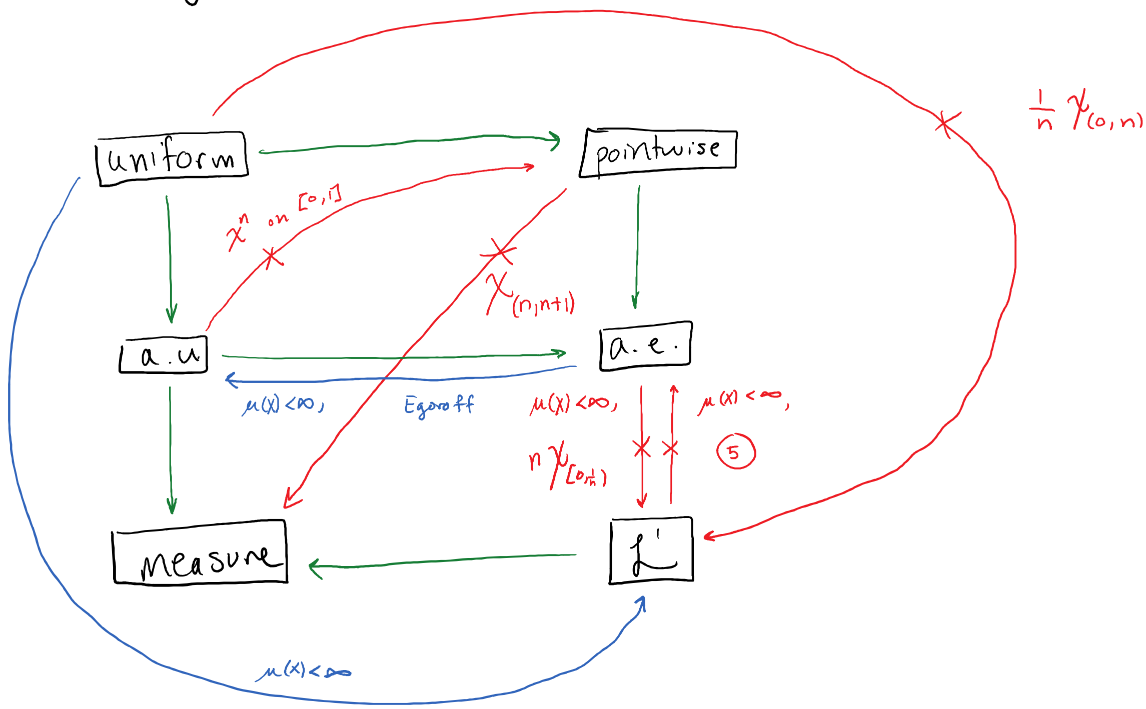
Lec 9/23

Monday, September 23, 2019 10:14

Modes of Convergence



Counterexample

① $f_n = \frac{1}{n}\chi_{(0,1)}$

② $f_n = \chi_{(n,n+1)}$

③ $f_n = n\chi_{[0,\frac{1}{n})}$

④ $f_n = x^n$ on $[0,1]$

⑤ $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_3 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{4}]}$,

typewriter sequence.

Lemma if $f_n \to f$ unif, and $\mu(X) < \infty$, $f_n \to f$ in $\mathcal{L}^1$.

Pf Let $\varepsilon > 0$. $\int |f - f_n| < \varepsilon\, \mu(X)$ for n large.

Thm (Egoroff) if $f_n \to f$ a.e. and $\mu(X) < \infty$, $f_n \to f$ a.u. $(N = \{f_n \not\to f\})$

Pf We may assume $f_n \to f$ everywhere by replacing X w/ $X \setminus N$, $\mu(N) = 0$.

$\forall k \in \mathbb{N}$, note that $E_{n,k} := \bigcup_{j=n}^{\infty} \{|f_n - f| \geq \frac{1}{k}\} \searrow \emptyset$ as $n \to \infty$.

As μ is cts from above $(\mu(X) < \infty)$, $\forall k$ $\mu(E_{n,k}) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$. For all $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ s.t. $\mu(E_{n,k}) < \frac{\varepsilon}{2^k}$ $\forall n \geq n_k$. Set $E := \bigcup_{k=1}^{\infty} E_{n_k,k}$.

$$\mu(E) \leq \sum \mu(E_{n_k,k}) \leq \sum \frac{\varepsilon}{2^k} = \varepsilon$$

and $\forall n \geq n_k$, $x \notin E \Rightarrow x \notin E_{n_k,k} \Rightarrow |f(x) - f_n(x)| < \frac{1}{k}$,

so $f_n \to f$ uniformly on E^c. $\square$

$$\boxed{\text{measure}} \xrightarrow[f_{n_k}]{\text{subsequence}} \boxed{\text{a.e}}$$

Def A seq. (f_n) is called Cauchy in measure if $\forall \varepsilon > 0$, $\mu(|f_n - f_m| \geq \varepsilon) \to 0$ as $m, n \to \infty$.

<u>Question</u> define $\rho_\varepsilon(f,g) := \mu(|f-g| \geq \varepsilon)$.

what's up with that?

<u>Theorem</u> If (f_n) is Cauchy in measure, $\exists!$ (up to μ-a.e.) $\mathcal{M}$-mble fn f s.t. $f_n \to f$ in measure. moreover, $\exists$ subseq $f_{n_k} \to f$ a.e.

<u>Remark</u> if $f_n \to f$ in measure, f is Cauchy in measure.

Let $\varepsilon, \delta > 0$. pick $N \in \mathbb{N}$ large so $\mu(|f-f_n| \geq \varepsilon') < \delta'$ $\forall n > N$.

$$|f_n - f_m| \leq |f_n - f| + |f_m - f|$$

So $\{|f_n - f_m| \geq \varepsilon\} \subset \{|f_n - f| + |f_m - f| \geq \varepsilon\} \longrightarrow 0$ in measure. $\square$

<u>pf of theorem</u>

<u>step 1</u>: $\exists$ subseq (g_k) of (f_n) s.t.

$$\mu(|g_k - g_{k+1}| \geq 2^{-k}) < 2^{-k}$$

<u>pf</u> $\forall k \in \mathbb{N}$, $\mu(|f_n - f_m| \geq 2^{-k}) \longrightarrow 0$, so pick k inductively

so $n_{k+1} > n_k$ and $m, n \geq n_k \Rightarrow \mu(|f_m - f_n| \geq 2^{-k}) < 2^{-k}$.

<u>Step 2</u>: (g_k) is pointwise cauchy off a μ-null set N.

<u>pf</u> for $k \in \mathbb{N}$, let $E_k = \{|g_k - g_{k+1}| \geq 2^{-k}\}$.

set $N_\ell = \bigcup_{k=\ell}^{\infty} E_k$. Then

$$\mu(N_\ell) \leq \sum_{k=\ell}^{\infty} \mu(E_k) \leq \sum_{k=\ell}^{\infty} 2^{-k} = 2^{1-\ell}$$

set $N = \bigcap N_\ell = \limsup E_k$. $\mu(N) = 0$.

If $x \notin N$, then $x \notin N_\ell$ for some ℓ, so $\forall i \geq j \geq \ell$,

$$(*) \qquad |g_i(x) - g_j(x)| \leq \sum_{k=i}^{j-1} |g_k(x) - g_{k+1}(x)| \leq \sum_{k=i}^{j-1} 2^{-k} \leq 2^{1-i}.$$

Step 3: define $f(x) = \begin{cases} 0 & x \in N \\ \lim_k g_k(x) & x \notin N \end{cases}$. Then f is measurable

and $g_k \longrightarrow f$ a.e.

~~pf~~ each g_k is mble, as is $g_k\big|_{N^c}$ w.r.t. $\mathcal{M}\big|_{N^c} = \{E \cap N^c \mid E \in \mathcal{M}\}$.

By (Exercise), $f\big|_{N^c} = \lim_k g_k\big|_{N^c}$ is measurable.

(Better: define $f = \lim_k g_k \chi_{N^c}$ obviously measurable).

Step 4: $g_k \longrightarrow f$ in measure.

pf $\forall x \notin N_\ell$ and $k \geq \ell$,

$$|g_k(x) - f(x)| = \lim_{j \to \infty} |g_k(x) - g_j(x)| \leq 2^{1-k}.$$

Let $\varepsilon > 0$. Pick $\ell \in \mathbb{N}$ s.t. $0 < \frac{1}{2^\ell} < \varepsilon$.

Then $\forall\ k \geq \ell$,

$$\mu(|g_k - f| \geq \varepsilon) \leq \mu(|g_k - f| \geq \tfrac{1}{2^k}) < \frac{1}{2^k} \longrightarrow 0.$$

Step 5: $f_n \longrightarrow f$ in measure.

pf $$\{|f_n - f| \geq \varepsilon\} \subseteq \underbrace{\{|f_n - g_k| \geq \tfrac{\varepsilon}{2}\}}_{\substack{\text{small by cauchy} \\ \text{in measure}}} \cup \underbrace{\{|g_k - f| \geq \tfrac{\varepsilon}{2}\}}_{\substack{\to 0 \text{ by} \\ \text{step 4}}}$$

Step 6: f is the unique (up to μ-a.e.) mble fn s.t. $f_n \longrightarrow f$ in measure.

pf If g is another such fn,

$$\{|f-g| \geq \varepsilon\} \subseteq \underbrace{\{|f - f_n| \geq \tfrac{\varepsilon}{2}\}}_{\to 0} \cup \underbrace{\{|g - f_n| \geq \tfrac{\varepsilon}{2}\}}_{\to 0}$$

$$\Rightarrow \mu(|f-g| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0.$$

So $f = g$ a.e.