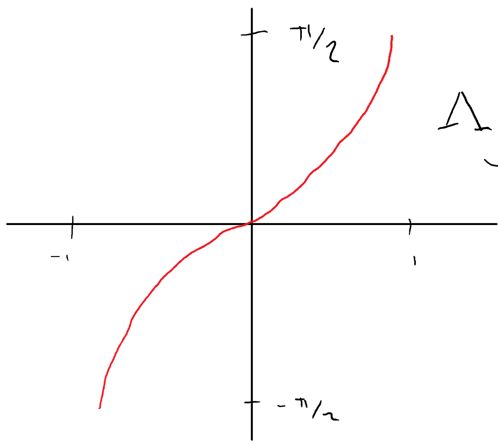
Lec 11/10

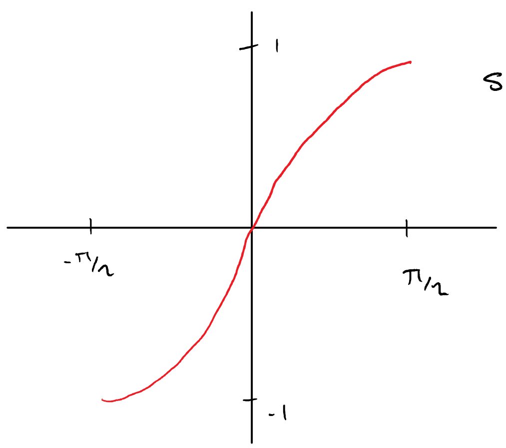
Thursday, November 10, 2016 9:09 AM

Constructing sin and cos

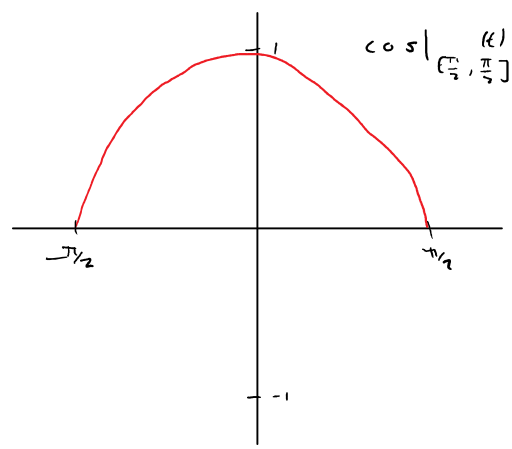


$\Lambda = \sin^{-1}$

→ defined by an integral



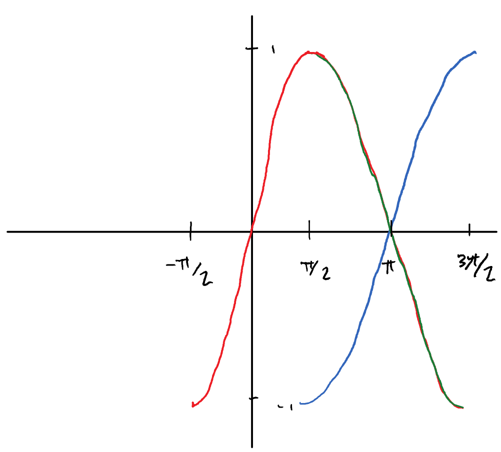
$\sin|_{[-\pi/2,\, \pi/2]}$

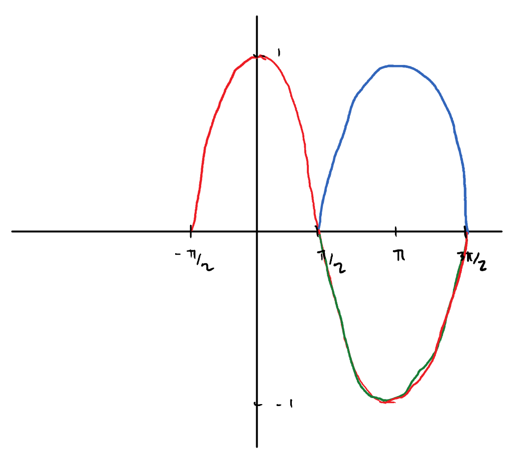


$\cos|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(t) = \sqrt{1-\sin^2(t)}$

shift

flip





Prop 5: Let $a < b < c$. Suppose f: ...

Prop 5: Let $a<b<c$, suppose $f:[a,b]\to\mathbb{R}$ and $g:[b,c]\to\mathbb{R}$ are continuous

(1) if $f(b)=g(b)$, then $h=f\cup g$ is cts on $[a,c]$

(2) If $\underbrace{f(b)=g(b)}_{}$ and f', g' are defined on (a,b) and (b,c) respectively

and $\lim_{x\to b^-} f'(x) = \lim_{x\to b^+} g'(x)$, then h' is defined on (a,c). → use L'H.

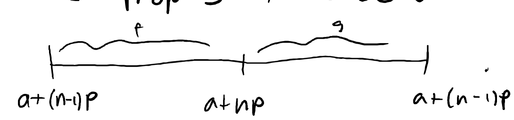
Theorem 8 (extending by periodicity): if $h:[a,a+P]\to\mathbb{R}$ is continuous and $h(a)=h(a+P)$ then h extends to a cts periodic function on $\mathbb{R}$. $\hat{h}$.

also, if h' exists on $(a,a+P)$ and $\lim_{x\to a^+} h'(x) = \lim_{x\to a+P^-} h'(x)$ then $\hat{h}$ is diffable on $\mathbb{R}$.

Proof sketch: Given $t\in\mathbb{R}$, find integer n s.t. $a+nP\le t< a+(n+1)P$

$n=\lfloor \frac{t-a}{P}\rfloor$. define $\hat{h}(t)=h(t-nP)$.

Use Prop 5 to show $\hat{h}$ is cont. and diffable.



$\sin'(t)=\cos(t)$
$\cos'(t)=-\sin(t)$ } for all t.

$\sin''(t)=\cos'(t)=-\sin(t)$

$\cos''(t)=-\sin'(t)=-\cos(t)$

both $\sin(t)$, $\cos(t)$ satisfy $f''(t)+f(t)=0$ $\forall t$.

Lemma if $(*)$ $f''(t)+f(t)=0$ and $f'(0)=f(0)=0$, then $f(t)=0$ $\forall t$.

Proof: multiply $(*)$ by $2f'(t)$:

$$0=2f'(t)f''(t)+2f'(t)f(t)$$

$$=\frac{d}{dt}\left(f'(t)^2+f(t)^2\right)$$

$\Rightarrow \quad f'(t)^2 + f(t)^2 = C.$

plug in $t=0 \Rightarrow C=0 \Rightarrow f(t)=0$ ∎

Theorem If $f''(t)+f(t)=0$ for all t then

$$f(t) = f(0)\cos(t) + f'(0)\sin(t)$$

Proof: Let $g(t) = f(t) - f(0)\cos(t) - f'(0)\sin(t)$

Then $g''(t) + g(t) = 0 \ \forall t$. Also, $g(0) = f(0) - f(0) = 0$

$g'(0) = f'(0) - f'(0) = 0$

So by lemma, $g(t) = 0 \ \forall t$ so

$f(t) = f(0)\cos(t) + f'(0)\sin(t)$ ∎

Theorem (1) $\sin^2(t) + \cos^2(t) = 1$

(2) $\sin(-t) = -\sin(t)$

(3) $\cos(-t) = \cos(t)$ $\quad \forall t, a.$

(4) $\sin(t+a) = \sin(a)\cos(t) + \sin(t)\cos(a)$

(5) $\cos(t+a) = \cos(t)\cos(a) - \sin(t)\sin(a)$

Proofs: (1) $\frac{d}{dt}\left(\sin^2(t) + \cos^2(t)\right) = 2\sin(t)\cos(t) - 2\cos(t)\sin(t) = 0$

so $\sin^2(t) + \cos^2(t) = C$, plug in $0 \Rightarrow$ ✓ $= 0$. ∎

(2) $f(t) = \sin(-t)$, $f'(t) = -\cos(-t)$ so $f(0)=0$, $f'(0)=-1$

using theorem above, $f(t) = -\sin(t)$. ∎

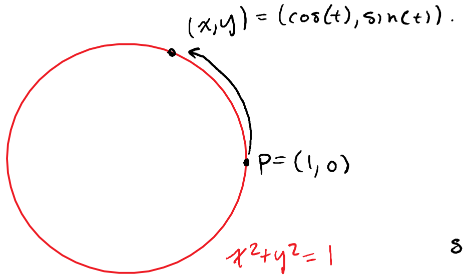
(3) Apply Theorem above. ∎

(4) $f(t) = \sin(t+a)$, $f'(t) = \cos(t+a)$, $f(0) = \sin(a)$, $f'(0) = \cos(a)$

applying thm above: $f(t) = \sin(a)\cos(t) + \cos(a)\sin(t)$. ∎

(5) Similar ∎

If we assume that $\cos(t)$, $\sin(t)$ exist as defined by physicists, we could proceed as follows:



$\frac{d}{dx}(x^2+y^2=1) \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

and $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1$ (velocity vector $= \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$)

$x\frac{dx}{dt} = -y\frac{dy}{dt} \Rightarrow x^2\left(\frac{dx}{dt}\right)^2 = y^2\left(\frac{dy}{dt}\right)^2$

$x^2\left(1-\left(\frac{dy}{dt}\right)^2\right) = y^2\left(\frac{dy}{dt}\right)^2$

$x^2 - x^2\left(\frac{dy}{dt}\right)^2 = y^2\left(\frac{dy}{dt}\right)^2$

$x^2 = \underbrace{(x^2+y^2)}_{=1}\left(\frac{dy}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2$

so $\frac{dy}{dt} = \pm x$

but $\frac{dy}{dt} > 0$ at $t=0$

so $\frac{dy}{dt} = x$ → extend to all t.

$\left(\frac{dx}{dt}\right)^2 = 1 - \left(\frac{dy}{dt}\right)^2$
$= 1 - x^2$
$= y^2$

Similarly, $\frac{dx}{dt} = -y$

Can do this for other curves $F(x,y) = 0$.