Lec 9/13

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$T \in L(V, V)$ $\dim V = n$

two bases of V $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_n\}$

$T \underset{\{v_i\}}{\rightsquigarrow} A$

$T \underset{\{w_i\}}{\rightsquigarrow} B$

$B = X^{-1} A X$ where X = transition matrix (σ_{ij})

X^{-1}: invertible s.t. $w_j = \sum_{i=1}^{n} \sigma_{ij} v_i$

$\delta_{kj} = 1$ if $k = j$, 0 if $k \neq j$

$\dim V = 2$, $T \in L(V, V)$ wrt $\{u_1, u_2\}$ basis of V.

$$\begin{cases} T(u_1) = u_1 - u_2 \\ T(u_2) = u_1 \end{cases}$$

determines T.

$V = \lambda_1 u_1 + \lambda_2 u_2$. $T(V) = \lambda_1 u_1 - \lambda_1 u_2 + \lambda_2 u_1$

$$T \underset{\{u_1, u_2\}}{\rightsquigarrow} A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

l.in. indep (can be checked)

$$\begin{cases} w_1 = 3u_1 - u_2 = u_1 + u_1 + u_1 - u_2 \\ w_2 = u_1 + u_2 \end{cases}$$

$$X = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

$u_1 = \frac{1}{4}w_1 + \frac{1}{4}w_2$

$u_2 = -\frac{1}{4}w_1 + \frac{3}{4}w_1$

$$X^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$B = \underbrace{\frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}_{X^{-1}} \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}}_{X}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 5 & 2 \\ -7 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 5/4 & 1/2 \\ -7/4 & -1/2 \end{pmatrix}$$ or maybe it should be $\begin{pmatrix} 5/4 & 3/4 \\ -7/4 & -1/4 \end{pmatrix}$

Precursors:

$$\mathcal{N}(T) = \{v \in V : T(v) = 0\}$$

$$T(V) = \{w \in W : \exists v \in V \text{ s.t. } T(v) = w\}$$

Fundamental Theorem of Linear Transformations

Let $T \in L(V, W)$, $\dim_F V = n$, $\dim_F W = m$

then $\dim_F \mathcal{N}(T) + \dim_F T(V) = \dim V$

Proof $\mathcal{N}(T)$ subsp. of V. choose a basis $\{v_1, \dots, v_p\}$

extend this to a basis of V by $\{u_1, \dots, u_{n-p}\}$.

$T(v_i) = 0$. $\{T(u_1), \ldots, T(u_{n-p})\}$ form a basis of $T(V)$. for one, they are generators since $w \in T(V)$ is $T(\lambda_1 v_1 + \cdots + \lambda_p v_p + \mu_1 u_1 + \cdots + \mu_{n-p} u_{n-p})$

$= \sum_{i=1}^{n-p} \mu_i T(u_i)$. Second, they are lin. indp.

Since if $\sum \mu_i T(u_i) = 0$ then $T(\sum \mu_i u_i) = 0$

so $\sum \mu_i u_i \in \mathcal{N}(T)$ and so equals $\sum \lambda_i v_i$, so

$\sum \mu_i u_i - \sum \lambda_i v_i = 0$ but $\{v_i, u_i \ldots\}$ are a basis so $\lambda_i, \mu_i = 0$ (in particular, $\mu_i = 0$). ∎

<u>Cor</u>: $T \in L(V,V)$ $\dim_F V = n$, then the following are equiv.

(1) T invertible

(2) T injective

(3) T surjective

Something to think about:

$T \in L(V, W)$

T injective iff $\exists L \in L(W,V)$ s.t. $L \circ T = I_V$

T surjective iff $\exists R \in L(W,V)$ s.t. $T \circ R = I_W$