January 14 - I forgot my electronic note-taking device. These are all of my notes for the day.

- Nullity / Rank of linear operators: $\mathrm{null}(L) = \dim(\mathrm{Ker}(L))$, $\mathrm{rank}(L) = \dim(\mathrm{image}(L))$. Theorem: $L: V \to W \Rightarrow \mathrm{rank}(L) + \mathrm{null}(L) = \dim(V)$.

10.5.10: If R is commutative & unital, any simple R-module $M \cong R/I$ where I is a maximal ideal (i.e. M has structure of a field).

Proof: M is cyclic so $M \cong R/I$ for some ideal I. if $\exists$ ideal J s.t. $I \subsetneq J \subsetneq R$, then J/I is a submodule in $R/I \cong M$, and $R/I \neq J/I \neq 0$.

10.3.2: If R is commutative, then $R^m \ncong R^n$ for $m \neq n$. $\quad R^n/I(R^n)$ ← from linear algebra.

Proof: Let I be a max'l ideal in R, let $F = R/I$ (a field). consider $R^n/I^n \cong (R/I)^n = F^n \ncong F^m$ if $m \neq n$.

If R is non-commutative, it may be that $M^2 \cong M$. (for $R^2 \cong R$ as rings, take $R = F^{\mathbb{N}}$. for modules it's not so easy).

Short Five Lemma: Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\varphi'"'] & B' \arrow[r, "\psi'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. (a hom'sm of two short exact sequences). if α & γ are $(*)$ then β is $(*)$.

(a): $(*)$ = surjective
(b): $(*)$ = injective
(c): $(*)$ = isomorphism.
((c) ⇐ (a) & (b)).

(b): Let $b \in B$, $\beta(b) = 0$. Then $\psi'(\beta(b)) = 0$, so $\gamma(\psi(b)) = 0$ so $\psi(b) = 0$ since γ is injective. Thus $\exists a \in A$ s.t. $b = \varphi(a)$, now $\beta(\varphi(a)) = 0$ so $\varphi'(\alpha(a)) = 0$, but φ' is injective so $\alpha(a) = 0$, but α is injective so $a = 0$. Thus $b = 0$.

(a): Let $b' \in B'$. find $c \in C$ s.t. $\gamma(c) = \psi'(b')$. find $b_0 \in B$ s.t. $\psi(b_0) = c$. $\psi'(\beta(b_0)) = \gamma(\psi(b_0)) = \gamma(c) = \psi'(b')$. So $\psi'(\beta(b_0) - b') = 0$. So $\exists\, a' \in A'$ s.t. $\varphi'(a') = \beta(b_0) - b'$. find $a \in A$ s.t. $a' = \alpha(a)$. Put $b = b_0 - \varphi(a)$. Then $\beta(b) = \beta(b_0) - \beta(\varphi(a)) = \beta(b_0) - \varphi'(\overbrace{\alpha(a)}^{a'}) = b'$.

Snake Lemma: hypothesis of short five lemma $\Rightarrow \exists$ unique $\delta: \mathrm{Ker}\,\gamma \to \mathrm{Coker}\,\alpha$ which makes the following diagram commutative & the snake exact. Note: $\forall \varphi$, $0 \to \mathrm{Ker}\,\varphi \to M \xrightarrow{\varphi} M' \to \mathrm{Coker}\,\varphi \to 0$ is exact.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathrm{Ker}\,\alpha \arrow[r] \arrow[d] & \mathrm{Ker}\,\beta \arrow[r] \arrow[d] & \mathrm{Ker}\,\gamma \arrow[d] \arrow[dddll, bend left, "\delta"] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ & \mathrm{Coker}\,\alpha \arrow[r] \arrow[d] & \mathrm{Coker}\,\beta \arrow[r] \arrow[d] & \mathrm{Coker}\,\gamma \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Def: Let M_1 & M_2 be R-modules. The direct product of M_1 & M_2 is

$M_1 \oplus M_2 = M_1 \times M_2 = \{(u_1, u_2) : u_1 \in M_1, u_2 \in M_2\}$ (componentwise ops).

eg $M \oplus 0 \cong M$. $M_1 \oplus M_2 \cong M_2 \oplus M_1$. $M_1 \oplus M_2 \oplus M_3$ is well-defined.

$M_1 \cong M_1 \times 0$, a submodule of $M_1 \oplus M_2$. $(M_1 \oplus M_2)/M_1 \cong M_2$.

"$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$" is an exact sequence.

Embeddings: $\eta_1: M_1 \to M_1 \oplus M_2$. $\eta_2: M_2 \to M_1 \oplus M_2$.

Projections: $\pi_1: M_1 \oplus M_2 \to M_1$. $\pi_2: M_1 \oplus M_2 \to M_2$.

Universal properties: ① if $\varphi_1: M_1 \to N$, $\varphi_2: M_2 \to N$ be homs, $\exists!\, \varphi: M_1 \oplus M_2 \to N$ s.t.

$$\begin{tikzcd} & M_1 \arrow[dl, "\eta_1"'] \arrow[dr, "\varphi_1"] & \\ M_1 \oplus M_2 \arrow[rr, "\varphi"] & & N \\ & M_2 \arrow[ul, "\eta_2"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes. $\varphi(u_1, u_2) = \varphi(u_1, 0) + \varphi(0, u_2) = \varphi_1(u_1) + \varphi_2(u_2)$.

② if $\varphi_1: N \to M_1$, $\varphi_2: N \to M_2$ are homs, $\exists!\, \varphi: N \to M_1 \oplus M_2$ s.t.

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ N \arrow[ur, "\varphi_1"] \arrow[rr, "\varphi"] \arrow[dr, "\varphi_2"'] & & M_1 \oplus M_2 \\ & M_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

commutes. $\varphi(u) = (\varphi_1(u), \varphi_2(u))$.

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eg $[1,\infty) \approx (1,\infty)$ by $x \mapsto x+1$ if $x \in \mathbb{N}$ and $x \mapsto x$ otherwise. $(1,\infty) \approx (0,1)$ by $x \mapsto \frac{1}{x}$. $[a,b] \approx [a,b) \approx (a,b)$. $(-1,1) \approx \mathbb{R}$ by $x \mapsto \frac{x}{1-|x|}$.

Theorem: $|X| < |\mathcal{P}(X)|$. pf: if $f: X \to \mathcal{P}(X)$ then let $N = \{x \in X : x \notin f(x)\}$. Thus $N \notin f(X)$ ($f(n) = N \Rightarrow n \in N \Leftrightarrow n \notin N$ ↯), so f is not onto. $\quad (\leq, 1\text{-}1)$. $x \mapsto \{x\}$

Russell's Paradox: Let $C = \{x : x \notin x\}$. ($f = id$ on the set of all sets). Conclusion: There is no set of all sets. $\quad |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |[0,1]|$