Lec 9/23

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Rings Intro I

Def a ring has 1, but it could be that $1 = 0$.

1. $(R, +, 0)$ is an abelian gr
2. $(R, \cdot, 1)$ is a monoid
3. $a(b+c) = ab + ac$
   $(a+b)c = ac + bc$

Example $\mathbb{Z}$.

Observation: $\forall a, b \in R$, $(1+1)(a+b) = 1(a+b) + 1(a+b) = a+b+a+b$

Also, $(1+1)(a+b) = (1+1)a + (1+1)b = a+a+b+b$

so $b+a = a+b$.

(additive gp must be abelian by dist. laws).

ex: $\mathbb{Z}_3$. Can $2^2 = 0$ or 2 ?

ex if $ab = ba$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

<u>Note</u> $R = 0$ iff $1=0$.

<u>Def</u> R is a domain (integral domain)

if $R^* = R \setminus \{0\}$ is a submonoid of $(R, \cdot, 1)$.

(so 0 is not a domain).

<u>Prop</u> R is a domain iff $R \neq 0$ and $\forall a, b \in R$, $ab = 0 \Rightarrow a=0$ or $b=0$.

[equivalently if $a, b \in R^*$ then $ab \neq 0$].

<u>Note</u> $M_n(R)$ is not commutative if $R \neq 0$ and $n > 1$.

<u>Note</u> $[0,1]^{\mathbb{R}}$ is not a domain.

<u>Prop</u> R is a domain iff $R \neq 0$ and the cancellation laws hold:

$$ab = ac, \quad a \neq 0 \implies b = c$$

$$ac = bc, \quad c \neq 0 \implies a = b.$$

<u>Pf</u> distributive law

<u>Def</u> $a \in R$ is a (left) zero divisor if $\exists b \neq 0$ s.t. $ab = 0$.

<u>Ex</u> 0 is a zero divisor, if $R \neq 0$.

<u>Def</u> a <u>subring</u> has 1.

<u>Def</u> R is a division ring if $R^* = R \setminus \{0\}$ is a subgroup of the monoid $(R, \cdot, 1)$ (every element $\neq 0$ is invertible)

<u>Ex</u> a subring of a division ring is a domain

<u>Def</u> a field is a commutative division ring.

<u>Def</u> the group of units $U = R^\times$ in R is the subset of $(R, \cdot, 1)$ consisting of invertible elements.