Lec 2/9

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Corollary: $[0,1]$ is connected. $\square$

Theorem: Let I be an interval in $\mathbb{R}$. I is connected.

Pf Let U be clopen in I. Let $U' = I \setminus U$, so U' is also clopen in I. We wish to show that $U = \emptyset$ or $U = I$. Suppose not so $U \neq \emptyset$ and $U' \neq \emptyset$. So let $u \in U$ and $u' \in U'$. Either $u < u'$ or $u' < u$. Wolog let's say $u < u'$. Define $f : [u, u'] \longrightarrow [0,1]$ by

$f(x) = \frac{x - u}{u' - u}$. Then f is continuous one-to-one from $[u, u']$ onto $[0,1]$.

$f^{-1} : [0,1] \longrightarrow [u, u']$ is also cts so f is a homeomorphism.

Let $A = f[\underbrace{U \cap [u, u']}_{\text{clopen in } [u,u']}]$. Then A is clopen in $[0,1]$ since f is a homeomorphism.

thus A is $\emptyset$ or $[0,1]$. but this means $U \cap [u, u'] = \emptyset$ or $[u, u']$. Either one is a contradiction since one says $u \notin U$ and one says $u' \in U$. $\square$

but $u \in U$ so $A \neq \emptyset$ so $A = [0,1]$. but the only point in $[u, u']$ which maps to 1 is u' so $u' \in U$, a contradiction $\square$

Corollary (The Intermediate value Theorem)

Let $f : [a,b] \rightarrow \mathbb{R}$ be cts, where $a, b \in \mathbb{R}$ with $a < b$. Suppose v is between $f(a)$ and $f(b)$. Then $\exists\, c \in [a,b]$ s.t. $f(c) = v$.

Proof $[a,b]$ is connected. Let $I = f([a,b])$. Then I is an interval.

Now $f(a), f(b)$ are in I and v is between them so $v \in I$, and thus $\exists\, c \in [a,b]$ s.t. $f(c) = v$. $\square$

Continuous images of connected spaces:

Reminder: Let X and Y be sets. Let $V \subseteq Y$ and $f: X \to Y$.

$$f[f^{-1}[V]] = V \cap f[X]. \text{ In particular, if } f \text{ is onto, } f[f^{-1}[V]] = V.$$

Theorem: Let X and Y be topological spaces. Let $f: X \to Y$ be cts. Suppose X is connected and f is onto Y. Then Y is connected.

Proof Let V be clopen in Y. WTS $V = \emptyset$ or $V = Y$. Let $U = f^{-1}[V]$. Then U is clopen in X so $U = \emptyset$ (so $f[f^{-1}[V]] = V = \emptyset$) or $U = X$ (so $f[f^{-1}[V]] = V = Y$) $\square$

using $f[U] = V$ since f onto

Components:

Theorem: Let X be a topological space, $p \in X$, and $\mathcal{C}$ be a set of connected subsets of X s.t. $\forall\, C \in \mathcal{C}$, $p \in C$. Let $E = \bigcup \mathcal{C}$. E is connected.

Proof: Let U be clopen in E. WTS $U = \emptyset$ or $U = E$. Either $p \in U$ or $p \notin U$.

Case 1: Suppose $p \in U$. Then $\forall\, C \in \mathcal{C}$, $U \cap C$ is clopen. Thus $U \cap C = \emptyset$ or $U \cap C = C$. But $p \in U \cap C$ so $U \cap C = C$.

Thus $U \cap E = U \cap (\bigcup \mathcal{C}) = \bigcup_{C \in \mathcal{C}} C \cap U = \bigcup \mathcal{C} = E$ so $U = E$.

Case 2: Suppose $p \notin U$. Then $p \notin U \cap C$ so $U \cap C = \emptyset$. Thus $U \cap E = \emptyset$ so $U = \emptyset$. $\square$

Corollary Every open ball in $\mathbb{R}^d$ is connected.

Pf Let $E = B(p,r)$ where $p \in \mathbb{R}^d$ and $r \in (0,\infty)$. $\forall q \in E$, let

$C_q = \{(1-t)p + tq : t \in [0,1]\}$. This is a continuous image of $[0,1]$ and so is connected containing p and q and is contained in E since

$$|(1-t)p + tq - p| = t|p-q| < tr \le r$$

thus $E = \bigcup_{q \in E} C_q$ and since every C_q is connected & contains p,

E is connected. □

Corollary: If $C_k \cap C_{k+1} \neq \emptyset$ and C_i is connected $\forall i$, $\bigcup_{k=1}^{n} C_k$ is connected.

Corollary Let X be a topological space. Define a relation $\sim$ by

$x \sim x'$ iff $\exists\, C$ connected subset of X s.t. $x, x' \in C$.

(a) This is an equivalence relation.

For each $x \in X$, let $[x] = \{x' \in X : x' \sim x\}$.

(b) $[x]$ is connected.

(c) for each non-empty connected $C \subseteq X$, $\exists\, x \in X$ s.t. $C \subseteq [x]$.

(since $\sim$ is an equivalence relation, $\{[x] : x \in X\}$ is a partition of X)

Proof: (a) $\sim$ is reflexive since $\{x\}$ is connected.

$\sim$ is symmetric since C contains both x and x'.

$\sim$ is transitive since if $x_1 \overset{C_1}{\sim} x_2$ and $x_2 \overset{C_2}{\sim} x_3$ then

C_1 and C_2 are connected & share point x_2 so $C_1 \cup C_2$ is connected and $x_1, x_3 \in C_1 \cup C_2$ so $x_1 \sim x_3$

(b) Let $C_{x'}$ = the set used if $x \sim x'$. Then $[x] \subseteq \bigcup C_{x'}$

Uses AoC $\rightarrow$ (b) Let $C_{x'}$ = the set used if $x \sim x'$. Then $[x] \subseteq \bigcup_{x' \in [x]} C_{x'}$ which is connected and if $x'' \in \bigcup_{x' \in [x]} C_{x'}$ then it's in some $C_{x'_0}$ which is a connected set w/ x and x''. So $x'' \in [x]$ so $[x] = \bigcup_{x' \in [x]} C_{x'}$ which is connected.

better $\rightarrow$ Let $x \in X$. Let $\mathcal{C} = \{C \subseteq X : C \text{ is connected } \& \ x \in C\}$. Then $[x] = \bigcup \mathcal{C}$ and $\bigcup \mathcal{C}$ is connected.

(c) Let C be a non-empty connected $\subseteq X$. Then $C \subseteq [x]$ where x can be any element of C. $\square$

eg $(1,2) \cup (2,3)$

eg The components of $\mathbb{Q}$ are $\{x\} : x \in \mathbb{Q}$

eg the only component of $\mathbb{R}$ is $\mathbb{R}$.

eg the components of the Cantor set are $\{x\} : x \in C$ (its singletons).

Components are always closed; and are sometimes open.