Lec 10/10

Monday, October 10, 2016 9:06 AM

Proposition (power rule)

If n is a positive integer

$f(x) = x^{1/n} = \sqrt[n]{x}$ for $x > 0$

$f'(a) = \frac{1}{n} a^{1/n - 1}$ for $a > 0$

Proof: $\lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{x - a} = \lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{(x^{1/n} - a^{1/n})\left(\sum_{j=0}^{n-1} x^{j/n} a^{\frac{n-j}{n}}\right)}$

Easier proof: (using chain rule)

Let $g(y) = y^n$. then $(g \circ f)(x) = (\sqrt[n]{x})^n$ for $x > 0$

chain rule:

$$(g \circ f)'(x) = 1 = g'(f(a))\, f'(a)$$

$$1 = n\, a^{1/n\; n-1} f'(a)$$

$$\frac{1}{n a^{\frac{n-1}{n}}} = f'(a) = \frac{1}{n} a^{\frac{1}{n} - 1}$$

This is only valid if we know $f'(a)$ exists beforehand.

Corollary: If n is odd and $f(x) = x^{1/n}$ for $x \neq 0$

then $f'(a) = \frac{1}{n} a^{1/n - 1}$ for $a \neq 0$.

Proof: if $a > 0$, already proved

if $a < 0$, then $f(x) = x^{1/n} = -(-x)^{1/n} = (-f \circ h)(x)$ where $h(x) = -x$

then $-a = h(a) = b > 0$ so by chain rule

$f'(a) = -f'(b) \cdot h'(a)$

$= -\left(\frac{1}{n} b^{1/n - 1}\right) \cdot (-1)$

$$= \frac{1}{n}(-a)^{1/n - 1}$$

$$= \frac{1}{n}(-a)^{\frac{1-n}{n}}$$

$$= \frac{1}{n}\left(\sqrt[n]{-a}\right)^{1/n} = \frac{1}{n}\left(-\sqrt[n]{a}\right)^{\overbrace{1-n}^{\text{even}}} = \frac{1}{n}\left(\sqrt[n]{a}\right)^{1-n}$$

$$f'(a) = \frac{1}{n} a^{\frac{1}{n}-1} \qquad \blacksquare$$

Theorem (Power rule for rational exponents.

Let $r = \frac{p}{q}$, $p \in \mathbb{Z}$ $q \in \mathbb{Z}^+$. Let $f(x) = x^r = \left(\sqrt[q]{x}\right)^p$ for $x > 0$ if q even, for $x \neq 0$ if q odd

Then $f'(a) = r a^{r-1}$ for all $a \in \operatorname{dom} f$.

Proof: $f(x) = g(h(x))$ where $g(x) = x^p$ and $h(x) = \sqrt[q]{x} = x^{1/q}$

$$f'(a) = g'(h(a))\, h'(a)$$

$$= p\left(a^{1/q}\right)^{p-1} \frac{1}{q} a^{\frac{1}{q}-1}$$

$$= \frac{p}{q} a^{\frac{p}{q} - \frac{1}{q}} a^{\frac{1}{q}-1}$$

$$= \frac{p}{q} a^{\frac{p}{q}-1}$$

$$= r a^{r-1} \qquad \blacksquare$$

Addendum: if $f(x) = x^r$, $r = \frac{p}{q}$, q odd, $r > 1$

Then $f'(0) = 0$

Proof: $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^r}{x}$$

$$= \lim_{x \to 0} x^{r-1} = 0. \qquad \blacksquare$$

Quotient Rule

if $f'(a)$, $g'(a)$ exist and $g(a) \neq 0$, then $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$

Proof: $\left(\frac{1}{g}\right)'(a) = \left(g^{-1}\right)'(a)$

$$\left(\frac{1}{g}\right)'(a) = \left(g^{-1}\right)'(a)$$
$$= -1\,(g(a))^{-2}\,g'(a)$$
$$= \frac{-g'(a)}{[g(a)]^2}.$$

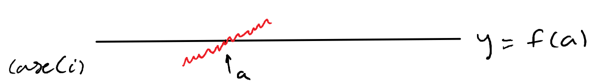
$$\left(\frac{f}{g}\right)'(a) = \left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$
$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2}$$
$$= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

Applications of Derivatives

Lemma: (i) if $f'(a)$ is positive, then $\exists\, \delta > 0$ s.t. $f(y) < f(a) < f(z)$ for $y \in (a-\delta, a)$, $z \in (a, a+\delta)$

(ii) " negative " " $f(y) > f(a) > f(z)$ ".

Note: it is not true in general that f is increasing over $(a-\delta, a+\delta)$ in case (i) or decreasing in case (ii).



Proof of Lemma: (case (ii)) $f'(a) < 0$.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = -L < 0$$

Hence $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow x \in \mathrm{dom}(f)$ & $\left|\frac{f(x)-f(a)}{x-a} + L\right| < \varepsilon$

so $$-L - \varepsilon < \frac{f(x)-f(a)}{x-a} < -L + \varepsilon.$$ Take $\varepsilon = \frac{L}{2}$

$f(x) - f(a)$ . . . .

$$\frac{}{x-a} < -\frac{L}{2} < 0 \quad \text{for } x \in (a-\delta, a) \cup (a, a+\delta)$$

Let $y \in (a-\delta, a)$ so that $\frac{f(y)-f(a)}{y-a} < 0$ $\quad y-a<0$

so $f(y)-f(a) > 0$ so $f(y) > f(a)$.

Let $z \in (a, a+\delta)$ so that $\frac{f(z)-f(a)}{z-a} < 0$ $\quad z-a>0$

so $f(z) - f(a) < 0$ so $f(z) < f(a)$ ∎

Theorem Suppose that $f: [a,b] \to \mathbb{R}$ takes a minimum at $c \in (a,b)$ and $f'(c)$ exists. Then $f'(c) = 0$.

Proof: if f takes a max. val. at $c \in (a,b)$ then $f(c) \geq f(x)$ $\forall x \in (a,b)$ so it cannot be the case that $f(y) < f(c) < f(z)$ or $f(y) > f(c) > f(z)$ for any $y, z \in (c-\delta, c+\delta) \subseteq (a,b)$. Therefore $f'(a) \not> 0$ and $f'(a) \not< 0$. So $f'(a) = 0$.
Similarly for min val. ∎

Combining this with EVT:

Theorem if $f: [a,b] \to \mathbb{R}$ is cts, then f takes on a max val at c which is one of the following:

(1) point $c \in (a,b)$ where $f'(c) = 0$ (critical point)

(2) point $c \in (a,b)$ where $f'(c)$ does not exist (singular point)

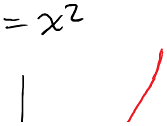
(3) $c \in \{a, b\}$, one of the endpoints.

(same for minimum).

Examples:

(1) $f: [-1, 2] \to \mathbb{R}$

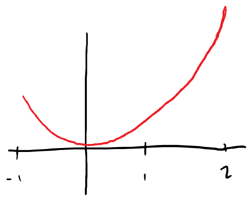
$f(x) = x^2$



min at $x=0$. critical pt.

max at $x=2$. endpoint

$f(x) = x$



min at $x = 0$. critical pt.

max at $x = 2$. endpoint

(2) $g : [-1, 2] \to \mathbb{R}$

$g(x) = |x|$

min at $x = 0$ singular pt

max at $x = 2$ endpt.