Lec 11/7

Monday, November 7, 2016 9:11 AM

$\ln : (0,\infty) \longrightarrow (-\infty,\infty)$ is 1-1 and onto

given $x \in (-\infty, \infty)$

Can find $n \in \mathbb{N}$ s.t. $\ln(2^{-n}) = -n\ln(2) \leq x \leq n\ln(2) = \ln(2^n)$

so by IVT $\exists\, y$ s.t. $\ln(y) = x$.

Inverse function:

$\exp : (-\infty,\infty) \to (0,\infty)$

Definition: $y = \exp(x) \Leftrightarrow x = \ln(y)$

recall: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$f = \ln$, $f^{-1} = \exp$

$$\frac{d}{dx}(\exp(x)) = \frac{1}{\ln'(\exp(x))}$$

but $\ln'(u) = \frac{1}{u}$

$$= \exp(x)$$

$\ln(uv) = \ln(u) + \ln(v)$

$\ln(u/v) = \ln(u) - \ln(v)$

$\ln(u^r) = r\ln(u)$ where $r \in \mathbb{Q}$

Corresponding properties:

(1) $\exp(x+y) = \exp(x)\exp(y)$

(2) $\exp(x-y) = \exp(x)/\exp(y)$

(3) $\exp(x)^r = \exp(rx)$ where $r \in \mathbb{Q}$

Proof: let $u = \exp(x)$ so $\ln(u) = x$
$v = \exp(y)$ $\ln(v) = y$

(1) $x+y = \ln(u) + \ln(v) = \ln(uv)$
So $\exp(x+y) = uv = \exp(x)\exp(y)$

(2) $x-y = \ln(u) - \ln(v) = \ln(u/v)$
so $\exp(x-y) = u/v = \exp(x)/\exp(y)$

(3) $rx = r\ln(u) = \ln(u^r)$
So $\exp(rx) = u^r = \exp(x)^r$

Want to define a^λ where $\lambda \notin \mathbb{Q}$ as "$\lim_{r\to\lambda}$" a^λ where $r \in \mathbb{Q}$

Theorem: If $a > 0$ and $\lambda \in \mathbb{R}$ then $\lim_{r\to\lambda} a^r = \exp(\lambda \ln(a))$

Proof: We have $a = \exp(\ln(a))$.
Hence by (3), $\forall r \in \mathbb{Q}$, $a^r = \exp(\ln(a))^r = \exp(r\ln(a))$
so $\lim_{r\to\lambda} a^r = \lim_{r\to\lambda} \exp(r\ln(a)) \underset{\uparrow \text{ exp cts.}}{=} \exp(\lim_{r\to\lambda} r\ln(a)) = \exp(\lambda\ln(a))$ ∎

Definition if $a > 0$ and $x \in \mathbb{R}$, $a^x = \exp(x\ln(a))$.

Definition $e = \exp(1) \Leftrightarrow \ln(e) = 1$ so $e^x = \exp(x)$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(\exp(x\ln(a))) = \exp(x\ln(a))\frac{d}{dx}(x\ln(a))$$
$$= \exp(x\ln(a))\ln(a) = a^x \ln(a)$$

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C \quad \text{provided } a \neq 1$$

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x \, dx = e^x + C$$

<u>Theorem</u>: $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ is an increasing sequence and $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

<u>Proof</u>: $(1+\frac{1}{n})^n$ extend to a function $f(x) = (1+\frac{1}{x})^x$ for $x > 0$.

it suffices to show $\lim_{x\to\infty} f(x) = e$ and f is increasing.

let $u = \frac{1}{x}$, $g(u) = (1+u)^{\frac{1}{u}}$ so $f(x) = g(\frac{1}{x})$

so the above $\Leftrightarrow$ g is decreasing and $\lim_{u\to 0^+} g(u) = e$

$$g'(u) = \frac{d}{du}\left(\exp\left(\tfrac{1}{u}\ln(1+u)\right)\right) = \exp(\tfrac{1}{u}\ln(1+u))\frac{d}{dx}\left(\tfrac{1}{u}\ln(1+u)\right)$$

$$= \exp(\tfrac{1}{u}\ln(1+u))\left(\frac{-1}{u^2}\ln(1+u) + \frac{1}{u}\frac{1}{1+u}\right)$$

$$= \exp(\tfrac{1}{u}\ln(1+u))\left[\frac{u-(1+u)\ln(1+u)}{u^2(1+u)^2}\right]$$

$h(u) := u - (1+u)\ln(1+u)$, g' has same sign as h.

$h'(u) = -\ln(1+u) < 0$ for $u \in (0,\infty)$

$h(0) = 0$, so $h(u) < 0$ on $(0,\infty)$

so $g' < 0$ so g is decreasing so f is increasing.

now $$\lim_{x\to\infty} f(x) = \lim_{u\to 0^+} g(u) = \lim_{u\to 0^+} \exp\left(\frac{\ln(1+u)}{u}\right) = \exp\left(\lim_{u\to 0^+}\frac{\ln(1+u)}{u}\right)$$

$$= \exp\left(\lim_{u\to 0^+}\frac{1}{1+u}\right) \quad (\text{L'H})$$

$$= \exp(1)$$

$$= e. \quad \blacksquare$$

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

<u>Theorem</u>: if $x \geq 0$, $n \in \mathbb{N}$, then $e^x \geq \sum_{j=0}^{n} \frac{x^j}{j!}$

Proof: By induction.

if $x \geq 0$ $e^x \geq 1$, so base case $(n=0)$ holds.

$n \Rightarrow n+1$

$$e^t \geq \sum_{j=0}^{n} \frac{t^j}{j!} \quad \text{for } t \in [0, x] \qquad x \text{ fixed}$$

$$\text{so} \quad \int_0^x e^t \, dt \geq \sum_{j=0}^{n} \int_0^x \frac{t^j}{j!} \, dt$$

$$e^x - 1 \geq \sum_{j=0}^{n} \frac{x^{j+1}}{(j+1)!}$$

$$e^x \geq \sum_{j=0}^{n+1} \frac{x^j}{j!}$$

so $\forall n \in \mathbb{N}$, this holds. ∎

Next time:

$$\underset{\downarrow \atop e}{\left(1+\frac{1}{n}\right)^n} < \underset{\downarrow \atop e}{\sum_{j=0}^{n} \frac{1}{j!}} \leq \underset{\downarrow \atop e}{e}$$

by sq. thm.