Lec 10/4

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$$(1+2^2)^{2^{n-2}} \equiv 1 \mod 2^n$$

$$(1+2^2)^{2^{n-3}} \not\equiv 1 \mod 2^n$$

enough to show $\quad 2^n \,\Big|\, \binom{2^{n-2}}{\ell} 2^{2\ell}$.

$$2^{n-2}\left(\frac{(2^{n-2}-1)(2^{n-2}-2)\cdots(2^{n-2}-\ell+1)}{\ell!}\right)2^{\ell}$$

in denominator we get $\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{\ell}{4} \rfloor + \cdots \le \ell - 1$

in numerator we get $\lfloor \frac{\ell-1}{2} \rfloor + \lfloor \frac{\ell-1}{4} \rfloor + \cdots + 2n-2+2\ell$

To prove: $n \le n + 2\ell - 2 + \sum_{j=1}^{\infty}\left(\lfloor \frac{\ell-1}{2^j} \rfloor - \lfloor \frac{\ell}{2^j} \rfloor\right)$

show $\lfloor \frac{\ell-1}{2^j} \rfloor - \lfloor \frac{\ell}{2^j} \rfloor = \begin{cases} -1 & \text{if } \ell \equiv 0 \mod 2^j \\ 0 & \text{otherwise.} \end{cases}$

use $(a^{2^n}-1) = (a^{2^{n-1}}-1)(a^{2^{n-1}}+1) = (a-1)(a+1)(a^2+1)(a^4+1)\cdots$

Solvable	$\longleftarrow$	nilpotent
commutator series	DEFN	central series
graded pieces abelian	Σ	$[G; H_i] \subset H_{i+1}$
sub & quotient solvable	Serre	sub & quotient nilpotent

$\Big\}$ Serre $\Big\{$ sub & quotient nilpotent

$A \leq Z(G)$, G/A nil $\Rightarrow$ G nil.

$\longleftarrow$ abelian, p-groups

same here.

Direct product of solvable is solvable

if $G \neq \{e\}$ is nilpotent, then

$C^n(G) \supsetneq C^{n+1}(G) = \{e\}$,
$\quad \not\!\!\!=\ \{e\}$

$[G, C^n(G)] = \{e\}$

$\Rightarrow C^n(G) \subset Z(G)$

$\Rightarrow Z(G) \neq \{e\}$

Thm: Nilpotent $\Longleftrightarrow$ direct product of p-groups

Pf ($\Leftarrow$) Yesterday ✓

($\Rightarrow$) WTS: if gp is nilpotent then every $P \in Syl_p(G)$ ($p \mid |G|$) is normal. i.e. $|Syl_p(G)| = 1$.

if true, then let $p_1, \dots, p_\ell$ be prime div. $|G|$, let $\{P_i\} = Syl_{p_i}(G)$.

$$P_1 \cdots P_\ell = G,\ P_i \cap P_j = \{e\},\ P_i \trianglelefteq G \Rightarrow G = P_1 \times P_2 \times \cdots \times P_\ell. \quad \square$$

"Self-normalizing lemma": H group, $P \leq H$ sylow p-subgp.

$$N_H(P) \leq L \Rightarrow N_H(L) = L.$$

Main Lemma: If G is nilpotent & finite, $H \lneq G$, then $H \lneq N_G(H)$.

Using this, we prove the Thm:

Let G be finite & nilpotent, $P \in Syl_p(G)$. If P is not normal then $\underbrace{N_G(P)}_{=H} \lneq G$. Main lemma says $N_G(H) \supsetneq H$, self-normalizing lemma says $N_G(H) = H$.

Contradiction $\Rightarrow$ P is normal. $\square$

Pf of main lemma: G is nilpotent $\Rightarrow$ $G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$ s.t. $[G, K_\ell] \subset K_{\ell+1}$. ($K_\ell \supset K_{\ell+1}$)

in particular, $K_\ell \trianglelefteq G \ \forall \ell$. Take $H_\ell = K_\ell \cdot H$.

$G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{m-1} \supseteq H_m = H$

<u>Claim</u>: $H_\ell \trianglerighteq H_{\ell+1} \ \forall \ell$.

Using this, Let j be $\overset{\text{largest}}{}$ s.t. $H_j \underset{\neq}{\trianglerighteq} H_{j+1} = \cdots = H_m = H$

has to happen since $H \neq G$. So $N_G(H) \supset H_j \supsetneq H_{j+1} = H$ □

<u>Lemma</u>: $N_2 \subset N_1 \subset G$ two normal subgps, $[G;N_1] \subset N_2$, $H \leq G \Rightarrow N_2 \cdot H \trianglelefteq N_1 \cdot H$.
$\qquad b \in N_2 \cdot H,\ a \in N_1 \cdot H$

<u>Pf</u>: T.S. $aba^{-1} \in N_2 \cdot H$.

if $a \in N_1 \subset N_1 H$, b arbitrary (in $N_2 \cdot H$), $b^{-1}aba^{-1} \in [G, N_1] \subset N_2$

$\Rightarrow aba^{-1} \in bN_2 \subset N_2 \cdot H$

if $a \in H \subset N_1 \cdot H$, $aba^{-1} = \underbrace{ana^{-1}}_{\in N_2} \underbrace{axa^{-1}}_{\in H}$ □

$$U_3 = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_p \right\}$$ has p^3 elements & is non-abelian.

<u>Ex</u>: compute its central series. at least figure out its center.