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Modules:

Def. Let R be a ring. An (Left) R-module is an abelian group M with binary operation $R \times M \to M$ $(a,u) \mapsto au$, called multiplication by a scalar in R.

Satisfying: $(ab)u = a(bu)$

$(a+b)u = au + bu$ $\quad \forall a,b \in R,\ u,v \in M$

$a(u+v) = au + av$

If $1 \in R$, then $1 \cdot u = u \quad \forall u \in M$

(Assume not, then $\forall u \in M$, $u = 1\cdot u + (u - 1\cdot u)$.

now $1\cdot(1\cdot u) = 1\cdot u$ so $1\cdot(u - 1\cdot u) = 0$

and so $\forall a \in R$, $a(u - 1\cdot u) = 0$, so $R\cdot(u - 1\cdot u) = 0$

So $M = M_1 \oplus M_0$

$M_1 = \{1\cdot u\}$, $M_0 = \{u - 1\cdot u\}$, $R\cdot M_0 = 0$)

In a right module, $(a,u) \longmapsto ua$

the difference is $u(ab) = \underline{(ua)b}$.

If R is commutative, there is no difference.

A bimodule is an abelian group with both structures (left & right modules)

s.t. $\forall a,b \in R$, $u \in M$, $(au)b = a(ub)$.

R is an R-module

$ab \neq ba$ if R is not commutative.

Def: A is an $\overset{\text{left}}{\vee}$ R-algebra if A is an $\overset{\text{left}}{\vee}$ R-module and a ring,

s.t. $\forall \alpha, \beta \in A$, $a \in R$, $a(\alpha\beta) = (a\alpha)\beta = (\alpha a)\beta$

(R is assumed to be commutative)

Examples: ⓪ · 0-module $= \{0\}$.

① : R itself is an R-bimodule

② : any left ideal is a left module, right ..... right .... .
two-sided ideal is a bimodule.

③ : any abelian group $\overset{G}{\vee}$ is a $\mathbb{Z}$-module.
$\forall u \in G$, $n \in \mathbb{N}$, $nu = u + \cdots + u$ n times, $(-n)u = -(nu)$

④ $\forall n \in \mathbb{N}$, $R^n = R \times \cdots \times R = \{(u_1, \ldots, u_n) : u_i \in R\}$ is an R-$\overset{\text{bi}}{\vee}$module
it's called a free module of rank n.

⑤ Let X be a set, let $F = \{f : X \longrightarrow R\}$ $(= R^X)$.
it is an R-bimodule

⑥ Polynomials over R : $R[x] = \{a_N x^N + \cdots + a_1 x + a_0 : a_i \in R\}$.
it's an R-bimodule, R-algebra if R is commutative.

⑦ $\mathrm{Mat}_{m,n}(R) =$ $m \times n$ matrices over R.
As an R-module, this is just R^{mn}.

⑧ Let V be an n-dim vector space over a field F.
Let $R = \mathrm{Mat}_{n\times n}(F)$. Then V is an R-module for $A \in R$, $u \in V$, $Au \in V$.

⑨ Let V be an F-vector space, let T be a linear transformation of V.
let $R = F[x]$. define an action of R on V by $x \cdot u = T(u)$, et cetera.
Then V is an R-module.

⑩ Let G be a group, R a commutative ring. Then the R-algebra of G is $R[G] = \{a_1 g_1 + \cdots + a_n g_n : a_i \in R, g_i \in G\}$.

Polynomials: RG where $G = \{1, x, x^2, \dots\}$ semigroup.

⑪ M : R-module, $S \subset R$ subring, M is an S-module as well. (restriction of scalars).

⑫ The module of formal power series $\{a_0 + a_1 x + a_2 x^2 + \cdots\}$.

(same as $R^{\mathbb{N}}$ if R is not commutative).

if R is commutative, this is an R-algebra.

Elementary Properties.

Theorem: Let M be an R-module.

Then (1) $\forall\, u \in M$, $0u = 0$

(2) $\forall\, a \in R$, $a0 = 0$

(3) $\forall\, a \in R, u \in M$, $(-a)u = -(au) = a(-u)$.

Proof: (1) $0u = (0+0)u = 0u + 0u$

(2) $a0 = a(0+0) = a0 + a0$

(3) $(-a)u + au = (-a+a)u = 0$

$a(-u) + au = a(-u+u) = 0$.

Submodules:

M : R-module, then $N \subset M$ is a submodule of N is an R-module under same operations.

This is so if $N \leq M$, $N - N \subseteq N$ and $RN \subseteq N$

Examples: ① 0-submodule $= \{0\}$.

② Continuous fns on $[0,1]$ form a submodule of all fns on $[0,1]$.