Regularity properties of L-S measures

$F : \mathbb{R} \to \mathbb{R}$ increasing & right cts

$$\mu_F(E) = \inf\left\{ \sum \underbrace{(F(b_j) - F(a_j))}_{\mu_F(a_j, b_j]} \;\middle|\; E \subset \bigcup (a_j, b_j] \right\} \qquad \forall E \in \mathcal{M}_F \supseteq \mathcal{B}_{\mathbb{R}}$$

Lemma: $\forall E \in \mathcal{M}_F$, $\mu_F(E) = \inf\left\{ \sum \mu_F(a_j, b_j) \;\middle|\; E \subset \bigcup (a_j, b_j) \right\}$.

Pf denote the inf on the RHS by $\nu(E)$.

Step 1: $\mu_F(E) \leq \nu(E)$.

If $E \subset \bigcup (a_j, b_j)$, Can write each

$$(a_j, b_j) = \coprod_i (a_j^i, b_j^i].$$

Then $E \subset \bigcup_j \left(\coprod_i (a_j^i, b_j^i] \right)$, so

$$\mu_F(E) \leq \sum_j \sum_i \mu_F(a_j^i, b_j^i] = \sum_j \mu_F(a_j, b_j)$$

$$\Rightarrow \mu_F(E) \leq \nu(E).$$

Step 2: $\nu(E) \leq \mu_F(E)$.

Let $\varepsilon > 0$. Then $\exists\, (a_j, b_j]$ s.t. $E \subset \bigcup (a_j, b_j]$

and $\sum \underbrace{\mu_F(a_j, b_j]}_{F(b_j) - F(a_j)} \leq \mu_F(E) + \frac{\varepsilon}{2}$.

By right cont. of F, $\forall j\ \exists\ \delta_j > 0$ s.t.

$$F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^{j-1}}.$$

Then $E \subset \bigcup (a_j, b_j + \delta_j)$ and

$$\sum \mu_F(a_j, b_j + \delta_j) \leq \sum \mu_F(a_j, b_j + \delta_j]$$

$$= \sum (F(b_j + \delta_j) - F(a_j))$$

$$< \sum (F(b_j) - F(a_j) + \frac{\varepsilon}{2^{j+1}})$$

$$= \sum (F(b_j) - F(a_j)) + \frac{\varepsilon}{2}$$

$$\leq \mu_F(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu_F + \varepsilon. \qquad \square$$

(X, τ) Hausdorff top. sp.

↳ $\forall x \neq y \in X\ \exists\ U, V \in \tau$ s.t. $x \in U$, $y \in V$, $U \cap V = \phi$.

$\mathcal{M} \subset P(X)$ σ-alg s.t. $\mathcal{B}_\tau \subset \mathcal{M}$. (i.e. $\tau \subset \mathcal{M}$).

A measure μ on $\mathcal{M}$ is called <u>outer regular</u>

if $\mu(E) = \inf\{\mu(U) \mid E \subset U \in \tau\}$

and <u>inner regular</u> if

$$\mu(E) = \sup\{\mu(K) \mid \text{compact } K \subset E\}.$$

it's <u>regular</u> if both inner & outer regular.

<u>Thm</u> μ_F on $\mathcal{M}_F$ is regular.

<u>Step 1</u> outer regular.

Pf. Let $E \in \mathcal{M}_F$. let $\varepsilon > 0$ By the lemma, ~~$\forall \varepsilon > 0$~~

$\exists\, (a_j, b_j)$ s.t. $E \subset \bigcup (a_j, b_j)$ and

$\sum \mu_F(a_j, b_j) \leq \mu_F(E) + \varepsilon$.

Set $U = \bigcup (a_j, b_j)$. by subadditivity & monotonicity,

$\mu_F(E) \leq \mu_F(U) \leq \sum \mu_F(a_j, b_j) \leq \mu_F(E) + \varepsilon$

Hence $\forall \varepsilon > 0$, $\exists$ open $U \supseteq E$ s.t. $\mu_F(U) \leq \mu_F(E) + \varepsilon$.

$\Rightarrow \mu_F(E) = \inf\{\mu_F(U) \mid E \subset U \text{ open}\}$.

<u>Step 2</u>:

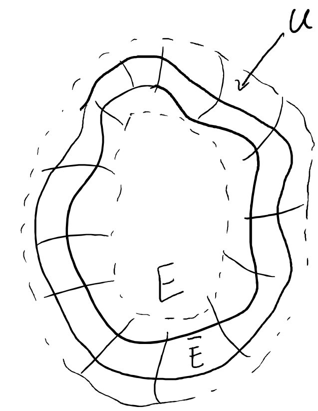
<u>step 2a</u>: assume E is bdd, so $\overline{E}$ is cpt,

and $\mu_F(\overline{E}) < \infty$.

Let $\varepsilon>0$. by step 1, $\exists$ open U

containing $\bar{E}\setminus E$ s.t.

$\mu_F(U) \leq \mu_F(\bar{E}\setminus E) + \varepsilon$.



Then $K := \bar{E}\setminus U$ is cpt &

contained in E.

Then $\mu_F(K) = \mu_F(E) - \mu_F(E\cap K^c)$

$$= \mu_F(E) - \mu_F(E\cap U)$$

$$= \mu_F(E) - [\mu_F(U) - \mu_F(U\setminus E)]$$

$$\geq \mu_F(E) - \mu_F(U) + \mu_F(\bar{E}\setminus E)$$

$\hookrightarrow U \supset \bar{E}\setminus E$

$$\geq \mu_F(E) - \varepsilon$$

So $\mu_F(E) = \sup\{\mu_F(K) \mid E\supset K \text{ compact}\}$.

Step 2b E arbitrary in $\mathcal{M}_F$.

$\mathbb{R} = \coprod (j, j+1]$, so $E = \coprod E_j$ where $E_j := E\cap(j,j+1]$.

Let $\varepsilon>0$. by step 2a, $\forall j$ $\exists$ cpt $K_j \subset E_j$

s.t. $\mu_F(K_j) \geq \mu_F(E_j) - \frac{\varepsilon}{2^{|j|+1}}$.

For $n\in\mathbb{N}$, let $F_n = \coprod_{-n}^{n} K_j$, cpt.

$\forall n$, $\mu_F(F_n) \geq \mu_F(\coprod_{-n}^{n} E_j) - \frac{\varepsilon}{2}$

Case 1 if $\mu_F(E) = \infty$, since $\mu_F(\coprod_{-n}^{n} E_j) \nearrow \infty$, eventually $\mu_F(F_n) > M$ for any fixed M.

Then $\sup\{\mu(K) \mid E \supset K \text{ cpt}\} = \infty = \mu_F(E)$.

<u>Case 2</u>: $\mu_F(E) < \infty$. Then $\exists\, N$ s.t.

$$\mu_F(E) \le \mu_F\Big(\bigsqcup_{-N}^{N} E_j\Big) + \frac{\varepsilon}{2}$$

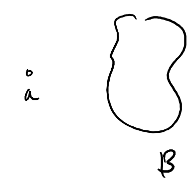
$$\le \mu_F(F_N) + \frac{\varepsilon}{2}$$

So $\forall\, \varepsilon > 0$, $\exists$ cpt $K \subset E$ s.t. $\mu_F(K) \le \mu_F(E) \le \mu_F(K) + \varepsilon$

So $\mu_F(E) = \inf\{\mu_F(K) \mid E \supset K \text{ cpt}\}$. $\square$

<u>Hausdorff Measure</u> (Ch 11).

Let (X, ρ) be a metric space



For $B \subset X$, $B \neq \emptyset$, and $a \in X$, define

$$\rho(a, B) = \inf\{\rho(a,b) \mid b \in B\}$$

If $A, B \subset X \setminus \{\emptyset\}$, $\rho(A,B) = \inf\{\rho(a,b) \mid a \in A,\ b \in B\}$.

<u>Definition</u>: An outer measure μ^* on $\mathcal{P}(X)$ is called a (Caratheodory) <u>metric</u> outer measure if

$$\underbrace{\rho(A, B) > 0}_{\Rightarrow\, A \cap B = \emptyset} \Rightarrow \mu^*(A \sqcup B) = \mu^*(A) + \mu^*(B).$$

Prop. If μ^* is a metric outer measure on $P(X)$,

then $\mathcal{B}_\rho \subset \mathcal{M}^*$

Def: If (X, ρ) metric space, $p \geq 0$ and $\varepsilon > 0$,

$\operatorname{diam}(S) = \sup_{x,y \in S} \rho(x,y)$.

convention: $\inf \emptyset = \infty$.

define $$\eta^*_{p,\varepsilon}(E) := \inf \left\{ \sum_1^\infty [\operatorname{diam} B_n]^p \;\middle|\; \begin{array}{l} (B_n) \text{ open balls,} \\ \operatorname{diam}(B_n) \leq \varepsilon \;\; \forall n, \\ \text{and } E \subset \bigcup B_n \end{array} \right\}.$$

Show $\underbrace{\lim_{\varepsilon \to 0} \eta^*_{p,\varepsilon}}_{\eta^*_p}$ is a metric outer measure.

restricts to $\mathcal{B}_\rho \subset \mathcal{M}^*_p$ for η^*_p

$\hookrightarrow$ Hausdorff measure η_p on $\mathcal{B}_\rho$ $\forall p > 0$.