Lec 10/18

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Recall – defn of an Ideal

R: ring
$\vee$
I: ideal (left/right/2-sided)

(1) $(I,+) \leq (R,+)$

(2) $rx \in I$ (left ideal)

Ideals generated by subsets

R: ring and $X \subset R$ a subset.

$\underbrace{I = (X)}_{\text{notation}} \subset R$ is smallest Ideal containing X

Lemma: If I_τ is an ideal in R $\forall \tau \in T$, $\bigcap_{\tau \in T} I_\tau$ is an ideal in R as well.

Pf: obvious.

In practice what could be in $(X)_L$?

$$(X)_L = FS\left(\bigcup_{x \in X} R \cdot x\right)$$

From now on R is commutative

Some ops on Ideals:

Intersection;
Sum; $\longrightarrow$ $\sum_{\alpha \in A} I_\alpha = \left(\bigcup_{\alpha \in A} I_\alpha\right)$
Product;

$I, J \subset R$ ideals

These are the same if one of I or J is principal

$$I \cdot J = \{ a_1 b_1 + a_2 b_2 + \cdots + a_n b_n : n \in \mathbb{N},\ a_i \in I, b_i \in J\ \forall i = 1, 2, \ldots, n \}$$

$$I * J = \{ ab : a \in I, b \in J \}$$

<u>Definition</u>: An ideal $\mathfrak{a} \subset R$ is principal if $\exists a \in \mathfrak{a}$ s.t. $\mathfrak{a} = \overset{(a) \text{ (notation)}}{\overset{\shortparallel}{(\{a\})}}$.

eg $n\mathbb{Z} = (n)$ (all ideals in $\mathbb{Z}$ are principal).

Take $R = K[X, Y]$ K: field

$$\begin{array}{c} R \\ \cup \\ J = I = (x, y) = \{ f(x,y) : f(0,0) = 0 \} \end{array}$$

$$Y^2, X^3 \in I * I \quad \text{but} \quad \underset{f \cdot g}{\underset{\shortparallel}{Y^2 - X^3}} \notin I * I$$

(can't factor — exercise)

So $I * I$ is not necessarily an ideal.

$$I \cdot I = (x^2, xy, y^2)$$

Some properties: $I \cdot J \subset I \cap J$

eg
$$\begin{array}{ccc} (m\mathbb{Z}) \cdot (n\mathbb{Z}) & & (m\mathbb{Z}) \cap (n\mathbb{Z}) \\ \shortparallel & & \shortparallel \\ mn\mathbb{Z} & \subsetneq & \text{lcm}(m,n)\,\mathbb{Z} \end{array}$$

except when $(m,n) = 1$

Definition R: commutative ring, $I, J \subset R$ two ideals are coprime if
$I + J = R$.

Lemma If I and J are coprime then $I \cdot J = I \cap J$

Pf $I \cdot J \subset I \cap J$ always.

Let $x \in I \cap J$. I & J are coprime $\Longrightarrow \exists\, a \in I$ s.t. $1 - a \in J$

$$x = x \cdot 1 = x \cdot (a + 1 - a) = \underset{J}{\underset{\psi}{x}} \cdot \underset{I}{\underset{\psi}{a}} + \underset{I}{\underset{\psi}{x}} \cdot \underset{J}{\underset{\psi}{(1-a)}} \in I \cdot J.$$

Lemma $I_1 + J = R$, $I_2 + J = R \implies I_1 \cdot I_2 + J = R$

pf $x_1 + y_1 = 1$ for some $x_1 \in I_1$, $y_1 \in J$
$x_2 + y_2 = 1$ for some $x_2 \in I_2$, $y_2 \in J$

$$1 = (x_1 + y_1)(x_2 + y_2) = \underset{I_1 \cdot I_2}{\underset{\psi}{x_1 x_2}} + \underbrace{x_1 y_2 + x_2 y_1 + y_1 y_2}_{\substack{\psi \\ J}}$$

Cor. $I_1, I_2, \ldots, I_\ell \subset R$ ideals, $I_i + I_j = R \quad \forall i \neq j$

$\Rightarrow I_1 \cdot I_2 \cdot \cdots \cdot I_\ell = I_1 \cap I_2 \cap \cdots \cap I_\ell$

Recall: $\mathbb{Z}/mn\mathbb{Z} \overset{\text{group iso}}{\cong} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ if $(m, n) = 1$.

(same cond. for finitely many)

(Direct) product of rings:

R_1, R_2 rings:

$R_1 \times R_2 = \{(a_1, a_2) : a_1 \in R_1, a_2 \in R_2\}$

all ops are component-wise

$1_{R_1 \times R_2} = (1_{R_1}, 1_{R_2})$

$0_{R_1 \times R_2} = (0_{R_1}, 0_{R_2})$

Theorem: Let R be a commutative ring, $I, J \subset R$ be two coprime ideals.

Then $R/_{I \cdot J} \cong R/_I \times R/_J$ (same cond. for finitely many).

Recall: $R \xrightarrow{\pi} R/\mathfrak{a}$ for any ideal $\mathfrak{a} \subsetneq R$

$$R \xrightarrow{f} R/_I \times R/_J$$

$$a \longmapsto (a \pmod I, \; a \pmod J))$$

$\mathrm{Ker}(f) = I \cap J = I \cdot J$ since I & J are coprime.

remains to show f is surjective

Is $(1,0) \in \mathrm{Im}(f)$? does there exist $a \in R$ s.t. $a \equiv 1 \pmod I$, $a \equiv 0 \pmod J$)

i.e. $a \in J$, $1 - a \in I$? yes! I & J are coprime.

Choose $a_1 \in J$ s.t. $1 - a_1 \in I$.

Choose $a_2 \in I$ s.t. $1 - a_2 \in J$.

$f(a_1) = (1, 0)$, $f(a_2) = (0, 1)$

So $f(x a_1 + y a_2) = (x, y) = \mathrm{Im}(f) = R/_I \times R/_J$ $\square$