Solvable Groups:

$$G^{(0)} = G, \quad G^{(i+1)} = [G^{(i)}; G^{(i)}]$$

G is solvable if $\exists n \geq 0$ s.t. $G^{(n)} = \{e\}$.

Note $G^{(i)}/G^{(i+1)}$ is abelian.

<u>Thm</u>: G is solvable iff $\exists$ composition series

Σ: $G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$ s.t. H_i/H_{i+1} is abelian

<u>Proof</u> $(\Rightarrow)$ ✓

$(\Leftarrow)$ Idea: prove by induction that $G^{(\ell)} \subset H_\ell$.

If true, $G^{(n)} \subset \{e\} \Rightarrow G$ solvable.

Base case: $\ell = 0$. $G^{(0)} = H_0 = G$.

Induction Step: $H_\ell/H_{\ell+1}$ is abelian $\Rightarrow [H_\ell ; H_\ell] \subset H_{\ell+1}$

$$G^{(\ell+1)} = [G^{(\ell)}; G^{(\ell)}]$$
$$\subset [H_\ell, H_\ell] \quad \text{inductive step.}$$
$$\subset H_{\ell+1}$$

$\square$

<u>Application</u>: Every p-group is solvable.

<u>Pf</u> We proved $|G| = p^n \Rightarrow Z(G) \neq \{e\}$

$\uparrow$ abelian, normal in G.

using $|G|=p^n$, $G \curvearrowright X \Rightarrow |X| = |X^G|$ (mod p)

and using $G \curvearrowright G$ by conj.

<u>So</u> $|G| = p^n \Rightarrow G$ has a composition series

$G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$ with abelian graded pieces

Pf Induction on n:

$n=1$ $\quad G = \mathbb{Z}/p\mathbb{Z}$ is abelian

$n>1$ replace G by $G/Z(G) = \bar{G}$.

by induction we have

$$\bar{\Sigma}: \bar{G} = \bar{K}_0 \trianglerighteq \bar{K}_1 \trianglerighteq \cdots \trianglerighteq \bar{K}_s = \{e\}$$

$\pi: G \longrightarrow G/Z(G)$ $\quad$ set $H_\ell = \pi^{-1}(\bar{K}_\ell)$

$$H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_s = Z(G) \trianglerighteq \{e\}$$

<u>Ex</u>: Show $H_\ell / H_{\ell+1} \cong \bar{K}_\ell / \bar{K}_{\ell+1}$. $\square$

<u>Application 2</u>: Let $N \trianglelefteq G$.

Then G is solvable $\Leftrightarrow$ N and G/N are solvable.

<u>Pf</u> ($\Rightarrow$) G is solvable means $G^{(n)} = \{e\}$. but $N^{(n)} \subset G^{(n)} \Rightarrow N$ is solvable.

true for any subgroup, not nec. normal.

$G \xrightarrow{\pi} G/N$ is a group hom so $\pi(G^{(\ell)}) = (\pi(G))^{(\ell)}$ $\quad (\pi(aba^{-1}b^{-1}) = \pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1})$ ...

so $\pi(G^{(n)}) = (\pi(G))^{(n)} \Rightarrow \pi(G) = G/N$ is solvable.

($\Leftarrow$) Assume N & G/N are solvable.

$\Sigma_1 : N = H_0 \trianglerighteq \cdots \trianglerighteq H_k = \{e\}$

$\mathcal{T} : G/N = K \trianglerighteq \cdots \trianglerighteq K_\ell = \{e\}$

w/ abelian graded pieces

$\Sigma_2 : G/N = K_0 \trianglerighteq \cdots \trianglerighteq K_\ell = \{e\}$

Aim: produce one such for G.

Set $G_j = \pi^{-1}(K_j) \quad 0 \le j \le \ell$

$G_\ell = N, \; G_{\ell+t} = H_t \quad (0 \le t \le k)$

$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_\ell = N = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_k = \{e\}$

Same ex: $G_i/G_{i+1} \cong K_i/K_{i+1}$ □

<u>Serre Property</u> is a property like this (true for sub & quotient $\Rightarrow$ true for whole)

If G is finite & solvable, graded pieces of any of its J-H series are both abelian & simple, so $\mathbb{Z}/p\mathbb{Z}$.

<u>Pf</u> $\Sigma : G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_s = \{e\}$ J-H series.

take $\Sigma' : G = K_0 \trianglerighteq \cdots \trianglerighteq K_t = \{e\}$ w/ abelian graded pieces (because G is solvable)

We can find a common refinement of Σ and Σ', Σ''.

the graded pieses of Σ'' are abelian since ... *

but the nontrivial gp are those in Σ □

$* : \quad K_0 \trianglerighteq K_1 \underset{\trianglerighteq J \trianglerighteq}{\trianglerighteq} K_2 \trianglerighteq \cdots \trianglerighteq K_t = \{e\}$ w/ K_j/K_{j+1} abelian

$J/K_2 \le K_1/K_2 \Rightarrow J/K_2$ is abelian as a subgp of abelian

$K_1/J \longleftarrow K_1/K_2$, $xJ \longleftarrow x$ $\Rightarrow K_1/J$ is abelian

$$K_1/J \underset{\text{2nd iso thm}}{\cong} \frac{\overbrace{(K_1/K_2)}^{\text{abelian}}}{\underbrace{(J/K_2)}_{\text{subgp}}} \Rightarrow K_1/J \text{ is abelian}$$

G finite:

G solvable $\iff$ graded pieces of any J-H series for G are $\mathbb{Z}/p\mathbb{Z}$.

Another construction of composition series using commutator subgps.

$C^1(G) := G \rightsquigarrow [G;G] =: C^2(G)$

$C^{n+1}(G) := [G, C^n(G)]$. $\quad A, B \trianglelefteq G \Rightarrow [A,B] \trianglelefteq G$

So each $C^n(G) \trianglelefteq G$.

Central series $C^1(G) \trianglerighteq C^2(G) \trianglerighteq \dots$

why is $C^{n+1}(G) \subset C^n(G)$?

because a generator of $C^{n+1}(G)$ is of the form

$\underbrace{gxg^{-1}}_{\substack{\in \\ C^n}} x^{-1}$ where $g \in G$, $x \in C^n(G)$.

since $C^n \trianglelefteq G \implies$ the generator is in C^n.

G is called nilpotent if $C^n(G) = \{e\}$ for some $n \geq 1$.