Lec 1/12

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<u>Propn</u> Let T be a normal operator on V. Let $\lambda_1 \neq \lambda_2 \in \mathbb{C}$.

Let $M_k = \{v \in V : Tv = \lambda_k v\}$ for $k = 1, 2$. Then $M_1 \perp M_2$

$$\left(\langle T^* x_1 \mid x_1 \rangle = \langle x_1 \mid T x_1 \rangle = \lambda \langle x_1 \mid x_1 \rangle = \langle \bar{\lambda} x_1 \mid x_1 \rangle \right).$$

<u>Pf</u>: Let $x_1 \in M_1$, $x_2 \in M_2$.

$$\lambda_2 \langle x_1 \mid x_2 \rangle = \langle x_1 \mid \lambda_2 x_2 \rangle = \langle x_1 \mid T x_2 \rangle = \langle T^* x_1 \mid x_2 \rangle = \langle \bar{\lambda}_1 x_1 \mid x_2 \rangle = \lambda_1 \langle x_1 \mid x_2 \rangle$$

so $\langle x_1 \mid x_2 \rangle = 0$.

The Spectral Theorem in a finite-dimensional inner product space $\overset{E}{\vee}$ over $\mathbb{C}$.

Let T be a normal operator on E.

Let Σ = the set of eigenvalues of T. $\Sigma \subset \mathbb{C}$ is finite, $|\Sigma| \leq \dim E$.

$\Sigma \neq \emptyset$ if $E \neq \{0\}$.

for each $\lambda \in \Sigma$, let $M_\lambda = \{x \in E : Tx = \lambda x\}$.

Let P_λ be the operator of orthogonal projection from E onto M_λ.

Then (a) $\sum_{\lambda \in \Sigma} M_\lambda = E$, (b) $\sum_{\lambda \in \Sigma} P_\lambda = I$, (c) $\sum_{\lambda \in \Sigma} \lambda P_\lambda = T$

<u>Pf</u> (a) Let $N = \left(\sum_{\lambda \in \Sigma} M_\lambda \right)^\perp = \bigcap_{\lambda \in \Sigma} M_\lambda^\perp$.

Hence $T[N] \subseteq \bigcap_{\lambda \in \Sigma} M_\lambda^\perp = N$, because $T[M_\lambda^\perp] \subseteq M_\lambda^\perp \ \forall \lambda \in \Sigma$.

Then $T : N \longrightarrow N$. Let $S = T|_N$. Then S is a linear operator on N.

<u>Claim</u> $N = \{0\}$. Suppose not. then $\exists \mu \in \mathbb{C}$, $\exists y \in N$, $y \neq 0$ and $Sy = \mu y$.

Then $Ty = \mu y$. Hence $\mu \in \Sigma$ and $y \in M_\mu$. but $N \perp M_\mu \Rightarrow y \perp y \Rightarrow y = 0$. ~~X~~

Thus $\sum_{\lambda \in \Sigma} M_\lambda = N^\perp = E$.

(b) Let $x \in E$. by (a), $\exists (x_\lambda)_{\lambda \in \Sigma}$ s.t. $\forall \lambda \in \Sigma$, $x_\lambda \in M_\lambda$ and $\sum_{\lambda \in \Sigma} x_\lambda = x$.

$$\left(\sum_{\lambda \in \Sigma} P_\lambda\right) x = \sum_{\lambda \in \Sigma} P_\lambda x = \sum_{\lambda \in \Sigma} P_\lambda \left(\sum_{\mu \in \Sigma} x_\mu\right) = \sum_{\lambda \in \Sigma} P_\lambda x_\lambda = \sum_{\lambda \in \Sigma} x_\lambda = x.$$

(c) $$\left(\sum_{\lambda \in \Sigma} \lambda P_\lambda\right) x = \sum_{\lambda \in \Sigma} \sum_{\mu \in \Sigma} \lambda P_\lambda x_\mu = \sum_{\lambda \in \Sigma} \lambda P_\lambda x_\lambda = \sum_{\lambda \in \Sigma} \lambda x_\lambda = \sum_{\lambda \in \Sigma} T x_\lambda = Tx. \quad \square$$

<u>Remarks</u>: $T^2 = \sum_{\lambda \in \Sigma} \lambda^2 P_\lambda$. $T^n = \sum_{\lambda \in \Sigma} \lambda^n P_\lambda$.

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!} = \sum_{n=0}^{\infty} \frac{\sum_{\lambda \in \Sigma} \lambda^n P_\lambda}{n!} = \sum_{n=0}^{\infty} \sum_{\lambda \in \Sigma} \frac{\lambda^n}{n!} P_\lambda = \sum_{\lambda \in \Sigma} \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) P_\lambda = \sum_{\lambda \in \Sigma} e^\lambda P_\lambda$$

$$f(T) = \sum_{\lambda \in \Sigma} f(\lambda) P_\lambda \quad \text{in general.}$$

Cross products of vectors in $\mathbb{R}^3$.

Let $u, v \in \mathbb{R}^3$. $u \times v = \begin{pmatrix} \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \\ \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix} \\ \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \end{pmatrix}$ where $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$u \cdot (u \times v) = \begin{vmatrix} u_1 & u_1 & v_1 \\ u_2 & u_2 & v_2 \\ u_3 & u_3 & v_3 \end{vmatrix} = 0$$

$$(u \times v) \cdot v = \begin{vmatrix} u_1 & v_1 & v_1 \\ u_2 & v_2 & v_2 \\ u_3 & v_3 & v_3 \end{vmatrix} = 0$$

Thus $u \times v \perp u$ and $u \times v \perp v$.

Let $w = u \times v$.

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \|w\|^2 \geqslant 0.$$

If u, v lin. indep. then $\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$ has column rank 2

So it has row rank 2 as well. So at least one pair of rows is linearly independent. thus $u \times v \neq 0$, so $\|w\|^2 > 0$.

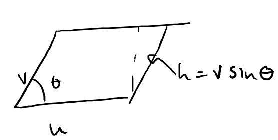
So $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} > 0$. so $u, v, u \times v$ form a positively oriented basis for $\mathbb{R}^3$.

So there is a continuous path in $\overbrace{GL(3, \mathbb{R})}^{\text{the set of bases}}$ that starts at $\overbrace{I}^{e_1, e_2, e_3}$ and ends at $\underbrace{\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}}_{u,\, v,\, u \times v}$.

if u, v linearly dependent then $u \times v = 0$.

$(u \times v) \cdot w$ is the volume of the parallelipiped spanned by u, v, w. Hence $\|u \times v\| =$ the area of parallelogram spanned by u & v.

$= \|u\| \|v\| \sin\theta$.



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if $u, v \in \mathbb{R}^n$, $u \times v \in$ an $\binom{n}{2}$-dim space

If $u_1, \ldots, u_k \in \mathbb{R}^n$, then $u_1 \wedge \cdots \wedge u_k \in$ an $\binom{n}{m}$-dim space

If $u_1, \ldots, u_{n-1} \in \mathbb{R}^n$, then $u_1 \wedge \cdots \wedge u_{n-1} \in \mathbb{R}^n$. (or some n-dim space)