Lec 4/18

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6.3 Orientability and angular variation

A surface M in $\mathbb{R}^3$ is called orientable when there is a continuous unit normal map $\nu : M \longrightarrow \mathbb{S}^2$.

Orientability can be defined intrinsically (it means the surface can be covered by consistently oriented coordinate patches).

In this way, we can define orientability for C^1 surfaces in $\mathbb{R}^n$.

eg Sphere, single-, double-, etc- torus orientable. Möbius band not orientable.

Thm A compact surface (without boundary) in $\mathbb{R}^3$ is orientable.

Idea of proof:

Define $\nu : M \longrightarrow \mathbb{S}^2$ by $\nu(p) =$ outward normal at p. □

Fact a compact orientable surface (without boundary) is homeomorphic to a sphere with a certain number of handles.

eg The Klein bottle (2mb) and the projective plane (1mb) are compact non-orientable surfaces (without boundary) in $\mathbb{R}^4$.

FACT a compact non-orientable surface is homeomorphic to a sphere with a certain # of disks replaced by Möbius bands.

Obvious fact A surface that is the range of a single C^1 patch is orientable.

Easy fact Let G open $\subseteq \mathbb{R}^3$. Let $f : G \longrightarrow \mathbb{R}$ be C^1. Let $M = \{p \in G : f(p) = 0\}$. Suppose that $\forall p \in G$, $\nabla f(p) \neq 0$. Then M is an orientable surface in $\mathbb{R}^3$.

pf Let $p \in M$. Since $\nabla f(p) \neq 0$, at least one of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ is nonzero at p. Say it's $\frac{\partial f}{\partial z}$.

Then for (x, y, z) near p, we can solve $f(x,y,z) = 0$ for z as a C^1 function of x, y.

this is why M is a surface. Define $\nu : M \longrightarrow S^2$ by $\nu(p) = \frac{\nabla F(p)}{|\nabla F(p)|}$.

Then ν is a cts unit normal field on M. $\square$

Let (M, ν) be an oriented surface in $\mathbb{R}^3$.

Let $p \in M$ and let E_1, E_2 be a basis for T_pM. Let $F = \frac{E_1 \times E_2}{|E_1 \times E_2|}$.

Then either $F = \nu(p)$ or $F = -\nu(p)$.

If $F = \nu(p)$ we say E_1, E_2 is a positively oriented basis for T_pM, otherwise it's a negatively oriented basis.

A C^1 patch $x : \mathcal{U}_{\text{open}} \subseteq \mathbb{R}^2 \longrightarrow V_{\text{open}} \subseteq M$ is called positively oriented when $\forall u_0 \in \mathcal{U}$, $x_1(u_0), x_2(u_0)$ is a positively oriented basis for $T_{x(u_0)}M$.

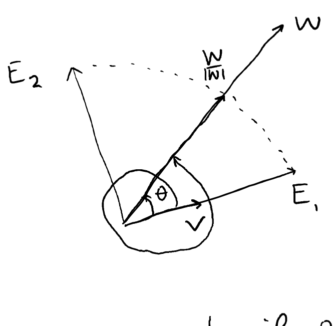
If $\mathcal{U}$ is connected then x is either $+$ or $-$ oriented.

For each $p \in M$, for each $V \in T_pM \setminus \{0\}$, define $E_1(V, p) = \frac{V}{|V|}$ and $E_2(V,p) = \nu(p) \times E_1(V,p)$, and note that $E_1(V,p), E_2(V,p)$ is a $+$'vely oriented ONB for T_pM.

Let $p, V, E_1(V,p), E_2(V,p)$ be as above. Let $W \in T_pM \setminus \{0\}$.

To say θ is a version of the oriented angle from V to W means $\theta \in \mathbb{R}$ and

$$\frac{W}{|W|} = E_1(V,p)\cos\theta + E_2(V,p)\sin\theta$$



if θ_1, θ_2 are two such versions, $\exists\, n \in \mathbb{Z}$ s.t. $\theta_1 - \theta_2 = 2\pi n$.

Note that if θ is such a version then $\cos\theta = \langle E_1, \frac{W}{|W|} \rangle = \frac{\langle V, W \rangle}{|V|\,|W|}$,

and $\sin\theta = \langle E_2, \frac{W}{|W|} \rangle = \frac{\langle \nu(p) \times V, W \rangle}{|V|\,|W|} = \frac{\langle V \times W, \nu(p) \rangle}{|V|\,|W|}$.

Now let x be a positively oriented C^1 patch about p on M.

then $\langle V, W \rangle = \sum_{i,j} g_{ij} V^i W^j$

and $\langle V \times W, \nu(p) \rangle = \langle (V^1 x_1 + V^2 x_2) \times (W^1 x_1 + W^2 x_2), \nu(p) \rangle$

$$= \langle (V^1 W^2 - V^2 W^1)(x_1 \times x_2), \nu(p) \rangle$$

$$= (V^1 W^2 - V^2 W^1) |x_1 \times x_2|$$

$$= (V^1 W^2 - V^2 W^1) \sqrt{g}$$

Thus θ is intrinsic.