Lec 1/27

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Maximum likelihood estimators:

$$\hat{\theta} = \arg\max L(\theta)$$

Where $L(\theta) = f(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$

Ex: $X_1, \dots, X_n \overset{iid}{\sim} f(x, \theta) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}$ $\quad (x, \theta) \in (0,1) \times (0, \infty)$

Find MLE. equiv to maximizing $\log(L(\theta))$.

$$\log(L(\theta)) = \log\left(\prod_{i=1}^{n} \frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}}\right) = \sum_{i=1}^{n} \log\left(\frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}}\right) = \sum_{i=1}^{n}\left(\frac{1-\theta}{\theta}\log(x_i) - \log(\theta)\right)$$

$$= \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log(x_i) - n\log(\theta) = (*)$$

$$\frac{\partial}{\partial\theta}(*) = 0 = -\frac{n}{\theta} + \frac{-\theta - 1 + \theta}{\theta^2} \sum_{i=1}^{n} \log(x_i)$$

$$\Rightarrow \frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} \log(x_i) = 0$$

$$\Rightarrow n + \frac{1}{\theta} \sum_{i=1}^{n} \log(x_i) = 0$$

$$\Rightarrow \theta = -\frac{\sum_{i=1}^{n} \log(x_i)}{n} \equiv \hat{\theta}.$$

$$\frac{\partial^2}{\partial\theta^2}(*) = \frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} \log(x_i)$$

$\hookrightarrow$ at $\hat{\theta} = \frac{1}{\hat{\theta}^3}\left(n\hat{\theta} + 2\sum_{i=1}^{n} \log(x_i)\right)$

$$= \frac{1}{\hat{\theta}^3} \sum_{i=1}^{n} \log(x_i) < 0 \qquad \text{since } \hat{\theta} > 0.$$

So $\hat{\theta}$ is a maximum & is the MLE for θ.

Midterm will have at least 1 MLE problem.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} f(x; \theta) = \frac{\theta}{x^2}$, $0 < \theta \le x \le \infty$. Find MLE for θ.

$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} = \theta^n \prod_{i=1}^{n} \frac{1}{x_i^2}$. Try to maximize:

$\cancel{L(\theta) \le \theta^n \prod_{i=1}^{n} \left(\frac{1}{\theta^2}\right) = \frac{\theta^n}{\theta^{2n}} = \frac{1}{\theta^n}}$

$\cancel{L(\theta) \le \theta^n \prod_{i=1}^{n} \left(\frac{1}{\theta^2}\right) = \frac{\theta}{\theta^{2n}} = \frac{1}{\theta^n}}$

to make θ as large as possible, take $\theta = \hat{\theta} = \min\{x_1, \dots, x_n\}$

Since $\theta \le x_i$ for all x_i this also makes $L(\theta)$ as large as possible.

Ex: $X_1, \dots, X_n \overset{iid}{\sim}$ gamma (α, β). α is known. Find MLE of β.

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}$$

$$= e^{-\frac{\sum x_i}{\beta}} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\right)^{n}$$

minimize by log.

$$(*)\!: \log(L(\beta)) = -\frac{\sum x_i}{\beta} + (\alpha-1)\log\left(\prod_{i=1}^{n} x_i\right) - n\log(\Gamma(\alpha)) - n\alpha\log(\beta)$$

$$\frac{\partial}{\partial\beta}(*) = \frac{\sum x_i}{\beta^2} - \frac{n\alpha}{\beta} = 0 \Rightarrow \sum x_i - n\alpha\beta = 0 \Rightarrow \beta = \frac{\sum x_i}{n\alpha} = \hat{\beta}$$

$$\left.\frac{\partial^2}{\partial\beta^2}(*)\right|_{\beta=\hat{\beta}} = \left.\frac{-2\sum x_i}{\beta^3} + \frac{n\alpha}{\beta^2}\right|_{\beta=\hat{\beta}} = \frac{-2(n\alpha)^3}{(\sum x_i)^2} + \frac{(n\alpha)^3}{(\sum x_i)^2} = \frac{-(n\alpha)^3}{(\sum x_i)^2} < 0 \Rightarrow \hat{\beta} \text{ maximizes.}$$

So $\hat{\beta}$ is MLE.

Invariance property of MLE:

If $\hat{\theta}$ is MLE of θ, and $g(\theta)$ is continuous, then $g(\hat{\theta})$ is MLE for $g(\theta)$.

Eg in the example above, let $\tau = (2\beta - 1)^2$, the MLE of τ is $\hat{\tau} = (2\hat{\beta} - 1)^2 = (2\frac{\bar{x}}{\alpha} - 1)^2$.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ i.e. $\theta = (\mu, \sigma^2)$. $\theta \in \mathbb{R} \times \mathbb{R}^+$. find MLE of θ.

sol. $$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x_i - \mu)^2\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right]$$

$$\log(L(\mu, \sigma^2)) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 = (*)$$

$$\frac{\partial}{\partial\mu}(*) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0 \Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0 \Rightarrow \mu = \bar{X} = \hat{\mu}.$$

$$\frac{\partial}{\partial\mu}(*) = \frac{1}{\sigma^2}\sum_{i=1}(x_i-\mu) = 0 \Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0 \Rightarrow \mu = \overline{X} = \hat{\mu}.$$

$$\frac{\partial}{\partial\sigma^2}(*) = \frac{-n}{2\sigma^2} + \frac{1}{(2\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \hat{\sigma}^2$$

HW: Check $\hat{\mu}$ and $\hat{\sigma}^2$ maximize likelihood function:

1: Second order derivative < 0.

2: $|\text{Jacobian matrix}| > 0$.