Lec 10/5

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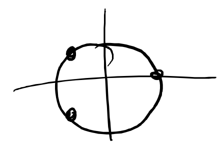
In P.D. Theorem, $V_i \neq 0$. if it was, $n(p_i(T)^{e_i}) = 0$, so m would not be minimal.

<u>Cor.</u> $T \in L(V,V)$ is diagonalizable iff $m(x) = (x-\xi_1)\cdots(x-\xi_r)$ for distinct $\xi_1, \dots, \xi_r$.

$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{R})$ ← or $\mathbb{C}$. is diagonalizable over $\mathbb{C}$ not $\mathbb{R}$.



$m(x) = x^3 - 1 = (x-1)\underbrace{(x^2+x+1)}_{\text{Prime in } \mathbb{R}[X]}$, $\frac{-1 \pm \sqrt{-3}}{2}$

$= \left(x - \underbrace{\left(i\frac{\sqrt{3}}{2} - \frac{1}{2}\right)}_{\alpha}\right)\left(x - \underbrace{\left(-i\frac{\sqrt{3}}{2} - \frac{1}{2}\right)}_{\bar{\alpha}}\right)$

find v_1, v_2, v_3 s.t.

$Av_1 = v_1$

$Av_2 = \alpha v_2$

$Av_3 = \bar{\alpha} v_3$

. In this basis, $\beta_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} \end{pmatrix}$

m cannot have degree $> n$ if $T \in L(V,V)$ and $\dim V = n$.

Ex: $Z \in L(V,V)$, $Z^r = 0$ and $Z^j \neq 0$ when $j < r$.

x^r annihilates Z, and x^r only has divisors $x, x^2, \dots$.

$0 = Z^r(V) \subset Z^{r-1}(V) \subset Z^{r-2}(V) \subset \cdots \subset Z(V)$

strict ← this one be strict o.w. $Z^{r-1}(V) = Z^{r-2}(V)$

$Z^r(V) = Z^{r-1}(V)$

contradiction

$\{v_1, \dots, v_{e_1}\}$ basis of

$\{v_1, \dots, v_{e_1}\}$ basis of

add vectors to get a basis of

$\{v_1, \dots, v_{e_1}, v_{e_1+1}, \dots, v_{e_2}\}$. until have a full basis of V.

... contradiction

Basis satisfies $Z(v_i) \in S(v_1, \dots, v_{i-1})$ $\forall i \in \{1, \dots, e_r\}$.

$$Z \mapsto A = \begin{pmatrix} 0 & * & * & \\ 0 & 0 & * & * \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 \cdots & 0 \end{pmatrix}$$ strictly upper triangular matrix.

$$(T - \lambda I)^r = 0 \Rightarrow T = \lambda I + \underset{\uparrow \text{nilpotent}}{Z}$$

$$T \mapsto \begin{pmatrix} \lambda & & * \\ & \lambda & \\ & & \ddots \\ 0 & & & \lambda \end{pmatrix}$$ upper triangular matrix.

$r \leq e_r = \dim V$.

Triangular Form Theorem

Let $T \in L(V, V)$ w/ $m(x) = (x - \lambda_1)^{e_1} (x - \lambda_2)^{e_2} \cdots (x - \lambda_r)^{e_r}$

Then $V = V_1 \oplus \cdots \oplus V_r$, $V_i = n((T - \lambda_i I)^{e_i})$

$\exists$ basis of V s.t. matrix A of T has the form

nilpotency degree $e_i \leq d_i$

$\dim V = d = d_1 + \cdots + d_r \geq \deg(m)$.

$$\begin{pmatrix} \boxed{A_1} & & 0 \\ & \boxed{A_2} & \\ & & \ddots \\ 0 & & & \boxed{A_r} \end{pmatrix}$$ where $\underbrace{A_i}_{d_i \times d_i}$ is upper triangular.

(in each block find λ_i s.t. $T - \lambda I$ is nilpotent)

__Corollary__: let F be a subfield of $\mathbb{C}$, V/F, $\dim V = n$, $T \in L(V, V)$.

... $\leq n$ where m is min. poly. of T.

then $\deg(m) \leq n$ where m is min. poly. of T.