Lec 4/6

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Reminders

$$\nabla_\gamma X = \text{the tangential part of } \frac{dX}{dt}$$

$$= \sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k X^i \frac{d\gamma^j}{dt} \right) x_k \qquad \text{is intrinsic.}$$

$$\frac{d}{dt}\langle X \mid Y \rangle = \langle \nabla_\gamma X \mid Y \rangle + \langle X \mid \nabla_\gamma Y \rangle$$

X is parallel along γ iff $\nabla_\gamma X \equiv 0$ iff $\forall k,\ \frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k X^i \frac{d\gamma^j}{dt} = 0$.

if X and Y are parallel along γ, then $t \longmapsto \langle X(t), Y(t) \rangle$ is constant.

In particular, if X is parallel along γ, then $t \longmapsto |X(t)|$ is constant.

<u>Propn</u> Let γ be a C^2 curve in V. Then γ is a constant speed geodesic iff $\frac{d\gamma}{dt}$ is parallel along γ.

if $V=0$, γ is a geodesic trivially.

<u>Proof</u> ($\Leftarrow$) Suppose $\frac{d\gamma}{dt}$ is parallel along γ. then $\left|\frac{d\gamma}{dt}\right|$ is constant, $\equiv V$, say.

And the tangential component of $\frac{d^2\gamma}{dt^2}$ is 0. But $\frac{dT}{ds} = \frac{d^2\gamma}{ds} = \frac{d}{ds}\left(\frac{d\gamma}{dt}\frac{dt}{ds}\right)$

$= \frac{d}{ds}\left(\frac{1}{V}\frac{d\gamma}{dt}\right) = \frac{1}{V}\frac{d^2\gamma}{dt^2}\frac{dt}{ds} = \frac{1}{V^2}\frac{d^2\gamma}{dt^2}$, so $\underset{\kappa N}{\underset{\parallel}{\frac{dT}{ds}}}$ is also normal to the surface

So γ is a geodesic.

($\Rightarrow$) reverse the steps $\square$

<u>Theorem 5.7 in Ch4</u> Let $X \in T_pM$ with $|X| = 1$, and $s_0 \in \mathbb{R}$. Then $\exists\ a, b \in \mathbb{R}$ with $a < s_0 < b$ and $\exists$ a unit-speed geodesic $\gamma : (a,b) \longrightarrow M$ so that $\gamma(s_0) = p$ and $\gamma'(s_0) = X$.

assume M is C^3.

Moreover, if $\gamma : (a,b) \longrightarrow M$ and $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \longrightarrow M$ are two such geodesics, Then $\forall s \in (a,b) \cap (\tilde{a}, \tilde{b})$, $\gamma(s) = \tilde{\gamma}(s)$.

Pf First, let's show that $\exists \varepsilon_0 > 0$ (depending on M, p, and X) such that $\forall \varepsilon \in (0, \varepsilon_0)$, there is a unique unit-speed geodesic $\gamma : (s_0 - \varepsilon, s_0 + \varepsilon) \longrightarrow M$ s.t. $\gamma(s_0) = p$ and $\gamma'(s_0) = X$.

Let $x : U$ open $\subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V$ open $\subseteq M$ be a C^3 coord patch on M with $p \in V$, $(0,0) \in U$, and $x(0,0) = p$. Since V is open in M, there is an $\varepsilon_1 > 0$ s.t. $V_1 \overset{\text{def}}{=} \{ q \in M : |q - p| < \varepsilon_1 \} \subseteq V$.

For each $\varepsilon \in (0, \varepsilon_1]$, and for each unit-speed curve $\alpha : (s_0 - \varepsilon, s_0 + \varepsilon) \longrightarrow M$ with $\alpha(s_0) = p$, we have range $(\alpha) \subseteq V_1 \subseteq V$. Thus, for each $\varepsilon \in (0, \varepsilon_1]$, for each C^2 curve $\gamma : (s_0 - \varepsilon, s_0 + \varepsilon) \longrightarrow M$, γ is a unit-speed geodesic with $\gamma(s_0) = p$ and $\gamma'(s_0) = X$ iff γ is of the form $x(\gamma^1(s), \gamma^2(s))$, where $\gamma^1, \gamma^2 : (s_0 - \varepsilon, s_0 + \varepsilon) \longrightarrow \mathbb{R}$ are C^2 functions satisfying $\left\{ \frac{d^2 \gamma^k}{ds^2} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} = 0 \text{ and } \gamma^k(s_0) = 0, \frac{d\gamma^k}{ds}(s_0) = X^k. \right\} (*)$

Note: take Γ_{ij}^k to be $\Gamma_{ij}^k(\gamma^1(s), \gamma^2(s))$

remember $x_{ij} = (\text{normal part}) + \sum_k \Gamma_{ij}^k x_k$. So if x is C^k then Γ_{ij}^k are C^{k-2}.

Since x is C^3, the Γ_{ij}^k's are C^1 so they are locally Lipschitz. Thus by picard's theorem, $\exists\, \varepsilon_2 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_2)$, the IVP $(*)$ has a unique solution on $(s_0 - \varepsilon, s_0 + \varepsilon)$, and thus we can take $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$.

Now let $\gamma : (a,b) \longrightarrow M$ and $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \longrightarrow M$ be two unit-speed geodesics with $s_0 \in (a,b) \cap (\tilde{a}, \tilde{b})$, $\gamma(s_0) = p = \tilde{\gamma}(s_0)$ and $\gamma'(s_0) = X = \tilde{\gamma}'(s_0)$. Let $\alpha = \max\{a, \tilde{a}\}$ and $\beta = \min\{b, \tilde{b}\}$. We WTS that $\forall s \in (\alpha, \beta)$ we have $\gamma(s) = \tilde{\gamma}(s)$.

review this induction thing.

First, lets show that $\gamma(s) = \tilde{\gamma}(s)$ $\forall s \in [s_0, \beta)$. Suppose not. Then $\exists\, r \in (s_0, \beta)$ s.t. $\gamma(r) \neq \tilde{\gamma}(r)$. Let $D = \{ t \in [s_0, \beta) : \gamma(s) = \tilde{\gamma}(s)\ \forall s \in [s_0, t] \}$. Then r is an upper bound for D, and $[s_0, s_0 + \varepsilon_0) \subseteq D$. Let $s_1 = \sup D$. Then $s_0 < s_0 + \varepsilon_0 \leq s_1 \leq r < \beta$. If $s \in [s_0, s_1)$, then s is not an upper bound for D so $s \leq t$ for some $t \in D$ so $s \in [s_0, t]$ so $\gamma(s) = \tilde{\gamma}(s)$. Thus $\gamma = \tilde{\gamma}$ on $[s_0, s_1)$ but $\gamma, \tilde{\gamma}$ are C^1 and so cts so $\gamma(s_1) = \tilde{\gamma}(s_1)$, and $\gamma'(s_1) = \tilde{\gamma}'(s_1)$. Then by the first part of pf (with s_0 replaced by s_1), there is an $\varepsilon > 0$ s.t. $\forall s \in (s_1 - \varepsilon, s_1 + \varepsilon)$, $\gamma(s) = \tilde{\gamma}(s)$. So $s_1 + \frac{\varepsilon}{2} \in D$, so s_1 is not an upper bound for D. (a similar argument works for $(\alpha, s_0]$). $\square$