

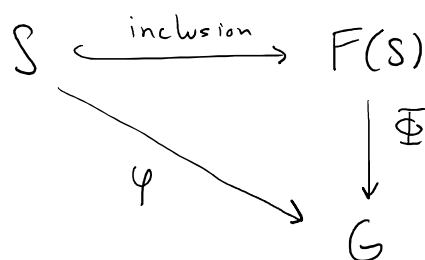
Group Presentations are a thing.

The Free Group on a set  $S$  exists. It's called  $F(S)$ .

Let  $G$  be a gp,  $\varphi: S \xrightarrow{\text{any set map}} G$ .

Then  $\exists!$  homomorphism  $\Phi: F(S) \rightarrow G$

s.t. the following diagram commutes:



Take  $S$  to be a set of generators for  $G$ .

i.e.  $S \subseteq G$ ,  $G = \langle S \rangle$ .

Def: A presentation of  $G$  is a pair  $(S, R)$  generates  $G$   
 $\downarrow$   
 $\hookrightarrow F(S)$

s.t. The normal closure of  $\langle R \rangle \leq F(S)$

is the kernel of  $\Phi$  (which extends  $S \hookrightarrow G$ ).

Def the normal closure<sup>of  $H$  in  $G$</sup>  is smallest normal subgroup of  $G$  containing  $H$ .

So  $G \cong_{\mathbb{F}} F(S)/K$  where  $K = \text{normal closure of } \langle R \rangle$ .

Def  $G$  is finitely presented if  $|S|, |R| < \infty$ .

ex  $D_n = \langle r, s \mid r^n, s^2, rsrs \rangle$

Group Actions:

$$G \curvearrowright S \longleftrightarrow \text{homomorphism } G \xrightarrow{T} \text{Sym}(S).$$

The action is said to be effective (or faithful) if the homomorphism  $T$  is injective.

The kernel of the action is the kernel of  $T$ .

Ex the action of  $G$  by left mult is faithful.

Ex the action of  $G$  by right mult:

$$g \cdot s = sg^{-1} \quad (g_1 g_2) \cdot s = s(g_1 g_2)^{-1} = s g_2^{-1} g_1^{-1} = g_1 \cdot (g_2 \cdot s).$$

(can't use  $g \cdot s = sg!$ ).

Ex  $G \curvearrowright G$  by conjugation:

$$g \cdot s = gsg^{-1} = \underbrace{gs}_{\text{jacobson's notation}}$$

The kernel of the action is  $Z(G)$ .

The action is effective if  $Z(G) = 1$ .

Ex The action of  $G$  on a space of left cosets of  $G$ :

$$\text{let } H \leq G, \quad G/H = \{xH \mid x \in G\}$$

$$g \cdot (xH) = gxH.$$

In general, this is not effective.

The kernel of this action is

$$\{g \in G \mid gxH = xH \quad \forall x \in G\}$$

$$= \{g \in G \mid x^{-1}gx \in H \quad \text{for all } x \in G\}$$

$$= \{g \in G \mid g \in xHx^{-1} \text{ for all } x \in G\}$$

$$= \bigcap_{x \in G} xHx^{-1}.$$

this is  
a normal subgroup of  $G$ .

It's also a subgroup of  $H$ .

Claim this is the largest normal  
subgroup of  $G$  contained in  $H$ .

~~iff~~ if  $N \leq H$  is normal in  $G$ ...