Lec 9/25

Monday, September 25, 2017 14:23

Q: $\{0,1\}^{\mathbb{N}} \approx (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$?

classical middle thirds

$C = \dot\bigcup_{i=1}^{K} C_i$ where C_i are cantor sets? Yes.

how about w/ $K \to \infty$? or $C = \dot\bigcup_{\tau \in I} C_\tau$?

$|I| = |[0,1]|$

Yes, can partition $\mathbb{N}$ into infinitely many infinite sets A_i of indices

C_i is in a space of sequences which are 0 on $\mathbb{N} \setminus A_i$.

$\dot\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$

so $\{0,1\}^{\mathbb{N}} \approx C_i \subset \{0,1\}^{\mathbb{N}}$

(Exercise: justify this $C = \dot\bigcup_{i=1}^{\infty} C_i$) (not an exercise any more lol)

$\boxed{C \approx C \times C}$ (exercise but I already did it)

google: Sapienti Sat.

this gives $\dot\bigcup_{i=1}^{\infty} C_i$ and $\dot\bigcup_{\tau \in I} C_\tau$.

$C - C = [-1, 1]$.

$\Updownarrow$

$C \cap (C - t) \neq \emptyset \quad \forall t \in [-1, 1]$

$x \in C \Rightarrow y = x + t \in C$.

$\Rightarrow \quad y-x \in C-C.$

$\Rightarrow \quad t \in C-C \Rightarrow \quad t = [-1,1]$

$C+C = [0,2]$

$\Updownarrow$

$C \cap (-C-t) \neq \emptyset \quad \forall t \in$?

Sum of 2 compact sets is compact.

then $C+C$ dense in $[0,2] \Rightarrow C+C = [0,2]$

(Exercise: Show $C+C=[0,2]$ by using $C+C$ dense)

Endpoints of removed intervals are dense in C. (Exercise)

Nevermind I got it.

Properties of C:

1. No isolated points
2. Nowhere dense in $[0,1]$.
3. Compact

$\Rightarrow$ uncountable?

do these things imply?

make more properties, enough to specify cantor sets.

Dense: Given $A \subseteq \mathbb{R}$, B is dense in A if

$\forall$ interval I where $I \cap A \neq \emptyset$, $B \cap A \cap I \neq \emptyset$.

Nowhere dense: $\forall\, [a,b] \subset [0,1]$, $\quad C \cap \overline{[a,b]} \neq [a,b]$.

Exercise $\quad \{0,1\}^{\mathbb{N}}$ w/ distance $d(x,y) = \sum \frac{|x_i - y_i|}{2^i}$ is a compact metric space.

Compact in metric spaces:

$$x_n \to x \in X \quad \forall \quad (x_n) \in X^{\mathbb{N}}$$

$$\bigcup_{\tau \in I} A_\tau = X \Rightarrow \bigcup_{i=1}^{n} A_{\tau_i} = X \qquad \text{where } A_\tau \text{ open.}$$

$$C[0,1], \quad d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

complete wrt

(exercise)

Let $B = \left\{ f \in C[0,1] : \|f\| := \max_{x \in [0,1]} |f(x)| < 1 \right\}$ (unit ball).

(Exercise: B is bounded, closed, not compact)

Def: A metric space is called separable if it has a dense countable subset.

Discrete metric space (X,d) is defined by:

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

$\ell_\infty(\mathbb{N})$ = all bdd sequences with $d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$

(Exercise: this is a metric space)

$D \subset \ell_\infty(\mathbb{N})$ which is all 0-1 sequences.

d on D is discrete metric.

(X,d) with d = discrete metric is separable iff $|X| \leq |\mathbb{N}|$

$\mathbb{Q}[x]$ is dense in $C[0,1]$.

think fourier

(exercise: $\mathbb{R}[x]$ dense $\Leftrightarrow$ trig polys dense in $C[0,2\pi]$).

T/F: $\exists$ cantor set made only of irrational #s? (exercise)

first digits of 2^n (keep thinking, it connects to u.d.)

Sárközy: $\bar{d}(A) > 0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $\exists\, x,y \in A$ s.t. $x - y = n^2$.

$\Leftrightarrow \quad A \cap (A - n^2) \neq \emptyset$.