

cyclic module $\cong R/I$

Thm If M is finitely generated $M = R\{u_1, \dots, u_n\}$ then M is a quotient module of R^n .

Pf homomorphism $\varphi: R^n \rightarrow M$ $\varphi(a_1, \dots, a_n) = a_1 u_1 + \dots + a_n u_n$
is surjective so $M \cong R^n / \ker(\varphi)$.

If R is comm., $\text{Hom}_R(M, N)$ is an R -module.

$$\text{Hom}_R(M, M) =: \text{End}_R(M).$$

Additional operation: \cdot = composition. $(\varphi\psi)(u) = \varphi(\psi(u))$.
multiplication

Then $\text{End}_R(M)$ is a ring & is an R -algebra if R is commutative.

If R is a field & M is an n -dim V.S. over R ,

then $\text{End}_R(M) \cong M_n(R) \leftarrow n \times n$ matrices over R .

M is also an $\text{End}_R(M)$ -module. by $\varphi \cdot u = \varphi(u)$.

Def. A module is irreducible, or simple, if it has no submodules (except 0 & itself).

Schur's Lemma: If M, N are simple then any $\varphi \in \text{Hom}_R(M, N)$ is either 0 or is invertible (isomorphism).

Pf. $\text{Ker}(\varphi)$ is a submodule of M .

$\varphi(M)$ is a submodule of N .

Example: Let T be a linear transformation of a vector space V over F .

Then V is an $F[x]$ module by $xu = T(u)$ & $P(x)u = P(T)(u)$.

Submodules of V : T -invariant subspaces of V .

(i.e. $T(W) \subseteq W$).

V is a simple $F[x]$ -module if $\forall u \in V$, $\text{Span}\{u, T(u), T^2(u), \dots\} = V$.

Let V, W be F -vector spaces, let $T \in \text{End}_F(V)$, $S \in \text{End}_F(W)$.

then V, W are $F[x]$ -modules by
$$\begin{aligned} xu &= T(u) & u \in V \\ xv &= S(v) & v \in W. \end{aligned}$$

What is $\text{Hom}_R(V, W)$?

$\varphi: V \rightarrow W$ $\left. \begin{aligned} \varphi(u_1 + u_2) &= \varphi(u_1) + \varphi(u_2) \quad \forall u_1, u_2 \in V \\ \varphi(au) &= a\varphi(u) \quad \forall a \in R, u \in V \end{aligned} \right\} \varphi \text{ is a linear transformation}$

$$\varphi(xu) = x\varphi(u)$$

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$$\varphi(T(u)) = S(\varphi(u)) \quad \forall u \in V.$$

$$\varphi T = S \varphi$$

$$\varphi T \varphi^{-1} = S$$

$$\begin{array}{ccc}
 V & \xrightarrow{T} & V \\
 \varphi \downarrow & & \downarrow \varphi \\
 W & \xrightarrow{S} & W
 \end{array}
 \quad S \circ \varphi = \varphi \circ T$$

(φ is an intertwining transformation $V \rightarrow W$)

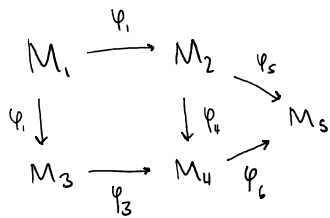
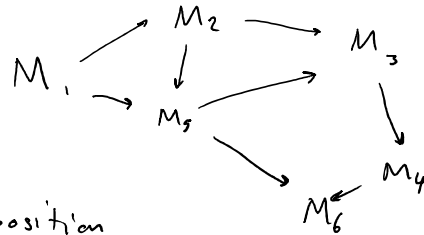
A diagram of mappings

is commutative if

\forall two vertices M_i, M_j , the composition

along any sequence of arrows from M_i to M_j

gives the same result.



is commutative if

$$\varphi_4 \circ \varphi_1 = \varphi_5 \circ \varphi_1$$

$$\varphi_5 = \varphi_6 \circ \varphi_4$$

\vdots

Let

$$\dots \rightarrow M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \rightarrow \dots$$

be a sequence of module homomorphisms.

The sequence is exact at M_i if

$$\varphi_{i-1}(M_{i-1}) = \text{Ker}(\varphi_i),$$

$$(\text{Then } \varphi_i \circ \varphi_{i-1} = 0).$$

A sequence is exact if it is exact at every step.

Examples: $0 \rightarrow N \xrightarrow{\varphi} M$ is exact at term N
iff φ is injective.

$M \xrightarrow{\varphi} K \rightarrow 0$ is exact at K iff
 φ is surjective.

$0 \rightarrow M \xrightarrow{\varphi} N \rightarrow 0$ is exact iff φ is an isomorphism.

Exact sequences of the form $0 \rightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} K \rightarrow 0$ are called short exact sequences.

it is exact iff φ is injective & ψ is surjective
 $N \subseteq M$ $K \cong M / \text{something}$

$$\varphi(N) = \ker(\psi)$$

$$K \cong M / \varphi(N) \cong M / N$$

The short five lemma.

Let the diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C' \longrightarrow 0 \end{array}$$

be commutative & have exact rows.

Then (a) if α and γ are epimorphisms (surjective), β is too.
(b) if α and γ are monomorphisms (injective), β is too.

(c) if α and γ are isomorphisms, ρ is too.

(c)

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \begin{array}{l} \nearrow B \\ \searrow B' \end{array} & \longrightarrow & C & \longrightarrow 0 \\ & & & \downarrow & & & \\ & & & B' & \longrightarrow & & \end{array} \quad \text{commutes \& has exactness} \Rightarrow B \cong B'.$$

Proof:

(a): let $b' \in B'$.