Lec 4/17

Monday, April 17, 2017 15:03

* 2 sheets of notes for final, 60% of content will be post-MT2.

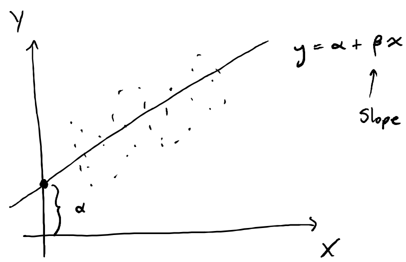
MLE Estimates for α, β: (for <u>normal</u> regression)

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}, \quad \hat{\beta} = \frac{s_{xy}}{s_{xx}}$$

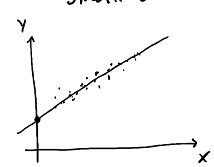
Makes sense since $\bar{y}$ estimates $\mathbb{E}Y$, $\bar{x}$ estimates $\mathbb{E}X$.

and $\frac{1}{n}S_{xy}$ estimates $Cov(X,Y)$, $\frac{1}{n}S_{xx}$ estimates $Var(X)$.

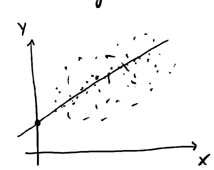
MLE Estimates are the same as Least-squares estimates $\hat{\alpha}^*$ and $\hat{\beta}^*$.



small σ



large σ



$\beta = 0 \Leftrightarrow$ no correlation. $(\rho = 0)$

People usually interested in testing:

$$H_0: \beta = 0 \quad \text{vs.} \quad H_1: \beta \neq 0$$

Sampling Theory: following based on conditional dist given $X_i = x_i$.

Let capital beta

$$\hat{B} = \frac{S_{x\overset{\text{constant}}{}}\overset{\text{RV}}{Y}}{S_{xx}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{S_{xx}} = \frac{1}{S_{xx}}\left[\sum_{i=1}^{n}(x_i - \bar{x})Y_i - \bar{Y}\underbrace{\sum_{i=1}^{n}(x_i - \bar{x})}_{=0}\right]$$

$$= \frac{1}{S_{xx}}\sum_{i=1}^{n}(x_i - \bar{x})Y_i$$

Now:

$$\mathbb{E}(\hat{B}) = \frac{1}{S_{xx}}\sum_{i=1}^{n}(x_i - \bar{x})\,\mathbb{E}(Y_i)^{\swarrow \alpha + \beta x_i}$$

$$= \frac{1}{S_{xx}}\left[\alpha\sum_{i=1}^{n}(x_i - \bar{x}) + \beta\sum_{i=1}^{n}(x_i - \bar{x})x_i\right]$$

$$= \frac{1}{S_{xx}} \left[\alpha \underbrace{\sum_{i=1}^{n} (x_i - \bar{x})}_{0} + \beta \sum_{i=1}^{n} (x_i - \bar{x}) x_i \right]$$

$$= \frac{\beta}{S_{xx}} \left[\sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) + \underbrace{\sum_{i=1}^{n} (x_i - \bar{x})}_{0} \bar{x} \right]$$

$$= \frac{\beta}{S_{xx}} S_{xx} = \beta \qquad (\hat{B} \text{ unbiased})$$

$$Var(\hat{B}) = \frac{1}{S_{xx}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \underbrace{Var(y_i)}_{\sigma^2} = \frac{1}{S_{xx}^2} \sigma^2 S_{xx} = \frac{\sigma^2}{S_{xx}}$$

$$\Rightarrow \hat{B} \sim N(\beta, \frac{\sigma^2}{S_{xx}})$$

Similarly, let $\hat{\Sigma}^2 = \frac{1}{n}(S_{yy} - \hat{B} S_{xy})$, whose values are $\sigma^2 = \frac{1}{n}(S_{yy} - \hat{\beta} S_{xy})$. We have:

$$\frac{n \hat{\Sigma}^2}{\sigma^2} \sim \chi^2_{n-2}$$

Also, $\frac{n \hat{\Sigma}^2}{\sigma^2}$ and $\hat{B}$ are independent.

Thus, we have:

$$\frac{\dfrac{\hat{B} - \beta}{\frac{\sigma}{\sqrt{S_{xy}}}}}{\sqrt{\dfrac{n \hat{\Sigma}^2}{\sigma^2} \Big/ (n-2)}} \sim t_{n-2} \quad \text{by Theorem 8.12.}$$

This simplifies to:

$$T = \frac{\hat{B} - \beta}{\hat{\Sigma}} \sqrt{\frac{(n-2) S_{xx}}{n}} \sim t_{n-2}$$ so we don't need σ^2, only $\hat{\Sigma}^2$.

So we can use T as our test statistic.

Ex:

hours studied x	test score y
$\vdots$	$\vdots$

Test $H_0: \beta = 3$ vs $H_1: \beta > 3$ at level $\alpha = 0.01$.

Sol. Yesterday we computed $\hat{\beta} = 3.471$, $S_{xx} = 376$, $S_{yy} = 4752.4$, $S_{xy} = 1305$

$$\hat{\sigma} = \sqrt{\tfrac{1}{10}\left(4752.4 - 3.471 \cdot 1305\right)}$$

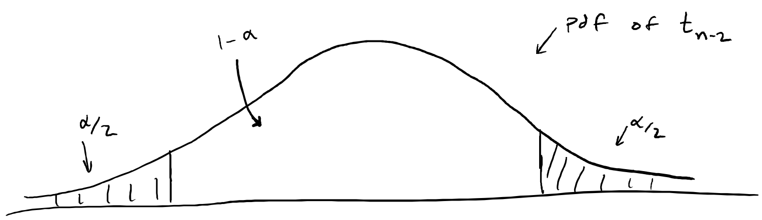
$$\Rightarrow T = \frac{\hat{\beta} - \beta_0}{\hat{\sigma}} \sqrt{\frac{(n-2) S_{xx}}{n}} = 1.73 \quad \text{value of test statistic.}$$

CR: Reject H_0 when $T > t_{.001, 8} = 2.896$

So fail to reject H_0.

We could not conclude that 1 extra hour of study would on average increase the score by 3 points.

Confidence Interval for β:



$$\mathbb{P}\left(-t_{\frac{\alpha}{2}, n-2} < T < t_{\frac{\alpha}{2}, n-2}\right) = 1 - \alpha$$

$$\Rightarrow \mathbb{P}\left(\hat{\beta} - t_{\frac{\alpha}{2}, n-2}\, \hat{\sigma} \sqrt{\frac{n}{(n-2) S_{xx}}} < \hat{\beta} < \beta + t_{\frac{\alpha}{2}, n-2}\, \hat{\sigma} \sqrt{\frac{n}{(n-2) S_{xx}}}\right) = 1 - \alpha$$

So $\hat{\beta} \pm t_{\frac{\alpha}{2}, n-2} \hat{\Sigma} \sqrt{\frac{n}{(n-2) S_{xx}}}$ is the $(1-\alpha)100\%$ CI for β.