Lec 1/8

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$\mathbb{R}^{d\times d}$ = the algebra of $d\times d$ matrices over $\mathbb{R}$.

$GL(d,\mathbb{R}) \subseteq \mathbb{R}^{d\times d}$

$\uparrow$ invertible $d\times d$ matrices.

(a group under matrix multiplication).

think of $\mathbb{R}^d$ as column vectors of size d. $\mathbb{R}^d = \mathbb{R}^{d\times 1}$

if $L:\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear map, then L is of the form $v \longmapsto M_L v$ for a suitable $d\times d$ matrix $M_L \in \mathbb{R}^{d\times d}$.

Let $v_1,\ldots,v_d \in \mathbb{R}^d$. This is a basis for $\mathbb{R}^d$ iff $\begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix} \in GL(d,\mathbb{R})$.

__Temporary__ __Notation__:

$\sim$ will be the binary relation on $GL(d,\mathbb{R})$ defined by $A\sim B$ iff there is a continuous map $M:[0,1] \longrightarrow GL(d,\mathbb{R})$ s.t. $M(0)=A$ and $M(1)=B$.

(can continuously deform one basis to another and have a basis at every intermediate point.

Remark: $\sim$ is an equivalence relation on $GL(d,\mathbb{R})$

reflexivity: $M(t)=A$

symmetricity: replace M by $t \mapsto M(1-t)$

transitivity: $M(t) = \begin{cases} M_1(2t) & t\in[0,1/2] \\ M_2(2t-1) & t\in[1/2,1] \end{cases}$

Also: If $A_1,\ldots,A_n, B_1,\ldots,B_n \in GL(d,\mathbb{R})$ and $A_j \sim B_j$ for $j=1,\ldots,n$, then $A_1\cdots A_n \sim B_1\cdots B_n$

$t \mapsto M_1(t)\cdots M_n(t)$ gives the mapping.

eg: In $GL(2,\mathbb{R})$, $\overset{A}{\overset{\shortparallel}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}} \not\sim \overset{B}{\overset{\shortparallel}{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}}$

Pf Suppose they are equivalent.

Let $M: [0,1] \to GL(2,\mathbb{R})$ be cts. w/ $M(0) = A$, $M(1) = B$

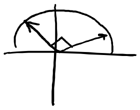
by IVT, $\det(M(\_))$ is cts.

Now $\det(A) = 1$, $\det(B) = -1$ So $\det(M(t)) = 0$ for some $t \in [0,1]$,

thus M does not take values in $GL(2,\mathbb{R})$ only. ⨳

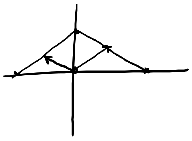
eg In $GL(3,\mathbb{R})$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \not\sim \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

eg $\begin{pmatrix}1&0\\0&1\end{pmatrix} \sim \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$. pf: Let $R(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$



then $t \mapsto R(\pi t)$ works. And $\operatorname{Det}(R(\theta)) = 1$.

eg $A = \begin{pmatrix}-1&0\\0&1\end{pmatrix} \sim \begin{pmatrix}0&1\\1&0\end{pmatrix} = B$



$t \mapsto \begin{pmatrix}t-1 & t\\ t & 1-t\end{pmatrix}$ works as a map.

$\begin{vmatrix}t-1 & t\\ t & 1-t\end{vmatrix} = -t^2 - 2t - 1 - t^2 < 0$ so it's invertible all the way thru.

Also: A is reflection about y-axis

B is reflection about $y = x$.

$R(\frac{-\pi}{4})\, A\, R(\frac{\pi}{4}) = B$

check:

$$\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\underbrace{\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}}_{\begin{pmatrix}-\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}}$$

$$\overbrace{\qquad\qquad}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\underbrace{\qquad\qquad}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So we can use $M(t) = R\left(-\frac{\pi}{4}t\right) A \, R\left(\frac{\pi}{4}t\right)$.

eg Similarly, in $GL(d, \mathbb{R})$,

$$A = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & 1 \\ & & & & 1 & \\ & & & & & \ddots \\ & & & 1 & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} = P_{kj} = \left(e_1 \cdots e_{j-1}\, \overset{\downarrow}{e_k}\, e_{j+1} \cdots e_{k-1}\, e_j\, e_{k+1} \cdots e_d\right)$$

k^{th}

$$\wr$$

$$\begin{array}{c} \\ \\ j \rightarrow \\ \\ \end{array}\begin{pmatrix} 0 & \cdots & 0 & \overset{j}{1} & 0 & \cdots & 0 \\ \vdots & 1 & & & & 0 & \\ 0 & & \ddots & & & & \\ 1 & & & 1 & & & \\ 0 & & & & 0 & & \\ \vdots & 0 & & & & 1 & \\ 0 & & & & & & \ddots\ 1 \end{pmatrix} = P_{1j} \quad \text{(permute 1 and } j^{th} \text{ standard basis vectors)}$$

eg in $GL(3, \mathbb{R})$

$$\begin{tikzcd}
(e_1\ e_2\ e_3) \arrow[r, mapsto, "P_{13}"] \arrow[ddr, bend right, "P_{23} = P_{13}P_{12}P_{13}"'] & (e_3\ e_2\ e_1) \arrow[d, mapsto, "P_{12}"] \\
& (e_2\ e_3\ e_1) \arrow[d, mapsto, "P_{13}"] \\
& (e_1\ e_3\ e_2)
\end{tikzcd}$$

(use first slot as a holding area for k^{th} vector to interchange k^{th} & j^{th} rows.

Let $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$P_{23} = \underbrace{P_{13} P_{12} P_{13}}_{\sim \text{ resp. products.}} \sim JJJ = J \quad \text{since } J^2 = I.$$

eg Similarly, in $GL(d, \mathbb{R})$, $\quad P_{kj} \sim J \quad$ (for $1 \le k < j \le d$).

$$eg\;\underbrace{\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}}\begin{pmatrix}1&a&d\\2&b&e\\5&c&f\end{pmatrix}=\begin{pmatrix}1&\cdots\\0&\cdots\\5&\cdots\end{pmatrix}$$

$$E(\underset{\overset{\uparrow}{c\in\mathbb{R}}}{-2},\underset{\overset{\uparrow}{j}}{1},\underset{\overset{\uparrow}{k}}{2})\sim I$$

in fact, any $E(c,j,k)\sim I$.

And multiplication by a constant (to a row) is $\sim I$ as well.

Thus every $A\in GL(d,\mathbb{R})$ has $A\sim I$ or $A\sim J$

(by gaussian elimination)

$\forall\, B\in GL(d,\mathbb{R})$, $\exists\, n$ s.t. $\exists\, A_1,\dots,A_n\in\{I,J\}$,

$$B = A_1\cdots A_n \sim \underset{\overset{\uparrow}{\text{even}}}{I}\ \text{or}\ \underset{\overset{\uparrow}{\text{odd}}}{J}.$$

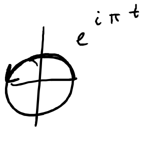
if $\#J^s$ is even or odd

$\det(B)=\det(A_1\cdots A_n)$, so $\det(B)>0 \Rightarrow B\sim I$

$\det(B)<0 \Rightarrow B\sim J$.

So there are two possible orientations for a basis of $\mathbb{R}^d$.

<u>Question</u>: what about $GL(d,\mathbb{C})$?



Again every $B\sim I$ or J.

But now $I\sim J$ by letting top left entry go around circle.

$$M(t)=\begin{pmatrix}e^{i\pi t}&&0\\&1\ddots&\\0&&1\end{pmatrix}$$ is cts, $M(0)=A$, $M(1)=B$.

$\det(M(t))=e^{i\pi t}$ so it's always invertible.

Norms and inner product

let $\langle \cdot | \cdot \rangle$ be an inner product on $V/\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

<u>Cauchy - Schwarz inequality</u> $\nearrow$ let $\|v\| = \sqrt{\langle v|v\rangle}$.

$$|\langle v|w\rangle| \leq \|v\|\|w\|$$

<u>Pf</u> $W = \alpha v + \beta v^{\perp}$, etc.

<u>Triangle inequality</u>

$$\|v+w\| \leq \|v\| + \|w\|$$

<u>Pf</u> $\|v+w\|^2 = \langle v+w | v+w\rangle = \langle v|v\rangle + \langle w|w\rangle + \langle v|w\rangle + \overset{\overline{\langle v,w\rangle}}{\overset{\shortparallel}{\langle w|v\rangle}}$

$$= \|v\|^2 + \|w\|^2 + 2\operatorname{Re}\langle v|w\rangle$$

$$\leq \|v\|^2 + \|w\|^2 + 2\langle v|w\rangle$$

$$\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$$

$$= (\|v\| + \|w\|)^2$$

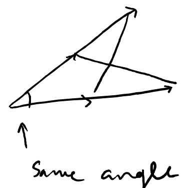
so taking sq. roots gives ∇ - ineq.

<u>Question:</u>

When does a norm arise from an inner product?

Answer: if SAS Δ-congruence works.

even in this special case:



same angle.