Lec 10/30

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Baire Category + Piccard's

Examples open & dense:

① $\mathbb{R}$, $\mathbb{R}\setminus\{0\}$.

② $\bigcup_{n=1}^{\infty}\left(r_n - \frac{\varepsilon}{2^n},\ r_n + \frac{\varepsilon}{2^n}\right)$ where $\{r_n\} \subset \mathbb{R}$ dense (eg $\mathbb{Q}$).

A Nowhere dense if $(\overline{A})^\circ = \phi$.

1^{st} category (meager): countable union of nowhere dense sets.

2^{nd} category (residual): complement of a 1^{st} category set.

Can a set be both 1^{st} & 2^{nd} category? No ← exercise.

Lemma:

Cantor's Intersection Thm: If X is a complete metric sp and $F_1 \supseteq F_2 \supseteq \cdots$ decreasing seq of non-empty closed sets in X w/ $\operatorname{diam}(F_n) \to 0$, then $\bigcap_{i=1}^{\infty} F_i$ is a singleton.

Proof: exercise.

- A open & dense $\rightsquigarrow$ A^c is nowhere dense & closed
- B nowhere dense & closed $\rightsquigarrow$ B^c is open & dense

<u>Baire Category Thm</u>: Let X be a complete metric sp.

(a) a meager set has empty interior.

(b) The complement of a meager set is dense

(c) A countable intersection of dense open sets is dense.

Further, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)

<u>Pf:</u> We prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) Then we'll prove (c).

<u>(a)$\Rightarrow$(b)</u>: Suppose E is meager. Let B be an open non-empty ball.

Then $E^c \cap B = \emptyset \Longleftrightarrow B \subseteq E$. Thus $\neg$(b) $\Rightarrow \neg$(a) & (b) $\Rightarrow$ (a)

<u>(b)$\Rightarrow$(c)</u>: First notice if $\mathcal{U}$ is open & dense, $\mathcal{U}^c$ is n.w. dense.

Thus if $\{\mathcal{U}_i\}$ open & dense $\Rightarrow \bigcup \mathcal{U}_i^c$ meager $\Rightarrow \bigcap \mathcal{U}_i$ dense.

<u>(c)$\Rightarrow$(b)</u>: Suppose E meager. Write $E = \bigcup E_i$, where E_i n.w. dense.

$\{E_1, E_2, \dots\}$ n.w dense $\rightarrow$ $\{\overline{E}_1, \overline{E}_2, \dots\}$ n.w. dense & closed

$E = \bigcup E_i \subseteq \bigcup \overline{E}_i \rightsquigarrow E^c \supseteq \bigcap (\overline{E}_i)^c \leftarrow$ dense $\Rightarrow E^c$ is dense.

<u>Pf of (c)</u>: Let $D_1, D_2, \dots$ be dense open sets.

Let $\mathcal{U}$ be open & non-empty.

$D_1 \cap \mathcal{U}$ is open & non-empty, and so $\exists\, x_1, r_1 > 0$ s.t.

$\overline{B(x_1, r_1)} \subseteq D_1 \cap \mathcal{U}$.

So $D_2 \cap B(x_1, r_1)$ is open & non-empty, so $\exists\, x_2, r_2 > 0$ s.t.

$\overline{B(x_2, r_2)} \subseteq D_2 \cap B(x_1, r_1) \subseteq D_1 \cap D_2 \cap \mathcal{U}$.

Continue the process to get a seq of nested closed balls. Apply Cantor intersection thm. ◻

Folland 5.28: The Baire Category thm is still true if X is LCH rather than complete m.s.

Idea: use Prop 4.21 in Folland.

- Steinhaus: $A, B \subseteq \mathbb{R}^d$, $\lambda(A), \lambda(B) > 0 \implies (A+B)^\circ \neq \emptyset$.

$$\lambda(A) > 0 \implies \text{for a.e. } x \in A,\ \exists\, \varepsilon_x > 0 \text{ s.t. } \frac{\lambda(B(x, \varepsilon_x) \cap A)}{\lambda(B(x, \varepsilon_x))} > 0.9.$$

Piccard's Thm: Let X be a complete metric space & let A, B be residual $\underbrace{\text{Baire sets}}_{\text{The symmetric diff of an open set \& a meager set.}}$. Then $(A - B)^\circ \neq \emptyset$.

Pf let X, A, B be such. Write $A = G_1 \,\Delta\, P_1$ and $B = G_2 \,\Delta\, P_2$ where G_i open & P_i meager. A, B residual $\Rightarrow$ G_1, G_2 non-empty.

Claim: $G_1 - G_2$ is open & contained in $A - B$.

Note $x \in G_1 - G_2 \iff (x + G_2) \cap G_1 \neq \emptyset$.

Let $x \in G_1 - G_2$. We wish to show $(x+B) \cap A \neq \emptyset$.

We'll show

① $\big((x + G_2) \cap G_1\big) \setminus \big((x + P_2) \cup P_1\big) \subseteq (x + B) \cap A$

② LHS $\neq \emptyset$

③ Conclude RHS $\neq \emptyset$.

Pf①: suppose $y \in \big((x+G_2)\cap G_1\big) \setminus \big((x+P_2)\cup P_1\big)$

$y \in (x+G_2)\setminus(x+P_2) = x+(G_2\setminus P_2) \Rightarrow y \in x+B.$

Similarly, $y \in A$

Pf② $\underbrace{(x+G_2)\cap G_1}_{\text{open \& nonempty}} \neq \emptyset$ by choice of x.

$(x+P_2)\cup P_1$ non-empty & first category, hence <u>not</u> open.

So $\big((x+G_2)\cap G_1\big)\setminus\big((x+P_2)\cup P_1\big)$ non-empty.

$\square$