

R - root system.

$$\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$$

$$\nabla = d - \sum_{\alpha \in R} \frac{d_{\alpha}}{\alpha} t_{\alpha} \quad , \quad t_{\alpha} \in \text{End}_{\mathbb{C}}(F), \quad t_{-\alpha} = t_{\alpha}.$$

Holonomy relns: $\forall Y \subseteq R_+$ max'l s.t. $\mathbb{C}\text{-span}(Y)$ is $\mathbb{Z}d$,

$$\text{we have } \left[\sum_{\alpha \in Y} t_{\alpha} \mid t_{\beta} \right] = 0 \quad \forall \beta \in Y.$$

W -equivariance: (assuming $W \subset F$)

$$w t_{\alpha} w^{-1} = t_{w(\alpha)}$$

D : (Dynkin) diagram of a root system, connected.

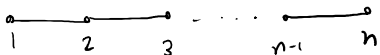
$B_1, B_2 \subset D$ subdiagrams (full subgraphs).

$$B_1 \perp B_2 \text{ if } \begin{cases} B_1, B_2 \text{ have no common vertices,} \\ \forall \alpha \in B_1, \beta \in B_2, \quad (\alpha, \beta) \notin D. \end{cases}$$

B_1 & B_2 are compatible if $B_1 \subset B_2$, $B_2 \subset B_1$, or $B_1 \perp B_2$.

- Nested set in D : collection of pairwise compatible connected subdiagrams.

- Maximal Nested sets.

Ex A_n 

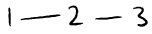
(or B_n or C_n)

Connected Subdiagram: 

Bracketings on $n+1$ variables x_1, x_2, \dots, x_{n+1}

$$i \text{ --- } \dots \text{ --- } j \longleftrightarrow x_1 \dots x_{i-1} (x_i \dots x_{j+1}) x_{j+2} \dots x_{n+1}$$

Max-l nested sets \longleftrightarrow complete bracketings of $n+1$ variables.

A_3 

$$\mathcal{F}_1 = \{ i ; i \text{ --- } 2 ; \text{ --- } 2 \text{ --- } 3 \} \longleftrightarrow (((x_1 x_2) x_3) x_4)$$

$$\frac{1}{n+1} \binom{2n}{n} = n^{th} \text{ catalan \#}$$

||

#M.N.S. of A_n .

Elementary Properties of MNS-.

Let \mathcal{F} be a max' nested set; and $B \in \mathcal{F}$.

Lemma $\mathcal{F}|_B = \{B' \in \mathcal{F} \mid B' \not\supset B\}$.

Let B_1, \dots, B_k be max' elements of $\mathcal{F}|_B$.

Then: (1) $B_i \perp B_j \quad \forall i \neq j$ (obvious)

(2) $\exists!$ vertex $\alpha \in B$ s.t. $\alpha \notin B_i \quad \forall i$.

i.e. B_1, \dots, B_k are the connected components of $B \setminus \alpha$.

In our case (D : Dynkin); $k=1, 2$, or 2 .

$\nexists \quad \bigcup B_i = B$ contradicts (i)

$|B \setminus \bigcup B_i| \geq 2$ contradicts maximality of \mathcal{F} . \square

Cor $|\mathcal{F}| = \dim \mathfrak{g}^* = |D|$

Def Let $\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}} = \mathcal{H}^*$.

$\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}}$ is called an adapted family if for every

maximal nested set \mathcal{F} ; $B \in \mathcal{F}$, the set

$\{x_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \subseteq B}}$ is a basis of $\mathcal{H}_B^* = \text{Span}(B) \subset \mathcal{H}^*$.

(in particular, $\{x_B\}_{B \in \mathcal{F}}$ is a basis of \mathcal{H}^*).

According to the Lemma, all we are requiring is that

$$x_B \in \mathcal{H}_B^* \setminus \left(\bigoplus_{i=1}^k \mathcal{H}_{B_i}^* \right).$$

e.g. (1) $x_B = \sum_{\alpha \in B} \alpha$ gives an adapted family.

$$(2) B \subset D \rightsquigarrow R_B = \mathcal{H}_B^* \cap R,$$

$$x_B = \sum_{\alpha \in (R_B)_+} \alpha \quad \text{also works}$$

eg $(D = \overline{1 \ 2 \ 3})$

$$X_1 = \alpha_1, \quad X_2 = \alpha_2, \quad X_{\overline{1 \ 2}} = \alpha_1 + \alpha_2, \quad \text{et cetera.}$$

$$\mathcal{F} = \{ \cdot \ ; \ \overline{1 \ 2} \ ; \ \overline{1 \ 2 \ 3} \}$$

\nearrow
 α_1

\uparrow
 B

$$\{X_1, X_{\overline{1 \ 2}}\}$$

should be a basis
of $\mathbb{C}\alpha_1 + \mathbb{C}\alpha_2$.

Change of Coordinates (depends on m.n.s. \mathcal{F})

$$\begin{array}{ccc} \mathbb{C}^{\mathcal{F}} & \xrightarrow{\rho_{\mathcal{F}}} & \mathfrak{h}^* \\ \downarrow \psi & & \downarrow \psi \\ (\underline{u} = (u_B)_{B \in \mathcal{F}}) & \longmapsto & h \end{array} \quad \begin{array}{l} \text{(birational map)} \\ \text{is uniquely determined} \end{array}$$

$$\text{by } \chi_B(h) = \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C \quad (\forall B \in \mathcal{F}).$$

$$\chi_B = \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C \quad \text{can be inverted as}$$

$$\chi_D \quad \text{if } B = D$$

$$u_B = \begin{cases} x_D & \text{if } B = D \\ \frac{x_B}{x_{c(B)}} & \text{o.w.} \end{cases}$$

where $c(B)$ = minimal element of \mathcal{F} which properly contains B .

eg $B_2 \quad R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$

$$\begin{matrix} \circ & \leftarrow & \circ \\ i & & j \end{matrix}$$

Adapted family : $x_1 = \alpha_1, x_2 = \alpha_2,$

$$x_{i \rightarrow j} = \alpha_1 + \alpha_2$$

$$\mathcal{F} = \left\{ \begin{matrix} i & ; & i \rightarrow j \end{matrix} \right\}$$

$$\begin{array}{ccc} \rightsquigarrow & \mathbb{C}^2 & \xrightarrow{\quad} \mathcal{H}^* \\ \downarrow & \downarrow & \downarrow \\ (u_1, u_2) & \xrightarrow{\quad} & h \\ \downarrow & & \text{s.t.} \\ u_{1 \rightarrow 2} & & \begin{aligned} (\alpha_1 + \alpha_2)(h) &= u_2, \\ \alpha_1(h) &= u_1 \cdot u_2. \end{aligned} \end{array}$$

$$\alpha_1 = u_1 \cdot u_2, \quad \alpha_1 + \alpha_2 = u_2$$

$$u_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}, \quad u_2 = \alpha_1 + \alpha_2.$$

Pull-back of a root (\mathcal{F} fixed M.N.S., $\{X_B\}_{\substack{B \in \mathcal{D} \\ \text{connected}}}$: adapted)

$\alpha \in R$, express $\mathbb{C}^{\mathcal{F}} \xrightarrow{\rho_{\mathcal{F}}} \int^* \xrightarrow{\alpha} \mathbb{C}$ in u variables

Pick $B \in \mathcal{F}$ minimal s.t. $\alpha \in R_B$.

$\{X_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \leq B}}$ is a basis of \int_B^* ,

we can write $\alpha = \sum_{\substack{B' \in \mathcal{F} \\ B' \leq B}} a_{B'} \cdot X_{B'}$,

$a_{B'} \in \mathcal{F}$, $a_B \neq 0$

$$\alpha = a_B \underbrace{X_B}_{\substack{\prod_{\substack{C \in \mathcal{F} \\ B \in C}} u_C}} \left(1 + \sum_{\substack{B' \in \mathcal{F} \\ B' \neq B}} \frac{a_{B'}}{a_B} \cdot \underbrace{\frac{X_{B'}}{X_B}}_{\substack{\prod_{\substack{C \in \mathcal{F} \\ B' \in C \neq B}} u_C}} \right)$$

Prop $\forall \alpha \in R$, \exists a polynomial $P_{\alpha}(u)$ s.t.

• $\alpha = a_B \prod_{\substack{C \in \mathcal{F} \\ B \in C}} u_C \cdot P_{\alpha}(u)$

• P_{α} depends only on $\{u_{\sim}\}$

• P_α depends only on $\{u_{B'}\}_{\substack{B' \in \mathcal{H} \\ B' \subseteq B}}$

• $P_\alpha(\emptyset) = 1$

B_2 example $(x_1 = \alpha_1, x_2 = \alpha_2, x_{1-2} = \alpha_1 + \alpha_2)$

$$\mathcal{H} = \{i; i-i\} \rightsquigarrow \begin{aligned} \alpha_1 &= u_1 \cdot u_2 \cdot 1 \\ \alpha_1 + \alpha_2 &= u_2 \cdot 1 \end{aligned}$$

$\nwarrow P_{\alpha_1}$
 $\nwarrow P_{\alpha_1 + \alpha_2}$

$$2\alpha_1 + \alpha_2 = u_2 \underbrace{(1 + u_1)}_{P_{2\alpha_1 + \alpha_2}}$$

$$\alpha_2 = u_2 \underbrace{(1 - u_1)}_{P_{\alpha_2}}$$