Lec 9/20

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Prop: $\forall f \in L^1$, $\left|\int f\right| \le \int |f|$.

cor: $\forall f, g \in L^1$, TFAE:

① $f = g$ a.e.

② $\int |f-g| = 0$

③ $\forall E \in \mathcal{M}$, $\int_E f = \int_E g$

Pf ①⟺② $f=g$ a.e. $\iff |f-g| = 0$ a.e. $\iff \int|f-g| = 0$.

②⟹③ $\forall E \in \mathcal{M}$, $\left|\int_E f - \int_E g\right| = \left|\int_E f-g\right| = \left|\int (f-g)\chi_E\right| \le \int |f-g|\chi_E \le \int |f-g| = 0$

③⟹① observe $f=g$ a.e. iff $\mathrm{Re}(f-g) = 0$ a.e. and $\mathrm{Im}(f-g) = 0$ a.e. prove this!

Recall $\int_E f - \int_E g = \int (f-g)\chi_E = 0 \ \forall E \in \mathcal{M}$

$$= \int \mathrm{Re}(f-g)\chi_E + i\int \mathrm{Im}(f-g)\chi_E.$$

Look at the following $E \in \mathcal{M}$:

$\{\mathrm{Re}(f-g) \ge 0\} \implies \mathrm{Re}(f-g)_+ = 0$ a.e

$\{\mathrm{Re}(f-g) \le 0\} \implies \mathrm{Re}(f-g)_- = 0$ a.e

⋮

similarly for Im. □

$\|\cdot\|_1 : L^1 \longrightarrow [0,\infty)$ by $\|f\|_1 = \int |f|$

- $\|\alpha f\|_1 = \int |\alpha f| = \int |\alpha||f| = |\alpha| \, \|f\|_1$

- $\|f+g\|_1 = \int |f+g| \leq \int |f|+|g| = \int |f| + \int |g| = \|f\|_1 + \|g\|_1$.

this is a seminorm

Define $\mathcal{L}^1 = L^1/\sim$ where $f \sim g$ if $f=g$ a.e

Observe $f \sim g \iff \int |f-g| = 0$

↳ $\|\cdot\|_1$ descends to a norm on $\mathcal{L}^1$.

$\rho_1(f,g) = \|f-g\|_1$ is a metric on $\mathcal{L}_1$.

HW: $\mathcal{L}^1$ is complete wrt ρ_1 .

- write $f \in \mathcal{L}^1$ to mean $f \in L^1$ representing its eq. class in $\mathcal{L}^1$.

Say $f_n \to f$ in $\mathcal{L}^1$ to mean $\int |f_n - f| \to 0$.

Remark $\mathcal{L}^1(\mu) \cong \mathcal{L}^1(\bar{\mu})$

Dominated Convergence Theorem Suppose $(f_n) \subset \mathcal{L}^1$ s.t. $f_n \to f$ a.e.

and $\exists\, g \in \mathcal{L}^1$ s.t. $g \geq 0$ and $|f_n| \leq g$ a.e. $\forall n$.

Then $f \in \mathcal{L}^1$ and $\int f = \lim \int f_n$.

(special case: Bdd CT: $\mu(X) < \infty$ and $|f_n| \leq M$ $\forall n$)

Pf wlog, f is $\mathcal{M}$-measurable. Taking limits,

$|f| \leq g$ a.e. so $f \in \mathcal{L}^1$. Taking Re, Im parts separately, we may assume f_n, f are $\mathbb{R}$-valued.

Then $-g \leq f_n \leq g$ a.e. so

$g + f_n \geq 0$ and $g - f_n \geq 0$ a.e.

By Fatou's Lemma.

$$\int g + \int f = \int g+f \leq \liminf \int g + f_n = \int g + \liminf \int f_n$$

$$\int g - \int f = \int g - f \leq \liminf \int g - f_n = \int g - \limsup \int f_n$$

so $\limsup \int f_n \leq \int f \leq \liminf \int f_n$. $\square$

<u>Modes of Convergence</u>: $(X, \mathcal{M}, \mu)$ fixed measure space.

(f_n), f are $\mathcal{M}$-$\mathcal{B}_{\mathbb{C}}$ mble fns.

$f_n \to f$ can mean:

- (pointwise) (everywhere) $f_n(x) \longrightarrow f(x)$ $\forall x \in X$.
- (a.e) $f_n(x) \longrightarrow f(x)$ for a.e. $x \in X$

- (a.e) $\quad f_n(x) \longrightarrow f(x)$ for a.e. $x \in X$

- (uniform) $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$n > N \Rightarrow |f_n(x) - f| < \varepsilon \quad \forall x \in X.$$

(*)

- (almost uniform) (a.u.) $\quad \forall \varepsilon > 0, \ \exists E \in \mathcal{M}$ w/ $\mu(E) < \varepsilon$ s.t.

$$f_n \chi_{E^c} \longrightarrow f \chi_{E^c} \text{ uniformly}$$

- (in $\mathcal{L}^1$) $\quad \int |f_n - f| \longrightarrow 0$

(*)

- (in measure) $\quad \forall \varepsilon > 0, \ \mu(\{|f - f_n| \geq \varepsilon\}) \longrightarrow 0$.

$$\begin{tikzcd}
\text{uniform} \arrow[r] \arrow[d] & \text{pointwise} \arrow[d] \\
\text{a.u.} \arrow[r] \arrow[d] & \text{a.e.} \\
\text{measure} & \mathcal{L}^1 \arrow[l]
\end{tikzcd}$$

Pf: $\underline{\text{a.u.} \Rightarrow \text{a.e.}}$ $\quad \forall n \in \mathbb{N}, \ \exists E_n \in \mathcal{M}$ s.t. $\mu(E_n) < \frac{1}{n}$ and

$$f_k \chi_{E_n^c} \longrightarrow f \chi_{E_n^c} \text{ unif.}$$

$$f_k \chi_{E_n^c} \longrightarrow f \chi_{E_n^c} \quad \text{unif.}$$

$$\text{so} \quad f_k \chi_{E_n^c} \longrightarrow f \chi_{E_n^c} \quad \text{pointwise.}$$

$$\text{so} \quad f_k \chi_{E^c} \longrightarrow f \chi_{E^c} \qquad \text{where } E^c = \bigcup E_n^c.$$

2. $\mu(E) = 0$ by cont. from above. $\square$

<u>$\mathcal{L}^1 \Rightarrow$ measure</u> Suppose $f_n \longrightarrow f$ in $\mathcal{L}^1$. Let $\varepsilon > 0$.

$$\mu(\{|f-f_n| \geq \varepsilon\}) = \int_{\{|f-f_n| \geq \varepsilon\}} 1 = \frac{1}{\varepsilon} \int_{\{|f-f_n| \geq \varepsilon\}} \varepsilon$$

$$\leq \frac{1}{\varepsilon} \int_{\{|f-f_n| \geq \varepsilon\}} |f-f_n| \leq \frac{1}{\varepsilon} \int |f-f_n| \longrightarrow 0 \qquad \square$$

<u>a.u. $\Rightarrow$ measure</u> Suppose $f_n \longrightarrow f$ a.u. Let $\varepsilon > 0$. Let $\delta > 0$.

find $N \in \mathbb{N}$ s.t. $n > N \Rightarrow \mu(\{|f-f_n| \geq \varepsilon\}) < \delta$.

Since $f_n \longrightarrow f$ a.u., $\exists\, E \in \mathcal{M}$ s.t. $\mu(E) < \delta$

and $f_n \chi_{E^c} \longrightarrow f \chi_{E^c}$ unif.

$$\{|f-f_n| \geq \varepsilon\} = \underbrace{\{|f-f_n| \geq \varepsilon\} \cap E}_{\mu \leq \delta} \sqcup \underbrace{\{|f-f_n| \geq \varepsilon\} \cap E^c}_{\text{uniformly to } 0, \text{ so } \emptyset \text{ for large } n.} \qquad \square$$