Lec 3/2

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Propn: Let f be a cts map from S^{d-1} into a top. sp. X. Then TFAE:

(a) f is null-homotopic in X

(b) f can be extended to a cts map from $\mathbb{B}^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$ into X.

Pf homework.

Cor. Let γ be a loop in a top. sp. X. Then TFAE:

(a) γ is null-homotopic in X

(b) γ can be extended to a cts map from $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$ into X.

Propn Let (X, ρ) be a metric space. Let $\emptyset \neq A \subseteq X$. $\forall x \in X$, let

$\rho(x, A) = \inf\{\rho(x, a) : a \in A\}$. Then $\forall x, y \in X$, $\rho(x, A) \leq \rho(x, y) + \rho(y, A)$.

Pf $\forall a \in A$, $\underset{\underset{\rho(x,A)}{\vee\!|}}{\rho(x,a)} \leq \rho(x,y) + \rho(y,a)$, ~~so taking inf of both sides gives result.~~

or give a proper pf.

So $\rho(x,A) - \rho(x,y) \leq \rho(y,a)$, so $\rho(x,A) - \rho(x,y)$ is a lower bound

dumb slut
blunts fuck
punk rock

for $\{\rho(y,a) : a \in A\}$, and so is $\leq \inf\{\rho(y,a) : a \in A\} = \rho(y,A)$ $\square$

Corollary Let (X, ρ) be a metric space. Let $\emptyset \neq A \subseteq X$. Then

a) $\forall x, y \in A$, $|\rho(x,A) - \rho(y,A)| \leq \rho(x,y)$

b) The fn $x \longmapsto \rho(x,A)$ is cts (in fact unif. cts. In fact lipschitz).

Pf (b) clearly follows from (a) so let's prove (a). Let $x, y \in X$.

by Propn, $\rho(x, A) \le \rho(x,y) + \rho(y,A) \Rightarrow \rho(x,A) - \rho(y, A) \le \rho(x,y)$

likewise, $\rho(y,A) - \rho(x,A) \le \rho(y,x) = \rho(x,y)$. $\square$

Corollary Let (X, ρ) be a metric space. Let A and B be disjoint closed subsets of X. Then there is a cts fn $f : X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$.

~~pf~~ to avoid trivialities, suppose $A \neq \emptyset$ and $B \neq \emptyset$. define f on X by (Notice that $\forall a \in A$, $\rho(a, B) > 0$ and $\rho(a, A) = 0$. Similarly, $\forall b \in B$, $\rho(b,A) > 0$ and $\rho(b,B) = 0$.)

$$f(x) = \frac{\rho(x,A)}{\rho(x,A) + \rho(x,B)}.$$

It's cts since all things are cts, and it takes values in $[0,1]$ since everything is ≥ 0 and $\rho(x,A) \le \rho(x,A) + \rho(x,B)$. $\square$

Urysohn's Lemma (1920s): Let X be a top. sp. Then TFAE:

(a) X is normal

(b) $\forall$ disjoint closed sets $A, B \subseteq X$, $\exists f : X \to [0,1]$ s.t. $f = 0$ on A, $f = 1$ on B.

(to say X is normal means $\forall$ disjoint closed $A, B \subseteq X$, $\exists U, V \subseteq X$ s.t. $U \cap V = \emptyset$, U, V are open, and $U \supseteq A$, $V \supseteq B$.)

(Pizza)

The Tietze Extension Theorem:

Let X be a metric space (or more generally a normal space).

Let $C \subseteq X$ be closed and let $f : C \to \mathbb{R}$ be continuous.

then f can be extended to a continuous fn $g: X \longrightarrow \mathbb{R}$.

Pf First let's suppose $f: C \longrightarrow [-a, a]$ where $a \in (0, \infty)$. Let $E = \{f \geq \frac{a}{3}\}$ and $F = \{f \leq \frac{a}{3}\}$. These are disjoint and closed in X so $\exists$ a cts fn $g_1: X \longrightarrow [-\frac{a}{3}, \frac{a}{3}]$ s.t. $g_1(E) = \{\frac{a}{3}\}$ and $g_1(F) = \{-\frac{a}{3}\}$.

$\forall x \in E$, $|f(x) - g_1(x)| = f(x) - \frac{a}{3} \leq a - \frac{a}{3} = \frac{2a}{3}$.

$\forall x \in F$, $|f(x) - g_1(x)| = -\frac{a}{3} - f(x) \leq -\frac{a}{3} + a = \frac{2a}{3}$.

$\forall x \in C \setminus (E \cup F)$, $|f(x) - g_1(x)| \leq |f(x)| + |g_1(x)| \leq \frac{a}{3} + \frac{a}{3} = \frac{2a}{3}$.

thus $\forall x \in C$, $|f(x) - g_1(x)| \leq \frac{2a}{3}$.

Now apply same argument to $f - g_1$. We find there is a cts fn $g_2: X \to [-\frac{2a}{3^2}, \frac{2a}{3^2}]$ s.t. $\forall x \in C$, $|f(x) - g_1(x) - g_2(x)| \leq \frac{2^2 a}{3^2}$.

Continue in this way constructing $\langle g_n \rangle$ s.t. $\forall x \in X$, $|g_n(x)| \leq \frac{2^{n-1}}{3^n} a$, so by Weierstrass M-test, $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly & absolutely for all $x \in X$. call limit fn g. also $|f(x) - g(x)| = \lim_{N \to \infty} |f(x) - \sum_{n=1}^{N} g_n(x)| \leq \lim_{n \to \infty} (\frac{2}{3})^n a = 0$.

So g is the extension of f to a fn on X (g is cts).

Also g maps X into $[-a, a]$.

Now suppose $f: C \longrightarrow (-1, 1)$. By what we showed, there is a cts fn $u: X \longrightarrow [-1, 1]$ s.t. $u = f$ on C.

Let $D = \{|u| = 1\}$. This is closed in X and $C \cap D = \emptyset$.

So there's a continuous fn $v: X \longrightarrow [0, 1]$ s.t. $v = 1$ on C and $v = 0$ on D.

then $uv = f$ on C and $uv = 0$ on D so $uv: X \longrightarrow (-1, 1)$ is an extension of f on X.

Now consider $f : C \longrightarrow \mathbb{R}$. then $x \xmapsto{\text{cts}} \underset{\displaystyle \frac{f(x)}{1+|f(x)|}}{\overset{\varphi(x)}{\shortparallel}}$ maps C into $(-1,1)$.

hence $\exists$ cts $\Psi : X \longrightarrow (-1,1)$ s.t. $\Psi = \varphi$ on C

then let $g(x) = \dfrac{\Psi(x)}{1-|\Psi(x)|}$. $g : X \longrightarrow \mathbb{R}$ and $g = f$ on C.