Lec 8/25

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Fact: any $x \in \mathbb{R}$ is a limit $\lim_{n\to\infty} \frac{p_n}{q_n}$ where $p_n, q_n \in \mathbb{P}$.

Is it true that $x = \lim_{n\to\infty} \frac{k_n^2}{m_n^2}$?

More Gaussian integers:

$\Gamma = \{a + bi : a, b \in \mathbb{Z}\}$ analogue of $\mathbb{Z}$.

$\mathbb{Z} \ni n = \pm 1 \, p_1^{c_1} \cdots p_n^{c_n}$ $c_i \geq 0$.

$\pi(n) \sim \frac{n}{\log n}$ $\pi_\Gamma(x) \sim ?$

$$(0,1) \ni x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \qquad a_i(x) \in \mathbb{N}$$

$\circledast$

$\mathbb{Q} \not\ni x \iff (a_i(x))$ infinite

Ex: if $x \in \mathbb{Q}$, then $\circledast$ is finite

$\lfloor x \rfloor$ is largest not exceeding x

Better formulation of exercise: $[(\sqrt{2} - 1)^n \to 0] \Rightarrow \sqrt{2} \notin \mathbb{Q}$

$$\frac{1 + \sqrt{5}}{2} = \varphi = \cfrac{1}{1 + \cfrac{1}{}}$$

$$1+\cfrac{1}{1+\cdots}$$

Lagrange theorem on quadratic continued fraction

Theorem: any quadratic "surd" has eventually periodic continued fraction (and vice versa).

Ex: find continued fraction expansions for $\sqrt{2}, \sqrt{3}$ (hint: clear irrationalities by $\times (1-\sqrt{2})$)

google: Brouncker's formula

Lambert: π is irrational

Lagrange: sum of four squares

Any $n \in \mathbb{N}$ is the sum of four squares, $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $\mathbb{Z} \ni x_i \geq 0$. (*)

Questions:

Don't submit but do

✓ 1: is representation (*) unique? No $4 = 2^2 + 0^2 + 0^2 + 0^2 = 1^2 + 1^2 + 1^2 + 1^2$

✓ 2: can any $n \in \mathbb{Z}$ be expressed as $x_1^2 + x_2^2 + x_3^2$? No. first 10 numbers.

✓ 3: can a sum of four nonnegative cubes represent any number $\in \mathbb{N}$? No

4: is any n a sum of k non-neg cubes?

Waring problem (solved positively by Hilbert):

Is it true that for any $k \in \mathbb{N}$ $\exists\, c(k)$ s.t. any $n \in \mathbb{N}$ is a sum of c k^{th} powers of nonnegative integers?

What is the rate of growth of $c(k)$? $c(2) = 4$, $c(3) = 9$

List of names:

Waring
Hilbert
Euler
Lagrange
Fermat
Liouville
Gauss
Wallis
Brouncker
Lambert
Minkowski
Mersenne
Pell
Szemeredi
Goldbach
Basel
Diophantus
Pythagoras
Basel
Plato
Cantor
Bernoullis
Green
Tao
Landau

Probabilistic NT:

A number $x \in (0,1)$ is base-2 normal if in its binary expansion every finite 0-1 word appears with "correct frequency"

$\mathbb{Q} \not\ni 0.01011011101111\ldots$ not normal

hint

<u>thm (E Borel)</u>: almost every $x \in (0,1)$ is normal in base 2

<u>Def.</u> a set $A \subset \mathbb{R}$ has "measure 0" if $\forall \varepsilon > 0$ it can be covered by a system of intervals w/ total length $< \varepsilon$.

So the set of non-normal numbers in $(0,1)$ is of measure 0.

Ex: the set of non-normal numbers is uncountable

What does "correct frequency" mean?

Champernowne's number is normal.

$\hookrightarrow$ $0.123456789101112\ldots$

normal $\rightarrow$ $0.235711\,1317\ldots$ Besicovitch

$\rightarrow$ $0.491625 36\ldots$ Erdős

Copeland

Davenport

Ex: changing squares in Champernowne's constant to 17 gives a normal number?
Just show that the squares have zero density.

D. Wall's Thm if (x_n) is normal then (x_{17n+8}) is too.