# Lec 10/16

Monday, October 16, 2017 14:21

<u>Theorem</u>: a version of Sárközy:

For any A w/ $\bar{d}(A)>0$ there $\{n: \underbrace{\bar{d}(A \cap (A-n^2))}_{\substack{x-y=n^2 \\ x,y\in A}} > 0\}$ is syndetic.

A finitary version of sárközy: $\forall \varepsilon>0$ $\exists L \in \mathbb{N}$ s.t. if $b-a>L$ then for any $A \subset [a, a+1, \dots, b]$ with $\frac{|A|}{b-a} > \varepsilon$, $\exists n \in \mathbb{N}$ & $x,y \in A$, st. $x-y=n^2$.

exercise prove equivalence to normal sárközy

$\exists n: A \cap (A-n^2) \neq \emptyset \iff \bar{d}(A \cap (A-n^2)) > 0$ (exercise)

✩ $\varphi(n) = \bar{d}(A \cap (A-n))$, $n \in \mathbb{Z}$ or $\mathbb{N}$, $A \subseteq \mathbb{Z}$ or $\mathbb{N}$, $\bar{d}(A)>0$.

<u>Def</u>: A sequence $\varphi(n)$ is positive definite if $\forall (\xi_i) \subset \mathbb{C}$ and any $n \in \mathbb{N}$, one has $\sum_{-N \le n,m \le N} \varphi(n-m)\, \xi_n \bar{\xi}_m \geq 0$

matrix $(a_{nm})_{n,m \in N}$ is called orthogonal (over $\mathbb{R}$ or unitary over $\mathbb{C}$.) if its rows and columns are orthonormal as vectors in $\mathbb{R}^N$

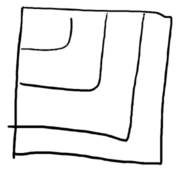
Quadratic forms:

Inertia $\Bigg\{$ Can be reduced to $\sum_{i=1}^{n} (-1)^{\delta_i} X_i^2$. Can it be reduced in different way (w/ diff # of $\pm 1$)? NO

Can it reduce to $X_1^2 + X_2^2 + \cdots + X_n^2$ ?

Sylvester criterion:

If all subdeterminants (minors) are positive then yes



**Exercise:** check that $\varphi(n)$ as defined above is positive definite

$$\varphi(n) = \langle U^n v, u \rangle \quad n \in \mathbb{Z}, \quad U \text{ is a unitary matrix, } v, u \in \mathbb{C}^m$$

$\|Ux\| = \|x\| \;\forall x, \; U^{-1} = U^*$ ← adjoint

**exercise** this $\varphi$ is also positive definite.

Herglotz theorem: Any positive definite sequence can be represented as a seq. of fourier coeffs of a measure on $\mathbb{T}$.

$$\varphi(n) = \int_{\mathbb{T}} e^{2\pi i n x} \, d\mu$$ ← **exercise**: this is P.D.

$\varphi(n) = \delta(A \cap (A-n))$. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(n^2) > 0 \Rightarrow$ Sárközy

So, $\varphi(n)$ P.D. $\Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_{\mathbb{T}} {}^{2\pi i n^2 x} \, d\mu$

$$= \int_{\mathbb{T}} \underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n^2 x}} \, d\mu$$

$$= \int_{\mathbb{T}} \underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1} e}_{=0 \text{ for } x \notin \mathbb{Q}} \, d\mu$$

Suppose we replace $n^2$ by $p_n$. Same result holds.

"Divisible": $\{n^2-1 : n\in\mathbb{Z}\} \cap a\mathbb{Z} \neq \emptyset \quad \forall a \in \mathbb{N}$ (seq. has all divisors)

so $n^2-1$ is divisible

$\equiv \{(n+1)^2-1\} = \{\underbrace{n^2+2n}_{\text{divisible}}\}$

Exercise $P-1$ and $P+1$ are divisible sets.

exercise $P \pm d$ is not divisible if $d \neq 1$.

What if we replace $N$ limits by $N-M$ limits?

Still works (not for $p_n$) since $n^2$ w.d.

try with $\bar{d}$ instead of $d^*$ if this is too hard.

Exercise if $d^*(A), d^*(B) > 0$ then $(A-A) \cap (B-B)$ is syndetic

$(x, x^2, x^3, \dots, x^n)$ curve of moments in $\mathbb{R}^n$.

Hurwitz Theorem: if $\omega \notin \mathbb{Q}$ then at least one among any 3 consecutive convergents to $\omega$ satisfies $\left|\frac{p_n}{q_n} - \omega\right| < \frac{1}{\sqrt{5}\, q_n^2}$

$\underbrace{\left|\frac{p_n}{q_n} - \omega\right| < \frac{1}{2q_n^2}}$

at least one of any 2

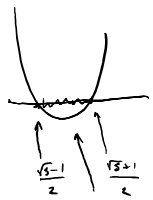
Proof: Assume thm is not true for $n = r-1, r, r+1$. Then

$$\underbrace{\frac{1}{q_{r-1}q_r}} = \left|\frac{p_{r-1}}{q_{r-1}} - \frac{p_r}{q_r}\right| = \left|\omega - \frac{p_{r-1}}{q_{r-1}}\right| + \left|\omega - \frac{p_r}{q_r}\right| \geq \underbrace{\frac{1}{\sqrt{5}}\left(\frac{1}{q_{r-1}^2} + \frac{1}{q_r^2}\right)}$$

So $\dfrac{1}{q_{r-1}q_r} \geqslant \dfrac{1}{\sqrt{5}}\left(\dfrac{q_{r-1}^2+q_r^2}{q_{r-1}^2 q_r^2}\right) \Rightarrow \left(\dfrac{q_{r-1}}{q_r}\right)^2 - \sqrt{5}\left(\dfrac{q_{r-1}}{q_r}\right)+1 \leq 0$

Also $\left(\dfrac{q_r}{q_{r+1}}\right)^2 - \sqrt{5}\left(\dfrac{q_r}{q_{r+1}}\right)+1 \leq 0$

$x^2 - \sqrt{5} + 1 = 0$ has roots $x = \dfrac{\sqrt{5}\pm 1}{2}$




length $= 1$ So

$$\frac{\sqrt{5}-1}{2} < \frac{q_{r-1}}{q_r}, \frac{q_r}{q_{r-1}}, \frac{q_r}{q_{r+1}}, \frac{q_{r+1}}{q_r} < \frac{\sqrt{5}-1}{2}$$

strict since these are rational

Exercise: see why $q_{r+1} > q_{r-1}$

$$\frac{q_{r+1}}{q_r} - \frac{q_{r-1}}{q_r} < 1$$

but $q_{r+1} = a_r q_r + q_{r-1}$

$q_r > q_{r+1} - q_r \Rightarrow q_{r+1} < q_r + q_{r-1} \Rightarrow a_r < 0$

contradiction. ∎

Exercise: see what you get when avoiding $a_r = 1$, replacing $\sqrt{5}$ by $\sqrt{8}$.

Exercise: find out what next number is $(\sqrt{5}, \sqrt{8}, \ldots)$

Reading: finish Ch 14 by monday.