Lec 10/3

Monday, October 3, 2016 9:11 AM

Problem A hints:

Extend the notion of sup, inf.

$S \neq \emptyset$

if S not bounded above, we write $\sup S = \infty$
below $\inf S = -\infty$

$$(\inf S, \sup S) \subseteq S \subseteq [\inf S, \sup S] \cap (-\infty, \infty)$$

use betweenness use definition of sup & inf.

Problem B: use problem A.

Ch 22 - Sequences.

Another way of looking at CA, continuity, and limits.

Definition $f: \mathbb{N}^+ \to \mathbb{R}$ is a sequence, or (more generally)

$f: \{n : n > m, n \in \mathbb{N}\} \to \mathbb{R}$.

Standard notation $\{a_n\}_{n=M}^{\infty}$ where a_n is an expression depending on n.

for example, $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{\frac{1}{n-3}\}_{n=4}^{\infty}$

Definition: Let $\{a_n\}$ be a sequence. We say $\lim_{n\to\infty} a_n = L$

if, $\forall \varepsilon > 0$, $\exists N$ s.t. $|a_n - L| < \varepsilon$ $\forall n > N$.

Remark: If $f: (M, \infty) \to \mathbb{R}$ is a function, then we can make a sequence by restricting the domain: $\{f(n)\}_{n=M}^{\infty}$

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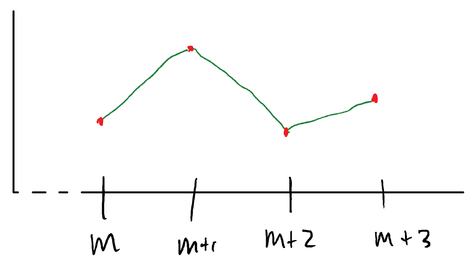
Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{n\to\infty} f(n)$ provided these limits exist.

$$\lim_{x\to\infty} f(x) = \lim_{u\to 0^+} f\left(\tfrac{1}{u}\right)$$

if $\{a_n\}_{n=m}^{\infty}$, you can extend it linearly to a function $f:[m,\infty)\to\mathbb{R}$



$$\lim_{n\to\infty} a_n = L \iff \lim_{x\to\infty} f(x) = L$$

Theorem if $\lim\limits_{n\to\infty} a_n = K$ and $\lim\limits_{n\to\infty} b_n = L$

Then
$$\lim_{n\to\infty} (a_n + b_n) = K + L$$
$$\lim_{n\to\infty} (a_n b_n) = KL$$
$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{K}{L} \quad \text{if } L \neq 0$$

Proof: This true for the linear extensions of these sequences. $\blacksquare$

Theorem (squeeze) If $a_n \le b_n \le c_n \; \forall n \geqslant K$ and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} b_n$

then $\lim\limits_{n\to\infty} b_n = L$ as well.

Theorem $\lim\limits_{n\to\infty} \frac{1}{n} = 0 \iff \mathbb{R}$ has no infinitesimals.

Proof: Recall: a positive infinitesimal $\varepsilon > 0$ is an element such that $\varepsilon < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ $\blacksquare$

Example (squeeze). $\lim \frac{n!}{n^n} = 0$

Example (squeeze): $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$$0 \le \frac{n!}{n^n} = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{2}{n}\right)\frac{1}{n} \le \frac{1}{n}$$

$$\underset{0}{\downarrow} \qquad \frac{n}{n} \le 1,\ \frac{n-1}{n} \le 1,\ \ldots,\ \frac{2}{n} \le 1 \qquad \underset{0}{\downarrow}$$

So $\frac{n!}{n^n}$ is squeezed between 0 and $\frac{1}{n}$.

We say that a sequence converges if it has a finite limit. otherwise, we say it diverges.

We say that $\lim_{n\to\infty} a_n = \infty$ if for any positive real M we can find an index N s.t. $a_n > M \ \forall n > N$

(similar definition for $\lim_{n\to\infty} a_n = -\infty$).

Theorem. A function $f: S \to \mathbb{R}$ ($S \subseteq \mathbb{R}$) is continuous at $c \in S$ iff. for any sequence $\{a_n\}$ taking values in S s.t. $\lim_{n\to\infty} a_n = c$, $\lim_{n\to\infty} f(a_n) = f(c)$

Proof: $\Leftarrow$ Suppose f cont. at c. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$|x - c| < \delta$ & $x \in S \Rightarrow |f(x) - f(c)| < \varepsilon$

Now suppose $\{a_n\}$ is a sequence in S s.t. $\lim_{n\to\infty} a_n = c$

then for some N,

$|a_n - c| < \delta$ for $n > N$

$\Downarrow$

$|f(a_n) - f(c)| < \varepsilon$

hence $\lim_{n\to\infty} a_n = f(c)$

$\Rightarrow$ Suppose $\lim_{n\to\infty} a_n = f(c)$ for any sequence $\{a_n\}$ s.t. $a_n \to c$

we want to show that f is continuous at c.

Prove this by contradiction: Suppose f is not continuous at c.

Then there is some $\varepsilon_0 > 0$ s.t. there is no $\delta > 0$ s.t. the implication holds

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon_0$$

So for any n, $\delta = \frac{1}{n}$ will not work.

So we can find $x_n \in S$ so that

$$|x_n - c| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(c)| \geq \varepsilon_0$$

then $\lim_{n \to \infty} x_n = c$ but $\lim_{n \to \infty} f(x_n) \neq f(c)$

which is a contradiction. ■

Definition: We say that a sequence is 'increasing' (nondecreasing) if $a_n \leq a_m$ whenever $n < m$

'strictly increasing' if $a_n < a_m$ when $n < m$

similar definitions for 'Decreasing' & 'strictly decreasing'

Definition: a sequence is said to be (strictly) monotonic if it is either (strictly) increasing or (strictly) decreasing.

Definition: we say $\{a_n\}$ is bounded above if there is a u s.t. $a_n \leq u$ for all n

below if there is an L s.t. $a_n \geq L$ for all n.

Monotone Convergence property:

any bounded increasing sequence converges.
or decreasing

$\Longleftrightarrow$ CA.