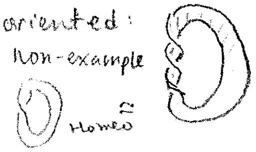
SEIFERT ALGORITHM Recall. Smooth $\times \rightsquigarrow)($, shift cycles $C \rightsquigarrow C\times\{\text{nesting \# of } C\}$. each $\hat{C}$ bounds $\hat{D} \subset \mathbb{R}^3$.
Remember where you cut it, remember orientation. $\rightsquigarrow$ oriented, disjoint

Seifert surface should be oriented: non-example

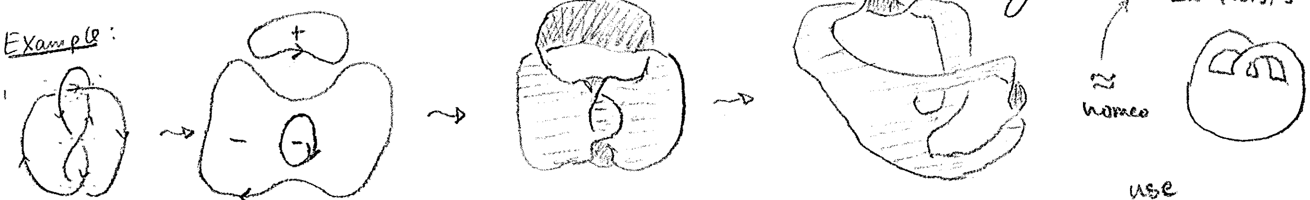


Homeo $\cong$

* Add in half-twist strips compatible w/ orientation (& over/under strands)

$\rightsquigarrow$ oriented surface w/ knot boundary.

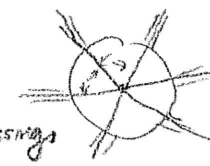
Example: 4_1



$\mathbb{R}^3$ can untwist 2π-twists $\cong$ homeo $= \Sigma_{g=1,n=1}$

genus of Diagram := genus of surface gotten from Seifert algorithm: $g(D)$. use χ = euler characteristic = # of 0-cells (vertices) − # of 1-cells (edges) + # of 2-cells (faces) − ...

$$\chi(\Sigma_{g,n}) = 2-2g-n. \quad \chi(\Sigma_{g,1}) = 1-2g$$



seifert surface $\to$ graph Γ

$\chi(\Gamma)$ = # vertices − # edges = # seifert cycles − # crossings

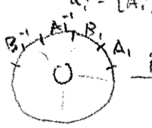
So $g(D) = \frac{1}{2}(1 + \#\text{crossings} - \#\text{cycles})$. $g(K)$ = smallest genus of a seifert surface for K (knot). $g(5_2) = 2$.

Lemma N = smooth, oriented 3-mfd. $f: S^1 \hookrightarrow N$ a smooth embedding (i.e. a knot). Then $f_*: H_1(S^1) \to H_1(N)$ is trivial iff f extends to a smooth embedding $F: \Sigma \hookrightarrow N$ of a surface w/ bdry S^1. ($j: S^1 \hookrightarrow \Sigma$, $F\circ j = f$)

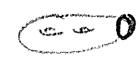
Pf $\Leftarrow$ $F\circ j = f$, $j: \partial\Sigma \hookrightarrow \Sigma$, $j_*: H_1(\partial\Sigma) \to H_1(\Sigma)$ is zero map. So $F_* \circ j_* = f_* = 0$.

$\Rightarrow$ $f_*: H_1(S^1) \to H_1(N)$ is zero map. $\pi_1(f)$ maps $[c] \mapsto [f\circ c] \in [\pi_1(N), \pi_1(N)]$. $[f\circ c] = \prod [\alpha_i, \beta_i]$, $\alpha_i, \beta_i \in \pi_1(N)$.
$\pi_1(f): \pi_1(S^1) \to \pi_1(N)$ ($\uparrow$ abelianize, $\parallel$). $\alpha_i = [A_i]$, $\beta_i = [B_i]$, A_i, B_i loops in N, $f \underset{\text{homotopic}}{\approx} A_1 * B_1 * A_1^{-1} * B_1^{-1} * \cdots * B_n^{-1}$.

$\tilde{F}: S^1\times[0,1] \to N$.

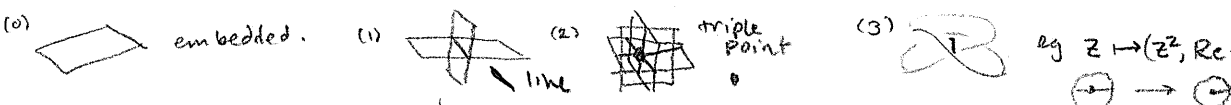


$\xrightarrow{\tilde{F}} N$. Identify stuff, get $\Sigma_{n,1}$ =



$\xrightarrow{f} N$.

* Whitney approx. Theorem: $\tilde{F}$ ε-homotopic to smooth map $\hat{F}$, coincides w/ f on bdry $= \partial\Sigma$. (smooth embed near $\partial\Sigma$)
* Put $\hat{F}$ in gen. pos. using Transversality Thms: $F': \Sigma \to N$. Local pictures are:



(0) embedded. (1) line (2) triple point (3) eg $z \mapsto (z^2, \operatorname{Re} z)$

Fix the non-embedded things:

get a diff surface w/ same bdry. (2) & (3) are fixable.

Moral of story: Seifert surfaces exist.