Lec 8/23

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$\mathcal{M}$ σ-algebra on X

0. $\mathcal{M} \neq \emptyset$
1. $\mathcal{M}$ is closed under countable union
2. $\mathcal{M}$ is closed under complements

<u>Observe</u> if $\mathcal{M}, \mathcal{N}$ σ-algs on X, so is $\mathcal{M} \cap \mathcal{N}$.

So, for $\mathcal{E} \subset P(X)$, we can define $\mathcal{M}(\mathcal{E})$ as the smallest σ-algebra on X containing $\mathcal{E}$.

$$\mathcal{M}(\mathcal{E}) := \bigcap_{\mathcal{E} \subset \mathcal{M}} \mathcal{M}$$

<u>Examples</u>: Let $(X, \underset{\text{topology}}{\tau})$ be a topological space

$\tau \subset P(X)$ s.t.

0. $\emptyset, X \in \tau$
1. τ closed under arbitrary unions
2. τ closed under finite intersections

Sets in τ are open sets.

$\mathcal{B} := \mathcal{M}(\tau)$ is called the Borel σ-algebra

Defn. a countable intersection of open sets is a G_δ set

" " union " closed " F_σ "

" " union " G_δ " $G_{\delta\sigma}$ "

" " intersection " F_σ " $F_{\sigma\delta}$ "

...

When $X = \mathbb{R}$, topology induced by $\rho(x,y) = |x-y|$,

$B_{\mathbb{R}}$ is Borel σ-alg.

<u>Prop</u>: $B_{\mathbb{R}} = \mathcal{M}(\mathcal{E})$ for the following $\mathcal{E}$:

1. open intervals (a,b)
2. closed intervals $[a,b]$ (3')
3. half-open intervals $[a,b)$ or $(a,b]$
4. open rays (a,∞) or $(-\infty,a)$ (4')
5. closed rays $[a,\infty)$ or $(-\infty,a]$ (5')

<u>Observation</u>: If $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$ with $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$,

then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

<u>proof of prop</u>: (1), (2), (4), ..., (5') are open or closed,

so they lie in $B_{\mathbb{R}}$. (3) are $(a,\infty) \cap (b,\infty)^c$

and (3') similarly, so they lie in $B_{\mathbb{R}}$, so

their σ-algebras lie in $B_{\mathbb{R}}$.

<u>lemma</u>: all open sets in $\mathbb{R}$ are ctble unions of disjoint open intervals.

<u>steps of proof of prop</u>: Let $U \subset \mathbb{R}$ be open.

1. $\forall x \in U$, let I_x be max'l interval $\subset U$ containing x. This exists.
2. $\{I_x \mid x \in U\}$ has disjoint elts: $I_x \cap I_y = \begin{cases} I_x \\ \emptyset \end{cases}$
3. $\overset{\shortparallel}{J}$ is countable & $\bigcup J = U$.
   ↳ inject $J \to \mathbb{Q}$.

So conversely, $B_{\mathbb{R}} \subset \mathcal{M}(1)$, and $\forall$ else, show

$① \subset \mathcal{M}(ⓘ) \quad \forall i > 1.$ $\square$

A set X equipped w/ a σ-algebra $\mathcal{M}$ is called a measurable space.

A measure on a measurable space $(X, \mathcal{M})$ is a fn

$$\mu : \mathcal{M} \longrightarrow [0, \infty] \quad \text{s.t.}$$

⓪ $\mu(\emptyset) = 0$

① $\forall$ seq of disjoint sets (E_n),

$$\mu\left(\coprod E_n\right) = \sum \mu(E_n).$$

call $(X, \mathcal{M}, \mu)$ a measure space.

- A measure space is called <u>finite</u> if $\mu(X) < \infty$,
- it's <u>σ-finite</u> if $X = \bigcup E_n$ where $\mu(E_n) < \infty \ \forall n$.
- It's semifinite if $\forall E \in \mathcal{M}$ w/ $\mu(E) = \infty$, $\exists F \subset E$ s.t. $0 < \mu(F) < \infty$.
- It's complete if $E \subset \mathcal{M}$ w/ $\mu(E) = 0$ and $F \subset E \implies F \in \mathcal{M}$ & $\mu(F) = 0$.
  (all subsets of null-sets are in σ-algebra).

<u>Examples</u>:

① counting measure on $P(X)$

② pick $x_0 \in X$, then on $P(X)$, define $\mu(E) = \begin{cases} 0 & \text{if } x_0 \notin E, \\ 1 & \text{if } x_0 \in E. \end{cases}$
(point mass / Dirac measure)

③ pick any $f : X \to [0, \infty]$ and on $P(X)$ define

$$\mu(E) = \sum_{x \in E} f(x).$$

④ on σ-algebra of countable or co-countable sets,

$$\mu(E) = \begin{cases} 0 & \text{for ctble sets} \\ 1 & \text{for co-ctble sets} \end{cases}$$

Basic Properties:

① (Monotonicity) $E, F \in \mathcal{M}$, $E \subset F$, then $\mu(E) \leq \mu(F)$.

pf $F = E \sqcup (F \setminus E)$.

② (subadditivity) $(E_n) \subset \mathcal{M} \Rightarrow \mu(\bigcup E_n) \leq \sum \mu(E_n)$.

pf disjointify.

③ (continuity from below) $E_1 \subset E_2 \subset E_3 \subset \cdots \Rightarrow \mu(\bigcup E_n) = \lim_{n\to\infty} \mu(E_n)$

pf disjointify easily: set $E_0 = \emptyset$,

$$\begin{aligned} \mu(\bigcup E_n) &= \mu\left(\bigcup (E_n \setminus E_{n-1})\right) \\ &= \sum \mu(E_n \setminus E_{n-1}) \\ &= \lim_K \sum_1^K \mu(E_n \setminus E_{n-1}) \\ &= \lim_K \mu\left(\bigcup_1^K E_n \setminus E_{n-1}\right) \\ &= \lim_K \mu(E_K). \end{aligned}$$