Lec 10/7

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Typo on HW: show $y \mapsto f(x-y)g(y) \in \mathcal{L}^1$ a.e.

Differentiation

We'll do $L^1(\lambda^n)$

extend to $L^1_{loc}(\lambda^n)$

$\{f: \mathbb{R}^n \to \mathbb{C} \mid f$ integ on bdd mble sets$\}$.

Def A cube in $\mathbb{R}^n$ is a set Q of the form $\prod_{i=1}^n I_i$ where each I_i is a closed nondegenerate bounded interval & all of them have the same length, denoted $\ell(Q)$.

- For $x \in \mathbb{R}^n$, $\mathcal{C}(x) := \{$cubes containing $x\}$
- If Q is a cube & $r>0$, rQ is the cube w/ same center but $\ell(rQ) = r\,\ell(Q)$.

Goal: Lebesgue differentiation thm: $\forall f \in L^1_{loc}(\lambda^n)$,

$$\lim_{\ell(Q)\to 0} \frac{1}{\lambda^n(Q)} \int_Q f \, d\lambda^n = f(x) \quad \text{a.e.}$$

$x \in Q$ $\qquad Q$

As a corollary when $n=1$, we get

FToC: Suppose $f \in L^1(\lambda)$. define

$$F(x) := \int_{(-\infty, x)} f \, d\lambda. \quad \text{Then } F' = f \quad \text{a.e.}$$

Proof observe

$$\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \lim_{\substack{h \to 0 \\ x \in [x-h, x+h]}} \frac{1}{\lambda(Q)} \int_Q f \, d\lambda$$

$$= f \quad \text{a.e.}$$

Def: for $f \in L^1_{loc}$, define $Mf : \mathbb{R}^n \longrightarrow [0, \infty]$ by

$$Mf := \sup \left\{ \frac{1}{\lambda^n(Q)} \int_Q |f| \, d\lambda^n \;\middle|\; Q \in \mathcal{C}(x) \right\}$$

Properties:

- $M(rf) = |r| \cdot M(f) \quad \forall r \in \mathbb{R}$
- $M(f+g) \leq Mf + Mg$

- $Mf > 0$ everywhere unless $f = 0$ a.e.

HW

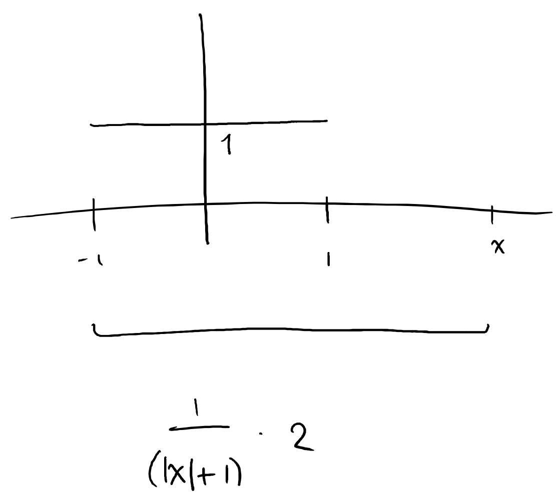
- Mf is lower semicontinuous ($\Leftrightarrow$ $\{Mf > a\}$ is open $\forall a \in \mathbb{R}$).

  $\Longrightarrow$ Mf is measurable

<u>Example</u> $f = \chi_{[-1,1]} : \mathbb{R} \longrightarrow \mathbb{C}$

$$Mf(x) = \begin{cases} 1 & x \in [-1, 1] \\ \frac{2}{1+|x|} & x \notin [-1, 1] \end{cases}$$

$\notin \mathcal{L}^1$



<u>Hardy-Littlewood Maximal Theorem</u>

$\exists$ $c > 0$ only depending on n s.t. $\forall f \in \mathcal{L}^1(\lambda^n)$ and $a > 0$,

$$\lambda^n(\{Mf > a\}) \leq \frac{c\,\|f\|_1}{a}.$$

<u>Remark</u>: Like a generalization of Chebyshev's Inequality:

$$\forall a>0, \quad f \in \mathcal{L}^1(\mu) \qquad \int_{\{|f|>a\}} |f| \, d\mu \geq a \, \mu(\{|f|>a\})$$

So $\mu(\{|f|>a\}) \leq \dfrac{\|f\|_1}{a}$.

<u>Exercise</u>: Let $E \subset \mathbb{R}^n$ and $\mathcal{C}$ a collection of cubes covering E

s.t. $\sup\{\ell(Q) \mid Q \in \mathcal{C}\} < \infty$.

Then $\exists$ sequence of disjoint cubes $(Q_k) \subset \mathcal{C}$ s.t.

$$\sum_k \lambda^n(Q_k) \geq 5^{-n} \underbrace{(\lambda^n)^*(E)}_{\text{outer measure.}}$$

This is a version of the <u>Vitali Covering Lemma</u>:

$\mathcal{B}$ is a collection of open balls in $\mathbb{R}^n$, let

$\mathcal{U} := \bigcup_{B \in \mathcal{B}} B$. If $c < \lambda^n(\mathcal{U})$, $\exists$ disjoint

$B_1, B_2, \ldots, B_k \in \mathcal{B}$ s.t. $\sum_1^k \lambda^n(B_j) > 3^{-n} c$.

<u>Pf of Vitali</u>: Since λ^n is regular, $\exists K \subset \mathcal{U}$ cpt s.t.

$c < \lambda^n(K)$. Observe $\exists A_1, \ldots, A_m \in \mathcal{B}$ s.t. $K \subset \bigcup_1^m A_i$.

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Inductively, define $B_1 =$ largest (largest radius) of the A_i, and

$B_j =$ largest of A_i which are disjoint from $B_1, \dots, B_{j-1}$.

Since there are finitely many A_i, this process terminates giving $B_1, \dots, B_K$.

<u>Trick</u>: If A_i is <u>not</u> one of $B_1, \dots, B_K$,

$\exists$ smallest $1 \le j \le K$ s.t. $A_i \cap B_j \neq \emptyset$.

Then $\mathrm{rad}(A_i) \le \mathrm{rad}(B_j) \Rightarrow A_i \subset 3B_j$.

Then $K \subset \bigcup_1^m A_i \subset \bigcup_1^K 3B_j$.

So $c < \lambda^n(K) \le \sum_1^K \lambda^n(3B_j) = 3^n \sum_1^K \lambda^n(B_j)$. $\square$

HLMT: $\exists\ c > 0$ only depending on n s.t. $\forall f \in \mathcal{L}^1(\lambda^n)$ and $a > 0$,

$$\lambda^n(\{Mf > a\}) \le \frac{c\,\|f\|_1}{a}.$$

Pf Suppose $f \in \mathcal{L}^1$ and $a > 0$. Let $E = \{Mf > a\}$.

Set $\mathcal{C} = \left\{ \text{cubes } Q \;\middle|\; \frac{1}{\lambda^n(Q)} \int_Q |f|\, d\lambda^n > a \right\}$.

Then $\mathcal{C}$ covers E!

By the exercise, $\exists$ seq. $(Q_k) \subset \mathcal{C}$ of disjoint cubes s.t. $\sum \lambda^n(Q_k) \geq 5^{-n} \lambda^n(E)$.

Then $\lambda^n(E) \leq 5^n \sum \underbrace{\lambda^n(Q_k)}_{< \frac{1}{a}\int_{Q_k} |f| d\lambda^n} < \frac{5^n}{a} \sum \int_{Q_k} |f| \, d\lambda^n \leq \frac{5^n}{a} \|f\|_1$

By the way, $\sup \{ \ell(Q) \mid Q \in \mathcal{C} \} < \infty$

Since $\frac{1}{a} \|f\|_1 \geq \lambda^n(Q) = \ell(Q)^n$. $\square$