Lec 9/20

Tuesday, September 20, 2016 9:08 AM

Theorem: If f, g continuous at $a \in dom(f) \cap dom(g)$ then so are (1) $f+g$ (proved yesterday)

(2) $f \cdot g$

Proof of (2): Let $\varepsilon > 0$ be arbitrary. Want to find $\delta > 0$ so that

$$|x-a| < \delta \text{ and } x \in dom(f \cdot g) \Rightarrow |(f \cdot g)(x) - (f \cdot g)(a)| < \varepsilon.$$

Strategy for proof: $|f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)|$

$$\leq \underbrace{|f(x) - f(a)||g(x)|}_{} + \underbrace{|f(a)||g(x) - g(a)|}_{}$$

want to make both less than $\frac{\varepsilon}{2}$

First need to control $|g(x)|$. Choose δ_0 so that

$|x-a| < \delta_0$ and $x \in dom(g) \Rightarrow |g(x) - g(a)| < 1$

$\Rightarrow |g(x)| \leq |g(a)| + |g(x) - g(a)| < |g(a)| + 1$

Then pick $\delta_1 > 0$ so that

$|x-a| < \delta_1$ and $x \in dom(f) \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2(|g(a)|+1)}$

pick $\delta_2 > 0$ so that

$|x-a| < \delta_2$ and $x \in dom(g) \Rightarrow |g(x) - g(a)| < \frac{\varepsilon}{2(|f(a)|+1)}$

Actual proof: So final choice of δ should be $\min(\delta_1, \delta_2, \delta_3)$

so $|x-a| < \delta$ and $x \in dom(f \cdot g) \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2(|g(a)|+1)}$

and $|g(x) - g(a)| < \frac{\varepsilon}{2(|f(a)|+1)}$

and $|g(x)| < |g(a)| + 1$

skipped some intermediate steps

so $|f(x) - f(a)||g(x)| + |f(a)||g(x) - g(a)| < \varepsilon$

so $|f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)| < \varepsilon$

so $|(f \cdot g)(x) - (f \cdot g)(a)| < \varepsilon$

so $f \cdot g$ is cts at a. ■

Theorem: If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = k$ then

$$\lim_{x \to a} (f(x) + g(x)) = l + k$$

$$\lim_{x \to a} (f(x) g(x)) = lk$$

(exercise)

Continuity & limits for function composition.

Theorem: If f is continuous at $a \in \mathrm{dom}(f)$ and g is continuous at $b = f(a)$, $b \in \mathrm{dom}(g)$ then $g \circ f$ is continuous at a

False analog for limits:

If $\lim_{x \to a} f(x) = L$ and $\lim_{u \to L} g(x) = K$ then $\lim_{x \to a} (g \circ f)(x) = K$

Counterexample: let $f(x) = 0$ let $g(u) = \begin{cases} u^2 & u \neq 0 \\ -1 & u = 0 \end{cases}$

then $\lim_{x \to 0} f(x) = \overset{L}{\overset{\shortparallel}{0}}$. $\lim_{u \to 0} g(u) = \overset{K}{\overset{\shortparallel}{0}}$

but $g(f(x)) = -1$ so $\lim_{x \to 0} g(f(x)) = -1 \neq K$.

problem: g is not continuous at L.

Possible fix:

if $\lim_{x \to a} f(x) = L$ and g is continuous at L then $\lim_{x \to a} g(f(x)) = g(L)$

Counterexample: let $f(x) = -x^2$. let $g(x) = \sqrt{x}$ $\forall x \in [0, \infty)$

cts at 0.

$\lim_{x \to 0} f(x) = 0$. but $g(f(x)) = \sqrt{-x^2}$ so $\mathrm{dom}(g \circ f) = \{0\}$

hence $\lim_{x \to 0} (g \circ f)(x)$ does not exist.

Correct theorem.

g cts at L and defined on an open interval containing L.

continu[illegible]

Correct theorem:

Theorem: If $\lim_{x\to a} f(x) = L$ and $\overbrace{\lim_{u\to L} g(u) = g(L)}$ then $\lim_{x\to a} (g\circ f)(x) = g(L) = K$

Proof: Let $\varepsilon > 0$ be arbitrary. want to find $\delta > 0$ so that

$$0 < |x-a| < \delta \Rightarrow x \in \text{dom}(g\circ f) \text{ and } |(g\circ f)(x) - g(L)| < \varepsilon$$

Since $\lim_{u\to L} g(u) = g(L)$. we can find $\delta_1 > 0$ s.t.

$$\cancel{0<}|u-L| < \delta_1 \Rightarrow u \in \text{dom}(g) \text{ and } |g(u) - g(L)| < \varepsilon \qquad (*)$$

can erase this since $L \in \text{dom}(g)$ and $g(L) - g(L) = 0 < \varepsilon$

Also, we can find $\delta > 0$ s.t.

$$0 < |x-a| < \delta \Rightarrow x \in \text{dom}(f) \text{ and } |f(x) - L| < \delta_1$$

now take $u = f(x)$. so

$0 < |x-a| < \delta \Rightarrow x \in \text{dom}(f) \Rightarrow u = f(x)$ is defined and

$|u-L| < \delta_1 \Rightarrow u \in \text{dom}(g)$ so $x \in \text{dom}(g\circ f)$

and $|g(f(x)) - g(L)| = |g(u) - g(L)| < \varepsilon$ ∎

Theorem about continuity left as an exercise.

Theorem (localization principle):

Suppose that $a \in (c,d)$ and $f(x) = g(x)$ $\forall x \in (c,a) \cup (a,d)$

and $\lim_{x\to a} g(x) = L$. Then $\lim_{x\to a} f(x) = L$ as well.

Proof: Let $\varepsilon > 0$. Then we can find $\delta_0 > 0$ such that

$$0 < |x-a| < \delta_0 \Rightarrow x \in \text{dom}(g) \text{ and } |g(x) - L| < \varepsilon$$

Now let $\delta = \min(\delta_0, a-c, d-a)$. Then

$0 < |x-a| < \delta \Rightarrow x \in (c,a) \cup (a,d) \subseteq \text{dom}(g) \cap \text{dom}(f)$

and $0 < |x-a| < \delta_1 \Rightarrow |g(x) - L| < \varepsilon$

and $g(x) = f(x)$ so $|f(x) - L| < \varepsilon$ ∎