Connection to Galois Thm

$$L_1 \cap L_2 \longleftrightarrow \langle H_1, H_2 \rangle \quad \text{NOT} \quad H_1 H_2 .$$

$$\begin{tikzcd}[column sep=small]
& \langle H_1, H_2 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
& H_1 \cap H_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& L_1 \cap L_2 \arrow[dl, dash] \arrow[dr, dash] & \\
L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] \\
& L_1 L_2 &
\end{tikzcd}$$

upside-down tower

$= \{h_1 h_2 : h_1 \in H_1, h_2 \in H_2\}$

If $H_1 \trianglelefteq G$, $H_1 H_2 = \langle H_1, H_2 \rangle$

if $H_1, H_2 \trianglelefteq G$, $H_1 H_2 \trianglelefteq G$

$$|H_1 H_2| = \frac{|H_1| \cdot |H_2|}{|H_1 \cap H_2|} .$$

Frobenius

<u>Example</u> ③ $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \Phi \rangle \cong \mathbb{Z}_n$

diagram of subgroups
$$\begin{tikzcd}[column sep=small]
& \cdots \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Z}_{d_1} \arrow[dr, dash] & \mathbb{Z}_{d_2} \arrow[d, dash] & \cdots \arrow[dl, dash] \\
& \mathbb{Z}_n &
\end{tikzcd}$$

for $n = 18$,

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z}_3 \arrow[d, dash] \arrow[dr, dash] & & \mathbb{Z}_2 \arrow[dl, dash] \\
\mathbb{Z}_9 \arrow[dr, dash] & \mathbb{Z}_6 \arrow[d, dash] & \\
 & \mathbb{Z}_{18} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & \mathbb{F}_{p^{18}} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{F}_{p^6} \arrow[d, dash] \arrow[dr, dash] & & \mathbb{F}_{p^9} \arrow[dl, dash] \\
\mathbb{F}_{p^2} \arrow[dr, dash] & \mathbb{F}_{p^3} \arrow[d, dash] & \\
 & \mathbb{F}_p &
\end{tikzcd}$$

$$\mathbb{Z}_{n/d} \longleftrightarrow \mathbb{F}_{p^d}$$

$$\parallel$$

$$\langle d \rangle$$

$\mathbb{F}_{p^d}$ is fixed by Φ^d: $\alpha : \alpha^{p^d} = \alpha$.

$$\begin{tikzcd}[column sep=small]
 & L_1L_2 = K \arrow[dl, equal, "n_2"'] \arrow[dr, equal, "n_1"] \arrow[dd, dash] & \\
L_1 \arrow[dr, equal, "n_1"'] \arrow[rr, leftrightarrow] & & L_2 \arrow[dl, dash, "n_2"] \\
 & F = L_1 \cap L_2 &
\end{tikzcd}$$

<u>Theorem</u> Let K/F be Galois. Let $F \subseteq L_1, L_2 \subseteq K$. Such that $K = L_1L_2$ and $L_1 \cap L_2 = F$. Let L_1/F be normal. Then $[K:F] = [L_1:F] \cdot [L_2:F]$,

$$Gal(K/F) = Gal(K/L_1) \rtimes Gal(L_1/F),$$

and $Gal(L_1/F) \cong Gal(K/L_2)$.

<u>Proof</u> Draw the Diagram of subgroups:

$$\begin{tikzcd}[column sep=small]
 & 1 = H_1 \cap H_2 \arrow[dl, equal] \arrow[dr, equal] & \\
 {} & & {}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& 1 = H_1 \cap H_2 \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
& G = H_1H_2 \text{ since } H_1 \trianglelefteq G &
\end{tikzcd}$$

So $G = H_1 \rtimes H_2$ and $H_2 \cong G/H_1$.

$\mathrm{Gal}(K/F) \cong \mathrm{Gal}(K/L_1) \rtimes \mathrm{Gal}(K/L_2)$ and $\mathrm{Gal}(K/L_2) \cong \mathrm{Gal}(L_1/F)$.

In particular, $[K:F] = |G| = |H_1| \cdot |H_2| = |G|/|H_2| \cdot |G|/|H_1| = [L_2 : F] \cdot [L_1 : F]$. $\square$

<u>Example</u> K is a splitting field of $x^n - 2$ over $\mathbb{Q}$.

Then $K = \mathbb{Q}(\alpha, \omega)$ where $\alpha = \sqrt[n]{2}$ and $\omega = e^{2\pi i/n} = \sqrt[n]{1}$.

$$\begin{tikzcd}[column sep=small]
& \mathbb{Q}(\alpha, \omega) = K \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\alpha) \arrow[dr, dash, "n"', "\text{not normal}"] & & \mathbb{Q}(\omega) \arrow[dl, dash, "\varphi(n)"] \\
& \mathbb{Q} &
\end{tikzcd}$$

If $n = 8$, $\omega = \frac{1+i}{\sqrt{2}}$ so $\mathbb{Q}(\omega) \ni \sqrt{2}$,

and $\mathbb{Q}(\sqrt[8]{2}) \ni \sqrt{2}$

So the theorem doesn't apply.

Assuming $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\omega) = \mathbb{Q}$,

we have $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}(\omega)) \rtimes \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$

$$\begin{array}{ccc} \mathrm{Gal}(K/\mathbb{Q}(\omega)) & & \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \\ \cong & & \cong \\ \mathbb{Z}_n & \rtimes & \mathbb{Z}_n^\times \end{array}$$

Theorem If $L_1, L_2 \subseteq K$, $L_1L_2 = K$, $L_1 \cap L_2 = F$, $L_1/F, L_2/F$ are Galois, then $[K:F] = [K:L_1]\cdot[K:L_2]$, $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(L_1/F) \times \mathrm{Gal}(L_2/F)$, and $\mathrm{Gal}(K/L_2) \cong \mathrm{Gal}(L_1/F)$, $\mathrm{Gal}(K/L_1) \cong \mathrm{Gal}(L_2/F)$.

$$\begin{tikzcd}
 & K = L_1L_2 \arrow[dl, equal] \arrow[dr, equal] & & & & 1 \arrow[dl, equal] \arrow[dr, equal] & \\
L_1 \arrow[dr, equal] & & L_2 \arrow[dl, equal] & \arrow[r, leftrightsquigarrow] & H_1 \arrow[dr, equal] & & H_2 \arrow[dl, equal] \\
 & F = L_1 \cap L_2 & & & & G = H_1H_2 &
\end{tikzcd}$$

so $G \cong H_1 \times H_2$, $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(K/L_1) \times \mathrm{Gal}(K/L_2)$

and $G/H_2 \cong H_1$, $G/H_1 \cong H_2$,

So $\mathrm{Gal}(L_1/F) \cong G/H_1 \cong H_2 = \mathrm{Gal}(K/L_2)$

$\mathrm{Gal}(L_2/F) \cong G/H_2 \cong H_1 = \mathrm{Gal}(K/L_1)$

So $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(L_1/F) \times \mathrm{Gal}(L_2/F)$. $\square$

Examples: $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$

Examples ① $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$.

② $K = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \omega = e^{2\pi i/3})$, splitting field of $(x^2-2)(x^3-2)$

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_2 \times S_3,$$ (if the intersection $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt[3]{2},\omega)$ is trivial. and it is.)

In the last theorem, if If $L_1 \cap L_2 \neq F$,

we have

$$\begin{tikzcd}
& K = L_1L_2 \arrow[dl, dash, "n_2/d"'] \arrow[dr, dash, "n_1/d"] & \\
L_1 \arrow[dr, dash, "n_1/d"] \arrow[ddr, dash, "n_1"'] & & L_2 \arrow[dl, dash, "n_2/d"'] \arrow[ddl, dash, "n_2"] \\
& L_1 \cap L_2 \arrow[d, dash, "d"] & \\
& F &
\end{tikzcd}$$

so $[K:F] = \frac{n_1 n_2}{d}$.

If $N_1 = \mathrm{Gal}(L_1/F)$
$N_2 = \mathrm{Gal}(L_2/F)$,

$N_1 = G/H_1$, $N_2 = G/H_2$

So $N = \mathrm{Gal}(L_1 \cap L_2/F)$ is a common factor of N_1 and N_2.

$$\mathrm{Gal}(K/L_1) \cong \mathrm{Gal}(L_2/(L_1 \cap L_2))$$

$$\mathrm{Gal}(K/L_2) \cong \mathrm{Gal}(L_1/(L_1 \cap L_2))$$

$$\mathrm{Gal}(K/F) = N_1 \times_N N_2$$

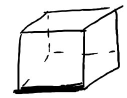
relative direct product over N.

N_1, N_2, N - common factor of N_1 & N_2.

Then $N_1 \times_N N_2 = \{(\varphi_1,\varphi_2) \in N_1 \times N_2 \text{ s.t. } \overline{\varphi_1} = \overline{\varphi_2} \text{ in } N\}$

$(N = N_1/_D \quad N = N_2/_D \qquad \varphi_1 \bmod P_1 = \varphi_2 \bmod P_2)$

$$\left(N = N_1/p_2,\quad N = N_2/p_2,\quad \varphi_1 \bmod p_1 \overset{?}{=} \varphi_2 \bmod p_2\right)$$



"multiplying 2 squares over an interval"

$$\begin{tikzcd}
N_1 \arrow[dr] & & N_2 \arrow[dl] \\
& N \arrow[d] & \\
& 0 &
\end{tikzcd}$$