Lec 2/22

Wednesday, February 22, 2017 14:49

Neyman-Pearson Lemma: only for simple-vs-simple tests.

Finish Example from last time:

$$L_0 = \begin{cases} 0.25 & x=0 \\ 0.5 & x=1 \\ 0.25 & x=2 \end{cases} \qquad L_1 = \begin{cases} 0.0625 & x=0 \\ 0.375 & x=1 \\ 0.5625 & x=2 \end{cases}$$

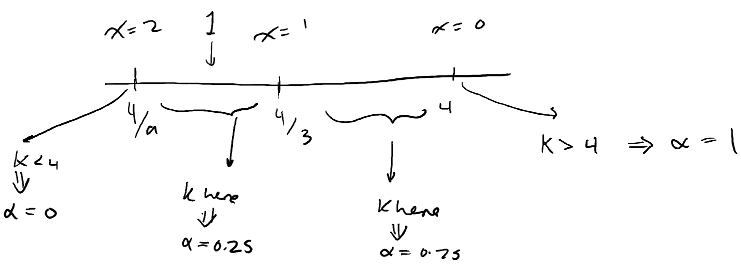
$$\frac{L_0}{L_1} = \begin{cases} 4 & x=0 \\ \frac{4}{3} & x=1 \\ \frac{4}{9} & x=2 \end{cases}$$

find k s.t. $\frac{L_0}{L_1} \le k$ inside C w/ size $\le \alpha$.

for $\alpha = 0.25$, $k = 1$ works.

for $\alpha = 0.5$, $k = 1$ works.

for $\alpha = 0.05$ There is no real good k.



Remark: For a Discrete distribution, the possible levels α of the test are not completely under our control (e.g. no test can have $\alpha = 0.5$ using only likelihood ratio). Can use randomness to remedy this.

Ex (NPL): Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$. Hyp: $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$ w/ $\theta_1 > \theta_0$. Use NPL.

$$\frac{L_0}{L_1} = \frac{\prod_{i=1}^n \frac{1}{\theta_0}\exp\left(-\frac{x_i}{\theta_0}\right)}{\prod_{i=1}^n \frac{1}{\theta_1}\exp\left(-\frac{x_i}{\theta_1}\right)} = \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left(n\left(\frac{1}{\theta_1}\sum_{i=1}^n x_i - \frac{1}{\theta_0}\sum_{i=1}^n x_i\right)\right) \le K.$$

$$n \log\left(\frac{\theta_1}{\theta_0}\right) + n\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\sum_{i=1}^{n} x_i \le \log(k)$$

See ex 7.14 p 221.

note $\sum x_i \sim \text{Gamma}(n, \theta_0)$

$\Rightarrow \sum_{i=1}^{n} x_i \ge \tilde{K}$. want $\mathbb{P}\left(\sum_{i=1}^{n} x_i \ge \tilde{K} ; \theta = \theta_0\right) = \alpha$. Use a computer to figure this $\tilde{K}$ out.

Remarks: Consider a test (i.e. critical region C)

When we reject H_0 (i.e. sample in C)

1. $\mathbb{P}(H_1 \text{ true}) = 1-\beta$? no. $\mathbb{P}(H_0 \text{ false}) = 1-\beta$ but H_0 false $\not\Rightarrow H_1$ true.
2. $\mathbb{P}(H_0 \text{ true}) = \alpha$? no, don't know $\mathbb{P}(H_0 \text{ true} \mid \text{reject } H_0)$.

The test & result make no references to $\mathbb{P}(H_0 \text{ true})$ or $\mathbb{P}(H_1 \text{ true})$.

§12.5 power of a test.

So far, we've only done simple vs simple. in general! we might want simple vs composite (eg $H_0: \theta = \theta_0$ and $H_1: \theta > \theta_0$). to evaluate these tests, use "power function"

Def: Power function of a test of H_0 against H_1 is given by

$$\pi(\theta) = \begin{cases} \alpha(\theta) & \text{for value of } \theta \text{ under } H_0 \\ 1-\beta(\theta) & \text{for value of } \theta \text{ under } H_1 \end{cases}$$

$$\pi(\theta) = \mathbb{P}(\text{reject } H_0 ; \theta).$$

Ex Wicking Example: we tested $H_0: p = 0.1$ vs $H_1: p = 0.4$.

Now consider test $H_0: p \le 0.1$ vs $H_1: p > 0.1$.

$X = \#$ successes in 20 trials $\sim \text{Bin}(20, p)$.

We reject H_0 if $X > 5$.

$$\pi(p) = \mathbb{P}(\text{reject } H_0 ; p) = \mathbb{P}(X > 5 ; p) = 1 - \mathbb{P}(X \le 5 ; p)$$

$$= 1 - \sum_{i=0}^{5} \mathbb{P}(X = i ; p)$$

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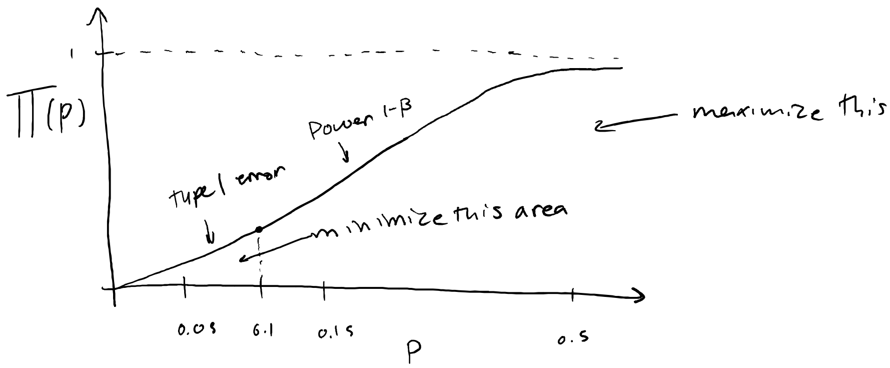
p	0.05	0.10	0.15	...	0.50
$\Pi(p)$	0.0003	0.01125	0.0693	...	0.9763
error type	type 1	type 1	no error	...	no error



Ideal: