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Wednesday, February 13, 2019 11:26

R – integral domain

$$(R^n)^* = \text{dual of } R^n = \mathrm{Hom}(R^n, R) \cong R^n$$

Elements of $(R^n)^*$ are linear forms / covectors.

$$f \in (R^n)^*, \quad f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} c_i a_i$$

$(c_1, \dots, c_n)$ are coordinates of f in basis:

$$\{f_1, \dots, f_n\} \quad \text{where} \quad f_i\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_i.$$

matrix of f is $(c_1 \cdots c_n)$ a $1 \times n$ matrix

if $e_1 = (1, 0, \dots, 0)$ then $c_i = f(e_i)$.
$e_2 = (0, 1, 0, \dots, 0)$
$\vdots$
$e_n = (0, \dots, 0, 1)$

$$f_i(e_j) = \delta_{ij}$$

$$f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum c_i a_i \Rightarrow f = \sum c_i f_i$$

$\{f_1, \dots, f_n\}$ is the dual basis for $\{e_1, \dots, e_n\}$.

Let $M \cong R^n$, let $\{u_1, \dots, u_n\}$ be a basis in M.

Then $M^* = \mathrm{Hom}(M, R) \cong R^n$ has a basis

$\{f_1, \dots, f_n\}$ defined by $f_i(u_j) = \delta_{ij}$

dual basis to $\{u_1, \dots, u_n\}$
in M^*

$f = (c_1 \cdots c_n) \Rightarrow f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (c_1 \cdots c_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ matrix mul.

$\underbrace{\{u_1, \dots, u_n\} \text{ - basis in } M \Rightarrow \text{dual basis is}}_{\text{book notation but not great since vector} \not\Rightarrow \text{covector.}}$ $\{u_1^*, \dots, u_n^*\}$.
$\hookrightarrow u_i^*(u_j) = \delta_{ij}$

$f(u) \in R$.

$(f, u) \in R$.

$$\begin{aligned} M &\longrightarrow M^{**} \\ u &\longmapsto u^{**} \end{aligned} \qquad \text{where } u^{**}(f) = f(u)$$

If $M \cong R^n$ then this is an isomorphism.

the dual of the dual basis to B is B itself.

$\{u_1, \dots, u_n\}$ - basis in M

$\{f_1, \dots, f_n\}$ - dual basis in M^*

$\{U_1, \dots, U_n\}$ - dual dual basis in M^{**}

$$U_i(f_j) = \delta_{ij} = f_j(u_i) = u_i^{**}(f_j)$$

$$\text{So } U_i = u_i^{**}$$

$$\begin{array}{ccc} \varphi : M & \longrightarrow & N \\ \cong & & \cong \\ R^m & & R^n \end{array}$$

A_φ is an $n \times m$ matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

in basis $\{u_1, \dots, u_m\} \subseteq M$
$\{v_1, \dots, v_n\} \subseteq N$

Dual hom-sm $\varphi^* : N^* \longrightarrow M^*$

defined by $\varphi^*(f) = f \circ \varphi$

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\varphi^*(f)"'] & & N \arrow[dl, "f"] \\ & R & \end{tikzcd}$$

$\{g_1, \dots, g_n\}$ $\{f_1, \dots, f_m\}$

Matrix A_{φ^*} in dual bases of N^* & M^* is $m \times n$ matrix:

Columns of A_{φ^*} are $\varphi^*(g_1), \dots, \varphi^*(g_n) \in M^*$

$$1^{st} \text{ column } \quad \varphi^*(g_1) = \begin{pmatrix} g_1 \circ \varphi(u_1) \\ \vdots \\ g_1 \circ \varphi(u_m) \end{pmatrix} = \begin{pmatrix} \text{1st coord of } \varphi(u_1) \\ \vdots \\ \text{1st coord of } \varphi(u_m) \end{pmatrix}$$

$$= 1^{st} \text{ row of } A_\varphi$$

So $\quad A_{\varphi^*} = A_\varphi^T$.

M, M^*

$(u, f) \in R$

$S^\perp = \text{Ann}(S) = \{ f \in M^* : f(u) = 0 \ \ \forall u \in S \}$

$P^\perp = \text{Ann}(P) = \{ u \in M : f(u) = 0 \ \forall f \in P \}$

$\underset{\text{submodule}}{N \subseteq M} \Rightarrow N \xhookrightarrow{\iota} M \Rightarrow M^* \xrightarrow{\iota^*} N^*$

$0 \longrightarrow N \longrightarrow M \qquad$ exact

$\text{Hom}(N, R) \longrightarrow \underbrace{\text{Hom}(M, R) \longrightarrow 0}$

exact if R is injective R-module.

(but not exact in general)

$$\operatorname{Ker}(\iota^*) = \{ f \in M^* : f|_N = 0 \}$$
$$= \operatorname{Ann}(N)$$

$2\mathbb{Z} \subseteq \mathbb{Z}$ $\quad \mathbb{Z}^* \longrightarrow (2\mathbb{Z})^*$ not surjective:

you don't get $2n \longmapsto n$ in the image.

any form here sends 2 to an even #.

If R is a field, V, W are ~~F~~ VS, $W \subseteq V$,
$R = F$

$0 \longrightarrow W \xrightarrow{\eta} V$, then

$\eta^* : V^* \longrightarrow W^*$ is surjective

(since R is injective module)

"any linear form on W can be extended to one on V."

$\forall f : W \to F$, $\exists \tilde{f} : V \to F$ s.t. $f = \tilde{f}|_W$.

namely, let W_2 be s.t. $V = W \oplus W_2$. Then

put $\tilde{f}(W_2) = 0$, $\tilde{f}|_W = f|_W$

So, in this case, $W^* \cong V^*/\mathrm{Ann}(W)$

If $\dim V = n$, $\dim W = m$, then $\dim V^* = n$, $\dim W^* = m$,

so $\dim(\mathrm{Ann}(W)) = n-m$

<u>Defn</u>: Rank of a module M is the cardinality of its maxl linearly independent subset.

if $\mathrm{Rank}(M) = n$ Then $0 \to R^n \to M \to M/R^n \to 0$

torsion module

Let F be field of fractions of R.

Then $0 \to F^n \to F \otimes M \to 0$ is exact since F is flat

Since M/R^n is torsion

So $V = F \otimes_R M \cong F^n$ is n-dim F-vector space.