Lec 10/29

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R comm ring
$\cup$
S mult. closed set $\rightsquigarrow$ $S^{-1}R$ ring of fractions

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$$\left[\begin{array}{c} j: R \longrightarrow S^{-1}R \\ a \longmapsto \frac{a}{1} \end{array}\right]$$ ring homomorphism.

$$\mathrm{Ker}(j) = \{ a \in R : \tfrac{a}{1} = 0 \iff ta = 0 \text{ for some } t \in S \}$$

<u>Additionally</u>

(1) $\forall I \subset R$ ideal, $\left\{ \frac{a}{s} : \begin{array}{c} a \in I \\ s \in S \end{array} \right\} =: S^{-1}I \leq S^{-1}R$ ← is an ideal in $S^{-1}R$

and $S^{-1}I = (j(I))$

(2) Every ideal $\tilde{I}$ of $S^{-1}R$ is of this form, i.e. $\tilde{I} = S^{-1}I$ for some $I \subset R$ ← ideal.

In fact, $I = j^{-1}(\tilde{I})$, so $\tilde{I} = S^{-1}(j^{-1}(\tilde{I}))$

More properties of $I \rightsquigarrow S^{-1}I$: $I_1, I_2 \subset R$ ideals.

(I)

(1) $S^{-1}(I_1 + I_2) = S^{-1}I_1 + S^{-1}I_2$

(2) $S^{-1}(I_1 \cap I_2) = S^{-1}I_1 \cap S^{-1}I_2$

(3) $S^{-1}(I_1 \cdot I_2) = S^{-1}I_1 \cdot S^{-1}I_2$

Proof is easy. use generators.

(II) $S^{-1}I = S^{-1}R \iff I \cap S \neq \emptyset$

Proof: $(\Leftarrow)$ Pick any $a \in I \cap S$. then
$1 = \frac{1}{a} \cdot a \in S^{-1}I \Rightarrow S^{-1}I = S^{-1}R$.

$(\Rightarrow)$ Know $1 \in S^{-1}I$. write $1 = \frac{1}{1} = \frac{a}{s}$ for some $a \in I$, $s \in S$.

by defn, $\exists t \in S$ s.t. $t(s-a) = 0$. but $ts \in S$ and $ta \in I$

So $ts = ta$ and so $\underset{\emptyset}{\overset{\#}{I \cap S}} \ni ts = ta$.

(III) $\forall$ ideal I in R we have $j^{-1}(S^{-1}I) = \{r \in R : tr \in I \text{ for some } t \in S\}$

Pf: pick $r \in j^{-1}(S^{-1}I) \iff j(r) = \frac{r}{1} \in S^{-1}I$ $\frac{r}{1} = \frac{a}{s}$ for some $a \in I$, $s \in S$.

so $t(rs - a) = 0 \iff (ts)r = ta \in I$.

(IV) Prime ideals in R Not intersecting S $\longleftrightarrow$ Prime ideals in $S^{-1}R$

$$P \longmapsto S^{-1}P$$
$$j^{-1}(p) \longleftarrow\!\shortmid\ \tilde{P}$$

Step 1 $\underset{\text{prime ideal}}{P \subsetneq R}$ and $P \cap S = \emptyset$ $\Longrightarrow S^{-1}P \subsetneq S^{-1}R$ is prime ideal

Proof pick $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in S^{-1}R$ w/ $\frac{a_1 a_2}{s_1 s_2} \in S^{-1}P$. So we can find
$p \in P$, $s \in S$ s.t. $\frac{a_1 a_2}{s_1 s_2} = \frac{p}{s} \iff tsa_1a_2 = tps_1s_2$ for some $t \in S$.

So $\underbrace{t s}_{\substack{\notin \\ S}} \underbrace{a_1 a_2}_{\substack{\notin \\ P}} \in P$ since $S \cap P = \emptyset$ so a_1 or $a_2 \in P$ $\Rightarrow$ $\frac{a_1}{s_1}$ or $\frac{a_2}{s_2} \in S^{-1}P$.

Step 2 $\underset{\text{prime ideal}}{\tilde{P} \subsetneq S^{-1}R} \Longrightarrow \overset{P}{j^{-1}(\tilde{P})} \subsetneq R$ is prime ideal and $P \cap S = \emptyset$.

pf it suffices to show $\underset{P \cap S = \emptyset}{P \subsetneq R \text{ prime}} \Rightarrow j^{-1}(S^{-1}P) = P$.

Clearly $P \subseteq j^{-1}(S^{-1}P)$. for the other inclusion,

pick $r \in j^{-1}(S^{-1}P) \underset{(\text{II})}{\Longleftrightarrow} tr \in P$ for some $t \in S$.

Since $P \cap S = \emptyset$, we conclude $r \in P$.

why it suffices $\longrightarrow$ Finally, $S^{-1}(j^{-1}\tilde{P}) = \tilde{P}$ (this is true for any $\tilde{P}$ ideal)

Hence $\begin{array}{c} P \longmapsto S^{-1}P \\ j^{-1}(P) \longleftarrow\!\shortmid\, \tilde{P} \end{array}$ are inverse to each other.

⚠ this correspondence breaks for non-prime ideals.

i.e. $j^{-1}(S^{-1}I) \neq I$ for some I with $I \cap S$.

Ex: $R = K[X,Y]$ for some field K.

$S = R \setminus (X)$ which is multiplicatively closed

$I = (XY)$

$$S^{-1}R = \left\{ \frac{f(x,y)}{g(x,y)} : g \text{ is NOT div. by } X \right\}$$

$Y \in S$. $YX \in I$. so $X \in j^{-1}(S^{-1}I)$ but $X \notin I$.

Ex: in fact, $j^{-1}(S^{-1}I) = (X) \supsetneq (XY) = I$

practice it!