Lec 9/11

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Integration

measurable functions

Let $f: X \longrightarrow Y$, get induced functions

$$f \quad \vec{f}: P(X) \longrightarrow P(Y) \qquad f(S) = \{f(s) \mid s \in S\}$$

$$f^{-1} \quad \overleftarrow{f}: P(Y) \longrightarrow P(X) \qquad f^{-1}(T) = \{x \in X \mid f(x) \in T\}$$

Recall $f: (X, \tau) \longrightarrow (Y, \theta)$ is cts iff $f^{-1}(T) \in \tau \;\; \forall T \in \theta$.

topologies

Suppose $f: X \longrightarrow (Y, \theta)$. f & θ induces a topology

on X by $f^{-1}(\theta) = \{f^{-1}(T) : T \in \theta\}$

smallest/weakest topology on X s.t. f is cts.

If $f: (X, \tau) \longrightarrow Y$, f & τ co-induce a top. on Y

by $\underbrace{S \subset Y \text{ open iff } f^{-1}(S) \in \tau.}_{\cdots}$

largest / strongest topology on Y s.t. f is cts

Now $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is $\mathcal{M}$-$\mathcal{N}$ measurable

iff $\forall E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.

If $f : X \longrightarrow (Y, \mathcal{N})$, get an induced σ-algebra

on X by $\underbrace{f^{-1}}_{}(\mathcal{N})$.

Smallest σ-alg. where f is mble.

If $f : (X, \mathcal{M}) \longrightarrow Y$, get a co-induced σ-alg

on Y by $\underbrace{E \subset Y \text{ mble iff } f^{-1}(E) \in \mathcal{M}}_{}$

largest σ-alg where f is mble.

Just as a composite of cts fns is cts,

if f is $\mathcal{M}$-$\mathcal{N}$ measurable and g is $\mathcal{N}$-$\mathcal{P}$ measurable,

then $g \circ f$ is $\mathcal{M}$-$\mathcal{P}$ measurable.

Lemma: Suppose $f:(X,\mathcal{M}) \to (Y,\mathcal{N})$ and $\mathcal{N} = \sigma(\mathcal{E})$ for some $\mathcal{E} \subset P(Y)$

Then f is $\mathcal{M}$-$\mathcal{N}$ measurable $\Leftrightarrow$ $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{E}$.

Pf: $\Rightarrow$ ✓

$\Leftarrow$ Look at co-induced σ-alg of $f, \mathcal{M}$

$\{E \subset Y \mid f^{-1}(E) \subset \mathcal{M}\}$

- strongest σ-alg on Y s.t. f is mble
- contains $\mathcal{E}$, so contains $\mathcal{N} = \sigma(\mathcal{E})$. □

Suppose $(X,\tau) \xrightarrow{f} (Y,\theta)$ we call f borel measurable

top. sp.

if f is $\mathcal{B}_\tau$-$\mathcal{B}_\theta$ measurable

Prop cts fns are Borel mble.

Pf use lemma.

Prop Suppose $f:(X,\mathcal{M}) \to \mathbb{R}$. TFAE:

1. f is $\mathcal{M}$ $(-\mathcal{B}_{\mathbb{R}})$ measurable
2. $f^{-1}(a,\infty) \in \mathcal{M}$ $\forall a \in \mathbb{R}$
3. $f^{-1}[a,\infty) \in \mathcal{M}$ "
4. $f^{-1}(-\infty,a] \in \mathcal{M}$ "

⑤ $f^{-1}(-\infty, a) \in \mathcal{M}$ "

⑥ (a,b) ⑦ $[a,b)$ ⑧ $(a,b]$ ⑨ $[a,b]$.

Pf use the lemma.

The topology on $\overline{\mathbb{R}}$ is induced by a cts bijection w/ $[0,1]$.

Borel σ-alg $B_{\overline{\mathbb{R}}}$ almost gen by ②–⑤ but Not ⑥–⑨

Corollary: Same as proposition, replace $\mathbb{R}$ by $\overline{\mathbb{R}}$,

and replace (a,∞) by $(a,\infty]$, etc.

If $f: X \longrightarrow \mathbb{R}$ (or $\overline{\mathbb{R}}$), write $\{f > a\} = \{x \in X : x > a\}$

$=, \geq, \leq, <, a \leq f < b, \vdots$ $\quad = f^{-1}((a,\infty) \text{ or }])$

Suppose $f, g : (X, \mathcal{M}) \longrightarrow \overline{\mathbb{R}}$ are $\mathcal{M}$-mble

[Remark if $f: (X,\mathcal{M}) \longrightarrow \overline{\mathbb{R}}$ takes values in $\mathbb{R}$, $\mathcal{M}-B_{\overline{\mathbb{R}}}$ mble $\Leftrightarrow \mathcal{M}-B_{\mathbb{R}}$ mble]

The following fns $X \longrightarrow \overline{\mathbb{R}}$ are $\mathcal{M}$-mble:

① $(f \vee g)(x) := \sup\{f(x), g(x)\}$

Pf $\{f \vee g > a\} = \{f > a\} \cup \{g > a\}$.

② $(f \wedge g)(x) := \inf\{f(x), g(x)\}$

Pf $\{f \wedge g > a\} = \{f > a\} \cap \{g > a\}$.

③ any well-defined linear combination of f & g.

Recall: $0 \cdot (\pm\infty) = 0$. $\pm\infty \mp \infty$ Not defined

pf: If $r \in \overline{\mathbb{R}}$, $\{rf > a\} = \begin{cases} X & \text{if } r = 0,\ 0 > a \\ \emptyset & \text{if } r = 0,\ 0 \leq a \\ \{f > 0\} & \text{if } r = +\infty \\ \{f < 0\} & \text{if } r = -\infty \\ \{f > \frac{a}{r}\} & r \in \mathbb{R} \setminus 0,\ r > 0 \\ \{f < \frac{a}{r}\} & r \in \mathbb{R} \setminus 0,\ r < 0 \end{cases}$

$f + g$ next time.