Lec 2/14

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<u>Defn</u> Let (X, ρ) be a metric space. Let $A \subseteq X$. To say A is Totally bounded is to say that for each $\varepsilon > 0$, A is a finite union of sets of diameter $< \varepsilon$. $\left(\text{diam}(S) = \sup_{x,y \in S} \rho(x,y)\right)$ in $[0,\infty)$

eg Let A be a bounded $\subseteq \mathbb{R}$. A is totally bounded.

Since $A \subseteq [a,b]$ for some $a, b \in \mathbb{R}$ w/ $a < b$. Let $\varepsilon > 0$, then we can divide the interval up & take intersection of A w/ divisions.

eg A bounded set in $\mathbb{R}^d$ is totally bounded. (same as above but w/ d-cubes).

eg Let X be any set. Define $\rho : X \times X \to [0,\infty)$ by $\rho(x,x') = \begin{cases} 1 & x \neq x' \\ 0 & x = x' \end{cases}$.

ρ is a metric on X under which X is bounded but the only totally bounded subsets of X are the finite ones.

eg Let X be an infinite dimensional normed linear space. Let $B = \{x \in X : \|x\| < 1\}$

Then B is bounded, but B is not totally bounded.

<u>Theorem</u> Let (X, ρ) be a complete, totally bounded metric space. Then X is compact.

<u>Proof</u> Let $\mathcal{U}$ be an open cover of X. if $A \subseteq X$ then to say that <u>A contains a lion</u> means that no finite subcollection of $\mathcal{U}$ covers A.

We wts X does not contain a lion. Suppose it does. Observe that if $A = \bigcup_{k=1}^{n} (A_k \subseteq X)$ and A contains a lion then some A_k contains a lion. Since X is totally bdd, if $A \subseteq X$ s.t. A contains a lion, and if $\varepsilon > 0$, then $\exists\, n \in \mathbb{N}$ and $A_1, \dots, A_n \subseteq A$ s.t. $\bigcup_{k=1}^{n} A_k = A$ and $\text{diam}(A_k) < \varepsilon$ $\forall k$. Some A_k contains a lion.

So there is a decreasing sequence (X_j) of subsets of X s.t. $\text{diam}(X_j) < \frac{1}{j}$ and X_j contains a lion. ~~$\bigcap_{j=1}^{\infty} X_j$ is a singleton~~

$\operatorname{diam}(X_j) < \frac{1}{j}$ and X_j contains a lion. ~~$\bigcap_{j=1} X_j$ is a singleton~~ ~~since each $X_j \neq \emptyset$ and diam$(X_j) \to 0$.~~ Pick $p_j \in X_j$. If $j < j'$ then $p_j \in X_j$ and $p_{j'} \in X_{j'} \subseteq X_j$ so $\rho(p_j, p_{j'}) < \frac{1}{j}$.

Thus (p_j) is cauchy in X. Hence, since X is complete wrt ρ, so $\exists p \in X$ s.t. $\rho(p_j, p) \to 0$. Now $\exists U \in \mathcal{U}$ s.t. $p \in U$. U is open so $\exists$ an $\varepsilon > 0$ s.t. $B(p, 2\varepsilon) \subseteq U$. Then since $p_j \to p$ $\exists$ an N_1 s.t. $\forall j > N_1$ $p_j \in B(p, \varepsilon)$. Since $\operatorname{diam}(X_j) \to 0$, $\exists N_2$ s.t. $\forall j > N_2$ $\operatorname{diam}(X_j) < \varepsilon$. Now let $J = \max\{N_1, N_2\}$, and let $x \in X_J$. $\rho(x, p) \le \rho(x, p_J) + \rho(p_J, p) < 2\varepsilon$

So $x \in U$. Thus X_J is completely covered by one set, $U \in \mathcal{U}$, a contradiction, a contradiction to X_J containing a lion. $\square$

Conversely one can show that a compact metric space (X, ρ) is totally bounded & complete.

Defn: Let $f: X \to \overline{\mathbb{R}}$ where X is a topological space. To say f is lower semicontinuous means that $\forall y \in \mathbb{R}$, $\{x : f(x) > y\}$ is open.

eg $\mathbb{1}_A$ is lsc when A is open.

eg continuous functions $f: X \to \overline{\mathbb{R}}$ are lsc

eg a pointwise sup of lsc fns is lsc, just as a union of open sets is open.

Theorem Let X be a topological space. Then each lower semicontinuous function on X achieves a minimum iff X is non-empty & $\underbrace{\text{countably compact}}$.

every countable $\mathcal{U}$ has finite subcover

Defn Let (X, ρ) and (Y, σ) be metric spaces. Let $f: X \to Y$. Let $H \subseteq X$.

To say f is uniformly continuous means that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall a \in H, \forall x \in X,$
$\rho(x,y) < \delta \Rightarrow \sigma(f(x), f(y)) < \varepsilon$

Theorem: Let (X,ρ) and (Y,σ) be metric spaces, $f : X \to Y$, $H \subseteq X$ be compact, and suppose $\underbrace{f \text{ is continuous on}}_{\forall a \in H, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in X,\ \rho(x,a) < \delta \Rightarrow \sigma(f(x), f(a)) < \varepsilon}$ H. Then f is uniformly cts on H.

Proof

Let $\varepsilon > 0$. $\forall a \in H$, let $\delta_a > 0$ s.t. $\forall x \in X$, $\rho(x,a) < \delta_a \Rightarrow \rho(f(x), f(a))$.
$\mathcal{U} = \{ B(a, \delta_a) : a \in H \}$ is an open cover of H. So $\exists$ a finite subcover
$\mathcal{V} \subseteq \mathcal{U}$. Let (Wait this uses AoC)

Let $\varepsilon > 0$, let $\mathcal{B} = \{ B(a,r) : a \in H, r > 0, \text{ and } \forall x \in B_X(a, 2r), \sigma(f(x), f(a)) < \frac{\varepsilon}{2} \}$.
$\mathcal{B}$ is an open cover of H, since f is cts. So there is a finite subcollection.
So $\exists n \in \mathbb{N}$ s.t. $\exists a_1, \dots, a_n \in H$, $\exists r_1, \dots, r_n \in (0, \infty)$ s.t. for $k = 1, \dots, n$,

$$f[B_X(a_k, 2r_k)] \subseteq B_Y(f(a_k), \tfrac{\varepsilon}{2}) \quad \text{and} \quad H \subseteq \bigcup_{k=1}^{n} B_X(a_k, r_k)$$

Let $\delta = \min_{1 \le k \le n} r_k$. Let $a \in H$, $x \in X$, and suppose $\rho(a,x) < \delta$.

Then $a \in B(a_k, r_k)$ for some k, so $\sigma(f(a), f(a_k)) < \frac{\varepsilon}{2}$. also,
$\rho(a_k, x) \le \rho(a_k, a) + \rho(a, x) < r_k + \delta \le 2 r_k$ So

$\sigma(f(a_k), f(x)) < \frac{\varepsilon}{2}$, so $\sigma(f(a), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$