Rec 11/19

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Prop (7.9) If μ is a Radon measure on X (LCH), $C_c(X)$ is dense in $L^1(\mu)$.

pf simple fns are dense (Assume $f \geq 0$). let $\varepsilon > 0$

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu \mid 0 \leq \phi \leq f \text{ is simple} \right\}.$$

Def'n $\phi = \sum_1^n c_i \chi_{E_i}$. $\exists \phi$ s.t. $\|f - \phi\| < \varepsilon$.

Problem is reduced to approximating χ_E by fns in $C_c(X)$; (E mble, $\mu(E) < \infty$).

Observation μ regular on E since $\mu(E) < \infty$.

So $\exists \overset{\text{cpt}}{K} \subset E \subset \overset{\text{open}}{U}$ w/ $\mu(U \setminus K) < \varepsilon$

LCH $\Rightarrow$ $\exists V$ s.t. $K \subset V \subset \overline{V} \subset U$. ($\overline{V}$ cpt)

Urysohn $\Rightarrow$ $\exists f \in C(X, [0,1])$ s.t. $f|_K = 1$, $f|_{X \setminus \overline{V}} = 0$.

So $f \in C_c(X)$, and $\mu(K) \leq \int f \, d\mu \leq \mu(U)$. $\square$

Lusin's Thm for Radon Measures

Given a Radon measure μ on X LCH and mble $f: X \to \mathbb{C}$ s.t. $f \equiv 0$ on $X \setminus E$ where $\mu(E) < \infty$. Then $\forall \varepsilon > 0$, $\exists \phi \in C_c(X)$ s.t. $\phi = f$ except for on a set of measure $< \varepsilon$.

(if $\|f\|_\infty < \infty$ then $\|\phi\|_\infty \leq \|f\|_\infty$).

Pf wlog $f \geq 0$. Assume f bdd (if not, $\mu(\{f > n\}) \to 0$).

Then $f \in L^1$, so $\exists$ seq $\{g_n\}$ in $C_c(X)$ s.t. $\|g_n - f\|_1 \to 0$.

By some thm, $\exists$ a subseq that converges a.e. (call it g_n still).

<u>Egoroff's Thm</u>: If $\{g_j\}$ mble & $g_j \to f$ a.e. on E w/ $\mu(E) < \infty$ then, given $\varepsilon > 0$, $\exists A \subset E$ s.t. $\mu(E \setminus A) < \varepsilon$ & $g_j \to f$ unif. on A.

So $g_n \to f$ unif on A, so $f|_A$ is cts.

$\exists \underset{\text{cpt}}{K} \subset A \subset \underset{\text{open}}{U}$ w/ $\mu(U \setminus K) < \varepsilon$, and $f|_K$ is cts.

Use Tietze extn thm to extend $f|_K$ to $\phi \in C_c(X)$.