Lec 4/9

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More about Parallelism along a curve.

Thm 6.7 (ch 4): Let M be a C^2 surface in $\mathbb{R}^3$.

Let $\gamma: (a,b) \longrightarrow M$ be a C^1 curve in M, let $t_0 \in (a,b)$.

Let $p = \gamma(t_0)$, and let $\tilde{X} \in T_pM$. Then there is a unique C^1 vector field X on M along γ s.t. X is parallel along γ (rel. to M) and $X(t_0) = \tilde{X}$.

Terminology X is called the parallel translate of $\tilde{X}$ along γ.

Pf On a coord patch x, X satisfies $\frac{dX^k}{dt} + \sum_{i,j} \underbrace{\Gamma_{ij}^k} X^i \frac{d\gamma^j}{dt} = 0 \quad (*)$

evaluated at $(\gamma^1(t), \gamma^2(t))$.

$(*)$ are linear in X^k's, so $\exists$ a unique global solution. (Picard's thm) $\square$

Remark Let M and N be C^2 surfaces in $\mathbb{R}^3$ which are tangent along a C^1 curve $\gamma: (a,b) \longrightarrow M \cap N$ $\left(T_{\gamma(t)}M = T_{\gamma(t)}N \quad \forall\, t \in (a,b)\right)$

Let $t_0 \in (a,b)$, let $p = \gamma(t_0)$, and let $\tilde{X} \in T_pM = T_pN$.

Then the parallel translates of $\tilde{X}$ along γ relative to M and to N are the same.

Reason Let $X: (a,b) \longrightarrow \mathbb{R}^3$ be C^1. Then $\forall\, t \in (a,b)$,

$X(t) \in T_{\gamma(t)}M$ iff $X(t) \in T_{\gamma(t)}N$, and the orthogonal proj of $X'(t)$ onto $T_{\gamma(t)}M$ is also the orthogonal proj of $X'(t)$ onto $T_{\gamma(t)}N$,

so $\nabla_{\dot\gamma}^M X = \nabla_{\dot\gamma}^N X$. $\qquad L_{ij} = \langle x_{ij} \mid n \rangle$

4-7 The Second Fundamental form and the Weingarten Map

Let M and N be C^2 surfaces in $\mathbb{R}^3$. Let $f: M \longrightarrow N$ be C^1.

Let $p \in M$. Let $X \in T_pM$. $Xf = \left.\frac{d f(\alpha(t))}{dt}\right|_{t=0} = \sum_i X^i \frac{\partial (f \circ x)}{\partial u^i}(0,0)$

where $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ is C^1, $\alpha(0) = p$, and $\alpha'(0) = X$.

and $x : U$ open $\subseteq \mathbb{R}^2 \longrightarrow V$ open $\subseteq M$ has $x(0,0) = p$, and $X = \sum X^i x_i(0,0)$.

$n : M \longrightarrow S^2$.

Weingarten Map $\longrightarrow$ $L_p(X) \overset{\text{def}}{=} -X_p n$. $\quad L(x_k) = \sum_\ell L^\ell_k \, x_\ell$ where $L^\ell_k = \sum_i L_{ik} g^{i\ell}$

$$L(x_k) = -\frac{\partial n}{\partial u^k} = -n_k .$$

(Weingarten's Equations)

eg spoze $M = S^2$, and choose n to be the outward pointing normal $n(p) = p$.

$$L(X) = -Xn = -\sum_{i=1}^{3} \tilde{X}^i \frac{\partial n}{\partial x^i} = -(\tilde{X}^1, \tilde{X}^2, \tilde{X}^3) = -X.$$

$(X = (\tilde{X}^1, \tilde{X}^2, \tilde{X}^3))$

Chloe
I don't need a reason

Or (without extending n to $\mathbb{R}^3$) Let $X \in T_p S^2$, let $t \longmapsto \alpha(t) = (x^1(t), x^2(t), x^3(t))$

be a C^1 curve on S^2 such that $\alpha(0) = p$ and $\alpha'(0) = X$.

then $L(X) = -n'(p)(X) = -\left[\dfrac{d}{dt} \underbrace{n(\alpha(t))}_{\alpha(t)}\right]_{t=0} = -\alpha'(0) = -X$

Reminders: $\langle L(x) | Y \rangle = \langle -Xn | Y \rangle = \langle -\sum_i X^i \frac{\partial n}{\partial u^i} \,|\, \sum_j Y^j x_j \rangle$

$= -\sum X^i Y^j \langle \frac{\partial n}{\partial u^i} | x_i \rangle$ but $0 = \frac{\partial}{\partial u^i} \overbrace{\langle n | x_i \rangle}^{0}$

$= \langle n | x_{ji} \rangle + \langle n_i | x_j \rangle$

So $-\langle \frac{\partial n}{\partial u^i} | x_j \rangle = \langle n | x_{ji} \rangle = \langle n | x_{ij} \rangle$.

thus $\langle L(X) | Y \rangle = \sum_{ij} X^i Y^j \underbrace{\langle n | x_{ij} \rangle}_{L_{ij}} = \sum_{ij} L_{ij} X^i Y^j = \mathrm{II}(X, Y)$.

Since $x_{ji} = x_{ij}$, $L_{ij} = L_{ji}$, so $\mathrm{II}(X,Y) = \mathrm{II}(Y,X)$.

Thus $L : T_pM \longrightarrow T_pM$ is self-adjoint.

Another reminder For a C^2 unit-speed curve $\gamma : (a,b) \longrightarrow M$ with $\gamma(s) = x(\gamma^1(s), \gamma^2(s))$, we have $K_n = \sum_{ij} L_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} = \mathrm{II}(\frac{d\gamma}{ds}, \frac{d\gamma}{ds})$

now let the L_k^l's be defined by $L(x_k) = \sum_l L_k^l x_l$.

thus for $X = \sum_k X^k x_k \in T_pM$, we have

$$L(X) = L(\sum_k X^k x_k) = \sum_k X^k L(x_k) = \sum_k X^k \sum_l L_k^l x_l = \sum_l (\sum_k L_k^l X^k) x_l$$

Thus (L_k^l) is the matrix of L wrt the basis x_1, x_2 for T_pM.

$$\text{Now } L_{jk} = \mathrm{II}(x_j, x_k) = \langle x_j | L(x_k) \rangle = \langle x_j | \sum_l L_k^l x_l \rangle$$

$$= \sum_l L_k^l \langle x_j | x_l \rangle = \sum_l g_{jl} L_k^l .$$

$$\text{Thus } \sum_j g^{ij} L_{jk} = \sum_j g^{ij} \sum_l g_{jl} L_k^l = \sum_l (\sum_j g^{ij} g_{jl}) L_k^l$$

$$= \sum_l \delta_l^i L_k^l = L_k^i$$

$$\text{Thus } L_k^l = \sum_j g^{lj} L_{jk}$$