Lec 11/18

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<u>Recall</u>: X LCH. A Radon meas on X is

- finite on cpt $K \subset X$,
- outer reg. on Borel sets

inner reg. on open sets

X σ-finite $\Rightarrow$ inner regularity.

"finite Radon measure" $\Leftrightarrow$ "finite regular Borel"

"σ-" "finite on compacta"

A Radon Integral on $C_c(X)$ is a linear ftl $\varphi: C_c(X) \to \mathbb{C}$

s.t. $\varphi(f) \geq 0 \ \forall f \geq 0$.

<u>Lemma</u>: Radon Integrals are bdd on $C_c(K) \subset C_c(X)$ compact.

<u>HW</u>: Every positive linear functional on $C_0(X)$ is bdd.

<u>Thm</u> (Riesz Rep): For a Radon integral φ on X, $\exists!$

Radon meas μ_φ on X s.t. $\varphi(f) = \int f \, d\mu_\varphi \ \forall f \in C_c(X)$.

Moreover, μ_φ satisfies:

(a) $\mu_\varphi(\mathcal{U}) = \sup\{\varphi(f) \mid \overbrace{f \prec \mathcal{U}}^{0 \leq f \leq 1 \text{ and } \overline{supp(f)} \subseteq \mathcal{U}}\}$

(a) $\mu_\varphi(\mathcal{U}) = \sup\{\varphi(f) \mid \widetilde{f \prec \mathcal{U}}\}$ $\quad\forall$ open $\mathcal{U}$.

$\left(\Rightarrow \mu_\varphi(\mathcal{U}) = \sup\{\varphi(f) \mid 0 \le f \le \chi_{\mathcal{U}}\}\right)$

(b) $\mu_\varphi(K) = \inf\{\varphi(f) \mid f \geq \chi_K\}$ $\quad\forall$ cpt K.

<u>Uniqueness</u>: Last time

<u>Existence</u>: For $\mathcal{U}$ open, define $\mu(\mathcal{U}) := \sup\{\varphi(f) \mid f \prec \mathcal{U}\}$

and define $\mu^*(E) = \inf\{\mu(\mathcal{U}) \mid E \subset \mathcal{U} \text{ open}\}$ for $E \subset X$.

Outline:

<u>Step 1</u> μ^* is an outer measure on $P(X)$.

<u>Step 2</u>: Every open set is μ^*-mble

$\Rightarrow$ By Carathéodory, $B_X \subset \underbrace{M^*}_{\mu^*\text{-mble sets}}$, and $\mu_\varphi := \mu^*|_{B_X}$ is a Borel meas.

By defn, μ_φ is <u>outer regular</u> and satisfies (a).

<u>Step 3</u>: μ_φ satisfies (b)

$\Rightarrow$ μ_φ is finite on cpt sets & inner regular on open sets,

[Since if $\mathcal{U} \subset X$ is open and $\alpha < \mu(\mathcal{U})$, choose $f \in C_c(X)$ s.t.

$f \prec \mathcal{U}$ and $\varphi(f) > \alpha$. Let $K := \overline{\operatorname{supp}(f)}$ cpt. Then $\forall g \in C_c(X)$

s.t. $g \geq \chi_K$, $g - f \geq 0$. So $\varphi(g) \geq \varphi(f) > \alpha$. Since (b) holds,

$\mu_\varphi(K) > \alpha$ so μ_φ is <u>inner regular</u> on $\mathcal{U}$.]

$\Rightarrow$ μ_φ is Radon.

<u>Step 4</u>: $\varphi(f) = \int f \, d\mu_\varphi$ $\quad\forall f \in C_c(X)$.

<u>Step 1</u>: It suffices to prove: if $(\mathcal{U}_n)$ is a seq of open sets,

$\mu^*(\bigcup \mathcal{U}_n) \le \sum \mu^*(\mathcal{U}_n)$. This will show that

$\mu^*(E) = \inf \left\{ \sum \mu(\mathcal{U}_n) \,\middle|\, \mathcal{U}_n \text{ open } \& \ E \subset \bigcup \mathcal{U}_n \right\}$, so μ^* is o.m.

If $f \in C_c(X)$ w/ $f \prec \bigcup \mathcal{U}_n$, let $K = \overline{\operatorname{supp}(f)}$ cpt. Then $K \subset \tilde{\bigcup} \mathcal{U}_n$.

partition of unity [Exercise: $\exists g_1, \dots, g_n \in C_c(X)$ s.t. $g_i \prec \mathcal{U}_i$ & $\sum^N g_i = 1$ on K

Then $f = f \sum_{i=1}^N g_i$. Also, $fg_i \prec \mathcal{U}_i$, and

$$\varphi(f) = \sum_{i=1}^N \varphi(fg_i) \le \sum_1^N \varphi(\chi_{\mathcal{U}_i}) = \sum_1^N \mu(\mathcal{U}_i) \le \sum^\infty \mu(\mathcal{U}_i).$$

Since f was arbitrary, $\mu(\mathcal{U}) = \sup\{\varphi(f) \mid f \prec \mathcal{U}\}$.

Step 2: Let $\mathcal{U} \subseteq X$ be open, and $E \subset X$ s.t. $\mu^*(E) < \infty$.

Show $\mu^*(E) \ge \mu^*(E \cap \mathcal{U}) + \mu^*(E \setminus \mathcal{U})$.

If E open, $E \cap \mathcal{U}$ open. So given $\varepsilon > 0$, $\exists f \prec E \cap \mathcal{U}$ s.t. $\varphi(f) > \mu(E \cap \mathcal{U}) - \frac{\varepsilon}{2}$. Since $E \setminus \overline{\operatorname{supp}(f)}$ is open, $\exists g \prec E \setminus \overline{\operatorname{supp}(f)}$ s.t. $\varphi(g) > \mu(E \setminus \overline{\operatorname{supp}(f)}) - \frac{\varepsilon}{2}$. Then $f + g \prec E$, so

$$\mu(E) \ge \varphi(f+g) = \varphi(f) + \varphi(g) > \mu(E \cap \mathcal{U}) + \mu(E \setminus \overline{\operatorname{supp}(f)}) - \varepsilon$$
$$\ge \mu(E \cap \mathcal{U}) + \mu(E \setminus \mathcal{U}) - \varepsilon.$$

Taking $\varepsilon \to 0$ give the inequality.

For general E, $\exists$ open $V \supset E$ s.t. $\mu(V) < \mu^*(E) + \varepsilon$.

So $\mu^*(E) + \varepsilon > \mu(V) \ge \mu^*(V \cap \mathcal{U}) + \mu^*(V \setminus \mathcal{U}) \ge \mu^*(E \cap \mathcal{U}) + \mu^*(E \setminus \mathcal{U})$

taking $\varepsilon \to 0$ gives ineq again.

$\in C_c(X)$

Step 3: For $K \subset X$ cpt & $\overset{\in C_c(X)}{f} \geq \chi_K$, set $\mathcal{U}_\varepsilon := \{f > 1-\varepsilon\}$ open.

If $g \prec \mathcal{U}_\varepsilon$, $(1-\varepsilon)^{-1} f - g \geq 0$. So $\varphi(g) \leq \frac{1}{1-\varepsilon}\varphi(f)$.

Hence $\mu_\varphi(K) \leq \mu_\varphi(\mathcal{U}_\varepsilon) \underset{\text{taking } \sup\{\varphi(g) \mid g \prec \mathcal{U}_\varepsilon\}}{\leq} (1-\varepsilon)^{-1}\varphi(f)$.

Letting $\varepsilon \longrightarrow 0$, $\mu_\varphi(K) \leq \varphi(f)$.

But $\forall$ open $\mathcal{U} \supset K$, $\exists f \prec \mathcal{U}$ s.t. $f \geq \chi_K$ by LCH Urysohn, and $\varphi(f) \leq \mu_\varphi(\mathcal{U})$. Since μ_φ is outer reg on K,

$\mu_\varphi(K) = \inf\{\mu_\varphi(\mathcal{U}) \mid \mathcal{U} \supset K \text{ open}\} = \inf\{\varphi(f) \mid 0 \leq f \leq \chi_K\}$. ✓

Step 4: We may assume $0 \leq f \leq 1$ since these fns span $C_c(X)$.

Fix $N \in \mathbb{N}$ and set $K_j = \{f \geq \frac{j}{N}\}$ and $K_0 = \overline{\operatorname{supp}(f)}$.

$(\emptyset = K_{N+1} \subset K_N \subset \cdots \subset K_1 \subset K_0)$.

Define f_j for $1 \leq j \leq N$ by $f_j := [(f - \frac{j-1}{N}) \vee 0] \wedge \frac{1}{N}$.

Then
$$f_j(x) = \begin{cases} 0 & x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & x \in K_{j-1} \setminus K_j \\ N^{-1} & x \in K_j \end{cases}.$$

Observe $\frac{\chi_{K_j}}{N} \leq f_j \leq \frac{\chi_{K_{j-1}}}{N}$ and $\sum_1^N f_j = f$.

This means
$$\boxed{\frac{1}{N}\mu_\varphi(K_j) \leq \int f_j \, d\mu_\varphi \leq \frac{1}{N}\mu_\varphi(K_{j-1}).} \quad (1)$$

$\forall$ open $\mathcal{U} \supset K_{j-1}$, $N f_j \prec \mathcal{U}$, so $\varphi(f_j) \le \frac{1}{N}\mu_\varphi(\mathcal{U})$.

By (b) & outer reg of μ_φ, $\boxed{\frac{1}{N}\mu_\varphi(K_j) \le \varphi(f_j) \le \frac{1}{N}\mu_\varphi(K_{j-1}).}$ ②

$$\Rightarrow \quad \frac{1}{N}\sum_{1}^{N}\mu_\varphi(K_j) \le \begin{matrix}\int f\,d\mu_\varphi \\ \varphi(f)\end{matrix} \le \frac{1}{N}\sum_{0}^{N-1}\mu_\varphi(K_j)$$

$$\Rightarrow \quad \left|\varphi(f) - \int f\,d\mu_\varphi\right| \le \frac{\mu_\varphi(K_0) - \mu_\varphi(K_N)}{N} \le \frac{\mu_\varphi(K_0)}{N}.$$

K_0 doesn't depend on N, $N \in \mathbb{N}$ was arbitrary. ✓