Lec 10/24

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Maximal $\Rightarrow$ Prime Ideal

(1) $\underset{\text{max'l}}{M_1, M_2} \subsetneq R \implies M_1 + M_2 = R$ or $M_1 = M_2$

(2)
$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ P_1 = f^{-1}(P_2) & \longrightarrow & P_2 \text{ prime} \end{array} \implies P_1 \text{ is prime}$$

Non-ex:
$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q} \supsetneq (0) \\ \psi & & \psi \\ n & \longmapsto & \frac{n}{1} \end{array}$$
(0) max'l, but $P_1 = f^{-1}((0))$ is not max'l

Defn A comm ring R is called local if it has only one max'l ideal

(R, M) is a local ring.

Propn: (R commutative) R is a local ring $\iff$ $\overset{M}{\overset{\parallel}{R \setminus R^\times}}$ is an ideal.

(in this case, M is the unique maximal ideal).

(This says $\mathbb{Z}$ is not local $(2-3=-1)$)

Proof: Recall: $I \subsetneq R \Rightarrow I \cap R^\times = \emptyset$. So if M is an ideal it is the max'l one.

($\implies$) if R is Local then $J \subsetneq R$ is the unique max'l ideal

$J \subset M$ since $J \subsetneq R$. And $x \in M \Rightarrow (x) \subsetneq R \Rightarrow (x) \subset \overset{\text{some max'l ideal}}{I} = J \leftarrow$ the only one.

so $J = M$.

Examples of local rings: $R = K[x]/(x^2) \ni \{a + bx : a, b \in K\}$

$R^\times$: $(a+bx)(c+dx) = ac + (bc+ad)x$

$a + bx \in R^\times$ iff $\begin{array}{l} ac = 1 \\ ad + bc = 0 \end{array}$ for some c, d

$c = \frac{1}{a}$, $d = -\frac{bc}{a}$ $\Rightarrow$ iff $a \in K^\times = K \setminus \{0\}$

$R^\times = \{a + bx : a \neq 0\}$

$R \setminus R^\times = \{bx : b \in K\} = (x)$ is an ideal

So $\left(K[x]/(x^2), (x) \right)$ is a local ring.

(2) $R = K[\![x]\!] = \{\underbrace{a_0 + a_1 x + a_2 x^2 + \cdots}_{\sum_{j=0}^{\infty} a_j x^j} : a_i \in K\}$

Addition is component-wise

Multiplication is distributive:

$$\left(\sum_{j=0}^{\infty} a_j x^j\right) \cdot \left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} a_k b_{n-k}\right)}_{\text{finite sum}} x^n$$

can switch bottom bound to $-M$

i.e. $\left\{ \sum_{j=-M}^{\infty} a_j x^j : M \in \mathbb{Z}, a_i \in K \right\} = K[x^{-1}, x]\!] = K((x))$

$\left\{ \sum_{j=-\infty}^{\infty} a_j x^j \right\} = K[\![x^{-1}, x]\!]$ abelian group ✓ but not a ring.

If there were a product then

$$\underbrace{\overbrace{\cdots + x^{-2} + x^{-1}}^{\frac{x^{-1}}{1-x^{-1}}} + \overbrace{1 + x + x^2 + \cdots}^{\frac{1}{1-x}}}_{\text{nonzero}}$$

and $\frac{x^{-1}}{1-x^{-1}} + \frac{1}{1-x}$

$= \frac{x^{-1} - 1 + 1 - x^{-1}}{(1-x^{-1})(1-x)} = 0$

$(K[\![x]\!], (x))$ is local ring.

$$\left(K[[x]]\right)^\times = \left\{ \sum_{j=0}^{\infty} a_j x^j \;:\; a_0 \neq 0 \right\}$$

$\Rightarrow$ $R \setminus R^\times = (x)$ is an ideal

(3) $p \in \mathbb{Z}_{\geq 2}$ prime. $R = \left\{ \frac{a}{b} \in \mathbb{Q} : \begin{array}{l} \gcd(a,b) = 1 \\ p \text{ does not divide } b \end{array} \right\}$

$R \subset \mathbb{Q}$ is a subring.

$R^\times = \left\{ \frac{a}{b} : \begin{array}{l} \gcd(a,b)=1 \\ a,b \notin p\mathbb{Z} \end{array} \right\}$

$R \setminus R^\times = (p)$ an ideal $\quad \mathbb{Z}_p$: standard notation for this ring.

$(R, (p))$ is a local ring.

<u>Geometric</u> viewpoint:

Comm R = ring of $\underset{\text{polynomial, cts}}{X}$ functions on $\underset{\text{vector, topological}}{Y}$ spaces valued in some field ($\mathbb{C}$ or $\mathbb{R}$ or something)

Ideals = functions vanishing on a subset of the space.

if topological, subset is closed.

Multiplicatively closed sets $\leftrightarrow$ Opens = "non-vanishing"

<u>Definition</u>. $S \subset \underset{\text{com ring}}{R}$ $\quad S$ is mult. closed if $0 \notin R$, $1 \in S$, $a,b \in S \Rightarrow ab \in S$.