Lec 1/23

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Recall: $\hat{\theta}$ is sufficient for θ if $f(x_1, \dots, x_n \mid \hat{\theta})$ is indp. of θ.

Ex: $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bernoulli}(\theta)$

Show that $\hat{\theta} = X_1 + 2X_2 + X_3$ is not a sufficient estimator of θ.

my try: $f(x_1, x_2, x_3 \mid \hat{\theta}) = \dfrac{f(x_1, x_2, x_3)}{g(\hat{\theta})} = \dfrac{\theta \cdot \theta \cdot \theta}{\quad}$

Sol: Try $(x_1, x_2, x_3) = (1, 0, 1) \Rightarrow \hat{\theta} = 2$.

$$f(x_1, x_2, x_3 \mid \hat{\theta}) = \frac{\theta^2(1-\theta)}{P(\hat{\theta}=2)} = \frac{\theta^2(1-\theta)}{(1-\theta)^2\theta + \theta^2(1-\theta)} = \frac{\theta}{(1-\theta)+\theta} = \theta \text{ depends on } \theta.$$

Theorem 10.4 The statistic $\hat{\theta}$ is a sufficient estimator of θ iff the joint pdf/pmf of the RS $X_1, \dots, X_n$ can be factored s.t.

$$f(x_1, \dots, x_n; \theta) = g(\hat{\theta}, \theta) \cdot h(x_1, \dots, x_n)$$

Where $g(\hat{\theta}, \theta)$ depends on only $\hat{\theta}, \theta$

$h(x_1, \dots, x_n)$ does not depend on θ.

$X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$.

$$f(x_1, \dots, x_n; \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \underbrace{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}_{g(\hat{\lambda}, \lambda)} \cdot \underbrace{\frac{1}{\prod_{i=1}^n x_i!}}_{h(x)} \Rightarrow \sum_{i=1}^n x_i \text{ is a sufficient estimator of } \lambda.$$

Ex: Let $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta) = \dfrac{\theta}{(1+x)^{\theta+1}}$ $\quad 0 < \theta < \infty$, $0 \le x < \infty$

$$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} \frac{\theta}{(1+x_i)^{\theta+1}} = \theta^n \cdot \frac{1}{\prod_{i=1}^n (1+x_i)^{\theta+1}} = \underbrace{\frac{\theta^n}{\left(\prod_{i=1}^n (1+x_i)\right)^{\theta+1}}}_{g(\hat{\theta}, \theta)} \cdot 1 \Rightarrow \prod_{i=1}^n (1+x_i) \text{ is sufficient for } \theta.$$

Remark: Sufficient statistic is <u>not</u> unique. any one-to-one function of a sufficient statistic is sufficient.

$$f(x_1,\dots,x_n;\theta) = g(\hat{\theta},\theta)\, h(x_1,\dots,x_n)$$

$$= g\big(\underbrace{T^{-1}(T(\hat{\theta})),\theta}_{\substack{|||\\ g(T(\hat{\theta}),\theta)}}\big)\, h(x_1,\dots,x_n) \quad \text{if } T \text{ is 1-1.}$$

For the previous example, can take $\hat{\theta} = \log\left(\prod_{i=1}^{n}(1-x_i)\right) = \sum_{i=1}^{n}\log(1-x_i)$

Ex: $X_1,\dots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$. Let σ^2 be a constant (known).

Show that $\bar{X}$ is sufficient for μ.

sol: $f(x_1,\dots,x_n;\mu) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{\frac{-(X_i-\mu)^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\tfrac{-1}{2\sigma^2}\sum_{i=1}^{n}x_i^2\right]}_{\substack{\shortparallel\\ h(x_1,\dots,x_n)}}\ \underbrace{\exp\left[\tfrac{-1}{2\sigma^2}(-2\mu n\bar{X}+n\mu^2)\right]}_{\substack{\shortparallel\\ g(\bar{x},\mu)}}$$

$$* = \sum_{i=1}^{n}(x_i-\mu)^2$$
$$= \sum_{i=1}^{n}x_i^2 - 2\sum_{i=1}^{n}x_i\mu + \sum_{i=1}^{n}\mu^2$$
$$= f(x_i) - 2\mu(n\bar{x}) + n\mu^2$$

so $\bar{X}$ is sufficient.

Section 10.7 method of moments:

Notation (4.3): r-th moment of a RV X about the origin is $E(X^r) \equiv \mu_r'$

The r-th sample moment of a sample $X_1,\dots,X_n$ is $\frac{X_1^r+\dots+X_n^r}{n} \equiv m_r'$

(MOM): Equate sample to population moments.

if a pop. has r parameters, then the MOM consists of solving the system of equations $m_k' = \mu_k'$ where $k\in\{1,\dots,r\}$. for the r params.

Example: Poisson dist w/ parameter λ. $r=1$ (one parameter).

$m_1' = \mu_1' \Rightarrow \bar{X} = \lambda$. So the MOM estimator for λ is $\bar{X}$.