Lec 1/24

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Propn: Let α be a C^2 unit-speed curve in $\mathbb{R}^3$, with κ never 0.

Then (a) and (b) below are equivalent:

(a) α is planar

(b) B is constant

if in addition α is C^3, then another equivalent condition is:

(c) $\tau \equiv 0$

Pf When α is C^3, then (b)$\Leftrightarrow$(c) since $B' = -\tau N$.

Now back to assuming α is C^2. Last time we showed that (b)$\Rightarrow$(a).

Now lets show (a)$\Rightarrow$(b)

Suppose α is planar. then $\exists\, x_0, n \in \mathbb{R}^3$ with $|n|=1$ s.t.

$\text{Range}(\alpha) \subseteq \Pi = \{x \in \mathbb{R}^3 : \langle x - x_0, n\rangle = 0\}$. Next, $\exists\, \ell, m \in \mathbb{R}^3$ s.t.

ℓ, m, n is a right-handed O.N.B. for $\mathbb{R}^3$. Define a, b by

$a(s) = \langle \alpha(s) - x_0, \ell\rangle$, $b(s) = \langle \alpha(s) - x_0, m\rangle$. a and b are C^2 and

$\alpha(s) - x_0 = a(s)\ell + b(s)m$ so $\alpha(s) = x_0 + a(s)\ell + b(s)m \quad \forall s$.

Then $T(s) = \alpha'(s) = a'(s)\ell + b'(s)m$.

$T'(s) = a''(s)\ell + b''(s)m$, so $\kappa(s) = |T'(s)| = \sqrt{a''(s)^2 + b''(s)^2}$

So $N(s) = \dfrac{T'(s)}{\kappa(s)} = \dfrac{a''(s)}{\kappa(s)}\ell + \dfrac{b''(s)}{\kappa(s)} m$

$$\text{And so } B(s) = T(s)\times N(s) = \big(a'(s)\ell + b'(s)m\big) \times \left(\frac{a''(s)}{\kappa(s)}\ell + \frac{b''(s)}{\kappa(s)} m\right) = \frac{a'(s)b''(s) - b'(s)a''(s)}{\kappa(s)}\, n.$$

$$= c(s)\, n$$

Now $|B(s)| = 1$ and $|n|=1$ so $|c(s)| = 1\ \forall s$. thus $c(s) = \pm 1$ but since c is continuous on a connected set (interval), $c(s)$ is constant (either 1 or −1.) thus $B(s)$ is constant (either $\pm n$) & the claim is proved. $\square$

Def: **Osculating Plane**: the plane spanned by $\{T(s), N(s)\}$.

Def: Let $\alpha: (a,b) \longrightarrow \mathbb{R}^3$ be a C^3 unit-speed curve in $\mathbb{R}^3$. Let $s_0 \in (a,b)$ and suppose $\kappa(s_0) \neq 0$.

(a) the Osculating plane to α at s_0 is the plane through $\alpha(s_0)$ spanned by $T(s_0)$ and $N(s_0)$ (or, equivalently, $\perp$ to $B(s_0)$).

(b) the Normal plane is $\perp$ to $T(s_0)$ (spanned by $N(s_0)$, $B(s_0)$) thru $\alpha(s_0)$.

(c) the rectifying plane to α at s_0 is $\perp$ to $N(s_0)$ thru $\alpha(s_0)$.

Significance of osculating plane.

If the curve lies in a plane, (which is unique if $\kappa(s_0) \neq 0$) it is the osculating plane.

More generally, the osculating plane is the plane that α is closest to lying inside of (for s near s_0) in the following sense:

Let $u \in \mathbb{R}^3$ with $|u| = 1$. Consider the plane

$$\Pi = \{ x \in \mathbb{R}^3 : \langle x - \alpha(s_0), u \rangle = 0 \}.$$

Define $f: (a,b) \longrightarrow \mathbb{R}$ by $f(s) = \langle \alpha(s) - \alpha(s_0), u \rangle$. Since $|u| = 1$, $f(s)$ is the signed distance from $\alpha(s)$ to Π. $\left(\frac{o(h^3)}{h^3} \to 0 \text{ as } h \to 0 \right)$

By Taylor's theorem,

$$f(s) = f(s_0) + f'(s_0)(s-s_0) + f''(s_0)\frac{(s-s_0)^2}{2} + f'''(s_0)\frac{(s-s_0)^3}{6} + \underbrace{o\left((s-s_0)^3\right)}_{\text{as } s \to s_0}$$

Obviously $f(s_0) = 0$. Next $f'(s) = \langle \alpha'(s), u \rangle = \langle T(s), u \rangle$

Hence $\alpha(s)$ has "first-order contact" with Π as $s \to s_0$

$\left(\text{meaning } \frac{f(s)}{s - s_0} \to 0 \text{ as } s \to s_0 \right)$

Iff $T(s_0) \perp u$. Now $f''(s) = \langle \kappa(s) N(s), u \rangle$ (recall $\kappa(s_0) \neq 0$ and $|N(s_0)| = 1$)

Hence $\alpha(s)$ has "second order contact" with Π as $s \to s_0$

(meaning $\frac{f(s)}{(s-s_0)^2} \to 0$ as $s \to s_0$)

iff $N(s_0) \perp u \perp T(s_0)$ The osculating plane thru α at s_0 (taking $u = B(s_0)$) is the only plane for which α has second-order contact.

Since $B' = -\tau N$ and B is normal to the osculating plane, τ measures how fast the osculating plane turns as s increases.

Note that if $u = B(s_0)$ then $f'''(s) = \langle K'(s)N(s) + K(s)N'(s), B(s) \rangle$

Hence $f'''(s_0) = \langle K(s) N'(s_0), B(s_0) \rangle = K(s_0)\,\tau(s_0)$

So $f(s) = K(s_0)\,\tau(s_0)\frac{(s-s_0)^3}{6} + o((s-s_0)^3)$ as $s \to s_0$

Therefore if $\tau(s_0) > 0$ then the curve moves toward the side of the osculating plane that $B(s_0)$ points towards. (as $s \nearrow$ near s_0). And the other way if $\tau(s_0) < 0$.

The Canonical representation of α near s_0.

(α C^3 unit-speed curve, $K(s_0) \neq 0$)

$$\alpha(s) = \alpha(s_0) + \left(s - K(s_0)^2\frac{(s-s_0)^3}{6}\right)T(s_0)$$
$$+ \left(K(s_0)\frac{(s-s_0)^2}{2} + K'(s_0)\frac{(s-s_0)^3}{6}\right)N(s_0)$$
$$+ K(s_0)\,\tau(s_0)\frac{(s-s_0)^3}{6}B(s_0)$$
$$+ o\left((s-s_0)^3\right)$$

(as $s \longrightarrow s_0$)

Pf (wolog take $s_0 = 0$). By taylor's theorem,

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + \alpha'''(0)\frac{s^3}{6} + o(s^3) \quad \text{as } s \longrightarrow s_0 .$$

$$\alpha' = T,\ \alpha'' = T' = \kappa N,\ \alpha''' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B)$$
$$= -\kappa^2 T + \kappa' N + \tau B$$

thus the representation holds. $\square$