$a \in R$, $M$: $R$-module:

$aM$ is a submodule if $a \in Z(R)$.

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$$\left| \bigoplus_{B} F \right| = \max\{|F|, |B|\} \quad \text{(assuming both are infinite).}$$

$$\rotatebox{90}{=}$$

$$\bigcup_{\substack{C \subseteq B \\ \text{finite}}} F^{|C|} \qquad \left| \prod_{B} F \right| = \left| F^{B} \right| = \max\{|F|, 2^{|B|}\}.$$

$\varphi: V \longrightarrow W$ finite-dimensional

$\dim(\varphi(V)) = \dim V - \dim(\ker\varphi)$.

If $\dim V = \dim W$ then $\varphi$ is surj. $\Leftrightarrow$ $\varphi$ is injective.

If infinite-dimensional, this isn't true.

$V = \bigoplus_{n \in \mathbb{N}} F$

$\varphi_1(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ is inj. but not surj.

$\varphi_2(x_1, x_2, \dots) = (x_2, x_3, \dots)$ is surj but not inj.

$2\mathbb{Z} \subsetneq \mathbb{Z}$ - same rank, isomorphic as $\mathbb{Z}$-modules.

$\varphi: \mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$ is injective but not surjective.

## Tensor Product of Modules.

Let $R$ be a commutative unital ring.

Let $M_1, M_2$ be $R$-modules.

$$M_1 \times M_2 = M_1 \oplus M_2$$

$$(u_1, u_2) = \underset{(u_1,0)}{\underset{\shortparallel}{u_1}} + \underset{(0,u_2)}{\underset{\shortparallel}{u_2}}$$

---

$u_1, u_2$: $(u_1 + v_1) u_2 = u_1 u_2 + v_1 u_2$.

write $u_1 \otimes u_2$.

$$(M_1 \times M_2 =)\ T = \{u_1 \otimes u_2 : u_1 \in M_1,\ u_2 \in M_2\}.$$

$M = RT$, linear combinations of the tensors

$$= \left\{ \sum_{i=1}^{n} a_i\, u_{1,i} \otimes u_{2,i} : n \in \mathbb{N},\ a_i \in R,\ u_{1,i} \in M_1,\ u_{2,i} \in M_2 \right\}.$$

We want: $(u_1 + v_1) \otimes u_2 = u_1 \otimes u_2 + v_1 \otimes u_2$

Let $N$ be the submodule of $M$ generated by elements of the form

$$(u_1 + v_1) \otimes u_2 - u_1 \otimes u_2 - v_1 \otimes u_2,$$
$$u_1 \otimes (u_2 + v_2) - u_1 \otimes u_2 - u_1 \otimes v_2,$$
$$(a u_1) \otimes u_2 - u_1 \otimes (a u_2),$$
$$(a u_1) \otimes u_2 - a(u_1 \otimes u_2).$$

The tensor product $M_1 \otimes_R M_2 := M/N$.

$u_1 \otimes u_2$ - simple tensor

elements of $M_1 \otimes_R M_2$ are called tensors

they are liner combinations of simple tensors

Def: Let $M_1, M_2, N$ be $R$-modules.

A mapping $\beta : M_1 \times M_2 \longrightarrow N$ is bilinear if it's liner in both inputs.

Consider the category of bilinear mappings from $M_1, M_2$: objects are $(N, \beta : M_1 \times M_2 \to N$ bilinear$)$

morphisms are

$$\begin{tikzcd} N_1 \arrow[rr, "\varphi"] & & N_2 \\ & M_1 \times M_2 \arrow[ul, "\beta_1"] \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

$\varphi : N_1 \to N_2$ s.t. the diagram commutes.

we have a "standard" bilinear mapping $\tau : M_1 \times M_2 \longrightarrow M_1 \otimes M_2$

$$(u_1, u_2) \longmapsto u_1 \otimes u_2$$

Claim: $M_1 \otimes M_2$ is a universal repelling object in this category.

Proof: Let $\beta : M_1 \times M_2 \longrightarrow N$ be bilinear.

define $\varphi : M \longrightarrow N$ by $\varphi(u_1 \otimes u_2) = \beta(u_1, u_2)$.

$M = R\{u_1 \otimes u_2 : u_1 \in M_1, u_2 \in M_2\} = RT$

Since $M$ is a free module generated by $T$, such a homomorphism $\varphi$ exists and is unique.

Let $K =$ submodule of $M$ generated by relations $(u_1 + v_1) \otimes u_2 - \cdots$

Since $\beta$ is bilinear, $\varphi(K) = 0$.

So $\varphi : M/K \longrightarrow N$.

$M/K = M_1 \otimes M_2$

Such a homism is unique since it must be that $\varphi(u_1 \otimes u_2) = \beta(u_1, u_2)$ for the diagram

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\varphi"] & N \\ M_1 \times M_2 \arrow[u, "\tau"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

to be commutative, and simple tensors generate $M_1 \otimes M_2$.

Examples: $0 \otimes M = \{\sum_{i=1}^{n} a_i(0 \otimes u_i)\} = \{\sum (a_i 0 \otimes u_i)\} = \{\sum 0 \otimes u_i\} = \{0 \otimes \sum u_i\} = \{0 \otimes u \text{ for some } u \in M = \sum u_i\}$.

and $0 \otimes u = (0+0) \otimes u = 0 \otimes u + 0 \otimes u$ so $0 \otimes u = 0$.

So $0 \otimes M = 0$.

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$R \otimes M \ni \sum a_i(b_i \otimes u_i) = \sum a_i b_i (1 \otimes u_i) = \sum 1 \otimes a_i b_i u_i$

$= 1 \otimes \sum a_i b_i u_i$

so $R \otimes M \ni$ only $1 \otimes u$ for $u \in M$.

claim: $R \otimes M \cong M$. proof: consider the mapping

$\beta : R \times M \longrightarrow M$ defined by $\beta(a, u) = au$.

$\beta$ is bilinear so $\exists!$ hom-sm $\varphi : R \otimes M \longrightarrow M$

s.t. $\varphi(a \otimes u) = au \quad \forall\ a \in R,\ u \in M$.

$\varphi$ is surj. since $u = \varphi(1 \otimes u) \quad \forall\ u \in M$.

define $\psi : M \longrightarrow R \otimes M$ by $\psi(u) = 1 \otimes u$.

Then $\forall u \in M$, $\varphi(\psi(u)) = u$,

$\forall a \in R, u \in M$, $\psi(\varphi(a \otimes u)) = \psi(au) = 1 \otimes au = a(1 \otimes u) = a \otimes u$.

Since simple tensors generate $R \otimes M$,

$$\psi \circ \varphi = \mathrm{Id}_{R \otimes M}.$$