Lec 9/8

Thursday, September 8, 2016 9:04 AM

Questions for future consideration:

Let $f : [0,\infty) \to \mathbb{R}$ and $f(x) = x^{3/2} = (\sqrt{x})^3$

Q1: is f continuous at 0?

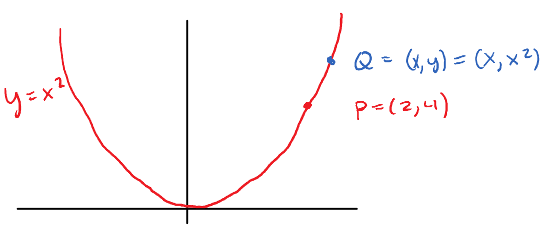
Q1': is it true that $\lim_{x\to 0} f(x) = 0$?

Q2: is f differentiable at 0?

Q2': is it true that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$?

Problem: what is the slope of the tangent line to the graph of the parabola $y = x^2$ at $(2,4)$?

Solution: approximate the tangent line at P by $\overline{PQ}$



$m \approx$ slope of $\overline{PQ}$

$m = \lim_{Q\to P}$ slope $\overline{PQ}$

$= \lim_{x\to 2} \frac{x^2-4}{x-2}$

Crucial step

$= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2}$

$= \lim_{x\to 2} x+2 = 2+2 = 4$

Implicit fallacy: $\frac{(x-2)(x+2)}{(x-2)} = (x+2)$ (the same function)

but dom $f(x) = \frac{(x-2)(x+2)}{(x-2)} \neq$ dom $g(x) = (x+2)$

dom $f = \mathbb{R}\setminus\{2\}$ dom $g = \mathbb{R}$

Rationale for crucial step:

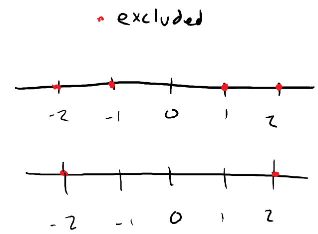
Some theorem. this one works:

Theorem: if $f(x) = g(x)$ $\forall x \in \mathbb{R}\setminus\{a\}$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ (provided the limits exist).

$\lim \frac{x^2-1}{\ldots} = \lim \frac{\cancel{x^2-1}}{\ldots} = \lim \frac{1}{\ldots} = \frac{1}{\ldots} = \frac{1}{\ldots}$

$$\lim_{x\to 1} \frac{x^2-1}{x^4-5x^2+4} = \lim_{x\to 1} \frac{\cancel{x^2-1}}{\cancel{(x^2-1)}(x^2-4)} = \lim_{x\to 1} \frac{1}{x^2-4} = \frac{1}{1-4} = \frac{1}{-3}$$

$$f(x) = \frac{x^2-1}{(x^2-1)(x^2-4)} \qquad g(x) = \frac{1}{x^2-4}$$



$\operatorname{dom} f = (-\infty,-2) \cup (-2,-1) \cup (-1,1) \cup (1,2) \cup (2,\infty)$

$\operatorname{dom} g = (-\infty,-2) \cup (-2,2) \cup (2,\infty)$

Localization principle:

Theorem: If $a < c < b$ and $g(x) = f(x)$ $\forall x \in (a,c) \cup (c,b)$ and $\lim_{x\to c} g(x) = L$ then $\lim_{x\to c} f(x) = L$.

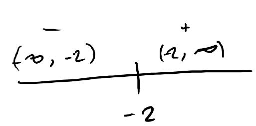
Ex: Let $f(x) = \dfrac{x^2 - 3|x+2| + 8}{3x^2 - |x^2-x-20| + 6}$

Find $\lim_{x\to 2} f(x)$, if it exists. Justify using Localization principle.

Note that $f(2) = \dfrac{4-3\cdot 4+8}{3\cdot 4-18+6} = \dfrac{0}{0}$

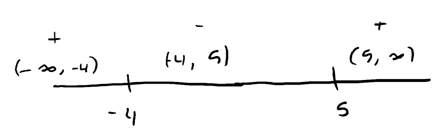
Intermediate value theorem: if $h(x) = 0$ at $x_1 < x_2 < \cdots < x_n$ then either $|h(x)| = h(x)$ for all x in (x_{i-1}, x_i) or $|h(x)| = -h(x)$ " "

$x+2 = 0$ at $x = -2$



$$|x+2| = \begin{cases} -(x+2) & \text{for } x \in (-\infty,-2) \\ x+2 & \text{for } x \in (-2,\infty) \end{cases}$$

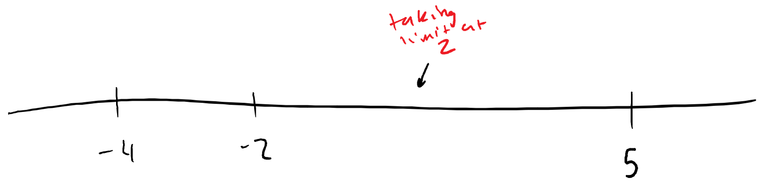
$x^2 - x - 20 = 0$ when $x = 5$ or $x = -4$



$$|x^2-x-20| = \begin{cases} x^2-x-20 & \text{for } x \in (-\infty,-4) \end{cases}$$

$x^2 - x - 20 = 0$ when $x = 5$ or $x = -4$

$$|x^2 - x - 20| = \begin{cases} x^2 - x - 20 & \text{for } x \in (-\infty, -4) \\ -(x^2 - x - 20) & \text{for } x \in (-4, 5) \\ x^2 - x - 20 & \text{for } x \in (5, \infty) \end{cases}$$



tentative simplification intervals: $(-2, 2) \cup (2, 5)$

if $x \in (-2, 2) \cup (2, 5)$

then the numerator is $x^2 - 3(x + 2) + 8 = x^2 - 3x + 2 = (x-1)(x-2)$

the denominator is $3x^2 + (x^2 - x - 20) + 6 = 4x^2 - x - 14 = (4x+7)(x-2)$

so $f(x) = \dfrac{(x-1)(x-2)}{(4x+7)(x-2)} \quad \forall x \in (-2, 2) \cup (2, 5)$

$g(x) = \dfrac{x-1}{4x+7}$ but $-\frac{7}{4} \in (-2, 2) \cup (2, 5)$ so $\neg\left(f(x) = g(x) \ \forall x \in (-2, 2) \cup (2, 5)\right)$

so $f(x) = g(x)$ for x in $(-\frac{7}{4}, 2) \cup (2, 5)$

so by localization principle, $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \frac{1}{15}$