A category consists of objects & morphisms.

for objects A, B in the category, we have a set of morphisms $A \to B$.

If $\varphi: A \to B$, $\psi: B \to C$ are morphisms, then $\psi \circ \varphi : A \to C$ is defined,

s.t. $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$

$\forall$ object A, $\exists$ morphism $1_A : A \to A$ s.t. $\forall \varphi : A \to B$, $\varphi \circ 1_A = \varphi$. $\forall \psi : B \to A$, $1_A \circ \psi = \psi$.

A & B are isomorphic if $\exists$ morphisms $\varphi : A \to B$, $\psi : B \to A$ s.t. $\varphi \circ \psi = 1_B$, $\psi \circ \varphi = 1_A$.

φ and ψ are called isomorphisms.

An object A is called a universal repelling* object if $\forall$ object B, $\exists!$ morphism $A \to B$*. *(attracting $\leadsto$ $B \to A$).

These objects may not exist, but

<u>Theorem</u>: if A_1 & A_2 are universal repelling (or attracting) objects, $A_1 \cong A_2$.

<u>Proof</u>: $\exists \varphi_1 : A_1 \to A_2$ & $\varphi_2 : A_2 \to A_1$ and $\varphi_1 \circ \varphi_2 : A_2 \to A_2$ but there is only one such morphism, 1_{A_2}.

<u>eg</u>: in Set, universal attractors are $\{x\}$, universal repellor is $\emptyset$.

eg Category of groups with n fixed elements $(g_1, g_2, \ldots, g_n)$ and morphisms

$(G, (g_1, \ldots, g_n)) \longrightarrow (H, (h_1, \ldots, h_n))$ being hom-sms $\varphi : G \to H$ s.t. $\varphi(g_i) = h_i \ \forall i$.

universal repellor $= (F_n = \langle a_1, \ldots, a_n \rangle, (a_1, \ldots, a_n))$.

eg Same as above but with abelian groups. Then $(\mathbb{Z}^n, (e_1, \ldots, e_n))$ is universal repellor eg Let R be a commutative ring. Consider the category of unital R-Algebras with a marked element; morphisms $(A, a) \longrightarrow (B, b)$ are R-algebra homs with $a \mapsto b$.

A universal repelling object is $(R[x], x)$ with $\begin{array}{c} R[x] \longrightarrow A \\ p(x) \longmapsto p(a) \end{array}$.

eg R: unital ring. Consider category of R-modules w/ n marked elements $(M, (u_1, \dots, u_n))$.
Universal repelling object: $(R^n, (e_1, \dots, e_n))$ with map $e_i \longmapsto u_i$.

eg Let M_1, M_2 be R-modules. Then $M_1 \oplus M_2 = M_1 \times M_2$ is a universal repelling object in the category of R-modules N with homomorphisms $\varphi_1 : M_1 \to N \leftarrow M_2 : \varphi_2$.
(diagrams $M_1 \xrightarrow{\varphi_1} N \xleftarrow{\varphi_2} M_2$) and morphisms

$(N, \varphi_1, \varphi_2) \longrightarrow (K, \psi_1, \psi_2)$ being homs $\pi : N \to K$ s.t. the diagram

$$\begin{tikzcd} & M_1 \arrow[dl, "\varphi_1"'] \arrow[dr, "\psi_1"] & \\ N \arrow[rr, "\pi"] & & K \\ & M_2 \arrow[ul, "\varphi_2"] \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

is commutative.

$M_1 \oplus M_2$ is also the universal attracting object in the category of $(N, \varphi_1, \varphi_2)$ where $\varphi_1 : N \to M_1$, $\varphi_2 : N \to M_2$

with morphisms $(N, \varphi_1, \varphi_2) \longrightarrow (K, \psi_1, \psi_2)$ being homs π

s.t. the diagram

$$\begin{tikzcd} & M_1 & \\ N \arrow[ur, "\varphi_1"] \arrow[rr, "\pi"] \arrow[dr, "\varphi_2"'] & & K \arrow[ul, "\psi_1"'] \arrow[dl, "\psi_2"] \\ & M_2 & \end{tikzcd}$$

commutes.

Back to direct sums:

$M_1, M_2 \rightsquigarrow M_1 \oplus M_2$.

<u>Definition</u>. Let M be an R-module & M_1, M_2 be its submodules. We say that M is a direct sum of M_1 and M_2, $M = M_1 \oplus M_2$, if $M \cong M_1 \oplus M_2$ so that the isomorphism is identical on M_1 & M_2. Or:

$$\begin{tikzcd} & M_1 \arrow[dl] \arrow[dr] & \\ M \arrow[rr] & & M_1 \oplus M_2 \\ & M_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

diagram of embeddings is commutative.

<u>Theorem</u>: Let M_1, M_2 be submodules of M. Then $M = M_1 \oplus M_2$ iff any of the following:

(i) $\forall u \in M$ is uniquely representable as $u = u_1 + u_2$ where $u_1 \in M_1$, $u_2 \in M_2$.

(ii) $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$.

(iii) The projection hom-sm $M \longrightarrow M/M_1$ is an isomorphism on M_2 (when restricted).

(iv) " " $M \longrightarrow M/M_2$ " " M_1 " "

<u>Proof</u> (i) define a mapping $\varphi : M \longrightarrow M_1 \oplus M_2$ by if $u = u_1 + u_2$ then $\varphi(u) = (u_1, u_2)$.
Prove this is an isomorphism & that it's identical on M_1 & M_2.

(ii) $\Rightarrow$ (iii): Let $\pi : M \longrightarrow M/M_1$. Then $Ker(\pi|_{M_2}) = M_1 \cap M_2 = 0$, and $\pi(M_2) = \pi(M_1 + M_2) = \pi(M) = M/M_1$.

(i) $\Leftrightarrow$ (ii)