Lec 9/16

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$(X, \mathcal{M}, \mu)$ measure space.

$$L^+ = L^+(X, \mathcal{M}, \mu)$$

$$= \{\mathcal{M}\text{-mble } f: X \to [0,\infty]\}$$

Def if $\overset{\sum_1^n c_k \chi_{E_k}}{\overset{\parallel}{\psi}} \in SF^+ \subset L^+$, define

$$\int \psi = \sum_1^n c_k \, \mu(E_k) \in [0, \infty].$$

$$\int_E \psi := \int \underbrace{\chi_E \psi}_{\text{first write in standard form.}}$$

Thm the fn $\int : SF^+ \to [0,\infty]$ satisfies

- $\forall r>0$, $\int r\psi = r\int \psi$.

  pf $\sum_1^n r c_k \chi_{E_k} = r \sum_1^n c_k \chi_{E_k}$. $\square$

- if $\varphi \le \psi$ everywhere, $\int \varphi \le \int \psi$.

  Pf write $\varphi = \sum_{j=1}^m a_j \chi_{E_j}$, $\psi = \sum_{k=1}^n b_k \chi_{F_k}$.

Trick: we may assume that $X = \bigsqcup E_j = \bigsqcup F_k$.

So $E_j = \bigsqcup_k (E_j \cap F_k)$, $F_k = \bigsqcup_j (E_j \cap F_k)$.

$$\underbrace{\varphi = \sum_{j,k} a_j \chi_{E_j \cap F_k} \le \sum_{j,k} b_k \chi_{E_j \cap F_k} = \psi}_{\text{write this backward}}$$

Thus $E_j \cap F_k \neq \emptyset \Rightarrow a_j \le b_k$.

So $\int \varphi = \sum_j a_j \mu(E_j) = \sum_{j,k} a_j \mu(E_j \cap F_k) \le \sum_{j,k} b_k \mu(E_j \cap F_k) = \sum_k b_k \mu(F_k) = \int \psi$. $\square$

- $\int \varphi + \psi = \int \varphi + \int \psi$.

Pf $\varphi + \psi = \sum_{j,k} (a_j + b_k) \chi_{E_j \cap F_k}$. Suppose $\varphi + \psi = \sum_\ell c_\ell \chi_{G_\ell}$.

(Similar to before: $E_j \cap F_k \cap G_\ell \neq \emptyset \Rightarrow a_j + b_k = c_\ell$).

"Follow your nose" $\square$

So $\int$ is an order-preserving $\mathbb{R}_{\geq 0}$-linear map.

Remark $\mathcal{M} \ni E \longmapsto \int_E d\mu$ is μ.

Lemma: for $\psi \in SF^+$, define $\mu_\psi : \mathcal{M} \longrightarrow [0, \infty]$ by

$\mu_\psi(E) = \int_E \psi$. μ_ψ is a measure.

Pf Write $\psi = \sum_1^n c_k \chi_{E_k}$ s.t. $\bigsqcup E_k = X$.

If (F_i) is a disjoint sequence in $\mathcal{M}$,

$$\int_{\sqcup F_i} \psi := \int \psi \chi_{\sqcup F_i} = \sum_k c_k \mu(E_k \cap \textstyle\bigsqcup_i F_i)$$

$$= \sum_{k,i} c_k \mu(E_k \cap F_i) = \sum_i \int_{F_i} \psi$$

Def for $f \in L^+$, define

$$\int f = \sup \left\{ \int \psi \;\middle|\; 0 \leq \underset{SF^+}{\overset{}{\underset{\cap}{\psi}}} \leq f \right\}$$

Remarks:

1. This extends $\int \psi$.
2. $f, g \in L^+$ w/ $f \leq g \Rightarrow \int f \leq \int g$.
3. if $f \in L^+$, $r \geq 0 \Rightarrow \int rf = r \int f$.

Monotone Convergence Theorem

Suppose $(f_n) \subset L^+$ is an increasing sequence.

Define $f := \lim f_n = \sup f_n$. Then $\int f = \lim \int f_n$.

Pf: observe $(\int f_n) \subset [0, \infty]$ is increasing, so it converges.

Since $\int f_n \leq \int f \;\; \forall n$, $\lim \int f_n \leq \int f$.

$\geq$: Pick any $0 \leq \overset{SF^+}{\underset{}{\psi}} \leq f$. Let $0 < \varepsilon < 1$.

Set $E_n := \{f_n > \varepsilon \psi\}$. Since $f_n \nearrow f$, (E_n) is increasing and $\bigcup E_n = X$.

Then $\int f_n \geq \int_{E_n} f_n \geq \varepsilon \int_{F_i} \psi \longrightarrow \varepsilon \int \psi$.

$$\overset{\overset{\sim n}{\|}}{\mu_\psi(E_n)} \longrightarrow \mu_\psi(X)$$

So $\forall\ 0 < \varepsilon < 1$, $\lim \int f_n \geq \varepsilon \int \psi$.

Since ε was arbitrary, let $\varepsilon \to 1$ to see that

$$\lim \int f_n \geq \int \psi.$$

Since ψ was arbitrary, $\lim \int f_n \geq \int f$. $\square$

<u>Corollaries of MCT</u>:

1. $\int f = \lim \int \psi_n$ for any $(\psi_n) \subset SF^+$ w/ $\psi_n \nearrow f$.

2. $\forall f, g \in L^+$, $\int f + \int g = \int f+g$.

   <u>Pf</u> Choose $\psi_n \nearrow f$, $\varphi_n \nearrow g$. Then $\psi_n + \varphi_n \nearrow f+g$.

3. $\forall (f_n) \subset L^+$, $\sum \int f_n = \int \sum f_n$.

   <u>Pf</u> by 2 & induction, it's true for finite sums.

   Then $\int \sum f_n = \int \lim_N \sum^N f_n = \lim_N \int \sum^N f_n = \lim_N \sum^N \int f_n = \sum \int f_n$.

4. Suppose $f \in L^+$. Then $\int f = 0 \Leftrightarrow f = 0$ a.e. (Doesn't actually require MCT. just a lemma for 5).

   <u>Pf</u> $\Rightarrow$: prove contrapositive. if not $f = 0$ a.e.,

   $\exists n > 0$ s.t. $\mu(\{f > \frac{1}{n}\}) > 0$. Then $f > \frac{1}{n} \chi_{\{f > \frac{1}{n}\}}$, so

   $\int f \geq \frac{1}{n} \mu(f > \frac{1}{n}) > 0$.

   $\Leftarrow$: if $f \in SF^+$ and $f = \sum c_k \chi_{E_k}$,

   $$\int f = 0 \Leftrightarrow \underset{\forall k}{\mu(E_k) = 0} \Leftrightarrow f = 0 \text{ a.e.}$$

   if $f \in L^+$ w/ $f = 0$ a.e., then $\forall\ 0 \leq \varphi \leq f$, $\varphi = 0$ a.e.

   So $\int \varphi = 0$ so $\int f = 0$.

⑤ If $(f_n) \subset L^+$, $f \in L^+$ s.t. $\underbrace{f_n \nearrow f \text{ a.e.}}_{\Rightarrow f}$,

Then $\int f_n \nearrow \int f$.