Lec 12/6

Tuesday, December 6, 2016 9:13 AM

Theorem If $f^{(n+1)}$ is defined on an open interval $(b,c) \ni a$, and $f^{(n+1)}$ is integrable over any closed finite subinterval of (b,c), then $\forall x \in (b,c)$, $R_{n,a,f}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$

Proof: Induction on n:

0th order taylor poly, so $f(x)-f(a) = R_{0,a,f}(x)$.

base $n=0$: $\int_a^x \frac{f'(t)}{0!}(x-t)^0\,dt = f(x)-f(a)$ by FTC.

induction: Assume $R_{n-1,a,f}(x) = \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\,dt$

so $\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt = \left[f^{(n)}(t)\frac{(x-t)^n}{n!}\right]_{t=a}^{x} - \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(-1)(x-t)^{n-1}\,dt$

$$\left[\begin{array}{ll} u = \frac{(x-t)^n}{n!} & du = \frac{(x-t)^{n-1}}{(n-1)!}(-1) \\ v = f^{(n)}(t) & dv = f^{(n+1)}(t)\,dt \end{array}\right]$$

$= 0 - \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\,dt$

$= R_{n-1,a,f}(x) - \frac{f^{(n)}(a)}{n!}(x-a)^n$

since $P_{n,a,f}(x) = P_{n-1,a,f}(a) + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$= R_{n,a,f}(x)$ ∎

Proving e is irrational

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \Rightarrow e = \sum_{j=0}^{\infty} \frac{1}{j!}$$

$$P_{n,0,e^x}(1) = \sum_{j=0}^{n} \frac{1}{j!}$$

$R_{n,0,e^x}(1) = \frac{f^{(n+1)}(x_0)}{(n+1)!} 1^{n+1} = \frac{e^{x_0}}{(n+1)!}$ for some x_0 btwn 0 and 1.

$< \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$

$(n+1)!$

Alternatively compare w/ geom. series to get remainder est.

$$R_{n,0,e^x}(1) = \sum_{j=n+1}^{\infty} \frac{1}{j!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$= \frac{1}{(n+1)!}\left[\frac{1}{1!} + \frac{1}{(n+2)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \cdots\right]$$

$$< \frac{1}{(n+1)!}\left[1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right]$$

$$= \frac{2}{(n+1)!}$$

Theorem e is irrational.

Proof: by contradiction. suppose $e = \frac{p}{q}$ $(p, q \in \mathbb{Z}^+)$

Choose $n \geq \max(q, 2)$

$$e = P_{n,0,e^x}(1) + R_{n,0,e^x}(1)$$

$$n!\, e^x = n!\left[2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}\right] + n!\, R_{n,0,e^x}(1)$$

$\hookrightarrow$ is an integer $= M$

$\|$

$n!\, \frac{p}{q}$

$\hookrightarrow$ is an integer since q is factor of $n!$

$$< M + n!\frac{2}{(n+1)!} = M + \frac{2}{n+1} < M + \frac{2}{3}$$

So $0 < n!\, e^x - M < \frac{2}{3}$

$\hookrightarrow$ is an integer. contradiction. ∎

Final exam Review

1) $\lim_{n \to 0^+} x \int_x^1 \frac{\cos(\frac{1}{t})}{t}\, dt$

use sq. thm.

$$-1 \leq \cos(\tfrac{1}{t}) \leq 1$$

$$\frac{-1}{t} \leq \frac{\cos(\frac{1}{t})}{t} \leq \frac{1}{t}$$

$$-\int_x^1 \frac{1}{t}\,dt \leq \int^1 \frac{\cos(\frac{1}{t})}{t}\,dt \leq \int^1 \frac{1}{t}\,dt$$

$$-\int_x^1 \frac{1}{t}\,dt \le \int_x^1 \frac{\cos(\frac{1}{t})}{t}\,dt \le \int_x^1 \frac{1}{t}\,dt$$

$$\begin{array}{ccc} \| & & \| \\ [-\ln(t)]_x^1 & & \\ \| & & \\ \ln(x) & & -\ln(x) \end{array}$$

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x) \to -\infty}{\frac{1}{x} \to \infty} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

So limit is 0.

2) Special case: f int and pos on $[a,b] \Rightarrow f^2$ int on $[a,b]$

Proof: Given $\varepsilon > 0$ want to find partition P of $[a,b]$ s.t. $U(f^2,P) - L(f^2,P) < \varepsilon$.

Given $\varepsilon_1 > 0$ can find partition P' of $[a,b]$ s.t. $U(f,P') - L(f,P') < \varepsilon_1$

$$U(f,P') - L(f,P') = \sum_{i=1}^{n} (M_i - m_i)(x_i' - x_{i-1}') \quad \text{where } M_i \text{ \& } m_i \text{ corresp. w/ } f.$$

$$U(f^2,P') - L(f^2,P') = \sum_{i=1}^{n} (M_i^2 - m_i^2)(x_i' - x_{i-1}')$$

$$= \sum_{i=1}^{n} (M_i + m_i)(M_i - m_i)(x_i' - x_{i-1}')$$

Note $M_i + m_i \le 2B$ for some bound on f.

$$\le 2B \sum_{i=1}^{n} (M_i - m_i)(x_i' - x_{i-1}')$$

$$< 2B\varepsilon_1, \quad \text{so take } \varepsilon_1 = \frac{\varepsilon}{2B}. \qquad \blacksquare$$

Lemma (More general) f = g not necessarily ≥ 0.

Pf. add positive const. to make $f + c \geq 0$ on $[a,b]$.

$$\Rightarrow \int_a^b (f+c)^2 \text{ exists.}$$

both exist

$$\int_a^b (f+c)^2 = \int_a^b f^2 + 2c\int_a^b f + c^2[b-a].$$

$$\Rightarrow \int_a^b f^2 \text{ exists.} \qquad \blacksquare$$

Most generally: $f \cdot g = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$.

integrable