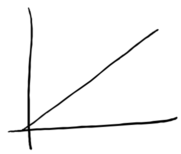
Lec 9/29

Friday, September 29, 2017 14:22

Cut the Knot

$\frac{1}{N}\sum 1_{(a,b)}(x_n) \longrightarrow \int_a^b f$ for some fixed f : 'f-distributed'

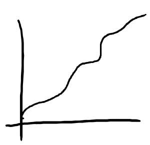


$\mu((a,b)) = |b-a|$ traditionally

$\int_0^1 f \cong \sum f(\xi_i)\,(x_{i+1}-x_i)$ traditionally.

instead of this take this

monotone



which sets in $\mathbb{R}$ can be points of Discontinuities of a monotone function

only countable sets. D_f

1. D_f is at most countable (exercise)
2. Any countable set can be D_f (including $\mathbb{Q}$)
3. any $f:\mathbb{R}\to\mathbb{R}$ monotone is differentiable A.E.
4. Which sets of measure zero can be points of nondiffability of monotone fns.

g monotone & nice enough

Can define: (distorted measure)

$\cancel{\int f\,dx} \to \int f\,g\,dx$

x_n is g - u.d. mod 1:

$$\frac{1}{N} \sum f(x_n) \longrightarrow \int f g \, dx$$

$$\underbrace{\int f \, dG(x)}_{\text{Stieltjes integral.}} \qquad (g = G')$$

Theorem: (Ishaan - Caleb) "Typical" $(x_n) \subset [0,1]$ is u.d.

$(x_n \ \{0,1\}^{\mathbb{N}}$ is normal iff $2^n x$ mod 1 is u.d. mod 1)

$x = \sum \frac{x_n}{2^n}$

$$X_n = n\alpha \mod 1$$

is u.d. for $\alpha \notin \mathbb{Q}$ (co-countable)

What can we say about α for which $2^n \alpha$ is u.d.? $\underbrace{\alpha \text{ normal}}_{\text{not co-countable}}$

Non-normal in base-2 are uncountable.

$(X_n) \subset [0,1]$ is "well-distributed" if

$$\lim_{N-M \to \infty} \frac{\#\{M \le n \le N-1, \ X_n \in (a,b)\}}{N-M} = b-a$$

$$\ldots\ldots \ \underset{2^n}{[}11\cdots\underset{2^n+n}{1]} \ \ldots$$

Not well-distributed

Claim: $n\alpha \mod 1$ is well distributed.

$$\left\{ \begin{array}{l} \forall (a,b) \subseteq [0,1], \quad \forall \varepsilon > 0, \exists C = C((x_n), \varepsilon, (a,b)) \text{ s.t. if} \\ I = \{M, \dots, N-1\} \text{ satisfies } N-M \geq C \text{ then} \\ \left| \frac{\# \{ n \in I : x_n \in (a,b) \}}{|I|} - (b-a) \right| < \varepsilon \end{array} \right.$$

Can shift $n\alpha \mod 1$ to get $(n-M)\alpha \mod 1$ u.d. $\to$ $n\alpha$ w.d.

$$\lim_{N-M \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} f(x_n) = \int f \quad \text{gives criterion}$$

$$\lim_{N-M \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} e^{2\pi i h x_n} = 0$$

$|\lambda| = 1$

$$\frac{1}{N-M} \sum_{n=M}^{N-1} \overbrace{e^{2\pi i h n \alpha}}^{\lambda^n} = \left| \frac{\lambda^{P_1} (\lambda^{P_2} - 1)}{\lambda - 1} \right|$$