Lec 8/30

Tuesday, August 30, 2016 9:03 AM

$.999\ldots \stackrel{?}{=} 1$

Proposition: $.999\ldots = 1$

Proof: $.999\ldots \in [\underbrace{.99\ldots9}_{n}, 1]$

Hence $0 \leq 1 - .999\ldots \leq 1 - .\underbrace{99..9}_{n} = \frac{1}{10^n} < \frac{1}{n} \; \forall n$

hence either $1 - .999\ldots = 0$ or $1 - .999\ldots$ is an infinitesimal

but there are no infinitesimals $\in \mathbb{R}$,

So $1 - .99\ldots = 0$. so $1 = .99\ldots$

Can enlarge $\mathbb{R}$ to a field that contains infinitesimals

Add an abstract positive infinitesimal ω to $\mathbb{R}$ & obtain a field $\mathbb{R}(\omega)$

$$\mathbb{R}(\omega) = \left\{ \frac{\sum_{i=0}^{n} a_i \omega^i}{\sum_{j=0}^{m} b_j \omega^j} \right\}$$

$$\frac{\omega - 1}{\omega^2 - 1} = \frac{1}{\omega + 1}$$

$$\frac{P_1(\omega)}{q_1(\omega)} = \frac{P_2(\omega)}{q_2(\omega)} \text{ iff } P_1(\omega) q_2(\omega) = P_2(\omega) q_1(\omega)$$

$$\underbrace{a_0 + a_1\omega + a_2\omega^2 + \cdots + a_n\omega^n}_{\text{has same sign as } a_0}$$

$$\sqrt{\omega} \notin \mathbb{R}(\omega)$$

$\mathbb{N} = \{0, 1, 2, \ldots\}$ are nonnegative integers

$\mathbb{N}^+ = \mathbb{N} \setminus \{0\} = \mathbb{Z}^+$

Principle of mathematical induction

(*) if $S \subseteq \mathbb{N}$ such that $0 \in S$ and $n \in S \Rightarrow n+1 \in S$ then $S = \mathbb{N}$

(Spivak version)

(**) if $S \subseteq \mathbb{N}^+$ such that $1 \in S$ and $n \in S \Rightarrow n+1 \in S$ then $S = \mathbb{N}^+$

(*) $\Rightarrow$ (**) Replace $S \subseteq \mathbb{N}^+$ by $S' = \{0\} \cup S$ then (*) implies $S' = \mathbb{N} \Rightarrow S = \mathbb{N}^+$

(**) $\Rightarrow$ (*) Replace $S \subseteq \mathbb{N}$ by $S' = S \setminus \{0\} \subseteq \mathbb{N}^+$

Proof by Induction: Given a sequence of statements

$P(0), P(1), P(2), \ldots$

or alternatively $P(1), P(2), \ldots$

Suppose we show that $P(0)$ holds and $P(n) \Rightarrow P(n+1)$ then we can conclude that $P(n)$ holds $\forall n \in \mathbb{N}$

This follows from the principle of M.I.:

let $S = \{n \in \mathbb{N} : P(n) \text{ holds}\}$

$P(0)$ true $\Leftrightarrow 0 \in S$

$P(n) \Rightarrow P(n+1) \Leftrightarrow n \in S \Rightarrow n+1 \in S$

Proposition: $2^n > n \quad \forall n \in \mathbb{N}$

Proof: Base Case: $n = 0$. $\underset{\underset{1}{\shortparallel}}{2^0} > 0$ ✓

$n = 1$. $\underset{\underset{2}{\shortparallel}}{2^1} > 1$ ✓

Induction: Suppose that $2^n > n$ (assume $n \geq 1$)

$2^{n+1} = 2^n \cdot 2 > 2n$

Lemma: $2n \geq n+1 \quad \forall n \geq 1$

Proof: $n \geq 1$

$n + n \geq n+1$

$2n \geq n+1$

so $2^{n+1} > n+1$ by lemma.

so $\forall n \in \mathbb{N}, 2^n > n$

Another application of Principle of M.I. is the definition of a sequence by recursion

Start by defining a_0

Define a_{n+1} in terms of $a_0, a_1, \dots, a_n$

then a_n is defined $\forall n$

This is true by following:

Let $S = \{n \in \mathbb{N} : a_n \text{ is defined}\}$

a_0 defined $\Leftrightarrow 0 \in S$

$a_0, a_1, \dots, a_n$ defined $\Rightarrow a_{n+1}$ defined $\Leftrightarrow n \in S \Rightarrow n+1 \in S$

Example: In completeness/noncompleteness discussion we used the following recursive definition

$$a_0 = 1 \qquad b_0 = 2$$

Having defined a_n and b_n with $a_n^2 < 2$ and $b_n^2 \geq 2$, let $c_n = \frac{a_n + b_n}{2}$

If $c_n^2 < 2$, let $a_{n+1} = c_n$ and $b_{n+1} = b_n$

else if $c_n^2 \geq 2$, let $a_{n+1} = a_n$ and $b_{n+1} = c_n$

if $S = \{n : a_n \text{ and } b_n \text{ are defined}\}$

Then $0 \in S$

and $n \in S \Rightarrow n+1 \in S$

so $S = \mathbb{N} \Leftrightarrow a_n, b_n$ defined $\forall n \in \mathbb{N}$

Proposition: $b_n - a_n = \frac{1}{2^n}$

Proof: by induction:

base case: $n=0$ $\quad b_0 - a_0 = 2-1 = 1 = \frac{1}{2^0}$

induction: let $b_n - a_n = \frac{1}{2^n}$

Then $b_{n+1} - a_{n+1}$ is either

$$b_n - c_n = b_n - \tfrac{1}{2}(a_n + b_n) = \tfrac{1}{2}(b_n - a_n) = \tfrac{1}{2}\left(\tfrac{1}{2^n}\right) = \frac{1}{2^{n+1}}$$

or $$c_n - a_n = \tfrac{1}{2}(a_n + b_n) - a_n = \tfrac{1}{2}(b_n - a_n) = \tfrac{1}{2}\left(\tfrac{1}{2^n}\right) = \frac{1}{2^{n+1}}$$

So $b_n - a_n = \frac{1}{2^n}$ $\forall n$

Induction "Paradox" All dogs are chihuahas

This follows from the following inductive statement

Proposition: If in a set $\{D_1, D_2, \dots, D_n\}$ of n dogs if D_1 is a chihuahua then $D_1, D_2, \dots, D_n$ are chihuahuas

Proof: base case: $\{D_1\}$ obviously true

$n=3 \Rightarrow n=4$ $\quad \{D_1, \dots, D_4\}$ D_1 ch...

$\{D_1, \dots, D_3\}$ by $D_1, \dots, D_3$ chi

$\{D_1, D_2, D_4\}$ D_4 is a chi...

so $D_1, \dots, D_4$ are chihuahuas

but $P(1) \not\Rightarrow P(2)$:

$\{D_1, D_2\}$ D_1 chihuahua

but nothing is to say D_2 is a chihuahua.

The only smaller set is $\{D_1\}$ which is trivially all chihuahuas but does not contain D_2.