Lec 11/15

Tuesday, November 15, 2016 9:09 AM

Integration by parts

$$\int P(x) \begin{cases} \exp(x) \\ \cos(x) \\ \sin(x) \\ \dots \end{cases} dx$$

$P(x)$ ↓ polynomial ↑ differentiate until polynomial disappears

↑ integrate

$$\int \lambda(x) \ln(u)\, dx \qquad \int \lambda(x) \arctan(u)\, dx$$

I ↑ $\lambda(x)$, D ↗ $\ln(u)$ / $\arctan(u)$

Indirect method of IBP

$$\int e^x \cos(x)\, dx = e^x \sin(x) - \int e^x \sin(x)\, dx$$

$u = e^x$, $dv = \cos(x)\,dx$; $v = e^x$, $du = \sin(x)\,dx$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$$

$$\Rightarrow 2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)$$

$$\int e^x \cos(x)\, dx = \frac{e^x(\sin(x)+\cos(x))}{2}$$

Partial fractions decomposition

$$\int \frac{P(x)}{q(x)}\, dx \qquad \frac{P}{q} \text{ is a rational function. } P, q \in \text{Poly}.$$

Fundamental Thm of Algebra: Any polynomial w/ complex coeffs can be factored into linear factors. $p(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ where n is $\deg(P)$.

- if Coeffs. are real, can be factored into linear & quadratic factors. $\left(\overline{P(x)} = P(x)\right)$
  (the complex roots occur in conj. pairs which multiply to give a real quadratic).

$$(x-\alpha)(x-\bar{\alpha}) = x^2 - \underbrace{(\alpha+\bar{\alpha})}_{\text{real}}x + \underbrace{\alpha\bar{\alpha}}_{\text{real.}}$$

$x^4 + 1 = 0.$ $\quad x^4 = -1 = e^{i\pi}$ (one root) $\quad x = e^{i\pi/4} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \alpha$

$\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \bar{\alpha}$

$$x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$$
$$= x^2 - \sqrt{2}x + 1 \quad \text{(one quadratic factor)}$$

$$x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$
$$= (x^2+1)^2 - 2x^2$$
$$= x^4+2x^2+1-2x^2$$
$$= x^4+1 \checkmark$$

PFD:

1) factor denominator
$$q(x) = c(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_k)^{m_k}(x+\beta_1+\gamma_1)^{n_1}\cdots(x+\beta_\ell+\gamma_\ell)^{n_\ell}$$

2) General form:
$$\frac{p}{q}(x) = \frac{A_{1,1}}{x-\alpha_1} + \cdots + \frac{A_{1,m_1}}{(x-\alpha_1)^{m_1}} + \frac{A_{2,1}}{x-\alpha_2} + \cdots + \frac{A_{2,m_2}}{(x-\alpha)^{m_2}} + \cdots + \frac{A_{k,m_k}}{(x-\alpha)^{m_k}}$$
$$+ \frac{B_{1,1}x + C_{1,1}}{x+\beta_1+\gamma_1} + \cdots + \frac{B_{\ell,n_\ell}x + C_{\ell,n_\ell}}{(x+\beta_\ell+\gamma_\ell)^{\ell}}.$$

3) solve for As, Bs, Cs

4) integrate.

Euclidean Algorithm

If $d(x)$ is the greatest common divisor of 2 polys $q_1(x), q_2(x)$, there are polys $a(x), b(x)$ so that $d(x) = a(x)q_1(x) + b(x)q_2(x)$

If q_1, q_2 have no common divisors, then $d(x) = 1$.

$$\frac{p(x)}{q_1(x)q_2(x)} = \frac{p(x)a(x)}{q_2(x)} + \frac{p(x)b(x)}{q_1(x)}$$

Repeatedly applying euclidean alg. $p(x)/q(x) = \sum_{\substack{\text{over} \\ \text{linear} \\ \text{factors}}} \frac{\text{Polynomial}}{\text{highest power}}$

How to integrate:

$$\int \frac{c}{(x-\alpha)^m}\,dx = -\frac{1}{m-1}\frac{c}{(x-\alpha)^{m-1}} \quad \text{if } m \neq 1$$
$$= c\log|x-\alpha| \quad \text{if } m = 1$$

$\int Dx + E$ [illegible]

$\int \frac{Dx + E}{(x^2 + px + q)^n} dx$ complete the square & use linear substitution $u = ax + b$ to rewrite this integral as

$$\int \frac{\tilde{D} u + \hat{E}}{(u^2 + \lambda^2)^n} du = \int \frac{\tilde{D} u}{(u^2+\lambda^2)^n} du + \int \frac{\tilde{E}}{(u^2+\lambda^2)^n} du$$

$\downarrow$ $v = u^2 + \lambda^2$

$\downarrow$ $u = \lambda \tan(t)$

$u^2 + \lambda^2 = \lambda^2 \sec^2(t)$

$du = \lambda \sec^2(t) dt$

$\tilde{c} \int \cos^{2n-2}(t)\, dt$

$\downarrow$

use half-angle formula.