Lec 10/27

Friday, October 27, 2017 14:19

Minkowski's ? function ← (google)

Smail's
Smale's horseshoe

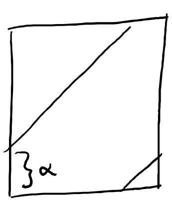
T measure preserving on (X, B, μ)

for any p.m.p.s. (X, B, μ, T), $\forall A \in B$, $\mu(A) > 0$, $\forall n_i \nearrow \infty$,

$\exists\, i<j$ s.t. $\mu(A \cap T^{-(n_j - n_i)} A) > 0$.

Proof Consider the sets $T^{-n_1}A, T^{-n_2}A, \dots, T^{-n_i}A, \dots, T^{-n_j}A, \dots$

Notice that $\exists\, i<j$ s.t. $0 < \mu(T^{-n_i}A \cap T^{-n_j}A) = \mu(A \cap T^{-(n_j - n_i)}A)$



$x \to x + \alpha \mod 1$ $\quad T^n 0 = n\alpha \mod 1$.

Cor. The set $R_A = \{ n : \mu(A \cap T^{-n}A) > 0 \}$ is syndetic

Pf if $\mathbb{N} \setminus R$ were thick, it would contain a set of differences

Def: $S \subseteq \mathbb{N}$ is a set of recurrence if $\forall$ pmps (X, B, μ, T) and $\forall A$ w/ $\mu(A) > 0$

$\exists\, n \in S$ s.t. $\mu(A \cap T^{-n}A) > 0$.

examples. $\{n_j - n_i, \ j > i\}$, n^2

$\lfloor n\alpha \rfloor$ $\forall\, \alpha > 0$ (exercise, show $\lfloor n\alpha \rfloor$ is a set of recurrence) examples. $\{n_j - n_i, j > i\}$, ..

$\lfloor n\alpha \rfloor$ $\forall$ $\alpha > 0$ (exercise, show $\lfloor n\alpha \rfloor$ is a set of recurrence)

$\lfloor p_n \alpha \rfloor$, $\lfloor p^c \alpha)$, $p - 1$

exercise: show that any thick set contains a set of differences

exercise:
Claim: if $d^*(A) > 0$ then $\forall n_i \nearrow \infty$ $\exists i < j$ s.t. $d^*(A \cap (A - (n_j - n_i))) > 0$

cor: $A - A$ is syndetic.

$\forall$ pmps $(X, \mathcal{B}, \mu, T)$, $\forall A, B \in \mathcal{B}$ w/ $\mu(A), \mu(B) > 0$,

$R_A \cap R_B = \{ n : \mu(A \cap T^{-n} A) \;\&\; \mu(B \cap T^{-n} B) > 0 \}$ is syndetic

$A - A \cap B - B$ is syndetic for $d^*(A), d^*(B) > 0$.

fact: $\forall C \subseteq \mathbb{Z}^2$ w/ $d^*(C) > 0$, $\forall n_i \nearrow \infty$, $\exists$ $i < j$ s.t.
$d^*(C \cap C - (n_j - n_i, n_j - n_i)) > 0$

$$d^*(C) = \limsup_{\substack{N_i - M_i \to \infty \\ \& \tilde{N}_i - \tilde{M}_i \to \infty}} \frac{C \cap ([M_i, N_i - 1] \times [\tilde{M}_i, \tilde{N}_i - 1])}{(N_i - M_i)(\tilde{N}_i - \tilde{M}_i)}$$

Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ be pmps.

The product system is $\left(X_1 \times X_2,\ \underbrace{\mathcal{B}_1 \otimes \mathcal{B}_2}_{\text{extension of product}},\ \underbrace{\mu_1 \otimes \mu_2}_{\text{not every}},\ T_1 \times T_2\right)$

extension of product (σ-algebra generated by measurable rectangles)

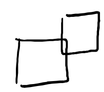
not every set in $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a rectangle, use rectangles to define $\mu_1 \times \mu_2$.

take $A \times B \in \mathcal{B}_1 \otimes \mathcal{B}_2$

$$R_{A\times B} = \{ n : \underbrace{\mu\left((A \times B) \cap (T \times T)^{-n}(A \times B)\right)}_{} > 0 \} \quad \text{is syndetic}$$

$$\| \qquad\qquad \|$$

$$R_A \cap R_B \qquad \mu\left((A \cap T^{-n}A) \times (B \cap T^{-n}B)\right)$$

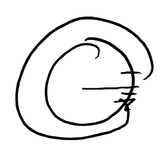


$$\| \quad \mu(A \cap T^{-n}A) \cdot \mu(B \cap T^{-n}B)$$

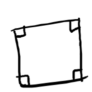
Claim: for any $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$,

$$\{ n : \|n\alpha_i\| < \varepsilon,\ i = 1, \dots, k \} \quad \text{is syndetic}$$

$\uparrow$ distance to nearest integer.



$T : x \to x + \alpha_i \mod 1$



$$(x, y) \xrightarrow{S^n} (x + n\alpha_1,\ y + n\alpha_2)$$

Lattice in $\mathbb{R}^n$. Let $e_1, \dots, e_n$ be independent vectors in $\mathbb{R}^n$.

Lattice generated by them is $\left\{ \sum_{i}^{n} n_i e_i,\ n_i \in \mathbb{Z} \right\}$

Lattice generated by them is $\left\{ \sum_{i=1}^{n} n_i e_i \ , \ n_i \in \mathbb{Z} \right\}$

A set $S \subset \mathbb{R}^n$ is discrete if $\forall r$, $B_r(0) \cap S$ is finite

exercise:

Theorem: An additive subgroup S of $\mathbb{R}^n$ is a Lattice iff S is discrete.