Lec 9/27

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<u>Last time</u> Suppose f is bdd and riemann integrable.

$\Downarrow$ $f \in \mathcal{L}^1$, $\int_{[a,b]} f \, d\lambda = \int_a^b f(x)\,dx$

$\forall \varepsilon > 0$, $\exists$ cts $g, h : [a,b] \longrightarrow \mathbb{R}$

s.t. $h \le f \le g$ and $\int_{[a,b]} (g-h)\, d\lambda < \varepsilon$

<u>Lebesgue's Theorem</u>:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be bdd. TFAE

① f is Riemann integrable

② f is continuous a.e.

Pf <u>① $\Rightarrow$ ②</u> Suppose f is R. int. by Lemma,

$\exists$ seq of cts fns $h_n \le f \le g_n$ s.t. $\int (g_n - h_n) < \frac{1}{n}$.

Since $\int g_{n+1} \wedge g_n - h_{n+1} \vee h_n \le \int g_{n+1} - h_{n+1} < \frac{1}{n+1}$,

We may assume $h_n \le h_{n+1} \le f \le g_{n+1} \le g_n$.

Let $h = \lim h_n$, $g = \lim g_n$.

Then $h \le f \le g$, and $\int h = \int f = \int g$ by MCT.

But $g - h \ge 0$, so $g = h$ a.e. on $[a,b]$.

Claim: Since $g_n \searrow g$, g is upper semicontinuous.

Similarly, h is lower semicontinuous.

$\Rightarrow$ when $f(x_0) = g(x_0) = h(x_0)$, f is cts at x_0!

$\Rightarrow$ f is cts a.e.

Pf of Claim: Let $x_0 \in [a,b]$, $\varepsilon > 0$. Pick $N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow g_n(x_0) - g(x_0) < \frac{\varepsilon}{2}$.

Pick $\delta > 0$ s.t. $x \in (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \cap [a,b]$

$\Rightarrow |g_n(x) - g_n(x_0)| < \frac{\varepsilon}{2}$,

Then $\forall x \in (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}) \cap [a,b]$,

$$g(x_0) \geq g_n(x_0) - \frac{\varepsilon}{2} \geq g_n(x) - \varepsilon \geq g(x) - \varepsilon.$$

so

$$f(x) - f(x_0) \leq g(x) - g(x_0) \leq \varepsilon$$
$$f(x_0) - f(x) \leq h(x_0) - h(x) \leq \varepsilon$$

② $\Rightarrow$ ① Suppose f is cts a.e. on $[a,b]$. Let E be the set of discontinuities, λ-null.

Let $\varepsilon > 0$. Construct a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$:

Take an open $U \supset E$ s.t. $\lambda(U) < \varepsilon'$

Let $K = [a,b] \setminus \mathcal{U}$ cpt. $f|_K$ is cts.

$\star$ $\forall x \in K$, $\exists\ \delta_x > 0$ s.t. $y \in [a,b]$ and $|x-y| < \delta_x \Rightarrow |f(x)-f(y)| < \varepsilon'$

Then $\{B_{\frac{\delta_x}{2}}(x)\}_{x \in K}$ is an open cover of K cpt, so

$\exists$ finite subcover say centred at $x_1, \dots, x_n \in K$.

Let $\delta = \min\left\{ \frac{\delta_{x_i}}{2} \,\middle|\, i = 1, \dots, n \right\}$.

Claim: If $x \in K$ and $y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x)-f(y)| < 2\varepsilon'$.

Pf wlog $x \in B_{\frac{\delta_1}{2}}(x_1)$. Then $y \in B_{\delta_1}(x_1)$, so use Δ-inequality.

Let P be a partition of $[a,b]$ whose intervals have length at most δ. Let P' consist of the intervals which intersect K, and P'' be the other ones. By the claim, if $J \in P'$,

$$M_J - m_J \le 4\varepsilon'$$

So
$$\mathcal{U}(f,P) - L(f,P) = \sum_{J \in P'} (M_J - m_J)\,\lambda(J) + \sum_{J \in P''} (M_J - m_J)\,\lambda(J)$$

$$\le 4\varepsilon'(b-a) + (\underset{\text{sup } f}{M} - \underset{\text{inf } f}{m})\,\lambda(\mathcal{U})$$

$$< \varepsilon'(4(b-a) + (M-m))$$

$$< \varepsilon,$$

So let $\varepsilon' = \dfrac{\varepsilon}{4(b-a) + (M-m)}$. $\square$

Product Spaces

X, Y top spaces,

$X \times Y$ has product topology.

- generated by $\{ U \times V \mid U \subset X,\ V \subset Y \text{ open} \}$.

<u>Exercise</u> Show that the product topology is the weakest topology s.t. canonical proj. maps

$$\pi_X : X \times Y \longrightarrow X$$
$$\pi_Y : X \times Y \longrightarrow Y$$

are cts.

$(X, \mathcal{M}), (Y, \mathcal{N})$ are mble spaces, give $X \times Y$ the product σ-algebra, $\mathcal{M} \otimes \mathcal{N}$ generated by $\{ E \times F \mid E \in \mathcal{M}, F \in \mathcal{N} \}$ set of mble rectangles

<u>Exercise</u> Show $\mathcal{M} \otimes \mathcal{N}$ is smallest σ-alg s.t. π_X, π_Y are mble.

Question: Recall that when X, Y are top sp, $\forall\, \mathcal{U} \subset X \times Y$ open, $\pi_X(\mathcal{U})$, $\pi_Y(\mathcal{U})$ are open.

When $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are mble sp, if $E \subset X \times Y$ is mble, are $\pi_X(E)$, $\pi_Y(E)$ mble?