R- non-commutative ring with no zero divisors.

M- R-module, Tor(M) is not a submodule of M.

Let $R = F\{x, y\}$ be the ring of polynomials in non-commuting x, y.
field

let $M = R / \underbrace{R\{x^2, y^2\}}_{\text{left ideal in } R \text{ generated by } x^2 \,\&\, y^2}$

Then $x, y \in \text{Tor}(M)$ since $x \cdot x = 0$ & $y \cdot y = 0$.

But $x+y \notin \text{Tor}(M)$, so Tor(M) is not a module.

direct product / sum.

If M_1, M_2 are R-modules then $M_1 \times M_2 = \{(u_1, u_2) : u_1 \in M_1, u_2 \in M_2\}$

is also an R-module by componentwise operations.

<u>Homomorphisms of Modules</u>

Let M_1, M_2 be (left) R-modules.

A mapping $\varphi : M_1 \longrightarrow M_2$ is an R-module homomorphism if

φ is a homomorphism of groups & $r\varphi(u) = \varphi(ru)$ $\forall\, u \in M_1, r \in R$.

If R is a field, R-module homomorphisms are called linear mappings/transformations.

Hom-sms $R \to R$ as rings is <u>not</u> the same as Hom-sms $R \to R$ as modules

<u>Ex:</u> $R = \mathbb{Z}$, $\varphi(n) = 2n$. $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is a $\mathbb{Z}$-module homomorphism but <u>not</u> a ring homomorphism.

<u>Ex:</u> $R = \mathbb{R}[x]$, $\varphi(p(x)) = p(x^2)$. φ is a ring hom but <u>not</u> a $\mathbb{R}[x]$-module homomorphism
$x \longmapsto x^2$
since $\varphi(x \cdot x) = x^2 \cdot x^2 \neq x \cdot \varphi(x) = x \cdot x^2$.

<u>Examples of homomorphisms</u>

① Let M be an R-module, let $u \in M$, $1 \in R$.

Then $\exists$ a unique hom-sm $\varphi : R \longrightarrow M$ s.t. $\varphi(1) = u$.

$\varphi(a) = a\varphi(1) = au$ $\forall\, a \in R$.

② If $u_1, \dots, u_n \in M$, $\exists$ unique hom $\varphi : R^n \to M$ s.t. $\varphi(1,0,\dots,0) = u_1$, $\vdots$, $\varphi(0,\dots,0,1) = u_n$

$$\varphi(a_1, \dots, a_n) = a_1u_1 + \cdots + a_nu_n$$

③ if N is a submodule of M then the embedding $\varphi : N \to M$ s.t. $\varphi(u) = u$ is a hom.

④ Let N be a submodule of M. Then $M \to M/N$, $\varphi(u) = \bar{u} = u + N \in M/N$ projection homomorphism.

⑤ Let F be the module of functions $f : X \to R$. Let $x_0 \in X$. The evaluation hom is $\varphi : F \to R$ given by $\varphi(f) = f(x_0)$.

<u>Proposition</u>: if $M = RS$ then any hom $\varphi : M \to N$ is uniquely determined by $\varphi(u) : u \in S$.

<u>proof</u> $\forall u \in M$, $u = a_1u_1 + \cdots + a_nu_n$ for $u_i \in S$ so $\varphi(u) = a_1\varphi(u_1) + \cdots + a_n\varphi(u_n)$.

<u>Proposition</u>: If $\varphi : M \to N$ is an invertible homomorphism, φ^{-1} is a homomorphism.

<u>Proof</u> $\forall v_1, v_2 \in N$, let $u_1 = \varphi^{-1}(v_1)$, $u_2 = \varphi^{-1}(v_2)$, so $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2)$

so $\varphi^{-1}(v_1 + v_2) = \varphi^{-1}(v_1) + \varphi^{-1}(v_2)$.

$\forall v \in N$, $a \in R$, let $u = \varphi^{-1}(v)$. $\varphi(au) = a\varphi(u)$ so $a\varphi^{-1}(v) = \varphi^{-1}(av)$.

<u>Proposition</u>: $\varphi_1 \circ \varphi_2$ is a homomorphism

<u>Proposition</u>: Let $\varphi : M \to N$ be a homomorphism. $\forall$ submodule $K \subseteq M$, $\varphi(K)$ is a submodule of N.

$\forall$ submodule $L \subseteq N$, $\varphi^{-1}(L)$ is a submodule of M.

<u>Proof</u>: Let $v_1, v_2 \in \varphi(K)$. Find $u_1, u_2 \in K$ s.t. $\varphi(u_1) = v_1$, $\varphi(u_2) = v_1$. Then $\varphi(\underset{\in K}{u_1 + u_2}) = \underset{\ni \varphi(u)}{v_1 + v_2}$

...

<u>def</u>: $\mathrm{Ker}(\varphi) = \varphi^{-1}(0)$. $\mathrm{Ker}(\varphi)$ is a submodule of M.

<u>def</u>: The factor module $N/\varphi(M)$ is called the co-kernel of φ, $\mathrm{Coker}(\varphi)$.

<u>def</u> injective homomorphism are called monomorphisms
surjective " " epimorphisms
bijective " " isomorphisms

homomorphisms $M \to M$ are called endomorphisms
isomorphisms " " automorphisms

<u>Proposition</u>: $\varphi : M \to N$ is a monomorphism $\iff$ $\mathrm{Ker}(\varphi) = 0$.

Proposition: $\varphi : M \longrightarrow N$ is a monomorphism $\Longleftrightarrow$ $\mathrm{Ker}(\varphi) = 0$.

$\varphi : M \longrightarrow N$ is an epimorphism $\Longleftrightarrow$ $\mathrm{Coker}(\varphi) = 0$.

Def: M & N are isomorphic if $\exists$ isomorphism $M \to N$.

Isomorphism Theorems:

① If $\varphi : M \longrightarrow N$ is a hom-sm then $M/\mathrm{Ker}(\varphi) \cong \varphi(M)$

$\bar{u} \longmapsto \varphi(u)$

(if φ is surjective, $M/\mathrm{Ker}(\varphi) \cong N$).

② If N_1, N_2 are submodules of M, then $(N_1 + N_2)/N_2 \cong N_1/(N_1 \cap N_2)$

$u_1 + u_2 \pmod{N_2} \cong u_1 \pmod{N_1 \cap N_2}$.

③ if K is a submodule of N which is a submodule of M, then $M/N \cong (M/K)/(N/K)$

$u \bmod N \longmapsto (u \bmod K) \bmod (N/K)$

Let $1 \in R$.

Theorem if M is a cyclic module, then $M \cong R/\mathrm{Ann}(u)$

$M = Ru$

Proof define $\varphi : R \longrightarrow M$ by $\varphi(a) = au$, $a \in R$. φ is surjective so $M \cong R/\mathrm{Ker}(\varphi) = R/\mathrm{Ann}(u)$.