Propn: Hom(K,0) is Left - exact:

if 
$$O \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow O$$
 is exact  
then so is  $O \longrightarrow Hom(K, A) \xrightarrow{\Phi} Hom(K, B) \xrightarrow{\Psi} Hom(K, C)$ .

T is surjective if Y \ i.k → C ] \ i.k → B s.t. \ = \ v. \ \

Not always tre: 
$$Z \xrightarrow{\text{mod } 2} Z_2 \longrightarrow 0$$

No homo-  $Z_2$ 

morphisms

So Hom (K,.) may not be right-exact.

Det: K i's projective if Hom(k,.) is right-exact.

This is the if 
$$\forall$$
 epinorphism  $B \xrightarrow{\psi} C \rightarrow o$   
and  $\forall$  how-sm  $\xi: K \rightarrow C$   $\xi \nearrow \gamma_{\xi}$   
 $\exists \ \tau \ s.t. \ \xi = \psi \cdot \tau$ 

Theorem: K is projective iff it's a direct summer of a free module

Proof: (€) Let  $K \oplus L = N$ , a free module gen-ed by S. Let  $B \xrightarrow{\Psi} C$  be surjective, let  $\xi: K \longrightarrow C$  be a hom-sm Define  $\tilde{\xi}: N \rightarrow C$  by  $\tilde{\xi}(L) = 0$ .

 $\forall u \in S$ , define  $\tau(u)$  to be any element of  $\psi^{-1}(\tilde{x}(u))$ .

Then T extends to a homomorphism N-B

S.t.  $\psi \circ \tau = \tilde{\beta}$  S.  $\psi \circ (\tau|_{\kappa}) = \gamma$ .

(⇒) Let K be projective. I free module N & a nomomorphism Y:N→K which is surjective.

we have

$$0 \longrightarrow L \longrightarrow N \xrightarrow{\psi} K \longrightarrow 0$$

$$|| \qquad \qquad || \qquad \qquad \uparrow \uparrow \qquad || \qquad || \qquad || \qquad \qquad ||$$

Since K is projective,  $\exists \tau$  as above sit the diagram country:  $\forall \circ \tau = idk$  so the sequence splits (from the right) So  $N \cong K \oplus L$ .

Examples I = (x,y) in F(x,y) = R is not a projective R-module. field of fractions of an ID R is not a projective R-module

If R is an integral domain, any projective module is torsion-free.

Functor Hom(·, K) is contravoriant functor from Category
of R-mobiles to itself.

A --- Hom (A,K), and

A -> B; Y: A -> B is mapped to D: Hom (B, K) -> Hom (A, K)

$$A \xrightarrow{\varphi} B$$
;  $\psi: A \to B$  is mapped to  $\Phi: Hom(B, K) \longrightarrow Hom(A, K)$ 

$$\beta \longmapsto \beta \circ \psi$$

Contravariant = "reverses arows"

s not necessorily right-exact Theorem: YK, Hom(., K) is (eft-exact: if 0 -> A B C -> o is exact, true

 $O \longrightarrow Hom(C,K) \xrightarrow{\Psi} Hom(B,K) \xrightarrow{\Phi} Hom(A,K)$  is exact

y is surjective so if 8: ( -> K is nonzero then  $Y \cdot Y$  is nonzero too since Y is surjective. So I is injective.

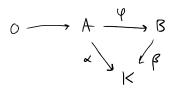
 $\Phi \circ \Psi (Y) = Y \circ \Psi \circ \varphi = 0$  so  $\text{Im } Y \subseteq \text{Ker } \Phi$ .

Now let  $\beta \in \text{Ker } \Phi$ . That is,  $\beta \circ \gamma = 0$ . so  $\beta(\gamma(\lambda)) = 0$ . So Ker y ⊆ Kerp. So & factorizes through Ker y to a homomorphism V: C-+ K so that B = Y · W = \$\P(y) So In Y = Ker D.

Why not exact at the right:

Hom (B, K) — Hom (A, K) is surjective if any

homomorphism  $\alpha : A \rightarrow K$  extends to a home:  $\alpha : B \rightarrow K$  s.t.  $\beta \circ \varphi = \alpha$ .



But:  $2Z \longrightarrow Z$  can't be extended to a hom-sm  $Z \longrightarrow Z$  (divide by 2)

definition: K is injective if Hom (., K) is right-exact (so exact).

this is so if  $\forall$  injective  $\forall: A \rightarrow B$ ,  $\forall \alpha: A \rightarrow K$ ,  $\exists \kappa \text{ s.t. } \alpha = \beta \circ \varphi$ .