Lec 2/23

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Propn Let $f : [0,1] \longrightarrow \mathbb{C}^\times$ be cts. Let w_0 be a logarithm of $f(0)$. Then there is a unique branch g of $\log f$ such that $g(0) = w_0$.

Pf By the last result from monday, there is a branch of $\log f$, say h. Then $e^{h(0)} = f(0) = e^{w_0}$ so $\exists\, k \in \mathbb{Z}$ s.t. $h(0) = w_0 + 2\pi i k$. let $g = h - 2\pi i k$.
Then g is cts and $\forall\, t \in [0,1]$ $e^{g(t)} = e^{h(t) - 2\pi i k} = e^{h(t)} e^{-2\pi i k} = f(t) \cdot 1 = f(t)$.
thus g is a branch of $\log f$.
g is unique since if $\tilde{g}$ is another one then $e^g = e^{\tilde{g}}$ so $g = \tilde{g} + 2\pi i n$. but $g(0) = w_0 = \tilde{g}(0)$ so $n = 0$ so $g = \tilde{g}$. $\square$

__Winding numbers:__

Let $\gamma : S^1 \longrightarrow \mathbb{C}^\times$ be continuous. Define $f : [0,1] \longrightarrow \mathbb{C}^\times$ by $f(t) = \gamma(e^{2\pi i t})$

let g, h be branches of $\log f$. Then since $[0,1]$ is connected $\exists$ an integer k s.t. $\forall\, t \in [0,1]$ $h(t) = g(t) + 2\pi i k$. hence $h(1) - h(0) = g(1) - g(0)$.

let $n = \dfrac{g(1) - g(0)}{2\pi i}$. $n \in \mathbb{Z}$ since $e^{g(1)} = f(1) = \gamma(e^{2\pi i}) = \gamma(e^0) = f(0) = e^{g(0)}$

so $g(1) - g(0) \in 2\pi i \mathbb{Z}$. furthermore, $\dfrac{h(1) - h(0)}{2\pi i} = n$ as well, so this n is well defined as a fn of γ, n is called the winding # of γ with respect to the origin, denoted $\operatorname{ind}(\gamma)$

eg Let $r : S^1 \xrightarrow{\text{cts}} (0, \infty)$. let $n \in \mathbb{Z}$. Define $\gamma : S^1 \longrightarrow \mathbb{C}^\times$ by $\gamma(z) = z^n r(z)$.

then $\operatorname{ind}(\gamma) = n$.

Pf Define $f : [0,1] \longrightarrow \mathbb{C}^\times$ by $f(t) = \gamma(e^{2\pi i t})$. Then $\forall\, t \in [0,1]$,

$f(t) = e^{2n\pi i t}\, r(e^{2\pi i t}) = e^{g(t)}$ where $g(t) = \ln(r(e^{2\pi i t})) + 2n\pi i t$.

g is cts so g is a branch of $\log f$, so $\operatorname{ind}(\gamma) = \dfrac{g(1) - g(0)}{2\pi i} = \dfrac{\ln(r(1)) + 2n\pi i - \ln(r(1))}{2\pi i} = n$.

Fact let $\beta, \gamma : S^1 \xrightarrow{cts} \mathbb{C}^\times$. Then $\operatorname{ind}(\beta) = \operatorname{ind}(\gamma)$ iff $\beta \simeq \gamma$ in $\mathbb{C}^\times$.

pf of ($\Rightarrow$) Suppose $\operatorname{ind}(\gamma) = \operatorname{ind}(\beta)$. Define $\tilde\beta, \tilde\gamma : [0,1] \longrightarrow \mathbb{C}^\times$ by $\tilde\beta(s) = \beta(e^{2\pi i s})$ $\tilde\gamma(s) = \dots$

let u and v be branches of $\log\tilde\beta$ and $\log\tilde\gamma$ respectively. $v-u$ is a branch of $\log\frac{\tilde\gamma}{\tilde\beta}$.

Define $\tilde H : [0,1]^2 \longrightarrow \mathbb{C}^\times$ by $\tilde H(s,t) = e^{(1-t)(v(s)-u(s))}$. Then $\tilde H$ is cts,

$\tilde H(s,0) = e^{v(s)-u(s)} = \frac{\tilde\gamma(s)}{\tilde\beta(s)}$ and $\tilde H(s,1) = e^0 = 1$.

Now define $H : S^1 \times [0,1] \longrightarrow \mathbb{C}^\times$ by $H(e^{2\pi i s}, t) = \tilde H(s,t)$. Since $\operatorname{ind}(\beta) = \operatorname{ind}(\gamma)$,

$v(1) - v(0) = u(1) - u(0)$ so $v(1) - u(1) = v(0) - u(0)$. Thus H is continuous (check), and

$H(z,0) = \frac{\gamma(z)}{\beta(z)}$ and $H(z,1) = 1$. So $H : \frac{\gamma}{\beta} \simeq 1$ in $\mathbb{C}^\times$, and so

$\beta H : \gamma \simeq \beta$ in $\mathbb{C}^\times$.

Can we simplify this???

Propn For all loops β and γ in $\mathbb{C}^\times$, the product $\beta\gamma$ is a loop in $\mathbb{C}^\times$ and $\operatorname{ind}(\beta\gamma) = \operatorname{ind}(\beta) + \operatorname{ind}(\gamma)$.

pf let $\tilde\beta(t) = \beta(e^{2\pi i t})$, $\tilde\gamma(t) = \gamma(e^{2\pi i t})$ as before. let u, v be branches of $\log\tilde\beta$ and $\log\tilde\gamma$ resp.

let $w = u + v$. $e^w = e^u e^v = \tilde\beta\tilde\gamma$. so w is a branch of $\log(\tilde\beta\tilde\gamma)$.

So $\operatorname{ind}(\beta\gamma) = \frac{w(1) - w(0)}{2\pi i} = \frac{(u(1)-u(0)) + (v(1)-v(0))}{2\pi i} = \operatorname{ind}(\beta) + \operatorname{ind}(\gamma)$. □

Corollary: $\operatorname{ind}\left(\frac{1}{\gamma}\right) = -\operatorname{ind}(\gamma)$, $\operatorname{ind}\left(\frac{\beta}{\gamma}\right) = \operatorname{ind}(\beta) - \operatorname{ind}(\gamma)$.

$\operatorname{ind}(\beta) = \operatorname{ind}(\gamma) \iff \operatorname{ind}\left(\frac{\beta}{\gamma}\right) = \operatorname{ind}\left(\frac{\gamma}{\beta}\right) = 0$.

Prop^n Let γ be a loop in $\mathbb{C}^\times$. Then $\operatorname{ind}(\gamma) = 0$ iff $\exists$ a branch of $\log(\gamma)$

pf $\Rightarrow$ Define $\tilde\gamma : [0,1] \longrightarrow \mathbb{C}^\times$ by $\tilde\gamma(t) = \gamma(e^{2\pi i t})$. Let $\tilde g$ be a branch of $\log\tilde\gamma$.

Then $\tilde g(1) - \tilde g(0) = 0$ so $\tilde g(1) = \tilde g(0)$. define $g : S^1 \longrightarrow \mathbb{C}^\times$ by $g(e^{2\pi i t}) = \tilde g(t)$

g is w.d. since and g is cts. also $e^g = \gamma$ so g is a branch of $\log\gamma$. ■

$\Leftarrow$ Suppose there is a branch of $\log\gamma$, say g. let $\tilde\gamma(t) = \gamma(e^{2\pi i t})$, $\tilde g(t) = g(e^{2\pi i t})$.

thus $\tilde g$ is a branch of $\tilde\gamma$. so $\operatorname{ind}(\gamma) = \frac{\tilde g(1) - \tilde g(0)}{2\pi i} = \frac{g(1) - g(1)}{2\pi i} = 0$.

__Corollary__ Let β, γ be loops in $\mathbb{C}^\times$. Suppose $\operatorname{ind}(\beta) = \operatorname{ind}(\gamma)$. Then $\beta \simeq \gamma$ in $\mathbb{C}^\times$.

__pf__ $\operatorname{ind}\left(\frac{\gamma}{\beta}\right)$ 0 so $\exists$ a branch of $\log \frac{\gamma}{\beta}$ in $\mathbb{C}^\times$, say w. Hence $\frac{\gamma}{\beta} \simeq 1$ in $\mathbb{C}^\times$.

let $H : \frac{\gamma}{\beta} \simeq 1$. then $\beta H : \gamma \simeq \beta$. $\square$