Lec 3/5

Monday, March 5, 2018 14:13

<u>Thm</u> Let X be a metric space (or a normal space)

Let $A \subseteq X$ be closed, and let $f: A \to \mathbb{C}^\times$ be cts.

(a) if f is null-homotopic then f can be extended to a cts map from X into $\mathbb{C}^\times$.

<u>Pf</u> Since f is null-homotopic in $\mathbb{C}^\times$, there's a continuous logarithm for f, say g. $g: A \to \mathbb{C}$ s.t. $e^g = f$. By the tietze extension thm (applied to $\mathrm{Re}(g)$, $\mathrm{Im}(g)$), g can be extended to $G: X \xrightarrow{cts} \mathbb{C}$ s.t. $g(x) = G(x)$ $\forall x \in A$. then let $F = e^G$. $F: X \xrightarrow{cts} \mathbb{C}^\times$ and $\forall x \in A$, $F(x) = e^{G(x)} = e^{g(x)} = f(x)$.

(b) f can be extended to a cts map from X into $\mathbb{C}^\times$ iff f is homotopic in $\mathbb{C}^\times$ to a continuous map which can be so extended.

<u>Pf</u> ($\Rightarrow$) $f \simeq f$ in $\mathbb{C}^\times$.

($\Leftarrow$) Suppose $f \simeq f_1: A \to \mathbb{C}^\times$ in $\mathbb{C}^\times$ s.t. $\exists$ a cts map $g_1: X \to \mathbb{C}^\times$ with $g_1(x) = f_1(x)$ $\forall x \in A$. Since $f \simeq f_1$, $\frac{f}{f_1} \simeq 1$ in $\mathbb{C}^\times$ so $\frac{f}{f_1}$ can be extended to a cts map $h: X \to \mathbb{C}^\times$.

let $g = hg_1$. then g is cts and $\forall x \in A$, $g(x) = h(x)g_1(x) = \frac{f(x)}{f_1(x)} f_1(x) = f(x)$.

<u>Corollary</u> (Borsuk, 1932).

Let $A \subseteq \mathbb{C}$ be closed. Let $f: A \to \mathbb{C}^\times$. Then TFAE:

(a) f is null-homotopic in $\mathbb{C}^\times$

(b) f has a continuous logarithm

(c) f can be extended to a cts map from $\mathbb{C}$ into $\mathbb{C}^\times$.

Pf (a) $\Rightarrow$ (b) see above thm

(b) $\Rightarrow$ (c) see pf of above thm

(c) $\Rightarrow$ (a) define $j: A \to \mathbb{C}$ by $j(z) = z$. $j \simeq 0$ in $\mathbb{C}$.

let f_1 be an extension of f to a cts map from $\mathbb{C}$ into $\mathbb{C}^\times$.

$f = f_1 \circ j$. j is null-homotopic in $\mathbb{C}$ so $f \simeq f_1(0)$ in $\mathbb{C}^\times$.

Defn Let X be a top. sp. Then $\pi(X, \mathbb{C}^\times)$ is the group of homotopy classes of cts maps from X into $\mathbb{C}^\times$.

$[f][g] = [fg]$.

$[f]^{-1} = [\frac{1}{f}]$

$[1]$ is the unit element.

$$\left(\begin{array}{c} \forall f \in C(X, \mathbb{C}^\times) \\ [f] = \{g \in C(X, \mathbb{C}^\times) : f \simeq g \text{ in } \mathbb{C}^\times\} \end{array} \right)$$

$\pi(X, \mathbb{C}^\times)$ is called the first cohomotopy group for X.

eg $\pi(S^1, \mathbb{C}^\times) \cong \mathbb{Z}$

pf $\forall n \in \mathbb{Z}$, define $\gamma_n : S^1 \longrightarrow \mathbb{C}^\times$ by $\gamma_n(z) = z^n$. Then

$$[\gamma_n][\gamma_m] = [\gamma_{n+m}]$$

$$[\gamma_n]^{-1} = [\gamma_{-n}]$$

$$[1] = [\gamma_0]$$

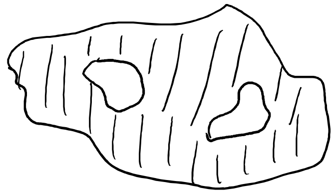
So $n \longmapsto [\gamma_n]$ is a homomorphism from $(\mathbb{Z}, +)$ into $\pi(S^1, \mathbb{C}^\times)$.

If $[\gamma_m] = [\gamma_n]$ then $\gamma_m \simeq \gamma_n$ so $\operatorname{ind}(\gamma_m) = \operatorname{ind}(\gamma_n) \Rightarrow n = m$.

So $n \longmapsto [\gamma_n]$ is 1-1. let $[\gamma] \in \pi(S^1, \mathbb{C}^\times)$. then $\gamma \simeq \gamma_{\operatorname{ind}\gamma}$ so $[\gamma] = [\gamma_{\operatorname{ind}\gamma}]$ so $n \longmapsto [\gamma_n]$ is onto $\pi(S^1, \mathbb{C}^\times)$.

On –

eg

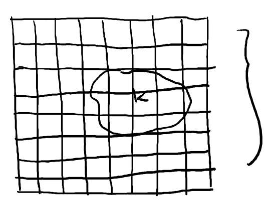
$K_2 =$  . $\pi(K_2, \mathbb{C}^\times) \cong \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$

$K_3 =$  $\pi(K_3, \mathbb{C}^\times) = \mathbb{Z}^3$

⋮

Thm Let K be a cpt $\subseteq \mathbb{C}$. Let $f : K \to \mathbb{C}^\times$ be cts. Then there is a finite set $E \subseteq \mathbb{C} \setminus K$ s.t. F can be extended to a cts map from $\mathbb{C} \setminus E$ to $\mathbb{C}^\times$.

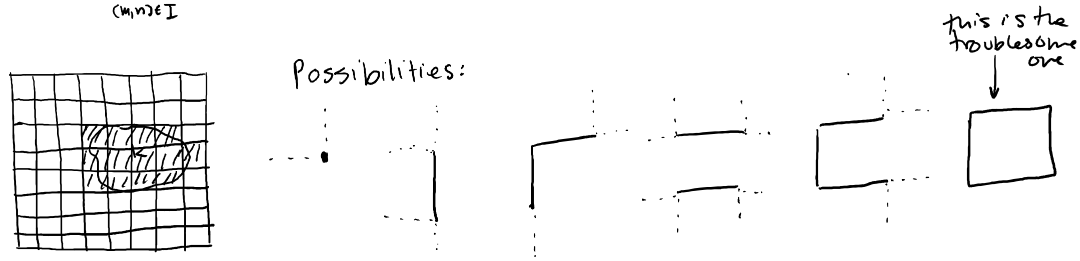
Pf by Tietze, there is a continuous extension of f to a map $g : \mathbb{C} \to \mathbb{C}$. $g(z) \neq 0 \ \forall z \in K$. Let $U = \{z \in \mathbb{C} : g(z) \neq 0\}$. U is open & contains K. If $U = \mathbb{C}$ we are done. If not, $\mathbb{C} \setminus U$ is non-empty & closed so $z \longmapsto \text{dist}(z, \mathbb{C} \setminus U)$ is cts on $\mathbb{C}$ and strictly positive on U. Let $\delta = \inf\{\rho(z, \mathbb{C} \setminus U) : z \in K\} > 0$. K is bounded so $\exists\ a \in (0, \infty)$ s.t. $K \subseteq [-a, a]^2$.



chop w/ diameter of little squares $< \delta$. Call squares $S_{m,n}$ with $1 \leq m, n \leq N$.

Note that if $S_{m,n} \cap K \neq \emptyset$, $S_{m,n} \subseteq U$. Let $J = \{1, \dots, N\}^2$. Let $I = \{(m,n) \in J : S_{m,n} \cap K \neq \emptyset\}$. Now let's describe the extension h of f to a map from $\mathbb{C} \setminus E$ into $\mathbb{C}^\times$, where $E \subseteq \mathbb{C} \setminus K$ is finite (we'll also describe E).

Let $\tilde{K} = \bigcup_{(m,n)\in I} S_{mn}$. Then $\tilde{K} \subseteq U$. $\forall z \in \tilde{K}$, let $h(z) = g(z)$.



Extend h square-by-square to obtain a rectangle (possibly w/ square holes) on which h is cts and nonzero. This is possible on squares w/ 3 or fewer edges already included, so we don't need to worry about this. If instead h is already defined on all edges of a square, let h be defined to be constant on rays emanating from center of square, undefined at center. Put the center of square in E.

There can only be finitely many such squares, so E is finite. Then extend outer rectangle to $\mathbb{C} \setminus E$ by making h constant on every ray emanating from rectangle. $\square$

or we could define h to be 1 outside of $[-a, a]^2$ and skip this step.