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Def The K^{th} Betti number of a sp. cpx X is

$$\beta_K(X) = \dim(H_K(X))$$

Prop: $\beta_0(X) = \#\{\text{path components of } X\}$

(Note: a path component is a max'l path connected subset).

Pf Let $\mathcal{C} = \{X_1, \dots, X_m\}$ be the path components of X.

we will show that $H_0(X) = V_{F_2}(\mathcal{C})$.

Start by defining a linear map $L: Z_0(X) \longrightarrow V_{F_2}(\mathcal{C})$
$\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\quad C_0(X)$

on basis vectors by

$L(v) = X_j$ s.t. $v \in X_j$. extend linearly. L is surj

If we show $Ker(L) = B_0 = Im(\partial_1)$ we are done.

1. $im(\partial_1) \subseteq Ker(L)$.

$\forall$ basis vector $vw \in C_1$, $L(\partial_1(vw)) = L(v) + L(w)$,
but $L(v) = L(w)$ since v is path connected to w
by vw, so $L(v) + L(w) = 0$.

2. $Ker(L) \subseteq Im(\partial_1)$.

let $v \in Ker(L)$. Then $v = v_1 + \cdots + v_\ell$ where

v_j are vertices of X. $L(v) = L(v_1) + \cdots + L(v_\ell) = 0$.

So each path component is mapped to an even number of times. rewrite $v = v_1^{(1)} + \cdots + v_{2k_1}^{(1)} + v_1^{(2)} + \cdots + v_{2k_2}^{(2)} + \cdots\cdots + v_1^{(m)} + \cdots + v_{2k_m}^{(m)}$

s.t. $v_p^{(j)} \in X_j$. k_j are nonnegative integers.

So $v_1^{(1)}, v_2^{(1)} \in X_j$. $\exists$ vertices $\overset{v_1^{(1)}}{\overset{\parallel}{w_1}}, w_2, \ldots, \overset{v_2^{(1)}}{\overset{\parallel}{w_p}}$

s.t. $w_i w_{i+1}$ is an edge $\forall\, i$. Then

$\partial_1 (w_1 w_2 + w_2 w_3 + \cdots + w_{p-1} w_p) = v_1^{(1)} + v_2^{(1)}$.

So $v_1^{(1)} + v_2^{(1)} \in \operatorname{Im}(\partial_1)$. Induct. $\square$

$\beta_1(X)$ counts "independent 1-d loops".

(Look up fundamental group of X).

$\beta_2(X)$ counts indp 2-d "voids".