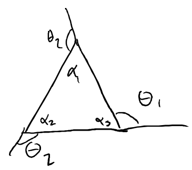
Lec 3/19

Monday, March 19, 2018 14:16



$\theta_i + \alpha_i = \pi$

$\theta_1 + \theta_2 + \theta_3 = 2\pi$

so $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.

Let $\gamma$ be a loop in $\mathbb{C}$.

This means $\gamma : S^1 \xrightarrow{\text{cts}} \mathbb{C}$

Define $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ by $\alpha(t) = \gamma(e^{it})$.

Then $\alpha$ is cts & $2\pi$-periodic.

To say $\gamma$ is $C^1$-regular means $\alpha$ is $C^1$-regular.

Suppose $\gamma$ is $C^1$-regular. Then $\alpha' : \mathbb{R} \longrightarrow \mathbb{C}^\times$ and $\alpha'$ is cts.

Remember $T_\alpha = \dfrac{\alpha'}{|\alpha'|}$ so $T_\alpha : \mathbb{R} \longrightarrow S^1$ and $T_\alpha$ is cts.

$\alpha$ is $2\pi$-periodic so $\alpha'$ and $T_\alpha$ are as well.

$\mathbb{R}$ is contractible so $T_\alpha$ has a continuous logarithm $L : \mathbb{R} \longrightarrow \mathbb{C}$.

$\forall t \in \mathbb{R}$, $e^{L(t+2\pi)} = T_\alpha(t+2\pi) = T_\alpha(t)$. So $t \longmapsto L(t+2\pi)$ is also a cts log of $T_\alpha$. Thus $\exists n \in \mathbb{Z}$ s.t. $\forall t \in \mathbb{R}$, $L(t+2\pi) = L(t) + 2\pi n i$.

$n$ is called the rotation index of $\gamma$, and is denoted by $i_\gamma$.

$i_\gamma$ is the net # of full turns that $T_\alpha$ performs around $S^1$ as $t$ increases by $2\pi$ (as $e^{it}$ travels around $S^1$).

**Lemma** let $f : \mathbb{R} \longrightarrow \mathbb{R}^d$ be $C^1$. Define $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^d$ by

$$g(u,v) = \begin{cases} \dfrac{f(v) - f(u)}{v - u} & \text{if } v \neq u \\ f'(u) & \text{if } v = u \end{cases}$$

Then $g$ is continuous.

Proof Since we can treat each component of $f$ separately, it suffices to consider the case where $d=1$. Then $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^2\to\mathbb{R}$.

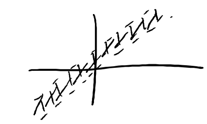
The MVT says $\forall u,v\in\mathbb{R}$, if $u\neq v$ then there is a number $w(u,v)$ between $u$ and $v$ s.t. $f'(w(u,v)) = \dfrac{f(v)-f(u)}{v-u}$. Let $u_0\in\mathbb{R}$. If $(u,v)\longrightarrow(u_0,u_0)$ with $u\neq v$, then $w(u,v)\longrightarrow u_0$ so $\dfrac{f(v)-f(u)}{v-u}\longrightarrow f'(u_0)$. Through points on the diagonal, $v=u$, $g(u,u)=f(u)\to f'(u_0)$. So $g$ is cts everywhere. □

# The turning tangents theorem

Let $\gamma$ be a simple $C^1$-regular loop in $\mathbb{C}$. Then $i_\gamma = \pm 1$.

Pf Define $\alpha:\mathbb{R}\longrightarrow\mathbb{C}$ by $\alpha(t)=\gamma(e^{it})$.

Let $D=\{(u,v)\in\mathbb{R}^2 : u<v<u+2\pi\}$

D: 

Since $\gamma$ is simple, for each $(u,v)\in D$, we have $\alpha(u)\neq\alpha(v)$.

Define $\beta_0: D\longrightarrow S^1$ by $\beta_0(u,v)=\dfrac{\alpha(v)-\alpha(u)}{|v-u|}$. Notice that $\forall (u,v)\in D$ we have $u<v$ so $|v-u|=v-u$ so $\beta_0(u,v)=\dfrac{\left(\frac{\alpha(v)-\alpha(u)}{v-u}\right)}{\left|\frac{\alpha(v)-\alpha(u)}{v-u}\right|}$. (1)

By the lemma and (1), $\forall u_0\in\mathbb{R}$, $\beta_0(u,v)\underset{(2)}{\longrightarrow}T_\alpha(u_0)$ as $(u,v)\longrightarrow(u_0,u_0)$ with $(u,v)\in D$.

Next, $\forall(u,v)\in D$ we have $v<u+2\pi$ so $v-2\pi<u$ so $|(v-2\pi)-u| = -((v-2\pi)-u)$, so $\beta_0(u,v)=\dfrac{\alpha(v-2\pi)-\alpha(u)}{|\alpha(v-2\pi)-\alpha(u)|} = -\dfrac{\left(\frac{\alpha(v-2\pi)-\alpha(u)}{(v-2\pi)-u}\right)}{\left|\frac{\alpha(v-2\pi)-\alpha(u)}{(v-2\pi)-u}\right|}$ (3)

Now for each $u_0\in\mathbb{R}$, as $(u,v)\longrightarrow(u_0,u_0+2\pi)$ in $D$, $\beta_0(u,v)\longrightarrow -T_\alpha(u_0)$. (4).

Let $\Delta=\{(u,u): u\in\mathbb{R}\}$. Let $\tilde{\Delta}=\{(u,u+2\pi): u\in\mathbb{R}\}$.

Let $E=\Delta\cup D\cup\tilde{\Delta}$ and define $\beta: E\longrightarrow S^1$ by $\beta_0(u,v)=\begin{cases}\beta_0(u,v) & \text{if } (u,v)\in D\\ T_\alpha(u) & \text{if } (u,v)\in\Delta\\ -T_\alpha(u) & \text{if } (u,v)\in\tilde{\Delta}\end{cases}$.

By (2) and (4), $\beta$ is continuous on $E$.

Since $E$ is convex, $\beta$ has a logarithm in $E$, call it $L$.

Now the continuous function $\mathrm{Im}(\gamma)$ achieves a minimum on $S^1$, say at $z_1\in S^1$. Let $t_1\in[0,2\pi)$ w/ $z_1=e^{it_1}$. Then $\mathrm{Im}(\alpha(t))\geq\mathrm{Im}(\alpha(t_1))$ $\forall t\in\mathbb{R}$.

hence $\mathrm{Im}(\alpha'(t_1)) = 0$ so $\mathrm{Re}(\alpha'(t_1)) \neq 0$ so $\mathrm{Re}(\alpha)$ does not achieve its minimum here.

either $\mathrm{Re}(\alpha(t_1)) > 0$ or $\mathrm{Re}(\alpha(t_1)) < 0$. In first case, $i_\gamma = 1$, $i_\gamma = -1$ in 2nd.

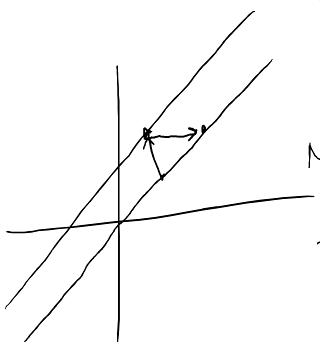
Its enough to consider first case (can reverse curve). WTS $i_\gamma = 1$.

The function $u \longmapsto L(u,u)$ is a continuous log of $T_\alpha$.

$$\text{So } i_\gamma = \frac{L(t_1+2\pi, t_1+2\pi) - L(t_1, t_1)}{2\pi i} = \frac{\theta(t_1+2\pi, t_1+2\pi) - \theta(t_1, t_1)}{2\pi} \quad (5)$$

Where $\theta = \mathrm{Im}(L)$. Now:

$$\theta(t_1+2\pi, t_1+2\pi) - \theta(t_1, t_1) = \theta(t_1+2\pi, t_1+2\pi) - \theta(t_1, t_1+2\pi)$$
$$+ \theta(t_1, t_1+2\pi) - \theta(t_1, t_1).$$

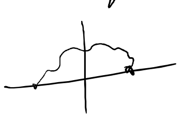


Note that $\beta(t_1, t_1) = T_\alpha(t_1) = 1$ and $\beta(t_1, t_1+2\pi) = -T_\alpha(t_1) = -1$

for $t_1 < v < t_1 + 2\pi$, the point $\beta(t_1, v) = \dfrac{\alpha(v) - \alpha(t_1)}{|\alpha(v) - \alpha(t_1)|} \in \{z \in \mathbb{C} : \mathrm{Im}\, z \geq 0\}$.

Hence $\theta(t_1, t_1+2\pi) - \theta(t_1, t_1) = \pi$

$\Delta\theta = \pi$



Next $\beta(t_1, t_1+2\pi) = -T_\alpha(t_1) = -1$, and $\beta(t_1+2\pi, t_1+2\pi) = T_\alpha(t_1) = 1$,

so for $t_1 < u < t_1 + 2\pi$, the point $\beta(u, t_1+2\pi) = \dfrac{\alpha(t_1) - \alpha(u)}{|\alpha(t_1) - \alpha(u)|} \in \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}$

So $\theta(t_1 - 2\pi, t_1 - 2\pi) - \theta(t_1, t_1 - 2\pi) = \pi$. So $i_\gamma = 1$. $\square$