$\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2$ $\qquad e_1 = (1,0), \quad e_2 = (0,1).$

$\cup$

$w = e_1 \otimes e_2 + e_2 \otimes e_1$. prove it's not simple.

$\neq$ ⟵ prove

$u \otimes v$ for any $u, v \in \mathbb{Z}^2$.

Basis in $\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2$ is $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$.

in these coords, $w \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ↙ non-degenerate (det $\neq 0$)

Any simple tensor $(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) \longleftrightarrow \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$ ↙ degenerate

$M$ : $R$-module, $I$ : ideal in $R$.

$$(R/I) \otimes M \cong M/IM \qquad \text{(Claim).}$$

<u>Proof</u> $R/I \times M \longrightarrow M/IM$

$(a \bmod I, u) \longmapsto au \bmod IM$

to show $a_1 = a_2 \bmod I \Rightarrow a_1 u = a_2 u \bmod IM$. obvious (subtract)

Bilinear. So we have a hom-sm $\varphi: R/I \otimes M \longrightarrow M/IM$.

Inverse: $\tilde{\psi}: M \longrightarrow R/I \otimes M$
$\qquad u \longmapsto (1 \bmod I) \otimes u$

$IM \subseteq \operatorname{Ker} \tilde{\psi}$: if $u = av$, $a \in I$, then $u \mapsto (1 \bmod I) \otimes av = (a \bmod I) \otimes v = 0$.

So $\tilde{\psi}$ induces a homomorphism $\psi$ $M/IM \longrightarrow (R/I) \otimes M$

And $\psi = \varphi^{-1}$:

$$u \xmapsto{\varphi} 1 \otimes u \xmapsto{\psi} u$$
$$a \otimes u \xmapsto{\psi} au \longmapsto 1 \otimes au = a \otimes u$$

$(\mathbb{Z}, \mathbb{Q})$

$R$: integral domain, $M$: $R$-module, $F$: field of fractions of $R$.

$F \otimes_R M$ ?

<u>Claim</u>: for the hom $\varphi: M \longrightarrow F \otimes_R M$ with $\varphi(u) = 1 \otimes u$,

$\operatorname{Ker} \varphi = \operatorname{Tor}(M)$. So if $M$ is torsion-free, this is injective.

<u>Proof</u>: If $u \in \operatorname{Tor}(M)$, let $a \neq 0$ s.t. $au = 0$. Then $\varphi(u) = 1 \otimes u = (a^{-1}a) \otimes u = a^{-1} \otimes au = 0$.

Every element of $F \otimes M$ can be written as $\frac{1}{d} \otimes u$ for $d \in F \setminus \{0\}$, $u \in M$.

indeed, for $w = \sum_{i=1}^{n} a_i \left(\frac{b_i}{c_i} \otimes u_i\right)$, let $d = c_1 \cdot \dots \cdot c_n$. Let $\tilde{c}_i = \prod_{j \neq i} c_j$

$$w = \sum_{i=1}^{n} a_i b_i \tilde{c}_i \left(\frac{1}{d} \otimes u_i\right) = \frac{1}{d} \otimes \left(\sum_{i=1}^{n} a_i b_i \tilde{c}_i \cdot u_i\right).$$

So $\forall u \in F \otimes_R M$, $u \in \left(\frac{1}{d} R\right) \otimes M$ for some $d = d(u)$ ($\frac{1}{d} R$ submodule of $F$).

Assume $\varphi(u) = 0$. That is, $1 \otimes u = 0$.

This means the pair $(1, u) \in K$ : submodule of distributivity rel$^{n}$s in the free module generated by $F \times M$.

So $(1, u)$ is a finite linear comb-ion of "relations" - elements of $K$.

All these rel$^{n}$s only use finitely many elements of $F$.

Let $d$ be a common denominator of these elements

Then $1 \otimes u = 0$ in the submodule $\frac{1}{d} R \otimes M \cong R \otimes M \cong M$

$\frac{a}{d} \otimes v \longmapsto a \otimes v \longmapsto av$

$$0 = 1 \otimes u = \frac{d}{d} \otimes u = \frac{1}{d} \otimes du \xrightarrow{\cong} du, \text{ so } \cong \Rightarrow du = 0.$$

<u>Theorem</u>: $M_1, M_2, N$ are $R$-modules.

Then $\mathrm{Hom}(M_1 \otimes M_2, N) \cong \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, N))$

$\cong \mathrm{Hom}(M_2, \mathrm{Hom}(M_1, N))$.

<u>proof</u>: Let $\varphi : M_1 \otimes M_2 \longrightarrow N$ -hom-sm

Then $\forall u \in M_1$, define $\varphi_u : M_2 \longrightarrow N$ by $\varphi_u(v) = \varphi(u \otimes v)$.

Then $\varphi_u \in \mathrm{Hom}(M_2, N)$ since $\forall v_1, v_2 \in M_2$, $\varphi_u(v_1 + v_2) = \varphi(u \otimes (v_1 + v_2))$

and since $\forall a \in R, v \in M_2$,

$$\begin{aligned} &= \varphi(u \otimes v_1 + u \otimes v_2) \\ &= \varphi(u \otimes v_1) + \varphi(u \otimes v_2) \\ &= \varphi_u(v_1) + \varphi_u(v_2). \end{aligned}$$

$\varphi_u(av) = \cdots a\varphi_u(v)$

So we have a mapping $\Phi : M_1 \longrightarrow \mathrm{Hom}(M_2, N)$

$\Phi(u) = \varphi_u$.

$\Phi$ is a hom-sm, so we have a map

$\varphi \longmapsto \Phi$.

Now let $\Psi : M_1 \longrightarrow \mathrm{Hom}(M_2, N)$.

define $\varphi : M_1 \otimes M_2 \longrightarrow N$ by

$$\varphi(u \otimes v) = \Psi(u)(v) \in N. \quad ?$$

bilinear wrt $u$ & $v$

So $\varphi$ is well-defined.

$\Psi \longmapsto \varphi$ is the inverse of $\varphi \longmapsto \Phi$ above.