Lec 9/13

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Prop $f, g : (X, \mathcal{M}) \longrightarrow \overline{\mathbb{R}}$ measurable

The following are mble:

① $f \vee g$ $\longleftarrow$ so is $\bigvee_{n \in \mathbb{N}} f_n$

② $f \wedge g$ $\longleftarrow$ so is $\bigwedge_{n \in \mathbb{N}} f_n$

(f_n) seq of mble fns.

Can also take limsup & liminf by these two $\overset{\shortparallel}{\inf_n \sup_{k>n}}$ $\overset{\shortparallel}{\sup_n \inf_{k>n}}$

③ any well-defined linear combination:

- $C \in \mathbb{R} \Rightarrow C \cdot f$ mble.
- suffices to show: if $f+g$ well-defined, $f+g$ mble

Pf $\{f+g > a\} = \bigcup_{\substack{r,s \in \mathbb{Q} \\ r+s>a}} \left[\{f > r\} \cap \{f > s\}\right]$ $\square$

④ fg

Pf Step1: Assume $f \geq 0$, $g \geq 0$. Then $\forall a \geq 0$,

$\{fg > a\} = \bigcup_{\substack{r,s \in \mathbb{Q} \\ r,s>0 \\ rs>a}} \left[\{f>r\} \cap \{g>s\}\right]$. $\forall a < 0$, $\{fg > a\} = X$.

Step2: for f, g arbitrary, can do:

Trick: $f = f_+ - f_-$ where $f_+ = \underbrace{\sup\{f, 0\}}_{f \vee 0}$, $f_- = -\underbrace{\inf\{f, 0\}}_{-(f \wedge 0)}$

$\operatorname{supp} f := \{f \neq 0\}$

$\text{supp} f_+ \cap \text{supp} f_- = \emptyset$.

$$fg = \underbrace{f_+ g_+ - f_+ g_- - f_- g_+ + f_- g_-}$$

all disjoint supports
- each is mble by step 1.
- apply 3.

$\square$

<u>Simple functions</u>: Let $(X, \mathcal{M}, \cancel{\mu})$ be a fixed measure[-able] space.

<u>Defn</u> an $\mathcal{M}$-mble fn $\psi : X \to \mathbb{R}$ is simple if it takes finitely many values:

$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$$

- can require $c_1, \dots, c_n$ are distinct and $\underbrace{\text{nonzero}}_{\text{choice}}$.
- and $E_1, \dots, E_n$ disjoint & nonempty.

— then this expression is unique.

<u>Book says</u>: $c_1, \dots, c_n$ distinct, $X = \coprod_1^n E_k$, $E_k \neq \emptyset$.

(also get unique expression).

<u>Remark</u>: The simple fns $SF = SF(X, \mathcal{M})$ form a

- $\mathbb{R}$-algebra $\quad (\chi_E \chi_F = \chi_{E\cap F})$.

- Lattice $\quad \chi_E \vee \chi_F = \chi_{E\cup F}$

$$\chi_E \wedge \chi_F = \chi_{E\cap F} = \chi_E \chi_F$$

<u>Define</u> $SF^+ = \{\varphi \in SF \mid \varphi \geq 0\}$

Observe SF^+ is:

Closed under $+, \cdot, r\cdot$ where $r \geq 0$

"$\mathbb{R}_{\geq 0}$-algebra"

- Positive <u>cone</u> in SF
- closed under $\cdot$
- sublattice

<u>Prop</u>: Suppose $f: (X, \mathcal{M}) \longrightarrow [0, \infty]$.

$\exists$ seq $(\varphi_n) \subset SF^+$ s.t.

- $\varphi_n \leq \varphi_{n+1} \ \forall n$,
- $\varphi_n \leq f \ \forall n$,
- $\forall M > 0$, $\varphi_n \to f$ <u>uniformly</u> on $\{f \leq M\}$

Proof: for $n \geq 0$, $1 \leq k \leq 2^{2n}$,

Define $E_n^k := \left\{ \frac{k-1}{2^n} < f \leq \frac{k}{2^n} \right\}$,

$$F_n := \{ f > 2^n \}$$

$$\psi_n := 2^n \chi_{F_n} + \sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} \chi_{E_n^k}$$

ψ_n approximates f, on $\{ f \leq 2^n \}$ within $\frac{1}{2^n}$. $\square$

from below

Integration of nonnegative fns:

fix measure space $(X, \mathcal{M}, \mu)$.

$$L^+ := L^+(X, \mathcal{M}, \mu)$$

$$:= \{ \mathcal{M}\text{-meas. } f : X \longrightarrow [0, \infty] \}$$

Defs for $\psi = \overbrace{\sum_{k=1}^{n} \underset{\substack{\uparrow \\ \text{distinct}}}{c_k} \underset{\substack{\uparrow \\ \text{nonempty, disjoint}}}{\chi_{E_k}}}^{\text{unique repn}} \in SF^+ \subset L^+$,

define $\int \psi = \int_X \psi \, d\mu = \int_X \psi(x) \, d\mu(x)$

$$:= \sum_{k=1}^{n} c_k \mu(E_k) \in [0, \infty].$$