Lec 9/7

Friday, September 7, 2018 11:31

$C_\lambda = \#$ of elements of $S_n$ of cycle type $\lambda$

$$= \frac{n!}{z_\lambda} = \frac{n!}{1^{l_1} \cdot 2^{l_2} \cdot \dots \cdot n^{l_n} \cdot l_1!\, l_2! \cdots l_n!} \qquad [\text{ex } 33 \text{ of } 4.3]$$

$$\# Z_{S_n}(\pi_\lambda) = z_\lambda \quad \text{where } \lambda \text{ is cycle type of } \pi_n.$$

$G \overset{\text{conjugation}}{\circlearrowright} G$ : orbits are called conjugacy classes

Stabilizers are called centralizers.

$$\sum_{\lambda:\text{ cycle type in } S_n} \frac{1}{z_\lambda} = 1 = \sum_{\lambda:\text{ cycle type in } S_n} \frac{l_1(\lambda)}{z_\lambda}$$

$$|X| = |G| \sum_{\mathcal{O}:\text{ orbit}} \frac{1}{|\mathrm{Stab}_G(x_{\mathcal{O}})|}$$

Burnside's Theorem:

$$|{}_G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

<u>Another Proof</u>:

$$f(x) := 1 + \sum_{n \geq 1} \left( \sum_{\lambda:\text{cycle type in } S_n} \frac{1}{z_\lambda} \right) x^n \qquad \left( z_\lambda = \prod_{k=1}^{n} (k^{l_k}\, l_k!) \right)$$

$$\sum k l_k = n$$

$$= 1 + \sum_{n \geq 1} \left( \sum_{\substack{l_1, \dots, l_n \geq 0 \\ \sum k l_k = n}} \left( \frac{1}{l_1!} \left( \frac{x^1}{1} \right)^{l_1} \right) \cdots \left( \frac{1}{l_n!} \left( \frac{x^n}{n} \right)^{l_n} \right) \right)$$

$$= \sum \left( \frac{1}{\ } \left( \ \right)^{l_1} \right) \left( \frac{1}{\ } \left( \frac{x^2}{\ } \right)^{l_2} \right.$$

$$= \sum_{\substack{l_1,\dots,l_n \geq 0 \\ \sum k l_k \text{ is finite}}} \left(\frac{1}{l_1}\left(\frac{x^1}{1}\right)^{l_1}\right)\left(\frac{1}{l_2}\left(\frac{x^2}{2}\right)^{l_2}\right)\cdots$$

$$= \sum_{l_2, l_3, \dots} \left(\prod_{k \geq 2} \left(\frac{x^k}{k}\right)^{l_k} \frac{1}{l_k}\right)\left(\sum_{l_1=0}^{\infty} \frac{1}{l_1}\left(\frac{x^1}{1}\right)^{l_1}\right)$$

$$= e^{x} \cdot e^{x^2/2} \cdot e^{x^3/3} \cdot \cdots$$

$$= e^{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots}$$

$$= e^{-\log(1-x)}$$

$$= \frac{1}{1-x}$$

$$= 1 + x + x^2 + \cdots$$

which is $\sum_{\lambda: \text{ cycle type in } S_n} \frac{1}{z_\lambda}$

So each coefficient of $x^n$ is $1$.

Can do the same with $l_1$ in the picture for second equality

mod. removing $1+$ in beginning

and you get $\frac{x}{1-x}$.

<u>Book</u>: Symmetric functions and Hall Algebras.

Now do something similar:

$$J(\underline{p}) = 1 + \sum_{n \geq 1} \left( \sum_{\substack{\lambda : \text{cycle} \\ \text{type in} \\ S_n}} \frac{1}{z_\lambda} \right) p_1^{\ell_1} p_2^{\ell_2} \cdots$$

$$= \sum_{\substack{\ell_1, \ell_2, \ldots \geq 0 \\ \sum k \ell_k \text{ is} \\ \text{finite}}} \left( \frac{1}{\ell_1} \left( \frac{p_1}{1} \right)^{\ell_1} \right) \left( \frac{1}{\ell_2} \left( \frac{p_2}{2} \right)^{\ell_2} \right) \cdots$$

$\vdots$

$$= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda = \prod_{r \geq 1} e^{p_r / r} \qquad \text{where } p_\lambda = \prod_{k \geq 1} p_k^{\ell_k}$$

So $$\sum_{\substack{\lambda : \text{cycle} \\ \text{type in} \\ S_n}} \frac{\ell_k}{z_\lambda} = \frac{1}{k} \quad \text{for} \quad k \leq n$$ (get this by taking derivatives).

Euler: $$\sum_{n \geq 1} \frac{1}{n} \overset{\text{in some sense}}{=} \prod_{p : \text{prime}} \frac{1}{1 - p^{-1}}$$

Application of $|X| = \sum_{\mathcal{O}} |\mathcal{O}|$ & $|\mathcal{O}| = \frac{|G|}{|\mathrm{Stab}_G(x_{\mathcal{O}})|}$ divides $|G|$.

If $p$ is a prime & $r, m \in \mathbb{Z}_{\geq 1}$

$$\binom{p^r m}{\ } \equiv m \pmod{p}$$

$p^r \mid -$

**Trick:** $G = \mathbb{Z}/p^r\mathbb{Z}$. $\quad X = \{x_1, \dots, x_m\}$

$E =$ set of all $p^r$-size subsets of $G \times X$.

$G \circlearrowright G \times X$ : $\quad g \cdot (h, x) = (g+h, x)$

So $G \circlearrowright E$ : $g \cdot \{e_1, \dots, e_{p^r}\} = \{g \cdot e_1, \dots, g \cdot e_{p^r}\}$.

So $|E| = \binom{p^r m}{p^r} =$ Sum of sizes of orbits in $E$

$\equiv$ (# of orbits of size 1) (mod $p$)

each orbit of size 1 is one of $\{(g, x_1) : g \in G\}$

$\{(g, x_2) : g \in G\}$

$\vdots$

So there are exactly $m$ of them.