Lec 4/13

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Recall: $L^{\ell}_{k} = \sum_j g^{\ell j} L_{jk}$

$\langle n|n\rangle \equiv 1$ so $2\langle n|n_i\rangle = 0$ so $n_i \perp n$ so $n_i \in T_pM$.

4-8 Principal, Gaussian, and Normal curvature.

Gaussian Curvature: $K = \det(L) = \det(L^{\ell}_{k}) = \det(L_{jk})/\det(g_{\ell j}) = K_1K_2$.

Mean Curvature: $H = \frac{1}{2}\mathrm{Tr}(L) = \frac{1}{2}\mathrm{Tr}(L^{\ell}_{k}) = \frac{1}{2}(L^1_1 + L^2_2) = \frac{1}{2}(K_1 + K_2)$.

L is symmetric so its eigenvalues K_1 and K_2 are real, and the corresponding unit eigenvectors $\underbrace{X_{(1)} \text{ and } X_{(2)}}_{\text{chosen to have same orientation as } x_1,\, x_2 \text{ (positively oriented)}}$ are orthogonal. (provided that $K_1 \neq K_2$).

It is customary to index K_1 and K_2 s.t. $K_1 \geq K_2$.
K_1 and K_2 are called the principal curvatures. The directions of $X_{(1)}$ and $X_{(2)}$ are called the principal directions. A point where $K_1 = K_2$ is called an umbilic.

All points of a plane are umbilics, as with a sphere.
No points of a surface with negative gaussian curvature everywhere are umbilics.

Thm 8.4 (Euler) Let $Y \in T_pM$ with $|Y| = 1$. Then $Y = X_{(1)}\cos\theta + X_{(2)}\sin\theta$ for some $\theta \in (-\pi, \pi]$. We have $\mathrm{II}(Y,Y) = K_1\cos^2\theta + K_2\sin^2\theta$.

Pf $\mathrm{II}(Y,Y) = \langle L(Y)\,|\,Y\rangle = \langle L(X_{(1)})\cos\theta + L(X_{(2)})\sin\theta \,|\, Y\rangle = \underbrace{K_1\cos^2\theta + K_2\sin^2\theta}_{\text{convex combination}}$ □.

Corollary $K_1 = \max \overbrace{\{\mathrm{II}_p(Y,Y) : Y \in T_pM,\ |Y| = 1\}}^{(*)}$, $K_2 = \min(*)$.
$= \mathrm{II}_p(X_{(1)}, X_{(1)})$ $\qquad = \mathrm{II}_p(X_{(2)}, X_{(2)})$.

Reminder Let $Y \in T_pM$ with $|Y| = 1$. Let γ be a C^2 unit speed curve on M with $\gamma(0) = p$ and $\gamma'(0) = Y$. Then $K_n(0) = \mathrm{II}(Y,Y)$.

4-9 Gauss's Theorema Egregium (1827) and Riemann's Curvature Tensor (1854, 1861).

Let M be a C^3 surface in $\mathbb{R}^3$ and let $x : U$ open $\subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V$ open $\subseteq M$ be a C^3 coordinate patch on M. Since x is C^3, we have $x_{ijk} = x_{ikj}$. By gauss's formulas.

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coordinate patch on M. Since x is C^3, we have $x_{ijk} = x_{ikj}$. By Gauss's formulas.

$$x_{ij} = \underbrace{L_{ij}\, n + \sum_\ell \Gamma_{ij}^\ell x_\ell}_{(*)}, \quad \text{so} \quad x_{ijk} = \frac{\partial}{\partial u^k}(*) = \frac{\partial L_{ij}}{\partial u^k} n + L_{ij} n_k + \sum_\ell \frac{\partial \Gamma_{ij}^\ell}{\partial u^k} x_\ell + \sum_\ell \Gamma_{ij}^\ell x_{\ell k}$$

But by Weingarten's Equations, $-n_k = \sum_\ell L_k^\ell x_\ell$ and by Gauss's formulae again, $x_{\ell k} = L_{\ell k} n + \sum_m \Gamma_{\ell k}^m x_m$. Hence

$$x_{ijk} = \frac{\partial L_{ij}}{\partial u^k} n - \sum_\ell L_{ij} L_k^\ell x_\ell + \sum_\ell \frac{\partial \Gamma_{ij}^\ell}{\partial u^k} x_\ell + \sum_\ell \Gamma_{ij}^\ell L_{\ell k} n + \sum_{\ell, m} \Gamma_{ij}^\ell \Gamma_{\ell k}^m x_m$$

$$= \left(\frac{\partial L_{ij}}{\partial u^k} + \sum_\ell \Gamma_{ij}^\ell L_{\ell k} \right) n + \sum_\ell \left(L_{ij} L_k^\ell + \frac{\partial \Gamma_{ij}^\ell}{\partial u^k} + \sum_p \Gamma_{ij}^p \Gamma_{pk}^\ell \right) x_\ell$$

Similarly,

$$x_{ikj} = \left(\frac{\partial L_{ik}}{\partial u^j} + \sum_\ell \Gamma_{ik}^\ell L_{\ell j} \right) n + \sum_\ell \left(L_{ik} L^\ell + \frac{\partial \Gamma_{ik}^\ell}{\partial u^j} + \sum_p \Gamma_{ik}^p \Gamma_{pj}^\ell \right) x_\ell$$

Since $x_{ikj} = x_{ikj}$ and since n, x_1, x_2 are linearly independent at every pt, we get

$$(9\text{-}4) \qquad \frac{\partial L_{ij}}{\partial u^k} + \sum_\ell \Gamma_{ij}^\ell L_{\ell k} = \frac{\partial L_{ik}}{\partial u^j} + \sum_\ell \Gamma_{ik}^\ell L_{\ell j}.$$

$$(9\text{-}5) \qquad L_{ij} L_k^\ell + \frac{\partial \Gamma_{ij}^\ell}{\partial u^k} + \sum_p \Gamma_{ij}^p \Gamma_{pk}^\ell = L_{ik} L^\ell + \frac{\partial \Gamma_{ik}^\ell}{\partial u^j} + \sum_p \Gamma_{ik}^p \Gamma_{pj}^\ell \qquad \text{for all } \ell.$$

eqn (9-4) can be rewritten as

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \sum_\ell \left(\Gamma_{ik}^\ell L_{\ell j} - \Gamma_{ij}^\ell L_{\ell k} \right)$$

and these are called the Codazzi - Mainardi eqn

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