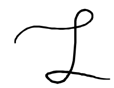
Lec 1/17

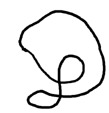
Wednesday, January 17, 2018 14:20

Def γ is a curve in $\mathbb{R}^d$ if it is a continuous map from an interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^d$.



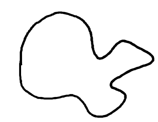
A similar definition works for curves in a metric space or, more generally, in a topological space.

Defn an arc is a curve which is one-to-one.

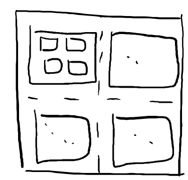


Defn a loop in $\mathbb{R}^d$ is a continuous map from $S^1 = \{z \in \mathbb{C} : |z|=1\}$ into $\mathbb{R}^d$ ← or a more general space. $= \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$

Defn A Jordan curve is a one-to-one loop.



A jordan curve w/ area > 0:



$K_0 = [0,1]^2$.

K_1 = Union of four disjoint congruent squares constructed inside K_0 as shown w/ total area 3/4.

K_2 = union of 16 squares, total area $\frac{7}{16}$.

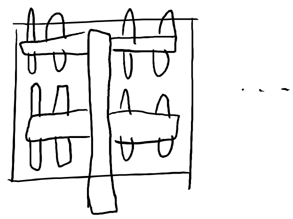
⋮

K_n = union of 4^n squares, total area $\frac{1}{2} + \left(\frac{1}{4}\right)^n$

⋮

$K = \bigcap_{n=1}^{\infty} K_n$. Area$(K) = \frac{1}{2}$ "fat Cantor set"

K is not a jordan curve but it is a subset of one:



There is a sequence (γ_n) of jordan curves s.t.

$\forall n, \ \forall z \in \mathbb{Z} \quad |\gamma_{n+1}(z) - \gamma_n(z)| < \frac{1}{2^n}$ (see picture).

this is Uniformly Cauchy (?)

So $\exists \gamma : S^1 \longrightarrow \mathbb{R}^d$ s.t. $\gamma_n \longrightarrow \gamma$ uniformly, so γ is continuous.

thus γ is a loop. Also, γ is 1-1 (think about it & check).

Claim: $K \subseteq \text{Range}(\gamma)$.

Let $p \in K$. then $\forall n$, $p \in K_n$, so $\exists z_n \in S_1$

so that $|p - \gamma_n(z_n)| < \frac{1}{2^n}$.

Now $\forall z \in S^1$, $|\gamma_n(z) - \gamma(z)| \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$

So $|p - \gamma(z_n)| < \frac{1}{2^n} + \frac{1}{2^{n-1}} = \frac{3}{2^n} \longrightarrow 0$.

Now $\exists$ subsequence z_{n_k} which converges since $\{z_n\} \subset S^1$ is bounded.

and since S^1 is closed, $z_{n_k} \longrightarrow z_0 \in S^1$.

on one hand, $\gamma(z_{n_j}) \to p$. also, $\gamma(z_{n_j}) \longrightarrow \gamma(z_0)$ since γ cts.

So $\gamma(z_0) = P$. Thus $K \subset \gamma(S^1)$. and so $\text{Area}(\gamma(S^1)) \geq \frac{1}{2}$.

<u>Defn</u> let $L \in (0, \infty)$. An L-periodic curve in $\mathbb{R}^d$ is a continuous map $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^d$ so that $\forall t \in \mathbb{R}$, $\gamma(t+L) = \gamma(t)$.

(generalizes to metric/topological spaces).

<u>Remark</u>. Let γ be a loop, let $L \in (0,\infty)$. define α on $\mathbb{R}$ by

$\alpha(t) = \gamma(e^{2\pi i t/L})$. α is L-periodic.

<u>Conversely</u>, let α be an L-periodic curve. Then $\exists$ a unique loop γ s.t. $\forall t \in \mathbb{R}$, $\alpha(t) = \gamma(\underbrace{e^{2\pi i t/L}}_{p(t)})$.

Check that γ is continuous.

(maybe $\mathbb{R}^d$)

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\alpha"] \arrow[d, "p"'] & X \\ S^1 \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

Check:

If $p(t_1) = p(t_2)$ then $\exists n \in \mathbb{Z}$ s.t. $t_2 = t_1 + nL$.

so $\alpha(t_1) = \alpha(t_2)$. This is why γ is well defined.

<u>Defns</u> let $k \in \mathbb{N}$.

(a) a C^k curve in $\mathbb{R}^d$ is a continuous $\alpha : I \longrightarrow \mathbb{R}^d$ ($I \subseteq \mathbb{R}$ is an interval) so that the first k derivatives of α; $\frac{d\alpha}{dt}, \dots, \frac{d^k\alpha}{dt^k}$; exist and are continuous in I.

(use approp 1-sided derivatives at any included endpoints of I).

(b) a C^k-<u>Regular</u> curve in $\mathbb{R}^d$ is a C^k curve $\alpha : I \longrightarrow \mathbb{R}^d$ such that $\frac{d\alpha}{dt}$ is never 0 in I.

(c) a regular curve in $\mathbb{R}^d$ is a C^1-regular curve (unless o.w. stated).

<u>Terminology</u> Let $\alpha : I \longrightarrow \mathbb{R}^d$ be a diff'ble curve.

(a) $\forall t_0 \in I$, $\frac{d\alpha}{dt}(t_0)$ is called the velocity vector of α at t_0.

(b) The vec-val'd fn $\frac{d\alpha}{dt}$ is called the velocity vector field of α.

(c) $\left|\frac{d\alpha}{dt}(t_0)\right|$ is the speed of α at t_0.

(d) if α is C^1-regular curve, the tangent vector field is

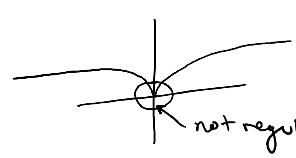
$$T(t_0) = \frac{\frac{d\alpha}{dt}(t_0)}{\left|\frac{d\alpha}{dt}(t_0)\right|}.$$ This is well-defined since α is regular.

(e) if α is C^1-regular and $t_0 \in I$ then the tangent line to α at t_0 is the straight line parallel to $T(t_0)$ through $\alpha(t_0)$

$$\ell = \{\alpha(t_0) + \lambda T(t_0) : \lambda \in \mathbb{R}\}.$$

$$= \{\alpha(t_0) + \lambda \frac{d\alpha}{dt}(t_0) : \lambda \in \mathbb{R}\}.$$

<u>eg</u> $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$. $\alpha(t) = (t^3, t^2)$. $\alpha(t)$ is C^∞ but $\alpha'(t) = (3t^2, 2t)$ which is 0 at 0.



not regular here (cusp)

image: $\{(t^3, t^2) : t \in \mathbb{R}\}$

$\{(x, x^{2/3}) : x \in \mathbb{R}\}$

Sup bruh

Suh dude.

Vlas is

Super cool

for letting me

Draw on this ♡