10/4

Friday, October 4, 2019 14:57

Knot Groups for Connected Sums

$$S_{K\#R} = S_K \underset{\text{Annulus}}{\cup} S_R$$

thickening of meridian = S^2 along which knots are connected

Seifert - van-Kampen:

$$\begin{tikzcd} \pi_1(A) \arrow[r] \arrow[d] & \pi_1(S_K) \\ \pi_1(S_R) & \end{tikzcd}$$

$$a \mapsto \tau_K, \quad a \mapsto \tau_R$$

$$\pi_{K\#R} \quad \underbrace{\pi_K * \pi_R}_{\text{free product}} / \langle \tau_K \cdot \tau_R^{-1} \rangle$$

$a \in \pi_K$, $\quad a = \tau_K^j c \quad c \in \pi_K'$

$b \in \pi_R$, $\quad b = \tau_R^i c \quad c \in \pi_R'$

$$\tau_K c \tau_K^{-1} = \hat{\tau}_K(c) \quad \ldots$$

any elt can be written as

$$\underset{\substack{\uparrow \\ \tau_k = \tau_R}}{\tau^j}\, c_1 * \tilde{c}_1 * \cdots * c_k * \tilde{c}_k \qquad \text{where } c_i \in \pi_K,\ \tilde{c}_i \in \pi_R$$

<u>Lemma</u> $\pi_{K\#R} = \mathbb{Z}_\tau \ltimes (\pi_K' * \pi_R')$

(Recall $\pi_K = \mathbb{Z}_\tau \ltimes \pi_K'$, $G' = [G,G]$).

<u>Lemma</u> If A, B are two gps $\neq 1$,

$$Z(A * B) = 1$$

<u>Corollary</u> If Q = composite knot,

$$Z(\pi_Q) \leqslant \langle \tau_Q \rangle$$

Torus Knots

$$S^3 = \{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2 \}$$

$$H_1 = \{ (z, w) \in S^3 : |z| \leq |w| \}$$

$$H_2 = \{ (z, w) \in S^3 : |z| \geq |w| \}$$

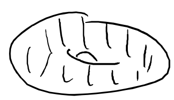
$$\Sigma = H_1 \cap H_2 = \{(z, w) \in S^3 : |z| = |w|\}$$

$$= S^1 \times S^1$$

$$D = \{ z \in \mathbb{C} : |z| < 1 \}$$

$$\rho_1 : D \times S^1 \longrightarrow H_1$$

$$(z, \xi) \longmapsto \rho_1(z, \xi) = (z, \xi \sqrt{2 - |z|^2})$$

$H_1 =$  $= D \times S^1,$

$$GL_2(\mathbb{Z}) = \{ A \in Mat_2(\mathbb{Z}) : \det(A) = \pm 1 \}$$

$$SL_2(\mathbb{Z}) = \{ A \in GL_2(\mathbb{Z}) : \det(A) = 1 \}$$

$$GL_2(\mathbb{Z}) \longrightarrow Homeom(\Sigma)$$

$$SL_2(\mathbb{Z}) \longrightarrow Homeom^+(\Sigma)$$

orientation preserving.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad a_{11}a_{22} - a_{12}a_{21} = \pm 1$$

$$A \cdot (z, w) = (z^{a_{11}} w^{a_{21}}, z^{a_{12}} w^{a_{22}})$$

$$\det A = 1 \Rightarrow A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Exercise: $A : \Sigma \to \Sigma$

When does it extend to $A : H_1 \xrightarrow{\approx} H_1$?

$$\Leftrightarrow \quad A = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

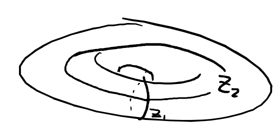
p, q coprime integers

$$\begin{bmatrix} p & s \\ q & r \end{bmatrix} = A$$

$pr - qs = 1$ (Bézout)

$$\xi_1 : t \longmapsto (e^{2\pi i t}, 1) \quad , \quad \xi_2 : t \longmapsto (1, e^{2\pi i t})$$

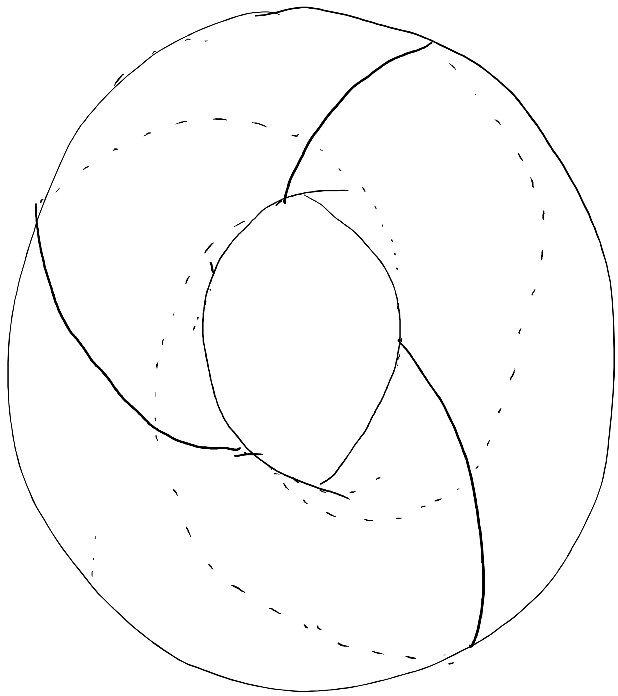
$\mathrm{im} = Z_1$ $\qquad$ $\mathrm{im} = Z_2$



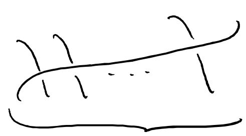
$$Z_1 \cap Z_2 = (1,1)$$

$$A \circ \xi_1 : t \longmapsto \left(e^{2\pi i p t}, e^{2\pi i q t}\right)$$

$\mathrm{im} = T_{p,q}$ torus knot.



$T_{3,2}$ = trefoil

$C =$ 

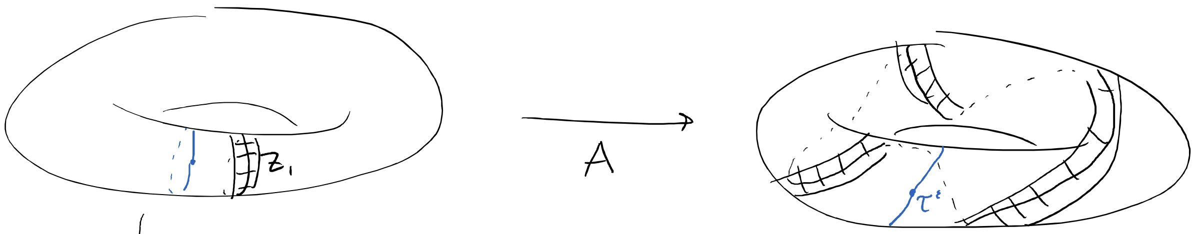
q

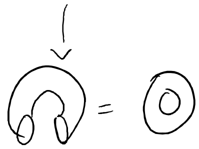
$$cl(C^p) = T_{p,q}$$

$$\underbrace{z^q - w^p = 0}$$

locus = variety in $\mathbb{C}^2$,

locus $\cap S^3 = T_{p,q}$





$$\hat{H}_i = H_i \setminus N(T_{p,q})$$

$$\hat{\Sigma} = \hat{H}_1 \cap \hat{H}_2 = \Sigma \setminus N(T_{p,q})$$

$$\hat{H}_1 \underset{\hat{\Sigma}}{\cup} \hat{H}_2 = S^3 \setminus N(T_{p,q})$$

$$z = [\tau^{\varepsilon}] \in \pi_1(\hat{\Sigma}) = \mathbb{Z}$$

$$\pi_1(\hat{H}_2) = \mathbb{Z} = \text{gen by core.}$$

$$t \longmapsto (\rho_1 e^{2\pi i a t}, \rho_2 e^{2\pi i b t})$$

$$H_1 = \{|z| \leq |w|\}$$

$$\begin{array}{ccc} \pi_1(\hat{\Sigma}) & \longrightarrow & \pi_1(H_1) \\ [\eta] & \longmapsto & u_1^b \end{array} \qquad u_1 = [(0, e^{2\pi i t})]$$

Seifert - Van-Kampen

$$\begin{array}{ccc} \pi_1(A) & \longrightarrow & \pi_1(H_1) \\ z & \longmapsto & u_1^q \\ \downarrow \quad \downarrow & & \end{array}$$

$$\begin{array}{cc} \big\downarrow & \big\downarrow \\ \pi_1(\hat{H}_2) & u_2^p \end{array}$$

$$\pi_1(\hat{H}_1 \underset{A}{\cup} \hat{H}_2) = \pi_1(\hat{H}_1) * \pi_1(\hat{H}_2) \big/ (u_1^q u_2^{-p})$$

$$= \langle u_1, u_2 \mid u_1^q = u_2^p \rangle .$$

$$z = u_1^q = u_2^p$$

$$z \in Z(\pi_{T_{p,q}})$$

$$\uparrow$$

center