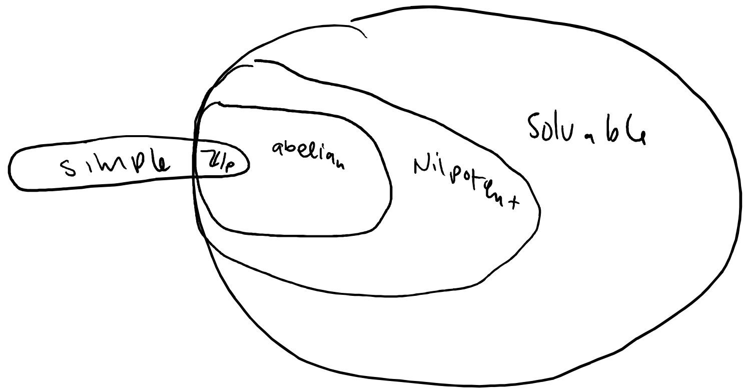
Lec 10/1

Monday, October 1, 2018 11:29



Reminder:

$$\Sigma: G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

$$G_i / G_{i+1} = gr_i^{\Sigma}(G_0) \quad \text{associated graded piece.}$$

Today building composition series w/ commutators.

Commutator subgroup generated by $\{xyx^{-1}y^{-1} : x, y \in G\}$.

$[x,y] = xyx^{-1}y^{-1}$ $\quad$ ($f([x,y]) = e$ $\forall$ gp hom f to an abelian gp).

For $A, B \subset G$ subsets,

$[A,B] =$ subgp of G generated by $\{[a,b] : a \in A, b \in B\}$.

Lemma: If $A, B \trianglelefteq G$, $[A,B] \trianglelefteq G$.

Pf $\mathrm{Conj}(g) : G \longrightarrow G$, $x \longmapsto gxg^{-1}$ is a gp hom.

wts $g[a,b]g^{-1} \in [A,B]$

$$\begin{array}{c} g[a,b]g^{-1} \\ \| \\ [gag^{-1}, gbg^{-1}] \quad \checkmark \end{array}$$

Define $G^{(0)} = G, \quad G^{(1)} = [G^{(0)}, G^{(0)}], \ \ldots$

$$G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}], \ \ldots \qquad \text{(may not stop)}$$

~~"derived"~~ or $\boxed{\text{"Commutator series"}}$ $\longrightarrow$ $G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \ldots$

Definition: G is called solvable if $G^{(n)} = \{e\}$ for some $n \geq 0$.

Properties of $[H,H] \trianglelefteq H$

(i) $H/[H,H]$ is abelian

(ii) $H' \trianglelefteq H$ any normal subgp s.t. H/H' is abelian, $H' \supset [H,H]$.

So $G^{(i)}/G^{(i+1)}$ is abelian $\forall i$.

If H is simple & non-abelian then $[H,H] \neq \{e\}$

So $[H,H] = H$ meaning chain never ends (H is not solvable)

If G is abelian, $[G,G] = \{e\}$ so G is solvable.

eg (solvable group)

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : ac \neq 0 \right\}$$

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & \frac{-b_1}{a_1 c_1} \\ 0 & \frac{1}{c_1} \end{bmatrix} \begin{bmatrix} \frac{1}{a_2} & \frac{-b_2}{a_2 c_2} \\ 0 & \frac{1}{c_2} \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$[B ; B] = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \text{ arbitrary} \right\}$$

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix}$$

$$B \trianglerighteq [B ; B] \trianglerighteq \{e\}.$$

<u>eg</u> S_3 is solvable

$$\underset{D_6}{\overset{S_3}{\parallel}} = \langle s, r : s^2 = r^3 = e ; srs = r^2 \rangle$$

$$srs^{-1}r^{-1} = r^{-1}r^{-1} = r^{-2} = r \quad \text{so } r \in [D_6, D_6] \Rightarrow \overset{\mathbb{Z}/3\mathbb{Z}}{\overset{\parallel}{\langle r \rangle}} \subset [D_6, D_6]$$

but $D_6 / \langle r \rangle$ is abelian so $\langle r \rangle \supset [D_6, D_6] \Rightarrow [D_6, D_6] = \langle r \rangle$.

So $D_6 \trianglerighteq \langle r \rangle \trianglerighteq \{e\}$