Lec 1/10

Friday, January 10, 2020 11:39

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "fg"'] & B \arrow[d, "f"] \\
& C
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
FA \arrow[r, "Fg"] \arrow[dr, "(Ff)(Fg) = F(fg)"'] & FB \arrow[d, "Ff"] \\
& FC
\end{tikzcd}
\quad \text{covariant}
$$

$$\Big\downarrow\!\!\wr$$

$$
\begin{tikzcd}
FA & FB \arrow[l, "Fg"'] \\
& FC \arrow[u, "Ff"'] \arrow[ul, "(Fg)(Ff) = F(f"]
\end{tikzcd}
\quad \text{contravariant.}
$$

Note: Contravariant F factors as
$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "op"] \arrow[rr, bend right, "F"'] & \mathcal{C}^{op} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd}
$$
where G is covariant.

Eg. $* : \underline{R\text{-mod}} \longrightarrow \underline{\text{mod-}R}$

$$M \longmapsto M^* = \mathrm{Hom}(M, R).$$

$$
\left(M \xrightarrow{f} N \right) \longmapsto
\left(
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varphi\circ f"'] & N \arrow[d, "\varphi"] \arrow[dl, "f^*"', bend right, phantom] \\
R \arrow[r, equal] & R
\end{tikzcd}
\right)
$$

$*$ is faithful & full.

<u>Natural transformations</u>: $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ covariant functors.

A natural transformation $\eta : F \longrightarrow G$ assigns $\eta_A \in \mathrm{Hom}(F(A), G(A))$ to each $A \in \mathrm{ob}\,\mathcal{C}$.

s.t.
$$
\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowleft"] & GA \arrow[d, "Gf"] \\
FB \arrow[r] & GB
\end{tikzcd}
.
$$

η_B

If η_A is an isomorphism $\forall A$, η is an isomorphism $F \simeq G$.

η can be described as a functor $\mathcal{C} \times 2 \longrightarrow \mathcal{D}$ $\quad (2 = (\cdot \longrightarrow \cdot))$

s.t.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[dr, "F"'] & \mathcal{C} \times (\cdot \to \cdot) \arrow[d, "\eta"] & \mathcal{C} \arrow[l, hook'] \arrow[dl, "G"] \\
& \mathcal{D} &
\end{tikzcd}$$

commutes.