2/18

Monday, February 18, 2019 11:34

$$\begin{tikzcd} 0 \arrow[r] & 2\mathbb{Z} \arrow[r, hook] \arrow[d, "\div 2"'] & \mathbb{Z} \arrow[dl, "\times"] \\ & \mathbb{Z} & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & aM \arrow[r] \arrow[d, "\div a"'] & M \arrow[dl] \\ & M & \end{tikzcd}$$

so injective $\Rightarrow$ divisible

$M \cong R^n$, $\varphi \in \mathrm{End}(M) \Rightarrow \det\varphi$

$$\Lambda^n\varphi : \underset{\substack{\cong \\ R}}{\Lambda^n M \longrightarrow \Lambda^n M} \quad \text{where} \quad u_1 \wedge \cdots \wedge u_n \longrightarrow \varphi(u_1) \wedge \cdots \wedge \varphi(u_n)$$

So $\Lambda^n\varphi(\omega) = c\,\omega$, $\quad c = \det\varphi$.

① det doesn't depend on basis

$\det A_\varphi = \det\varphi$ is the same $\forall$ bases in M

② $\det(\varphi \circ \psi) = \det\varphi \cdot \det\psi$

③ φ is invertible iff $\det\varphi \in R^\times$

($\Rightarrow$) obvious

($\Leftarrow$) $A = \begin{pmatrix} \cdot & | & \cdot \\ \hline & + & \\ \cdot & | & \cdot \end{pmatrix} \Rightarrow \mathrm{Adj}\,A = \begin{pmatrix} \vdots & \det\text{Minor} & \vdots \end{pmatrix}$, divide by det to get inverse.

$$A^{-1} = \frac{1}{\det A} (\operatorname{Adj} A)^T$$

Pairing: bilinear mapping $M \times \Lambda^{n-1} M \longrightarrow \Lambda^n M \cong R$

$$(u, \omega) \longmapsto u \wedge \omega$$

So, informally $M \cong (\Lambda^{n-1} M)^*$

If $\{u_1, \dots, u_n\}$ is a basis in M, then

$\{u_2 \wedge \cdots \wedge u_n, -u_1 \wedge u_3 \wedge \cdots \wedge u_n, \dots, \pm u_1 \wedge \cdots \wedge u_{n-1}\}$ is the dual basis in $\Lambda^{n-1} M$

Let $\varphi \in \operatorname{End}(M)$. Then we have $\Lambda^{n-1}\varphi \in \operatorname{End}(\Lambda^{n-1} M)$.

Let $\psi = (\Lambda^{n-1}\varphi)^* : M \longrightarrow M$ w.r.t this pairing.

$$\psi(u)(\omega) = u(\Lambda^{n-1}\varphi(\omega))$$

$$\varphi^*(f)(u) = f(\varphi(u))$$

$$\langle Au, v \rangle = \langle u, A^* v \rangle$$

$$\psi(\varphi(u))(\omega) = \varphi(u)(\Lambda^{n-1}\varphi(\omega))$$

$$= \varphi(u) \wedge (\Lambda^{n-1}\varphi(\omega))$$

$$= \Lambda^n \varphi(u \wedge \omega)$$

$$= \det\varphi \cdot (u \wedge \omega) = \det\varphi \cdot u(\omega)$$

$\psi(\varphi(u)) = \det\varphi \cdot u$, $\psi \circ \varphi = \det\varphi \cdot \mathrm{Id}_M$.

$\varphi \in \mathrm{End}(M) \Rightarrow \psi \in \mathrm{End}(M)$ s.t. $\psi \circ \varphi = \det\varphi \cdot \mathrm{Id}_M$

So $\det\varphi \in R^\times \Rightarrow \dfrac{1}{\det\varphi} \cdot \psi = \varphi^{-1}$.

Left inverse $\Rightarrow$ right inverse.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] & M \arrow[r, "\tilde{\psi}"] & M \arrow[r] & 0 \quad ??? \\ & & M \arrow[u] \arrow[ur] & & \end{tikzcd}$$

A - $n \times n$ matrix, $\det A = \det\varphi$, $\varphi(u) = Au \ \forall u \in R^n$.

$\det A = v_1 \wedge \cdots \wedge v_n$ when $A = (v_1 | \cdots | v_n)$.

$= \det A \cdot e_1 \wedge \cdots \wedge e_n$.

$\det A = 0$ if v_i are lin. dep.

Elementary Column operations

1. Multiply a column by a number a – then $\det A$ is multiplied by a.
2. Switch 2 columns: $\det A$ changes sign.
3. Add a multiple of 1 column to another column: $\det A$ doesn't change.

$\det A^T = \det A$ since $\det \varphi^* = \det \varphi$.

$\varphi \in \mathrm{End}(M) \Rightarrow \Lambda^n \varphi$

$\varphi^* \in \mathrm{End}(M^*) \Rightarrow \Lambda^n \varphi^* = (\Lambda^n \varphi)^*$

$= \det \varphi \in R$

$= \det \varphi^* \in R$

$\exists$ pairing $\Lambda^n M \times \Lambda^n (M^*) \longrightarrow R$

s.t. $\Lambda^n(M^*) = (\Lambda^n M)^*$

$\Lambda^n \varphi^* \in \mathrm{End}(\Lambda^n M^*)$, $(\Lambda^n \varphi)^* \in \mathrm{End}((\Lambda^n M)^*)$

Rows of A = columns of A^T

$\Rightarrow$ row ops have same properties as column ops.

$(AB)^T = B^T A^T$ since $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$

$$\begin{tikzcd}
N \arrow[r] \arrow[dr, bend right] & M \arrow[r] \arrow[d] \arrow[l, phantom, "\leftarrow" below] & K \arrow[dl, bend left] \arrow[l, phantom, "\leftarrow" below] \\
 & R &
\end{tikzcd}$$