Lec 9/25

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$T \in L(V,V)$ $\quad V/_F$ fin. dim.

① T invertible $\leftrightarrow$ $\det T \neq 0$

<u>Proof</u> fix a basis of V: $\{v_1, \dots, v_n\}$ The matrix of T is $\overset{(\alpha_{ij})}{\overset{\shortparallel}{A}} \in M_n(F)$

$$T(v_i) = \sum_{j=1}^{n} \alpha_{ij} v_j \,. \qquad T \text{ has an inverse iff } A \text{ does.}$$

$\rightarrow$: so A invertible. $AA^{-1} = I$ so $\det(A)\det(A^{-1}) = 1$ so $\det(A) \neq 0$.

$\leftarrow$: $\det A = \Phi_{st}(Ae_1, \dots, Ae_n) \neq 0 \Rightarrow \{Ae_1, \dots, Ae_n\}$ lin. indp. If not, $Ae_1 = \sum_{j=2}^{n} \lambda_j Ae_j$

Then $\Phi_{st}(Ae_1, \dots, Ae_n) = \lambda_2 \Phi_{st}(Ae_2, Ae_2, \dots, Ae_n) + \cdots + \lambda_n \Phi_{st}(Ae_n, \dots, Ae_n) = 0$.

② $Ax = b$, $A \in M_n(F)$, $b \in F^n$ has a unique solution iff $\det A \neq 0$.

<u>Cramer's Rule</u>

$$\left(X_i = \frac{\det\left(\left[c_1 \cdots c_{i-1} \; b \; c_{i+1} \cdots c_n\right]\right)}{\det([c_1 \cdots c_n])} \right) \quad \text{is the solution} \quad (A = [c_1 \cdots c_n]).$$

<u>Proof</u>: $\det(b, c_2, \dots c_n) = \Phi_{st}(X_1 c_1 + X_2 c_2 + \cdots + X_n c_n, c_2, \dots, c_n) = X_1 \Phi_{st}(c_1, c_2, \dots, c_n)$

$= X_1 \det A$

and so on.

① $T \in O(V) \longrightarrow \det(T) = \pm 1$

Fix o.b. $\{u_1, \dots, u_n\}$. Let A be the matrix assoc. to T wrt this basis.

Then $A^T A = I$ so $\det(A^T)\det(A) = 1 \Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$.

② Let $V/_{\mathbb{R}}$; $\langle , \rangle$, let W_1, W_2 be subspaces of V with $\dim W_1 = \dim W_2 = r$

Then $\exists\, T: V \to V$ s.t. $T(W_1) = W_2$.

Proof $\{u_1, \ldots, u_r\}$ OB of W_1

$\{v_1, \ldots, v_r\}$ OB of W_2

extend these to $\{u_1, \ldots, u_r, w_{r+1}, \ldots, w_n\}$, $\{v_1, \ldots, v_r, w'_{r+1}, \ldots, w'_n\}$ OBs of V — by extending to an arbitrary basis, then applying Gram-Schmidt.

Define $T(u_i) = v_i$, $T(w_i) = w'_i$, extend by linearity.

T is orthogonal since it takes an orthonormal basis to another one.

And $T(S(u_1, \ldots, u_r) = W_1) = S(v_1, \ldots, v_r) = W_2$.

Thm (Hadamard). Let $\{a_1, \ldots, a_n\} \subseteq \mathbb{R}^n$. Then $|\det(a_1, \ldots, a_n)| \leq \|a_1\| \cdots \|a_n\|$.

Proof: Wolog assume $a_i \neq 0 \ \forall i$. if one is 0, both sides are 0.

Moreover assume $a_1, \ldots, a_n$ are lin. indp. o.w. $\det(a_1, \ldots, a_n) = 0$.

So hadamard is equiv. to $\left|\det\left(\frac{a_1}{\|a_1\|}, \ldots, \frac{a_n}{\|a_n\|}\right)\right| \leq 1$ or $|\det(v_1, \ldots, v_n)| \leq 1$.

where $v_i = \frac{a_i}{\|a_i\|}$ is an arbitrary unit vector.

We proceed by induction on n. base case $n=1$ obvious.

Assume $\forall k < n$. look at $W_1 = S(v_2, \ldots, v_n)$, $W_2 = S(e_1, \ldots, e_n)$

so $\exists\, T \in O(\mathbb{R}^n)$ s.t. $T(W_1) = W_2$.

$T(v_1), \ldots, T(v_n)$ are unit vectors.

Let $\psi(v_1, \ldots, v_n) = \det(T(v_1), \ldots, T(v_n)) = \Phi_{st}(T(v_1), \ldots, T(v_n))$

$\Rightarrow \psi = \lambda \Phi_{st} = \det(T)\, \Phi_{st}$ (since $\lambda = \psi(e_1, \ldots, e_n)$).

so $\psi(v_1, \ldots, v_n) = \det(T)\, \Phi_{st}(v_1, \ldots, v_n)$

$= \det(T) \det(v_1, \ldots, v_n)$

so $|\psi(v_1, \ldots, v_n)| = |\det(v_1, \ldots, v_n)|$

Why did we do this?

Now $|\det(T(v_1), \ldots, T(v_n))| = \begin{vmatrix} \beta_{11} & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & & \beta_{2n} \\ \vdots & & \ddots & \vdots \end{vmatrix} = |\beta_{11} \det(T(v_2), \ldots, T(v_n))|$

$\leq |\beta_{11}|$

Now $\left|\det(T(v_1), \dots, T(v_n))\right| = \begin{vmatrix} \beta_{21} & \beta_{22} & & \beta_{2n} \\ \vdots & & \ddots & \vdots \\ \beta_{n1} & \cdots & & \beta_{nn} \end{vmatrix} = \left|\beta_{11} \det(T(v_2), \dots, T(v_n))\right|$

$\leq |\beta_{11}|$

≤ 1 since $\|T(v_1)\| \leq 1$.