Lec 12/1

Friday, December 1, 2017 14:20

Book Recommendations:

Apostol – Intro. to Analytic Number Theory.

Burger & Tubbs – Making transcendency transparent.

↳ for 0.12345678910111213... is transcendental

Task for monday: find out what is: Roth Theorem on Diophantine approximations.

Arnold Ross   Kurt Mahler – P-adic analysis.

Oppenheim's Conjecture, Problem 7 (almost periodicity).

Theorem: if $\alpha \notin \mathbb{Q}$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\mathbb{N} = \{\lfloor n\alpha \rfloor, n \in \mathbb{N}\} \cup \{\lfloor n\beta \rfloor, n \in \mathbb{N}\}$.
(exercise)

Hint: Can get density of the sets

Ostrowski: any nontrivial norm on $\mathbb{Q}$ is given either by
$|x|^{\alpha}$ $(\alpha \in (0,1])$ or $\rho^{\nu_p(x)}$ (where $\rho \in (0,1)$, $x = p^{\nu_p(x)} \frac{a}{b}$)

Proof: Let φ be a nontrivial norm. Consider two cases:

(i): $\exists a \in \mathbb{N}$ s.t. $\varphi(a) > 1$.

(ii): $\forall n \in \mathbb{N}$, $\varphi(n) \leq 1$

since $1 < \varphi(a) \leq a$

Consider (i). $\varphi(n) = \varphi(1 + \dots$

Consider (i). $\varphi(n) = \varphi(1+\cdots+1) \le \varphi(1) + \cdots + \varphi(1) = n \Rightarrow \exists! \alpha \in (0,1]$ s.t. $\varphi(a) = a^\alpha$

Let $N \in \mathbb{N}$. write $N = x_0 + x_1 a + \cdots + x_{k-1} a^{k-1}$, $x_i \in \mathbb{N} \cup \{0\}$, $0 \le x_i < a$, $x_{k-1} \ge 1$

$a^{k-1} \le N < a^k$. $\varphi(N) \le \varphi(x_0) + \varphi(x_1)\varphi(a) + \cdots + \varphi(x_{k-1})(\varphi(a))^{k-1}$

$$\le (a-1)\left(1 + a^\alpha + \cdots + a^{(k-1)\alpha}\right)$$

$$= (a-1)\frac{a^{k\alpha}-1}{a^\alpha - 1} < (a-1)\frac{a^{k\alpha}}{a^\alpha - 1} = \frac{(a-1)a^{k\alpha}}{a^\alpha - 1} \quad a^{(k+1)\alpha}$$

$$\le \frac{(a-1)a^\alpha}{a^\alpha - 1} N^\alpha = C N^\alpha.$$

So $\varphi(N) < CN^\alpha$ $\forall N \in \mathbb{N}$. Replace N by N^m.

So $\varphi(N)^m = \varphi(N^m) < CN^{\alpha m}$

so $\varphi(N) < \sqrt[m]{C}\, N^\alpha$, so $\varphi(N) \le N^\alpha$.

Let $N = a^k - b$. $0 < b \le a^k - a^{k-1}$.

we have $\varphi(b) \le b^\alpha \le (a^k - a^{k-1})^\alpha$

$\varphi(n) \ge \varphi(a^k) - \varphi(b) \ge a^{\alpha k} - (a^k - a^{k-1})^\alpha = \left(1 - \left(1 - \frac{1}{a}\right)^\alpha\right) a^{\alpha k} = C_1 a^{\alpha k} > C_1 N^\alpha$.

So $\varphi(N) > C_1 N^\alpha$ $\forall N \in \mathbb{N}$, so $\varphi(N)^m = \varphi(N^m) > C_1 N^{\alpha m}$

So $\varphi(N) > \sqrt[m]{C_1}\, N^\alpha$ so $\varphi(N) \ge N^\alpha$.

Thus $\varphi(N) = N^\alpha$

$\varphi(x) = \varphi(\frac{N_1}{}$

Consider (ii). If $\varphi(p) = 1$ $\forall p \in P$, then $\varphi(n) = 1$ $\forall n$, so φ is trivial! then $\exists p \in P$ s.t. $\varphi(p) < 1$.

Claim. $\forall$ other $q \in P$, $\varphi(q) = 1$.

Pf. Let $q \in P$, $q \neq p$, s.t. $\varphi(q) < 1$. Now let $k, \ell \in \mathbb{N}$ s.t. $\varphi(p)^k < \frac{1}{2}$, $\varphi(q)^\ell < \frac{1}{2}$.

$\exists u, v \in \mathbb{Z}$ s.t. $up^k + vq^\ell = 1$. $(\varphi(u), \varphi(v) \leq 1)$

$$1 = \varphi(up^k + vq^\ell) \leq \varphi(up^k) + \varphi(vq^\ell) < \tfrac{1}{2} + \tfrac{1}{2} = 1. \text{ ※}$$

So $\varphi(n = p^{\nu_p(n)} \prod q^\alpha) = \varphi(p)^{\nu_p(n)}$. So letting $\rho = \varphi(p)$, the theorem is proved. ■

Things to look at

- On dividing a square into triangles
  (Paul Monsky, AMM 1970 p.p. 161-164)

here first ☆ R.J. Stroeker

How to solve a diophantine equation
(AMM, 1984 pp 385-392)

- Skolem-Mahler-Lech Theorem (Terry Tao blog)

Statement: The zero set of a linear recurrence set is eventually periodic. Moreover it is a union of a finite set and a finite number of residue classes, $\{n \in \mathbb{N} : n \equiv r \bmod m\}$

final: sums of squares, Lagrange thm on quad surds
irrationality, möbius inversion, etc.

Oppenheim's Conjecture

Prelim: $\{ n^2\alpha + m,\ n,m \in \mathbb{Z}, \alpha \notin \mathbb{Q} \}$ is dense in $\mathbb{R}$.

exercise: find one for which it doesn't work

What about $n^2\alpha - m^2$ is not always dense in $\mathbb{R}$. depends on α.

exercise: characterize these α.

Actual conjecture: if $Q(x_1, \dots, x_5)$ is an indefinite quadratic form whose values are not confined to a multiple of $\mathbb{Z}$. (like $\alpha n^2 - m^2\alpha$ are)

Then $\overline{Q(\mathbb{Z}^5)} = \mathbb{R}$

Margolis proved: true already for 3 variables!

$$\overline{\alpha n^2 + \beta m^2 - k^2}_{n,m,k \in \mathbb{Z}} = \mathbb{R}$$

$SL(3,\mathbb{R}) / SL(3,\mathbb{Z})$ is behind it.

Exercise: Write down all bonus problems assigned

exercise: Create a list of 2 dozen or fewer juicy theorems that may be asked for proof.

Exam: 2:15 pm on thurs. 254.