Lec 11/8

Wednesday, November 8, 2017 14:24

$x \xmapsto{T^n} x+n$, $x \in \mathbb{R}$, is a $\mathbb{Z}$-action

(by shifts of $\mathbb{R}$)

$(T^n)_{n \in (\mathbb{Z},+)}$ acts on $\mathbb{R}$ by shifts or translation.

Fundamental domain (say, for this action) is <u>any</u> $S \subset \mathbb{R}$ which contains exactly one element of every orbit

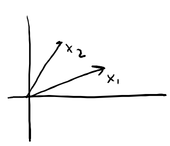
orbit of $x \in \mathbb{R}$ is $\{x+n,\ n \in \mathbb{Z}\}$

$S = [0,1)$ works.

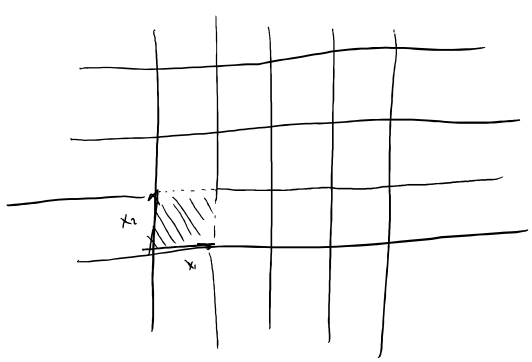
$$\bigsqcup_{n \in \mathbb{Z}} S+n = \mathbb{R}$$

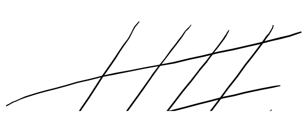
exercise: Cantor set is nowhere dense & closed.

the countable union of such sets cannot cover $\mathbb{R}$ (or $[0,1)$).



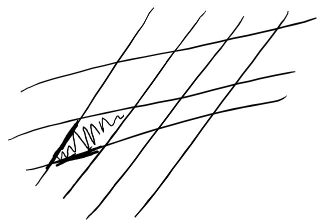
$\underset{\mathbb{R}^2 \,\ni}{x} \mapsto x + nx_1 + mx_2$ $\quad n,m \in \mathbb{Z},\ x \in$ $\mathbb{R}$





~~$\mathrm{Conv}(X_1, X_2)$ = convex span of~~ x_1, x_2

$\Delta_1 = \{\alpha_1 x_1 + \alpha_2 x_2\},\quad 0 \le \alpha_i < 1$



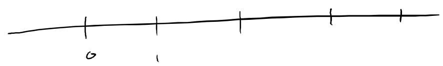
$\mathrm{conv}(x_1, x_2)$

$\Delta_L = \{\alpha_1 x_1 + \alpha_2 x_2\}, \quad 0 \le \alpha_i < 1$

<u>fact</u>: if $S \subset \mathbb{R}$ and $\mu(S) > 1$ then $S - S \ni n \in \mathbb{Z} \setminus \{0\}$.

Hint: $S \cap S - n \neq \emptyset$

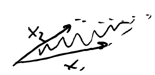
$S_n = S \cap [n-1, n), \quad \bigcup s_n = S$



Now let $\tilde{S}_n$ be translate of S_n into $[0,1)$. $\mu(\bigcup \tilde{S}_n) > 1$

Now use pigeonhole principle: two $\tilde{S}_n$ intersect nontrivially: $\tilde{S}_n \cap \tilde{S}_m \neq \emptyset$ (exercise: finish the proof)

Formulation for $\mathbb{R}^2$:

Let $S \subset \mathbb{R}^2$. Let L be a lattice in $\mathbb{R}^2$ w/ fundamental domain $\Delta_L =$ .

If $\mu(S) > \mu(\Delta_L)$ then $(S - S) \cap (L \setminus \{0\}) \neq \emptyset$.

Exercise: prove this

exercise: write a list of theorems which could be useful for midterm (which could be asked to prove) and provide proofs (maybe, if they could be asked for proof)

Let K be a symmetric convex body in $\mathbb{R}^2$.

Let L be a lattice. if $\mu(K) > 4\mu(\Delta_L)$ then $\exists (x,y) \in L \cap K \setminus \{0\}$

K is symmetric if $v \in K \Rightarrow -v \in K$.

Hint: consider $S = \frac{1}{2}K$. $\mu(\frac{1}{2}k) = \frac{1}{4}\mu(K) > 1$.

So $(\frac{1}{2}k - \frac{1}{2}k) \cap L \neq \emptyset$ but $\frac{1}{2}K - \frac{1}{2}k = \frac{1}{2}k + \frac{1}{2}k \overset{?}{=} K$

exercise: finish the proof

Let $p = 4k+1$. Then $\exists\, u \in \underbrace{\mathbb{Z}/_{p\mathbb{Z}}}_{\text{"}\mathbb{Z}_p}$ s.t. $u^2 = -1 \bmod p$. know this for midterm

Fact. for any prime p, the multiplicative group $\mathbb{Z}_p^*$ of invertible elements in $\mathbb{Z}_p$ is cyclic. (true for all finite fields!)
(exercise – not for general field just for $\mathbb{Z}_p$)

Suppose $p=5$: $\mathbb{Z}_p^* = 1,2,3,4$: $a=2$, $a^2=4$, $a^3=3$, $a^4=1$

$e, a, a^2, a^3, \cdots, a^{11}$ $\qquad a^6 = -1$.

$x^2 = -1$, $x^4 = 1$, $(x^2-1)(x^2+1) = 1$

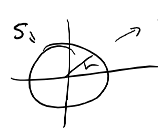
Let $u \in \mathbb{Z}$ be such that $u^2 \equiv -1 \bmod p$.

$L = \{(x,y) \in \mathbb{Z}^2 : y \equiv ux \bmod p\}$

1: Prove this is a lattice

2: Prove that $\mathrm{Vol}(\Delta_L) = p$

exercise (hint: find generators).

Now adjust radius of  to apply $\mu(S) > 4p$ to get $0 < x^2+y^2$ s.t. $x,y \in L$.

thus we get sums of 2 squares for p.

$u^2 + v^2 = -1 \mod p$ all that is needed for sums of 4 squares

formulations of PNT:

$\frac{1}{N} \sum \mu(n) \to 0$

T. Apostol. - "Intro to ..."

painless equivalent forms of PNT.

Von Mangoldt function

A.E. # is normal via ergodic thm.

$2x \mod 1$ ergodic,

$\{0,1\}^{\mathbb{N}}$ cylinders are independent

$A \subset \mathbb{N}$, $d(A) > 0$, $A - A$ syndetic follows from ergodicity (exercise)

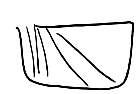
Continued fractions via ergodic theorem

$$(0,1) \ni x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}}$$

$\uparrow$

$\mathbb{N}^{\mathbb{N}}$, shift is ergodic transformation $Tx = \begin{cases} \{\frac{1}{x}\}, & x \in (0,1) \\ 0, & x = 0 \end{cases}$

$$\frac{a_1(x) + a_2(x) + \cdots + a_k(x)}{k} \xrightarrow[\text{a.e.}]{} \infty$$



Gauss: T preserves $\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}$

for a.e. x

Now $\sqrt[k]{a_0(x)\, a_1(x) \cdots a_k(x)} = K$

Khinchmer's constant < 3.