

# Fundamental Galois Thm

$K/F$  : Galois

$$G = \text{Gal}(K/F).$$

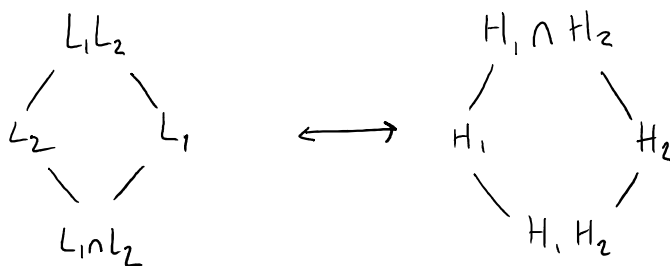
$$K \supseteq L \longmapsto \text{Gal}(K/L) = H \leq G$$

$$\text{Fix } H \longleftarrow H \leq G$$

$$\textcircled{1} |H| = [K:L], \quad |G:H| = [L:F]$$

$$\textcircled{2} L_1 \leq L_2 \text{ iff } H_1 \geq H_2$$

$\textcircled{3}$  The Group corresponding to  $L_1 \cap L_2$  is  $H_1 H_2$   
 the group corresponding to  $L_1 L_2$  is  $H_1 \cap H_2$



④ Any embedding  $\varphi: L/F \rightarrow K/F$  is defined by an element of  $G$ .

Embeddings are in 1-1 correspondence w/  $\uparrow$  cosets of  $H$  in  $G$ .  
left.

$G \longrightarrow \text{Embeddings.}$

elements of  $\text{Gal}(K/L) = H$  (and only those) define trivial embeddings.

two elements of  $G$  define the same embedding

if they are  $\varphi\psi_1$  and  $\varphi\psi_2$  where  $\psi_1, \psi_2 \in H$ .

⑤  $L' = \varphi(L)$  - conjugate of  $L$ .

Then  $H' = \varphi H \varphi^{-1}$  - conjugate of  $H$ .

Conjugates of  $L$  are in 1-1 correspondence w/  $G/N(H)$ .

⑥  $L/F$  is normal iff  $H \leq G$  is normal.

iff any embedding  
of  $L$  is an automorphism;  
it has no conjugates  
except itself.

iff it has  
no conjugates  
except itself

In this case,  $\text{Gal}(L/F) = G/H$

$$G \longrightarrow \text{Gal}(L/F) \quad \text{Kernel is } \text{Gal}(K/L) = H.$$

$$\varphi \longmapsto \varphi|_L$$

$$\varphi(L) = L$$

Example

①  $\mathbb{Q}(\sqrt{2}, \sqrt{3}) / \mathbb{Q}$

$\parallel$   
K

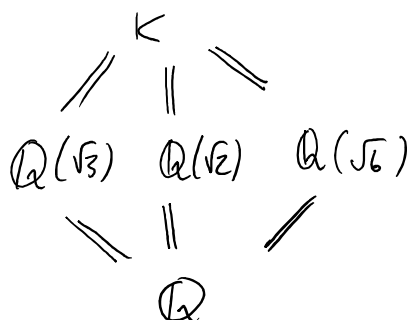
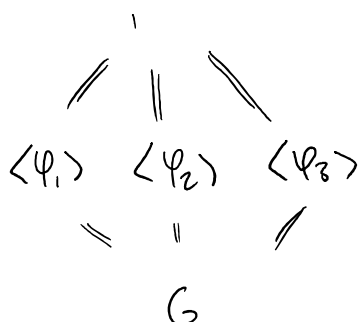
$$G = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$= \{1, \varphi_1, \varphi_2, \varphi_3\}$$

$$\varphi_1 : \begin{array}{l} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{array}$$

$$\varphi_2 : \begin{array}{l} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{array}$$

$$\varphi_3 = \varphi_1 \varphi_2$$



②  $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ ,  $\omega = e^{2\pi i/3}$ ,  $G \cong S_3$

$$G = \{1, \sigma, \sigma^2, \tau_1, \tau_2, \tau_3\}$$

permutes  $\{\alpha_1, \alpha_2, \alpha_3\}$

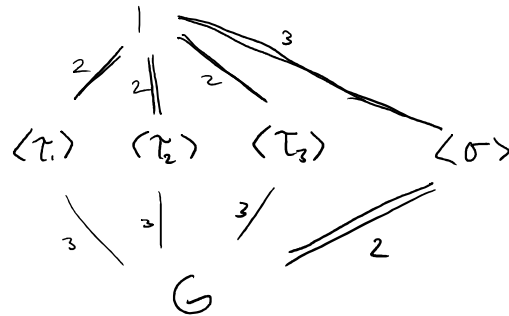
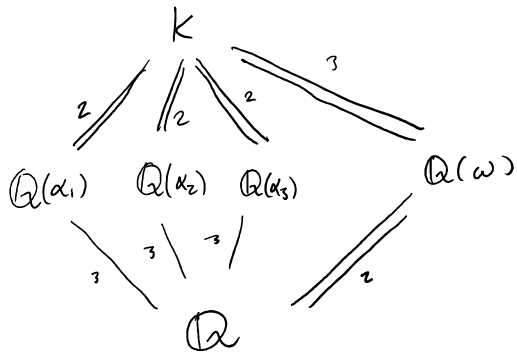
Where  $\alpha_1 = \sqrt[3]{2}, \alpha_2 = \omega \sqrt[3]{2}, \alpha_3 = \omega^2 \sqrt[3]{2}$ .

$$\tau_1: \alpha_2 \leftrightarrow \alpha_3$$

$$\tau_2: \alpha_1 \leftrightarrow \alpha_3$$

$$\tau_3: \alpha_1 \leftrightarrow \alpha_2$$

$$\sigma: \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_1$$



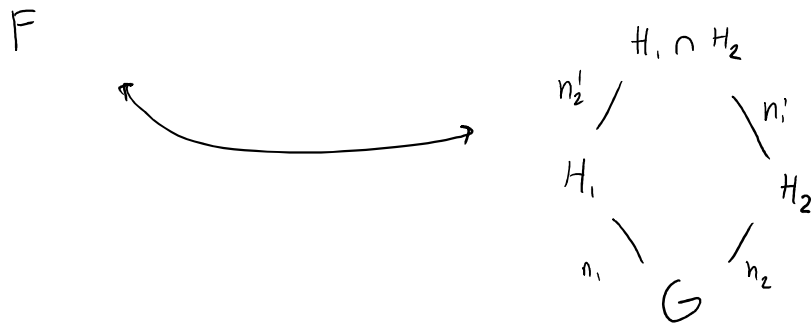
$\mathbb{Q}(\alpha_i)$  are conjugate.

$$\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = S_3/\mathbb{Z}_3 \cong \mathbb{Z}_2.$$

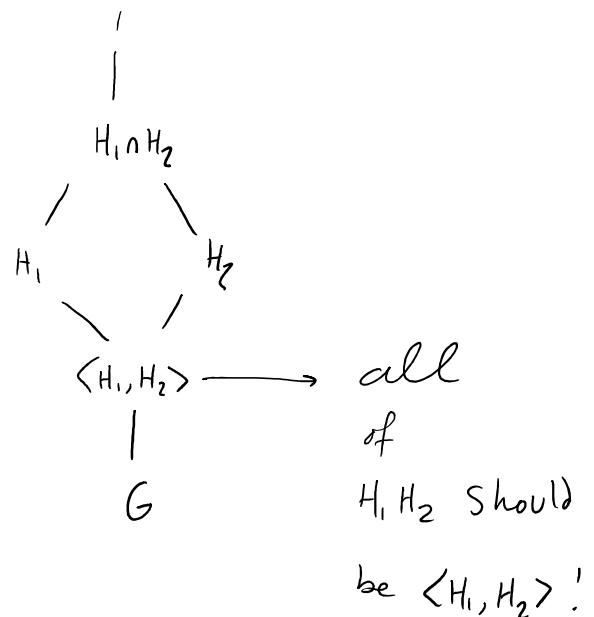
$$\begin{array}{c} L_1, L_2 \\ n'_2 / \quad \backslash n'_1 \\ L_1 \quad L_2 \\ n_1 \backslash \quad / n_2 \\ F \end{array}$$

$\mathbb{R}$

$$\begin{array}{c} H_1 \cap H_2 \\ n'_2 / \quad \backslash \dots \end{array}$$



$$[L_1 L_2 : F] = [L_1 : F][L_2 : F] \quad \text{iff} \quad |G : H_1 \cap H_2| = |G : H_1| \cdot |G : H_2|.$$

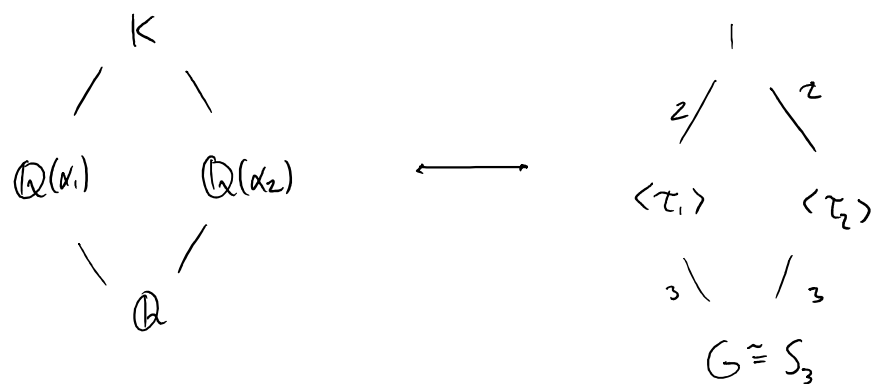


We have  $H_1 H_2 = \langle H_1, H_2 \rangle$   
if at least one of  
 $H_1, H_2$  is normal.

Eg  $K = \mathbb{Q}(\alpha_1, \alpha_2)$  where  $\alpha_1 = \sqrt[3]{2}$ ,  $\alpha_2 = \omega \sqrt[3]{2}$ .

K

1



Theorem if  $L_1/F$ ,  $L_2/F$  are subextensions of a separable extension and  $L_1/F$  is normal,

~~$$\text{then } [L_1 L_2 : F] = [L_1 : F] \cdot [L_2 : F]$$

$$L_1 \cap L_2 = F$$~~