Tensor Product of Algebras:

A_1 & A_2 are R-algebras.

$A_1 \otimes_R A_2$ is an R-algebra:

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$$

This works bc $((a_1, a_2), (b_1, b_2)) \mapsto$ is bilinear

Examples:

① $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R} \oplus \mathbb{R}i$ as a module.

$\cong$

$\mathbb{C}$ as a $\mathbb{Z}$-algebra.

② A: R-algebra. Then $R[x] \otimes_R A \cong A[x]$ as R-algebras

this is clear for their isomorphism as R-modules:

$$R[x] = R \oplus Rx \oplus Rx^2 \oplus \cdots$$

and $\otimes$ distributes over $\oplus$ and $RA \cong A$.

③ $R[x] \otimes R[y] \cong R[x,y]$ (free with basis $x^n \otimes y^m$).

④ X, Y – top. spaces. $C(X) \otimes_R C(Y)$

$$\sum_{i=1}^{n} f_i(x) \otimes g_i(y) \in C(X \times Y)$$

$C(X) \otimes C(Y)$ is a dense subset of $C(X \times Y)$ (for compact X & Y).

$$f(x) g(y) = (f \otimes g)(x,y)$$

Tensor Algebra of a Module:

Let M : R-module

Put $\mathcal{T}_1(M) = M$, $\mathcal{T}_2(M) = M \otimes M$, ... , $\mathcal{T}_n(M) = \underbrace{M \otimes \cdots \otimes M}_{n \text{ times}}$, ...

$$\mathcal{T}(M) = \underset{\substack{\shortparallel \\ \mathcal{T}_0(M)}}{R} \oplus \mathcal{T}_1(M) \oplus \mathcal{T}_2(M) \oplus \cdots \oplus \mathcal{T}_n(M) \oplus \cdots$$

$$= R \oplus M \oplus M \otimes M \oplus M \otimes M \otimes M \oplus \cdots$$

Element:

$$a + u + u_1 \otimes u_2 + v_1 \otimes v_2 \otimes v_3 + \cdots + w_1 \otimes \cdots \otimes w_k + \cdots$$

Let $(u_1 \otimes \cdots \otimes u_n) \cdot (v_1 \otimes \cdots \otimes v_m) = u_1 \otimes \cdots \otimes u_n \otimes v_1 \otimes \cdots \otimes v_m$

$\mathcal{T}(M)$ is called Tensor algebra of M.

$\mathcal{T}(M)$ is universal in the category

objects $\left(A : \text{unital } R\text{-algebra}, \ \varphi : M \to A \text{ is an } R\text{-module hom}\right)$

morphisms:

$$\begin{tikzcd} M \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & A_1 \\ & A_2 \arrow[u, "\text{algebra homomorphism}"'] \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A \\ \mathcal{T}(M) \arrow[ur] & \end{tikzcd}$$

$u_1 \otimes \cdots \otimes u_n \longmapsto \varphi(u_1) \cdots \varphi(u_n)$

M generates $\mathcal{T}(M)$ as an R-algebra

An algebra A is called <u>graded</u> if

$$A = A_0 \oplus A_1 \oplus \cdots = \bigoplus_{n=0}^{\infty} A_n$$

where A_n are submodules s.t. $\forall\, n, m$

$$A_n \cdot A_m \subseteq A_{n+m}$$

<u>Example</u>: $R[x] = R \oplus Rx \oplus Rx^2 \oplus \cdots$

$\mathcal{T}(M)$ is a graded algebra.

<u>Example</u>: $M = F^2$, where F is a field & $\{x, y\}$ is a basis.

$$\mathcal{T}(M) = F \oplus (Fx \oplus Fy) \oplus (F(x \otimes x) \oplus F(x \otimes y) \oplus F(y \otimes x) \oplus F(y \otimes y)) \oplus \cdots$$

$\cong$ polynomials in non-commuting x, y over F.

"Symmetrization" of $\mathcal{T}(M)$ want: $u_1 \otimes u_2 = u_2 \otimes u_1$

Let $\mathcal{C}(M)$ be the two-sided ideal in $\mathcal{T}(M)$ generated by tensors of the form $u_1 \otimes u_2 - u_2 \otimes u_1$, $u_i \in M$.

$$S(M) = \mathcal{T}(M) / \mathcal{C}(M)$$ - Symmetric tensor algebra of M.

ideal I in a graded algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded ideal if $I = \bigoplus_{n=0}^{\infty} (I \cap A_n)$.

in this case A/I is still a graded algebra.

$\mathcal{C}(M)$ is a graded ideal, so $S(M)$ is a commutative graded R-algebra.