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K/F is never a quotient field

K/F is an extension of fields, $F \subseteq K$.

If K/F is an extension, then K is an F-vector space

$\dim_F K$ is called the degree of the extension,

and is denoted $[K:F]$.

If $[K:F] < \infty$, we say that K/F is a finite extension.

Quadratic, cubic, quartic extensions — of degrees 2, 3, 4 resp.

Theorem: Let $E/K/F$ be a tower of extensions ($F \subseteq K \subseteq E$).
Let B be a basis of K/F and C be a basis of E/K. Then $CB = \{\gamma\beta : \gamma \in C, \beta \in B\}$ is a basis of E/F.

Corollary: if E/K, K/F are finite, then

$[E:F] = [E:K] \cdot [K:F] < \infty$ and E/F is finite.

$\{\beta_1, \dots, \beta_n\}$ is basis of K/F

$\{\gamma_1, \dots, \gamma_m\}$ is basis of E/K

$\Rightarrow$ $\{\gamma_1\beta_1, \gamma_2\beta_1, \gamma_1\beta_2, \dots, \gamma_m\beta_n\}$ is a basis of E over F.

<u>Proof</u>: Let $\alpha \in E$. Then $\alpha = \sum_{i=1}^{k} a_i \gamma_i$ for some $\gamma_1, \dots, \gamma_k \in C$, $a_i \in K$.

$\forall i$, $a_i = \sum_{j=1}^{l_i} b_{ij} \beta_j$ for some $\beta_j \in B$, $b_{ij} \in F$.

Then $\alpha = \sum_i \left(\sum_{j=1}^{l_i} b_{ij}\beta_j\right)\gamma_i = \sum_{i,j} b_{ij}\gamma_i b_j$, $b_{ij} \in F$.

So BC generates E/F.

Now Assume that $\sum b_{ij} \beta_i \gamma_j = 0$.

Then $\sum_j \left(\sum_i b_{ij}\beta_i\right)\gamma_i = 0$.

So each $\sum_i b_{ij}\beta_i = 0$.

So each $b_{ij} = 0$.

$$\left.\begin{array}{c} E \\ {\scriptstyle m}\,| \\ K \\ {\scriptstyle n}\,| \end{array}\right) mn$$

$$\begin{array}{c} K \\ n\,\Big| \\ F \end{array} \quad \cdots$$

<u>Corollary</u>: If $E/K/F$ and $[E:F] < \infty$, then

$$[E:K],\ [K:F] \ \Big|\ [E:F].$$

$$\begin{array}{c} K_r \\ n_r\,\Big| \\ \vdots \\ \Big| \\ K_2 \\ n_2\,\Big| \\ K_1 \\ n_1\,\Big| \\ F \end{array} \qquad \Rightarrow \qquad [K_r : F] = n_1 \cdot n_2 \cdot \cdots \cdot n_r .$$

subset

If K/F is an extension, $S \subseteq K$,

then $F(S)$ is the subextension of K/F generated by S. It's the minimal subfield of K containing $F \cup S$.

If $K = F(S)$ for finite S, we say that

K is finitely generated.

Compose $K_1 K_2 = K_1(K_2) = K_2(K_1)$

An extension of the form $F(\alpha)/F$ (generated by $\overset{\text{one element}}{\overset{\downarrow}{\alpha}}$)

is called simple.

Let K/F be extension, $\alpha \in K$. Consider $F(\alpha)$.

we have a hom-sm $\begin{aligned} F[x] &\xrightarrow{\varphi} F(\alpha) \\ x &\longmapsto \alpha \end{aligned}$ of rings (of F-algebras).

$$\varphi|_F = id_F$$

$$\varphi(f(x)) = f(\alpha).$$

Let $I = \ker \varphi$.

Case 1: $I \neq 0$ (that is, $\exists \overset{\not{x}^0}{f} \in F[x]$ s.t $f(\alpha) = 0$)

Then $F[x]/I \cong$ a subring of K

Since K has no zero-divisors, I must be prime.

$F[x]$ is a pid, so $I = (m_\alpha)$, m_α is an irreducible polynomial.

and (m_α) is maximal. Then $F[x]/I$ is a field.

So, $\varphi(F[x]) \subseteq K$ is a subfield of K.

So $\varphi(F[x]) = F(\alpha)$.

So $F(\alpha) = F[\alpha] \cong F[x]/(m_\alpha)$

m_α is called the minimal polynomial of α.

$[F(\alpha) : F] = \dim_F \left(F[x]/(m_\alpha) \right) = \deg(m_\alpha)$.

$\varphi(m_\alpha) = 0$ so $m_\alpha(\alpha) = 0$, and

$f(\alpha) = 0$ iff $\alpha \in \operatorname{Ker} \varphi = I$ iff $m_\alpha \mid f$.

All this is true if $f(\alpha) = 0$ for some $f \in F[x]$, $f \neq 0$

In this case, α is <u>algebraic</u> over F.

<u>Examples</u>:
① $F = \mathbb{Q}$, $\alpha = \sqrt{2}$ - algebraic since

$f(\sqrt{2}) = 0$ where $f(x) = x^2 - 2$.

$\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}$.

$$\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

② $\mathbb{Q}$, $\sqrt[3]{2} = \alpha$, $\mathbb{Q}(\alpha) = \{ a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q} \}$.

$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

③ $\mathbb{R}$, $\alpha = i$, $m_i = x^2 + 1$, $[\underset{\mathbb{R}(i)}{\underset{\parallel}{\mathbb{C}}} : \mathbb{R}] = 2$.

④ $\mathbb{Q}$, $\alpha = \pi$ - not algebraic (transcendental) over $\mathbb{Q}$

If α is algebraic over F, then $\deg_F \alpha = \deg m_\alpha = [F(\alpha) : F]$

$\alpha = \sqrt{2} + \sqrt{3}$

$\alpha^2 = 2 + 3 + \sqrt{6}$ so $(\alpha^2 - 5)^2 = 24$ so $\alpha^4 - 10\alpha^2 + 1 = 0$.

So $f(\alpha) = 0$ for $f(x) = x^4 - 10x^2 + 1$.

Is $f = m_\alpha$? If f is irreducible, $m_\alpha \mid f \Rightarrow f = m_\alpha$.

$K = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{ a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q} \}$

$(\mathbb{Q}(\sqrt{2}))(\sqrt{3})$ $\{1, \sqrt{2}\} \cdot \{1, \sqrt{3}\}$

$\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

either $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, then $\deg_{\mathbb{Q}} \alpha = 4$ and $f = m_\alpha$

or $\mathbb{Q}(\alpha) \subsetneq \mathbb{Q}(\sqrt{2}, \sqrt{3})$, then $\deg_{\mathbb{Q}} \alpha = 2$.

α acts on K by multiplication: $\beta \longmapsto \alpha\beta$

it has a matrix. the minimal pol-l of the matrix is the same as m_α.