Lec 11/21

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$\sum_{j=1}^{\infty} a_j$ $\qquad$ $q =$ largest cluster point of $\{|a_n|^{1/n}\}_{n=1}^{\infty}$

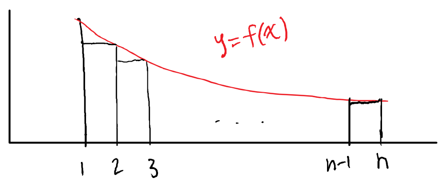
$q<1 \Rightarrow$ convergence.

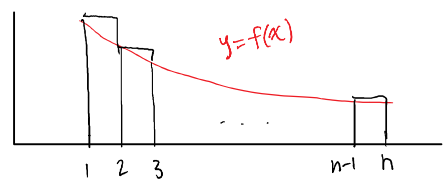
$q>1 \Rightarrow$ divergence.

$q=1 \Rightarrow$ inconclusive. $\rightarrow$ need additional convergence tests.

Integral Test: Suppose that $f:[1,\infty) \to \mathbb{R}$ is continuous, positive, and decreasing. Then $\sum_{j=1}^{\infty} f(j)$ converges iff $\int_1^{\infty} f(t)\,dt$.

Proof by picture:





lower approx: $\sum_{j=2}^{n} f(j)$ $\qquad$ upper approx: $\sum_{j=1}^{n-1} f(j)$

So $\sum_{j=2}^{n} f(j) \le \int_1^n f(t)\,dt \le \sum_{j=1}^{n-1} f(j)$ (all increasing as $n\to\infty$)

They converge iff they are bounded.

So if $\int_1^{\infty} f(t)\,dt$ converges, $\sum_{\substack{j=2 \\ \downarrow 1}}^{\infty} f(j)$ converges. ($\int_1^{\infty} f(t)\,dt$ is a bound)

and if $\sum_{j=1}^{\infty} f(j)$ converges, $\int_1^{\infty} f(t)\,dt$ converges ($\sum_{j=1}^{\infty} f(j)$ is a bound). $\blacksquare$

P-series: $\sum_{j=1}^{\infty} \frac{1}{j^p}$

Corollary of integral test: P-series converges iff $p>1$, diverges iff $p<1$.

<u>Proof:</u> if $p \le 0$, p-series diverges since $\frac{1}{n^p} \not\to 0$ as $n\to\infty$.

if $p>0$, then can apply integral test:

$$\sum_{j=1}^{\infty} \frac{1}{j^p} \text{ converges} \Leftrightarrow \int_1^{\infty} \frac{1}{t^p}\,dt \text{ converges}$$

$\int t^{-p}\,dt = \frac{t^{-p+1}}{-p+1}$ if $p \neq 1$ $\qquad$ $\log t$ if $p=1$.

So $\int_1^{\infty} \frac{1}{t^p}\, dt = \frac{1}{1-p} - \lim_{n\to\infty} \frac{p^{-p+1}}{-p+1}$ if $p \neq 1$ { this diverges if $p<1$, goes to 0 if $p>1$.

$= \lim_{n\to\infty} \log p \to$ diverges

Note: convergence parameter for p-series is 1 for any p.

$$\lim_{n\to\infty} \left|\frac{1}{n^p}\right|^{1/n} = \lim_{n\to\infty} n^{-p/n} = \lim_{n\to\infty} e^{-p/n \log(n) \to 0 \text{ by L'H}} \} \to 1$$

From proof of the integral test, for $p>1$ we have

$$\int_{n+1}^{\infty} \frac{1}{t^p}\, dt < R_n = \sum_{j=n+1}^{\infty} \frac{1}{j^p} < \int_n^{\infty} \frac{1}{t^p}\, dt$$

$$\left(\frac{1}{p-1}\right) \frac{1}{(n+1)^{p-1}} < R_n < \left(\frac{1}{p-1}\right) \frac{1}{n^{p-1}}$$

So if $p=3$, need 100001 terms to compute $\sum_{j=1}^{\infty} \frac{1}{j^p}$ to 10 digits.

Limit Comparison Test: suppose that $\sum_{j=1}^{\infty} a_j$, $\sum_{j=1}^{\infty} b_j$ are series w/ positive terms, (after a point). and suppose that $\lim_{n\to\infty} a_n/b_n = L > 0$. (*)

Then $\sum_{j=1}^{\infty} a_j$ converges iff $\sum_{j=1}^{\infty} b_j$ converges, and they have same convergence parameter.

Proof: (*) implies that for some N, $\frac{1}{2}L \le \frac{a_n}{b_n} \le \frac{3}{2}L$ for $n \ge N$.

So $\frac{1}{2}L b_n \le a_n \le \frac{3}{2}L b_n$. If $\sum_{j=1}^{\infty} b_j$ converges, so does $\sum_{j=1}^{\infty} \frac{3}{2}L b_j$.

So by comparison test, $\sum_{j=1}^{\infty} a_j$ converges.

if $\sum_{j=1}^{\infty} a_j$ converges, so does $\sum_{j=1}^{\infty} \frac{2}{L} a_j$ so $\sum_{j=1}^{\infty} b_j$ converges.

Also, $\underbrace{\left(\tfrac{1}{2}L\right)^{1/n}}_{\downarrow\ 1 \text{ as } n\to\infty} b_n^{1/n} \le (a_n)^{1/n} \le \underbrace{\left(\tfrac{3}{2}L\right)^{1/n}}_{\downarrow\ 1 \text{ as } n\to\infty} b_n^{1/n}$ so conv. params. are same. ∎

Variation of LCT: If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges and $\quad cp(\sum a_n) \leq cp(\sum b_n)$.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+5n^2-6n+12}{2n^5-3n^2+4n+5} = a_n \qquad$ let $b_n = \dfrac{1}{n^2}$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$. $\quad b_n$ converges by p-series test. $\quad cp(b_n)=1 \quad$ so $cp(a_n)=1$.

Absolute vs. Conditional Convergence:

Definition $\displaystyle\sum_{j=1}^{\infty} a_j$ converges absolutely if $\displaystyle\sum_{j=1}^{\infty} |a_n|$ converges.

Theorem: if a series converges absolutely, it converges

Proof: let $\quad b_n = \begin{cases} a_n = |a_n| & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases}$

$c_n = \begin{cases} -a_n = |a_n| & \text{if } a_n < 0 \\ 0 & \text{if } a_n \geq 0 \end{cases}$

Then $\forall n$, $\quad |a_n| = b_n + c_n \quad$ and $\quad a_n = b_n - c_n$

[observation: a series w/ nonnegative terms converges iff $\{S_n\}$ is bounded (MCP).

if $\displaystyle\sum_{j=1}^{\infty} |a_j| < \infty, \Rightarrow \sum_{j=1}^{\infty} b_j < \infty$ and $\displaystyle\sum_{j=1}^{\infty} c_j < \infty$

so $\quad \displaystyle\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} b_j - c_j < \infty$. $\quad\blacksquare$

Definition: A series converges conditionally if it converges but not absolutely