R: principal ideal domain - integral domain (no zero divisors), revery ideal is generated by one element.

Lema: (a) = (b) iff a=ub for some ue Rx

 $\not\sqsubseteq$ (a) \subset (b) means $\alpha \in$ (b) i.e. $\alpha = \times b$

(b) c (a) . . . b= ya

So $\alpha = xb = xya$ so $(1-xy)a = 0 \Rightarrow xy = 1$ if $a \neq 0$.

Lenner: R: PID, PFR prime ideal (P = (0)). Thun P is maximal.

If PFMCR hun M=R (to show).

P = (p), M = (m) for some peP, meM.

PCM => P=XM for some XER. P=M means m&P,

So xeP since Pispine thus x=yp.

P = y p m so $(1 - y m) = 0 \Rightarrow y m - 1$ so $m \in \mathbb{R}^{\times}$ meaning $(m) = \mathbb{R}$.

A general statement

If Risa commitative ring and

MER is a maximal ideal then MER is primary YnzI

Reason: in R/M^n every element is a unit or nilpotent. $X=a_0+M$. Zero Jiv in $R/M^n=nilpotent$ in R/M^n i.e. M^n is primary.

IN P.1.D's non-zoo printy = power of newero sent isent max's

Prop: R: P.I.D. $Q \subseteq R$ non-zero princing ideal Rad(Q) = P = (p), then $\exists n \ge 1 \le t$. $Q = (p^n)$

Proof: P = (P), Q = (Q), P = Rcd(Q) = QChoose smallest n s.t. $P^n \in Q$ $(P^{n-1} \notin Q)$ Let $P^n = xq$ and $Q = yP \Rightarrow P^n = xyP \Rightarrow P^{n-1}(1-xy)P = 0$ but $P \neq 0$ so $P^{n-1} = xy$. by minimality of n. $X \notin (P)$. $xy \in (P^{n-1})$, $x \notin Rad((P^{n-1}))$ i.e. $x \notin (P^{n-1})$ for any K,

So $Q \in (P^n)$, and $(P^n) \in Q$ by defined n.

Recall Rad (p") = P for a prime ideal P.

in P.I.D. hum $(p) \cdot (q) = (pq)$

Noether's Primary Decomposition Theorem (for P.I.D.'s)

Recall: R: Noetherian & I \subseteq R ideal \Longrightarrow I = Q, n ... n Q₁ where Q_i are all privary, Rad(Q_i) = P_i are all distinct Min(I) \subset {P_i: $1 \le i \le p$ } and Min(I) is uniquely determined by I.

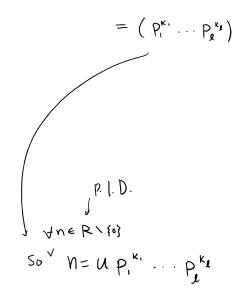
In P.I.D.: I = (n). $Q_i = (p_i^{k_i})$, P_i are all nonero princes \Rightarrow maximal So no embedded primes $(Min(I) = \{P_1, ..., P_2\})$ is uniquely determined). f $P_i \notin Min(I)$ then f $P_j \in Min(I)$ s.t. $P_j \in P_i$. but this contradicts "all primes are maximal".

i.e. (Pi) are uniquely determined.

and Qi is uniquely determined of Pi is minimal

So all Q: are uniquely determined.

$$(\mathsf{N}) = (\mathsf{P}_{\iota}^{\mathsf{K}_{\iota}})_{\mathsf{n}} \dots_{\mathsf{n}} (\mathsf{P}_{\iota}^{\mathsf{K}_{\iota}})$$



(hitese remainder thin: $M \neq M_2 \implies M_1 + M_2 = R$ $I_1, J_1 \iff J_2 \implies L_2 \implies L_3 = R$ If (oprime: $I_1 = I_2 = I_3 \implies L_4 = R$ $I_2 = I_3 \implies L_4 = R$ $I_3 = I_4 \implies L_4 = R$ $I_4 = I_5 \implies L_4 = R$ $I_4 = I_5 \implies L_4 = R$ $I_4 = I_5 \implies L_4 = R$

where (1) u is a unit

- (2) (Pi), ..., (Pe) are distinct non-zero prime ideals in R
- (3) (P1),..., (P,) we uniquely determined by n

Our examples of PID's:

 \mathbb{Z}

 $\bigvee (n) = |n|$

K[x] N(f(x)) = deg(f(x))

 $\mathbb{Z}[I_{-1}]$ $\mathbb{N}(a+bi) = a^2+b^2$

in each case we had a function

N: R - Zo s.t. VaibeR, b+o, 3 qire R

. κ_{20}). τ . $\forall \alpha, b \in K$, $b \neq 0$, $\exists q, r \in K$ So that $\alpha = qb + r$ where exter v = 0 or N(r) < N(b)

Defn: A domain R is alled enclidean if this function exists.

Easy lame: Every enclidean domain is a P.I.D. \not f: $\vec{I} \neq (0)$. choose $b \in I \setminus \{0\}$ of smallest N(b). Prove $\vec{I} = (b)$.

Reading for tomorrow. § 7.1 example 229