__Theorem__: Suppose μ is σ-finite and $1 \le p < \infty$ $\left(q = \text{conj} = \frac{p}{p-1}\right)$.
Let ϕ be a bounded linear operator on L^p. Then $\exists g \in L^q$ s.t.

$$\phi(f) = \int fg \quad \text{and} \quad \|\phi\| = \|g\|_q \,.$$

__Proof__ (Ovidiu Costin's notes)

__claim__: ϕ induces a complex measure λ where $\lambda \ll \mu$ and $\frac{d\lambda}{d\mu} \in L^q$.

Let's assume μ is finite. Define $\lambda(E) = \phi(\chi_E)$.

suppose $E_1, E_2, \ldots$ are mutually disjoint and $E = \bigcup_1^\infty E_i$,

Then $\chi_E = \sum \chi_{E_i}$ and $\lambda(E) = \sum \lambda(E_i)$ by ... something.

__Radon-Nikodym__: $\exists\, g \in L^1(\mu)$ s.t. $\lambda(E) = \phi(\chi_E) = \int \chi_E \, g \, d\mu$.

Linearity + density of simple functions + continuity of ϕ

$\rightsquigarrow$ $\phi(f) = \int fg$ for all $f \in L^p$.

__Next we show__: $\|g\|_q \le \|\phi\|$.

Let $E_n = \{x : g(x) \le n\}$ and

$$G = \chi_{E_n} \operatorname{sign}(g)\, |g|^{q-1}$$

Observe $|G|^p = |g|^q$ on E_n. Thus

$$\int_{E_n} |g|^q = \int Gg = |\phi(G)| \le \|\phi\| \left(\int_{E_n} |g|^q \right)^{1/p}$$

[illegible] $\left(\int |g|^q \right)^{1/q} \le \|\phi\|$

dividing gives $\left(\int_{E_n} |g|^q\right)^{1/q} \le \|\phi\|$.

$\Rightarrow \|g\|_q \le \|\phi\|$

<u>Now</u>: assume $\mu(X) = \infty$ and μ is σ-finite.

<u>Lemma</u> $\exists\, \omega \in L^1(\mu)$ s.t. $0 < \omega < 1$ in X,

$d\tilde{\mu} := \omega\, d\mu$ is finite

and $\mu \ll \tilde{\mu}$, $\tilde{\mu} \ll \mu$.

furthermore, the map $f \mapsto \omega^{1/p} f$ is an isometric isomorphism $L^p(\tilde{\mu}) \longleftrightarrow L^p(\mu)$

Let $d\tilde{\mu} = \omega\, d\mu$ as in the lemma.

Let ϕ be a bounded linear operator on L^p.

Define $\psi(f) := \phi(\omega^{1/p} f)$.

$$\Rightarrow \|\psi\|_{(L^p(\tilde{\mu}))^*} = \|\phi\|_{(L^p(\mu))^*}.$$

Since $\tilde{\mu}$ is finite, $\exists\, G \in L^q(\tilde{\mu})$ s.t.

$$\psi(f) = \int f G\, d\tilde{\mu} \quad \text{for } f \in L^p(\tilde{\mu}).$$

Let $g = \omega^{1/p} G$. Then

$$\int |g|^q\, d\mu = \int |G|^q\, d\tilde{\mu} = \|\psi\|^q = \|\phi\|^q.$$

And since $G\, d\tilde{\mu} = \omega^{1/p} g\, d\mu$,

$$\phi(f) = \psi(\omega^{-\frac{1}{p}} f) = \int \omega^{-\frac{1}{p}} f\, G\, d\tilde{\mu}$$

$$= \int fg \, d\mu .$$

$\square$

For $1 < p < \infty$, $L^q \xleftrightarrow[\text{isomorphism}]{\text{isometric}} (L^p)^*$.

This is also true for $p=1$ if μ is σ-finite.

if not σ-finite, $L^\infty \longrightarrow (L^1)^*$ fails to be injective.

Some Useful Inequalities

Theorem (Chebyshev's Inequality): for $1 \le p < \infty$, $\alpha > 0$, $f \in L^p$,

$$\mu(|f| > \alpha) \le \frac{1}{\alpha^p} \|f\|_p^p$$

pf $\|f\|_p^p = \int |f|^p \geq \int_{\{|f|>\alpha\}} |f|^p \geq \int_{\{|f|>\alpha\}} \alpha^p = \mu(|f|>\alpha) \cdot \alpha^p$. $\square$

An operator T of the form $Tf(x) = \int_Y K(x,y) \, f(y) \, d\nu(y)$ is called a linear integral and K is called the Kernel of T.

Theorem let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and $K : X \times Y \longrightarrow \mathbb{C}$ a Kernel which is uniformly L^1 wrt μ and ν, i.e. $\| K(\cdot, y) \|_{(L^1(\mu))^*} \le C$ for a.e. y and $\| K(x, \cdot) \|_{(L^1(\nu))^*} \le C$ for a.e. x (for some C).

Then for $1 \le p < \infty$, T is a bounded operator from L^p to L^p . . . and something else is true.