Lec 9/7

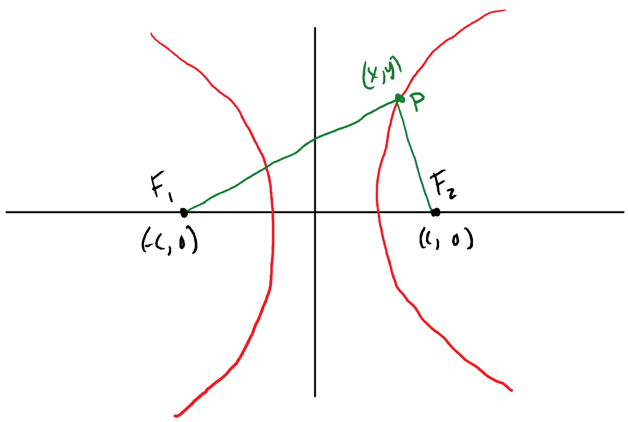
Wednesday, September 7, 2016 9:05 AM

Diagnostic Quiz next week tues
(does not count for grade)

Hyperbola

Geometric definitions

Set of points in the plane s.t. the absolute value of the difference between two fixed points (the foci) is constant.



$$|PF_1 - PF_2| = 2a$$

$$0 < a < c$$

(1) $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$

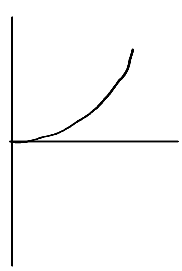
$$\Downarrow$$

(10) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b = \sqrt{c^2 - a^2}$

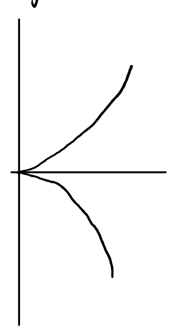
if (x,y) satisfies (1) then (x,y) satisfies (10).

- But converse is not automatically true:

$y = x^{3/2}$



$y^2 = x^3$



does (10) $\Rightarrow$ (1) ?

In general, $A^2 = B^2 \not\Rightarrow A = B$

$$\Downarrow$$

$$A = \pm B$$

$$(3) \Rightarrow \sqrt{(x+c)^2+y^2} = \pm\left(\pm 2a + \sqrt{(x-c)^2+y^2}\right)$$

Need to rule out this case:

$$\sqrt{(x+c)^2+y^2} = -\left(\pm 2a + \sqrt{(x-c)^2+y^2}\right)$$

Case 1: $\pm 2a$ is $2a$

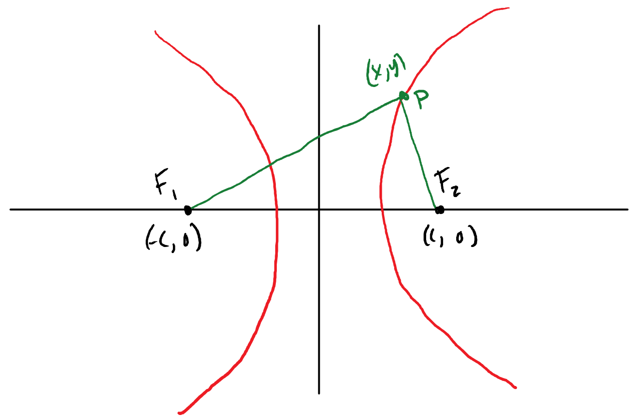
$$\sqrt{(x+c)^2+y^2} = -2a - \sqrt{(x-c)^2+y^2}$$

ruled out. left ≥ 0, right < 0

Case 2: $\pm 2a$ is $-2a$

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

$$PF_1 = 2a - PF_2$$



$$2a = PF_1 + PF_2 \geq F_1F_2 = 2c$$

$$2a \geq 2c$$

$$a \geq c$$

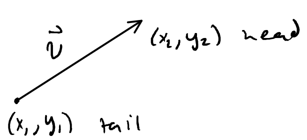
but hypothesis was $a < c$

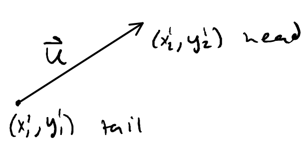
So contradiction and this case is ruled out.

Vectors algebraic way of encoding geometric information about the plane. $\mathbb{R}^2$ (and higher dimentional space $\mathbb{R}^n$)

Informal definition:

A vector is an equivalence class of arrows in the plane





We regard two such arrows as equivalent (essentially the same)

if $x_2' - x_1' = x_2 - x_1$

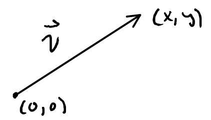
and $y_2' - y_1' = y_2 - y_1$

(geometrically: parallel, same length, same dir).

"Standard position" means that the tail is at the origin.

so the vector is determined only by position of its head.

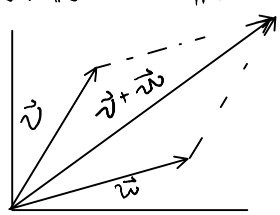
(a single point in the plane, (x,y))



Formal Definition: A vector is a point in the plane overlaid with certain operations:

(1) Vector addition: $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

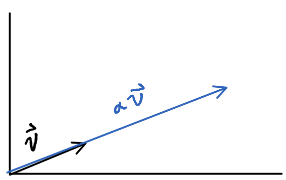
given $\vec{v} = (x_1, y_1)$, $\vec{w} = (x_2, y_2)$ $\quad \vec{v} + \vec{w} = (x_1 + x_2, y_1 + y_2)$



Vector addition is clearly associative and commutative.

(2) Scalar multiplication: $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

given $\vec{v} = (x_1, y_1)$, $\alpha \in \mathbb{R}$ $\quad \alpha\vec{v} = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$



note: if $\alpha < 0$ then reverse the direction of $\vec{v}$

Scalar multiplication distributes over vector addition in two ways:

$\alpha(\vec{v} + \vec{u}) = \alpha\vec{v} + \alpha\vec{u}$

$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

associative law:

$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

(3) dot product: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

given $\vec{v}_1 = (x_1, y_1)$, $\vec{v}_2 = (x_2, y_2)$ $\quad \vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2$

note that $\vec{v} \cdot \vec{v} = x^2 + y^2 = (\sqrt{x^2+y^2})^2 = \|\vec{v}\|^2$

btw $\|\vec{v}\| = \sqrt{x^2+y^2}$ = length of $\vec{v}$

dot product is commutative: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

no associative law.

distributive law: $\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$

Cauchy-Schwarz Inequality: $\vec{v} \cdot \vec{w} \le \|\vec{v}\| \|\vec{w}\|$

(we will prove $(\vec{v} \cdot \vec{w})^2 \le (\|\vec{v}\| \|\vec{w}\|)^2$) $(*)$

if $\vec{v} = (x_1, y_1)$ and $\vec{w} = (x_2, y_2)$

if $\vec{v} = (x_1, y_1)$ and $\vec{w} = (x_2, y_2)$

$$(\vec{v}\cdot\vec{w})^2 = (x_1x_2 + y_1y_2)^2 = x_1^2x_2^2 + 2x_1y_1x_2y_2 + y_1^2y_2^2$$

$$(\|\vec{v}\|\|\vec{w}\|)^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) = x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2$$

to prove (*) it suffices to prove that $2x_1y_1x_2y_2 \le x_1^2y_2^2 + x_2^2y_1^2$

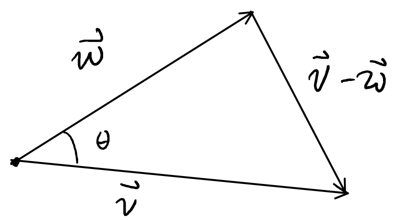
$0 \le x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2 \Rightarrow (x_1y_2 - x_2y_1)^2 \geq 0$ ✓

so $2x_1y_1x_2y_2 \le x_1^2y_2^2 + x_2^2y_1^2$

Geometric Connection

$$\|\vec{v} - \vec{w}\| = (\vec{v} - \vec{w})\cdot(\vec{v} - \vec{w}) = \vec{v}\cdot\vec{v} - 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$



law of cosines:

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$

therefore, $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

so Cauchy-Schwarz says $|\cos\theta| \le 1$