Quiz

if $x \in f^{-1}((a,\infty))$ and $y > x$,

then $f(y) > f(x) > a$, so $y \in f^{-1}((a,\infty))$.

Therefore $f^{-1}((a,\infty))$ is an interval. $\square$

Review of Riemann Integral

$f : [a,b] \longrightarrow \mathbb{R}$ bounded.

A partition of $[a,b]$ is a finite set P

of subsets of $[a,b]$ which are intervals

and which intersect at ≤ 1 point, and

$$\bigcup_{J \in P} J = [a,b].$$

Alternatively: $\{a = s_0 < s_1 < \cdots < s_n = b\}$,

$$\updownarrow$$

$$P = \{ [s_{i-1}, s_i] \mid 1 \leq i \leq n \}$$

Write $m_J = \inf_{x \in J} f(x)$, $M_J = \sup_{x \in J} f(x)$

<u>Lower sum</u>: $L(f,P) = \sum_{J \in P} m_J \, \lambda(J)$

$\wedge|$

<u>Upper sum</u>: $U(f,P) = \sum_{J \in P} M_J \, \lambda(J)$

A refinement of P is a partition Q s.t. $\forall J \in P$, $\exists Q_J \subset Q$ s.t. Q_J partitions J.

If Q refines P,

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

If P_1, P_2 are two partitions and Q is a common refinement,

$$\max_{i=1,2} L(f,P_i) \le L(f,Q) \le U(f,Q) \le \min_{i=1,2} U(f,P_i)$$

<u>upper integral</u> $\overline{\int_{[a,b]}} f := \inf_P U(f,P)$

<u>lower integral</u> $\underline{\int}_{[a,b]} f := \sup_{P} L(f,P)$

<u>Def</u> f is Riemann integrable if $\overline{\int} f = \underline{\int} f$.

<u>Exercise</u>: $f:[a,b] \longrightarrow \mathbb{R}$ is bdd. TFAE

① f is Riemann integrable

② $\forall \varepsilon>0$, $\exists$ partition P s.t. $U(f,P) - L(f,P) < \varepsilon$.

<u>Theorem</u> if f is Riemann integrable
then f is Lebesgue integrable, and

$$\int f \, d\lambda = \int_a^b f(x)\,dx$$

~~pf~~ Let (P_n) be a seq. of partitions for which P_{n+1} refines P_n.

s.t. $\underbrace{U(f,P_n)}_{\int \Psi} - \underbrace{L(f,P_n)}_{\int \varphi_n} < \frac{1}{n}$.

<u>Trick</u>: set $\varphi_n = \sum_{J \in P_n} m_J \chi_J$, $\Psi_n = \sum_{J \in P_n} M_J \chi_J$

then $\varphi_n \le \varphi_{n+1} \le f \le \Psi_{n+1} \le \Psi_n$

Let $\varphi = \lim \varphi_n$, $\Psi = \lim \Psi_n$.

all fns are bounded.

by DCT, φ and Ψ are both Integrable

$$\int \varphi = \lim \int \varphi_n = \int_a^b f(x)\,dx = \lim \int \Psi_n = \int \Psi.$$

but, $\int(\underbrace{\Psi - \varphi}_{\geq 0}) = 0$ so $\Psi = \varphi = f$ a.e.

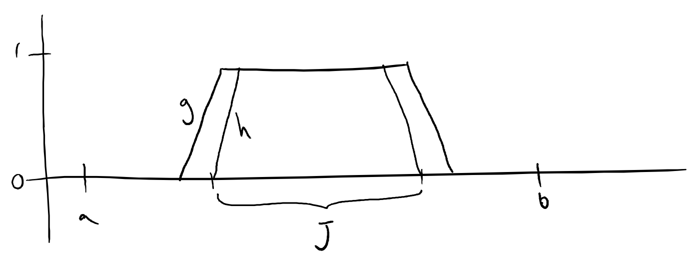
so f is Lebesgue integrable, and $\int f = \int_a^b f(x)\,dx$ $\square$

<u>Lemma</u> Suppose $f : [a,b] \to \mathbb{R}$ bdd, Riemann-integrable.

$\forall \varepsilon > 0$, $\exists$ cts $g, h : [a,b] \to \mathbb{R}$ s.t.

- $h \leq f \leq g$
- $\int_{[a,b]} (g-h)\,d\lambda \leq \varepsilon$

<u>Pf Step1</u>: Suppose $f = \chi_J$ for some interval $J \subseteq [a,b]$.



<u>Step 2</u>: WLOG $f \geq 0$. Let $\varepsilon > 0$. Take a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \frac{\varepsilon}{3}$.

As in the previous trick,

$$\psi = \sum_{J \in P} m_J \chi_J \quad , \quad \Psi = \sum_{J \in P} M_J \chi_J .$$

Perform Step 1 to each χ_J to obtain

$$h_J \le \chi_J \le g_J \quad \text{s.t.} \quad \int g_J - h_J < \frac{\varepsilon}{3|P|M}$$

Where $|P| = \#$ intervals of P and $M = \sup_{x \in [a,b]} f(x)$.

Set $g = \sum M_J g_J$, $h = \sum m_J h_J$,

$$h = \sum_J m_J h_J \le \sum_J m_J \chi_J \le f \le \sum_J M_J \chi_J \le \sum_J M_J g_J = g .$$

$$\int g - h = \sum_J M_J \int g_J - m_J \int h_J$$

$$= \sum_J \underbrace{M_J}_{\le M} \underbrace{\left(\int g_J - \lambda(J) \right)}_{\le \frac{\varepsilon}{3|P|M}} \qquad \le \quad \frac{\varepsilon}{3}$$

$$+ \sum_J \underbrace{m_J}_{\le M} \underbrace{\left(\lambda(J) - \int h_J \right)}_{\le \frac{\varepsilon}{3|P|M}} \qquad \le \quad \frac{\varepsilon}{3}$$

$$+ \left[U(f,P) - L(f,P) \right] \quad \le \quad \frac{\varepsilon}{3}$$

$$\le \varepsilon .$$