Lec 9/1

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$$(*)\begin{cases} \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n = \beta_1 \\ \vdots \qquad \ddots \qquad \vdots \\ \alpha_{m1}x_1 + \cdots + \alpha_{mn}x_n = \beta_m \end{cases}$$

$\alpha_{ij} \in$ Field

Non-homogeneous system when $\beta_i \neq 0$ for some i.

↳ called so b.c. $\underset{\mathbb{F}^n}{\underset{\uparrow}{X = (x_1, \dots, x_n)}}$ a soln $\not\Rightarrow$ λX a soln.

$$A = (\alpha_{ij}) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix}$$

$$\tilde{A} = \left[\begin{array}{c|c} A & \begin{matrix} \beta_1 \\ \vdots \\ \beta_n \end{matrix} \end{array}\right]$$ augmented matrix for system.

$$= \begin{bmatrix} c_1 & \cdots & c_n & b \end{bmatrix} \qquad c_1, \dots, c_n, b \in \mathbb{F}^m$$

$(*) \Leftrightarrow (**)$ $\quad x_1 c_1 + \cdots + x_n c_n = b_n$

<u>Thm 1</u> The following properties are equivalent

(i) $(*)$ has a solution

(ii) $b \in S(c_1, \dots, c_n) \subseteq \mathbb{F}^m$

(iii) $\underbrace{\dim S(c_1, \dots, c_n)} = \dim S(c_1, \dots, c_n, b)$

(iii) $\underbrace{\dim S(c_1, \dots, c_n)}$ $= \dim S(c_1, \dots, c_n, b)$

(column) rank of $A = [c_1, \dots, c_n]$

$$
\begin{aligned}
x_1 + x_2 + x_3 \quad &= 8 \\
x_1 + x_2 + \quad x_4 &= 1 \\
x_2 + x_3 + x_4 &= 14 \\
x_1 + \quad x_3 + x_4 &= 14
\end{aligned}
$$

$$
\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 14 \\ 1 & 0 & 1 & 1 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 0 & 0 & -1 & 1 & -7 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & -1 & 0 & 1 & 6 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 0 & 0 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & 20 \\ 0 & -1 & 0 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 0 & -1 & 0 & 1 & 6 \\ 0 & 0 & -1 & 1 & -7 \\ 0 & 0 & 0 & 3 & 13 \end{bmatrix}
$$

$$
\Rightarrow x_4 = \frac{13}{3} \quad \Rightarrow \quad x_3 = \frac{34}{3} \quad \Rightarrow \quad x_2 = \frac{5}{3} \quad \Rightarrow \quad x_1 = -5
$$

$$
\begin{cases} 3x_1 + 4x_2 = -1 \\ -x_1 - x_2 = 1 \\ x_1 - 2x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases}
$$

$$
\begin{bmatrix} -1 & -1 & 1 \\ 3 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}
$$

$$\begin{vmatrix} 3 & 4 & -1 \\ 1 & -2 & 0 \\ 2 & 3 & 0 \end{vmatrix} \sim \begin{vmatrix} 0 & 1 & 2 \\ 0 & -3 & 7 \\ 0 & 1 & 2 \end{vmatrix} \sim \begin{vmatrix} 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{vmatrix}$$

$\Rightarrow \quad 0x_1 + 0x_2 = 7 \Rightarrow$ no ~~so~~ solution.

$(*_{hom}) = x_1 c_1 + \cdots x_n c_n = 0$

<u>Thm 2</u> Let $X_p = (\lambda_1, \dots, \lambda_n)$ be a (particular) solution of $(*)$

(i) if $X_h = (\mu_1, \dots, \mu_n)$ is a solution of $(*_{hom})$, Then $X_p + X_h$ is a solution of $(*)$.

<u>Pf</u> $\lambda_1 c_1 + \cdots + \lambda_n c_n = b$, $\mu_1 c_1 + \cdots + \mu_n c_n = 0 \Rightarrow (\lambda_1 + \mu_1) c_1 + \cdots + (\lambda_n + \mu_n) c_n = b$

(ii) Conversely any soln x of $(*)$ is $X_p + X_h$ for some X_h.

<u>Pf</u> $X = (x_1, \dots, x_n)$ a soln, i.e.

$$\begin{array}{r} x_1 c_1 + \cdots + x_n c_n = b \\ - \quad \lambda_1 c_1 + \cdots + \lambda_n c_n = b \\ \hline (x_1 - \lambda_1) c_1 + \cdots + (x_n - \lambda_n) c_n = 0 \end{array}$$

so $X_h = (x_1 - \lambda_1, \dots, x_n - \lambda_n)$ is a soln of $(*_{hom})$

and $X = X_p + X_h$.

how to find all solns of $(*_{hom})$?