Lec 2/28

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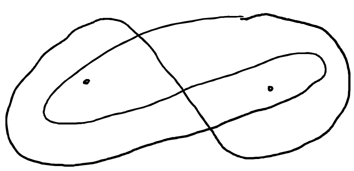
Corollary

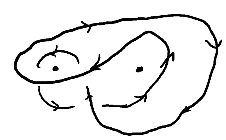
Let α and β be loops in $\mathbb{C}^{\times}$. if $\beta \simeq \gamma$ in $\mathbb{C}^{\times}$ then

$\operatorname{ind}(\beta) = \operatorname{ind}(\gamma)$ and conversely.

Pf of ($\Rightarrow$) Suppose that $\beta \simeq \gamma$ in $\mathbb{C}^{\times}$. then $\frac{\beta}{\gamma} \simeq 1$ in $\mathbb{C}^{\times}$, so $\frac{\beta}{\gamma}$ has a continuous log in $\mathbb{C}^{\times}$, so $\operatorname{ind}\left(\frac{\beta}{\gamma}\right) = 0$, but $\operatorname{ind}\left(\frac{\beta}{\gamma}\right) = \operatorname{ind}(\beta) - \operatorname{ind}(\gamma)$ so the two are equal. $\square$

Q: Suppose γ is a loop in $\mathbb{C} \setminus \{0, 1\}$. If γ is null-homotopic in $\mathbb{C} \setminus \{0,1\}$, then $\operatorname{ind}(\gamma, 0) = 0$ and $\operatorname{ind}(\gamma, 1) = 0$ as well. but $\operatorname{ind}(\gamma, 0) = 0 = \operatorname{ind}(\gamma, 1) \not\Rightarrow \gamma$ is null-homotopic in $\mathbb{C} \setminus \{0,1\}$.





how could we prove these loops are not null-homotopic?

A generalisation of the fundamental theorem of algebra.

Let $h: \mathbb{C} \longrightarrow \mathbb{C}$ be cts, and let $n \in \mathbb{Z} \setminus \{0\}$.

Suppose that $\frac{h(z)}{z^n} \longrightarrow 1$ as $|z| \longrightarrow \infty$.

then h has at least one 0.

Pf Suppose not. then $h : \mathbb{C} \longrightarrow \mathbb{C}^\times$. Since $\frac{h(z)}{z^n} \longrightarrow 1$ as $|z| \longrightarrow \infty$,

$\exists\, R \in (0,\infty)$ s.t. $\forall z \in \mathbb{C}$ with $|z| = R$ we have $\left|\frac{h(z)}{z^n} - 1\right| < 1$.

define $\gamma : S^1 \longrightarrow \mathbb{C}$ by $\gamma(w) = Rw$. Let $\alpha = h \circ \gamma$ and

let $\beta = \gamma^n$. Then α and β are loops in $\mathbb{C}^\times$.

now $\left|\frac{\alpha}{\beta} - 1\right| < 1$ so $\operatorname{ind}(\frac{\alpha}{\beta}) = 0$. But then $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$.

Now $\operatorname{ind}(\beta) = n$ and $\operatorname{ind}(\alpha) = 0$. $\times$ so h must have a zero □

γ is null-homotopic in $\mathbb{C}$

so $h \circ \gamma$ is null-homotopic in $\mathbb{C}^\times$.

Defn Let X be a top. sp. To say X is contractible means

id_X is null-homotopic in X.

iff $\exists\, H : X \times [0,1] \xrightarrow{cts} X$ s.t. $\forall x \in X$, $H(x,0) = x$, $\forall x, x' \in X$, $H(x,1) = H(x',1)$

eg $\mathbb{R}^d$ is contractible

pf define $H : \mathbb{R}^d \times [0,1] \longrightarrow \mathbb{R}^d$ by $H(x,t) = (1-t)x$. $H : id_{\mathbb{R}^d} \simeq 0$ in $\mathbb{R}^d$ □

eg Let $X \subseteq \mathbb{R}^d$. Suppose X is star-shaped wrt 0 ($\forall x \in X$, the line segment $[0,x] \subseteq X$).

then X is contractible using $H : X \times [0,1] \longrightarrow X$ by $H(x,t) = (1-t)x$. $H : id_X \simeq 0$ in X.

eg let $X \subseteq \mathbb{R}^d$ and $p \in X$. Suppose X is star-shaped wrt p. then X is contractible.

Remark let $X \subseteq \mathbb{R}^d$. then X is convex iff $\forall p \in X$, X is star-shaped wrt p.

Propn Let X be a contractible space. Let Y be any top. sp.

then each continuous fn $f : X \to Y$ is null-homotopic in Y.

Pf let $H : \mathrm{id}_X \simeq c_p^X$ (constant on X valued p). Then $f \circ H : f \circ \mathrm{id}_X \simeq f \circ c_p^X$, i.e. $f \circ H : f \simeq c^X_{f(p)}$

Propn Let X, Y be top. sp. with Y contractible. Then each cts fn $f : X \to Y$ is null-homotopic in X.

Propn Let X, Z be top. sp. Let $h : X \to Z$ be cts. Suppose h factors through a contractible space. then h is null-homotopic in Z.

Pf By assumption, $\exists$ a contractible space Y and continuous maps $f : X \to Y$, $g : Y \to Z$ s.t. $h = g \circ f$. Since Y is contractible, there is a point $y_0 \in Y$ and a continuous map $\Phi : Y \times [0,1] \to Y$ s.t. $\forall y \in Y$, $\Phi(y,0) = y$, $\Phi(y,1) = y_0$.

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, dashed, "f"'] & & Z \\ & Y \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Commutative diagram

define $\Psi \; X \times [0,1] \to Z$ by $\Psi(x,t) = g(\Phi(f(x), t))$. Ψ is continuous.

And $\forall x \in X$, $\Psi(x,0) = g(\Phi(f(x),0)) = g(f(x)) = h(x)$

$$\Psi(x,1) = g(\Phi(f(x),1)) = g(y_0)$$

Thus $\Psi : h \simeq \underset{X \times \{g(y_0)\}}{\underset{\parallel}{c^X_{g(y_0)}}}$ in Z. $\square$

Corollary Let $h : X \to \mathbb{C}^\times$ be cts. Suppose h has a cts log.

then h is null-homotopic in $\mathbb{C}$.

Pf h factors through $\mathbb{C}$ Which is contractible:

$$X \xrightarrow{h} \mathbb{C}^\times$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, dashed, "\log\circ h"'] & \mathbb{C}^\times \\ & \mathbb{C} \arrow[u, dashed, "\exp"'] \end{tikzcd}$$

eg let X be contractible let $h: X \longrightarrow \mathbb{C}^\times$ be cts. h is null-homotopic in $\mathbb{C}^\times$

Since X is contractible so $h \simeq 1$ so there is a continuous log of h in X.

eg Let X be a star-shaped subset of $\mathbb{R}^d$. Let $h: X \xrightarrow{cts} \mathbb{C}^\times$. then h has a cts log.

<u>Thm</u> Let $f: \mathbb{S}^{n-1} \longrightarrow X$ be cts, where X is a top. sp. Then TFAE:

$\{x \in \mathbb{R}^d : |x| \leq 1\}$, $\partial \mathbb{B}^d = \mathbb{S}^{d-1}$.

(a) f is null-homotopic in X

(b) f can be extended to a continuous map $F: \mathbb{B}^d \longrightarrow X$