Lec 3/28

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Reminders

M a C^2 surface in $\mathbb{R}^3$

$x : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^2 \longrightarrow V \overset{\text{open}}{\subseteq} M$ a C^2 patch

$x_i = \dfrac{\partial x}{\partial u^i}$ $\quad g_{ij} = \langle x_i \mid x_j \rangle$ $\quad g = \det(g_{ij})$

$$n = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{x_1 \times x_2}{\sqrt{g}}$$

$p \in V$, $u_0 \in \mathcal{U}$, $x(u_0) = p$.

$L_p(X) = -n'(p)(X) \quad \forall X \in T_pM$

This is "$\underbrace{\text{the other}}$ n" composed w/ x^{-1}.

$$L_p(x_i(u_0)) = -\frac{\partial n}{\partial u^i}(u_0)$$

$n : V \overset{\text{open}}{\subseteq} M \longrightarrow \mathbb{S}^2 \qquad n'(p) : T_pM \longrightarrow T_{n(p)}\mathbb{S}^2 = T_pM$

$\mathbb{II}_p(X, Y) = \langle L_p(X) \mid Y \rangle \quad \forall X, Y \in T_pM.$

If $X = \sum_i X^i x_i(u_0)$, $Y = \sum Y^j x_j(u_0)$

$$\mathbb{II}_p(X, Y) = \sum_{i,j} L_{ij}(u_0) X^i Y^j \text{ where } L_{ij}(u_0) = \mathbb{II}_p(x_i(u_0), x_j(u_0))$$
$$= \langle x_{ij}(u_0) \mid n(u_0) \rangle$$

Since $x_{ij} = x_{ji}$, $L_{ij}(u_0) = L_{ji}(u_0)$, so $\langle L_p(X) \mid Y \rangle = \langle X \mid L_p(Y) \rangle$.

4-4 Normal Curvature, Geodesic Curvature, and Gauss's Formulae.

M a C^2 surface in $\mathbb{R}^3$. $x : u \overset{open}{\subseteq} \mathbb{R} \xrightarrow{onto} V \overset{open}{\subseteq} M$.

$\gamma : (a,b) \longrightarrow V$ a C^2 unit-speed curve in $V \subseteq M$.

T is the unit tangent vector field for γ. $T : (a,b) \longrightarrow S^2$.

$S = n \times T$, $S(s) = n(\gamma(s)) \times T(s)$.

S is called the intrinsic normal of γ relative to M.

For each $s \in (a,b)$, $S(s)$ is tangent to M at $\gamma(s)$, and orthogonal to γ at s.

$S(s) \in T_{\gamma(s)} M$

Now $T'(s) = \gamma''(s) = W(s) + X(s)$

where $W(s)$ is normal to M at $\gamma(s)$ and $X(s) \in T_{\gamma(s)} M$.

$W(s) = k_n(s)\, n(\gamma(s))$.

k_n is called the normal curvature of γ.

$\langle T | T \rangle = 1$ so $\langle T | T \rangle' = 0$ so $\langle T' | T \rangle = \frac{1}{2}(\langle T' | T \rangle + \langle T | T' \rangle)$

$= \frac{1}{2} \langle T | T \rangle' = 0$

so $\langle W + X | T \rangle = 0$ but $\langle W | T \rangle = 0$ so $\langle X | T \rangle = 0$.

Thus X is perpendicular to both n and T, so $X(s) = K_g(s)\, S(s)$.

K_g is called the geodesic curvature of γ (relative to M).

We started with $T' = \gamma'' = W + X$.

Thus $\kappa N = T' = \gamma'' = \kappa_n n + \kappa_g S$

So $\kappa = \sqrt{\kappa_n^2 + \kappa_g^2}$

to say that γ is a <u>geodesic</u> on M

means that $\kappa_g \equiv 0$.

Digression (light) Done

$$x_{ij} = \frac{\partial^2 x}{\partial u^i \partial u^j}, \quad L_{ij} = \langle x_{ij} \mid n \rangle.$$

Propn 4.2

(a) (Gauss's Formulae)

$$x_{ij} = L_{ij}\, n + \sum_k \Gamma_{ij}^k x_k$$

where the functions $\Gamma_{ij}^k = \sum_l \langle x_{ij} \mid x_l \rangle g^{lk}$ are called the Christoffel Symbols (of the second kind, also denoted $\left\{ {k \atop ij} \right\}$.

(b) for any C^2 unit speed curve $s \longmapsto \gamma(s) = x(\gamma^1(s), \gamma^2(s))$

we have $\kappa_n = \sum_{i,j} L_{ij} \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}$

and $$\kappa_g S = \sum_k \left[\frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} \right] x_k$$

It follows that all C^2 unit-speed curves in V passing through a point p in a given tangential direction $T \in (T_pM) \cap S^2$ have the same normal curvature at that point, so this normal curvature does <u>not</u> depend on γ except through p and T, so it's a property of the surface M, point p, and direction T.

It also follows that a C^2 unit-speed curve in M is a geodesic iff its curvature κ is as small as possible, consistent with the curve remaining on M.

<u>Pf</u> (a) $x_{ij} = a_{ij}\, n + \sum_m b_{ij}^m x_m$. $\langle n, n\rangle = 1$, $\langle x_m | n\rangle = 0$,

so $a_{ij} = \langle x_{ij} | n\rangle = L_{ij}$.

Also
$$\langle x_{ij} | x_\ell\rangle = \langle a_{ij} + \sum_m b_{ij}^m x_m | x_\ell\rangle$$
$$= 0 + \sum_m b_{ij}^m \langle x_m | x_\ell\rangle$$
$$= \sum_m b_{ij}^m g_{m\ell}$$

so $\langle x_{ij} | x_\ell\rangle g^{\ell k} = \sum_m b_{ij}^m g_{m\ell}\, g^{\ell k}$

so $\sum_\ell \langle x_{ij} | x_\ell\rangle g^{\ell k} = \sum_m b_{ij}^m \delta_m^k = b_{ij}^k$,

so $b_{ij}^k = \Gamma_{ij}^k$ as required. ◪