Lec 10/17

Monday, October 17, 2016 8:34 AM

Midterm 2 - Wed Oct 26

Appendix to chap 11.

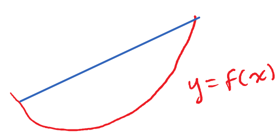
Convexity & concavity & 2nd derivative

$\parallel$ $\parallel$

concave up concave down

f convex $\longleftrightarrow$ $-f$ concave

Geometric Definition of convexity



a function f is convex over an interval I if, $\forall a<b \in I$,

the graph of the function lies below the secant line from $(a, f(a))$ to $(b, f(b))$.

Eqn of secant line: $y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$

Convexity condition is:

$$f(x) < \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \forall x \in (a,b)$$

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a} \quad \begin{matrix} \forall a<b \in I \\ \forall x \in (a,b) \end{matrix}$$

Eqn of line is also: $y = \frac{f(b)-f(a)}{b-a}(x-b) + f(b) > f(x)$

so $$\frac{f(x)-f(b)}{x-b} > \frac{f(b)-f(a)}{b-a} \quad \begin{matrix} \forall a<b \in I \\ \forall x \in (a,b) \end{matrix} \text{ as well.}$$

(note that $(x-b) < 0$)

Theorem 1 If f is convex on an interval I, $a<b \in I$, and $f'(a)$ and $f'(b)$ are defined, then $f'(a) < f'(b)$

Proof: ① $f'(a) = \lim_{x \to a^+} \dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(b)-f(a)}{b-a}$ (strong inequality not preserved by limit).

② $f'(b) = \lim_{x \to b^-} \dfrac{f(x)-f(b)}{x-b} \geq \dfrac{f(b)-f(a)}{b-a}$

so $f'(a) \leq f'(b)$.

we want a strict inequality.

let $y \in (a,b)$, let $x \in (a,y)$

from part ①, $f'(a) = \lim_{x \to a^+} \dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(y)-f(a)}{y-a}$

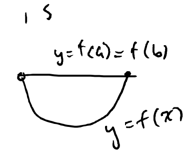
and by defn. of convexity, $\dfrac{f(y)-f(a)}{y-a} < \dfrac{f(b)-f(a)}{b-a}$

so $f'(a) < f'(b)$. (from pt. 2). $\blacksquare$

Corollary if f is convex on an interval, I and $f'(x)$ exists $\forall x \in I$, then f' is increasing on I.

Theorem 2: if f' is defined and increasing on an interval I, then f is convex on I.

Lemma (special case): if f is continuous on $[a,b]$ and f' is defined & increasing on $[a,b]$ and $f(a)=f(b)$, then $f(x) < f(a) = f(b)$ $\forall x \in (a,b)$.



Proof of Lemma: By EVT, f has a maximum on $[a,b]$. if the only maximum occur at endpoints, then $f(x) < f(a) = f(b)$ $\forall x \in (a,b)$.

otherwise, there is a maximum in the interior at x_0.

Then $f'(x_0) = 0$. But on the other hand, by MVT, for some $y \in (x_0, b)$ we have $f(y) - f(x_0) \le 0 = f'(c)(y - x_0)$ for some $c \in (x_0, y)$. so $f'(c) \le 0$, which is a contradiction since $f'(c) \le f'(x_0)$ so f' is not increasing. therefore, there is no maximum in the interior and $f(x) < f(a) = f(b)$ ∎

Proof of Thm 2: Let $g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$

Then $g'(x) = f'(x) - c$, so g' is increasing.

and $g(a) = f(a) - f(a) = 0$

$$\begin{gathered} g(a) = f(a) - f(a) = 0 \\ \parallel \\ g(b) = f(b) - f(b) + f(a) - f(a) = 0 \end{gathered}$$

so by lemma, $g(x) < g(a) = g(b) \quad \forall x \in (a,b)$.

so $f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right) < 0$

$$\frac{f(x) - f(a)}{x-a} < \frac{f(b) - f(a)}{b-a},$$

So f is convex on I. ∎

Corollary: Suppose f' is defined on an interval I. Then f is convex on I iff f' is increasing on I.

Corollary: if f'' is defined and $f'' > 0$ over an interval, then f is convex over that interval.