Lec 10/7

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Differentiation Rules:

Theorem: Suppose $f'(a)$ and $g'(a)$ both exist. Then

(1) $(f+g)'(a) = f'(a) + g'(a)$

(2) $(f\cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$

(3) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$ provided $g(a) \neq 0$

Theorem: (Chain rule): suppose $f'(g(a))$ and $g'(a)$ are well defined. Then

$$(f\circ g)'(a) = f'(g(a))\,g'(a)$$

Theorem: (power rule): suppose $f(x) = x^n$. Then $f'(a) = n a^{n-1}$

(1) if n is a positive number

(2) if n is any integer (if $a \neq 0$ or $n \neq 0$)

(3) if n is a rational (if $a > 0$)

(4) if $n = p/q$, q odd

Proof of Theorem 1 part 2

$$(f\cdot g)'(a) = \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a}$$

$$= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}\right]$$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a} g(x) + f(a)\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)$$

$$= f'(a)\,g(a) + f(a)\,g'(a) \qquad (g \text{ is cont.}) \quad ■$$

Proof of chain rule

$$(g\circ f)'(a) = \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a}$$

$$\text{"="} \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{\underbrace{f(x)-f(a)}_{\text{possibly } 0.}} \frac{f(x)-f(a)}{x-a}$$

To avoid this, define $G(y) = \begin{cases} \frac{g(y)-g(b)}{y-b} & \text{if } y \in \operatorname{dom}(g) \text{ and } y \neq b \\ g'(b) & \text{if } y = b \end{cases}$

Then $\lim_{y\to b} G(y) = G(b) = g'(b)$

$$G(f(x)) = \begin{cases} \frac{g(f(x))-g(b)}{f(x)-b} & x \in \operatorname{dom} g\circ f \\ g'(b) & f(x) = b \end{cases}$$

assume $x \neq a$ and $x \in \operatorname{dom} f$

$$G(f(x)) \frac{f(x) - \overbrace{f(a)}^{=b}}{x-a} = \begin{cases} \frac{g(f(x))-g(b)}{f(x)-b} \frac{f(x)-b}{x-a} & \text{if } x \in \operatorname{dom} g\circ f \text{ and } f(x) \neq b \\ g'(b) \frac{f(x)-b}{x-a} = 0 & \text{if } f(x) = b,\ x \neq a \end{cases}$$

Important: this formula covers both cases.

$$= \frac{g(f(x))-g(b)}{f(x)-b} \frac{f(x)-b}{x-a} \qquad \text{if } x \in \operatorname{dom} g\circ f \text{ and } x \neq a$$

So $\lim_{x\to a} \frac{g(f(x))-g(b)}{f(x)-b} \frac{f(x)-b}{x-a} = \lim_{x\to a} G(f(x)) \frac{f(x)-f(a)}{x-a}$

$$= \lim_{x\to a} G(f(x)) \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= G\left(\lim_{x\to a} f(x)\right) f'(a)$$

letting $y=f(x)$, $\lim_{x\to a} G(f(x)) = G(b) = g'(b)$

$$= G(f(a))\, f'(a)$$

$$= G(b)\, f'(a)$$

$$= g'(b)\, f'(a)$$

$$= g'(f(a))\, f'(a) \qquad \blacksquare$$

Lemma If n is a positive integer, then $u^n - v^n = (u-v)\sum_{j=0}^{n-1} u^j v^{n-1-j}$

for ex: $u^2 - v^2 = (u-v)(u+v)$

$u^3 - v^3 = (u-v)(u^2 + vu + v^2)$

Proof: $$(u-v)\left(\sum_{j=0}^{n-1} u^j v^{n-1-j}\right) = u\sum_{j=0}^{n-1} u^j v^{n-1-j} - v\sum_{j=0}^{n-1} u^j v^{n-1-j}$$

$$= \underbrace{\sum_{j=0}^{n-1} u^{j+1} v^{n-1-j}}_{k=j+1} - \underbrace{\sum_{j=0}^{n-1} u^j v^{n-j}}_{k=j}$$

$$= \sum_{k=1}^{n} u^k v^{n-k} - \sum_{k=0}^{n-1} u^k v^{n-k}$$

$$= u^n + \sum_{k=1}^{n-1} u^k v^{n-k} - v^n - \sum_{k=1}^{n-1} u^k v^{n-k}$$

$$= u^n - v^n \qquad \blacksquare$$

Proof of Theorem 3: $f(x) = x^n$ $n \in \mathbb{N}^+$, $f'(a) = nx^{n-1}$

$$f'(a) = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{(x-a)\left(\sum_{j=0}^{n-1} x^j a^{n-1-j}\right)}{(x-a)} \qquad \text{(lemma)}$$

$$= \lim_{x\to a} \sum_{j=0}^{n-1} x^j a^{n-1-j} \qquad \text{(LP)}$$

$$= \sum_{j=0}^{n-1} a^j a^{n-1-j}$$

$$= \sum_{j=0}^{n-1} a^{n-1} = n\, a^{n-1} \qquad \blacksquare$$

☆ for $n=-1$: $f(x) = \frac{1}{x}$ $\quad x \neq 0$

if $a \neq 0$, $f'(a) = \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x-a} = \lim_{x \to a} \frac{\frac{a-x}{ax}}{x-a}$

$$= \lim_{x \to a} \frac{-1}{ax} \qquad (LP: (-\infty, a) \cup (a, 0) \text{ if } a < 0$$
$$(0, a) \cup (a, \infty) \text{ if } a > 0$$

$$= \frac{-1}{a^2} = -1 \cdot a^{-2} \qquad \blacksquare$$

☆ for $n = -m$, $m \in \mathbb{N}^+$, $f(x) = x^n = x^{-m} = \frac{1}{x^m} = (x^m)^{-1} = g(x^m)$

Chain rule: $f'(a) = g'(a^m)(m a^{m-1})$

$$= \frac{-1}{a^{m2}}(m a^{m-1})$$

$$= \frac{-m}{a^{m+1}} = -m\, a^{-m-1} = n\, a^{n-1} \qquad \blacksquare$$

☆ for $f(x) = x^{1/n}$ for $n \in \mathbb{N}^+$, $x > 0$

Use Lemma with $u = x^{1/n}$ $\quad V = a^{1/n}$ $\qquad a > 0$

$u^n = x \qquad V^n = a$

$$x - a = (x^{1/n} - a^{1/n}) \sum_{j=0}^{n-1} x^{\frac{j}{n}} a^{\frac{n-1-j}{n}}$$

First show f is continuous at a.

$$f(x) - f(a) = x^{1/n} - a^{1/n} = \frac{x-a}{\sum_{j=0}^{n-1} x^{\frac{j}{n}} a^{\frac{n-1-j}{n}}} < \frac{x-a}{a^{n-1/n}} < \varepsilon$$

letting $\delta = \min(a, \varepsilon a^{n-1/n})$,

$|x-a| < \delta \Rightarrow x \in \text{dom}(f)$ and $|f(x) - f(a)| < \varepsilon$

$$f'(a) = \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{x-a} = \lim_{x \to a} \frac{(x-a) / \sum_{j=0}^{n-1} x^{j/n} a^{\frac{n-1-j}{n}}}{x-a} \quad \text{by above}$$

$$= \lim_{x \to a} \frac{1}{\sum_{j=0}^{n-1} x^{\frac{j}{n}} a^{\frac{n-1-j}{n}}} \qquad \text{by LP over } (0, a) \cup (a, \infty)$$

1 $\qquad\qquad j/n$

$$x \to a \quad \sum_{j=0} x^{n} \, a^{n}$$

Note: if n odd

$-(-x)^{1/n} = x^{1/n}$

$$= \frac{1}{\sum_{j=0}^{n-1} a^{\frac{j}{n}} a^{\frac{n-1-j}{n}}}$$ because $x^{j/n}$ is cts on $(0,\infty)$

$$= \frac{1}{\sum_{j=0}^{n-1} a^{\frac{n-1}{n}}}$$

$$= \frac{1}{n a^{\frac{n-1}{n}}} = \frac{1}{n} a^{\frac{-n+1}{n}}$$

$$= \frac{1}{n} a^{\frac{1}{n}-1} \quad \blacksquare$$