(18) V : $\mathbb{Q}$-vector space, $\dim_{\mathbb{Q}}(V) = n$.

$T \in \mathrm{End}(V)$ s.t. $T^{-1} = T^2 + T$.

Prove that $3 \mid n$.

$T^3 + T^2 - 1 = 0 \Rightarrow p(T) = 0$ for $p(x) = x^3 + x^2 - 1$.

p is irreducible, so $m_T = p$.

Invariant factors divide p, so all are equal to p.

$n = \sum$ degrees of invariant factors $\Rightarrow n = \sum 3$ so $3 \mid n$.

- A is similar to A^T

Smith Normal forms of A and A^T are the same.

$$xI - A \longrightarrow \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_m \end{pmatrix}$$

same ops, just on rows instead of columns & v.v.

$$xI - A^T \longrightarrow \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_m \end{pmatrix}^T = \underbrace{\begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_m \end{pmatrix}}_{\text{Smith of } A^T.}$$

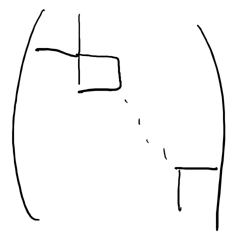
Jordan Normal Form of a transformation.

$\varphi \in End(V)$, $\dim(V) = n$. Assume $m_\varphi(x) = (x-\lambda_1)\cdots(x-\lambda_k)$

(this is so iff $C_\varphi(x) = (x-\lambda_1)^{r_1}\cdots(x-\lambda_k)^{r_k}$ s.t. $\sum r_i = n$).

Elementary divisors of φ are $(x-\lambda_1)^{d_1}, \dots, (x-\lambda_\ell)^{d_\ell}$, λ_i are not necessarily distinct.

Corresponding normal form of the matrix φ is



each block is companion of $(x-\lambda_i)^{d_i}$.

Let V_i = Subspace corresponding to i^{th} block.

Let $\lambda = \lambda_i$, $d = d_i$, s.t. $\varphi|_{V_i}$ has min. poly $(x-\lambda)^d$.

Let $\psi = \varphi|_{V_i} - \lambda I$. Then $m_\psi(x) = x^d$.

ψ is called a nilpotent matrix.

$x \mapsto x - \lambda = y$

$F[x] \longmapsto F[y]$.

If u is a cyclic vector in V_i, then our basis was $\{u, \psi(u), \psi^2(u), \dots, \psi^{d-1}(u)\}$.

$$\underset{u_1}{u},\ \underset{u_2}{\psi(u)},\ \underset{u_3}{\psi^2(u)},\ \dots,\ \underset{u_d}{\psi^{d-1}(u)}$$

$$\psi : u_1 \longmapsto u_2 \longmapsto u_3 \longrightarrow \cdots \longmapsto u_d \longmapsto \psi^d(u) = 0.$$

So matrix of ψ in this basis is $\begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots\ 1 & 0 \end{pmatrix}$ (companion of x^k).

Put $v_1 = u_d$, $v_2 = u_{d-1}$, $\dots$, $v_d = u_1$.

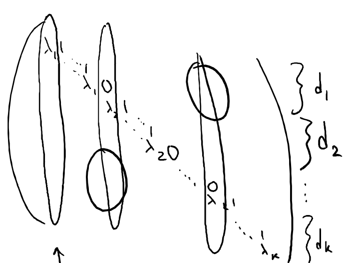
Then $\psi : v_1 \mapsto 0$, $v_2 \mapsto v_1$, $\dots$, $v_d \mapsto v_{d-1}$

Matrix in this basis $\{v_1, \dots, v_d\}$ is $\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$.

Matrix of φ = this $+ \lambda I = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ - Jordan Cell.

Jordan Normal form: if elem. divisors of φ are

$(x-\lambda_1)^{d_1}, \dots, (x-\lambda_k)^{d_k}$ then the Jordan Normal form is



eigenvectors

λ_i are the roots of C_φ. These are eigenvalues of C_φ.

for each i, $\exists\, u \in V$ s.t. $\varphi(u) = \lambda_i u$, u is an eigenvector corr. to λ_i.

In this form it is circled.

$\det(\lambda_i - \varphi) = C_\varphi(\lambda_i) = 0$. So $\varphi - \lambda_i$ is degenerate & $\ker(\varphi - \lambda_i) \neq 0$.

$\forall\, u \in \ker(\varphi - \lambda_i)$, $\varphi(u) = \lambda_i u$.

Smith Normal form: $xI - A \rightsquigarrow \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{pmatrix}$

Smith Normal form: $xI - A \rightsquigarrow \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_m \end{pmatrix}$

$\underline{123}$ ① $V = V_1 \oplus \cdots \oplus V_4$ s.t. V_i are cyclic $F[X]$-modules,

$$\mathrm{Ann}(V_1) = (X+1)^2$$
$$\mathrm{Ann}(V_2) = (X-1)(X^2+1)^2$$
$$\mathrm{Ann}(V_3) = X^4 - 1$$
$$\mathrm{Ann}(V_4) = (X+1)(X^2+1)$$

Inv factors and elementary divisors of V.

Elem divisors are $(X+1)^2$, $X-1$, $(X^2+1)^2$, $X-1$, $X+1$, X^2+1, $(X+1)^2$, $X-1$,

$\underbrace{(X^2+1)^2, \ X^2+1}$, $\underbrace{X-1, \ X-1, \ X-1}$, $\underbrace{(X+1)^2, \ (X+1)^2, \ X+1}$