Lec 1/6

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people.math.harvard/~mazur/preprints/when_is_one.pdf

Category Theory

1945: $V \simeq V^*$; $V \simeq V^{**}$ more naturally.

give a rigorous defn to "natural"

A category $\mathcal{C}$ consists of

- A class of objects, denoted $Ob(\mathcal{C})$
- A set of morphisms between each pair of objects A, B denoted $Hom_{\mathcal{C}}(A,B)$ or $Hom(A,B)$
- $\forall A, B, C \in Ob(\mathcal{C})$, a set map

$$Hom(A,B) \times Hom(B,C) \longrightarrow Hom(A,C)$$
$$(f, g) \longmapsto g \cdot f$$

Satisfying 3 axioms:

C1: $(A,B) \neq (C,D) \Rightarrow Hom(A,B) \cap Hom(C,D) = \emptyset$.

C2: $\exists\ id_A \in Hom(A,A)$ $(\forall A \in Ob(\mathcal{C}))$ s.t.

$$\forall f \in Hom(A,B), \quad f \cdot id_A = f,$$
$$\forall g \in Hom(B,A), \quad id_A \cdot g = g.$$

C3: Associativity: $\forall f \in Hom(A,B)$, $g \in Hom(B,C)$, $h \in Hom(C,D)$, $h(gf) = (hg)f =: hgf$.

Class: Gödel-Bernays axioms for set theory. (keep it vague).

Good book by Paul Cohen: Set theory & the continuum hypothesis.

Note 1. $\mathrm{Hom}(A,B)$ could be empty.

$\mathrm{Hom}(A,A)$ always has id_A.

2. We will use the usual notions of commutative diagrams

if $f \in \mathrm{Hom}(A,B) \rightsquigarrow A \xrightarrow{f} B$ or $f: A \to B$.

eg

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "id_A"'] & B \\ A \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "g"'] & A \arrow[d, "id_A"] \\ & A \end{tikzcd}$$

unit diagrams

$$\begin{tikzcd} & B \arrow[r, "g"] \arrow[drr, "h \cdot g"] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[urr, "g \cdot f"] \arrow[rrr, "h \cdot g \cdot f"'] & & & D \end{tikzcd}$$

associativity diagram

Examples

1. $\underline{\mathbf{Set}}$ = category of sets & set maps
2. $\underline{\mathbf{Mon}}$ = category of monoids & monoid homomorphisms

} product is compositions

$\underline{\mathbf{Grp}}$, $\underline{\mathbf{Ab}}$, etc.

Subcategory: $\mathcal{D}$ subcategory of $\mathcal{C}$ if

- $Ob(\mathcal{D})$ subclass of $Ob(\mathcal{C})$
- $\mathrm{Hom}_{\mathcal{D}}(A,B) \subset \mathrm{Hom}_{\mathcal{C}}(A,B)$ if $A,B \in Ob(\mathcal{D})$
- product of morphisms in $\mathcal{D}$ agrees w/ $\mathcal{C}$.

A subcategory is <u>full</u> if $\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ $\forall A,B \in Ob(\mathcal{D})$.

(Note: axiom: if X, Y are classes & $X \in Y$, then X is a set).

<u>Ab</u> is a full subcategory of <u>Grp</u>.

<u>Grp</u> is a full subcategory of <u>Mon</u>.

<u>Mon</u> is not even a subcategory of <u>Set</u> (the objects are different).

<u>Note</u>: $\mathrm{Hom}(A,A)$ is a monoid.

eg let M be any monoid. form a category $\underline{M}$ with

- $Ob(\underline{M}) = \{A\}$,
- $\mathrm{Hom}(A,A) = M$.

This is a small category ($ob(\mathcal{C})$ forms a set).

<u>Isomorphisms</u>: $\mathcal{C}$ is a category, $A,B \in Ob(\mathcal{C})$, $f \in \mathrm{Hom}(A,B)$.

f is called an isomorphism if $\exists\, g \in \mathrm{Hom}(B,A)$ s.t. $f \cdot g = id_B$, $g \cdot f = id_A$.

<u>Defn</u> the opposite or dual category:

if $\mathcal{C}$ is a category, $\mathcal{C}^{op}$ is the category w/ the same objects, and if $A,B \in Ob(\mathcal{C}) = Ob(\mathcal{C}^{op})$, then

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\cdot f"'] & B \arrow[d, "g"] & & & A & B \arrow[l, "f"'] \\
& C & {} \arrow[r, rightsquigarrow] & {} & & C \arrow[u, "g"'] \arrow[ul, "f\cdot g"]
\end{tikzcd}$$

in $\mathcal{C}$ in $\mathcal{C}^{op}$ (so $f^{op} \cdot g^{op} = (g \cdot f)^{op}$).