Lec 9/8

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$T: F^n \longrightarrow F^m$

$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

$x = x_1 e_1 + \cdots + x_n e_n$

$T(x) = x_1 T(e_1) + \cdots + x_n T(e_n)$

$T(x) = y$

$T(e_i) = \sum_{j=1}^{m} \lambda_{ij} e_j$ for $1 \le i \le n$

$$T(x) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \lambda_{ij} e_j$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \lambda_{ij} x_i \right) e_j$$

so $y_j = \sum_{i=1}^{n} \lambda_{ij} x_i$ for $1 \le j \le m$

this looks like a system of equations,

which $\Leftrightarrow$ T being linear if $T: F^n \to F^m$

$L(V)$ is a ring (non commutative)

$V = F^2$. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ $S\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

$(T \circ S)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$ $(S \circ T)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ not the same

$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ has no inverse.

T has an inverse iff it is bijective

$T^{-1}(w_1 + w_2) \stackrel{?}{=} T^{-1}(w_1) + T^{-1}(w_2)$

$v' = T^{-1}(\lambda w)$

$v = T^{-1}(w)$

$T^{-1}(\lambda w) \stackrel{?}{=} \lambda T^{-1}(w)$.

so $v' = \lambda v$ since T is injective.

so $T^{-1}(\lambda w) = \lambda T^{-1}(w)$ if T^{-1} exists. $\Rightarrow T(v') = \lambda w = \lambda T(v) = T(\lambda v)$

$v' = T^{-1}(w_1) + T^{-1}(w_2)$

$v = T^{-1}(w_1 + w_2)$

$T(v) = w_1 + w_2$

$T(v') = T(T^{-1}(w_1)) + T(T^{-1}(w_2)) = w_1 + w_2$

so $v' = v$ and $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$ if T^{-1} exists

Prop. $T \in L(V)$ is invertible iff T is bijective.

$[T^{-1} \circ T = I = T \circ T^{-1}, \quad T^{-1} \in L(V)]$.

Proof above.

Remark: $T : V \to W$ is injective (1-1) iff $T(v) = 0$ implies $v = 0$.

Proof $\Rightarrow$ obvious $T(v) = 0 = T(0) \to v = 0$.

$\Leftarrow$ $T(v_1) = T(v_2) \Rightarrow T(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$. ∎

Vectors satisfying $T(v) = 0$ is null space $\mathcal{N}$ of T.

Injectivity $\Leftrightarrow$ $\mathcal{N}ull(T) = \{0\}$.

Prop 3: Let $T \in L(V)$ s.t. T injective. Then T bijective.

Proof Let $\{v_1, \dots, v_n\}$ be a basis of V.

then $\{T(v_1), \dots, T(v_n)\}$ is lin. indp:

if $\lambda_1 T(v_1) + \cdots + \lambda_n T(v_n) = 0$

then $T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = 0$

so $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$

so $\lambda_i = 0 \quad \forall i$.

but there are n so they generate V

so $v \in V = \mu_1 T(v_1) + \cdots + \mu_n T(v_n)$

$= T(\mu_1 v_1 + \cdots + \mu_n v_n)$

meaning T surjective.

(Can replace $L(V)$ by $L(V, W)$ as long as $\dim V = \dim W$). ∎

Not true when V

, W infinite dimensional.