Lec 10/28

Monday, October 28, 2019 10:24

X – vector space over $\mathbb{R}$ or $\mathbb{C}$.

Def seminorm is

① (homogeneous) $\|\lambda x\| = |\lambda|\,\|x\|$

② (subadditive) $\|x+y\| \le \|x\| + \|y\|$

Norm also has

③ (definite) $\|x\| = 0 \to x = 0$

Recall: $\rho(x,y) = \|x-y\|$ is a metric if $\|\cdot\|$ is a norm.

This induces the norm topology.

$\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent if $\exists\, C>0$ s.t.

$$\frac{1}{C}\|x\|_1 \le \|x\|_2 \le C\|x\|_1$$

Exercises:

① Show all norms on $\mathbb{R}^n$ are equivalent

② Call two norms top. equivalent if they induce the same topology. Find two top. equiv norms that aren't equiv.

Defn A Banach space is a complete normed v.s.

Examples: $C_0(X)$ if X LCH

$L^1(X, M, \mu)$ if X LCH

<u>Defn</u> Suppose $(x_n) \subset X$ is a sequence.

Say $\sum x_n$ converges to x if $\sum^N x_n \to x$ as $N \to \infty$

Say $\sum x_n$ converges absolutely if $\sum \|x_n\| < \infty$.

<u>Prop</u>: For a normed space X, TFAE:

① X Banach

② Every absolutely convergent series converges.

<u>Proof</u>: Suppose X is Banach & $\sum \|x_n\| < \infty$.

Let $\varepsilon > 0$ and pick $N > 0$ s.t. $\sum_{n \geq N} \|x_n\| < \varepsilon$.

Then $\forall\, m, n > N$,

$$\left\| \sum_1^m x_i - \sum_1^n x_i \right\| = \left\| \sum_{n+1}^m x_i \right\| \leq \sum_{n+1}^m \|x_i\| < \infty$$

so $\left(\sum_{i=1}^m x_i \right)_m$ is cauchy & hence converges.

Conversely, suppose (x_n) is Cauchy & abs. conv. series converge.

Choose $n_1 < n_2 < n_3 < \cdots$ s.t. $\|x_m - x_n\| < 2^{-k}$ when $m, n > n_k$

define $y_0 = 0 = "x_{n_0}"$ and $\forall k \in \mathbb{N}$, $y_k = x_{n_k} - x_{n_{k-1}}$.

Now $\sum^k y_i = x_{n_k}$ and $\sum \|y_k\| \leq \|x_{n_1}\| + \sum 2^{-k} < \infty$.

Now $\sum_{j=1}^{k} y_j = x_{n_k}$ and $\sum \|y_k\| \le \|x_{n_1}\| + \sum 2^{-k} < \infty$.

So Hence $x = \lim_{k \to \infty} x_{n_k}$ exists in X, since (x_n) is Cauchy, $x_n \to x$. □

Prop: Suppose X and Y are normed spaces & $T : X \to Y$ is linear. Then TFAE:

① T is continuous

② T is continuous at 0

③ T is bounded: $\exists M > 0$ s.t. $\forall x \in X$, $\|Tx\| \le M \|x\|$.

pf: ①⇒② trivial

②⇒③ Suppose T is cts at 0.

Then $\exists \delta > 0$ s.t. $\|x\| \le \delta \Rightarrow \|Tx\| \le 1$.

Then $\frac{\delta}{\|x\|}\|x\| \le \delta$ so $\frac{\delta}{\|x\|}\|Tx\| \le 1 \Rightarrow \|Tx\| \le \frac{1}{\delta}\|x\|$.

③⇒① Let $\varepsilon > 0$. if $\|x_1 - x_2\| \le \frac{\varepsilon}{M}$ then $\|Tx_1 - Tx_2\| \le M\|x_1 - x_2\| \le \varepsilon$.

Def: let $\mathcal{L}(X,Y) := \{\text{Bdd linear maps } X \to Y\}$

Define $\|T\| := \sup\{\|Tx\| \mid \|x\| = 1\} = \sup\{\|Tx\| \mid \|x\| = 1\}$

Operator Norm

$= \sup\left\{\frac{\|Tx\|}{\|x\|} \;\middle|\; x \ne 0\right\} = \inf\{c > 0 \mid \|Tx\| \le c\|x\| \;\forall x \in X\}$.

exercise: These are equivalent

Observe: if $S \in \mathcal{L}(Y, Z)$ & $T \in \mathcal{L}(X, Y)$ then

$ST \in \mathcal{L}(X, Z)$ w/ operator norm $\|ST\| \leq \|S\| \cdot \|T\|$.

(since $\|STx\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\|$)

Prop If Y is complete, so is $\mathcal{L}(X, Y)$.

pf: If $(T_n) \subset \mathcal{L}(X, Y)$ is Cauchy, so is $(T_n x)$ $\forall x \in X$.

Set $Tx := \lim_{n \to \infty} T_n x$. Then verify that

1. T is linear
2. T is bdd
3. $T_n \to T$ in $\mathcal{L}(X, Y)$.