Lec 11/6

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X v.s / K. $\mathcal{P} = \{ p : X \to [0,\infty)$ seminorms$\}$

$$\mathcal{U}_{x,p,\varepsilon} := \{ y \in X \mid p(y-x) < \varepsilon \}$$

τ = top. generated by $\mathcal{U}_{xp\varepsilon}$.

Locally convex TVS.

- Hausdorff iff $\mathcal{P}$ separates pts.
- Hausdorff & $\mathcal{P}$ ctble $\Rightarrow$ $\exists$ translation invariant metric ρ inducing same TVS structure.

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{P_n(x-y)}{2^n[1+P_n(x-y)]}$$

$$x_n \xrightarrow{\rho} x \quad \text{iff} \quad P_k(x_n - x) \longrightarrow 0 \quad \forall k$$

$$\text{iff} \quad x_n \xrightarrow{\tau} x$$

families of seminorms

Prop Suppose $(X, \mathcal{P}, \tau)$ and $(Y, \mathcal{Q}, \theta)$ are locally convex TVS, and $T : X \to Y$ is linear. TFAE:

① T cts

② T cts at 0

③ $\forall q \in \mathcal{Q}$ $\exists$ $p_1, \dots, p_n \in \mathcal{P}$ and $c > 0$ s.t. $q(Tx) \le c \sum_{i=1}^{n} p_i(x)$ $\forall x$.

Pf ③ ⇒ ①: Suppose ③ for fixed q. if $x_\lambda \to x$ Then

$p(x - x_\lambda) \to 0 \ \forall p \in \mathcal{P}$. So

$$q(T(x - x_\lambda)) \le c \sum_{i=1}^{n} p(x - x_\lambda) \longrightarrow 0$$

So $\forall q$, $q(Tx - Tx_\lambda) \to 0$, so $Tx_\lambda \to Tx$.

②⇒③: Suppose T cts at 0 & $q \in \mathcal{Q}$. Then $\exists p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ s.t. $\forall x \in V = \bigcap_{i=1}^{n} \mathcal{U}_{0 p_i \varepsilon}$, $q(Tx) < 1$.

Fix $x \in X$. If $p_i(x) = 0 \ \forall i = 1, \dots, n$, then $rx \in V \ \forall r > 0$. Hence $q(Trx) = r\, q(Tx) < 1 \ \forall r$. So $q(Tx) = 0$. Assume $p_1(x) > 0$. Then

$$y := \frac{\varepsilon x}{2 \sum_{i=1}^{n} p_i(x)} \in V. \quad \text{Thus} \quad q(Tx) = \left[2 \varepsilon^{-1} \sum_{i=1}^{n} p_i(x)\right] \underbrace{q(Ty)}_{<1}$$

so $q(Tx) < \frac{2}{\varepsilon} \sum_{i=1}^{n} p_i(x)$ as desired. □

Example: X normed space. Recall X^* separates pts by HB. Consider $\mathcal{P} = \{ p_\varphi(x) := |\varphi(x)| \}_{\varphi \in X^*}$. $\mathcal{P}$ is a separating family of seminorms. $\tau_{\mathcal{P}}$ is a locally convex TVS structure on X which is Hausdorff. This is called the weak topology on X.

Prop If $\mathcal{U} \subseteq X$ is weakly open, $\mathcal{U}$ is norm open

so $x_n \to x$ in $\|\cdot\| \implies x_n \to x$ weakly

[$x_n \to x$ in norm $\Rightarrow$ $|\varphi(x_n - x)| \longrightarrow 0 \ \forall \varphi \in X^* \Rightarrow x_n \to x$ weakly].

pf: Every basic open set $U_{x\varphi\varepsilon} = \{ y \in X \mid |\varphi(x-y)| < \varepsilon \}$ is norm open since $\varphi \in X^*$ is cts & $|\cdot|$ is cts.

Ex: Show that weak top = $\|\cdot\|$ top $\Leftrightarrow$ X finite dim'l.

Prop: A linear functional $\varphi : X \longrightarrow \mathbb{C}$ is cts wrt weak topology iff $\varphi \in X^*$.

pf: · Suppose $\varphi \in X^*$. Then $\varphi^{-1}(B_\varepsilon^{\mathbb{C}}(0)) = \{ x \mid |\varphi(x)| < \varepsilon \} = U_{0\varphi\varepsilon}$ is weakly open. So φ cts at $0 \Rightarrow \varphi$ weakly cts.

· Now suppose φ weakly cts and let $\varepsilon > 0$. Then $\exists\, \varphi_1, \ldots \varphi_n \in X^*$ s.t. $\bigcap_{i=1}^n U_{0\varphi_i\varepsilon} \subseteq \varphi^{-1}(B_\varepsilon^{\mathbb{C}}(0))$. Hence

$$|\varphi(x)| \leq \max_{i=1}^n |\varphi_i(x)| \quad \forall x. \qquad \text{Claim}$$

pf of claim: if $\max_{i=1}^n |\varphi_i(x)| < r < |\varphi(x)|$ for some x,

$$x \in \bigcap_{i=1}^n U_{0\varphi_i r} \qquad r = \left(\tfrac{r}{\varepsilon}\right)\varepsilon.$$

$$\tfrac{\varepsilon}{r} x \in \bigcap_{i=1}^n U_{0\varphi_i\varepsilon} \subseteq \varphi^{-1}(B_\varepsilon^{\mathbb{C}}(0))$$

$$\varepsilon > \left|\varphi\left(\tfrac{\varepsilon}{r}x\right)\right| = \left|\tfrac{\varepsilon}{r}\right| \cdot |\varphi(x)| > \left|\tfrac{\varepsilon}{r}\right| |r| = \varepsilon \qquad \text{(contradiction)}.$$

Then by future HW, $\varphi \in \operatorname{Span}\{\varphi_1, \ldots, \varphi_n\} \subset X^*$. $\square$

$X^{**} \supseteq X$. X^* has a weak top. induced by X^{**}.

$ev_x \xleftarrow{\text{isometry}} x$

But $X \subset X^{**}$ separates pts of X^* by defn!

Weak* topology on X^* is induced by

$$\mathcal{P} \ \{\varphi \longmapsto |\varphi(x)| \mid x \in X\}$$

• sep family of seminorms on X^*.

Thm (Banach-Alaoglu) The norm-closed unit ball $B^* \subset X^*$ is wk* cpt.

Warning: not necessarily wk* sequentially cpt.

Pf: for $x \in X$, let $D_x = \{z \in \mathbb{C} \mid |z| \leq \|x\|\}$ cpt $\forall x$.

By Tychonoff's Thm, $D = \prod_{x \in X} D_x$ is cpt.

Let elts $d \in D$ are precisely fns $d : X \longrightarrow \mathbb{C}$ s.t. $|d(x)| \leq \|x\|$ $\forall x \in X$.

The topology is ptwise convergence.

Observe $B^* \subset D$ are linear fns.

Relative top on B^* is top. of ptwise convergence:

$$\varphi_\lambda \longmapsto \varphi \quad \text{iff} \quad \varphi_\lambda(x) \longrightarrow \varphi(x) \quad \forall x$$

$$\text{iff} \quad |\varphi_\lambda(x) - \varphi(x)| \longrightarrow 0 \quad \forall x$$

$$\text{iff} \quad \varphi_\lambda \longrightarrow \varphi \quad \text{weak}^*$$

It remains to show $B^* \subset D$ is closed $\Rightarrow$ cpt.

If $\langle \varphi_\lambda \rangle \subset B^*$ is a net w/ $\varphi_\lambda \longrightarrow \varphi \in D$, then

$$\varphi(\alpha x+y)=\lim_\lambda \varphi_\lambda(\alpha x+y)=\lim_\lambda\left(\alpha\varphi_\lambda(x)+\varphi_\lambda(y)\right)=\alpha\varphi(x)+\varphi(y). \qquad \square$$

Hilbert spaces: H is a $\mathbb{K}$-VS. a fn $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{K}$ is

a sesquilinear form if it's linear in 1^{st} var & conj. linear in 2^{nd}.

- self-adjoint: $\langle x,y\rangle=\overline{\langle y,x\rangle}$
- non-deg: $\langle x,y\rangle=0\ \forall y \Rightarrow x=0$
- positive: $\langle x,x\rangle\geq 0\ \forall x$

  ↳ pos. def: in addition, $\langle x,x\rangle=0\Rightarrow x=0$.

Ex: $\langle\cdot,\cdot\rangle$ sesquilinear form on H.

① polarization $\forall x,y\in H$,

$$4\langle x,y\rangle=\begin{cases}\sum_{l=0}^{3} i^k\langle x+i^k y,\ x+i^k y\rangle & \text{if } \mathbb{K}=\mathbb{C}\\ \langle x+y,x+y\rangle-\langle x-y,x-y\rangle & \text{if } \mathbb{K}=\mathbb{R}\end{cases}$$

This may be wrong.