Lec 9/9

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(X, ρ) metric space

An outer measure $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ is a metric outer measure if

$$\rho(A, B) > 0 \Rightarrow \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

<u>Prop</u> if μ^* is a metric outer measure on (X, ρ), then $\mathcal{B}_\rho \subset \mathcal{M}^*$.

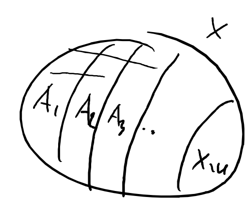
~~pf~~ Let $U \subset X$ be open.

<u>Step 1</u>: We may assume $\rho(U, X \setminus U) = 0$.

otherwise, $\forall F \subset X$, $\rho(F \cap U, F \setminus U) > 0$, so

$$\mu^*(F) = \mu^*(U \cap F) + \mu^*(F \setminus U),$$

so $U \in \mathcal{M}^*$.



<u>Step 2</u>: For $n \in \mathbb{N}$, define $A_n = \{x \in U \mid \rho(x, X \setminus U) > \frac{1}{n}\}$

Then $A_n \subset A_{n+1}$ $\forall n$ and $U = \bigcup A_n$.

Set $A_0 = \emptyset$, define $B_n = A_n \setminus A_{n-1}$ $\forall n \in \mathbb{N}$.

then $\coprod B_n = U$, and $B_n \neq \emptyset$ <u>frequently</u> $\leftarrow$ infinitely many times

Step 3: If $|m-n|>1$ & $B_m \neq \emptyset \neq B_n$, then $\rho(B_m, B_n) > 0$.

Suppose w.o.l.o.g $1 \leq m < n-1$. let

$x \in B_m$, $y \in B_n$. Then $y \notin A_{n-1} \supset A_{m+1}$, so

$\exists\, z \in X \setminus U$ s.t. $\rho(y,z) \leq \frac{1}{m+1}$.

But $x \in B_m$, so $\rho(x,z) > \frac{1}{m}$. by Δ-ineq,

$$\rho(x,y) \geq \rho(x,z) - \rho(y,z) > \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}.$$

Hence $\rho(B_m, B_n) \geq \frac{1}{m(m+1)} > 0$.

Step 4: Let $F \subset X$. If $\mu^*(F) = \infty$, then

$\mu^*(F) \geq \mu^*(F \cap U) + \mu^*(F \setminus U)$, so assume $\mu^*(F) < \infty$.

Then $\sum_{n=k}^{\infty} \mu^*(F \cap B_n) \longrightarrow 0$ as $k \to \infty$.

Pf by step 3, $\forall k \in \mathbb{N}$,

$$\sum_1^k \mu^*(F \cap B_{2n}) = \mu^*\left(\coprod_1^k (F \cap B_{2n})\right) \leq \mu^*(F).$$

$$\sum_1^k \mu^*(F \cap B_{2n+1}) = \mu^*\left(\coprod_1^k (F \cap B_{2n+1})\right) \leq \mu^*(F).$$

So taking $k \to \infty$,

$$\sum \mu^*(F \cap B_n) \leq 2\mu^*(F) < \infty,$$

So tail $\longrightarrow 0$.

<u>Step 5:</u> Now $\forall n \in \mathbb{N}$,

$$\mu^*(F \cap U) + \mu^*(F \setminus U) \leq \mu^*(F \cap A_n) + \mu^*(F \cap (U \setminus A_n)) + \mu^*(F \setminus U)$$

$$\rho(F \cap A_n, F \setminus U) \geq \rho(A_n, X \setminus U) \geq \frac{1}{n} > 0.$$

$$\text{So RHS} = \mu^*(F \cap (A_n \cup F \setminus U)) + \mu^*(F \cap \underbrace{(U \setminus A_n)}_{= \coprod_{n+1}^{\infty} B_k})$$

$$\leq \mu^*(F) + \sum_{n+1}^{\infty} \mu^*(F \cap B_k) \to \mu^*(F) \text{ as } n \to \infty. \qquad \square$$

(X, ρ) metric space. $p \geq 0$, $\varepsilon > 0$, define $\forall E \subset X$

$$\eta^*_{p,\varepsilon}(E) := \inf \left\{ \sum_{1}^{\infty} (\operatorname{diam} B_n)^p \;\middle|\; \begin{array}{l} (B_n) \text{ set of open balls} \\ \text{with } \operatorname{diam} B_n \leq \varepsilon \\ \text{such that } E \subset \bigcup B_n \end{array} \right\}$$

(convention: $\inf \emptyset = \infty$)

<u>Claim</u> $\eta^*_{p,\varepsilon}$ is an outer measure $\forall \varepsilon > 0$.

<u>Observe</u>: if $\varepsilon < \varepsilon'$ then $\eta^*_{p,\varepsilon}(E) \geq \eta^*_{p,\varepsilon'}(E)$

define $\eta^*_p(E) = \lim_{\varepsilon \to 0} \eta^*_{p,\varepsilon}(E)$. This is well defined.

$\hookrightarrow = \sup_{\varepsilon > 0} \eta^*_{p,\varepsilon}(E)$

Lemma: Suppose $(\mu_i^*)_{i\in I}$ is a family of outer measures on X. Then $\mu^*(E) := \sup_{i\in I} \mu_i^*(E)$ is an outer measure

maybe not true, think about this

Moreover, if $S \subset \mathcal{M}_i^*$ $\forall i \in I$, then $S \subset \mathcal{M}^*$ for μ^*.

Pf Exercise.

Prop: η_p^* is a metric outer measure

Pf suppose $\rho(E,F) > \varepsilon > 0$.

Choose an ε-covering (B_n) of $E \amalg F$.

Then $\forall n$, B_n intersects at most one of E or F.

$\to$ partition (B_n) into (B_n^E) & (B_n^F) s.t.

$E \subset \bigcup B_n^E$ and $B_n^E \cap F = \emptyset$ $\forall n$ (and vice versa).

Thus $\eta_{p,\varepsilon}^*(E) + \eta_{p,\varepsilon}^*(F) \leq \sum (\operatorname{diam} B_n^E)^p + \sum (\operatorname{diam} B_n^F)^p$

$= \sum (\operatorname{diam} B_n)^p$

$\Rightarrow \forall \varepsilon < \rho(E,F)$, $\eta_{p,\varepsilon}^*(E) + \eta_{p,\varepsilon}^*(F) \leq \eta_{p,\varepsilon}^*(E \amalg F)$

$\leq \eta_{p,\varepsilon}^*(E) + \eta_{p,\varepsilon}^*(F)$.

Let $\varepsilon \to 0$. $\square$

Def Let $\eta_p := \eta_p^* \big|_{\mathcal{B}_p \subset \mathcal{M}^* \text{ for } \eta_p^*}$

Called p-dimensional Hausdorff measure

Properties:

① $\forall$ isometry $f: X \longrightarrow X$, $\eta_p(E) = \eta_p(f(E))$

pf $\forall \varepsilon > 0$, $\eta^*_{p,\varepsilon}(E) = \eta^*_{p,\varepsilon}(f(E))$ as $E \subset \bigcup B_n \Leftrightarrow f(E) \subset \bigcup f(B_n)$ ...

② η_1 on $\mathcal{B}_{\mathbb{R}}$ is $\lambda\big|_{\mathcal{B}_{\mathbb{R}}}$

pf follow by ① & uniqueness of λ on $\mathcal{B}_{\mathbb{R}}$.

③ if $\eta_p(E) < \infty$ then $\eta_q(E) = 0 \quad \forall q > p$

pf Let $\varepsilon > 0$. $\exists\, (B_n)$ s.t. $E \subset \bigcup B_n$, $\operatorname{diam} B_n \le \varepsilon$, and $\sum (\operatorname{diam} B_n)^p \le \eta_p(E) + 1$

But if $q > p$ then
$$\sum (\operatorname{diam} B_n)^q = \sum \underbrace{(\operatorname{diam} B_n)^{q-p}}_{\le \varepsilon^{q-p}} (\operatorname{diam} B_n)^p$$
$$\le \varepsilon^{q-p} \sum (\operatorname{diam} B_n)^p$$
$$\le \varepsilon^{q-p} \left[\eta_p(E) + 1\right]$$

So $\forall \varepsilon > 0$, $\eta^*_{p,\varepsilon}(E) \le \varepsilon^{q-p}(\eta_p(E) + 1) \longrightarrow 0$ as $\varepsilon \to 0$.

④ if $\eta_p(E) > 0$, then $\eta_q(E) = \infty \quad \forall q < p$

pf contrapositive of ③.

Def If $E \in \mathcal{B}_p$, its Hausdorff dimension is

$$\inf \{ p \ge 0 \mid \eta_p(E) = 0 \} = \sup \{ p \ge 0 \mid \eta_p(E) = \infty \}$$

<u>example</u> Hausdorff dim of C is $\frac{\ln 2}{\ln 3}$.