Lec 10/4

Friday, October 4, 2019 10:21

Last time Fubini-Tonelli for χ_E.

Thm Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite. Then $\forall E \in \mathcal{M} \otimes \mathcal{N}$,

① $x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$ are mble, and

② $(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$

Thm (Tonelli): Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite.

for $f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$,

① $x \mapsto \int_Y f_x \, d\nu$ is $\mathcal{M}$-mble

$y \mapsto \int_X f^y \, d\mu$ is $\mathcal{N}$-mble

② $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f_x \, d\nu \right] d\mu = \int_Y \left[\int_X f^y \, d\mu \right] d\nu$

Remark: If $f \in L^+ \cap L^1(\mu \times \nu)$, then

- $\underbrace{\int_Y f_x \, d\nu < \infty}_{f_x \in L^1(\nu)}$ for a.e. $x \in X$.

- $\int_X f^y \, d\mu < \infty$ for a.e. $y \in Y$.

Pf of Tonelli: If $f = \chi_E$ for some $E \in \mathcal{M} \otimes \mathcal{N}$, we are done by prev. thm.

Since $(cf+g)_x = cf_x + g_x$ (Exercise),

we get the result for $L^+ \cap SF = SF^+$ by linearity.

Suppose $(\varphi_n) \subset SF^+$ s.t. $\varphi_n \nearrow f$ everywhere.

Then $(\varphi_n)_x \nearrow f_x$ and $(\varphi_n)^y \nearrow f^y$.

So by MCT $\int_Y (\varphi_n)_x \, d\nu \nearrow \int_Y f_x \, d\nu$, $\int_X (\varphi_n)^y d\mu \nearrow \int_X f^y d\mu$.

This implies ①.

Again, by MCT, a few times

$$\int_X \left[\int_Y f_x \, d\nu \right] d\mu(x) = \int_X \lim \left[\int_Y (\varphi_n)_x \, d\nu \right] d\mu$$

$$= \lim \int_X \left[\int_Y (\varphi_n)_x \, d\nu \right] d\mu$$

$$= \lim \int_{X \times Y} \varphi_n \, d(\mu \times \nu)$$

$$= \int_{X \times Y} f \, d(\mu \times \nu).$$

similarly $\longrightarrow$ $\cdots = \int_Y \left[\int_X f^y d\mu \right] d\nu(y)$

Corollary (Fubini): If $f \in L^1(\mu \times \nu)$, then

① $f_x \in L^1(\nu)$ for a.e. $x \in X$

② $f^y \in L^1(\mu)$ for a.e. $y \in Y$

③ $\left[x \longmapsto \int_Y f_x \, d\nu \right] \in L^1(\mu)$

④ $\left[x \longmapsto \int_X f^y d\mu \right] \in L^1(\nu)$

④ $\left[x \longmapsto \int_X f^y \, d\mu\right] \in L^1(\nu)$

⑤ $\int_{X\times Y} f \, d(\mu\times\nu) = \int_X \left[\int_Y f_x \, d\nu\right] d\mu = \int_Y \left[\int_X f^y \, d\mu\right] d\nu$

Pf: write $f = Re(f)_+ - Re(f)_- + i\,Im(f)_+ - i\,Im(f)_-$,

where $Re(f)_\pm,\ Im(f)_\pm \in L^+ \cap L^1$.

Hence Tonelli's Thm applies to these 4 functions, as does the Remark.

n-dim'l Lebesgue integral:

Defn $(\mathbb{R}^n, \mathcal{L}^n, \lambda^n)$ is the completion of

$(\mathbb{R}^n, \mathcal{L}\otimes\cdots\otimes\mathcal{L}, \lambda\times\cdots\times\lambda)$

Properties:

① λ^n is σ-finite

② λ^n is regular

③ $\forall E \in \mathcal{L}^n$ with $\lambda^n(E) < \infty$, $\forall \varepsilon > 0$, $\exists R_1, \dots, R_K$ disjoint rectangles whose sides are intervals s.t.

$\lambda^n\left(E \,\Delta\, \coprod_1^K R_j\right) < \varepsilon$

④ $SF \cap \mathcal{L}^1(\lambda^n)$ is dense in $\mathcal{L}^1(\lambda^n)$.

⑤ $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^1(\lambda^n)$

⑥ Suppose $E \in \mathcal{L}^n$.

- $\forall\, r \in \mathbb{R}^n$, $r + E \in \mathcal{L}^n$ and $\lambda^n(r+E) = \lambda^n(E)$.

- $\forall\, t \in \underbrace{GL_n(\mathbb{R})}_{\text{or } M_n(\mathbb{R})}$, $TE \in \mathcal{L}^n$ and $\lambda^n(TE) = |\det T|\, \lambda^n(E)$

⑦ $\forall\, \mathcal{L}^n$-mble $f : \mathbb{R}^n \to \mathbb{C}$,

- $x \longmapsto f(x+r)$ is mble $\forall\, r \in \mathbb{R}^n$

- $x \longmapsto f(Tx)$ is mble $\forall\, T \in \underbrace{GL_n(\mathbb{R})}_{\text{or } M_n(\mathbb{R})}$

If moreover $f \in \mathcal{L}^+$ or $\mathcal{L}^1$, then

- $\int f(x+r)\, d\lambda^n(x) = \int f\, d\lambda^n$

- $\int f(Tx)\, d\lambda^n(x) = |\det T| \int f\, d\lambda^n$