4.13.

```
In:  from sage.rings.polynomial.generalized_order import
     GeneralizedOrder
In:  GeneralizedOrder(3)
Out: Generalized order in 3 variables using (lex, min)
```

Another example, using this time the score function `degmin` of Example 1.4.12:

```
In:  G = GeneralizedOrder(2, score_function='degmin'); G
Out: Generalized order in 2 variables using (lex, degmin)
```

Now we can compare tuples:

```
In:  G.greatest_tuple((-2,3), (1,2))
Out: (−2, 3)
In:  G.greatest_tuple_for_cone(2, (1,3), (-1,2), (-4,-3))
Out: (−4, −3)
```

The `LaurentPolynomialRing` constructor has been updated to accept instances of the new `GeneralizedOrder` class for the keyword `order`. Elements have new methods:

```
In:  L.<x,y> = LaurentPolynomialRing(QQ, order=G)
In:  f = 2*x^2*y^-1 + x^-3*y - 3*y^-5
In:  f.leading_monomial()                        // lm(f)
Out: y^-5
In:  f.leadin_monomial_for_cone(1)               // lm_1(f)
Out: x^-3y
In:  f.generator_for_cone(2)                     // T_2(f)
Out: xy^2
```

We can reduce an element using multivariate division (Algorithm 9):

```
In:  L = [x^-2*y^-1 + x*y, x^-2*y + x^2*y^-1]
In:  f.generalized_reduction(L)
Out: (-y^3 + 2x^2y^-1 - 3x^-1y^-1, [x^-1y^2 + 3x^-2y^-2, -3x^-2y^-4])
```

Lastly, within the `LaurentPolynomialIdeal` class, the method `groebner_basis` has been modified so that it uses Algorithm 10 when a generalized order is specified:

```
In:  G = GeneralizedOrder(3, score_function='degmin')
In:  L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In:  I = L.ideal([x^-3*y^-4 + x*y*z, x^3*y^-2 + y^-1*z])
In:  I.groebner_basis()
Out: (x^3y^-4 + xyz,
      x^3y^-2 + y^-1z,
      -y^-4 + x^-1y^-2z^-1)
```

Another example of Gröbner basis computation for an ideal in the ring $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$:

```
In:  G = GeneralizedOrder(3, score_function='min')
In:  L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In:  g1 = 1/2*x^-1*y + 3*y^-4*z^2 + y
In:  g2 = 2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6
In:  I = L.ideal([g1, g2])
In:  I.groebner_basis()
```

Out:
$$\begin{aligned}
&(y + \tfrac{1}{2}x^{-1}y + 3y^{-4}z^2,\\
&2x^2y^3z^{-1} - \tfrac{1}{3}x^{-1}y^3z^{-6},\\
&\tfrac{1}{4}y^5z^5 - 3x^2z^7 + \tfrac{3}{2}xz^7 + \tfrac{1}{3}y^5 + z^2,\\
&\tfrac{1}{4}y^{10}z^5 - \tfrac{3}{4}y^5z^7 + \tfrac{1}{3}y^{10} - \tfrac{9}{2}xz^9 + 2y^5z^2 + 3z^4,\\
&\tfrac{1}{4}y^{15}z^5 + \tfrac{1}{3}y^{15} + 3y^{10}z^2 + 9y^5z^4 + 9z^6,\\
&6x^2y^4z^4 + 3xy^4z^4 + 3x^{-1}y^{-1}z)
\end{aligned}$$

Same ideal as above, but using the score function `degmin`:

```
In:  G = GeneralizedOrder(3, score_function='degmin')
In:  L.<x,y,z> = LaurentPolynomialRing(QQ, order=G)
In:  g1 = 1/2*x^-1*y + 3*y^-4*z^2 + y
In:  g2 = 2*x^2*y^3*z^-1 - 1/3*x^-1*y^3*z^-6
In:  I = L.ideal([g1, g2])
In:  I.groebner_basis()
```

Out:
$$\begin{aligned}
&(y + \tfrac{1}{2}x^{-1}y + 3y^{-4}z^2,\\
&2x^2y^3z^{-1} - \tfrac{1}{3}x^{-1}y^3z^{-6},\\
&y^5z^3 + \tfrac{1}{3}x^{-2}y^5 * z^{-2} + x^{-2},\\
&\tfrac{-1}{16}y^5z^6 - \tfrac{1}{12}y^5z - \tfrac{1}{4}z^3 + \tfrac{1}{8}x^{-1}z^3 - \tfrac{1}{16}x^{-2}z^3,\\
&-\tfrac{1}{6}xy^3z^{-1} + \tfrac{1}{24}x^{-1}y^3z^{-1} - \tfrac{1}{12}x^{-2}y^{-2}z^{-4} + \tfrac{1}{24}x^{-3}y^{-2}z^{-4},\\
&-\tfrac{1}{36}y^3z^{-1} - \tfrac{1}{72}x^{-1}y^3z^{-1} - \tfrac{1}{72}x^{-3}y^{-2}z^{-4})
\end{aligned}$$

An example in the ring $\mathbb{F}_9[x^{\pm 1}, y^{\pm 1}]$:

```
In:  G = GeneralizedOrder(2, score_function='degmin')
In:  F = GF(9)
In:  L.<x,y> = LaurentPolynomialRing(F, order=G)
In:  g1 = x^2*y + y^-6
In:  g2 = x^3*y^-2 + x^-6*y
In:  g3 = x^-2*y + x^-1*y^-2
In:  I = L.ideal([g1, g2, g3])
In:  I.groebner_basis()
```

Out: $(x^2y + y^{-6},$
$x^3y^{-2} + x^{-6}y,$
$x^{-2}y + x^{-1}y^{-2},$
$-xy + x^{-2}y^{-3},$
$x^2y + x^{-2},$
$y^{-1} + x^{-1},$
$-y^2 + x^{-1},$
$x^{-1}y^{-1} + x^{-2}y^{-2})$

Glossary

$K\langle x\rangle$ Tate algebra. 31

$K_r\langle x\rangle$ Tate algebra, log-radii r. 32

$K_r^\circ\langle x\rangle$ integer ring of $K_r\langle x\rangle$. 40

$K_P\{x\}$ polyhedral algebra. 33

$K_P^+\{x\}$ polyhedral algebra, positive exponents. 53

$K_r\{x\}$ polyhedral algebra, $P = \{r\}$. 37

$K_P^\circ\{x\}$ integer ring of $K_P\{x\}$. 40

I_P indices for the vertex ordering of P. 55

$I_P(f)$ indices i from I_P s.t $\mathrm{val}_{r_i}(f) = \mathrm{val}_P(f)$. 55

$V_{i,<}$ the set of monomials m s.t $\mathrm{val}_{r_i}(m) = \mathrm{val}_P(m)$ and $i = \min(I_P(m))$. 57

$V_{i,<}(f)$ the set of monomials $\{m \in \mathcal{M}_+,\ \mathrm{lm}(mf) \in V_{i,<}\}$. 62

$\mathrm{LM}^i(f)$ the set of monomials $\{\mathrm{lm}(mf),\ m \in \mathcal{M}_+\} \cap V_{i,<}$. 62

$\mathrm{LM}_i(f)$ the set of monomials $\mathrm{LM}_i(f) := \{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_i$. 79

$M_{i,j}^<$ the set of monomials $M_{i,j} \cap V_{i,<}$. 101

$M_{i,j}^<(f)$ the set of monomials $\{m \in \mathcal{M},\ \mathrm{lm}(mf) \in M_{i,j}^<\}$. 101

$\mathrm{LM}_j^i(f)$ the set of monomials $\{\mathrm{lm}(mf),\ m \in \mathcal{M}\} \cap M_{i,j}^<$. 101

Bibliography

[BLV24] Moulay Barkatou, Lucas Legrand, and Tristan Vaccon. "Gröbner Bases Over Polytopal Affinoid Algebras". In: *Proceedings of the 2024 International Symposium on Symbolic and Algebraic Computation*. Raleigh NC USA: ACM, July 16, 2024, pp. 188–197. DOI: 10.1145/3666000.3669690.

[BLV25] Moulay A. Barkatou, Lucas Legrand, and Tristan Vaccon. "Gröbner bases over polytopal affinoid algebras". Extended version. May 2025. URL: https://hal.science/hal-04510822.

[Bos14] Siegfried Bosch. *Lectures on Formal and Rigid Geometry*. Vol. 2105. Lecture Notes in Mathematics. Springer International Publishing, 2014.

[Buc06] Bruno Buchberger. "Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal". In: *Journal of Symbolic Computation* 41 (Mar. 2006), pp. 475–511. DOI: 10.1016/j.jsc.2005.09.007.

[CCM25] Cyrille Chenavier, Thomas Cluzeau, and Adya Musson-Leymarie. "Topological closure of formal powers series ideals and application to topological rewriting theory". In: *Journal of Symbolic Computation* 129 (2025), p. 102416. ISSN: 0747-7171. DOI: https://doi.org/10.1016/j.jsc.2024.102416. URL: https://www.sciencedirect.com/science/article/pii/S0747717124001202.

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. en. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-16721-3.

[CM18] Andrew J. Chan and Diane Maclagan. "Gröbner bases over fields with valuations". In: *Mathematics of Computation* 315 (Apr. 6, 2018), pp. 467–483. DOI: 10.1090/mcom/3321.

[CVV19] Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "Gröbner Bases Over Tate Algebras". In: *Proceedings of the 2019 International Symposium on Symbolic and Algebraic Computation*. ACM, July 2019, pp. 74–81. DOI: 10.1145/3326229.3326257.

[CVV20] Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "Signature-based algorithms for Gröbner bases over Tate algebras". In: *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*. July 2020, pp. 70–77. DOI: 10.1145/3373207.3404035.

[CVV21] Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "On FGLM Algorithms with Tate Algebras". In: *Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation*. ACM, July 18, 2021, pp. 67–74. DOI: 10.1145/3452143.3465521.

[CVV22] Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "On Polynomial Ideals and Overconvergence in Tate Algebras". In: *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*. Villeneuve-d'Ascq France: ACM, July 4, 2022, pp. 489–497. DOI: 10.1145/3476446.3535491.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. "Non-archimedean amoebas and tropical varieties". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2006.601 (Jan. 19, 2006). DOI: 10.1515/CRELLE.2006.097.

[FP04] Jean Fresnel and Marius van der Put. *Rigid Analytic Geometry and Its Applications*. Progress in Mathematics 218. 2004. DOI: 10.1007/978-1-4612-0041-3.

[GB09] Joseph Gubeladze and Winfried Bruns. *Polytopes, Rings, and K-Theory*. Springer Monographs in Mathematics. Springer New York, 2009. ISBN: 978-0-387-76355-2 978-0-387-76356-9. DOI: 10.1007/b105283.

[Gub07a] Walter Gubler. "The Bogomolov conjecture for totally degenerate abelian varieties". en. In: *Inventiones mathematicae* 169.2 (June 2007), pp. 377–400. DOI: 10.1007/s00222-007-0049-y.

[Gub07b] Walter Gubler. "Tropical varieties for non-archimedean analytic spaces". en. In: *Inventiones mathematicae* 169.2 (June 2007), pp. 321–376. DOI: 10.1007/s00222-007-0048-z.

[Knu97] Donald Ervin Knuth. *The art of computer programming*. 3rd ed. Reading, Mass: Addison-Wesley, 1997. ISBN: 978-0-201-89683-1 978-0-201-89684-8 978-0-201-89685-5.

[MR20] Thomas Markwig and Yue Ren. "Computing Tropical Varieties Over Fields with Valuation". In: *Foundations of Computational Mathematics* 20.4 (Aug. 2020), pp. 783–800. DOI: 10.1007/s10208-019-09430-2.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. en. Graduate studies in mathematics volume 161. Providence (R. I.): American mathematical society, 2015.

[PU99] Franz Pauer and Andreas Unterkircher. "Gröbner Bases for Ideals in Laurent Polynomial Rings and their Application to Systems of Difference Equations". In: *Applicable Algebra in Engineering, Communication and Computing* 9.4 (1999), pp. 271–291. DOI: 10.1007/s002000050108.

[PZ96] Franz Pauer and Sandro Zampieri. "Gröbner Bases with Respect to Generalized Term Orders and their Application to the Modelling Problem". en. In: *Journal of Symbolic Computation* 21.2 (1996), pp. 155–168. DOI: 10.1006/jsco.1996.0007.

[Rab12] Joseph Rabinoff. "Tropical analytic geometry, Newton polygons, and tropical intersections". en. In: *Advances in Mathematics* 229.6 (Apr. 2012), pp. 3192–3255. DOI: 10.1016/j.aim.2012.02.003.

[Sch98] Alexander Schrijver. *Theory of linear and integer programming*. Nachdr. Wiley-Interscience series in discrete mathematics and optimization. Chichester Weinheim: Wiley, 1998. 471 pp. ISBN: 978-0-471-98232-6.

[Tat71] John Tate. "Rigid analytic spaces". In: *Inventiones Mathematicae* 12.4 (1971), pp. 257–289. DOI: 10.1007/BF01403307.

[Vac15] Tristan Vaccon. "Matrix-F5 Algorithms and Tropical Gröbner Bases Computation". In: *Proceedings of the 2015 ACM International Symposium on Symbolic and Algebraic Computation*. Bath United Kingdom: ACM, June 24, 2015, pp. 355–362. DOI: 10.1145/2755996.2756665.

[VV23] Tristan Vaccon and Thibaut Verron. "Universal Analytic Gröner Bases and Tropical Geometry". In: *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation*. July 2023, pp. 517–525. DOI: 10.1145/3597066.3597110.

[VVY21] Tristan Vaccon, Thibaut Verron, and Kazuhiro Yokoyama. "On affine tropical F5 algorithms". In: *Journal of Symbolic Computation* 102 (Jan. 2021), pp. 132–152. DOI: 10.1016/j.jsc.2019.10.012.

[Zie95] Günter M. Ziegler. *Lectures on Polytopes*. Graduate texts in mathematics 152. New York, NY: Springer, 1995. ISBN: 978-1-4613-8431-1.

Gröbner bases over polyhedral algebras

Résumé : Cette thèse étend la théorie des bases de Gröbner aux algèbres polyédriques, définies comme des anneaux de séries formelles soumis à des conditions de convergence dictées par un polyèdre. Ces structures se rencontrent naturellement en géométrie tropicale et en géométrie rigide analytique. Nous adaptons l'algorithme de Buchberger à ce cadre en introduisant des ordres de termes adaptés, des procédures de réduction généralisées et des décompositions prenant en compte la géométrie polyédrique sous-jacente. Ce travail soulève plusieurs défis, notamment l'absence d'ordre bien fondé dans les monoïdes de Laurent et la sous-multiplicativité des valuations. La théorie est développée progressivement sur des cas particuliers, avant d'être généralisée. Une implémentation dans SageMath illustre l'effectivité des algorithmes proposés.

Mots clés : Bases de Gröbner, algèbres polyédriques, algorithme de Buchberger, géométrie rigide, géométrie tropicale, algèbres affinoïdes, ordres généralisés, SageMath.

Gröbner bases over polyhedral algebras

Abstract: This thesis develops a Gröbner basis theory for polyhedral algebras, which are power series rings defined by convergence constraints arising from polyhedra. These algebras appear naturally in tropical and rigid analytic geometry. We generalize Buchberger's classical algorithm to this setting by introducing adapted term orderings, generalized reductions, and polyhedral-aware decompositions. Challenges include the lack of well-ordering in Laurent monoids and submultiplicativity of valuations. The theory is built step-by-step for restricted classes of polyhedral algebras, and ultimately extended to the general case. We provide computational methods and a SageMath implementation illustrating the effectivity of the developed algorithms.

Keywords: Gröbner bases, polyhedral algebras, Buchberger algorithm, tropical geometry, rigid geometry, Tate algebras, generalized orders, SageMath.

Univ. de Limoges, CNRS, XLIM, UMR 7252, F-87000 Limoges, France