Lec 11/15

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dictionary ordering on monomials ~~an~~ order on $x_1 < x_2 < \cdots < x_n$.

all we need is a total order on $\{ \underline{x}^{\underline{\alpha}} : \underline{\alpha} \in \mathbb{Z}^n_{\geq 0} \}$

s.t. $x^\alpha \leq x^\beta \Longrightarrow x^{\alpha+\gamma} \leq x^{\beta+\gamma}$. $\forall \gamma$.

(grlex also works: degree first, then lexicographical).

(also can pick $w_1, \ldots, w_n \in \mathbb{R}$, $x^\alpha < x^\beta \Leftrightarrow \langle \alpha, w \rangle < \langle \beta, w \rangle \rightsquigarrow$ tropical geometry).

$$f(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}}^{\text{finite}} c(\alpha) x^\alpha \rightsquigarrow LT(f) = c(\alpha_0) x^{\alpha_0} \text{ s.t. } c(\alpha_0) \neq 0 \text{ and if } c(\alpha) \neq 0, \ x^\alpha \leq x^{\alpha_0}.$$

$\underset{\text{ideal}}{I} \subset R = K[x_1, \ldots, x_n] \rightsquigarrow \underset{\text{ideal}}{LT(I)} \subset R$ $\longrightarrow$ generated by $\{LT(f) : f \in I\}$.

Note $I = (f_1, \ldots, f_k) \not\Longrightarrow LT(I) = (LT(f_1), \ldots, LT(f_k))$.

Grobner Basis of I is a (finite) set of generators of I, $\{g_1, \cdots, g_m\}$ s.t. $LT(I) = (LT(g_1), \ldots, LT(g_m))$.

- does it exist (yes, use (1) HBT (2) division algorithm).
- why care?

<u>Division Algorithm:</u>

Fix. monomial order

$g_1, \cdots, g_m \in R$.

<u>Input</u>: $f \in R$.

<u>Output</u>: $q_1, \cdots, q_m, r \in R$ satisfying:

(1) $f = q_1 g_1 + \cdots + q_m g_m + r$

(2) No term in r is divisible by any $LT(g_i)$.

(3) $LT(q_i g_i) \leq LT(f)\ \forall i$. ← monomial order

Procedure: Set $q_1, \dots, q_m, r = 0$.

while $f \neq 0$; do

if $LT(f) = a_i LT(g_i)$ ⟵ check for each i

Then $f \longmapsto f - a_i g_i$

$q_i \longmapsto q_i + a_i$

else $f \longmapsto f - LT(f)$

$r \longmapsto r + LT(f)$

Example: $R = K[x,y]$, $x > y$ ⇝ lex order on monomials

$f(x) = x^3y^3 + 3x^2y^4$

$g(x) = xy^4$

- $LT(f) = x^3y^3$ not div by $LT(g) = xy^4$

$r \longmapsto x^3y^3$

$f \longmapsto 3x^2y^4$

- $LT(f) = 3x^2y^4 = 3x\, LT(g)$

$q \longmapsto 3x$

$f \longmapsto 0$

$$f(x) = \underbrace{3x}_{q}\, g(x) + \underbrace{x^3y^3}_{r}$$

fixed : $R = K[x_1, \dots, x_n]$
$\leq$: monomial order

Prop: Let $I \subset R$ be an ideal.

(1) if $g_1, \dots, g_m \in I$ s.t. $LT(I) = (LT(g_1), \dots, LT(g_m))$

then $(g_1, \dots, g_m) = I$ (i.e. $\{g_1, \dots, g_m\}$ is a gröbner basis).

(2) I has a gröbner basis.

<u>proof</u> (1): $(g_1, \dots, g_m) \subset I$ obviously. Let $f \in I$.

$f = q_1 g_1 + \dots + q_m g_m + r$ where no term in r is divisible by any $LT(g_i)$.

Then $r \in I$ so $LT(r) \in LT(I) = (LT(g_1), \dots, LT(g_m))$ so any monomial in r is divisible by one of $LT(g_i)$. so $r = 0$. $\square$

(2): $LT(I) \subset K[x_1, \dots, x_n]$ is an ideal so by HBT, $I = (p_1, \dots, p_\ell)$ for some $p_i \in R$. Each p_i is a finite sum of monomials, taking them all gives finitely many monomials a_m. Choose $g_1, \dots, g_m$ s.t. we get these monomials from $LT(g_1), \dots, LT(g_m)$. This is a Gröbner Basis. $\square$

membership Test

<u>Theorem</u> (Gröbner) $I \subset R$ ideal. $\{g_1, \dots, g_m\}$ is a Gröbner Basis.

$\forall f \in I$, we can write $f = f_I - r$ where $f_I \in I$ and no term in r is divisible by any $LT(g_i)$. $f \in I \Leftrightarrow r = 0$.

<u>Pf</u>: the first statement is just the division algorithm.

If $r = 0$ then $f = f_I \in I$. To show $(\Rightarrow)$, we prove $f_I + r$ is unique!

If $f = f_I + r = \tilde{f}_I + \tilde{r}$ then $r - \tilde{r} = \tilde{f}_I - f_I \in I$, so take $LT(r - \tilde{r}) \in LT(I)$.

Since $g_1, \dots, g_m$ is a grobner basis, $r - \tilde{r} = 0$.