Lec 11/4

Friday, November 4, 2016 9:07 AM

Napier's Algorithm

Let $b > 1$. Chose large n (10^k)

$Naplog_{10}(2) = 7$

$\vdots$

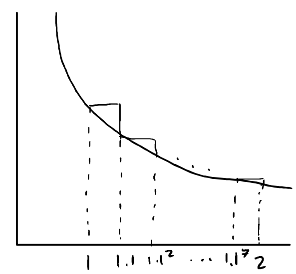
$Naplog_{1000}(2) = 693$

Define $Naplog_n(b) = \max\{i : (1+\frac{1}{n})^i \leq b\}$

exists by WOP. $\lim_{i\to\infty}(1+\frac{1}{n})^i = \infty$

Theorem: $\lim_{n\to\infty}\frac{Naplog_n(b)}{n}$ exists and equals $\int_1^b \frac{1}{x}\,dx$

Idea of proof: $\int_1^2 \frac{1}{x}\,dx$. Let $P = \{1, 1.1, 1.1^2, 1.1^3, \ldots, 1.1^7, 2\}$



$\frac{1}{x}$ decreasing so sup at left endpoint

$$U(f,P) = \sum_{i=1}^{7}\frac{1}{1.1^{i-1}}(1.1^i - 1.1^{i-1}) + \frac{1}{1.1^7}(2 - 1.1^7)$$

$$= \sum_{i=1}^{7}(1.1-1) + \frac{1}{1.1^7}(2-1.1^7)$$

$$= .7 +$$

$$\int_1^2 \frac{1}{x}\,dx \approx .7 = \frac{7}{10} = \frac{Naplog_{10}(2)}{10}$$

Proof: fix n, let $N = naplog_n(b)$

Then $(1+\frac{1}{n})^N \leq b < (1+\frac{1}{n})^{N+1}$

$$\int_1^{(1+\frac{1}{n})^N}\frac{1}{x}\,dx \leq \int_1^b \frac{1}{x}\,dx < \int_1^{(1+\frac{1}{n})^{N+1}}\frac{1}{x}\,dx \quad \text{since } \frac{1}{x} > 0.$$

$P = \{(1+\frac{1}{n})^i : 0 \leq i \leq N\}$ partition of $[1, (1+\frac{1}{n})^N]$

$$Q = \{(1+\tfrac{1}{n})^i : 0 \le i \le N+1\} \quad \text{partition of} \quad [1, (1+\tfrac{1}{n})^{N+1}]$$

$$U(f,Q) = \sum_{i=1}^{N+1} \frac{1}{(1+\frac{1}{n})^{i-1}} \left[(1+\tfrac{1}{n})^i - (1+\tfrac{1}{n})^{i-1}\right]$$

$$= \sum_{i=1}^{N+1} \left[(1+\tfrac{1}{n}) - 1\right]$$

$$= \sum_{i=1}^{N+1} \frac{1}{n} = \frac{N+1}{n}$$

$$L(f,P) = \sum_{i=1}^{N} \frac{1}{(1+\frac{1}{n})^i} \left[(1+\tfrac{1}{n})^i - (1+\tfrac{1}{n})^{i-1}\right]$$

$$= \sum_{i=1}^{N} \frac{1}{1+\frac{1}{n}} \left(1+\tfrac{1}{n} - 1\right) = \frac{1}{1+\frac{1}{n}} \sum_{i=1}^{N} \frac{1}{n} = \frac{n}{n+1} \frac{N}{n}$$

hence $\displaystyle\int_1^b \frac{1}{x}\,dx \in \left[\frac{n}{n+1}\frac{N}{n}, \frac{N}{n} + \frac{1}{n}\right]$

$\in$

$\frac{N}{n}$

hence $\displaystyle\left|\int_1^b \frac{1}{x}\,dx - \frac{N}{n}\right| \le \frac{N}{n} + \frac{1}{n} - \frac{n}{n+1}\frac{N}{n}$

$$= \left(1 - \frac{n}{n+1}\right)\frac{N}{n} + \frac{1}{n}$$

$$= \frac{1}{n+1}\frac{N}{n} + \frac{1}{n}$$

$$1 + \frac{N}{n} \le \left(1+\frac{1}{n}\right)^N \le b$$

$$\frac{N}{n} \le b-1$$

$$\le \frac{1}{n+1}(b-1) + \frac{1}{n}$$

which goesto 0 as $n \to \infty$.

So $\displaystyle\int_1^b \frac{1}{x}\,dx = \lim \frac{\text{Narlog}_n(b)}{n}$.

So $\int_1^b \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{\mathrm{Narlog}_n(b)}{n}$.

Definition Define $\ln : (0,\infty) \to \mathbb{R}$ by $\ln(x) = \int_1^x \frac{1}{t}\,dt$

Then by FTC, $\frac{\partial}{\partial x}(\ln(x)) = \frac{1}{x}$

basic properties of ln:

(0) $\ln(1) = 0$

(1) $\ln(xy) = \ln(x) + \ln(y)$

(2) $\ln(x/y) = \ln(x) - \ln(y)$

(3) $\ln(x^r) = r\ln(x)$ if $r \in \mathbb{Q}$ (not yet defined for irrational r)

(4) ln is increasing on $(0,\infty)$

(5) $\lim_{x\to\infty} \ln(x) = \infty$

(6) $\lim_{x\to 0^+} \ln(x) = -\infty$

(7) ln is 1-1 and onto, so it has an inverse $\exp = (\ln)^{-1} : (-\infty,\infty) \to (0,\infty)$

Proofs:

(0) ✓ Definition

(1) $\frac{\partial}{\partial x}(\ln(xy)) = \frac{1}{xy}\cdot y = \frac{1}{x} = \frac{d}{dx}(\ln(x))$

$\Rightarrow \ln(xy) = \ln(x) + C$

if $x=1$, $\ln(y) = \ln(1) + C = C \Rightarrow \ln(y) = C$ ∎

(2) (1)$\Rightarrow$ $\ln(x/y) + \ln(y) = \ln(x) \Rightarrow \ln(x/y) = \ln(x) - \ln(y)$ ∎

(3) $\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}\frac{d}{dx}(x^r) = \frac{1}{x^r}\, r x^{r-1} = r\frac{1}{x} = \frac{d}{dx}(r\ln(x))$

So $\ln(x^r) = r\ln(x) + C$, let $x=1$, $C=0$. ∎

(4) $\frac{d}{dx}(\ln(x)) = \frac{1}{x} > 0$ ∎

(5) By (3), $\ln(2^n) = n\ln(2)$. $2^n \to \infty$ as $n\to\infty$. ∎

(6) $u = \frac{1}{x}$, $\ln\left(\frac{1}{u}\right)$ as $u\to\infty$ $= \lim_{u\to\infty} \ln(1) - \ln(u) = -\infty$ ∎

(7) 1-1 bc increasing, onto bc goes to $-\infty$ and ∞. ∎

$\int \frac{1}{x} \partial x = \ln(x) + C$ for positive x.

let $f(x) = \ln(-x)$, $f : (-\infty, 0) \to \mathbb{R}$

$f'(x) = \frac{1}{-x} \frac{\partial}{\partial x}(-x) = \frac{1}{x}$

$\int \frac{1}{x} \partial x = \ln(-x) + C$ for negative x

So $\int \frac{1}{x} \partial x = \ln|x| + C$ for $x \neq 0$