Lec 1/13

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Natural transformations

Example (E-M)

duality

$$\underline{R\text{-mod}} \xrightarrow{D} \underline{\text{mod-}R}$$

$$M \longmapsto M^*$$

$$\begin{array}{ccccc} M & & M & & M^{**} \\ L \downarrow & \xrightarrow{D} & L^* \uparrow & \longrightarrow & L^{**} \downarrow \\ N & & N^* & & N^{**} \end{array}$$

D^2 is covariant

$D^2(M) = M^{**}$

defn of L^*

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[dr, "g \cdot L = L^*(g)"'] & N \arrow[d, "g"] \\ & R \end{tikzcd}$$

So defn of L^{**} is

$$\begin{tikzcd} N^* \arrow[r, "L^*"] \arrow[dr, "L^{**}(\varphi) = \varphi \cdot L^*"'] & M^* \arrow[d, "\varphi"] \\ & R \end{tikzcd}$$

If $g : N \longrightarrow R$

$(g \in N^*)$

$$(L^{**}\varphi)(g) = \varphi(L^*(g)) = \varphi(gL).$$

• If $x \in M$ then x defines an elt $\eta_M(x) \in M^{**}$ by $\eta_M(x)(f) = f(x)$, $f \in M^*$.

Claim: η defines a natural transf. $1_{\underline{R\text{-mod}}} \longrightarrow D^2_{\underline{R\text{-mod}}}$

$$
\begin{tikzcd}
M \arrow[d, "L"'] \\
N
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "L"'] & M^{**} \arrow[d, "L^{**}"] \\
N \arrow[r, "\eta_N"'] & N^{**}
\end{tikzcd}
$$

η defines a natural transf if that square commutes. $(\forall M, N, L)$.

Let $x \in M$. $L(x) \in N$ and $\eta_N(L(x))(g) = g(L(x)) \in R \quad \forall g \in N^*$.

Also, $L^{**}(\eta_M(x))(g) = \eta_M(x)(L^* g) = \eta_M(x)(gL) = g(L(x)) \quad \forall g \in N^*$.

So the square commutes. ✓

<u>Note</u> if $R = K$ then $\underline{R\text{-mod}} = \underline{\text{vect}}_K$, and if we restrict to $\underline{\text{vect}}_K^{fd}$ then η is in fact a natural isomorphism $\text{id} \xrightarrow{\sim} (-)^{**}$.

<u>Example</u> Abelianization. $F : \underline{\text{Grp}} \longrightarrow \underline{\text{Ab}}$

$$G \longmapsto G/[G,G] =: G^{ab}$$

We can compose this w/ the injection functor $\underline{\text{Ab}} \hookrightarrow \underline{\text{Grp}}$ to get $F' : \underline{\text{Grp}} \longrightarrow \underline{\text{Grp}}$.

Let $\nu_G : G \longrightarrow G^{ab}$ is the natural projection then

$G \xrightarrow{\nu_G} G^{ab}$

$$\begin{tikzcd} G \arrow[r, "\nu"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & G^{ab} \arrow[d, "F'(f)"] \\ H \arrow[r, "\nu_H"'] & H^{ab} \end{tikzcd}$$

ν is a natural trans

$$1_{Grp} \xrightarrow{\nu} F'.$$

Isomorphisms and equivalences of categories

Isomorphism: C and D are isomorphic if $\exists$ functors

$$F : C \longrightarrow D \quad \text{and} \quad G : D \longrightarrow C \quad \text{s.t.}$$

$$FG = 1_D \quad \text{and} \quad GF = 1_C.$$

Examples $\underline{Ab} \cong \mathbb{Z}\text{-}\underline{mod}$

$R\text{-}\underline{mod} \cong \underline{mod\text{-}R}^{op}$ reversing order of multiplication in R.

Equivalence: C and D are equivalent if $\exists$ functors

$$F : C \longrightarrow D \quad \text{and} \quad G : D \longrightarrow C \quad \text{s.t.}$$

$$FG \cong 1_D \quad \text{and} \quad GF \cong 1_C$$

natural isomorphism

Remarks
- isomorphic categories are equivalent.
- Equivalence of categories is an equivalence relation.
- If you have an isomorphism, F & G are really inverses and F determines G. This is not the case for equivalences.

Can you tell, just given F, whether F determines an eq. of categories?

If F is part of an eq. of categories with G, then

(i) since $GF \simeq 1_C$, then $\mathrm{Hom}_C(A,B) \simeq \mathrm{Hom}_C(GFA, GFB)$

Then F must be injective on morphisms (faithful).

(ii) since $FG \simeq 1_D$, F must be surjective on morphism (full).

(iii) If $A \in ob(D)$, $FG \simeq 1_D$, so $\exists$ isomorphism $\eta_{A'}$ in $\mathrm{Hom}_D(A', FGA')$.

if we let $A = GA'$, then $\exists A \in C$

s.t. $A' \simeq FA$.

(this is called **essential surjectivity**).

$$\begin{tikzcd} A' \arrow[r, "\eta_{A'}"] \arrow[d, "f"'] & FGA' \arrow[d, "FGf"] \\ B' \arrow[r, "\eta_{B'}"'] & FGB' \end{tikzcd}$$

<u>Theorem</u>: These three things are sufficient. i.e., if $F : C \longrightarrow D$ is a functor then ($\exists$ a functor $G : D \longrightarrow C$ s.t. (F,G) determines an equivalence of categories) iff
(F is faithful (i), F is full (ii), and F is essentially surjective (iii)).

Pf (sketch) use (iii) to define G (using axiom of large choice) on objects.
for morphisms,

$$\begin{tikzcd} A' \arrow[r, "\eta_{A'}", "\sim"'] \arrow[d, "f'"'] & F(A) \arrow[d, dotted, "F(f)"] & = & F(G(A')) \arrow[d, dotted, "F(G(f'))"] \\ B' \arrow[r, "\sim", "\eta_{B'}"'] & F(B) & = & F(G(B')) \end{tikzcd}$$

$\eta_{B'} \cdot f' \cdot \eta_{A'}^{-1} \in \mathrm{Hom}(F(A), F(B))$, use faithfull & full

to say $\exists!\ f \in \mathrm{Hom}(A,B)$ s.t. $\eta_{B'} \cdot f' \cdot \eta_{A'}^{-1} = F(f)$.

So $G(f') = f$.

Then check.

$\square$

<u>Example</u> Let R be a ring, $M_n(R) =$ $n\times n$ matrices over R.

Then $\underline{\text{mod}}\text{-}R$ and $\underline{\text{mod}}\text{-}M_n(R)$ are equivalent.

<u>Pf</u> Here's $F: \underline{\text{mod-}R} \longrightarrow \underline{\text{mod -}}M_n(R)$

$M \in ob(\underline{\text{mod}}\text{-}R)$, let $F(M) = M^{(n)} = \underbrace{M \oplus \cdots \oplus M}_{n \text{ times}}$

$f \in Hom(M, N)$, let $F(f) = f^{(n)}$,

where $f^{(n)}(m_1, \ldots, m_n) = (f(m_1), \ldots, f(m_n))$.

Now check F is faithful, full, ess. surj.

(find out about ideals in $M_n(R)$).