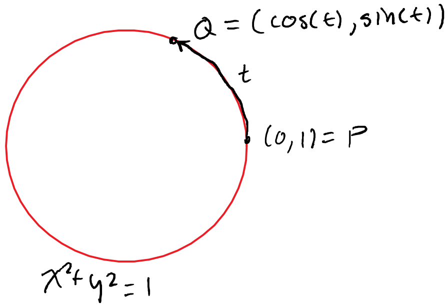
Lec 11/9

Wednesday, November 9, 2016 9:08 AM



$$t = \int_0^{\sin(t)} \frac{1}{\sqrt{1-y^2}}\,dy \quad \text{for } t \in (-1,1) \text{ (acute angles)}$$

Defined $\Lambda(\omega) = 2\int_0^{\omega} \sqrt{1-y^2}\,dy - \omega\sqrt{1-\omega^2}$ for $\omega \in [-1,1]$

and $\Lambda(\omega) = \int_0^{\omega} \frac{1}{\sqrt{1-y^2}}\,dy$ for $\omega \in (-1,1)$.

Theorem (1) $\Lambda(-\omega) = -\Lambda(\omega)$

(2) $\Lambda'(\omega) = \frac{1}{\sqrt{1-\omega^2}}$ for $\omega \in (-1,1)$

(3) Λ increasing on $[-1,1]$

(4) $\Lambda([-1,1]) = [-\frac{\pi}{2}, \frac{\pi}{2}]$ where $\pi = 4\int_0^1 \sqrt{1-y^2}\,dy$

Proof (1): $\Lambda(-\omega) = 2\int_0^{-\omega} \sqrt{1-y^2}\,dy \underbrace{- (-\omega)\sqrt{1-(-\omega)^2}}_{\longrightarrow\; + \omega\sqrt{1-\omega^2}}$

Show that: $\int_0^{-\omega} \sqrt{1-y^2}\,dy = -\int_0^{\omega} \sqrt{1-y^2}\,dy$

by diff.ing: $-\sqrt{1-\omega^2} = -\sqrt{1-\omega^2}$ and when $\omega = 0$ they are both 0.

so equality holds.

Definition $\sin\big|_{[-\pi/2,\, \pi/2]} := \Lambda^{-1}$

Corollary (1) For $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin(-t) = -\sin(t)$

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(2) $\sin'(t) = \sqrt{1-\sin^2(t)}$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Proof (1): Let $\omega = \sin(t) \Leftrightarrow t = \Lambda(\omega)$, $-t = -\Lambda(\omega) = \Lambda(-\omega)$

$$\Rightarrow \sin(-t) = -\omega = -\sin(t)$$

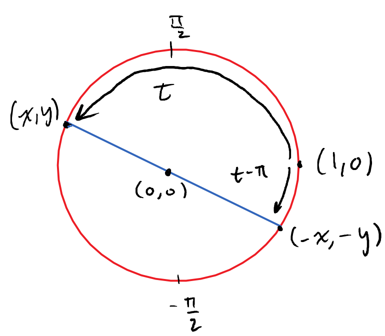
(2): $$\sin'(t) = \frac{1}{\Lambda'(\sin(t))} = \sqrt{1-\sin^2(t)}$$

Definition: $\cos(t) := \sqrt{1-\sin^2(t)}$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Then $\frac{d}{dt}(\sin(t)) = \cos(t)$ by definition $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\cos(-t) = \sqrt{1-\sin^2(-t)} = \sqrt{1-(-\sin(t))^2} = \sqrt{1-\sin^2(t)} = \cos(t).$$

Now extend domain of sin & cos to $[-\pi/2, 3\pi/2]$



if $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ then $t - \pi = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Definition for $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$,

$$\sin(t) = -\sin(t-\pi)$$
$$\cos(t) = -\cos(t-\pi)$$

Note: $\frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \cap [\frac{\pi}{2}, \frac{3\pi}{2}]$ so $\sin(\pi/2)$ should equal $-\sin(\overset{\frac{\pi}{2}-\pi}{-\frac{\pi}{2}})$

$\cos(\pi/2)$ should equal $-\cos(-\frac{\pi}{2})$

and indeed $\sin(\pi/2) = 1$ and $-\sin(-\frac{\pi}{2}) = 1$.

$\cos(\pi/2) = 0$ and $-\cos(-\pi/2) = 0$.

is sin continuous on $(\frac{\pi}{2}, \frac{3\pi}{2}]$? Yes, $\sin(t) = -\sin(t-\pi)$ which is a composite of cts. functions.

Check continuity at $\frac{\pi}{2}$:

$$\lim_{t \to \frac{\pi}{2}^-} \sin(t) = \underset{\substack{\| \\ 1}}{\sin\left(\frac{\pi}{2}\right)} \text{ by continuity of } \sin\big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$$

$$\lim_{t \to \frac{\pi}{2}^+} \sin(t) = \lim_{t \to \frac{\pi}{2}^+} -\sin(t - \pi) = -\sin\left(\lim_{t \to \frac{\pi}{2}^+} (t - \pi)\right) \quad \text{by cont. of}$$
$$= -\sin\left(-\frac{\pi}{2}\right)$$
$$= 1$$

so sin is continuous at $\frac{\pi}{2}$.

Similar arg shows these facts for cos.

Proposition $\frac{\partial}{\partial t}(\sin(t)) = \cos(t)$ for $t \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Proof: If $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ then

(if $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ this already holds).

$$\frac{\partial}{\partial t}(\sin(t)) = \frac{\partial}{\partial t}(-\sin(t - \pi))$$
$$= -\cos(t - \pi)$$
$$= \cos(t).$$

$$\sin'\left(\frac{\pi}{2}\right) = \lim_{s \to \frac{\pi}{2}} \frac{\sin(s) - \sin\left(\frac{\pi}{2}\right)}{s - \pi/2}$$

$$= \lim_{s \to \frac{\pi}{2}} \frac{\sin'(s)}{1} \quad \text{by L'H}$$

$$= \lim_{s \to \frac{\pi}{2}} \cos(s) \quad \text{by LP}$$

$$= \cos\left(\frac{\pi}{2}\right)$$

$\cos'(t) \ldots$ if $t \in (-\pi/2, \pi/2)$ this $= \frac{\partial}{\partial t}\left(\sqrt{1 - \sin^2(t)}\right)$

$$= \frac{1}{2\sqrt{1 - \sin^2(t)}} \cdot -2\sin(t) \cdot \cos(t)$$

$$= \frac{-\sin(t)\cos(t)}{\cos(t)}$$

$$= -\sin(t)$$

If otoh $t \in (\pi/2, 3\pi/2)$ then $\cos'(t) = \frac{d}{dt}(-\cos(t-\pi))$

$$= -(-\sin(t-\pi))$$
$$= -\sin(t)$$

$$\cos'(\tfrac{\pi}{2}) = \overset{\text{by L'H}}{\lim_{t \to \frac{\pi}{2}}} -\sin(t) = -\sin(\tfrac{\pi}{2}).$$

Theorem If $f : [a, a+p] \to \mathbb{R}$ is continuous, diffable, and

(1) $f(a) = f(a+p)$

(2) $\lim_{t \to a^+} f'(t) = \lim_{t \to a+p^-} f'(t)$

Then f extends to a periodic diffable function $\hat{f} : \mathbb{R} \to \mathbb{R}$ with $f(x+p) = f(x)$

Proof: for any $x \in \mathbb{R}$, find $n \in \mathbb{Z}$ so that $n \leq \frac{x-a}{p} < n+1$ (well-ordering principle).

Then $a + np \leq x < a + (n+1)p$

$$a \leq x - np < a + p$$

define $\hat{f}(x) := f(x - np)$.

If $x = a + (n+1)p$, then $\lim_{t \to x^-} \hat{f}(t) = \lim_{t \to a+(n+1)p^-} f(t-np) = f(a+p)$

$\qquad ||$

and $\lim_{t \to x^+} \hat{f}(t) = \lim_{t \to a+(n+1)p^+} f(t-np) = f(a)$

Check that these conditions hold for sin and cos.