$N \subseteq M \Rightarrow \mathrm{Ann}(N) \subseteq M^*$

$\varphi: R^n \longrightarrow R^m$

$\tilde{\varphi}: F^n \longrightarrow F^m$

$M \cong R^n$ $\{u_1, \dots, u_n\}$

$V = F \otimes M \cong F^n$ $\downarrow$ same basis since dim is same

and $1 \otimes u_1, \dots, 1 \otimes u_n$ span V.

$\tilde{\varphi} = \mathrm{Id}_F \otimes \varphi$, same matrix

$\mathrm{rank}\,\varphi := \mathrm{rank}(\varphi(M))$

$= \text{column rank of } A_\varphi.$

$$\begin{array}{ccc} \Bigg(& \Bigg| \quad \Bigg| \quad \Bigg| & \Bigg) \\ & \varphi(u_1) \dots \varphi(u_n) & \end{array}$$

(columns of A_φ generate $\varphi(M)$).

$\text{row rank } A_\varphi = \text{column rank } A_{\varphi^*}$

M, N free of finite rank.

$\{u_1, \dots, u_m\}$ basis in M, $\{v_1, \dots, v_n\}$ basis in N.

$$M \otimes N \cong R^m \otimes R^n \cong R^{mn} \quad \text{w/ basis } \left\{ u_i \otimes v_j : \begin{smallmatrix} 1 \le i \le m \\ 1 \le j \le n \end{smallmatrix} \right\}.$$

$\forall\, w \in M \otimes N$, $w = \sum a_{ij}\, u_i \otimes v_j$ uniquely.

these coords form a matrix:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$M \otimes N \otimes K$ basis $\{ u_i \otimes v_j \otimes w_\ell \}$, coordinates of a tensor form a 3-dimensional matrix

$$w = \sum a_{ij\ell}\, u_i \otimes v_j \otimes w_\ell$$

eg $\Gamma_k^{i,j}$ - Cristoffel's Symbols

covector

$$M^* \otimes N \otimes K^* \quad \text{basis} \quad \{ f_i \otimes v_j \otimes g_\ell \}$$

vector

So coordinats here would be $a^j_{i,\ell}$

$$w = \sum_{i,j,\ell} a^j_{i,\ell}\, f^i \otimes v_j \otimes g^\ell$$

$\exists$ natural homomorphism

$$N \otimes M^* \xrightarrow{\Phi} \mathrm{Hom}(M, N)$$

$$v \otimes f \longmapsto \varphi \quad \text{where } \varphi(u) = f(u) \cdot v$$

if N, M are free of finite rank, then this is an isomorphism.

If $\{u_1, \dots, u_m\}$ is a basis in M which gives $\{f_1, \dots, f_m\}$ a basis in M^*, and $\{v_1, \dots, v_n\}$ is a basis in V,

Then $\{v_i \otimes f_j\}$ is a basis in $N \otimes M^*$

And $\Phi(v_i \otimes f_j)(u_k) = f_j(u_k) \cdot v_i = \begin{cases} v_i & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$,

and so $\{\Phi(v_i \otimes f_j)\}$ is the natural basis for $\mathrm{Hom}(M, N)$.

So Φ is an isomorphism which is <u>very</u> natural.

So homomorphisms from M to N are tensors in $N \otimes M^*$

$$\underset{\substack{\cap \\ N\otimes M^*}}{\left(\sum a_{ij}\, v_i \otimes f_j\right)} \underset{\substack{\cap \\ M}}{\left(\sum b_\ell u_\ell\right)} = \sum_i \left(\sum_j a_{ij} b_j\right) v_i = \begin{pmatrix} \sum a_{1j} b_j \\ \vdots \\ \sum a_{nj} b_j \end{pmatrix}$$

$$M^* \otimes M \longrightarrow R \quad \text{contraction}$$

$$f \otimes u \longmapsto f(u)$$

$$\left(\sum c_i f_i\right) \otimes \left(\sum a_i u_i\right) = \sum c_i a_i$$

$$N \otimes M^* \otimes M \longrightarrow N$$

$$(a_{ij\ell}) \longmapsto \left(\sum_j a_{ijj}\right)$$

$$\cdots \otimes M \otimes \cdots \otimes M^* \otimes \cdots$$

$$\sum_{\cdots i = i}$$

$$\underset{\substack{\cap \\ N\otimes M^*}}{\varphi} : M \longrightarrow N, \quad \underset{\substack{\cap \\ K \otimes N^*}}{\psi} : N \longrightarrow K$$

$\psi \circ \varphi \in K \otimes M^*$

by $\quad K \otimes N^* \otimes N \otimes M^* \xrightarrow{\text{Contract}} K \otimes M^*$

$\quad w \otimes g \qquad v \otimes f$

$u \in M \quad (w \otimes g)\Big((v \otimes f)(u)\Big) = w \otimes g\,(f(u) \cdot v) = f(u)\, g(v) \cdot w$

$K \otimes M^* : \quad u \longmapsto g(v)\, \underbrace{f(u) \cdot w}_{(w \otimes f)(u)}$

$(w \otimes g) \otimes (v \otimes f) \xmapsto{\text{Contract}} g(v)\; w \otimes f$

$\varphi : M \longrightarrow N \quad \Longrightarrow \quad \varphi^* : N^* \longrightarrow M^*$

$\varphi \in N \otimes M^* \quad , \ \varphi^* \in M^* \otimes N^{**} = M^* \otimes N$

φ^* is (probably) obtained from φ by transposing components.

$\underbrace{M^* \otimes M^*}_{\text{bilinear mapping.}} \otimes M \otimes M \xrightarrow{\text{contract}} R$

$$(\cdots)\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}\begin{pmatrix} \vdots \end{pmatrix}$$

$$(u,v) \longmapsto R$$

$$(f \otimes g)(u,v) = (f \otimes g)(u \otimes v) = f(u)\,g(v)$$

$M^* \otimes \cdots \otimes M^*$ — multilinear forms