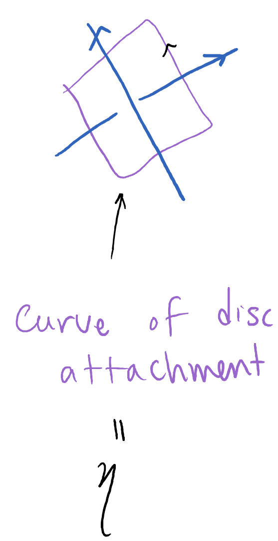
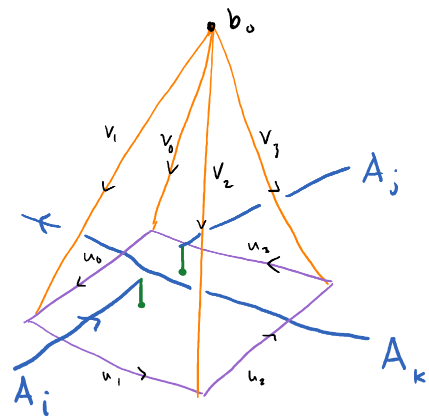
10/9

Wednesday, October 9, 2019 15:01

gluing in a disc along a curve $\longleftrightarrow$ dividing π_1 by the curve

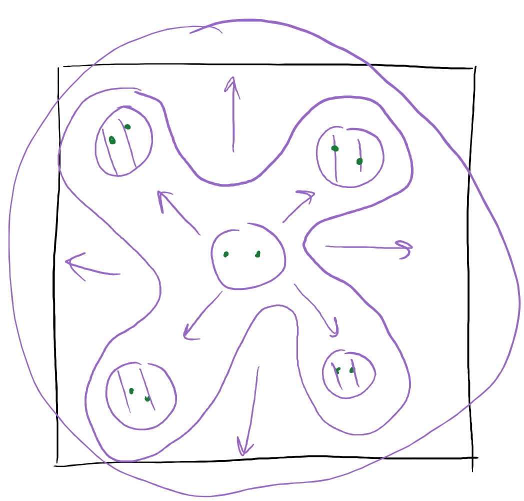




$x_i \in \pi_1$ corresponding to A_i

so $\eta = x_j \; x_k^{-1} \; x_i^{-1} \; x_k$

(inverse rel'n for positive crossing)

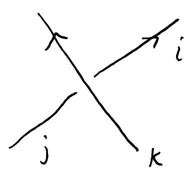


- the last relation is trivial, unnecessary

Wirtinger Presentation

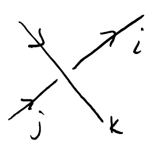
Given link-diagram $\hat{L}$ w/ N crossings

* Choose an orientation on $\hat{L}$.
* Denote each max'l over-crossing arc by a label $j \in \mathcal{A}$, $|\mathcal{A}| = N$.
* For each $j \in A$ introduce generator x_j
* Choose subset $\mathcal{C}$ w/ $(N-1)$ crossings.
* for each $c \in \mathcal{C}$ assign relators:

$c =$ 

$$\rightsquigarrow \quad r_c = x_j x_k^{-1} x_i^{-1} x_k$$

$$\left(x_i = x_k x_j x_k^{-1} \right)$$

$c =$ 

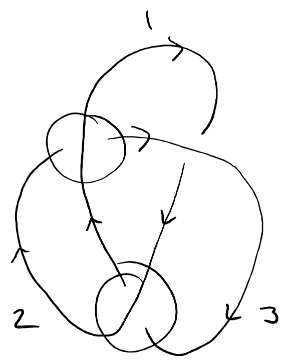
$$\rightsquigarrow \quad r_c = x_j x_k x_i^{-1} x_k^{-1}$$

$$\left(x_i = x_k^{-1} x_j x_k \right)$$

Thm $\pi_L \cong \langle x_j : j \in \mathcal{A} \mid r_c : c \in \mathcal{C} \rangle$

(defect $= 1$)

Example



$$\pi \cong \langle x_1, x_2, x_3 \mid x_3 = x_1 x_2 x_1^{-1},\ x_1 = x_2 x_3 x_2^{-1} \rangle$$

$$\cong \langle x_1, x_2 \mid x_1 = x_2 x_1 x_2 x_1^{-1} x_2^{-1} \rangle$$

$$\cong \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

$$\cong B_3 \leftarrow \text{Braid gp}$$

Conjugacy Classes of π_{Trefoil} = [S', Trefoil gp element]

$\downarrow$

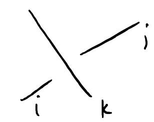
{lass of links w/ Braid index ≤ 3}

Trefoil $= T_{3,2}$, so

$$\pi \cong \langle u, v \mid u^2 = v^3 \rangle \quad \text{as well.}$$

Exercise: find explicit isomorphism.

Abelianization:



$$x_i = x_k x_j x_k^{-1}$$

$$\rightsquigarrow$$

$$x_i = x_j$$

So each part in the link gives $\mathbb{Z}$ to homology.

Pick any finite group G.

$$X_L^G = \mathrm{Hom}(\pi_L, G)$$

$$\begin{array}{ccc} \rho : \pi_L & \longrightarrow & G \\ \uparrow & & \\ \langle x_1, \dots, x_N \rangle & & \end{array}$$

given by $(\rho(x_1), \dots, \rho(x_N)) \in G^{\times N}$
(maybe, we need
$g_j = g_k g_i g_k^{-1}$

$$\text{So } \left|\chi_L^G\right| \leq N \cdot |G|$$

Knot Invariant

$$\begin{tikzcd}
\pi_L \arrow[r, "\rho"] \arrow[d] & G \\
\frac{\pi_L}{\pi_L'} = H_1 = \mathbb{Z}^k \arrow[ur] &
\end{tikzcd}$$

abelian

components

$$\text{So } \left|\chi_L^G\right| = k \cdot |G|$$

So abelian gps are boring.

Smallest non-ab gp $S_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3 = D_6$

<u>Theorem</u>: $\left|\chi_K^{S_3}\right| = 3 + (\text{Fox -3-coloring } \#)$

Knot

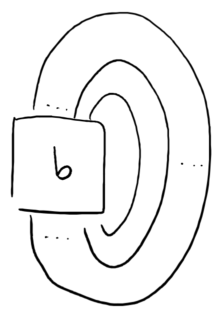
<u>Fox 3-coloring</u> #:

# of proper colorings

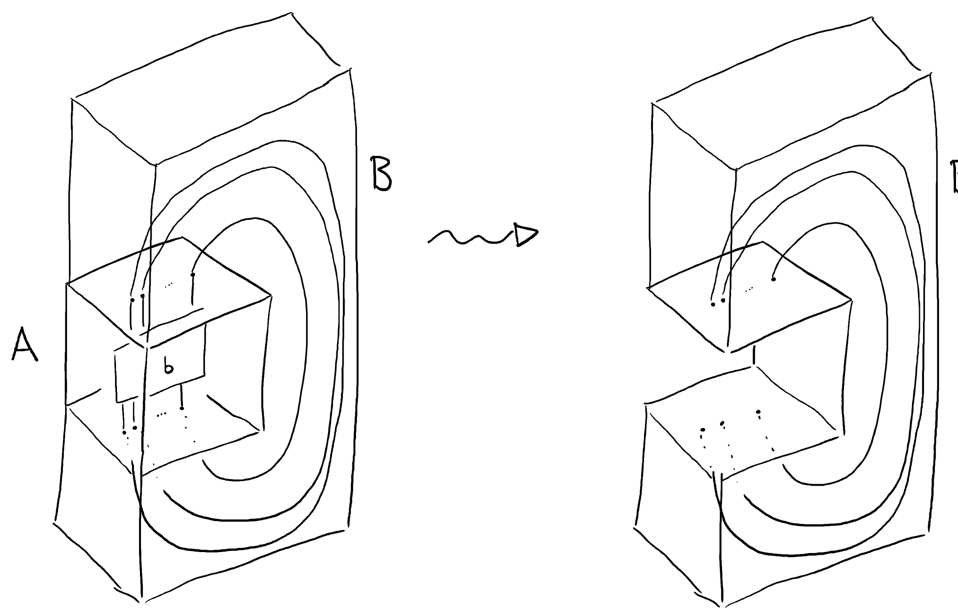
$$\begin{tikzcd}
\pi_K \arrow[r, "\rho"] \arrow[dr] & \mathbb{Z}_2 \ltimes \mathbb{Z}_3 \arrow[r, two heads] & \mathbb{Z}_2 \\
& \mathbb{Z} = H_1 \arrow[ur] &
\end{tikzcd}$$

hint for proof of theorem

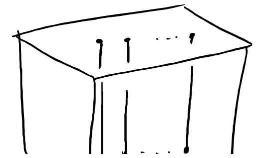
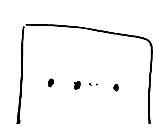
Getting π_1 thru braids

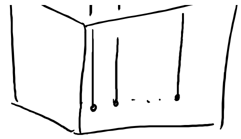
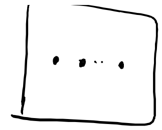
$L = cl(b) =$  .

$A = \square \times [0,1] \setminus b$

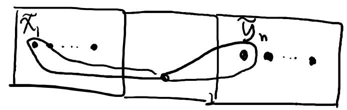


$(\text{box} - cl(b) = A \cup B)$

$B \approx$  $\approx$ 

$B \approx$  $\approx$ 

So $\pi_1(B) = F_n$

$P = A \cap B =$ 

$$\pi_1(P) = F_{2n}$$

$$\begin{tikzcd} \pi_1(P) \arrow[r] \arrow[d] & \pi_1(B) \\ \pi_1(A) & \end{tikzcd}$$

$$\tilde{x}_i,\ \tilde{y}_i \longmapsto \bar{x}_i$$

$$\tilde{x}_i \mapsto x_i \qquad \tilde{y}_i \mapsto y_i$$

$$\pi_1(A \underset{P}{\cup} B) = \pi_1(A) * \langle \bar{x}_1, \dots, \bar{x}_n \rangle \Big/ \left\langle \begin{array}{l} x_i = \bar{x}_i \\ y_i = \bar{y}_i \end{array} \right\rangle$$

$$= \pi_1(A) \big/ \langle x_i = y_i \rangle$$