Lec 11/1

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Dual spaces:

Let X, Y be normed v.s.

$\mathcal{L}(X, Y) = \{$ bdd linear maps $X \to Y \}$

Operator norm: $\|T\| = \sup\{ \|Tx\| \mid \|x\| = 1\}$.

$f: X \to \mathbb{K}$ is a linear functional ($\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$)

<u>Def</u> Dual space $X^* := \mathcal{L}(X, \mathbb{K})$

- Banach space
- example: L^q is dual to L^p for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Prop</u>: Let X be a $\mathbb{C}$ vs. & $\varphi: X \to \mathbb{C}$ linear. Then

① $Re(\varphi): X \to \mathbb{R}$ is $\mathbb{R}$-linear, & $\forall x \in X$,

$$\varphi(x) = Re(\varphi)(x) - i\, Re(\varphi)(ix)$$

② if $f: X \to \mathbb{R}$ is $\mathbb{R}$-linear then setting

$$\psi(x) = f(x) - i f(ix)$$

defines a $\mathbb{C}$-linear functional.

⑤ if X is normed,

- Case 1: $\|\varphi\| < \infty \Rightarrow \|Re(\varphi)\| \leq \|\varphi\|$
- Case 2: $\|Re(\varphi)\| < \infty \Rightarrow \|\varphi\| \leq \|Re(\varphi)\|$

$\Big\}$ $\|\varphi\| = \|Re(\varphi)\|$

Pf ① $Im(\varphi) = -Re(i\varphi(x)) = -Re(\varphi(ix)) = -Re(\varphi)(ix)$.

② $\Psi(ix) = f(ix) - if(-x) = if(x) + f(ix) = i(f(x) - if(ix)) = i\Psi(x)$.

③ case 1: $|Re(\varphi)(x)| \leq |\varphi(x)| \ \forall x$

case 2: If $\varphi(x) \neq 0$, $|\varphi(x)| = \overline{sign(\varphi(x))}\,\varphi(x) = \varphi(\overline{sign(\varphi(x))}\,x) = Re(\overline{sign\,\varphi(x)}\,x) \leq \|Re\varphi\|\,\|x\|$.

$\square$

<u>Def</u> X is a $\mathbb{R}$-vs. A <u>sublinear functional</u> on X is a fn $p: X \to \mathbb{R}$ s.t.

- (Positive homogeneity) $\forall x \in X$ & $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$
- (subadditivity) $\forall x, y \in X$, $p(x+y) \leq p(x) + p(y)$

<u>ex</u> seminorm.

<u>Hahn-Banach Thm</u>: Let X be a $\mathbb{R}$-vs, $p: X \to \mathbb{R}$ a sublinear functional, $M \subseteq X$ a subspace, $f: M \to \mathbb{R}$ a linear functional s.t. $f(m) \leq p(m)$ $\forall m \in M$. Then $\exists$ linear functional $\varphi: X \to \mathbb{R}$ s.t. $\varphi|_M = f$ and $\varphi(x) \leq p(x)$ $\forall x \in X$.

Pf: <u>Step 1</u>: $\forall x \in X \setminus M$, $\exists g: M + \mathbb{R}x \to \mathbb{R}$ lin. ftl, st. $g|_M = f$ and $g(y) \leq p(y)$ $\forall y \in M + \mathbb{R}x$.

Pf If $g(m + \lambda x) = f(m) + \lambda\alpha$, We want α s.t. $f(m) + \lambda\alpha \leq p(m + \lambda x)$.

Suffices to check $\lambda = \pm 1$. so we want

$$f(m) - \alpha \le p(m-x), \quad f(n) + \alpha \le p(n+x) \qquad \forall\, m,n \in \mathcal{M}.$$

i.e. $\quad f(m) - p(m-x) \le \alpha \le p(n+x) - f(n).$

Now $p(n+x) - f(n) - f(m) + p(m-x)$

$$\begin{aligned} &= p(n+x) + p(m-x) - f(n+m) \\ &\ge p(n+m) - f(n+m) \\ &\ge 0, \end{aligned}$$

so such an α exists.

<u>Step 2</u>: We can apply Step 1 to any ext g of f to $\mathcal{N} \supset \mathcal{M}$ s.t. $g|_{\mathcal{M}} = f$ & $g(y) \le p(y)\ \forall y \in \mathcal{N}$. Thus any max'l ext φ of f s.t. $\varphi|_{\mathcal{M}} = f$ and $\varphi \le p$ has domain X.

Now $\{(\mathcal{N}, g) \mid \mathcal{M} \subseteq \mathcal{N} \subseteq X,\ g : \mathcal{N} \to \mathbb{R} \text{ s.t. } g|_{\mathcal{M}} = f \ \&\ g \le p\}$

is partially ordered by extension

ex: every chain has an upper bound, so there is a max'l elt by Zorn's Lemma. $\square$

<u>Remark</u>: Suppose p is a seminorm on X & $f : X \to \mathbb{R}$ is $\mathbb{R}$-linear. Then $f \le p \Leftrightarrow |f| \le p$

$(\,|f(x)| = \pm f(x) = f(\pm x) \le p(\pm x) = p(x)\,)$.

<u>$\mathbb{C}$ Hahn-Banach Thm</u>. X is $\mathbb{C}$-v.s., $p : X \to [0,\infty)$ seminorm, $\mathcal{M} \subseteq X$ subsp, $f : \mathcal{M} \to \mathbb{C}$ is a $\mathbb{C}$-linear ftl s.t. $|f| \le p$ on $\mathcal{M}$. Then $\exists$ $\mathbb{C}$-linear

$\exists ! \ \varphi : X \to \mathbb{C}$ s.t. $\varphi|_m = f$ and $|\varphi| \le p$.

Pf $\mathbb{R}$-HB applied to $\mathrm{Re}(f)$: $\exists \ g : X \to \mathbb{R}$ $\mathbb{R}$-linear s.t. $|g| \le p$ and $g|_m = f$.

Let $\varphi(x) = g(x) - i g(ix)$. $\mathbb{C}$-linear ext, $\varphi|_m = f$ and $\forall x$,

$|\varphi(x)| = \overline{\mathrm{sign}\,\varphi(x)}\ \varphi(x) = \varphi(\overline{\mathrm{sign}\,\varphi(x)}\ x) = g(\overline{\mathrm{sign}\,\varphi(x)}\ x) \le p(\overline{\mathrm{sign}\,\varphi(x)}\ x) = p(x)$. □

<u>Cor of H-B thm</u>: X is a normed v.s.

① If $x \in X$, $x \ne 0$, $\exists \ \varphi \in X^*$ s.t. $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$.

Pf $f : \mathbb{K}x \to \mathbb{K}$ s.t. $f(\lambda x) = \lambda \|x\|$. Note $\|f\| \le \|\cdot\|$ apply H.B. □

② If $m \subseteq X$ closed & $x \in m^c$, $\exists \ \varphi \in X^*$ s.t. $\varphi(x) = \inf\limits_{m \in m} \|x - m\|$, $\|\varphi\| = 1$

Pf apply ① to $x + m \in X/m$ to get $\tilde{\varphi} \in (X/m)^*$ s.t. $\|\tilde{\varphi}(x+m)\| = \|x+m\|$

$= \inf\limits_{m \in m} \|x+m\|$,

$\|\varphi\| = 1$.

By HW, $Q : X \to X/m$ cts and $\|x+m\| \le \|x\|$ so $\|Q\| \le 1$.

Then $\varphi = \tilde{\varphi} \circ Q$ □

③ X^* separates points of X

Pf if $x \ne y \in X$, $\exists \ \varphi \in X^*$ s.t. $\varphi(x-y) = \|x-y\| \ne 0$ by ①. □

④ for $x \in X$, def $ev_x : X^* \to \mathbb{K}$ by $ev_x(f) = f(x)$.

Then $ev : X \to X^{**}$ is a linear isometry.

Pf $\forall \varphi \in X^*$, $\|ev_x \varphi\| = |\varphi(x)| \le \|\varphi\| \|x\|$ so $\|ev_x\| \le \|x\|$

also, if $x \ne 0$, apply ①. Then $\|ev_x\| = \|x\|$.

Def $\hat{X} := \overline{ev(X)} \subseteq X^{**}$ is a Banach space.

"Completion" of X. If X is banach, $ev(X)$ is closed.

If $ev(X) = X^{**}$ then X is called reflexive.