Read about Symmetric Tensors

Alternating Tensors:

Let $A(M)$ be the two-sided ideal in $\mathcal{T}(M)$ generated by $\{u \otimes u : u \in M\}$.

The exterior algebra of M is

$$\Lambda(M) = \mathcal{T}(M)/A(M).$$

$A(M)$ is a graded ideal, so $\Lambda(M)$ is a graded algebra.

$$\Lambda(M) = \bigoplus_{n=0}^{\infty} \Lambda^n(M) = R \oplus M \oplus \Lambda^2(M) \oplus \cdots$$

instead of $\otimes$, the operation on $\Lambda(M)$ is denoted by $\wedge$:

$u_1 \wedge u_2 \wedge \cdots \wedge u_n$ instead of $u_1 \otimes u_2 \otimes \cdots \otimes u_n$.

$\wedge$ is called "exterior product" instead of tensor product.

$\forall u_1, u_2 \in M$, we have $(u_1 + u_2) \wedge (u_1 + u_2) = 0$

$$= \cancel{u_1 \wedge u_1} + u_1 \wedge u_2 + u_2 \wedge u_1 + \cancel{u_2 \wedge u_2}$$

$$\Rightarrow \quad u_1 \wedge u_2 = -u_2 \wedge u_1$$

$$(u_1 \wedge u_2) \wedge u_3 = -u_1 \wedge u_3 \wedge u_2 = u_3 \wedge (u_1 \wedge u_2).$$

If $\omega_1 \in \Lambda^n(M)$, $\omega_2 \in \Lambda^m(M)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{nm} \omega_2 \wedge \omega_1.$$

Example: differential forms on a smooth manifold:

$f(x)\, dx_1 \wedge \dots \wedge dx_k$: elements of the exterior algebra of the module of covectors (1-forms) df over the ring of smooth functions

Let $n \in \mathbb{N}$. define $\mathrm{Alt}_n : \mathcal{T}^n(M) \longrightarrow \mathcal{T}^n(M)$

$$\omega \longmapsto \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot \sigma(\omega)$$

eg: $\mathrm{Alt}_2(u_1 \otimes u_2) = u_1 \otimes u_2 - u_2 \otimes u_1$

$$\begin{aligned} \mathrm{Alt}_3(u_1 \otimes u_2 \otimes u_3) = {} & u_1 \otimes u_2 \otimes u_3 \\ & - u_2 \otimes u_1 \otimes u_3 - u_1 \otimes u_3 \otimes u_2 - u_3 \otimes u_2 \otimes u_1 \\ & + u_2 \otimes u_3 \otimes u_1 + u_3 \otimes u_1 \otimes u_2 \end{aligned}$$

Def: $\Lambda T^n(M)$ is the submodule of $\mathcal{T}^n(M)$ consisting of antisymmetric tensors: alternating

$$\sigma(\omega) = \mathrm{sign}(\sigma) \cdot \omega$$

$\mathrm{Alt}_n : \mathcal{T}^n(M) \longrightarrow \Lambda\mathcal{T}^n(M)$, $\quad \mathrm{Ker}(\mathrm{Alt}_n) = A(M)$

So $\Lambda^n(M) = \mathcal{T}^n(M)/A^n(M) \cong \mathrm{Alt}_n(\mathcal{T}^n(M))$ $\left(\approx \Lambda\mathcal{T}^n(M), \text{ equal if } n! \in R^\times \right)$.

One more Construction:

If $\varphi_1 : M_1 \longrightarrow N_1$ & $\varphi_2 : M_2 \longrightarrow N_2$ are R-mod hom.

Then $\varphi_1 \otimes \varphi_2 : M_1 \otimes M_2 \longrightarrow N_1 \otimes N_2$ is defined by

$$\varphi_1 \otimes \varphi_2 (u_1 \otimes u_2) = \varphi_1(u_1) \otimes \varphi_2(u_2).$$

$\varphi_1 \in \mathrm{Hom}(M_1, N_1)$, $\varphi_2 \in \mathrm{Hom}(M_2, N_2)$

So we should have: $\varphi_1 \otimes \varphi_2 \in \mathrm{Hom}(M_1, N_1) \otimes \mathrm{Hom}(M_2, N_2)$?

What if R is not commutative?

$$M_1 \otimes M_2 = ?$$

is it $R(M_1 \times M_2) / \text{relations}$? maybe.

Suppose it is: for $a, b \in R$, $u_1 \in M_1, u_2 \in M_2$

$$ab(u_1 \otimes u_2) = au_1 \otimes bu_2 = ba(u_1 \otimes u_2).$$

So $M_1 \otimes M_2$ as defined earlier is over

the commutative ring $R/(\{ab-ba\})$.

instead, we throw out the relation

$a(u \otimes v) = au \otimes v = u \otimes av$.

and we say:

M_1 : right R-module

M_2 : left R-module,

and relation is

$$u_1 a \otimes u_2 = u_1 \otimes a u_2$$

So $M_1 \otimes_R M_2$ is just an abelian group (not a module).

$\beta: M_1 \times M_2 \longrightarrow N$ is balanced if

$\beta(u_1 a, u_2) = \beta(u_1, a u_2) \quad \forall a \in R, u_1 \in M_1, u_2 \in M_2$.

$\beta(u_1 + v_1, u_2) = \beta(u_1, u_2) + \beta(v_1, u_2)$ et. cetera.

if M_1 is a 2-sided R-module,

$M_1 \otimes M_2$ is an R-module: $a(u_1 \otimes u_2) = a u_1 \otimes u_2$.