Lec 10/6

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Triangular Form Theorem

Let $T \in L(V,V)$. if $m(x) = (x-\lambda_1)^{e_1}(x-\lambda_2)^{e_2}\cdots(x-\lambda_r)^{e_r}$

$\exists$ a basis of V s.t. $T \sim A$ where

$$A = \begin{pmatrix} \boxed{A_1} & & 0 \\ & \boxed{A_2} \ddots & \\ 0 & & \boxed{A_r} \end{pmatrix}$$ where A_i is upper triangular.

$\exists v \neq 0$ s.t. $T(v) = \lambda_1 v$. $V_i = n\left((T-\lambda_i I)^{e_i}\right) \iff (T-\lambda_i I)^{e_i} v = 0$

$e_i \leq V_i$. so $\deg m \leq \dim V$.

Working with $F \subset \mathbb{C}$. Examples, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q} + \mathbb{Q}\sqrt{2} = \mathbb{Q}[\sqrt{2}]$.

$(r+t\sqrt{2})(r-t\sqrt{2}) = r^2 - 2t^2 \Rightarrow (r+t\sqrt{2})^{-1} = \dfrac{r-t\sqrt{2}}{r^2-2t^2}$.

$F[X] \xhookrightarrow{\text{injective map.}} \mathcal{P}(F)$ = polynomial functions. Since f has more than n 0s $\Leftrightarrow f = 0$.

Characteristic polynomial of T: $h(x) = \det(xI - T)$, $x \in F$.

$h(x) = \det(xI - A) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$ where $\lambda_i = \alpha_{ii}$ if $A = (\alpha_{ij})$.

a is tf matrix

$= m(x)$ wow!

A ω det is unchanged by basis so $m(x) = \det(xI - T)$ irregardless of basis.

$\text{Trace}(T) = \text{Tr}(T) = \beta_{11} + \beta_{22} + \beta_{33} + \cdots + \beta_{nn}$ where $T \sim B = (\beta_{ij})$

well defined (irrelevant of basis).

since $\text{Trace}(T) = -$ the coefficient of x^{n-1} in $h(x)$.

P.S. $\det(T) = (-1)^n$ constant term of $h(x)$. $(= (-1)^n h(0))$.

$\det(T \cdot S) = \det(T)\det(S)$

$\mathrm{Trace}(T \cdot S) = \mathrm{Trace}(S \cdot T)$

To check: $A, B \in M_n(F)$. $\Rightarrow Tr(AB) = Tr(BA)$

$\Rightarrow \mathrm{Trace}(A^{-1} \cdot B \cdot A) = \mathrm{Trace}(B)$ showing Trace is well defined.

$T \in L(V,V)$, $m(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_r) \Leftrightarrow T$ diagonalizable.

<u>Lemma</u> Let $S, T \in L(V,V)$ s.t. $ST = TS$ both diagonalizable

Then one can find a basis of V for which S and T are both diagonal!

<u>Proof</u> $m_T(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_r)$, $\lambda_i \neq \lambda_j$ when $i \neq j$.

$V = V_1 \oplus \cdots \oplus V_r$ $\quad V_i = \{v \in V \,;\, T(v) = \lambda_i v\}$

Now $S(V_i) \subseteq V_i$ since $v \in V_i \Rightarrow S(v) \in V_i$ since $T(S(v)) - S(T(v)) = S(\lambda_i v) = \lambda_i S(v)$

restrict S: $S_i : V_i \to V_i$ s.t. $\Sigma S_i = S$, $m_{S_i} \mid m_S$ so S_i is diagonalizable.

So $\exists$ a basis of eigenvectors of S_i in V_i.

More generally: Let $T_1, \ldots, T_m \in L(V,V)$ with $T_i T_j = T_j T_i$ and all diagonalizable.

<u>Jordan Decomposition Theorem</u>

Let $T \in L(V,V)$. ① $\exists\, D, N \in L(V,V)$ so that

$T = D + N$, D is diagonalizable, N is nilpotent, $D = f(T)$, $N = g(T)$

for some $f, g \in F[x]$. (so $DN = ND$).

② If $T = D' + N'$ where D' diagonalizable, N' nilpotent w/ $D'N' = N'D'$, $D' = D$, $N' = N$.

Make $D(v_1) = \lambda_1 v_1, \ldots, D(v_n) = \lambda_r v_n$ for a basis of V, $N = T - D$.