Lec 11/2

Wednesday, November 2, 2016 9:06 AM

Integrability Condition:

f is integrable over $[a,b]$ iff $\forall \varepsilon > 0$ $\exists P_\varepsilon$ a partition of $[a,b]$ so that $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$.

Lemma 1 If f is integrable over $[a,b]$ then it is integrable over any subinterval $[c,d] \subseteq [a,b]$.

Proof: Let $\varepsilon > 0$ be given. Can find a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Let P' be $P \cup \{c,d\}$. Then we have the following inequalities:

$$U(f, P' \cap [c,d]) - L(f, P' \cap [c,d]) \leq U(f,P') - L(f,P') \leq U(f,P) - L(f,P) < \varepsilon. \quad \blacksquare$$

Lemma 2 If f is integrable over $[a,c]$ & $[c,b]$ then f is integrable over $[a,b]$ and $\int_a^c f + \int_c^b f = \int_a^b f$

Proof: given $\varepsilon > 0$, find partitions P, Q of $[a,c]$ and $[c,b]$ so that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$, $U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}$

then
$$U(f, P \cup Q) = U(f,P) + U(f,Q)$$
$$L(f, P \cup Q) = L(f,P) + L(f,Q)$$
$$U(f,P\cup Q) - L(f,P \cup Q) = U(f,P) - L(f,P) + U(f,Q) - L(f,Q)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(1) $\int_a^b f \in [L(f,P\cup Q), U(f,P\cup Q)]$

(2) $\int_a^c f \in [L(f,P), U(f,P)]$

(3) $\int_c^b f \in [L(f,Q), U(f,Q)]$

(2)+(3): $\int_a^c f + \int_c^b f \in [L(f,P) + L(f,Q), U(f,P) + U(f,Q)]$

$= [L(f,P\cup Q), U(f,P\cup Q)]$ (4)

(1), (4) $\Rightarrow$ $\left| \left(\int_a^c f + \int_b^c f \right) - \int_a^b f \right| \leq$ length of interval $< \varepsilon$ for all $\varepsilon > 0$

So they must be equal! $\blacksquare$

Definition If f is integrable over $[a,b]$ we define $\int_b^a f = -\int_a^b f$.

Physics justification: going backwards through time.

Lemma 3 (exercise) If f is integrable over an interval I and $a,b,c \in I$ then physics justification going backwards ... f ...

Lemma 3 (exercise) If f is integrable over an interval I and $a,b,c \in I$ then
$$\int_a^b f = \int_a^c f + \int_c^b f$$ no matter how a,b,c are positioned.

Theorem (Fundamental Thm of Calculus V2) Spoze that F is continuous on $[a,b]$ and diffable on (a,b) and F' is integrable over $[a,b]$ (Define $F'(a), F'(b)$ arbitrarily). Then, $\int_a^b F' = F(b) - F(a)$.

Proof: Let $\varepsilon > 0$ be given. Pick a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

so that $U(F',P) - L(F',P) < \varepsilon$.

By MVT, $F(x_i) - F(x_{i-1}) = F'(y_i)(x_i - x_{i-1})$ for some $y \in (x_{i-1}, x_i)$

and $m_i \le F'(y_i) \le M_i$

Then
$$L(F',P) = \sum_{i=1}^n m_i (x_i - x_{i-1}) \le \sum_{i=1}^n F'(y_i)(x_i - x_{i-1}) \le \sum_{i=1}^n M_i (x_i - x_{i-1}) = U(F',P)$$
$$\sum_{i=1}^n F'(y_i)(x_i - x_{i-1}) = \sum_{i=1}^n (F(x_i) - F(x_{i-1}))$$
$$= F(x_0) - \cancel{F(x_1)} + \cancel{F(x_1)} - \cancel{F(x_2)} + \cancel{\cdots} + \cancel{F(x_{n-1})} - F(x_n) = F(b) - F(a)$$

so $F(b) - F(a) \in [L(F',P), U(F',P)]$

but $\int_a^b F' \in [L(F',P), U(F',P)]$

and length of interval $< \varepsilon$ arbitrary so $\int_a^b F' = F(b) - F(a)$. ∎

Note: There are examples where F' is not integrable. (perhaps a bonus problem on final).

FTC V2 tells how to compute integrals if we can find an antiderivative.

But, do antiderivatives really exist?

Theorem (FTC V1) Suppose f is defined on an interval (possibly not closed, maybe infinite).

Suppose f is integrable over any finite closed subinterval.

Let $a \in I$ and define $F(x) = \int_a^x f$ for $x \in I$

Suppose $c \in$ interior of I and f is continuous at c.

Then $F'(c) = f(c)$.

Corollary Continuous functions have antiderivatives.

example (exercise): if $f(x) = \text{sign}(x)$ then $F(x) = \int_0^x f = |x|$.

Proof of FTC V1 Since $c \in$ interior of I, $c+h \in I$ if h is small enough.
$$F(c+h) = \int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f = F(c) + \int_c^{c+h} f$$

so $F(c+h)-F(c) = \int_c^{c+h} f$

If $h>0$, then $m_h h = L(f, \{c, c+h\}) \le \int_c^{c+h} f \le U(f, \{c, c+h\}) = M_h h$

so $m_h \le \dfrac{F(c+h)-F(c)}{h} \le M_h$.

given $\varepsilon>0$ we can find δ so that $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

if we pick $|h|<\delta$ then

$f(c)-\varepsilon < M_h < f(c)+\varepsilon$

$f(c)-\varepsilon < m_h < f(c)+\varepsilon$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{F(c+h)-F(c)}{h} = f(c)$