Lec 11/16

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Infinite Series

Paradox 1: what is the sum of the infinite series

$$\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{4}{5}\right)+\dots$$

a: $1/2$. but each term is negative:

this equals $-\frac{1}{6}-\frac{1}{2}-\frac{1}{20}-\frac{1}{30}-\dots$

Paradox 2: $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\dots$

$\frac{1}{2}\ln 2 = 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\dots$

$\frac{3}{2}\ln 2 = 1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\dots$

$? = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\dots$

P1: failure of Assoc. Law P2: failure of Comm. Law

Conclusion: be careful.

$$\sum_{i=1}^{\infty} a_i = a_1+a_2+a_3+\dots$$

add left-to-right. $a_1,\ a_1+a_2,\ a_1+a_2+a_3, \dots$

$S_n = \sum_{i=1}^{n} a_i$ partial sums

Definition $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} S_n$, provided this limit exists.

if $\lim_{n\to\infty} S_n$ exists, series converges. o.w. series diverges.

Apply: $\sum^{\infty}\left(\frac{j}{j+1}-\frac{j+1}{j+2}\right)$. $S_n = \left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\dots+\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right) = \frac{1}{2}-\frac{n+1}{n+2}$

Apply: $\sum_{j=1}^{\infty}\left(\frac{j}{j+1}-\frac{j+1}{j+2}\right)$. $S_n = \left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\cdots+\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right) = \frac{1}{2}-\frac{n+1}{n+2}$

$$\lim_{n\to\infty}\left(\frac{1}{2}-\frac{n+1}{n+2}\right) = \frac{1}{2}-1 = -\frac{1}{2}$$

two sequences associated w. inf. series: $\underbrace{\{a_i\}_{i=1}^{\infty}}_{\text{terms}}$, $\underbrace{\{S_n\}_{n=1}^{\infty}}_{\text{partial sums}}$

Theorem Fundamental Divergence Test: If series converges, then $\lim_{n\to\infty} a_n = 0$.
(only useful to show a series diverges).

Note: Converse does not hold: $\frac{1}{\sqrt{n}} \to 0$ as $n\to\infty$, but $\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}$ DNE:

$$S_n = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$$
$$\geq \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\cdots+\frac{1}{\sqrt{n}} = \sqrt{n} \to \infty \text{ as } n\to\infty$$

Question: Are infinite series associative? when are they?

Given an infinite series $\sum_{j=1}^{\infty} a_j$ and a sequence of indices $N_1 < N_2 < N_3 < \cdots$

we can group the terms $(a_1+a_2+\cdots+a_{N_1})+(a_{N_1+1}+\cdots+a_{N_2})+(a_{N_2+1}+\cdots a_{N_3})+\cdots$

and still get the same result?

$b_1 = a_1+a_2+\cdots+a_{N_1}$

$b_j = a_{N_{j-1}+1}+a_{N_{j-1}+2}+\cdots+a_{N_j}$

Theorem If the series $\sum_{j=1}^{\infty} a_j$ converges, so does the grouped series $\sum_{j=1}^{\infty} b_j$

Proof: let S_n, S_n' be partial sums for each series.

$S_n' = S_{N_n}$ so $\{S_n'\}$ is a subsequence of $\{S_n\}$.

we have shown that if $\{S_n\}$ converges, any subsequence $\{S_{n_j}\}$ converges to the same limit. $\blacksquare$

Back to paradox 1:

$$-\frac{1}{2} = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots$$

$$\frac{1}{2} = \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) + \cdots$$

but $\frac{1}{2} - \frac{2}{3} + \frac{2}{3} - \frac{3}{4} + \frac{3}{4} - \cdots$ diverges:

S_n when n odd $= 1/2 \to 1/2$

S_n when n even $= \frac{1}{2} - \frac{n+1}{n+2} \to -\frac{1}{2}$

So S_n diverges by FDT.

Simpler example: $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges

$$(1-1) + (1-1) + (1-1) + \cdots = 0$$

$$1 + (-1+1) + (-1+1) + \cdots = 1$$

Theorem $\sum_{j=1}^{\infty} a_j$ converges iff for any $\varepsilon > 0$ we can find an index N s.t.

$$|a_m + a_{m+1} + \cdots + a_n| < \varepsilon \quad \text{if} \quad n > m > N$$

Proof: $\{S_n\}$ is cauchy, so S_n converges. $\blacksquare$

Given an infinite series $\sum_{j=1}^{\infty} a_j$ there are 2 basic questions:

1) does it converge?

2) if so, how fast? - analyze remainder / estimate size of tail

$\sum_{j=1}^{n} a_j + \sum_{j=n+1} a_j$ → compare this to a geom. series