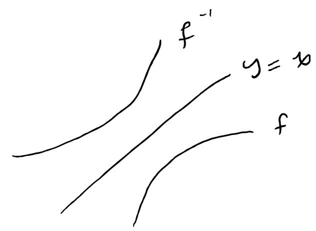
Lec 10/20

Thursday, October 20, 2016 9:06 AM

Inverse functions

If (x,y) point in $\mathbb{R}^2$ then (y,x) is the reflection of (x,y) around $y=x$



Theorem 1 f defined, continuous, on interval I $\Rightarrow$ f 1-1 $\Leftrightarrow$ f increasing on I or f decreasing on I.

Proof. Yesterday

Corollary T + IVT: (using hyp of thm 1) $f(I)$ is an interval. If a, b are endpoints of I, $f(a), f(b)$ are endpoints of $f(I)$. If a, b are $\pm\infty$ then $f(a), f(b)$ should be interp'd as a limit. Also, $f(a)$ or $f(b)$ are included in $f(I)$ iff a or b included in I.

Theorem 2 f^{-1} continuous on $f(I)$ (using same hyp as thm 1).

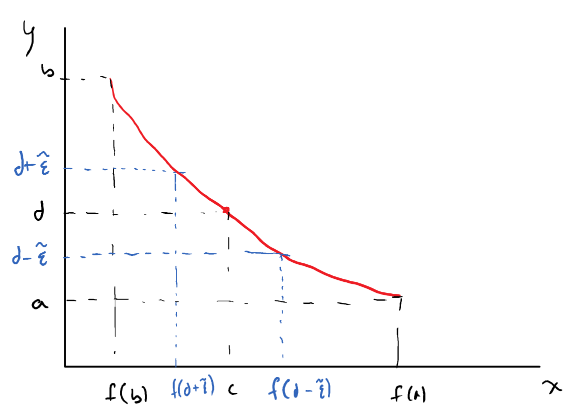
Proof: Let $f(d)=c$, $f^{-1}(c)=d$. $d \in I$, $c \in f(I)$.

Consider the case when f is decreasing on I. ($\Rightarrow f^{-1}$ decreasing on $f(I)$)

and $c \in$ interior of I $\Leftrightarrow$ $d \in$ interior of $f(I)$.

Let $\varepsilon > 0$ be given. WTF $\delta > 0$ st $|x-c| < \delta \Rightarrow |f^{-1}(x) - \underset{f^{-1}(c)}{d}| < \varepsilon$

let $\tilde{\varepsilon} = \min(\varepsilon, b-d, d-a)$



Let $\delta = \min(f(d-\tilde{\varepsilon}) - c,\ c - f(d+\tilde{\varepsilon}))$.

Then $|x-c|<\delta \Rightarrow x \in (f(d+\tilde{\varepsilon}), f(d-\tilde{\varepsilon}))$

$$\Rightarrow f^{-1}(x) \in (d-\tilde{\varepsilon}, d+\tilde{\varepsilon})$$

$$\Rightarrow |f^{-1}(x) - d| < \tilde{\varepsilon} \le \varepsilon \qquad \blacksquare$$

Theorem 3: Same hypothesis as in Thm 1, If $f(d) = c \Leftrightarrow f^{-1}(c) = d$, $c \in \text{int } I$, $d \in \text{int } f(I)$, and $f'(d) \neq 0$, Then $(f^{-1})'(c)$ exists and $(f^{-1})'(c) = \frac{1}{f'(d)} = \frac{1}{f'(f^{-1}(c))}$.

Proof: assume f is increasing on I ($\Rightarrow f^{-1}$ increasing on $f(I)$).

we have $\lim_{y\to d} \frac{f(y)-f(d)}{y-d} = f'(d)$

$$\Longrightarrow \lim_{y\to d} \frac{y-d}{f(y)-f(d)} = \frac{1}{f'(d)}$$

Let $\varepsilon > 0$ be given. Then there is a $\tilde{\delta} > 0$ s.t.

$$0 < |y-d| < \tilde{\delta} \Rightarrow \left| \frac{y-d}{f(y)-\underset{c}{\overset{\shortparallel}{f(d)}}} - \frac{1}{f'(d)} \right| < \varepsilon$$

$$\Updownarrow$$

$$y \in (d-\tilde{\delta}, d) \cup (d, d+\tilde{\delta})$$

Let $x = f(y) \Leftrightarrow y = f^{-1}(x)$

So $x \in (f(d-\tilde{\delta}), f(d)) \cup (f(d), f(d+\tilde{\delta}))$.

Now let $\delta = \min(\overset{c}{\overset{\shortparallel}{f(d)}} - f(d-\tilde{\delta}), f(d+\tilde{\delta}) - \overset{c}{\overset{\shortparallel}{f(d)}})$.

Then $0<|x-c|<\delta \Rightarrow x \in (f(d-\tilde{\delta}), f(d)) \cup (f(d), f(d+\tilde{\delta}))$

$$\Rightarrow y = f^{-1}(x) \in (d-\tilde{\delta}, d) \cup (d, d+\tilde{\delta})$$

$$\Rightarrow \left| \frac{f^{-1}(x) - f^{-1}(c)}{x-c} - \frac{1}{f'(d)} \right| < \varepsilon$$

$$\Rightarrow \lim_{x\to c} \frac{f^{-1}(x) - f^{-1}(c)}{x-c} = \frac{1}{f'(d)} = \frac{1}{f'(f^{-1}(c))} = (f^{-1})'(c) \qquad \blacksquare$$

Implicit functions/differentiation

Ex: Find the equation of the tangent line to the graph of $x^3 + xy + 2y^3 = 4$

at the point $(1,1)$.

Assume we can solve this equation for y in terms of x. i.e. $y = f(x)$ for some f defined on an open interval containing $x=1$ with $f(1)=1$.

Then $x^3 + xy + 2y^3 - 4 = 0$ holds $\forall x \in$ this open interval.

We can differentiate this wrt x, treating y as a function of x. (using chain rule).

$$\frac{\partial}{\partial x}\left(x^3 + xy + 2y^3 - 4\right) = 0$$

$$3x^2 + x\frac{\partial y}{\partial x} + y + 6y^2\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} = -\frac{3x^2 + y}{6y^2 + x}$$

So the slope at $(1,1)$ is $-\frac{3+1}{6+1} = -\frac{4}{7}$

So the equation of the tangent line is $y = -\frac{4}{7}(x-1) + 1$.

Next Semester: Implicit function Theorem: if f is a differentiable function of x, y, $f(x_0, y_0) = 0$ & $\left.\frac{\partial f}{\partial y}\right|_{y=y_0} \neq 0$, then we can solve $f(x,y)=0$ for y in terms of x:

$y = f(x)$ defined on an open interval containing x_0, with $f(x_0) = y_0$.

Then $f'(x_0) = -\dfrac{\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)}}{\left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)}}$

Inverse function thm is a special case: $x - f(y) = 0$

$y = f^{-1}(x)$