Lec 8/29

Monday, August 29, 2016 9:11 AM

Def: Let $\mathbb{F}$ be an ordered field

A **Dedekind Cut** of $\mathbb{F}$ is a pair of subsets (A, B) of $\mathbb{F}$ satisfying

(1) $A \neq \phi$, $B \neq \emptyset$

(2) $\mathbb{F} = A \cup B$

(3) if $a \in A$ then $\forall b \in B$ $a < b$

remark: $(3) \Rightarrow A \cap B = \phi$. if $c \in A \cap B$ then $c < c$ X.

Def if (A, B) is a dedekind cut of $\mathbb{F}$, we say that c is a **cut point** of (A, B) if either c is the least element of B or c is the greatest element of A $\Rightarrow$ $A = (-\infty, c)$, $B = [c, \infty)$

or $A = (-\infty, c]$, $B = (c, \infty)$

P13: Completion axiom

Any Dedekind cut of $\mathbb{R}$ has a cut point.

Remark: $\mathbb{Q}$ does not satisfy P13

Indeed if an ordered field $\mathbb{F}$ satisfies P13 then $\mathbb{F} = \mathbb{R}$

Why does $\mathbb{Q}$ not satisfy P13?

Let $A_{\mathbb{Q}} = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$

$B_{\mathbb{Q}} = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$

$(A_{\mathbb{Q}}, B_{\mathbb{Q}})$ is a Dedekind cut of $\mathbb{Q}$

① $A_{\mathbb{Q}} \neq \phi$ $\quad (0 \in A_{\mathbb{Q}})$

$B_{\mathbb{Q}} \neq \phi$ $\quad (2 \in B_{\mathbb{Q}})$

② $\mathbb{Q} = A_{\mathbb{Q}} \cup B_{\mathbb{Q}}$ $\quad (x^2 = 2 \Rightarrow x \notin \mathbb{Q})$

③ $a \in A_{\mathbb{Q}}$, $b \in B_{\mathbb{Q}} \Rightarrow a < b$

<u>Case 1</u> $a \le 0$ then $a < b$ since $b > 0$

<u>Case 2</u> $a > 0$ and $a^2 < 2$ $a < b$ since $b^2 > 2$ and $b^2 > a^2$ and $b, a > 0$

Suppose that $(A_{\mathbb{Q}}, B_{\mathbb{Q}})$ has a cut point $c \in \mathbb{Q}$.

This will lead to contradiction.

We will construct a sequence $\{a_0, a_1, a_2, \dots\} \subseteq A_{\mathbb{Q}}$
and $\{b_0, b_1, b_2, \dots\} \subseteq B_{\mathbb{Q}}$

satisfying $a_0 \le a_1 \le a_2 \le \dots < \dots \le b_2 \le b_1 \le b_0$

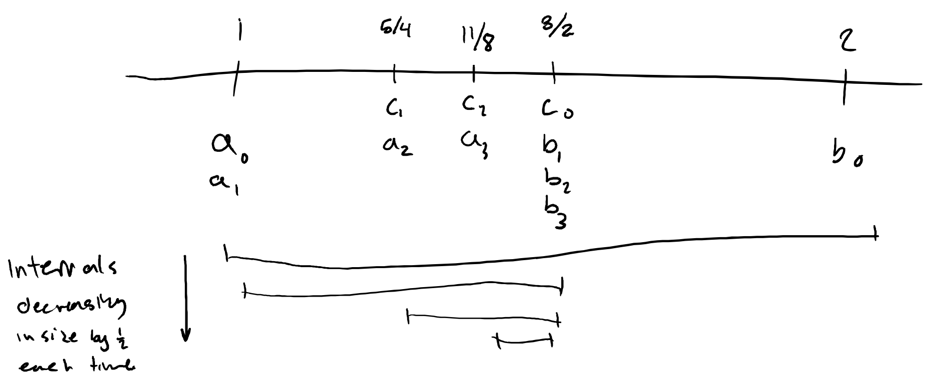
Let $a_0 = 1 \in A_{\mathbb{Q}}$

$b_0 = 2 \in B_{\mathbb{Q}}$

based on $a_n < b_n$ let $c_n = \frac{1}{2}(a_n + b_n)$

If $c_n \in A_{\mathbb{Q}}$ let $a_{n+1} = c_n$ $b_{n+1} = b_n$

else if $c_n \in B_{\mathbb{Q}}$ let $a_{n+1} = a_n$ $b_{n+1} = c_n$



Continue this indefinitely.

$[a_{n+1}, b_{n+1}]$ is either the left half or the right half of $[a_n, b_n]$.

$b_n - a_n = \left(\frac{1}{2}\right)^n \to 0$

$c > 0$ because $c \in [1, 2]$

The cut point of $(A_{\mathbb{Q}}, B_{\mathbb{Q}})$, $c \in [a_n, b_n]$ so $c^2 \in [a_n^2, b_n^2]$

The cut point of $(A_{\mathbb{Q}}, B_{\mathbb{Q}})$, $c \in [a_n, b_n]$ so $c^2 \in [a_n^2, b_n^2]$

also $2 \in [a_n^2, b_n^2] \; \forall n$ by definition of $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$.

$c^2, 2 \in [a_n^2, b_n^2] \; \forall n \Rightarrow |c^2-2| \leq b_n^2 - a_n^2 = (b_n - a_n)(b_n + a_n) = \frac{1}{2^n}(b_n + a_n) \leq \frac{1}{2^n}\cdot 4$

$\underbrace{|c^2-2|}_{\text{positive rational \# } \frac{p}{q} \text{ with } p,q \in \mathbb{Z}^+} \leq \frac{1}{2^{n-2}} < \frac{1}{n-2}$ (because $2^{n-2} \geq n-2$) $\forall n$.

$\frac{1}{q} \leq \frac{p}{q} \leq \frac{1}{n-2}$ can't be true for all n

$\frac{p}{q} \geq \frac{1}{q}$ so take $n = q+3$, $\frac{1}{q} > \frac{1}{q+1}$

for reals take $A = \{x \in \mathbb{R} : x \leq 0 \text{ or } x^2 < 2\}$

$B = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 \geq 2\}$

by P13 a cut point c exists

Define a_n, b_n as before

$|c^2-2| \leq \frac{1}{n-2} \quad \forall n > 2$

infinitesimal $< \frac{1}{n} \; \forall n$

infinitesimals don't exist in $\mathbb{R}$

Conclusion: $|c^2-2|$ is either 0 or an infinitesimal

if we show that $\mathbb{R}$ has no infinitesimals then we must have $c^2 = 2$, $c \in \mathbb{R}$

A pseudo-infinite element Ω in an ordered field $\mathbb{F}$ satisfies:

$\Omega > n = 1+1+\dots+1$ for any $n \in \mathbb{Z}^+$

Ω pseudo-infinite $\Rightarrow \frac{1}{\Omega}$ infinitesimal

Theorem $\mathbb{R}$ contains no pseudo-infinite elements

Proof: assume pseudo-infinite elements exist.

let $B \subseteq \mathbb{R}$ be the pseudo-infinite elements in $\mathbb{R}$

let $A = \mathbb{R} \setminus B$

then (A, B) is a dedekind cut of $\mathbb{R}$

by P13 (A, B) has a cut point c in either A or B

Case 1: $c \in A$

c not pseudo-infinite

but $c+1$ is pseudo-infinite

$\underset{\text{not p.i.}}{c < N}$ for some integer $N \Rightarrow \underset{\text{not p.i.}}{c+1 < N+1}$

Case 2: $c \in B$

c pseudo-infinite

but $c-1$ is not pseudo-infinite

$\underset{\text{not p.i.}}{c-1 < N-1}$ for some $N \in \mathbb{Z}^+ \Rightarrow \underset{\text{not p.i.}}{c < N}$

So because $|c^2-2|$ is either 0 or infinitesimal and infinitesimals $\notin \mathbb{R}$, $|c^2-2| = 0$ so $c^2 = 2 \Rightarrow c = \sqrt{2} \in \mathbb{R}$.