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Tuesday, February 5, 2019 11:45

<u>Propn</u>: $\mathrm{Hom}(K,\cdot)$ is left-exact:

if $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ is exact

then so is $0 \longrightarrow \mathrm{Hom}(K,A) \xrightarrow[\Phi]{} \mathrm{Hom}(K,B) \xrightarrow[\Psi]{} \mathrm{Hom}(K,C)$.

Ψ is surjective if $\forall\ \xi : K \to C\ \ \exists\ \tau : K \to B$ s.t. $\xi = \psi \circ \tau$

Not always true:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\mathrm{mod}\,2"] & \mathbb{Z}_2 \arrow[r] & 0 \\
& \mathbb{Z}_2 \arrow[ul, "\times"] \arrow[u, "\sim"'] &
\end{tikzcd}$$

no homo-morphisms

So $\mathrm{Hom}(K,\cdot)$ may not be right-exact.

<u>Def</u>: K is projective if $\mathrm{Hom}(K,\cdot)$ is right-exact (so exact).

This is true if $\forall$ epimorphism

$$\begin{tikzcd}
B \arrow[r, "\psi"] & C \arrow[r] & 0 \\
& K \arrow[ul, "\tau"] \arrow[u, "\xi"'] &
\end{tikzcd}$$

and $\forall$ hom-sm $\xi : K \to C$

$\exists\ \tau$ s.t. $\xi = \psi \circ \tau$

<u>Theorem</u>: K is projective iff it's a direct summand of a free module

<u>Proof</u>: $(\Leftarrow)$ Let $K \oplus L = N$, a free module gen-ed by S.

Let $B \xrightarrow{\psi} C$ be surjective, let $\xi : K \to C$ be a hom-sm

Define $\tilde{\xi} : N \longrightarrow C$ by $\tilde{\xi}(L) = 0$.

$\forall u \in S$, define $\tau(u)$ to be any element of $\psi^{-1}(\tilde{\xi}(u))$.

Then τ extends to a homomorphism $N \longrightarrow B$

s.t. $\psi \circ \tau = \tilde{\xi}$. So $\psi \circ (\tau|_K) = \xi$.

($\Rightarrow$) Let K be projective. $\exists$ free module N & a homomorphism $\psi : N \longrightarrow K$ which is surjective.

we have

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & N \arrow[r, "\psi"] & K \arrow[r] & 0 \\
 & \operatorname{Ker}\psi & & K \arrow[ul, "\tau"] \arrow[u, "\iota"'] &
\end{tikzcd}$$

Since K is projective, $\exists\ \tau$ as above s.t. the diagram commutes: $\psi \circ \tau = \mathrm{id}_K$. So the sequence splits (from the right)

So $N \cong K \oplus L$.

<u>Examples</u> $I = (x, y)$ in $F[x, y] = R$ is not a projective R-module.

field of fractions of an ID R is not a projective R-module

If R is an integral domain, any projective module is torsion-free.

Functor $\operatorname{Hom}(\cdot, K)$ is contravariant functor from Category of R-modules to itself.

$A \longmapsto \operatorname{Hom}(A, K)$, and

$A \xrightarrow{\varphi} B$; $\varphi : A \to B$ is mapped to $\Phi : \operatorname{Hom}(B, K) \longrightarrow \operatorname{Hom}(A, K)$

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\Phi"] & B \arrow[dl, "\beta"] \\ & K & \end{tikzcd}$$; $\varphi: A \to B$ is mapped to $\Phi: \mathrm{Hom}(B,K) \longrightarrow \mathrm{Hom}(A,K)$
$\beta \longmapsto \beta \circ \varphi$

Contravariant = "reverses arrows"

<u>Theorem</u>: $\forall K$, $\mathrm{Hom}(\cdot, K)$ is left-exact: if

not necessarily right-exact

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$ is exact, then

$$0 \longrightarrow \mathrm{Hom}(C,K) \xrightarrow{\Psi} \mathrm{Hom}(B,K) \xrightarrow{\Phi} \mathrm{Hom}(A,K)$$ is exact

<u>proof</u>

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[dr, "\alpha"'] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[dl, "\gamma"] & 0 \\ & & K & & \end{tikzcd}$$

ψ is surjective so if $\gamma: C \to K$ is nonzero

then $\underbrace{\gamma \circ \psi}_{}$ $= \Psi(\gamma)$ is nonzero too since ψ is surjective.

So Ψ is injective.

$\Phi \circ \Psi(\gamma) = \gamma \circ \psi \circ \varphi = 0$ so $\mathrm{Im}\,\Psi \subseteq \mathrm{Ker}\,\Phi$.

Now let $\beta \in \mathrm{Ker}\,\Phi$. That is, $\beta \circ \varphi = 0$. so $\beta(\varphi(A)) = 0$.

So $\overset{\varphi(A)}{\overset{\|}{\mathrm{Ker}\,\psi}} \subseteq \mathrm{Ker}\,\beta$. So β factorizes through $\mathrm{Ker}\,\psi$

to a homomorphism $\gamma: C \to K$ so that $\beta = \gamma \circ \psi = \Psi(\gamma)$

So $\mathrm{Im}\,\Psi = \mathrm{Ker}\,\Phi$.

Why not exact at the right:

$\mathrm{Hom}(B,K) \xrightarrow{\Phi} \mathrm{Hom}(A,K)$ is surjective if any homomorphism $\alpha : A \to K$ extends to a hom-sm $\beta : B \to K$ s.t. $\beta \circ \varphi = \alpha$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[dr, "\alpha"'] & B \arrow[d, "\beta"] \\ & & K \end{tikzcd}$$

<u>But</u>:

$$\begin{tikzcd} 2\mathbb{Z} \arrow[r] \arrow[dr, "2"'] & \mathbb{Z} \arrow[d, "\cancel{\ }"] \\ & \mathbb{Z} \end{tikzcd}$$

$2n \mapsto n$

(divide by 2)

can't be extended to a hom-sm $\mathbb{Z} \to \mathbb{Z}$

<u>definition</u>: K is injective if $\mathrm{Hom}(\cdot, K)$ is right-exact (so exact).

This is so if $\forall$ injective $\varphi : A \to B$, $\forall \alpha : A \to K$, $\exists \beta$ s.t. $\alpha = \beta \circ \varphi$.