

people.math.harvard/~mazur/preprints/when\_is\_one.pdf

## Category Theory

1945:  $V \cong V^*$ ;  $V \cong V^{**}$  more naturally.

give a rigorous def<sup>n</sup> to "natural"

A category  $\mathcal{C}$  consists of

- A class of objects, denoted  $\text{Ob}(\mathcal{C})$
- A set of morphisms between each pair of objects  $A, B$   
denoted  $\text{Hom}_{\mathcal{C}}(A, B)$  or  $\text{Hom}(A, B)$
- $\forall A, B, C \in \text{Ob}(\mathcal{C})$ , a set map  

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

$$(f, g) \mapsto g \circ f$$

Satisfying 3 axioms:

$$C1: (A, B) \neq (C, D) \Rightarrow \text{Hom}(A, B) \cap \text{Hom}(C, D) = \emptyset.$$

$$C2: \exists \text{id}_A \in \text{Hom}(A, A) \quad (\forall A \in \text{Ob}(\mathcal{C})) \text{ s.t.}$$

$$\forall f \in \text{Hom}(A, B), \quad f \circ \text{id}_A = f,$$

$$\forall g \in \text{Hom}(B, A), \quad \text{id}_A \circ g = g.$$

$$C3: \text{Associativity: } \forall f \in \text{Hom}(A, B), g \in \text{Hom}(B, C), h \in \text{Hom}(C, D), \quad h(gf) = (hg)f =: hgf.$$

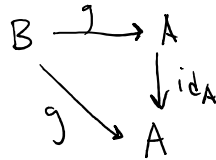
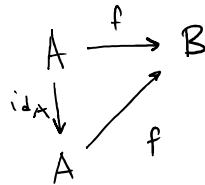
Class: Gödel-Bernays  
axioms for set theory.  
(keep it vague).

Good book by Paul Cohen:  
Set theory & the continuum  
hypothesis.

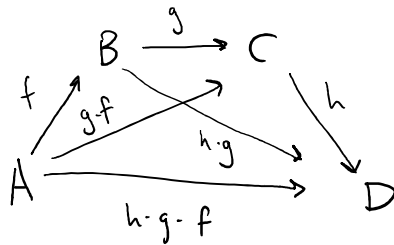
Note 1.  $\text{Hom}(A, B)$  could be empty.  
 $\text{Hom}(A, A)$  always has  $\text{id}_A$ .

2. We will use the usual notions of commutative diagrams  
 if  $f \in \text{Hom}(A, B) \rightsquigarrow A \xrightarrow{f} B$  or  $f: A \rightarrow B$ .

eg



unit  
diagrams



associativity  
diagram

### Examples

1. Set = category of sets & set maps

2. Mon = category of monoids & monoid homomorphisms

Grp, Ab, etc.

} product is  
compositions

Subcategory:  $\mathcal{D}$  subcategory of  $\mathcal{C}$  if

- $\text{Ob}(\mathcal{D})$  subclass of  $\text{Ob}(\mathcal{C})$
- $\text{Hom}_{\mathcal{D}}(A, B) \subset \text{Hom}_{\mathcal{C}}(A, B)$  if  $A, B \in \text{Ob}(\mathcal{D})$
- product of morphisms in  $\mathcal{D}$  agrees w/  $\mathcal{C}$ .

A subcategory is full if  $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) \quad \forall A, B \in \text{Ob}(\mathcal{D})$ .

(Note: axiom: if  $X, Y$  are classes &  $X \in Y$ , then  $X$  is a set).

Ab is a full subcategory of Grp.

Grp is a full subcategory of Mon.

Mon is not even a subcategory of Set (the objects are different).

Note:  $\text{Hom}(A, A)$  is a monoid.

eg let  $M$  be any monoid. form a category  $\underline{M}$  with

$$\cdot \text{Ob}(\underline{M}) = \{A\},$$

$$\cdot \text{Hom}(A, A) = M.$$

This is a small category ( $\text{ob}(\mathcal{C})$  forms a set).

Isomorphisms:  $\mathcal{C}$  is a category,  $A, B \in \text{Ob}(\mathcal{C})$ ,  $f \in \text{Hom}(A, B)$ .

$f$  is called an isomorphism if  $\exists g \in \text{Hom}(B, A)$  s.t.  $f \cdot g = \text{id}_B$ ,  $g \cdot f = \text{id}_A$ .

Defn the opposite or dual category:

if  $\mathcal{C}$  is a category,  $\mathcal{C}^{\text{op}}$  is the category w/ the

same objects, and if  $A, B \in \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}})$ , then

$$\text{Hom}_{\mathcal{C}^{op}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A).$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \cdot f & \downarrow g \\ & & C \end{array}$$

in  $\mathcal{C}$



$$\begin{array}{ccc} A & \xleftarrow{f} & B \\ & \nwarrow f \cdot g & \uparrow g \\ & & C \end{array}$$

in  $\mathcal{C}^{op}$

(so  $f^{op} \cdot g^{op} = (g \cdot f)^{op}$ ).