Rec 10/8

Tuesday, October 8, 2019 11:35

<u>2.4.10</u> D : div ring, $S^\times \trianglelefteq D^\times \Rightarrow S$ is central in D.

<u>Notation</u> $x, y \in D^\times$ are "S - indep" if $s_1 x + s_2 y = 0 \Rightarrow s_1 = s_2 = 0$. ($s_i \in S$)

<u>Lemma</u>: If $x, y \in D^\times$ are S-indep, $d \in D^\times \setminus \{x, y\}$, Then

either x & d are S-indep or y & d are S-indep.

<u>pf</u>: $s_1 x + s_2 d = 0 = s_3 y + s_4 d$, $s_i \neq 0 \Rightarrow x = -s_1^{-1} s_2 d$, $y = -s_3^{-1} s_4 d$.

<u>Problem Pf</u>: $S \subsetneq D$, fix $s \in S$. $s = 0$ clear, assume $s \neq 0$.

Define $\rho : D \to D$ by $\rho(d) = ds$

<u>Step 1</u>: ρ is an S-homomorphism (by distributivity & associativity)

<u>Step 2</u>: $\forall\, d \in D^\times\ \exists\, \lambda_d \in S$ s.t. $\rho(d) = \lambda_d d$

$(dsd^{-1} \in S \Rightarrow dsd^{-1} = \lambda_d)$.

<u>Step 3</u>: If x, y are S-indep, then $\lambda_x = \lambda_y$.

$$\left(\begin{aligned} &\lambda_x x + \lambda_y y = \rho(x) + \rho(y) = \rho(x+y) = \lambda_{x+y}(x+y) = \lambda_{x+y} x + \lambda_{x+y} y \\ &0 = (\lambda_{x+y} - \lambda_x) x + (\lambda_{x+y} - \lambda_y) y \rightsquigarrow \lambda_x = \lambda_{x+y} = \lambda_y . \end{aligned} \right)$$

<u>Step 4</u>: $\exists\, \lambda \in S$ s.t. $\lambda = \lambda_d\ \forall d \in D^\times$. Take $x \in D \setminus S$, $y \in S^\times$.

$$\left(\begin{aligned} &\text{Suppose } s_1, s_2 \in S \ \& \ s_1 \neq 0, \text{ but } s_1 x + s_2 y = 0. \\ &\text{so } x = -s_1^{-1} s_2 y \in S, \text{ contradiction.} \\ &\text{so } \forall d \in D^\times, \ \lambda_d = \lambda_x = \lambda_y = \lambda. \end{aligned} \right)$$

Step 5: $\rho(1) = 1s = s1$ so $\lambda = s$. $\square$

Unique factorization domain: An integral domain where every nonzero elt factors uniquely into the product of "atoms":

i.e. $x = a_1 \cdots a_n = b_1 \cdots b_m \Rightarrow m = n$ & $a_i \sim b_{\sigma(i)}$ (i.e. $(a_i) = (b_{\sigma(i)})$) for some $\sigma \in S_n$.

atoms: $a = bc \Rightarrow b$ or c is a unit.

Thm $R[x]$ is a UFD iff R is a UFD.

α is an algebraic integer if α satisfies a monic polynomial over $\mathbb{Z}$.

- i.e. $\sqrt{-5}$: $x^2 + 5 = 0$.

- $\mathbb{Z}[\sqrt{-5}]$ is not a u.f.d: $6 = \underbrace{2 \cdot 3}_{\ell(6)=2} = \underbrace{(1+\sqrt{-5})(1-\sqrt{-5})}_{\ell(6)=2}$.

half-factorial Domain:

$x \in R^{\times}$, any two factorizations have same length.

ex $\mathbb{F}_2[x^2, x^3]$: $x^6 = x^2 x^2 x^2 = x^3 x^3$.

$\hookrightarrow$ Not a half-factorial domain.

bounded factorization domain

for $x \in R^*$, $\exists N_x$ s.t. $\ell(x) \leq N_x$

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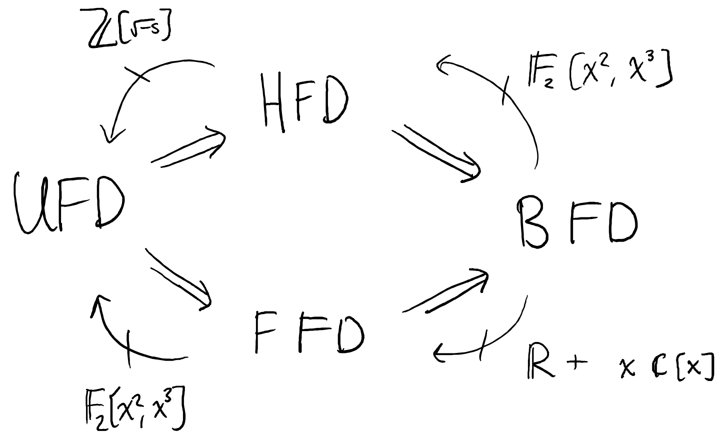
$\forall$ factorization

<u>finite factorization doman</u>

$x \in R^*$ has finitely many factorizations.

$\mathbb{R} + x\,\mathbb{C}[x]$



$x^2 = \alpha x \, \alpha^{-1} x$

$\forall \alpha \in S^1$.

and αx is atomic.