Lec 10/3

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Properties of solvable groups

(1) DFN $\underbrace{G^{(0)} = G;\ G^{(i+1)} = [G^{(i)}, G^{(i)}]}_{\text{Commutator series}}$,

G is solvable if $G^{(n)} = \{e\}$ for some $n > 0$.

(2) THMs G is solvable $\Leftrightarrow$ $\exists\ \Sigma : G = H_0 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$ s.t. H_i/H_{i+1} is abelian

(3) Sub & Quotient gps of solvable G are solvable

(4) $N \trianglelefteq G$, N & G/N solvable $\Rightarrow$ G solvable

(5) If G is finite & $\Sigma : G = K_0 \ntrianglerighteq \cdots \ntrianglerighteq K_\ell = \{e\}$

is a J-H series, G solvable $\Leftrightarrow$ $K_i/K_{i+1} \cong \mathbb{Z}/p_i\mathbb{Z}$.

Properties of Nilpotent groups

(1) DFN $\underbrace{C^1(G) = G,\ C^{n+1}(G) = [G, C^n(G)]}_{\text{Central Series}}$, nilpotent if $C^m(G) = \{e\}$ for some $m > 0$.

(Note: $C^i(G) \trianglelefteq G \quad \forall i$.)

Abelian $\Rightarrow$ Nilpotent $\Rightarrow$ solvable

Ex: $G^{(\ell)} \subset C^{2^\ell}(G)$

(2) $C^{n+1}(G) \subset C^n(G)$

$$\begin{array}{ccc} C^{n+1}(G) & \subset & C^n(G) \\ \| & & \cap \\ [G, C^n(G)] & \ni & \overbrace{g\underbrace{x}_{\in C^n}g^{-1}x^{-1}} \\ \cup & & \end{array}$$

$[G, C^n(G)] \ni \underbrace{gxg^{-1}}_{\in C^n} x^{-1}$

$\cup$

$[C^n(G), C^n(G)]$

$\Rightarrow C^n(G) / C^{n+1}(G)$ is abelian

<u>Lemma</u>: H: group & $N \trianglelefteq H$, H/N is abelian $\iff N \supset [H;H]$

<u>pf</u> $(\Rightarrow)$ $H \xrightarrow{\pi} H/N \leftarrow$ abelian

$\pi(aba^{-1}b^{-1}) = e$

$(\Leftarrow)$ $H/N \cong \dfrac{H/[H;H]}{N/[H;H]}$

$$\left.\begin{array}{ccc} H & \xrightarrow{\pi} & H/[H;H] \\ \triangledown & & \triangledown\mathrm{I} \\ N & \longleftrightarrow & N/[H;H] \\ \subset & & \\ {[H;H]} & & \end{array}\right\} 2^{nd} \text{ iso}$$

<u>Ex</u>: $H/[H;H]$ is abelian. $\Rightarrow H/N$ is abelian.

<u>Thm</u> G is nilpotent $\iff \exists \Sigma: G = H_0 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$

s.t. $[G : H_\ell] \subset H_{\ell+1}$ $(*)$

$\left([G;[G;[G; \cdots [G;G]] \cdots]] = \{e\} \right)$

<u>pf</u>: $(\Rightarrow)$ $H_\ell = C^{\ell+1}(G)$ ✓

$(\Leftarrow)$ Given Σ satisfying $(*)$,

Show by induction $C^{\ell+1}(G) \subset H_\ell \Rightarrow C^{m+1}(G) = \{e\}$.

<u>Example</u> $B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ac \neq 0 \right\}$ is solvable but not nilpotent

$$C^1(B) = B,\ C^2(B) = [B;B] = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$$

$$C^3(B) = [B, C^2(B)] \ni \begin{bmatrix} 1 & (z-1)x \\ 0 & 1 \end{bmatrix}$$

$$= C^2(B) \ \ldots$$

We can have $N \trianglelefteq G$ s.t. N is nilpotent & G/N is nilpotent but G is not nilpotent

Nilpotency is not a Serre property.

i.e. $G = B$, $N = [B;B]$ is abelian so nilpotent,

$B/N = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : xy \neq 0 \right\}$ also abelian $\Rightarrow$ nilpotent.

Lemma: Let G be a group, $A \leq Z(G)$ s.t. G/A is nilpotent then G is nilpotent.

(recall $Z(G) = \{ x \in G : gx = xg \ \forall g \in G \}$ is abelian & nilpotent)

Corollary: every p-group is nilpotent $\Rightarrow$ solvable

Pf of Lemma: Aim: construct a comp series $G = H_0 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$ s.t. $[G; H_\ell] < H_{\ell+1}$

Since G/A is nilpotent, we have $\exists$ $G/A = \overline{H}_0 \trianglerighteq \cdots \trianglerighteq \overline{H}_n = \{e_{G/A}\}$ with $[G/A ; \overline{H}_\ell] < \overline{H}_{\ell+1}$.

take $\pi : G \longrightarrow G/A$, define $H_i = \pi^{-1}(\overline{H}_i)$ for $0 \leq i \leq n$.

we get $G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = A \trianglerighteq \{e\} = H_{n+1}$

and notice $[G ; A] = \{e\} = H_{n+1}$ since $A \subset Z(G)$. $\square$

Pf of Corollary: Induction on n in $|G| = p^n$, note $Z(G)$ is nontrivial if $|G| = p^n$.

$$\text{Abelian} \underset{\not\Leftarrow}{\Rightarrow} \text{nilpotent} \underset{\not\Leftarrow}{\Rightarrow} \text{solvable.}$$

direct products of $\mathbb{Z}/p^l\mathbb{Z}$

direct products of p-groups

<u>Lemma</u>: G_1, G_2 nilpotent $\Rightarrow$ $G_1 \times G_2$ nilpotent.

<u>Pf</u>

$$\Sigma_1 : G_1 = H_0 \trianglerighteq \cdots \trianglerighteq H_n = \{e_1\} \qquad [G ; H_i] \subset H_{i+1}$$

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$$\Sigma_2 : G_2 = K_0 \trianglerighteq \cdots \trianglerighteq K_m = \{e_2\} \qquad [G_2 ; K_j] \subset K_{j+1}$$

$$\Sigma : G = G_1 \times G_2 = L_0 \trianglerighteq H_1 \times G_2 = L_1 \trianglerighteq H_2 \times G_2 = L_3 \trianglerighteq \cdots \trianglerighteq H_n \times G_2 = \{e_1\} \times G_2 = L_n$$

$$\trianglerighteq \{e_1\} \times K_1 = L_{n+1} \trianglerighteq \cdots \trianglerighteq \{e_1\} \times K_m = \{e_{G_1 \times G_2}\} = L_{n+m}$$

$(0 \le i < k)$

$$[G_1 \times G_2 ; L_i] = [G_1 \times G_2 ; H_i \times G_2] \subset [G_1, H_i] \times G_2 \subset H_{i+1} \times G_2 = L_{i+1}$$

$$[G_1 \times G_2 ; L_{n+j}] = [G_1 \times G_2 ; \{e_1\} \times K_j] = \{e_1\} \times [G_2 ; K_j] \subset \{e_1\} \times K_{j+1} = L_{n+j+1}$$

Let G be a group, $P \in Syl_p(G)$. $N_G(P) \le L$ (another subgp of G)

then $N_G(L) = L$.

<u>Pf</u> if $g \in N_G(L)$ then $gPg^{-1} \subset gLg^{-1} = L$

P & gPg^{-1} are two sylow p-subgps of L, so $\exists\ l \in L$ s.t.

$$lPl^{-1} = gPg^{-1} \Rightarrow g^{-1}l \in N_G(P) \subset L \Rightarrow g \in L.$$