Lec 10/25

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<u>Recall</u>: A compactification of X is an embedding $e : X \hookrightarrow K$ (compact)
where $e(X)$ is dense in K.

- When X is LCH but <u>not</u> cpt, there is a smallest (one-pt) compactification.

- When X is Tychonoff (completely regular & T_1) there is a largest compactification (Stone-Čech).

<u>Recall</u>: X tychonoff $\iff \exists$ embedding $X \hookrightarrow [0,1]^I$.

<u>SČC</u>: Suppose X is tychonoff. Let $\Phi = C(X, [0,1])$ and consider $e : X \hookrightarrow [0,1]^{\Phi}$ by $e(x) = (f(x))_{f \in \Phi}$.
define $\beta X = \overline{e(X)}$.

<u>Theorem</u>: The compactification $(\beta X, e)$ satisfies:

①
$$\begin{tikzcd} \beta X \arrow[dr, dotted, "\tilde{f}"] & \\ X \arrow[u, hook, "e"] \arrow[r, "f"'] & Z \end{tikzcd}$$

$\forall$ cpt Hausdorff Z and cts $f : X \to Z$, $\exists!$ cts $\tilde{f} : \beta X \to Z$ s.t. $\tilde{f} \circ e = f$.

② Remark $\exists$ implies $!$ by density of $e(X) \subset \beta(X)$.

③ βX is uniquely characterized by univ. property ①.

④ β is a functor $\{\text{Tych } sp\} \longrightarrow \{\text{Cpt Hausdorff } sp\}$.

Pf of ③: Suppose (φ, Y) and (ψ, Z) satisfy univ. prop in ①.

$$\begin{tikzcd}
Y \arrow[r, "\exists!\,\tilde\psi"] \arrow[rr, bend left, "\exists!\, id_Y"] & Z \arrow[r, "\exists!\,\tilde\varphi"] \arrow[rr, bend left, "\exists!\, id_Z"] & Y \arrow[r, "\exists!\,\hat\varphi"] & Z \\
 & X \arrow[ul, "\varphi"] \arrow[u, "\psi"] \arrow[ur, "\varphi"] \arrow[urr, "\psi"'] & &
\end{tikzcd}$$

so $\tilde\psi \circ \tilde\varphi = id_Z$

$\hat\varphi \circ \hat\psi = id_Y$

$Y \cong Z$ homeom by canonical maps.

Pf of ④: suppose $f: X \longrightarrow Y$ cts & X, Y are tychonoff.

Let $\Phi_X = C(X, [0,1])$ and $\Phi_Y = C(Y, [0,1])$.

Define $F: [0,1]^{\Phi_X} \longrightarrow [0,1]^{\Phi_Y}$

componentwise: for $g \in \Phi_Y$, $\pi_g \circ F := \pi_{\underbrace{g \circ f}_{\text{cts}}}$,

where $g \circ f \in \Phi_X$

so F cts.

$$\begin{tikzcd}
 & {[0,1]^{\Phi_X}} \arrow[d] \arrow[ddl, "\pi_{g\circ f}"'] \\
 & {[0,1]^{\Phi_Y}} \arrow[dl, "\pi_g"] \\
{[0,1]} &
\end{tikzcd}$$

Now $\forall x \in X$,

$$\pi_g[F[e_X(x)]] = \pi_{g\circ f}(e_X(x))$$

$$= (g\circ f)(x)$$

$$= \pi_g[e_Y(F(x))]$$

Hence $F \circ e_X = e_Y \circ f : X \longrightarrow [0,1]^{\Phi_Y}$

Thus $\text{Image}(F|_{\beta X}) \subset \overline{e_Y(Y)} = \beta Y$

($x \in \beta X$ and $\underset{\underset{e(x)}{\uparrow}}{x_\lambda} \longrightarrow x$ then $F(x_\lambda) \in e_Y(Y) \Rightarrow F(x) \in \beta Y$).

define $\beta f = F|_{\beta X} : \beta X \longrightarrow \beta Y$ cts

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \beta X \arrow[r, hook] \arrow[d, "\beta f"] \arrow[dr, phantom, "\circlearrowleft"] & {[0,1]^{\Phi_X}} \arrow[d, "F"] \\ Y \arrow[r, hook] & \beta Y \arrow[r, hook] & {[0,1]^{\Phi_Y}} \end{tikzcd}$$

Observe:

$$\begin{tikzcd} \beta X \arrow[drr, "\exists! \beta f"] & & \\ X \arrow[u, hook, "e_X"] \arrow[r, "f"'] & Y \arrow[r, hook, "e_X"'] & \beta Y \end{tikzcd}$$

$\beta f \circ e_X = e_Y \circ f$

Functoriality:

id: show $\beta[\mathrm{id}_X] = \mathrm{id}_{\beta X}$.

$$\begin{tikzcd}
\beta X \arrow[rrd, bend left, "\mathrm{id}_{\beta X} = \beta[\mathrm{id}_X]"] & & \\
X \arrow[u, "e_X"] \arrow[r, "\mathrm{id}_X"'] & X \arrow[r, "e_X"'] & \beta X
\end{tikzcd}$$

by uniqueness

$\circ$: suppose $f : X \to Y$ & $g : Y \to Z$.

$$\beta(g \circ f) \circ e_X = e_Z \circ (g \circ f) = (e_Z \circ g) \circ f$$

$$= \beta g \circ e_Y \circ f = \beta g \circ \beta f \circ e_X$$

$\Rightarrow \beta(g \circ f) = \beta g \circ \beta f$ by uniqueness

$$\begin{tikzcd}
\beta X \arrow[r, "\beta f"'] \arrow[rr, bend left, "\beta(g \circ f)"] & \beta Y \arrow[r, "\beta g"'] & \beta Z \\
X \arrow[u] \arrow[r, "f"'] & Y \arrow[u] \arrow[r, "g"'] & Z \arrow[u]
\end{tikzcd}$$

pf of ① Suppose $f : X \to Z$ cts and Z cpt Hausdorff

observe: $\beta Z = Z$ by the universal property!

by functoriality, $\exists!\ \beta f : \beta X \to \beta Z = Z$.

s.t. $\underset{\displaystyle f}{\underset{\parallel}{e_Z \circ f}} = \beta f \circ e_X$

$\square$

Corollary: X is Tychonoff, $e: X \hookrightarrow K$ cptification.

① $\beta e : \beta X \longrightarrow K$ is surjective

② if $\forall f \in C_b(X)$, $\exists g \in C(K)$ s.t. $f = g \circ e$,

then $\beta e : X \to K$ is a homeomorphism.

$$\begin{tikzcd} \beta X \arrow[dr, "\beta e"] & \\ X \arrow[u, hook, "e_X"] \arrow[r, "e"] \arrow[d, "f"'] & K \arrow[dl, dotted, "g"] \\ \mathbb{C} & \end{tikzcd}$$

Pf of ① since $\beta e \circ e_X = e$ and $e(X)$ is dense in K,

$\beta e\,[\beta X]$ is dense in K since $\beta X \supset e(X)$.

but βX cpt & βe cts so $\beta e[\beta X]$ is cpt so closed, so it $= K$.

②: It suffices to prove βe is injective

Recall: $\beta e := F|_{\beta X}$.

claim: if every $f \in C_b(X)$ factorizes thru $e: X \hookrightarrow K$, F is injective.

Indeed, every $f \in \Phi_X = C(X, [0,1]) \subset C_b(X)$ factorizes as $g \circ e$

So $F(x) = F(x') \iff x = x'$

$$\begin{matrix} \pi_g(F(x)) & = \pi_{g\circ e}(x) & \\ \| & & \Rightarrow \pi_f(x) = \pi_f(x') \\ \pi_g(F(x')) & = \pi_{g\circ e}(x') & \forall f \in \Phi_X. \end{matrix}$$

$\square$