Lec 8/26

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Basic Properties of measures: $(X, \mathcal{M}, \mu)$ measure space.

① $E \subset F$ (both in $\mathcal{M}$) $\Rightarrow \mu(E) \leq \mu(F)$

② $(E_n) \subset \mathcal{M} \Rightarrow \mu(\bigcup E_n) \leq \sum \mu(E_n)$

③ $E_1 \subset E_2 \subset E_3 \subset \cdots$ (all in $\mathcal{M}$) $\Rightarrow \mu(\bigcup E_n) = \lim_n \mu(E_n)$

④ (continuity from above) $E_1 \supset E_2 \supset E_3 \supset \cdots$ (all in $\mathcal{M}$), w/ $\mu(E_1) < \infty$

$\Rightarrow \mu(\bigcap E_n) = \lim_n \mu(E_n)$

Pf Let $F_n = E_1 \setminus E_n$. Then since $E_1 \supset E_2 \supset E_3 \supset \cdots$,

$F_1 \subset F_2 \subset F_3 \subset \cdots$. Moreover, $\mu(E_1) = \mu(E_n) + \mu(F_n)$.

Then $\bigcup F_n = \bigcup (E_1 \setminus E_n) = E_1 \setminus (\bigcap E_n)$.

So $\lim_n \mu(F_n) = \mu(\bigcup F_n) = \mu(E_1) - \mu(\bigcap E_n)$

$\qquad \|$

$\lim_n \mu(E_1) - \mu(E_n) = E_1 - \lim_n \mu(E_n)$

So $\mu(\bigcap E_n) = \lim_n \mu(E_n)$.

Everything is $< \infty$ so subtraction is ok.

Cor: If $E \in \mathcal{M}$ s.t. $\mu(E) = 0$, and $F \in \mathcal{M}$ s.t. $F \subset E$, $\mu(F) = 0$ too.

Thm (Completion): Spose $(X, \mathcal{M}, \mu)$ is a measure space. define

$$\overline{\mathcal{M}} = \{ E \cup F \mid E \in \mathcal{M} \text{ and } \underbrace{\exists N \in \mathcal{M} \text{ s.t. } \mu(N) = 0}_{N \text{ is } \mu\text{-null}} \text{ and } F \subset N \}.$$

Then ① $\overline{\mathcal{M}}$ is a σ-algebra w/ $\mathcal{M} \subset \overline{\mathcal{M}}$

② there is a unique measure $\bar{\mu}$ on $\overline{\mathcal{M}}$ s.t. $\bar{\mu}\big|_{\mathcal{M}} = \mu$.

Pf

⓪ Clear $\mathcal{M} \subset \overline{\mathcal{M}}$, so $\emptyset \in \overline{\mathcal{M}} \neq \emptyset$

① Suppose $(E_n \cup F_n)$ is a sequence in $\overline{\mathcal{M}}$. Then

$\bigcup (E_n \cup F_n) = \underbrace{(\bigcup E_n)}_{\in \mathcal{M}} \cup (\bigcup F_n)$. For each n, choose

$N_n \in \mathcal{M}$ s.t. $\mu(N_n) = 0$ and $F_n \subset N_n$. Then

$\bigcup F_n \subset \bigcup N_n$ and $\mu(\bigcup N_n) \leq \sum \mu(N_n) = 0$.

So $\bigcup (E_n \cup F_n) \in \overline{\mathcal{M}}$.

② Suppose $E \cup F \in \overline{\mathcal{M}}$ and $F \subset N$ μ-null

WTS $(E \cup F)^c \in \overline{\mathcal{M}}$.

Trick: $X = N \amalg N^c$

$$\begin{aligned}
(E \cup F)^c &= E^c \cap F^c \cap X \\
&= E^c \cap F^c \cap [N \amalg N^c] \\
&= \underbrace{\underbrace{(E^c \cap \underbrace{F^c \cap N^c}_{= N^c})}_{\in \mathcal{M}} \amalg \underbrace{(E^c \cap F^c \cap N)}_{\subset N}}
\end{aligned}$$

$$\underbrace{\in \mathcal{M} \qquad\qquad\qquad}_{\in \overline{\mathcal{M}}}$$

Hence $\overline{\mathcal{M}}$ is a σ-algebra, $\mathcal{M} \subset \overline{\mathcal{M}}$.

<u>$!\bar{\mu}$</u>: Suppose $\bar{\mu}|_{\mathcal{M}} = \mu$. Then $\forall\ \overset{\mathcal{M}}{\overset{\cup}{E}} \cup F \subset \overline{\mathcal{M}}$ with $F \subset N$ μ-null,

$$\mu(E) = \bar{\mu}(E) \leq \bar{\mu}(E \cup F) \leq \bar{\mu}(E) + \bar{\mu}(F) \leq \bar{\mu}(E) + \bar{\mu}(N) = \mu(E) + \mu(N) = \mu(E).$$

$\rightsquigarrow$ so $\bar{\mu}(E \cup F) = \mu(E)$. Let $\bar{\mu}(E \cup F) := \mu(E)$

This is <u>well-defined</u>:

if $E_1 \cup F_1 = E_2 \cup F_2$, then

$$E_1 \subset E_1 \cup F_1 = E_2 \cup F_2 \subset E_2 \cup N_2$$

so $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$.

Similarly $\mu(E_2) \leq \mu(E_1)$, so

$\bar{\mu}$ is ind. of choice of representative $E \cup F$.

⓪ $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$.

① Suppose $(E_n \cup F_n) \subset \overline{\mathcal{M}}$ are disjoint.

Then so are (E_n).

$$\begin{aligned}\bar{\mu}\left(\coprod (E_n \cup F_n)\right) &= \bar{\mu}\left(\amalg E_n \cup \amalg F_n\right) \\ &= \mu(\amalg E_n) \\ &= \sum \mu(E_n) \\ &= \sum \bar{\mu}(E_n \cup F_n).\end{aligned}$$

$\square$

Outer Measures:

$$\mu^* : P(X) \longrightarrow [0,\infty] \quad \text{s.t.}$$

⓪ $\mu^*(\emptyset) = 0$

① $E \subset F \Rightarrow \mu^*(E) \leq \mu^*(F)$

② $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$.

Strategy:

$\underbrace{\text{pre-measure}}_{\text{easy to define}} \rightsquigarrow$ outer measure $\rightsquigarrow$ measure

Prop: Suppose $\mathcal{E} \subset P(X)$ s.t. $\emptyset, X \in \mathcal{E}$.

Suppose $\rho : \mathcal{E} \longrightarrow [0,\infty]$ s.t. $\rho(\emptyset) = 0$.

Define $\mu^*(E) := \inf\left\{ \sum_{n=1}^{\infty} \rho(E_n) \,\middle|\, E_n \in \mathcal{E},\ E \subset \bigcup E_n \right\}$

Then μ^* is an outer measure.

Pf ⓪ Taking $E_n = \emptyset\ \forall n$, $\mu^*(\emptyset) = 0$.

① Suppose $E \subset F \subset \bigcup F_n$. Then $E \subset \bigcup F_n$.

So Inf for $E \leq$ inf for F.

② Trick: $\varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$.

Trick: $r \leq s$ iff $r \leq s + \varepsilon \quad \forall\, \varepsilon > 0$.

Suppose (E_n) is a seq. of sets. Let $\varepsilon > 0$.

For each n, $\exists \left(F_j^n\right)_{j=1}^{\infty}$ s.t. $E_n \subset \bigcup_j F_j^n$

and $\sum_j \rho(F_j^n) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$.

Then $\bigcup E_n \subset \bigcup_n \bigcup_j F_j^n$. So

$$
\begin{aligned}
\mu^*\left(\bigcup E_n\right) &\leq \sum_n \sum_j \rho(F_j^n) \\
&\leq \sum_n \left(\mu^*(E_n) + \frac{\varepsilon}{2^n}\right) \\
&= \sum_n \mu^*(E_n) + \sum_n \frac{\varepsilon}{2^n} \\
&= \sum_n \mu^*(E_n) + \varepsilon .
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, $\mu^*\left(\bigcup E_n\right) \leq \sum \mu^*(E_n)$. $\square$