Rec 11/19

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Prime Avoidance Lemma

Let R be a commutative ring and $P_1, P_2, \dots, P_n$ be ideals such that at most 2 are not prime. If there is an ideal $I \subseteq \bigcup_{i=1}^n P_i$, Then $I \subseteq P_i$ some i.

Contrapositive: if $I \not\subseteq P_i$ for some i, then $I \not\subseteq \bigcup_{i=1}^n P_i$.

(McCoy: $f \in R[x]$ s.t. $fg = 0$ for some $g \in R[x]$, then $\exists c \in R$ s.t. $cf = 0$).

Pf sketch: Induction. Base case is obvious.

Assume if $I \not\subseteq P_i$ for all $i \in \{1, \dots, n\}$ then $I \not\subseteq \bigcup_{i=1}^n P_i$.

Suppose $I \not\subseteq P_i$ for all $i \in \{1, \dots, n+1\}$. Consider $\bigcup_{j \neq i}^{n+1} P_j$.

This is a union of n prime ideals. We know $I \not\subseteq P_j$ by assumption. By ind. hyp. Pick $x_i \in I \setminus \bigcup_{j \neq i}^{n+1} P_j$ for each i.

If $x_k \notin P_k$ for some k, we are done since $x_k \in I \setminus \bigcup_{j=1}^{n+1} P_j$.

Otherwise, let $x = x_1 \cdots x_n + x_{n+1}$. Suppose $x \in P_i$ for some i.

case 1: $i < n+1$: $x_{n+1} = x - x_1 \cdots x_n \in P_i$. contradiction since $i \neq n+1$.

case 2: $i = n+1$: $x_1 \cdots x_n = x - x_{n+1} \in P_{n+1}$.

since P_{n+1} is prime, some $x_i \in P_{n+1}$, a contradiction

(if $n=1$, the product is x_1 so it's ok if P_{n+1} isn't prime). □

Thm: If R is commutative & $M \neq 0$ is f.g. R-module, $S \subseteq Z(M)$,

Then $\exists$ a single $m \in M$ s.t. $Sm = 0$.

"In a module over a noetherian ring, every ideal of zero divisors is contained in the annihilator of a single element"

where $Z(M) = \{ r \in R \mid \exists x \in M \setminus 0 \text{ w/ } rx = 0 \}$.