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<u>Theorem</u>: Let M_1 & M_2 be submodules of M. Then the following are equivalent:

(0) $M = M_1 \oplus M_2$ (internal)

(1) any $u \in M$ is uniquely representable as $\overset{M_1}{\underset{\cup}{u_1}} + \overset{M_2}{\underset{\cup}{u_2}}$

(2) $M = M_1 + M_2$ & $M_1 \cap M_2 = \emptyset$.

(3) if π is the projection $M \longrightarrow M/M_1$ then $\pi|_{M_2}$ is an isomorphism $M_2 \longrightarrow M/M_1$.

<u>Proof</u>:

(1) $\Rightarrow$ (2) $\forall u = u_1 + u_2 \Rightarrow M = M_1 + M_2$. if $u \in M_1 \cap M_2$ then $u = \overset{M_1}{0} + \overset{M_2}{u} = \overset{M_1}{u} + \overset{M_2}{0}$ unique so $u = 0$.

(2) $\Rightarrow$ (1) $M = M_1 + M_2 \Rightarrow u = u_1 + u_2$. if $u_1 + u_2 = v_1 + v_2$ then $u_1 - v_1 = u_2 - v_2 \in M_1 \cap M_2$ so $u_1 = v_1$, $u_2 = v_2$.

(2) $\Rightarrow$ (3) $\operatorname{Ker}(\pi|_{M_2}) = M_2 \cap M_1 = 0$. $\pi(M_2) = \pi(M_1 + M_2) = \pi(M)$ so $\pi|_{M_2}$ is isomorphism $M_2 \longrightarrow M/M_1$.

(3) $\Rightarrow$ (2) $M_1 \cap M_2 = \operatorname{Ker}(\pi|_{M_2}) = 0$. $\pi(M_2) = \pi(M_1 + M_2)$. if $\underbrace{M_1 + M_2 \neq M \text{ then } \pi(M_1+M_2) \neq \pi(M)}_{\text{3rd iso thm (both contain } M_1)}$.

So if $\pi|_{M_2}$ is isomorphism, then $M_1 + M_2 = M$.

(0) $\Rightarrow$ (1), (2), (3).

(2) $\Rightarrow$ (0) Consider $\varphi: M_1 \oplus M_2 \longrightarrow M$ by $\varphi(u_1, u_2) = u_1 + u_2$. Then φ is surjective

& $\operatorname{Ker}\varphi = (u, -u)$ with $u \in M_1$, $-u \in M_2$ so $\operatorname{Ker}\varphi = 0$.

If $M_1 \subseteq M$. <u>Def</u> M_1 is a direct summand of M if $\exists M_2$ (submodule of M.) s.t. $M = M_1 \oplus M_2$

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r, "\pi"'] & M/M_1 \arrow[r] \arrow[l, bend right, "\sigma"'] & 0 \\ & & M_2 \arrow[u, phantom, "\cup"] \arrow[ur, phantom, "\cong"] & & \end{tikzcd}$$

<u>Def</u>. for a surjective mapping $M \underset{\sigma}{\overset{\pi}{\rightleftarrows}} K \to 0$,

$\sigma: K \to M$ is a section of π if

$\pi \circ \sigma = id_K$.

Then $K \cong \sigma(K)$, $\pi|_{\sigma(K)}$ is an isomorphism.

<u>Def</u> A short exact sequence $0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} K \longrightarrow 0$

splits (from the right) if π has a section.

<u>Thm</u> N is a direct summand of M iff

$0 \to N \to M \to M/N \to 0$ splits from the right.

eg $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ doesn't work. $\nexists$ section.

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$$

$\mathbb{R} \neq \mathbb{Z} \oplus S^1$

$\mathbb{Z} \times S^1 =$  $\quad \left(0 \to \mathbb{Z} \to \mathbb{Z} \times S^1 \to S^1 \to 0 \right)$

Proof if $M = N \oplus K$ then $0 \to N \to M \xrightarrow{\pi} M/N \to 0$.

$$M \supseteq K \xleftarrow{\sigma}$$

Now let $0 \to N \to M \xrightarrow{\pi} M/N \to 0$ split, let $\sigma : M/N \to M$ be a section of π. Let $K = \sigma(M/N)$.

Then $\forall u \in N \cap K$, $u = \sigma(v)$ for some $v \in M/N$.

Then $\pi(u) = \pi(\sigma(v)) = v$. but $\pi(u) = 0$ since $u \in N$, so $v = 0$ so $u = 0$.

So $N \cap K = 0$.

Let $u \in M$. Let $v = \pi(u)$, and let $u_1 = u - \sigma(v)$.

then $u = u_1 + \sigma(v)$, $\sigma(v) \in K$, and $\pi(u_1) = \pi(u) - \pi(\sigma(v)) = 0$.

So $u_1 \in N$. So $M = N + K$, so $M = N \oplus K$.

Def Let $\eta : N \to M$ be an injection.

A hom-sm $\tau : M \to N$ is a projection for η if $\tau \circ \eta = id_N$

$$0 \to N \xrightarrow{\eta} M, \quad M \xrightarrow{\tau} N$$

Def: $0 \to N \xrightarrow{\eta} M \to K \to 0$ splits from the left if $\exists$ projection for η.

Theorem: Let $N \subset M$ be a submodule.

Then N is a direct summand of M iff

$\exists$ projection for the embedding $N \to M$.

<u>Theorem</u>: Let $0 \longrightarrow N \underset{\subseteq}{\xrightarrow{\iota}} M \longrightarrow K \longrightarrow 0$ be exact. then if it splits from the left then $M = N \oplus \underset{\text{of } K}{\text{a copy}}$

$$M = M_1 \oplus \cdots \oplus M_n = M_1 \times \cdots \times M_n$$

is universal attracting object in Category of $(N, \varphi_1, \ldots, \varphi_n)$ $\quad \varphi_1 : M_1 \to N, \ \ldots, \ \varphi_n : M_n \to N$

it's universal repelling object in ${}^{\text{another}}$ Category

$M_1, \ldots, M_n$ - submodules of M.

$\Rightarrow$ M = internal direct sum of M_i

if $M \cong M_1 \oplus \cdots \oplus M_n$.

So that

$$\begin{tikzcd} & M_i \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{embeddings}"] & \\ M \arrow[rr] & & M_1 \oplus \cdots \oplus M_n \end{tikzcd}$$

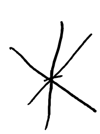
is commutative $\forall i$.

Theorem: Let $M_1, \ldots, M_n$ be submodules of M.

Then $M = M_1 \oplus \cdots \oplus M_n$

iff ① $\forall u \in M$, $u = u_1 + \cdots + u_n$ uniquely $(u_i \in M_i)$.

iff ② $M = M_1 + \cdots + M_n$, and $\forall i$, $M_i \cap \left(\sum_{j \neq i} M_j\right) = 0$

$(\mathbb{R}^2 \neq M_1 \oplus M_2 \oplus M_3$  $)$

M_α; $\alpha \in \Lambda$ - family of modules

$$\prod_{\alpha\in\Lambda} M_\alpha = \{ (u_\alpha)_{\alpha\in\Lambda} : u_\alpha \in M_\alpha \ \forall \alpha \}$$

but if Λ is infinite, $\underbrace{\prod_{\alpha\in\Lambda} M_\alpha \neq \bigoplus_{\alpha\in\Lambda} M_\alpha}_{\text{submodule}}$ ← all but finitely many entries must be zero

<u>Def</u>: direct sum $\bigoplus_{\alpha\in\Lambda} M_\alpha = \{ (u_\alpha)_{\alpha\in\Lambda} : u_\alpha \in M_\alpha \ \forall \alpha,\ u_\alpha = 0 \text{ for all but finitely many } \alpha \}$.

<u>Example</u> F : field.

as F-modules

$$F \times F \times \cdots = \prod_{i=0}^{\infty} F \cong F[[x]] \text{ - power series}$$

$$F \oplus F \oplus \cdots = \bigoplus_{i=0}^{\infty} F \cong F[x] \text{ - power series}$$