Lec 10/9

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Lebesgue's Differentiation Thm

If $f \in L^1_{loc}(\lambda^n)$, then

L.D. $$\left\{ \lim_{\substack{\ell(Q) \to 0 \\ Q \in C(x)}} \frac{1}{\lambda^n(Q)} \int_Q f \, d\lambda^n = f(x) \quad \text{a.e.} \right.$$

HLMT: $\exists c > 0$ depending only on n s.t. $\forall f \in L^1(\lambda^n)$ and $a > 0$,

$$\lambda^n(\{Mf > a\}) \leq \frac{c \, \|f\|_1}{a}$$

Tchebychev's Inea: $\forall a > 0$, $\mu(\{|f| > a\}) \leq \frac{\|f\|_1}{a}$.

Pf of L.D.

Step1 The result for $f \in L^1 \Rightarrow$ result for $f \in L^1_{loc}$.

(replace $f \in L^1_{loc}$ by $f \chi_Q \in L^1$).

Step2 The result for $f \in C_c(\mathbb{R}^n) \Longrightarrow$ result for $f \in L^1$.

~~pf~~ For $Q \in C(0)$ & $f \in L^1$, define

$$I_Q f(x) := \frac{1}{\lambda^n(Q)} \int_{Q+x} f \, d\lambda^n.$$

Observe I_Q is <u>linear</u> and $|I_Q f| \leq Mf$ everywhere

Now fix $f \in L^1$ and $\varepsilon > 0$. let

$$E = \{ x \in \mathbb{R}^n \mid \limsup_{\substack{\ell(Q) \to 0 \\ Q \in \mathcal{C}(0)}} | I_Q f(x) - f(x) | > \varepsilon \}$$

We'll show $(\lambda^n)^*(E) = 0 \rightsquigarrow E \in \mathcal{L}^n, \ \lambda^n(E) = 0$.

Fix $\delta > 0$. Since $C_c(\mathbb{R}^n) \subset L^1$ is dense [exercise]

$\exists \, g \in C_c(\mathbb{R}^n)$ s.t. $\| f - g \|_1 < \delta$. Then:

$$|I_Q f - f| = |I_Q(f-g) + [I_Q g - g] + (g - f)|$$

$$\leq \underbrace{|I_Q(f-g)|}_{\substack{\wedge\!\!| \\ M(f-g)}} + \underbrace{|I_Q g - g|}_{\substack{\downarrow \\ 0 \\ \text{as } \ell(Q) \to 0}} + |g - f|$$

Hence $E \subset \{ M(f-g) > \frac{\varepsilon}{2} \} \cup \{ |g - f| > \frac{\varepsilon}{2} \}$

By HLMT & Tchebyshev,

$$(\lambda^n)^*(E) \leq \lambda^n(\{ M(f-g) > \tfrac{\varepsilon}{2} \}) + \lambda^n(\{ |g-f| > \tfrac{\varepsilon}{2} \})$$

$$\leq \frac{c\|f-g\|_1}{\varepsilon/2} + \frac{\|f-g\|_1}{\varepsilon/2}$$

$$= \frac{2(C+1)}{\varepsilon}\|f-g\|_1 < \underbrace{\frac{2(c+1)}{\varepsilon}\delta}_{\text{can be made small w/ } \delta.}$$

So $(\lambda^n)^* = 0$, and we are done.

<u>Step 3</u>: The result holds for $C_c(\mathbb{R}^n)$

Pf f is uniformly continuous. So $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$\|x-y\|_\infty\ \delta \Longrightarrow |f(x)-f(y)| < \varepsilon$.

let $\varepsilon>0$. Choose $\delta>0$ s.t. $x,y\in Q$ w/ $\ell(Q)<\delta$

$\Rightarrow |f(x)-f(y)| < \varepsilon$.

fix y, fix $Q \in \mathcal{C}(y)$ w/ $\ell(Q) < \delta$

$$\left| \frac{1}{\lambda^n(Q)} \int_Q f(x)\, d\lambda^n(x) - f(y) \right|$$

$$= \left| \frac{1}{\lambda^n(Q)} \int_Q (f(x)-f(y))\, d\lambda^n(x) \right|$$

$$\leq \frac{1}{} \int |f(x)-f(y)|\, d\lambda^n(x)$$

$$= \left| \frac{1}{\lambda^n(Q)} \int_Q (f(x)-f(y))\, d\lambda^n(x) \right|$$

$$\leq \frac{1}{\lambda^n(Q)} \int_Q |f(x)-f(y)|\, d\lambda^n(x)$$

$$< \frac{1}{\lambda^n(Q)} \int_Q \varepsilon\, d\lambda^n = \varepsilon.$$

In fact, the result holds for $C(\mathbb{R})$. $\square$

Suppose $E \in \mathcal{L}^n$. $x \in E$ is called a lebesgue point of density of E if

$$\lim_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{\lambda^n(Q\cap E)}{\lambda^n(Q)} = 1.$$

Corollary: for $E \in \mathcal{L}^n$, almost all points of E are LP.D's

Pf Apply L.D.T. to χ_E.

<u>Defn</u> for $f \in L^1(\lambda^n)$, $x \in \mathbb{R}^n$ is called a Lebesgue point of f if

$$\lim_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)|\, d\lambda^n = 0$$

<u>Corollary</u>: If $f \in L^1_{loc}$, a.e. $x \in \mathbb{R}^n$ is a Lebesgue pt.

<u>Pf</u> As in pf of LDT, we may assume $f \in L^1$.

Let $D \subset \mathbb{C}$ be a countable dense subset ($D = \mathbb{Q} + i\mathbb{Q}$ will suffice)

for $d \in D$, write $E_d = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim\limits_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f-d| - |f(x)-d| \, d\lambda^n = 0 \right\}$

By LDT for $|f-d|$, E_d^c is λ^n-null.

Set $E = \bigcap_{d\in D} E_d$. Then E^c is still λ^n-null.

We claim every $x \in E$ is a Lebesgue pt.

If $x \in E$, $\forall d \in D$, $|f - f(x)| \le |f-d| + |f(x) - d|$

$$= \big[|f-d| - |f(x)-d| \big] + 2|f(x)-d|$$

$$\limsup_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)|\, d\lambda^n$$

$$\le 2|f(x)-d| + \limsup_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f - f(x)|\, d\lambda^n \longrightarrow 0$$

$$\le 2|f(x)-d| + \limsup_{\substack{\ell(Q)\to 0 \\ Q\in \mathcal{C}(x)}} \frac{1}{\lambda^n(Q)} \int_Q |f-d|\,dx$$

D is dense, so make $2|f(x)-d|$ small. $\square$