

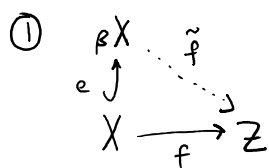
Recall: A compactification of  $X$  is an embedding  $e: X \hookrightarrow K$  <sup>compact</sup>  
 where  $e(X)$  is dense in  $K$ .

- When  $X$  is LCH but not cpt, there is a smallest (one-pt) compactification.
- When  $X$  is Tychonoff (completely regular & T<sub>1</sub>)  
 there is a largest compactification (Stone-Čech).

Recall:  $X$  tychonoff  $\iff \exists$  embedding  $X \hookrightarrow [0, 1]^I$ .

SČC: Suppose  $X$  is tychonoff, let  $\Phi = C(X, [0, 1])$   
 and consider  $e: X \hookrightarrow [0, 1]^\Phi$  by  $e(x) = (f(x))_{f \in \Phi}$ .  
 define  $\beta X = \overline{e(X)}$ .

Theorem: The compactification  $(\beta X, e)$  satisfies:



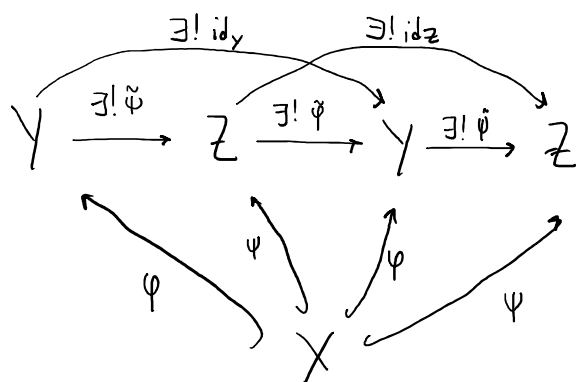
$\forall$  cpt Hausdorff  $Z$  and cts  $f: X \rightarrow Z$ ,  
 $\exists!$  cts  $\tilde{f}: \beta X \rightarrow Z$  s.t.  $\tilde{f} \circ e = f$ .

② Remark  $\exists$  implies! by density of  $e(X) \subset \beta(X)$ .

③  $\beta X$  is uniquely characterized by univ. property ①.

④  $\beta$  is a functor  $\{\text{Tych } S_p\} \longrightarrow \{\text{cpt Hausdorff } S_p\}$ .

Pf of ③: Suppose  $(\varphi, \gamma)$  and  $(\psi, \zeta)$  satisfy univ. prop in ①.



$$\text{so } \tilde{\psi} \circ \tilde{\varphi} = id_Z$$

$$\tilde{\varphi} \circ \tilde{\psi} = id_Y$$

$Y \cong Z$  homeom by  
canonical maps.

Pf of ④: Suppose  $f: X \rightarrow Y$  cts &  $X, Y$  are Tychonoff.

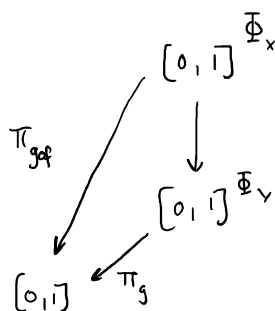
Let  $\Phi_X = C(X, [0, 1])$  and  $\Phi_Y = C(Y, [0, 1])$ .

Define  $F: [0, 1]^{\Phi_X} \rightarrow [0, 1]^{\Phi_Y}$

componentwise: for  $g \in \Phi_Y$ ,  $\pi_g \circ F := \pi_{\underbrace{g \circ f}_{cts}}$ ,

where  $g \circ f \in \Phi_X$

so  $F$  cts.



Now  $\forall x \in X$ ,

$$\begin{aligned}\pi_g[F(e_x(x))] &= \pi_{g \circ f}(e_x(x)) \\ &= (g \circ f)(x) \\ &= \pi_g[e_y(f(x))]\end{aligned}$$

Hence  $F \circ e_x = e_y \circ f : X \longrightarrow [0,1]^{\Phi_Y}$

Thus  $\text{Image}(F|_{\beta X}) \subset \overline{e_Y(Y)} = \beta Y$

( $x \in \beta X$  and  $x_x \xrightarrow{e(x)} X$  then  $F(x_x) \in e_Y(Y) \Rightarrow F(x) \in \beta Y$ ).

define  $\beta f = F|_{\beta X} : \beta X \longrightarrow \beta Y$  etc

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & \beta X & \xrightarrow{\quad} & [0,1]^{\Phi_X} \\ f \downarrow & \circlearrowleft & \beta f \downarrow & \circlearrowleft & F \downarrow \\ Y & \xrightarrow{\quad} & \beta Y & \xrightarrow{\quad} & [0,1]^{\Phi_Y} \end{array}$$

Observe:

$$\begin{array}{ccccc} & & \beta X & & \\ & e_x \uparrow & \searrow \beta f & & \\ X & \xrightarrow{f} & Y & \xrightarrow{e_x} & \beta Y \end{array}$$

$$\beta f \circ e_x = e_y \circ f$$

Functoriality:

id: show  $\beta[id_X] = id_{\beta X}$ .

$$\begin{array}{ccccc}
 & \beta X & & & \\
 e_X \uparrow & & \searrow id_{\beta X} = \beta[id_X] & & \text{by uniqueness} \\
 X & \xrightarrow{id_X} & X & \xrightarrow{e_X} & \beta X
 \end{array}$$

=o: suppose  $f: X \rightarrow Y$  &  $g: Y \rightarrow Z$ .

$$\begin{aligned}
 \beta(g \circ f) \circ e_X &= e_Z \circ (g \circ f) = (e_Z \circ g) \circ f \\
 &= \beta g \circ e_Y \circ f = \beta g \circ \beta f \circ e_X
 \end{aligned}$$

$\Rightarrow \beta(g \circ f) = \beta g \circ \beta f$  by uniqueness

$$\begin{array}{ccccc}
 & \beta(g \circ f) & & & \\
 & \curvearrowright & & & \\
 \beta X & \xrightarrow{\beta f} & \beta Y & \xrightarrow{\beta g} & \beta Z \\
 \uparrow & & \uparrow & & \uparrow \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
 \end{array}$$

pf of ①

Suppose  $f: X \rightarrow Z$  cts and  $Z$  cpl Hausdorff

observe:  $\beta Z = Z$  by the universal property!

by functoriality,  $\exists! \beta f: \beta X \rightarrow \beta Z = Z$ .

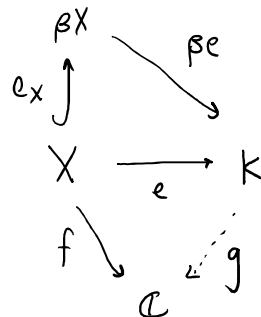
$$\begin{aligned}
 \text{s.t. } e_Z \circ f &= \beta f \circ e_X \\
 \parallel & \\
 f &
 \end{aligned}$$

□

Corollary:  $X$  is Tychonoff,  $e: X \hookrightarrow K$  cptification.

①  $\beta e: \beta X \rightarrow K$  is surjective

② if  $\forall f \in C_b(X), \exists g \in C(K)$  s.t.  $f = g \circ e$ ,  
then  $\beta e: X \rightarrow K$  is a homeomorphism.



pf of ① since  $\beta e \circ e_x = e$  and  $e(X)$  is dense in  $K$ ,

$\beta e[\beta X]$  is dense in  $K$  since  $\beta X \supset e(X)$ .

but  $\beta X$  cpt &  $\beta e$  cts so  $\beta e[\beta X]$  is cpt so closed, so it =  $K$ .

②: It suffices to prove  $\beta e$  is injective

Recall:  $\beta e := F|_{\beta X}$ .

claim: if every  $f \in C_b(X)$  factorizes thru  $e: X \hookrightarrow K$ ,  $F$  is injective.

Indeed, every  $f \in \Phi_x = C(X, [0,1]) \subset C_b(X)$  factorizes as  $g \circ e$

so  $F(x) = F(x') \iff x = x'$

$$\pi_g(F(x)) = \pi_{g \circ e}(x)$$

||

$$\pi_g(F(x')) = \pi_{g \circ e}(x')$$

$$\Rightarrow \pi_f(x) = \pi_f(x')$$

$$\forall f \in \Phi_x.$$

□