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Fundamental Galois Thm

K/F : Galois

$G = \mathrm{Gal}(K/F)$.

$K \geq L \longmapsto \mathrm{Gal}(K/L) = H \leq G$

$\mathrm{Fix}\, H \longleftarrow\!\shortmid \quad H \leq G$

① $|H| = [K:L]$, $|G:H| = [L:F]$

② $L_1 \leq L_2$ iff $H_1 \geq H_2$

③ the Group corr'ing to $L_1 \cap L_2$ is $H_1 H_2$

the Group corr'ing to $L_1 L_2$ is $H_1 \cap H_2$

$$\begin{tikzcd}[column sep=small]
 & L_1L_2 \arrow[dl, dash] \arrow[dr, dash] & & & & & H_1 \cap H_2 \arrow[dl, dash] \arrow[dr, dash] & \\
L_2 \arrow[dr, dash] & & L_1 \arrow[dl, dash] & & \longleftrightarrow & H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
 & L_1 \cap L_2 & & & & & H_1H_2 &
\end{tikzcd}$$

④ Any embedding $\varphi : L/F \to K/F$ is defined by an element of G.

Embeddings are in 1-1 correspondence w/ left cosets of H in G.

$G \longrightarrow$ Embeddings.

elements of $Gal(K/L) = H$ (and only those) define trivial embeddings.

two elements of G define the same embedding if they are $\varphi\psi_1$ and $\varphi\psi_2$ where $\psi_1, \psi_2 \in H$.

⑤ $L' = \varphi(L)$ - conjugate of L.

Then $H' = \varphi H \varphi^{-1}$ - conjugate of H.

Conjugates of L are in 1-1 corr-ence w/ $G/N(H)$.

⑥ $\underbrace{L/F \text{ is normal}}_{\text{iff any embedding of } L \text{ is an automorphism; it has no conjugates except itself.}}$ iff $\underbrace{H \le G \text{ is normal}}_{\text{iff it has no conjugates except itself}}$.

In this case, $\mathrm{Gal}(L/F) = G/H$

$$\begin{aligned} G &\longrightarrow \mathrm{Gal}(L/F) \\ \varphi &\longmapsto \varphi|_L \end{aligned} \qquad \text{Kernel is } \mathrm{Gal}(K/L) = H.$$

$$\varphi(L) = L$$

<u>Example</u> ① $\underset{\substack{\shortparallel \\ K}}{\mathbb{Q}(\sqrt{2}, \sqrt{3})} / \mathbb{Q}$

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$= \{1, \varphi_1, \varphi_2, \varphi_3\}$

$\varphi_1 : \begin{array}{l} \sqrt{2} \longmapsto -\sqrt{2} \\ \sqrt{3} \longmapsto \sqrt{3} \end{array}$

$\varphi_2 : \begin{array}{l} \sqrt{2} \longmapsto \sqrt{2} \\ \sqrt{3} \longmapsto -\sqrt{3} \end{array}$

$\varphi_3 = \varphi_1 \varphi_2$

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \langle \varphi_1 \rangle \arrow[dr, dash] & \langle \varphi_2 \rangle \arrow[d, dash] & \langle \varphi_3 \rangle \arrow[dl, dash] \\ & G & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbb{Q}(\sqrt{3}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{2}) \arrow[d, dash] & \mathbb{Q}(\sqrt{6}) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

② $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$, $\omega = e^{2\pi i/3}$. $G \cong S_3$

$\tau_1 : \alpha_2 \longleftrightarrow \alpha_3$

$\tau_2 : \alpha_1 \longleftrightarrow \alpha_3$

$\tau_3 : \alpha_1 \longleftrightarrow \alpha_2$

$$\sigma : \begin{tikzcd} \alpha_1 \arrow[r, mapsto] & \alpha_2 \arrow[dl] \\ \alpha_3 \arrow[u] & \end{tikzcd}$$

$G = \{1, \sigma, \sigma^2, \tau_1, \tau_2, \tau_3\}$

permutes $\{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = \sqrt[3]{2}$, $\alpha_2 = \omega\sqrt[3]{2}$, $\alpha_3 = \omega^2\sqrt[3]{2}$.

$$\begin{tikzcd}[column sep=small] & & K \arrow[dll, dash, "2"'] \arrow[dl, dash, "2"] \arrow[d, dash, "2"] \arrow[dr, dash, "3"] & \\ \mathbb{Q}(\alpha_1) \arrow[drr, dash, "3"'] & \mathbb{Q}(\alpha_2) \arrow[dr, dash, "3"] & \mathbb{Q}(\alpha_3) \arrow[d, dash, "3"] & \mathbb{Q}(\omega) \arrow[dl, dash, "2"] \\ & & \mathbb{Q} & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & & 1 \arrow[dll, dash, "2"'] \arrow[dl, dash, "2"] \arrow[d, dash, "2"] \arrow[dr, dash, "3"] & \\ \langle\tau_1\rangle \arrow[drr, dash, "3"'] & \langle\tau_2\rangle \arrow[dr, dash, "3"] & \langle\tau_3\rangle \arrow[d, dash, "3"] & \langle\sigma\rangle \arrow[dl, dash, "2"] \\ & & G & \end{tikzcd}$$

$\mathbb{Q}(\alpha_i)$ are conjugate.

$$\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = S_3/\mathbb{Z}_3 \cong \mathbb{Z}_2.$$

$$\begin{tikzcd} & L_1L_2 \arrow[dl, dash, "n_2'"'] \arrow[dr, dash, "n_1'"] & \\ L_1 \arrow[dr, dash, "n_1"'] & & L_2 \arrow[dl, dash, "n_2"] \\ & F & \end{tikzcd}$$

$n_2' \le n_2$

$n_1' \le n_1$

$H_1 \cap H_2$

n_2'

$$\begin{tikzcd}[column sep=small]
F \arrow[rrrr, leftrightarrow, bend right] & & & & & H_1 \cap H_2 \arrow[dl, dash, "n_2'"'] \arrow[dr, dash, "n_1'"] & \\
& & & & H_1 \arrow[dr, dash, "n_1"'] & & H_2 \arrow[dl, dash, "n_2"] \\
& & & & & G &
\end{tikzcd}$$

$$[L_1L_2 : F] = [L_1 : F][L_2 : F] \quad \text{iff} \quad |G : H_1 \cap H_2| = |G : H_1| \cdot |G : H_2|.$$

$$\begin{tikzcd}[column sep=small]
& K \arrow[d, dash] & & & & 1 \arrow[d, dash] & \\
& L_1L_2 \arrow[dl, dash] \arrow[dr, dash] & & & & H_1 \cap H_2 \arrow[dl, dash] \arrow[dr, dash] & \\
L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] & & H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
& L_1 \cap L_2 \arrow[d, dash] & & & & \langle H_1, H_2 \rangle \arrow[d, dash] \arrow[r] & \text{all of } H_1H_2 \text{ should be } \langle H_1, H_2 \rangle ! \\
& F & & & & G &
\end{tikzcd}$$

We have $H_1 H_2 = \langle H_1, H_2 \rangle$ if at least one of H_1, H_2 is normal.

<u>Eg</u> $K = \mathbb{Q}(\alpha_1, \alpha_2)$ where $\alpha_1 = \sqrt[3]{2}$, $\alpha_2 = \omega \sqrt[3]{2}$.

K $\qquad\qquad\qquad\qquad$ 1

$$
\begin{tikzcd}[column sep=small]
 & K \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\alpha_1) \arrow[dr, dash] & & \mathbb{Q}(\alpha_2) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
\langle \tau_1 \rangle \arrow[dr, dash, "3"'] & & \langle \tau_2 \rangle \arrow[dl, dash, "3"] \\
 & G \cong S_3 &
\end{tikzcd}
$$

<u>Theorem</u> If L_1/F, L_2/F are subextensions of a separable extension and L_1/F is normal, then $\cancel{\left([L_1L_2 : F] = [L_1 : F]\cdot[L_2 : F]. \quad L_1 \cap L_2 = F \right)}$