Lec 10/26

Friday, October 26, 2018 11:31

Ring of fractions

$$S^{-1}R = \left\{ \frac{r}{s} : r \in R, s \in S \right\} / \sim$$

one rel^n: $\frac{r}{s} = \frac{0}{1} \Leftrightarrow \exists\, t$ s.t. $rt = 0$.

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \iff \exists\, t \text{ s.t. } t(r_1 s_2 - s_1 r_2) = 0 .$$

Warning: book assumes there are no zero divisors in a mult-closed set.

Yesterday: $R = \mathbb{Z}$ (some integral domain)

$$\mathbb{Z} \rightsquigarrow \mathbb{Q}$$

$1 + (p-1)^2 = p^2 - 2p$

$F(R)$ = field of fractions of integral domain R.

R comm ring, P prime ideal

$R_P = (R \setminus P)^{-1} R$

Ex: $R = \mathbb{Z}/6\mathbb{Z}$ $\quad S = \{1, 2, 4\}$

$$S^{-1}R = \frac{S \times R}{\sim}$$

18 elts in $S \times R$

$\frac{0}{s} \sim \frac{x}{y}$ iff $xt = 0$ for some t

has 3 elements $\Downarrow$ $= \mathbb{Z}/3\mathbb{Z}$

6 elts in each equiv class

$\left\{ (s,0), (s,3) \right\}_{s = 1,2,4}$ (6 elements)

$\left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{2} \right\}$

$\frac{2}{1} = \frac{1}{2}$ since $4 - 1 = 3$ and $2 \cdot 3 = 0$

$\frac{1}{4} \neq \frac{1}{2}$ since 2 is in S.

Lemma: (1) R = integral domain
then $F(R) = (R\setminus\{0\})^{-1}R$ is a field.

pf $\left(\begin{array}{l} x = \frac{a}{b},\ b \neq 0. \quad \text{if } a=0 \text{ then } x=0. \quad \text{if } a\neq 0 \text{ then } \frac{b}{a} \in F(R) \\ \text{so } F(R)^{\times} = F(R)\setminus\{0\} \Rightarrow \text{a field} \end{array} \right)$

(2) $\overset{\text{comm ring}}{R} \supsetneq P$ prime ideal $\Longrightarrow (R\setminus P)^{-1}R = R_P$ is
local w/ max'l Ideal $\left\{ \frac{a}{s} : \begin{array}{l} a\in P \\ s \notin P \end{array} \right\}$

pf $\left(\begin{array}{ll} (R_p)^{\times} = R_p \setminus M : & x = \frac{a}{b} \in R_p,\ a \in p \Rightarrow x \in M \\ (\text{Ex}: M \text{ is an ideal}) \Rightarrow M = R_p \setminus R_p^{\times} & a \notin P \Rightarrow \frac{b}{a} \in R_p \Rightarrow x \in (R_p)^{\times} \end{array} \right)$

Lemma: $|R| < \infty$ and R commutative integral domain $\Rightarrow R$ field
consider the map $\sigma_x : y \longmapsto xy$. it's injective if $x \neq 0$. (Kernel is $\{0\}$)
but this mean's it's surjective so some $y \longmapsto 1 \Rightarrow xy = 1$.

eg: $R = \overset{\text{field}}{K}[X]$, $F(R) = K(x) = \left\{ \frac{f(x)}{g(x)} : g(x) \neq 0 \right\}$ field of rat'l functions.

$$\begin{array}{lll} R & \text{comm ring} & \\ \cup\!\!| & & \overset{\text{"invert" } S}{\rightsquigarrow} S^{-1}R \\ S & \text{mult-closed set} & \end{array}$$

we get a ring hom $j : R \longrightarrow S^{-1}R$
$x \longrightarrow \frac{x}{1}$

$$\text{Ker}(j) = \{ x \in R : \exists y \in S \text{ s.t. } xy = 0 \}$$

Observation $j(S) \subset (S^{-1}R)^{\times}$ since $\frac{s}{1}^{-1} = \frac{1}{s}$

Similarities w/ $\begin{array}{l} R \\ \cup\!\!| \\ I \end{array} \rightsquigarrow R/I$

Similarities w/ $\begin{matrix} R \\ \cup \\ I \end{matrix} \rightsquigarrow R/I$

- Ring hom $\pi : R \twoheadrightarrow R/I$, $\operatorname{Ker}(\pi) = I$,

1^{st} iso thm: $f : R \longrightarrow R'$ ring hom s.t. $\operatorname{Ker}(f) \supset I$

$$\rightsquigarrow \bar{f} : R/I \longrightarrow R'$$
$$(r \bmod I) \longrightarrow f(r)$$

<u>Prop</u> $\forall$ ring hom $f : R \longrightarrow R'$ s.t. $f(S) \subset (R')^{\times}$

we get a unique hom $R^{-1}S \longrightarrow R'$

$$\frac{r}{s} \longmapsto f(s)^{-1} f(r)$$

Let $I \subset R$ be an ideal. $S^{-1}I = \{ \frac{a}{s} : a \in I, s \in S \} \subset S^{-1}R$

$(j(I))_{S^{-1}R}$ = ideal in $S^{-1}R$ generated by $j(I)$

<u>Lemma</u>. (1) $(j(I))_{S^{-1}R} = S^{-1}I$

(2) Every ideal in $S^{-1}R$ is of this form

options $S^{-1}I = S^{-1}R$ even tho $I \neq R$ when $I \cap S \neq \emptyset$

<u>Pf</u> (1) $\{ \frac{a}{1} : a \in I \} \subset S^{-1}I$, $S^{-1}I$ is an ideal*

- $S^{-1}I \subset$ ideal generated by $j(I)$ since $\frac{a}{s} = \frac{1}{s} \frac{a}{1}$

*

$$\frac{a_1}{s_1} \pm \frac{a_2}{s_2} = \frac{a_1 s_2 \pm a_2 s_1}{s_1 s_2} \in S^{-1}I, \quad \frac{0}{1} \in S^{-1}I$$

$$\frac{r_1}{s_1} \cdot \frac{a_1}{s_2} = \frac{r_1 a_1}{s_1 s_2} \in S^{-1}I \quad ✓$$

(2) $j : R \longrightarrow S^{-1}R$ inverse image of an ideal is ...

(2) $j : R \longrightarrow S^{-1}R$ inverse image of an ideal is an ideal.

$$\begin{array}{ccc} R & \longrightarrow & S^{-1}R \\ \cup & & \cup \\ I = j^{-1}(\tilde{I}) & \longrightarrow & \tilde{I} \text{ ideal} \end{array}$$

$$\Rightarrow \quad S^{-1}I = (j(I))_{S^{-1}R} \subset \tilde{I}$$

Conversely if $\frac{r}{s} \in \tilde{I}$ then $\frac{1}{s}\frac{r}{s} = \frac{r}{1} \in \tilde{I} \Rightarrow r \in I = \frac{r}{s} \in S^{-1}I$

So $S^{-1}I = \tilde{I}$

$R \xrightarrow{j} S^{-1}R$ ring hom, every ideal is an image of an ideal.

Ideals in $S^{-1}R \longleftrightarrow$ ideals in $S^{-1}(R/\mathrm{Ker}(j))$

$$\| $$

$$\{ S^{-1}I \mid I \subset R \overset{\text{ideal}}{\text{containing}} \ \mathrm{Ker}(j) \}$$

(remember ideals in R/J = ideals of R containing J)

$$S^{-1}I = S^{-1}R \iff I \cap S \neq \emptyset$$

Distinct $\overset{\text{proper}}{\vee}$ ideals in $S^{-1}R$

$$= \left\{ S^{-1}I : I \subset R \text{ is an ideal s.t. } \mathrm{Ker}(j) \subset I, \ I \cap S = \emptyset \right\}$$

$R = \mathbb{Z}/6\mathbb{Z}, \quad S = \{1, 2, 4\}$

$$S^{-1}R = \mathbb{Z}/3\mathbb{Z} = \{ \tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{2} \}$$

$$
\begin{tikzcd}
\text{ideals: } (\text{in } \mathbb{Z}/6\mathbb{Z}) & \{0\} \arrow[d] & \{0,2,4\} \arrow[d] & \{0,3\} \arrow[dr] & (1) \arrow[dl] \\
\text{ideals: } (\text{in } S^{-1}R) & (0) & (0) & (1) & (1)
\end{tikzcd}
$$

$$\mathrm{Ker}\,(j) = \{0, 3\}$$