Lec 11/28

Monday, November 28, 2016 9:10 AM

Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = f(x) \qquad a: \text{ center}$$

constant

Trivial example: polynomial: (consists of a finite # terms, no convergence issues)

$p(x) = 3x^2 - 5x + 7 = 7 - 5(x-0) + 3(x-0)^2 + 0(x-0)^3 + \dots$

Can recenter it to -2. Let $u = x - a = x+2$, $x = u-2$

$$\begin{aligned} p(x) &= 3(u-2)^2 - 5(u-2) + 7 \\ &= 3u^2 - 17u + 29 \\ &= 29 - 17(x+2) + 3(x+2)^2 \end{aligned}$$

When does a power series converge?

Theorem Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series,

Let q be the convergence parameter of $\sum_{n=0}^{\infty} c_n$.

Then the convergence parameter q_x of $\sum_{n=0}^{\infty} c_n (x-a)^n$ is

$$q_x = \begin{cases} 0 & \text{if } x = a \\ q\,|x-a| & \text{if } q \neq \infty \\ \infty & \text{if } q = \infty,\ x \neq a. \end{cases}$$

Proof: q is the largest cluster point of $\left\{ |c_n|^{1/n} \right\}_{n=1}^{\infty}$

q_x is the largest cluster point of $\left\{ |c_n (x-a)^n|^{1/n} \right\}_{n=1}^{\infty}$

$$|c_n (x-a)^n|^{1/n} = |c_n|^{1/n}\,|x-a|$$

if $x = a$, all terms are 0 ✓

of this

if $q \neq \infty$, then cluster points are the same as those of this but multiplied by $|x-a|$

if $q = \infty$, then the divergent subsequence still diverges. ∎

Definition: The radius of convergence of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is

$$R = \begin{cases} \infty & \text{if } q = 0 \\ \frac{1}{q} & \text{if } 0 < q < \infty \\ 0 & \text{if } q = \infty \end{cases} \qquad R = \text{"}\frac{1}{q}\text{"}$$

Theorem Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$. Then,

(a) if $|x-a| < R$, the series converges absolutely $(q_x < 1)$

(b) if $|x-a| > R$, the series diverges $(q_x > 1)$

(c) if $0 < |x-a| = R < \infty$ the series may or may not converge. $(q_x = 1)$ absolutely/ conditionally.

(d) if $R = 0$, the series only converges if $x = a$, converges absolutely.

Note: $|x-a| < R \iff x \in (a-R, a+R)$.

Corollary The values of x for which a power series converges form an interval called the interval of convergence.

If $R = \infty$, then interval is $(-\infty, \infty)$

If $R = 0$, interval is degenerate: $[a, a] = \{a\}$

if $0 < R < \infty$, interval is open, closed, or half open w/ endpoints $a-R$ and $a+R$.

Example:

$\sum_{n=0}^{\infty} x^n$ geometric series, interval of convergence $= (-1, 1)$.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ interval of convergence $= [-1, 1)$. Cond @ -1, Ast., Harmonic/p-series.

$\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$ " " " $= (-1, 1]$. Cond @ +1

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ " " " $= [-1, 1]$. abs. @ ±1

absolute convergence @ one endpoint $\Rightarrow$ absolute convergence at both endpoints.

Mistake to avoid: What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$

$\hookrightarrow \quad c_n = \frac{1}{4^n} \Rightarrow c_n^{1/n} = \frac{1}{4}$ so $q = \frac{1}{4}$, $R = 4$.

actually, $c_n = 0$ if n odd, $\frac{1}{4^m}$ if $n = 2m$. so $c_n^{1/n} = \begin{cases} 0 & n \text{ odd} \\ \left(\frac{1}{4^m}\right)^{1/2m} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2} & n \text{ even.} \end{cases}$

so $q = \frac{1}{2}$, $R = 2$ (cluster pts are $0, \frac{1}{2}$).

More Reliable: Compute q_x directly w/ root or ratio test.

determine for which x $q_x < 1$

$$q_x = \lim_{n\to\infty} \left|\frac{x^{2n}}{4^n}\right|^{1/n} = \frac{x^2}{4} < 1 \iff |x|^2 < 4 \iff |x| < 2$$

Consistency question: $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n} = 1 + \frac{x^2}{4} + \frac{x^4}{16} + \frac{x^6}{64} + \cdots$

If we think of this as a special case of a general power series, then should think of it as $1 + 0x + \frac{x^2}{4} + 0x^3 + \frac{x^4}{16} + \cdots$

This converges "more slowly" than other series.

Proposition If a series $\sum_{n=0}^{\infty} a_n$ is obtained by interpolating 0s into a series $\sum_{n=0}^{\infty} b_n$, both series either converge or diverge.

Suppose $\sum_{n=0}^{\infty} a_n = b_0 + 0 + b_1 + 0 + \cdots$

Look at partial sums:

$S_0 = b_0$	$S_0' = S_0$	they both approach the same limit
$S_1 = b_0 + b_1$	$S_1' = S_0$	
$S_2 = b_0 + b_1 + b_2$	$S_2' = S_1$	
	$S_3' = S_1$	
$\vdots$	$\vdots$	$\downarrow$

Proposition let $\{S_n\}$ be a sequence and $f: \mathbb{N} \to \mathbb{N}$ and suppose that $\lim_{n\to\infty} f(n) = \infty$. If $\lim_{n\to\infty} S_n = L$, then $\lim_{n\to\infty} (S \circ f) = L$ as well.

$$(S \circ f) \,\|\, S_{f(1)}, S_{f(2)}, \ldots$$