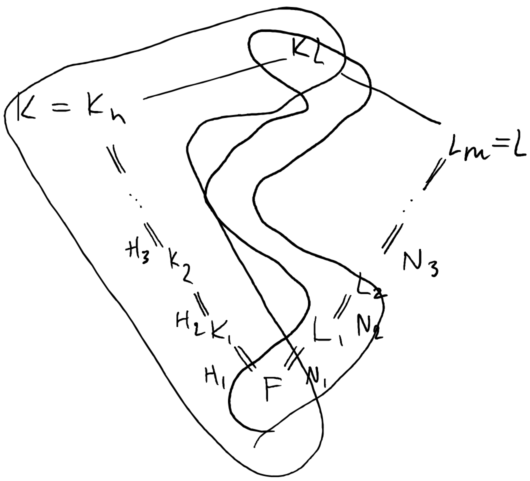
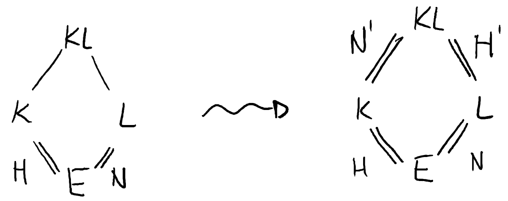


$\forall\, i, j$, $K_iL_j / K_{i-1}L_j$ and K_iL_j / K_iL_{j-1} are galois,
and their galois groups $\leq H_i, N_j$

~~pf~~: induction:
construct the lattice.



and $H' \leq H$
$\underbrace{N' \leq N}$
equality
if diagram
is minimal: $E = K \cap L$.

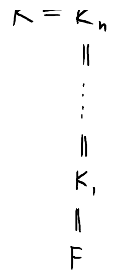
$$Gal(KL/K) \cong Gal(L/L\cap K) \leq Gal(L/E) = N.$$

All neighboring extensions have a subgroup relationship.

<u>Thm1</u> if K/F, L/F are towers of extensions with groups $H_1, \dots, H_n$, $N_1, \dots, N_m$, Then KL/F is a tower of extensions whose groups are subgroups of N_i, H_j.

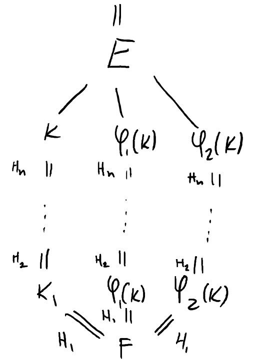
$K = K_n$
$\|$
$\vdots$

Galois closure of K/F = Composite of conjugates of K
$\|$
in the Galois closure of K/F



Galois closure of K/F = Composite of conjugates of K in the Galois closure of K/F.

$= E$



$\varphi_i \in Gal(E/F)$

<u>Theorem 2</u> If K/F is a tower of Galois extensions w/ groups $H_1, \dots, H_n$, Then the Galois Closure of K/F is also a tower of Galois extensions whose groups are isomorphic to subgroups of $H_1, \dots, H_n$.

<u>More General Defn</u> K/F is a p-extension iff it is a subextension of a Galois p-extension.

Then, if this is so, $[K:F] = p^r$ for some r.

But converse is not true.

eg $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, but it's not a 3-extension because its Galois Closure, $\mathbb{Q}(\sqrt[3]{2}, \omega = e^{2\pi i/3})/\mathbb{Q}$, has degree 6.

<u>Theorem</u> if K/F is a p-extension, then it is a tower of Galois extensions of degree p (so, with $Gal \cong \mathbb{Z}_p$).

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash] \arrow[dd, dash, "G:\ p\text{-group}"] \\
K \arrow[dr, dash] & \\
 & F
\end{tikzcd}$$

$H \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$

$H_{i+1}/H_i \cong \mathbb{Z}_p$

E/F is a tower of Galois extensions of degree p.

$$E = E_n / E_{n-1} / \cdots / E_0 = F,$$

$$K = (K \cap E_n)/(K \cap E_{n-1})/\cdots/(K \cap E_0) = F$$

__Theorem__ converse is also true: If K is a tower of Galois extensions of degree p, then it is a p-extension.

__Proof__ by Thm 2, if E/F is Galois closure of K/F, then it is a tower of Galois ext-ns whose Galois groups are subgroups of $\mathbb{Z}_p$, so E/F is a p-extension.

__Theorem__ K/F is a 2-extn iff it is a tower of extensions of degree 2.

__Note__: $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ - not a 2-extn.

Constructions with ruler and compass

__Operations__: ① given two points, construct straight line containing them

② given three points a, b, c, draw a circle centered at a of radius $|b-c|$.

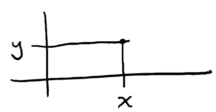
(3,4,5) find intersections of lines & circles (make more points)

Given $S \subseteq \mathbb{R}^2$, construct new points using (1-5).

A point is constructible (from S) if it's obtainable in this way.

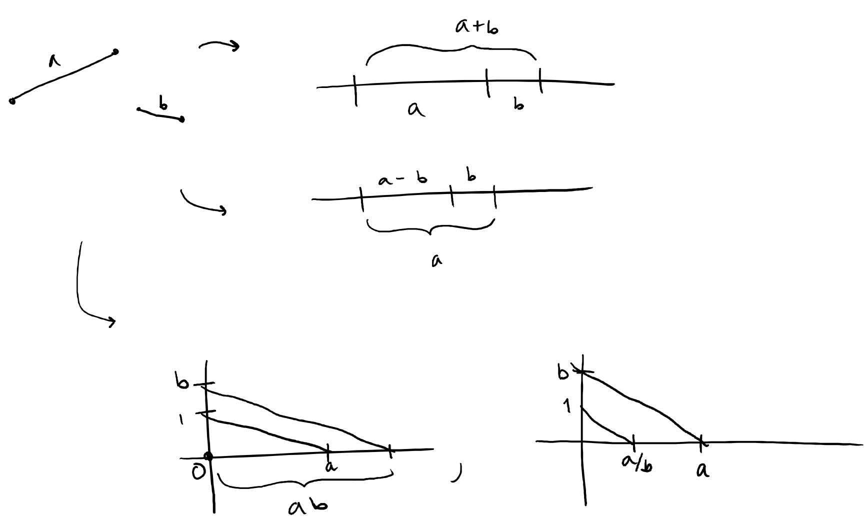
Given just 2 points, (0,0) and (1,0), you can construct all of $\mathbb{Q}$.

A point is constructible iff its coordinates are constructible.



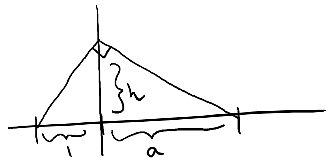
A number a is constructible if there are 2 constructible points a distance a from each other.

If a, b are constructible then $a+b$, $a-b$, ab, a/b are constructible.



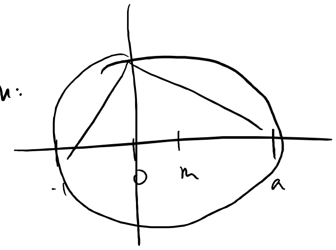
So constructible #s form a field.

Also, if a is constructible, then $\sqrt{a}$ is too.



$\frac{a}{h} = \frac{h}{1} \Rightarrow h = \sqrt{a}.$

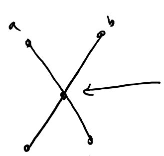
Construction:



<u>Theorem</u> Let F be the field generated by the coordinates of the points in S.

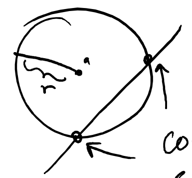
Then α is constructible from S iff α is contained in a tower of quadratic extensions.

(recall $[K:F] = 2 \Rightarrow K = F(\sqrt{a})$ for some $a \in F$).

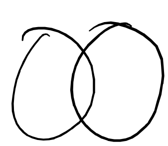


don't escape field generated by abcd

$F \ni a, r$



coordinates given by square roots of stuff in F (quadratic extn)



again quadratic extn.