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Friday, March 29, 2019 12:03

Missed a lot...

If F is a field, K is a prime subfield ($\to \mathbb{Z}_p$ or $\mathbb{Q}$),

then $\forall \varphi \in \mathrm{Aut}(F)$, $\varphi\big|_K = id_K$.

Aut(K/F)

$$\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{1, \varphi\} \text{ where } \varphi(z) = \bar{z}.$$

$$\cong \mathbb{Z}_2.$$

because $i \longmapsto \pm i$ and $\mathbb{C} = \mathbb{R}(i)$

any automorphism is identical on prime subfield

$$\mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \mathrm{Aut}(\mathbb{Q}(\sqrt{2}))$$

$$= \{1, \varphi\} \quad \text{where } \varphi(a+b\sqrt{2}) = a-b\sqrt{2}$$

$$\cong \mathbb{Z}_2$$

since $\sqrt{2} \mapsto \pm\sqrt{2}$.

$$\mathrm{Aut}(\mathbb{Q}(\sqrt{2},\sqrt{3})) \cong \mathbb{Z}_2^2 = \{1, \varphi_1, \varphi_2, \varphi_1\varphi_2\}$$

$\sqrt{2} \longrightarrow \pm\sqrt{2}$ $\qquad$ $\varphi_1 : \sqrt{2} \longmapsto -\sqrt{2}$

$\sqrt{3} \longrightarrow \pm\sqrt{3}$ $\qquad$ $\varphi_2 : \sqrt{3} \longmapsto -\sqrt{3}$

$$\begin{tikzcd}
\mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[d, dash, "x^2-3 \mapsto x^2-3"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[r] \arrow[d, dash] & \mathbb{Q}(\sqrt{2}) \\
\mathbb{Q} \arrow[ur, dash] &
\end{tikzcd}$$

and $m_{\sqrt{3},\mathbb{Q}(\sqrt{2})}$ is still x^2-3

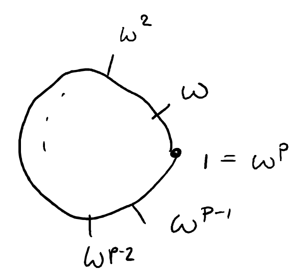
So all four choices are possible

($\sqrt{2}$ choice is independent of $\sqrt{3}$)

$$\left(\left(\text{Proved later maybe}\right) \operatorname{Aut}(F(t)) = \left\{ t \longmapsto \frac{at+b}{ct+d} \text{ where } ad-bc \neq 0 \right\}\right)$$

p is prime

$\operatorname{Aut}(\mathbb{Q}(\omega))$, $\quad \omega = \sqrt[p]{1} = e^{2\pi i/p}$



$\forall\, k \in \{1, \ldots, p-1\}$, ω^k is conjugate to ω.

$m_\omega = 1 + x + \cdots + x^{p-1} = \Phi_p(x)$.

$\forall\, k$, let $\varphi_k(\omega) = \omega^k$.

$|\mathrm{Aut}| = p-1$

$$\varphi_k \circ \varphi_\ell = \varphi_{k\ell}$$

So $\mathrm{Aut} \cong \mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$

$\varphi_k \longleftrightarrow k$

for $p=3$, group is $\mathbb{Z}_3^\times = \mathbb{Z}_2$

$\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})) = 1$ $\quad$ $\sqrt[3]{2}$ has conjugates $\omega\sqrt[3]{2}$, $\omega^2\sqrt[3]{2}$, $\omega = e^{2\pi i/3}$.

$\omega \notin \mathbb{Q}(\sqrt[3]{2})$ so no other auts.

$\mathbb{Q}(\sqrt[3]{2}, \omega)$ - splitting field of x^3-2

$\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \omega))$

$$\sqrt[3]{2} \to \sqrt[3]{2}, \quad \sqrt[3]{2} \to \omega\sqrt[3]{2}, \quad \sqrt[3]{2} \to \omega^2\sqrt[3]{2}$$

$$\omega \to \omega, \quad \omega \to \omega^2$$

all are independent,

So $\mathrm{Aut} \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathrm{Aut}\left(\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}, \omega)\right)$

$\sqrt[3]{2}$

$\sqrt[3]{3}$

ω

this time, the choices are not independent.