

Prop: $\forall f \in L^1, \int |f| \leq \|f\|_1.$

Cor: $\forall f, g \in L^1, \text{ TFAE:}$

① $f = g \text{ a.e.}$

② $\int |f - g| = 0$

③ $\forall E \in \mathcal{M}, \int_E f = \int_E g$

Pf ① \Leftrightarrow ② $f = g \text{ a.e.} \Leftrightarrow |f - g| = 0 \text{ a.e.} \Leftrightarrow \int |f - g| = 0.$

② \Rightarrow ③ $\forall E \in \mathcal{M}, \left| \int_E f - \int_E g \right| = \left| \int_E (f - g) \right| = \left| \int (f - g) \chi_E \right| \leq \int |f - g| \chi_E \leq \int |f - g| = 0$

③ \Rightarrow ① observe $f = g \text{ a.e.}$ iff $\text{Re}(f - g) = 0 \text{ a.e.}$ and $\text{Im}(f - g) = 0 \text{ a.e.}$ ← prove this!

Recall $\int_E f - \int_E g = \int (f - g) \chi_E = 0 \quad \forall E \in \mathcal{M}$
 $= \int \text{Re}(f - g) \chi_E + i \int \text{Im}(f - g) \chi_E.$

Look at the following $E \in \mathcal{M}$:

$\{\text{Re}(f - g) \geq 0\} \Rightarrow \text{Re}(f - g)_+ = 0 \text{ a.e.}$

$\{\text{Re}(f - g) \leq 0\} \Rightarrow \text{Re}(f - g)_- = 0 \text{ a.e.}$

\vdots

similarly for $\text{Im}.$

□

$\|\cdot\|_1 : L^1 \rightarrow [0, \infty)$ by $\|f\|_1 = \int |f|$

- $\|\alpha f\|_1 = \int |\alpha f| = \int |\alpha| |f| = |\alpha| \|f\|_1$

- $\|f+g\|_1 = \int |f+g| \leq \int |f| + |g| = \int |f| + \int |g| = \|f\|_1 + \|g\|_1$

this is a seminorm

Define $L' = L'/\sim$ where $f \sim g$ if $f=g$ a.e.

Observe $f \sim g \iff \int |f-g| = 0$

$\hookrightarrow \|\cdot\|_1$ descends to a norm on L' .

$\rho_1(f, g) = \|f-g\|_1$ is a metric on L_1 .

HW: L' is complete wrt ρ_1 .

• write $f \in L'$ to mean $f \in L'$ representing its eq. class in L' .

Say $f_n \rightarrow f$ in L' to mean $\int |f_n - f| \rightarrow 0$.

Remark $L'(\mu) \cong L'(\bar{\mu})$

Dominated Convergence Theorem Suppose $(f_n) \subset L'$ s.t. $f_n \rightarrow f$ a.e.

and $\exists g \in L^1$ s.t. $g \geq 0$ and $|f_n| \leq g$ a.e. $\forall n$.

Then $f \in L^1$ and $\int f = \lim \int f_n$.

(special case: Bdd CT: $\mu(X) < \infty$ and $|f_n| \leq M$ $\forall n$)

pf wlog, f is \mathcal{M} -measurable. Taking limits,

$|f| \leq g$ a.e. so $f \in L^1$. Taking Re, Im parts

separately, we may assume f_n, f are \mathbb{R} -valued.

Then $-g \leq f_n \leq g$ a.e. so

$$g + f_n \geq 0 \quad \text{and} \quad g - f_n \geq 0 \quad \text{a.e.}$$

By Fatou's Lemma.

$$\int g + \int f = \int g + f \leq \liminf \int g + f_n = \int g + \liminf \int f_n$$

$$\int g - \int f = \int g - f \leq \liminf \int g - f_n = \int g - \limsup \int f_n$$

$$\text{so } \limsup \int f_n \leq \int f \leq \liminf \int f_n.$$

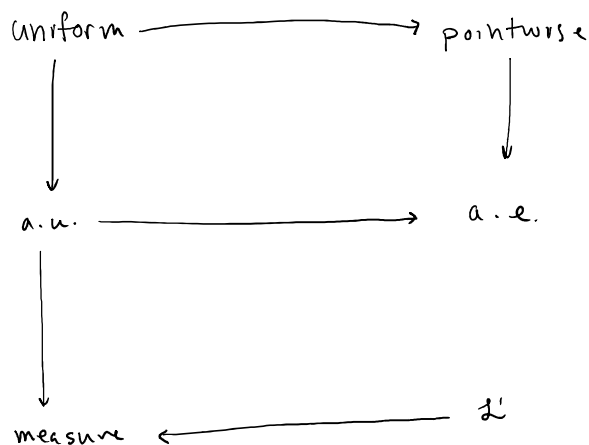
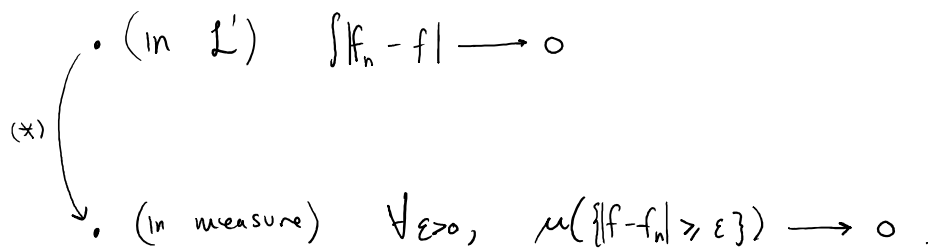
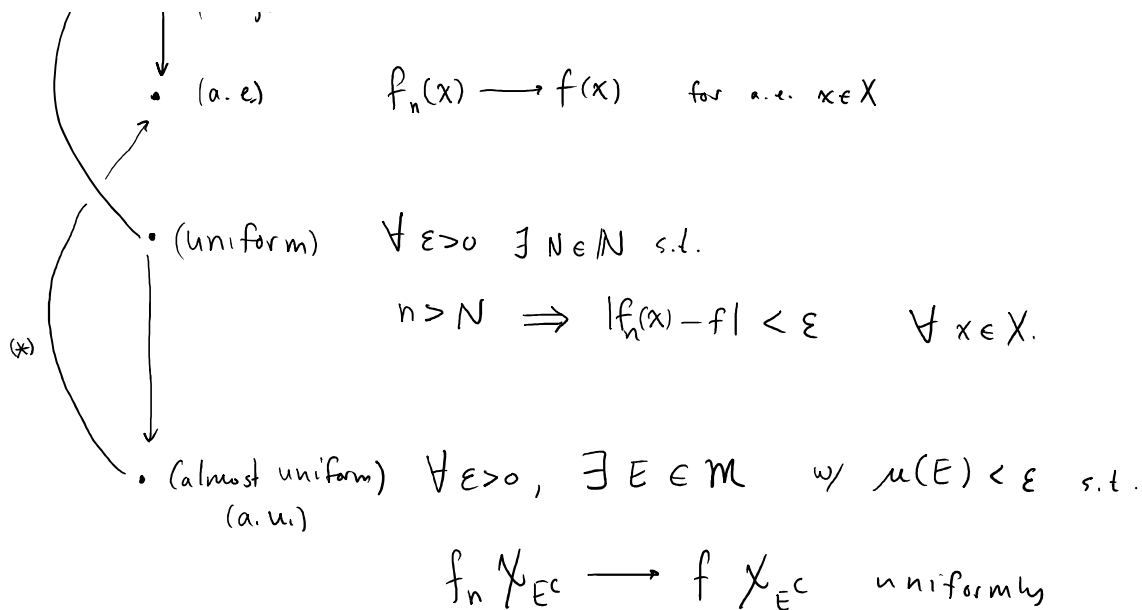
□

Modes of Convergence: (X, \mathcal{M}, μ) fixed measure space.

$(f_n), f$ are \mathcal{M} -B_C mble fns.

$f_n \rightarrow f$ can mean:

- (pointwise everywhere) $f_n(x) \rightarrow f(x) \quad \forall x \in X.$
- (a.e.) $f_n(x) \rightarrow f(x) \quad \text{for a.e. } x \in X$



Pf: a.u. \Rightarrow a.e. $\forall n \in \mathbb{N}, \exists E_n \in \mathcal{M}$ s.t. $\mu(E_n) < \frac{1}{n}$ and

$f_n \nearrow_{E_n^c} f \nearrow_{E_n^c}$ unif.

$$f_k \chi_{E_n^c} \longrightarrow f \chi_{E_n^c} \quad \text{unif.}$$

$$\text{so } f_k \chi_{E_n^c} \longrightarrow f \chi_{E_n^c} \quad \text{pointwise.}$$

$$\text{so } f_k \chi_{E^c} \longrightarrow f \chi_{E^c} \quad \text{where } E^c = \bigcup E_n^c.$$

$\hookrightarrow \mu(E) = 0$ by cont. from above. \square

$L^1 \Rightarrow \text{measure}$ Suppose $f_n \rightarrow f$ in L^1 . Let $\varepsilon > 0$.

$$\begin{aligned} \mu(\{|f-f_n| \geq \varepsilon\}) &= \int_{\{|f-f_n| \geq \varepsilon\}} 1 = \frac{1}{\varepsilon} \int_{\{|f-f_n| \geq \varepsilon\}} \varepsilon \\ &\leq \frac{1}{\varepsilon} \int_{\{|f-f_n| \geq \varepsilon\}} |f-f_n| \leq \frac{1}{\varepsilon} \int |f-f_n| \longrightarrow 0 \end{aligned} \quad \square$$

$\text{a.u.} \Rightarrow \text{measure}$ Suppose $f_n \rightarrow f$ a.u. Let $\varepsilon > 0$. Let $\delta > 0$.

find $N \in \mathbb{N}$ s.t. $n > N \Rightarrow \mu(\{|f-f_n| \geq \varepsilon\}) < \delta$.

Since $f_n \rightarrow f$ a.u., $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \delta$

and $f_n \chi_{E^c} \rightarrow f \chi_{E^c}$ unif.

$$\{|f-f_n| \geq \varepsilon\} = \underbrace{\{|f-f_n| \geq \varepsilon\} \cap E}_{\mu \leq \delta} \sqcup \underbrace{\{|f-f_n| \geq \varepsilon\} \cap E^c}_{\text{uniformly to 0, so } \rightarrow 0 \text{ for large } n.} \quad \square$$