Lec 10/19

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$I \leq R$ s.t. $RI = I$ (left-ideal)

$IR = I$ (right ideal)

$R = K[X]$ ← K a field.

"Euclidean algorithm works" meaning if $g(X) \in R \setminus \{0\}$, $f(X) \in R$

then $f(x) = q(x)g(x) + r(x)$ where $\deg(r(x)) < \deg(g(x))$

$g(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0$. $d = \deg(g)$

assume $a_d = 1$

<u>Pf</u>. Induction on $\deg(f) =: n$.

if $n < d$ then $r = f$, $q = 0$ works.

o.w. if $n \geq d$ then $f(X) = b_n X^n + \cdots + b_0$ and

$f(x) - b_n X^{n-d} g(X)$ has smaller degree than f $\square$

So every ideal of $K[X]$ is $\overset{(g(x))}{\overset{\shortparallel}{g(x)K[X]}}$ for some $g(X) \in K[X]$,

i.e. every ideal is principal.

<u>Pf</u> if $\overset{K[X]}{\overset{*}{I}} \neq \{0\}$ then choose $g(x) \in I \setminus \{0\}$ of smallest degree. ← say d

We may assume $g(x)$ is monic since $\frac{1}{a_d} g(x)$ is one of these.

Then $(g(x)) \subset I$ and conversely: let $f(x) \in I$

$\Rightarrow f(X) = q(X)g(X) + r(X)$ where $0 \leq \deg(r) < \deg(g)$

and $r(X) \in I$ so $r(X) = 0$ or $r(X) = \alpha$ for some $\alpha \in K^\times$.

but $I \neq K[X]$ excludes the second case so $g(x) \mid f(x)$

i.e. $f(x) \in (g(x))$.

So $I = (g(x))$ $\square$

2: Eg $R = \mathbb{Z}[\overset{i}{\sqrt{-1}}] \subset \mathbb{C}$. $R \xrightarrow{\text{Norm}} \mathbb{Z}_{\geq 0}$

$z = a+bi \longmapsto a^2+b^2 = |z|^2$

Note
$\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$
$\text{Norm}(1) = 1$
$\Rightarrow$ if something is invertible, its norm must be 1.
i.e. it's ± 1 or $\pm i$.

"Euclidean Algorithm Works": Meaning $\forall z \in R \setminus \{0\}$, $w \in R$, $w = qz + r$ w/ $\text{Norm}(r) < \text{Norm}(z)$

$\frac{w}{z} \in \mathbb{C}$. $\frac{w}{z} = s + it$. $\exists$ integers a, b s.t. $s-a, t-b \in [-\frac{1}{2}, \frac{1}{2}]$.

$\Rightarrow \text{Norm}(s+it - (a+ib)) \leq \frac{1}{2}$

$w = (a+bi)z + r$ where $r = (s+it - (a+ib))z$ so $\text{Norm}(r) \leq \frac{1}{2}\text{Norm}(z) < \text{Norm}(z)$ $\square$

Cor: every ideal is principal.

Read about $\mathbb{Z}[\sqrt{D}]$ pg 229, §7.1

(3) $R = \mathbb{Z}[x] \supset I = \{f(x) : f(0) \in 2\mathbb{Z}\}$ is an ideal.

$I = (2, x) \ni \overset{\psi}{2 \cdot k + x \cdot (\dots)}$

Ex: I is not a principal ideal.

$$\begin{array}{ccccc} R & \xrightarrow{\varphi} & \mathbb{Z} & = & \mathbb{Z}[x]/(x) \\ f(x) & \longmapsto & f(0) & & \\ \cap & & \cap & & \\ I & \longrightarrow & 2\mathbb{Z} & & \\ \| & & & & \\ \{f(x) : \varphi(f(x)) \in 2\mathbb{Z}\} & & & & \end{array}$$

Another "proof" that $I \subset \mathbb{Z}[x]$ is an ideal.

Lemma: if $\varphi : R_1 \longrightarrow R_2$ is a ring hom & $I_2 \subset R_2$ is an ideal, then $I_1 := \varphi^{-1}(I_2)$ is an ideal in R_1.

proof $(I_1, +) \leq (R_1, +)$ since $\varphi(0_{R_1}) = 0_{R_2} \in I_2$, $\varphi(a \pm b) = \varphi(a) \pm \varphi(b) \Rightarrow a \pm b \in I_1\ \forall a, b \in I_1$.

$r_1 \in R$, $x_1 \in I_1 \Rightarrow \varphi(r_1 \cdot x_1) = \varphi(r_1) \cdot \varphi(x_1) \in I_2 \Rightarrow r_1 \cdot x_1 \in I_1$.

$f(I)$ is not necessarily an ideal: $\mathbb{Z} \xrightarrow{id} \mathbb{Q}$

Lemma cont. but if f is surjective then it does take ideals to ideals.

Pf $\tilde{f} : R/\mathrm{Ker}(f) \longrightarrow \mathrm{Im}(f)$ is a bij, $\tilde{f}^{-1}$:

$I_2 = f(I_1) \leq R_2$ obviously.

$\forall r_2 \in R_2$, $x_2 \in I_2$, $\exists r_1 \in R_1, x_1 \in I_1$ s.t. $f(r_1) = r_2$, $f(x_1) = x_2$

So $r_2 x_2 = f(r_1 x_1) \in I_2$ since $r_1 x_1 \in I_1$.

In concl. if $f : R_1 \longrightarrow R_2$ is a surj ring hom,

then set of ideals in R_1 containing $\mathrm{Ker}(f)$ (L/R/2) $\longleftrightarrow$ set of ideals in R_2 (L/R/2)

$$I_1 \longmapsto f(I_1)$$

$$f^{-1}(I_2) \longleftarrow I_2$$

the bijection preserves all operations w/ ideals.

for instance if $I_2 \subset R_2$ is a 2-sided ideal

$I_1 = f^{-1}(I_2) \subset R$ is a 2-sided ideal

Then $R_1/I_1 \cong R_2/I_2 \cong (R_1/J)/(I_1/J)$

(analogue of $(G/K)/(N/K) \cong G/N$)

<u>Pf</u>

$$\begin{tikzcd} R_1 \arrow[r, two heads, "f"] \arrow[rr, bend right, "\pi \circ f"'] & R_2 \arrow[r, two heads, "\pi"] & R_2/I_2 \end{tikzcd}$$

$I_1 = \mathrm{Ker}(\pi \circ f)$.