Rec 11/7

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Sequence Space $X = \{ x = (x_n)_{n=1}^{\infty} \}$

ℓ^1: $\sum |x_i| =: \|x\|_1 < \infty$

$\left.\begin{array}{ll} c_0: & x_i \longrightarrow 0 \\ c: & x_i \longrightarrow \text{limit} \\ \ell^\infty: & \{x_i\} \text{ bdd} \end{array}\right\} \|x\|_\infty = \sup \|x_i\|$

$\ell^1 \subset c_0 \subset c \subset \ell^\infty$ (1) these spaces are complete

$\ell^1 = \underbrace{L^1(\mathbb{N}, P(\mathbb{N}), \text{counting measure})}$

complete, since absolutely convergent sequences converge by DCT.

ℓ^∞: $\underbrace{(x^{(k)})}_{\text{Cauchy sequence}}$, $x^{(k)} = (x_i^{(k)})_{i=1}^{\infty}$

Cauchy sequence: $\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $k, \ell > N_\varepsilon \Rightarrow \| x^{(k)} - x^{(\ell)} \| < \varepsilon$

$\Rightarrow \| x_j^{(k)} - x_j^{(\ell)} \| < \varepsilon \quad \forall j$

$\Rightarrow \quad x_j^{(k)} \longrightarrow x_j$ as $k \longrightarrow \infty$.

and $\|x\| \leq \|x^{(N_\varepsilon)}\| + \varepsilon < \infty$. ✓

So ℓ^∞ is also complete.

c_0, c are closed in ℓ^∞ and so are complete:

if $x^{(k)} \in c$, show $x \in c$ as well

$x_j^{(N)} \xrightarrow[j]{} L^{(N)}$, $\quad \| x_j^{(k)} - x_j^{(\ell)} \| < \varepsilon$ if $k, \ell > N, \ldots$

$L^{(n)} \longrightarrow L$ as $n \longrightarrow \infty$. ...

(2) Dual space $f : \ell' \longrightarrow \mathbb{C}$

$f = (f^j) \in$ Sequence Space

$f(x) = \sum f^j x_j$.

Need $\left| \sum f^j x_j \right| < \infty \quad \forall (x_i) \in \ell'$

Clearly true if $\sup_j |f^j| < \infty$

So $(\ell')^* \supset \ell^\infty$.

<u>Claim</u>: $(\ell')^* \cong \ell^\infty$

if (f^j) is unbounded, $\exists j$ s.t. $|f^j| > 1$ (suppose it's f'), $|f^2| > 2, \ldots, |f^j| > j$, etc.

Let $x_n = \frac{1}{n^2} \overline{\text{sign} f^n} \rightsquigarrow \sum x_n < \infty$ but $\left| \sum f^n x_n \right| = \infty$.