Lec 2/10

Friday, February 10, 2017 15:00

Rao-Blackwell Theorem & Minimum Variance Unbiased Estimators (MVUE) (not in text)

Recall Conditional Expectation:

$$\mathbb{E}(X \mid Y=y) = \int_{-\infty}^{\infty} x\, f(x|y)\, dx \quad \text{or} \quad \sum_{X} x\, f(x|y)$$

Notice this is a function of y, and if we let y range over all possible values, we get $E(X|Y)$ as a function of Y, so it's a RV.

similarly, $Var(X|Y=y)$ is a func of y, so $Var(X|Y)$ is a RV.

Note: ① $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$

② $Var(X) = \mathbb{E}(Var(X|Y)) + Var(\mathbb{E}(X|Y))$

Thm (Rao-Blackwell):

Let $X_1, \dots, X_n$ be a random sample and let $\hat{\theta} = \hat{\theta}(x_1, \dots, x_n)$ be an unbiased estimator of parameter θ s.t. $Var(\hat{\theta})$ is finite. If U is a sufficient statistic for θ and $\hat{\theta}^* = \mathbb{E}(\hat{\theta} \mid U)$, then for all θ,

$\mathbb{E}(\hat{\theta}^*) = \theta$ and $Var(\hat{\theta}^*) \le Var(\hat{\theta})$.

Proof:
$$\begin{aligned} Var(\hat{\theta}) &= \mathbb{E}(Var(\hat{\theta}|U)) + Var(\mathbb{E}(\hat{\theta}|U)) \\ &\ge Var(\mathbb{E}(\hat{\theta}|U)) \\ &= Var(\hat{\theta}^*) \end{aligned}$$

$$\mathbb{E}(\hat{\theta}^*) = \mathbb{E}(\hat{\theta}|U) = \mathbb{E}(\hat{\theta}) = \theta \quad \blacksquare$$

Remarks: ① by doing conditional expectation of an unbiased estimator given a sufficient statistic, we can obtain a better unbiased estimator.

② if $\hat{\theta}$ is a function of the sufficient statistic U, say $\hat{\theta} = h(U)$, then Then there is no new information in $\hat{\theta}^*$ so $\hat{\theta}^* = \hat{\theta}$.
($E(h(U)|U) = h(U)$ since if you know U you know $h(U)$).

③ $\hat{\theta}^*$ is a function of U. So $\hat{\theta}^{**} = \mathbb{E}(\hat{\theta}^* | U) = \hat{\theta}^*$ by ②.

(4) many sufficient statistics for θ. which to use? The minimal sufficient statistic. The statistic that best summarizes info in a sample about the parameter θ. The minimal sufficient statistic can be identified usually by the factorization theorem.

Def u is a minimal statistic for θ if for any $\tilde{u}$, u is a function of $\tilde{u}$.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta) = \frac{1}{Z_\theta} |x|^\theta \mathbb{1}_{\{x \in [-1,1]\}}$

Sol: $f(x_1, \dots, x_n; \theta) = \prod_{i=1}^n \frac{1}{Z_\theta} |x_i|^\theta \mathbb{1}_{\{x_i \in [-1,1]\}} = \underbrace{\frac{1}{Z_\theta^n} \left| \prod_{i=1}^n x_i \right|^\theta}_{g(\prod_{i=1}^n x_i, \theta)} \underbrace{\prod_{i=1}^n \mathbb{1}_{\{x_i \in [-1,1]\}}}_{h(x_1, \dots, x_n)}$

so $\prod_{i=1}^n x_i$ is a sufficient statistic, but is not minimal.

Since $\left|\prod_{i=1}^n x_i\right|$ is also sufficient but there is no function $|z| \to z$.

↳ minimal sufficient statistic.

In fact, if u is a minimal sufficient statistic & $\hat{\theta}$ is an unbiased estimator, under a completeness condition,

$E(\hat{\theta} \mid u)$ is MVUE for θ.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$. find the MVUE for θ.

Sol: $f(x_1, \dots, x_n; \theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-(\sum_{i=1}^n x_i)/\theta}$ so $\sum_{i=1}^n x_i$ is a minimal sufficient stat.

so find an unbiased estimator $\hat{\theta}$. take $\hat{\theta} = \bar{X}$ as unbiased estimator.

but $\bar{X}$ is a function of $\sum_{i=1}^n x_i$, so $E(\bar{X} \mid u) = \bar{X}$ must be MVUE for θ.

Problem 3 on practice exam:

$$f(x_1, \dots, x_n; \theta) = \underbrace{(3\theta)^{\sum x_i} e^{-3n\theta}}_{g(\sum x_i, \theta)} \underbrace{\frac{1}{\prod x_i!}}_{h(x_1, \dots, x_n)}$$

take $u = \sum_{i=1}^n x_i$, it best summarizes the data.

$\hat{\theta} = \frac{1}{3}\bar{X}$ is unbiased for θ, so $\mathbb{E}(\frac{1}{3}\bar{X} \mid \Sigma X_i)$ is MVUE.

but $\frac{1}{3}\bar{X}$ is a function of ΣX_i so $\mathbb{E}(\frac{1}{3}\bar{X} \mid \Sigma x_i) = \frac{1}{3}\bar{X}$ is MVUE.

<u>Exercise</u>: $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta) = \begin{cases} \frac{2x}{\theta} e^{-x^2/\theta} & x > 0 \\ 0 & o.w. \end{cases}$

find MVUE for θ.

$$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} \frac{2x_i}{\theta} e^{-x_i^2/\theta} = \underbrace{\left(\frac{2}{\theta}\right)^n}_{g} \underbrace{\left(\prod_{i=1}^{n} x_i\right)}_{h} \underbrace{e^{-\Sigma x_i^2/\theta}}_{g}$$

so $\sum_{i=1}^{n} x_i^2$ is a min suff. est.

Note that:

$$\mathbb{E}(\Sigma x_i^2) = n\,\mathbb{E}(x_i^2) = n\int_0^\infty \frac{2x^3}{\theta} e^{-x^2/\theta}\, dx = n\left(-e^{-x^2/\theta}(t - x^2)\right)\Big|_0^\infty = nt$$

$$\left[\begin{array}{ll} \int \frac{2x^3}{\theta} e^{-x^2/\theta} dx = \int u \frac{1}{\theta} e^{-u/\theta}\, dx & = -ue^{-u/\theta} + \int e^{-u/\theta} du \\ u = x^2,\ du = 2x\, dx & = -ue^{-u/\theta} - \theta e^{-u/\theta} \\ \text{Side Calculation} & = -e^{-x^2/\theta}(t - x^2) \end{array}\right]$$

So $\frac{\Sigma x_i^2}{n}$ is an unbiased estimator for θ.

therefore $\mathbb{E}\left(\frac{\Sigma x_i^2}{n}, \Sigma x_i^2\right)$ is MVUE. But $\frac{\Sigma x_i^2}{n}$ is a func of Σx_i^2 so this $= \frac{\Sigma x_i^2}{n}$ is MVUE.