Lec 8/29

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Thm 1. Let V/F be finitely generated, $V = S(a_1, \dots, a_n)$.

If $W \leq V$ is a subspace then W is fin. gen & has a basis.

Proof Let $W \neq \{0\}$ so there is some vector $b_1 \in W$ nonzero. If $S(b_1) = W$, we are done.

else pick a vector $b_2 \in W \setminus S(b_1)$. Since $b_2 \neq \lambda b_1$, the two are lin. indp.

$\hookrightarrow \begin{cases} \lambda_1 b_1 + \lambda_2 b_2 = 0, \ \lambda_1 \neq 0 \Rightarrow b_2 \in S(b_1) \\ \lambda_2 \neq 0 \Rightarrow \lambda_1 \text{ cannot be } 0. \end{cases}$

continue in this way to get $\{b_1, \dots, b_m\} \subset W$

linearly independent, look at $S(b_1, \dots, b_m) \leq W$.

If $\neq W$, pick $b_{m+1} \in W \setminus S(b_1, \dots, b_m)$ and $\{b_1, \dots, b_{m+1}\}$ are linearly independent.

$\left\{ \begin{array}{l} \lambda_1 b_1 + \dots + \lambda_m b_m + \lambda_{m+1} b_{m+1} = 0. \text{ If } \lambda_{m+1} \neq 0, \ b_{m+1} = \text{a linear combination of } \{b_1, \dots, b_m\} \\ \text{meaning } b_{m+1} \in S(b_1, \dots, b_m) \text{ so } \lambda_{m+1} = 0 \text{ so } \lambda_1 b_1 + \dots + \lambda_m b_m = 0 \Rightarrow \lambda_i = 0 \ \forall i. \end{array} \right.$

lin. indp.

this process must stop at some point, since V is finitely generated.

Cor1 Every finitely generated v.s. V/F has a basis.

Cor 2. If $\{b_1, \dots, b_m\}$ all belong to V and are lin. indp. then one can find $\{b_{m+1}, \dots, b_n\} \subset V$ s.t. $\{b_1, \dots, b_n\}$ is a basis of V.

Thm 2 Assume $V = S(a_1, \dots, a_n)$. Then one can extract a basis for V from $\{s_1, \dots, s_n\}$, say $\{a_1, \dots, a_m\}$ where $m \leq n$.

Proof reorder the a_is s.t. the first m are linearly independent and anything after m is a liner combo of the first m. Then $\{a_1, \dots, a_m\}$ generates V and is lin. indp.

so it's a basis.

$\hookrightarrow$ requires thm 7.1.

other pf: If $a_n \in S(a_1, \dots, a_{n-1})$ then $S(a_1, \dots, a_{n-1}) = V$.

If $a_n \notin S(a_1, \dots, a_{n-1}) = S(a_1, \dots, a_m)$, $m \leq n-1$, $\{a_1, \dots, a_m\}$ lin. indp.

Then $\{a_1, \dots, a_m, a_n\}$ lin. indp. and generates V.

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Let V/F be finitely generated.

Let S, T be subspaces of V.

$S \cap T = \{v \in V ; v \in S, v \in T\}$ is a subspace.

$v_1, v_2 \in S \cap T$
$\lambda_1, \lambda_2 \in F \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in S$ and $\in T$ so $\in S \cap T$.

$S \cup T = \{v \in V ; v \in S, v \in T\}$ is not a subspace.

$S(e_1) \cup S(e_2) \not\ni \langle 1, 1\rangle = e_1 + e_2$.

$S + T = \{v \in V ; v = s + t, s \in S, t \in T\}$ is a subspace.

$v_1, v_2 \in S+T$
$\lambda_1, \lambda_2 \in F \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 (s_1 + t_1) + \lambda_2 (s_2 + t_2) = (\lambda_1 + \lambda_2)(s_1 + s_2) + (\lambda_1 + \lambda_2)(t_1 + t_2)$
$= s_3 + t_3 \in S+T$.

When V is finitely generated, $S = S(v_1, \ldots, v_m)$, $T = S(w_1, \ldots, w_p)$

$S + T = S(v_1, \ldots, v_m, w_1, \ldots, w_p)$

Thm Let V/F be finitely generated, S, T subspaces of V. Then

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$$

Proof $S \cap T \subseteq S, \subseteq S+T, \subseteq T$. $S \cap T$ has a basis $\{u_1, \ldots, u_r\}$. extend it to $\{u_1, \ldots, u_r, v_1, \ldots, v_p\}$, a basis of S.

and extend it to $\{u_1, \ldots, u_r, w_1, \ldots, w_q\}$, a basis of T.

Claim: $\{u_1, \ldots, u_r, v_1, \ldots, v_p, w_1, \ldots, w_q\}$ is a basis of $S+T$.

it's clear that they generate $S+T$. They are also lin. indp.