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Christoffel symbols actually don't form a tensor.

R^i_{jkl} - riemann curvature tensor is though.

PAP^{-1} linear transfm $M \otimes M^*$

$(P^{-1})^T A P^{-1}$ bilinear form $M^* \otimes M^*$.

M_1, M_2, N. Bilinear mappings $M_1 \times M_2 \longrightarrow N$.

$$\cong$$

$$\mathrm{Hom}(M_1 \otimes M_2, N)$$

$\cong$ - if M_1, M_2, N are free of finite rank.

$$N \otimes (M_1 \otimes M_2)^* = N \otimes M_1^* \otimes M_2^*$$

Bilinear forms on M are bilinear maps $M \times M \to R$,

So they are in $R \otimes M^* \otimes M^* = M^* \otimes M^*$ - 2 times covariant.

$$\overset{u}{(\ b_i\ )} \cdot (\ a_{ij}\ ) \cdot \overset{v}{(c_j)} = \sum a_{ij} b_i c_j$$

bilinear form $\beta : M^2 \longrightarrow R$ is symmetric if

$$\beta(u,v) = \beta(v,u) \quad \forall u, v \in M.$$

it's antisymmetric if

$$\beta(u,v) = -\beta(v,u) \quad \forall u, v \in M.$$

This is so iff β is represented by a symmetric (or antisymmetric) tensor from $M^* \otimes M^*$.

$$\mathrm{Hom}(M_1 \otimes M_2, N) \cong \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, N))$$

$$N \otimes (M_1 \otimes M_2)^* = (N \otimes M_2^*) \otimes M_1^*$$ for free modules of finite rank

$\varphi \in \mathrm{End}(M)$ where $M \cong R^n$

$\varphi \in M \otimes M^*$ contraction maps $M \otimes M^* \longrightarrow R$
$u \otimes f \longmapsto f(u)$

$$A_\varphi = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \Rightarrow \varphi = \sum a_{ij}\, u_i \otimes f_j$$

where $\{u_1, \dots, u_n\}$ – basis in M
$\Rightarrow \{f_1, \dots, f_n\}$ – basis in M^*

So contraction of φ is trace φ.

if $A_\varphi = (a_{ij})$ is the matrix of φ in <u>any</u> basis (& its dual),

then trace φ = trace $A_\varphi = \sum_i a_{ii}$

$\det \varphi = ?$

Let $M \cong R^n$, $\mathcal{T}^k(M) = \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$ - free of rank n^k.

basis $\{u_1, \ldots, u_n\}$

basis is $\{u_{i_1} \otimes \cdots \otimes u_{i_k} : 1 \le i_1, \ldots, i_k \le n\}$.

kth component of symmetric tensor algebra $\rightarrow$ $S^k(M) = \mathcal{T}^k(M) / C^k(M)$ where $C^k(M)$ is gen. by $u \otimes v - v \otimes u$

$\uparrow$

free w/ basis $\{u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_k} : 1 \le i_1 \le i_2 \le \cdots \le i_k \le n\}$.

kth component of exterior algebra $\rightarrow$ $\Lambda^k(M) = \mathcal{T}^k(M) / A^k(M)$ where $A^k(M)$ is generated by $u \otimes u$.

$\uparrow$

free w/ basis $\{u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$

Rank of $\Lambda^k(M)$ is $\binom{n}{k}$, which is 0 if $k > n$.
1 if $k = n$.

Exterior algebra of M is $\Lambda(M) = R \oplus M \oplus \Lambda^2(M) \oplus \cdots \oplus \Lambda^n(M)$

Now $\Lambda^n(M) \cong R$, basis is $\{u_1 \wedge \cdots \wedge u_n\}$.

$\forall \omega \in \Lambda^n(M)$, $\omega = a(u_1 \wedge \cdots \wedge u_n)$.

Let $\varphi : M \to N$. Then $\forall k$, $\varphi^{\otimes k} : M^{\otimes k} \to N^{\otimes k}$

$$\varphi^{\otimes k} = \underbrace{\varphi \otimes \cdots \otimes \varphi}_{k} : u_1 \otimes \cdots \otimes u_k \mapsto \varphi(u_1) \otimes \cdots \otimes \varphi(u_k)$$

$\varphi: M \to N,\quad \psi: N \to K, \Rightarrow (\psi \circ \varphi)^{\otimes l} = \psi^{\otimes l} \circ \varphi^{\otimes l}$

the ideals $C^k_{(M)} \longmapsto C^k_{(N)}$ $A^k_{(M)} \longmapsto A^k_{(N)}$ under $\varphi^{\otimes k}$, and $S^k = T^k/C^k$, $\Lambda^k = T^k/A^k$

So we have hom-sms $S^k(M) \to S^k(N)$, $\Lambda^k(M) \xrightarrow{\Lambda^k\varphi} \Lambda^k(N)$.

Let $\varphi: \underset{\substack{\cong\\ R^n}}{M} \to M$. Then $\Lambda^n\varphi : \underset{\substack{\cong\\ R}}{\Lambda^n(M)} \longrightarrow \underset{\substack{\cong\\ R}}{\Lambda^n(M)}$, and any of these is $a \mapsto ca$ for some $c \in R$.

$$\det(\varphi) = \Lambda^n\varphi(1)$$

Again, doesn't depend on choice of basis.

if $\{u_1, \dots, u_n\}$ is a basis in M, then $\Lambda^n\varphi(u_1 \wedge \cdots \wedge u_n) = \varphi(u_1) \wedge \cdots \wedge \varphi(u_n)$

$= c\,(u_1 \wedge \cdots \wedge u_n)$

$c = \det \varphi$.

eg $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad \varphi(u_1) = a_{11}u_1 + a_{21}u_2$

$\varphi(u_2) = a_{12}u_1 + a_{22}u_2$

and $\varphi(u_1) \wedge \varphi(u_2) = \cancel{a_{11}a_{12}u_1 \wedge u_1} + a_{11}a_{22}\,u_1 \wedge u_2 + a_{21}a_{12}\,u_2 \wedge u_1 + \cancel{a_{21}a_{22}u_2 \wedge u_2}$

$= (a_{11}a_{22} - a_{21}a_{12})\, u_1 \wedge u_2$

① $\det\varphi$ depends on φ, not on choice of basis. $\det(PAP^{-1}) = \det(A)$.

② $\det \varphi \circ \psi = \det \varphi \cdot \det \psi$ since $\Lambda^n(\varphi \circ \psi) = \underbrace{\Lambda^n \varphi}_{a \mapsto ca} \circ \underbrace{\Lambda^n \psi}_{a \mapsto da} = a \mapsto cda$

③ if φ is invertible, then $\det \varphi \in R^\times$

(converse is also true).