Lec 11/18

Friday, November 18, 2016 9:13 AM

$\sum_{j=1}^{\infty} a_j$ $\quad q =$ convergence parameter $\in [0, \infty]$

$= $ largest clusterpoint of $\{\sqrt[n]{a_n}\}_{n=1}^{\infty}$

$= \limsup \sqrt[n]{a_n}$

Main convergence thm: (refined version of root test)

1) if $q=0$, $\sum a_j$ converges faster than any geom. series
2) if $0< \; <1$, then $\sum a_j$ converges faster than $\sum b s^{n-1}$ where $q < s < 1$
3) if $q > 1$, $\sum a_j$ diverges
4) if $q = 1$, indeterminate (if cvgs, is slower than any geom. series)

<u>Proof</u>: for (1) and (2), see yesterday.

In case (3), use fundamental divergence theorem.

$\lim_{i \to \infty} |a_{n_i}|^{1/n_i} = q > 1$ for some subsequence

$|a_{n_i}| > 1$ for i large enough.

$\lim_{n \to \infty} a_n \neq 0$, so series diverges.

(4) • $\sum \frac{1}{\sqrt{j}} = \sum a_n$. $\quad |a_n|^{1/n} = |n^{-1/2}|^{1/n}$

$\hookrightarrow$ diverges.

$= \left[e^{-\ln(n)/2n \to 0}\right] \to 1$

• $\sum \left(\frac{1}{j} - \frac{1}{j+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots$

$S_n = 1 - \frac{1}{n+1} \to 1$ as $n \to \infty \to$ converges.

$$= \sum \left(\frac{1}{j(j+1)}\right), \qquad |a_n|^{1/n} = \left(\frac{1}{n(n+1)}\right)^{1/n} = e^{\frac{-\ln(n(n+1))}{n} \to 0} \Big] \to 1$$

To compute the value of a series $\sum_{j=1}^{\infty} a_j$ with $q<1$:

(1) First pick an $s \in (q,1)$, any convenient choice (e.g. $\frac{q+1}{2}$)

(2) then determine N so that $|a_n|^{1/n} < s$ for $n > N$.

(3) find $n > N$ large enough s.t. $R_n^{\text{geom}} = \frac{s^{n+1}}{1-s} < \varepsilon$ - tolerance you want in answer.

See notes packet p. 33 for an example

not lim sup.

The Ratio Test: (Theorem): If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = q$, then q is the convergence parameter.

Proof Sketch: Pick $r < q < s$. Then for $n \geq N$ (some N), $r < \frac{|a_{n+1}|}{|a_n|} < s$.

so $|a_n| r < |a_{n+1}| < |a_n| s$ for $n \geq N$.

iterating this for $n = N, N+1, N+2, \ldots$

we obtain

$$|a_N| r^{n-N} \leq |a_n| \leq |a_N| s^{n-N} \quad \text{for } n \geq N.$$

$$|a_N| r^{-N} r^n \leq |a_n| \leq |a_N| s^{-N} s^n$$

$$\underbrace{(|a_N| r^{-N})^{1/n}}_{\downarrow\ 1 \text{ as } n\to\infty} r \leq |a_n|^{1/n} \leq \underbrace{(|a_N| s^{-N})^{1/n}}_{\downarrow\ 1 \text{ as } n\to\infty} s$$

This means $r \leq |a_n|^{1/n} \leq s$, so $r \leq \limsup |a_n|^{1/n} \leq s$

but r, s can be as close to q as you like, so

$\limsup |a_n|^{1/n} = q$. ∎