Lec 4/20

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Given $V, W \in T_pM \setminus \{0\}$

$$\langle\langle V, W, p\rangle\rangle = (\langle V \mid W\rangle, \langle V \times W \mid \nu(p)\rangle)$$

$$= \left(\sum_{i,j} g_{ij} V^i W^j, (V^1W^2 - V^2W^1)\sqrt{g}\right).$$

$$= |V||W| (\cos\theta, \sin\theta)$$

$\uparrow$ iff $\theta \in \measuredangle(V, W, p)$

if $\theta_1 \in \measuredangle(V, W, p)$ then $\measuredangle(V, W, p) = \{\theta_1 + 2\pi k : k \in \mathbb{Z}\}$.

Let X be a (connected) contractible space (such as $[a,b]$ or $[0,1]\times[0,1]$ or $\bar{B}(r,0)$)

Let $f : X \longrightarrow M$ be cts. Let V and W be cts fields of nonzero vectors on M along f. (This means that V is a cts fn on X st. $\forall x \in X$, $V(x) \in T_{f(x)}M$ (and W)).

To say θ is a cts version of the oriented angle from V to W along f (denoted by $\theta \in \measuredangle(V, W, f)$) means θ is a cts map from X to $\mathbb{R}$ and $\forall x \in X$, $\theta(x) \in \measuredangle(V(x), W(x), f(x))$. equivalently, $\theta \in \measuredangle(V, W, f)$ iff θ is a cts argument of the map $x \longmapsto \langle\langle V(x), W(x), f(x)\rangle\rangle$ from X to $\mathbb{R}^2_*$. $\measuredangle(V, W, f) \neq 0$ because X is contractible.

If $\theta_0 \in \measuredangle(V, W, f)$, then $\measuredangle(V, W, f) = \{\theta_0 + 2\pi n : n \in \mathbb{Z}\}$ because a contractible space is connected.

Suppose U is another cts field of nonzero vectors on M along f. Let $\theta_1 \in \measuredangle(U, V, f)$. Let $\theta_2 \in \measuredangle(V, W, f)$. Then $\theta_1 + \theta_2 \in \measuredangle(U, W, f)$, because $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

Let $R \subseteq M$. Let V be a cts field of nonzero vectors in R on M. Let $\gamma : [a,b] \longrightarrow R$ be cts. and let Z be a cts field of nonzero vectors on M along γ.

Then the angular variation of Z with respect to V along γ is $\delta(V, Z, \gamma) = \theta(b) - \theta(a)$ where θ is any element of $\sphericalangle(\underset{\downarrow}{V\circ\gamma}, Z, \gamma)$. (any other $\tilde{\theta} \in \sphericalangle(V\gamma, Z, \gamma)$ will work). Let γ be a loop in R.

Then the angular variation of Z wrt V around γ is

$$\delta(V, Z, \gamma) = \delta(V, Z \circ \tilde{e}, \gamma \circ \tilde{e}) \quad \text{where } \tilde{e}: [0, 2\pi] \to S^1$$

defined by $\tilde{e}(t) = e^{it}$.

Lemma: Let $R \subseteq M$ and let V and W be cts fields of non-zero vectors in R on M, let γ be a loop in R, and let $\psi \in \sphericalangle(V \circ \gamma \circ \tilde{e}, W \circ \gamma \circ \tilde{e}, \gamma \circ \tilde{e})$.

(a) Let γ_0 be a loop in R which is homotopic to γ in R. Let $\psi_0 \in \sphericalangle(V \circ \gamma_0 \circ \tilde{e}, W \circ \gamma_0 \circ \tilde{e}, \gamma_0 \circ \tilde{e})$. Then $\psi(2\pi) - \psi(0) = \psi_0(2\pi) - \psi_0(0)$.
i.e. $\delta(V, Z, \gamma) = \delta(V, Z, \gamma_0)$

(b) Suppose γ is null-homotopic in R. Then $\psi(2\pi) - \psi(0) = 0$.

pf (a) $\dfrac{\psi(2\pi) - \psi(0)}{2\pi}$ is the winding number of the loop $\langle\langle V \circ \gamma, W \circ \gamma, \gamma \rangle\rangle = |V\circ\gamma||W\circ\gamma| (\cos\psi, \sin\psi)$ in $\mathbb{R}^2_*$ with respect to the origin. Similarly for ψ_0. Remember that for each $z \in S^1$, $\langle\langle V\circ\gamma(z), W\circ\gamma(z), \gamma(z)\rangle\rangle = \big(\langle V(\gamma(z)) \mid W(\gamma(z))\rangle, \langle V(\gamma(z)) \times W(\gamma(z)) \mid \nu(\gamma(z))\rangle\big)$ and similarly for γ_0.

There's a homotopy between γ and γ_0 in R

gives rise to a homotopy between
$\langle\!\langle V\circ\gamma, W\circ\gamma, \gamma \rangle\!\rangle$ and $\langle\!\langle V\circ\gamma_0, W\circ\gamma_0, \gamma_0 \rangle\!\rangle$ in $\mathbb{R}^2_*$.

hence these two loops in $\mathbb{R}^2_*$ have the same winding # abt the origin.

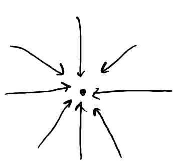
therefore $\psi(2\pi) - \psi(0) = \psi_0(2\pi) - \psi_0(0)$.

(b) if γ_0 is constant then $\psi_0(2\pi) - \psi_0(0) = 0$ $\square$

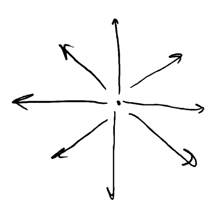
<u>Thm</u> Let $R \subseteq M$, let γ be a loop in R, and let Z be a cts field of nonzero vectors on M along γ. Suppose γ is null-homotopic in R. Then $\delta(V, Z, \gamma)$ does not depend on the choice of reference field V in R of non-zero vectors on M.

<u>Pf</u> Let V and W be two such reference fields. let $\psi \in \measuredangle(V\circ\gamma\circ\tilde{e}, W\circ\gamma\circ\tilde{e}, \gamma\circ\tilde{e})$ and let $\theta \in \measuredangle(W\circ\gamma\circ\tilde{e}, Z\circ\tilde{e}, \gamma\circ\tilde{e})$. Note that $V\circ\gamma\circ\tilde{e}$, $W\circ\gamma\circ\tilde{e}$, and $Z\circ\tilde{e}$ are cts fields of nonzero vectors on M along $\gamma\circ\tilde{e}$. Hence,
$\psi + \theta \in \measuredangle(V\circ\gamma\circ\tilde{e}, Z\circ\tilde{e}, \gamma\circ\tilde{e})$. by the lemma, since γ is null-homotopic in R, $\psi(2\pi) - \psi(0) = 0$. Hence

$$(\psi+\theta)(2\pi) - (\psi+\theta)(0) = \theta(2\pi) - \theta(0), \text{ i.e. } \delta(V, Z, \gamma) = \delta(W, Z, \gamma). \quad \square$$



$i_p = 1$



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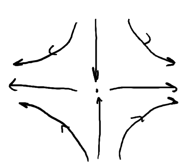

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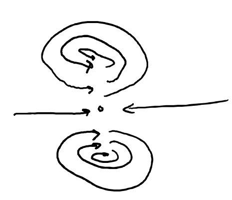

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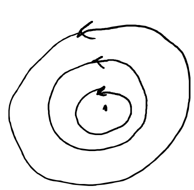

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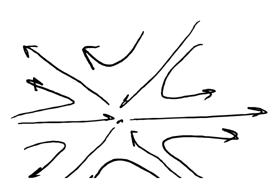

$i_p = -2$.

$c_p = -2.$