Lec 10/2

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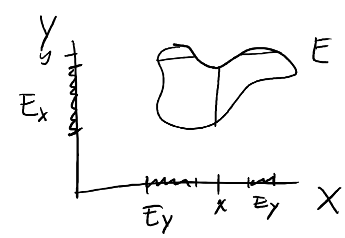
Cross Sections

Fix measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

For $E \subset X \times Y$, define



X-section $E_x = \{y \in Y \mid (x,y) \in E\}$

$= \pi_Y(E \cap \{x\} \times Y)$

Y-section $E^y = \{x \in X \mid (x,y) \in E\}$

$= \pi_X(E \cap X \times \{y\})$

Exercise: suppose $(E_n) \subset \mathcal{M} \otimes \mathcal{N}$.

① $\left[\bigcup E_n\right]_x = \bigcup (E_n)_x$

② $\left[\bigcap E_n\right]_x = \bigcap (E_n)_x$

③ $\left[E_i \setminus E_j\right]_x = (E_i)_x \setminus (E_j)_x$

④ $\chi_{E_i}(x, y) = \chi_{(E_i)_x}(y)$

(similarly for y-sections)

Prop Let $E \in \mathcal{M} \otimes \mathcal{N}$. $\forall x \in X$, $E_x \in \mathcal{N}$, $\forall y \in Y$, $E^y \in \mathcal{M}$.

pf We claim $\mathcal{S} = \{E \subset X \times Y \mid E_x \in \eta\}$ is a σ-alg containing mble rectangles.

⓪ $\phi \in \eta \Rightarrow \phi \in S$.

① if $(E_n) \subset \mathcal{S}$ so $(E_n)_x \in \eta$, $(\bigcup E_n)_x = \bigcup (E_n)_x \in \eta$.

② if $E \in \mathcal{S}$ so $E_x \in \eta$, $(E^c)_x = (E_x)^c \in \eta$. □

Exercise Use the proposition to prove $\mathcal{L} \otimes \mathcal{L} \neq \mathcal{L}^2 = (\lambda \times \lambda)^*$-mble sets in $\mathbb{R}^2$.

for $f: X \times Y \longrightarrow \mathbb{R}, \overline{\mathbb{R}}$ or $\mathbb{C}$, define

x-section $f_x : Y \longrightarrow \mathbb{R}$ by $f_x(y) = f(x, y)$

y-section $f_y : X \longrightarrow \mathbb{R}$ by $f^y(x) = f(x, y)$

Corollary if f is $\mathcal{M} \otimes \eta$-measurable, then $\forall x \in X$, f_x is η-mble, $\forall y \in Y$, f^y is $\mathcal{M}$-mble.

pf $\forall x \in X$, $G \in \mathcal{B}_{\text{codomain}}$, $f_x^{-1}(G) = f^{-1}(G)_x \in \eta$.

Thm: Suppose $(X, \mathcal{M}, \mu)$ and (Y, η, ν) are σ-finite.

Then $\forall E \in \mathcal{M} \otimes \eta$,

① the fns $x \mapsto \nu(E_x)$ and $y \longrightarrow \mu(E^y)$ are mble

② $(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$.

~~pf:~~ First, assume μ, ν finite.

Let $\Lambda \subset \mathcal{M} \otimes \mathcal{N}$ be the subset for which ① & ② hold.

Step 1 $\Pi :=$ set of mble rectangles $\subseteq \Lambda$

~~pf~~ clear. $E = F \times G \Rightarrow [x \mapsto \nu(E_x)] = \nu(G)\, \chi_F$ is mble.

$(\mu \times \nu)(E) = \mu(F)\nu(G) = \int \nu(G)\, \chi_F\, d\mu$.

Step 2 Π is a π-system

~~pf~~ $(F_1 \times G_1) \cap (F_2 \times G_2) = (F_1 \cap F_2) \times (G_1 \cap G_2)$ maybe

Step 3 Λ is a λ-system, so $\Lambda = \mathcal{M} \otimes \mathcal{N}$ by the π-λ thm

~~pf~~ ◦ $X \times Y \in \Pi \subset \Lambda$

◦ If $E \in \Lambda$, so ① & ② hold. Then $x \mapsto \nu((E^c)_x) = \nu((E_x)^c) = \underbrace{\nu(Y) - \nu(E_x)}_{\text{mble fn.}}$ finiteness

similarly for $y \mapsto \mu((E^c)^y)$.

Moreover, $(\mu \times \nu)(E^c) = (\mu \times \nu)(X \times Y) - (\mu \times \nu)(E)$

$$= \int \nu(Y)\, d\mu(x) - \int \nu(E_x)\, d\mu(x)$$

$$= \int \nu(Y) - \nu(E_x)\, d\mu(x)$$

$$= \int \nu((E_x)^c)\, d\mu(x)$$

$$= \int \nu((E^c)_x)\, d\mu(x),$$

and similarly for the other one. So $E^c \in \Lambda$.

② Suppose $(E_k) \subset \Lambda$ is a disjoint sequence.

Then $\forall k$, $x \mapsto \nu((E_k)_x)$ is mble, and so are

$$\underbrace{x \mapsto \sum \nu((E_k)_x)}_{\text{sup of finite sums = mble.}} = \nu\left(\bigsqcup (E_k)_x\right) = \nu\left(\left(\bigsqcup E_k\right)_x\right)$$

similarly for $y \mapsto \mu\left(\left(\bigsqcup E_k\right)^y\right)$.

Now
$$(\mu\times\nu)\left(\bigsqcup E_k\right) = \sum (\mu\times\nu)(E_k) = \sum \int \nu((E_k)_x)\, d\mu(x)$$
$$= \int \sum \nu((E_k)_x)\, d\mu(x)$$
$$= \int \nu\left(\left(\bigsqcup E_k\right)_x\right) d\mu(x)$$

and similarly for other integral.

Step 4 When μ and ν are σ-finite, write $X \times Y$ as an increasing union $X\times Y = \bigcup_n X_n \times Y_n$ ← finite measure.

Then for $E \in \mathcal{M}\otimes\mathcal{N}$, write $E_n = E \cap X_n \times Y_n$.

$x \mapsto \nu(E_x) = \nu((\cup E_n)_x) = \nu(\cup (E_n)_x) = \lim \nu((E_n)_x)$ mble.

similarly for Y.

and
$$(\mu\times\nu)(E) = \lim (\mu\times\nu)(E_n) = \lim \int \nu((E_n)_x)\, d\mu(x)$$
$$= \int \lim \nu((E_n)_x)\, d\mu(x)$$
$$= \int \nu(E_x)\, d\mu(x). \qquad \square$$