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Friday, April 5, 2019 11:28

$$\begin{tikzcd}[column sep=small]
 & K = L_1L_2 \arrow[dl, no head] \arrow[dr, no head] & \\
L_1 \arrow[dr, no head] & & L_2 \arrow[dl, no head] \\
 & L_1 \cap L_2 \arrow[d, no head] & \\
 & F &
\end{tikzcd}$$

$\leftarrow$ this is normal bc L_1 and L_2 / F / F.

$$\mathrm{Gal}(K/L_1) \cong \mathrm{Gal}(L_2/(L_1 \cap L_2)) \leq \mathrm{Gal}(L_2/F)$$

relative direct product

$$\mathrm{Gal}(K/F) \cong \mathrm{Gal}(L_1/F) \times_{\mathrm{Gal}(L_1L_2/F)} \mathrm{Gal}(L_2/F)$$

<u>Example</u>: K = spl. field of $(x^3-2)(x^3-3) \in \mathbb{Q}[x]$.

$$K = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}, \omega = e^{2\pi i/3})$$

$$\begin{tikzcd}[column sep=small]
 & K \arrow[dl, no head, "3"'] \arrow[dr, no head, "3"] & \\
\mathbb{Q}(\sqrt[3]{2}, \omega) \arrow[dr, no head, "3"'] & & \mathbb{Q}(\sqrt[3]{3}, \omega) \arrow[dl, no head, "2"] \\
 & \mathbb{Q}(\omega) \arrow[d, no head, "2"] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\mathrm{Gal}(x^3-3) \cong S_3$$

$$\mathrm{Gal}(x^3-2) \cong S_3$$

$$\mathrm{Gal}(K/\mathbb{Q}) = S_3 \times_{\mathbb{Z}_2} S_3$$

$$= \{(\varphi_1, \varphi_2) \text{ s.t. } \varphi_1 = \varphi_2 \bmod \mathbb{Z}_3\}$$

$\boxed{14.2}$ ① $m_{\alpha,\mathbb{Q}}$, $\alpha = \sqrt{2} + \sqrt{5}$

Galois Group of some Galois.

Galois group of ... Galois extension containing α.

Conjugates of α are $\pm\sqrt{2} \pm \sqrt{5}$ $\longrightarrow$ conjugates are $G \cdot \alpha$.

since Spl. field is $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ and

$$m_{\alpha,\mathbb{Q}} = (x-(\sqrt{2}+\sqrt{5}))(x-(\sqrt{2}-\sqrt{5}))(x-(-\sqrt{2}+\sqrt{5}))(x-(-\sqrt{2}-\sqrt{5}))$$

(2) Same but $\alpha = 1+\sqrt[3]{2}+\sqrt[3]{4}$.

Spl. field of x^3-2 is $\mathbb{Q}(\sqrt[3]{2}, \omega = e^{2\pi i/3})$.

$$\sqrt[3]{2} \longmapsto \sqrt[3]{2},\ \omega\sqrt[3]{2},\ \omega^2\sqrt[3]{2}.$$

$$\sqrt[3]{4} \longmapsto \sqrt[3]{4},\ \omega^2\sqrt[3]{4},\ \omega\sqrt[3]{4} \longleftarrow \underline{\text{not}} \text{ independently.}$$

$$m_{\alpha,\mathbb{Q}}(x) = (x-(1+\sqrt[3]{2}+\sqrt[3]{4}))(x-(1+\omega\sqrt[3]{2}+\omega^2\sqrt[3]{4}))(x-(1+\omega^2\sqrt[3]{2}+\omega\sqrt[3]{4}))$$

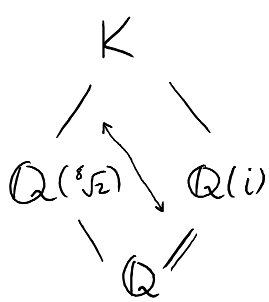
(6,7) $\mathrm{Gal}(x^8-2)$. $K = \mathbb{Q}(\sqrt[8]{2}, \omega = e^{2\pi i/8} = \frac{1+i}{\sqrt{2}}) = \mathbb{Q}(\sqrt[8]{2}, i)$. ← deg 16

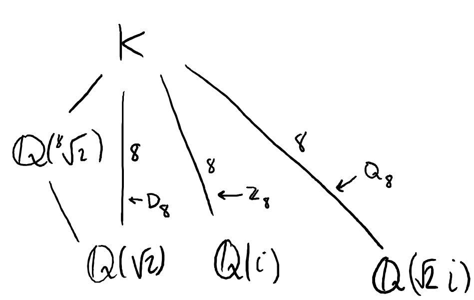
$$= \mathbb{Q}(\sqrt[8]{2}) \cdot \mathbb{Q}(\omega) \longleftarrow \text{but they intersect}$$

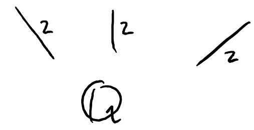
$$8 \cdot 4 = 32 > 16$$

$$= \mathbb{Q}(\sqrt[8]{2}) \cdot \mathbb{Q}(i) \longleftarrow \text{no intersection}$$

$$8 \cdot 2 = 16$$







$$\mathrm{Gal}(K/\mathbb{Q}) = \underset{\overset{?\,\Vert}{\mathbb{Z}_8}}{\mathrm{Gal}(K/\mathbb{Q}(i))} \rtimes \underset{\overset{\Vert}{\mathbb{Z}_2}}{\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})}$$

$$\mathrm{Gal}(\mathbb{Q}(\overset{\overset{\alpha}{\Vert}}{\sqrt[8]{2}}, i)/\mathbb{Q}(i)) = ?$$

8

$$m_{\alpha, \mathbb{Q}(i)} = x^8 - 2$$

$$\varphi : \alpha \mapsto \alpha\omega, \quad \varphi(i) = i. \qquad \omega = \frac{1+i}{\sqrt{2}}$$

$\sqrt{2} = \alpha^4 \longmapsto \alpha^4\omega^4 = -\alpha^4 = -\sqrt{2}$, so $\omega \mapsto \frac{1+i}{-\sqrt{2}} = -\omega$

$$\varphi : \alpha \mapsto \alpha\omega \mapsto -\alpha\omega^2 \mapsto -\alpha\omega^3 \mapsto \alpha\omega^4$$
$$\mapsto \alpha\omega^5 \mapsto -\alpha\omega^6 \mapsto -\alpha\omega^7 \mapsto \alpha$$

but $-\omega = \omega^5$, so the sequence is

$$\alpha \mapsto \alpha\omega \mapsto \alpha\omega^6 \mapsto \alpha\omega^7 \mapsto \alpha\omega^4 \mapsto \alpha\omega^5 \mapsto \alpha\omega^2 \mapsto \alpha\omega^3 \mapsto \alpha$$

So φ has order 8, so $\mathrm{Gal}(\mathbb{Q}(\sqrt[8]{2}, i)/\mathbb{Q}(i)) = \langle\varphi\rangle \cong \mathbb{Z}_8$.

(it's enough to check that order of φ is > 4).

Now, what sort of Semidirect product do we have?

let $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) = \mathbb{Z}_2 = \langle \psi \rangle$.

then what is $\psi \varphi \psi^{-1}$?

it is φ^k for some k.

$\psi : i \longmapsto -i$.

$\psi : \alpha \longmapsto \alpha$

$\psi : \omega \longmapsto \frac{1-i}{\sqrt{2}} = \omega^{-1} = \omega^7$

So $\psi \varphi \psi^{-1} : \alpha \longmapsto \alpha \longmapsto \alpha\omega \longmapsto \alpha\omega^{-1} = \alpha\omega^7$

So $\psi \varphi \psi^{-1} = \varphi^3$

So $\mathrm{Gal}(K/\mathbb{Q}) = \langle \varphi, \psi : \varphi^8 = \psi^2 = 1, \ \psi\varphi\psi = \varphi^3 \rangle$

$= \mathbb{Z}_8 \rtimes \mathbb{Z}_2$