Lec 9/13

Wednesday, September 13, 2017 14:22

$SL(2,\mathbb{Z})$ = 2x2 invertible integer matrices (det 1)

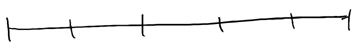
$\cap$ preserve Lattices in $\mathbb{R}^2$

$SL(2,\mathbb{R})$

$\mathbb{Z}_2$

$\uparrow$ example of a Lattice in $\mathbb{R}^2$

Lattice in $\mathbb{R}^2$: 2 linearly independent vectors & their integer multiples.



$\mathbb{R}/\mathbb{Z}$

Check: Definition of coset in subgroup

$G \supset H$, aH, Hb are L/R cosets.

$\mathbb{R} \supset \mathbb{Z}$, $\mathbb{R} \supset \mathbb{Z}+t$ for any t.

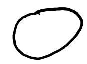
$\bigcup_{t\in[0,1)} (\mathbb{Z}+t) = \mathbb{R} \longrightarrow$  0 mod 1 = 1 mod

General Result: (Exercise)

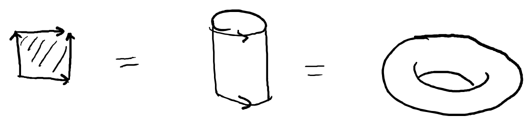
Any Group G with a subgroup H is representable as a disjoint union of cosets of H:

$$G = \bigcup_{a\in I_a} Ha = \bigcup_{b\in I_b} bH \quad \text{for some } I_a, I_b < G$$

Hint: $\forall a_1, a_2 \in G$, either $a_1H \cap a_2H = \emptyset$ or $a_1H = a_2H$.

$$\mathbb{R}/\mathbb{Z} \approx \underset{\text{1 dim torus}}{\mathbb{T}} = S^1 =$$ 

$$\mathbb{R}^2/\mathbb{Z}^2 = \bigcup_{(t_1,t_2)\in[0,1)\times[0,1)} (\mathbb{Z}^2 + (t_1,t_2))$$

$$\underbrace{}_{\mathbb{T}^2} =$$ 

G/H is the set of cosets of H in G.

$\mathbb{Z}/p\mathbb{Z}$ is a finite group

take Γ instead of $\mathbb{Z}$, → $m\mathbb{Z}$ replaced by a Lattice. Factor & see when you get a field.

Integral domain has no divisors of 0.

<u>Fact</u>: every vector space has a basis.

$\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Question: What is cardinality of a basis in $\mathbb{R}/\mathbb{Q}$

- Not finite o/w $\mathbb{R}$ ctble
- Not ctble o/w $\mathbb{R}$ ctble (countable union of $\overbrace{\text{n-tuples of } \mathbb{Q}}^{\text{countable}}$)
- Uncountable, not cardinality > $|\mathbb{R}|$ so $\aleph_1$

$\mathrm{Span}_{\mathbb{Q}}(B)$ is finite linear combinations of vectors in B w/ coeffs in $\mathbb{Q}$.

Hamel basis is uncountable

Let H be a basis in $\mathbb{R}_{\mathbb{Q}}$ → $\mathbb{R}$ vs over $\mathbb{Q}$

$\forall x \in \mathbb{R} \quad x = \sum a_i h_i \qquad \{h_i\} = \ldots$

$\forall x \in \mathbb{R}, \quad x = \sum_{\alpha \in I} r_\alpha h_\alpha \qquad \{h_\alpha\}_{\alpha \in I} = H.$

$\mathbb{Q} \ni r_\alpha \neq 0$ for only finitely many α.

Hamel Bases are either measure 0 or is immeasurable

Can integers be products of two small sets?

(Exercise) is $\{n \in \mathbb{N} : n > N\} = AB$ for two density 0 sets A, B?

(exercise) $\{n_1^2 + n_2^2 : n_1, n_2 \in \mathbb{N}\}$ has 0 density

Hint: primes 4n+1 are in this set, what about others?
↳ see handout 1.

$f : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$

$f(x+y) = f(x) + f(y)$

$f(x) = 17x \quad \forall 17$

Fact: if $f(x+y) = f(x) + f(y)$ where $f : \mathbb{R} \to \mathbb{R}$ is cts,
then $f(x) = cx$ for some $c \in \mathbb{R}$

monotone
measurable

But there are uncountably many other non-cts examples.

$x = \sum_{\alpha \in I} r_\alpha h_\alpha$

$f(x) = \sum_{\alpha \in I} r_\alpha$ check to see this isn't cx, since it takes no irrational values.