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K/F is Galois, K is a tower of Galois extensions.

$$G = Gal(K/F)$$

$$\begin{array}{c} K=L_n \\ | \\ \vdots \\ | \\ L_2 \\ | \\ L_1 \\ | \\ F=L_0 \end{array}$$

$\forall i$, there is

$$Gal(L_i/L_{i-1})$$

$$\begin{array}{c} H_n=1 \\ | \\ \vdots \\ | \\ H_2 \\ | \\ H_1 \\ | \\ G=H_0 \end{array},$$

$\forall i$, $H_i = Gal(K/L_i)$

$$1 = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq H_0 = G.$$

subnormal series

$$\forall i, \; H_{i-1}/H_i \cong Gal(L_i/L_{i-1}).$$

Conversely, if

$1 = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G$ is a subnormal series in G,

Then K is a tower of Galois extensions:

$$\begin{array}{c} K=L_n \\ | \\ L_{n-1} \\ | \\ \vdots \end{array}$$

where $\forall i$, $L_i = Fix(H_i)$

$$L_{n-1}$$
$$\|$$
$$\vdots$$
$$\|$$
$$L_1$$
$$\|$$
$$L_0 = F$$

where $\forall i$, $L_i = \mathrm{Fix}(H_i)$,

$$\mathrm{Gal}(L_i/L_{i-1}) \cong H_{i-1}/H_i$$

Abelian Extensions

Galois extension is <u>abelian</u> if $\mathrm{Gal}(K/F)$ is abelian.

If K/F is abelian, then any subextension L/F $(L \subseteq K)$ is normal.

Let $\alpha \in K$. Let $L = F(\alpha)$. Then $F(\alpha)/F$ is normal, i.e. all conjugates of α are in $F(\alpha)$.

primitive $\sqrt[n]{1}$

$\mathbb{F}_{p^n}/\mathbb{F}_p$ - abelian. Cyclotomic extensions $\mathbb{Q}(\omega)/\mathbb{Q}$.

$\mathbb{Q}(\sqrt[n]{a}, \omega)/\mathbb{Q}(\omega)$ - abelian.

$\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ - abelian.

G - finite abelian group

$$\Longrightarrow \quad G \cong H_1 \times \dots \times H_k$$

$\nwarrow \quad \nearrow$
cyclic

Let $G = H_1 \times \cdots \times H_k \Longrightarrow \forall i,$ let $N_i = H_1 \times \cdots \times H_{i-1} \times H_{i+1} \times \cdots \times H_k,$

$\parallel$
$Gal(K/F)$

let $L_i = Fix(N_i).$

Then $K = L_1 \cdots L_k$, and $[K:F] = [L_1:F] \cdot \cdots \cdot [L_k:F]$,

and $\forall i$, $\underset{\underset{N_i}{\Updownarrow}}{L_i} \cap \underset{\underset{H_i}{\Updownarrow}}{(L_1 \cdots L_{i-1} L_{i+1} \cdots L_k)} = F$

(since $H_i = \bigcap_{j \neq i} N_j$, and $H_i \cdot N_i = G$)

$K \cong L_1 \otimes_F \cdots \otimes_F L_k.$

K/F is abelian

$\Rightarrow K = L_1 \cdots L_k,$

where each L_i/F is a cyclic extension

(i.e. $Gal(L_i/F)$ is cyclic)

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p-groups (2-groups)

$|G| = p^r$, p is prime.

<u>Theorem</u> if $|G| = p^r$, then there is a sequence

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = G \quad \text{s.t. } \forall i, \quad H_i/H_{i-1} \cong \mathbb{Z}_p$$

$$(\forall i, \ H_i \trianglelefteq G)$$

If K/F is a "p"-extension (K/F is Galois & $[K:F] = p^r$),

Then K is a tower

$$\begin{array}{c} K = L_r \\ \| \\ L_{r-1} \\ \| \\ \vdots \\ \| \\ L_1 \\ \| \\ L_0 = F \end{array} \qquad \text{where } \forall i, \quad Gal(L_i/L_{i-1}) \cong \mathbb{Z}_p.$$

<u>F.T. Algebra</u>: $\mathbb{C}$ is algebraically closed, $\mathbb{C} = \overline{\mathbb{R}}$.

<u>To prove</u>: any irreducible pol-l $f \in \mathbb{R}[x]$ splits completely in $\mathbb{C}$.

<u>Facts we know</u>: ① if $f \in \mathbb{R}[x]$, $\deg f$ is odd, then f has a root in $\mathbb{R}$.

② any quadratic pol-l from $\mathbb{C}[x]$ splits in $\mathbb{C}$.

<u>Proof</u> Let $f \in \mathbb{R}[x]$ be irreducible. Let K be the

Splitting field of f over $\mathbb{C}$.

Let $G = Gal(K/\mathbb{R})$. Let H be the sylow 2-subgroup of G.

$(|G| = 2^r \cdot m$, m odd. $|H| = 2^r)$. Let $L = Fix(H)$.

Then $[L:\mathbb{R}] = m$ - odd. Let $\alpha \in L$,

Then $\deg m_{\alpha,F}$ is odd. So $m_{\alpha,F}$ has a root in $\mathbb{R}$.

This is impossible unless $\alpha \in \mathbb{R}$.

So $L = \mathbb{R}$ and $H = G$ is a 2-group.

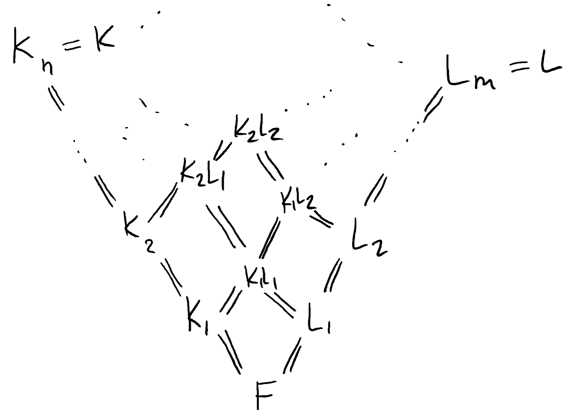
So K is a tower of quadratic extensions.

$$\begin{array}{c} K = L_r \\ \| \, 2 \\ L_{r-1} \\ \| \, 2 \\ \vdots \\ \| \, 2 \\ L_1 \\ \| \, 2 \\ L_0 = \mathbb{R} \end{array}$$

But any quadratic subextension of $K/\mathbb{R}$ is equal to $\mathbb{C}$,

and $\mathbb{C}$ has no quadratic extensions,

So $r = 1$ and $K = L_1 = \mathbb{C}$. $\square$



$$\begin{tikzcd} & K_1L_1 \arrow[dl, dash] \arrow[dr, dash] & \\ K_1 \arrow[dr, dash] & & L_1 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

KL

$$\begin{tikzcd} & K_1L_1 \arrow[dl, dash] \arrow[dr, dash] & \\ K_1 \arrow[dr, dash] & & L_1 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

$Gal(K_1L_1/L_1) \lesssim Gal(K_1/F)$