Lec 9/12

Monday, September 12, 2016 9:11 AM

Definition 1 We say that f is continuous at $a \in dom(f)$

bad good

if, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$|x-a|<\delta$ and $x \in dom(f)$ $\Rightarrow$ $|f(x)-f(a)|<\varepsilon$ and $x \in dom(f)$

Definition 2 We say that $\lim_{x \to a} f(x) = L$ if for any $\varepsilon > 0$ there

bad good

is a $\delta > 0$ such that

$0<|x-a|<\delta$ and $x \in dom(f)$ $\Rightarrow$ $|f(x)-L|<\varepsilon$ and $x \in dom(f)$

Intermediate Value Theorem

if $f:[a,b] \to \mathbb{R}$ continuous on $[a,b]$

then f takes on every value in $[f(a), f(b)]$

Note: With bad definition, no function satisfies this condition, so the theorem is useless. (function would not be cts at endpoints).

let $f:[0,\infty] \to \mathbb{R}$ such that $f(x) = x^{3/2} = (\sqrt{x})^3$

with bad definition, we can say that $\lim_{x \to -1} f(x) = \pi$.

$\Rightarrow$ for any $\varepsilon > 0$, take $\delta = 1$:

$$\underbrace{0<|x-(-1)|<1 \quad \& \quad x \in dom(f)}_{\Updownarrow \; x \in \emptyset} \Longrightarrow |f(x)-\pi|<\varepsilon$$

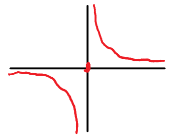
hypothesis always false so implication holds

Understanding discontinuity

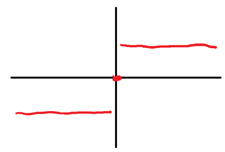
Suppose $a \in dom(f)$. What does it mean for f to be discontinuous at a?

Take $a=0$. look at examples of functions discts at 0.

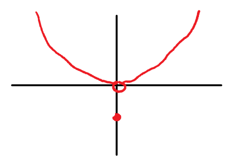
1) $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



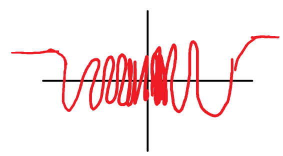
2) $\text{signum}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



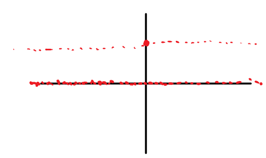
3) $g(x) = \begin{cases} x^2 & x \neq 0 \\ -1 & x = 0 \end{cases}$



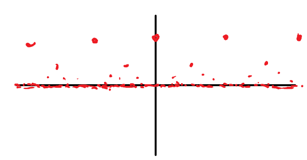
4) $h(x) = \begin{cases} \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



5) $\chi(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{o.w.} \end{cases}$



6) $\text{pop}(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, \ p/q \text{ ints with no common factors} \\ 0 & \text{otherwise} \end{cases}$



in all examples, there is some kind of jump in the graph near $x=0$.

jump interval at $x=0$ for

(1) is $(-\infty, \infty) \times [-\delta, \delta]$

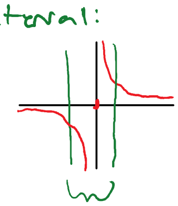
(2) is $[-1, 1]$

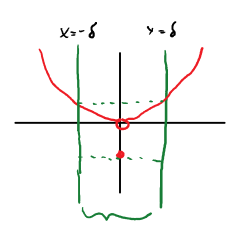
(3) is $[-1, 0]$

(4) is $[-1, 1]$

(5) is $[0, 1]$

Jump interval:





(5) is $[0,1]$

(6) is $[0,1]$

how much does the graph extend if we keep squishing this strip

Measuring discontinuity of a function at 0.

Definition Let $J_{f,0,\delta}$ $(\delta > 0)$ = smallest closed interval containing $\{f(x) : -\delta \le x \le \delta,\ x \in \mathrm{dom}(f)\}$

if $\delta' < \delta$ then $J_{f,0,\delta'} \subseteq J_{f,0,\delta}$

Jump interval at 0 is "$\lim_{\delta \to 0} J_{f,0,\delta}$" $= \bigcap_{\delta > 0} J_{f,0,\delta} = J_{f,0}$

Consider prev. examples.

(1). $J_{f,0,\delta} = (-\infty, \infty)$ $\forall \delta > 0$ $\qquad J_{f,0} = (-\infty, \infty)$

(2). $J_{\text{signum},0,\delta} = [-1, 1]$ $\forall \delta > 0$ $\qquad J_{\text{signum},0} = [-1, 1]$

(3). $J_{g,0,\delta} = [-1, \delta^2]$ $\qquad J_{g,0} = \bigcap_{\delta > 0} J_{f,0,\delta} = [-1, 0]$

(4). $J_{h,0,\delta} = [-1, 1]$ $\forall \delta > 0$ $\qquad J_{h,0} = [-1, 1]$

(5). $J_{x,0,\delta} = [0,1]$ $\forall \delta > 0$ $\qquad J_{x,0} = [0, 1]$

(6) $J_{\text{pop},0,\delta} = [0,1]$ $\forall \delta > 0$ $\qquad J_{\text{pop},0} = [0, 1]$

if function is cts at 0, $J_{f,0} = \{f(0)\}$

Theorem let $\{J_\alpha\}$ be an arbitrary collection of closed intervals. Then $\bigcap_\alpha J_\alpha$ is one of the following:

(1) a closed interval

(2) a single point

(3) the empty set

$[0,1] \cap [1,2] = \{1\}$ $\qquad \bigcap_{n \geq 1} [0, \frac{1}{n}] = \{0\}$ $\qquad \bigcap_{n \geq 0} [n, \infty) = \emptyset$