Reminder (Exercise 7):

Let I be an interval in $\mathbb{R}$ having non-zero length, let $t_0 \in I$, and

Let $U, A : I \longrightarrow \mathbb{R}^{n \times n}$ with U diffable & $U' = UA$. Then

(1) $U(t)$ is orthogonal $\forall t \in I$

$\Updownarrow$

(2) $U(t_0)$ is orthogonal & $A(t)$ is skew-symmetric $\forall t \in I$.

Recall the dimension of $\{A \in \mathbb{R}^{n \times n} : A \text{ is skew-symmetric}\} = \text{skew}(n, \mathbb{R})$

is $1 + 2 + \cdots + (n-1) = \dfrac{n(n-1)}{2}$.

If $t \longmapsto U(t)$ is a diffable curve in $O(n, \mathbb{R})$ with

$U(0) = I$ then $U'(0) \in \text{Skew}(n, \mathbb{R})$.

Conversely, if $A \in \text{Skew}(n, \mathbb{R})$, $\exists\, U : \mathbb{R} \longrightarrow O(n, \mathbb{R})$ s.t.

$U(0) = I$ and $U'(0) = A$. (e^{At} will do)

thus $\text{Skew}(n, \mathbb{R})$ is tangent to $O(n, \mathbb{R})$ at I.

(Note A exists in $U' = UA$ since U' exists so $A = U^{-1}U'$)

The Fundamental Theorem of Curves

Let I be an interval in $\mathbb{R}$ with nonzero length, let $\kappa : I \longrightarrow (0, \infty)$ be continuously differentiable, and let $\tau : I \longrightarrow \mathbb{R}$ be continuously differentiable. Then $\exists$ a C^3 unit speed curve $\alpha : I \longrightarrow \mathbb{R}^3$, which is unique up to its position in space, whose curvature is K and whose torsion is τ.

<u>Pf</u> fix $s_0 \in I$. to simplify notation, suppose $s_0 = 0$.

If β is a C^3 unit-speed curve in $\mathbb{R}^3$ with $K_\beta = K$ and $\tau_\beta = \tau$, and let $x_0 = \beta(0)$ if M is the proper orthogonal matrix whose columns are $T_\beta(0)$, $N_\beta(0)$, $B_\beta(0)$, and if $\alpha : I \longrightarrow \mathbb{R}^3$ is defined by $\alpha(s) = M^{-1}(\beta(s) - x_0)$,

$$\left.\begin{array}{l} \text{then } \alpha \text{ is a } C^3 \text{ unit-speed curve in } \mathbb{R}^3 \text{ with } \alpha(0) = 0,\ K_\alpha = K,\ \tau_\alpha = \tau. \\ T_\alpha(0) = M^{-1}(\beta'(s))\big|_{s=0} = M^{-1}T_\beta(0) = e_1 \\ N_\alpha(0) = e_2, \quad \text{and} \quad B_\alpha(0) = e_3. \end{array}\right\} \quad (*)$$

and also conversely.

It suffices to show that $\exists$ a unique C^3 unit-speed curve $\alpha : I \longrightarrow \mathbb{R}^3$ satisfying $(*)$.

Consider the initial value problem

$$(\dagger) \quad \left\{\begin{array}{l} a' = u_1, \quad u_1' = K u_2, \quad u_2' = -K u_1 + \tau u_3, \quad u_3' = -\tau u_2, \\ a(0) = 0, \quad u_1(0) = e_1, \quad u_2(0) = e_2, \quad u_3(0) = e_3. \end{array}\right.$$

By Picard's theorem on differential equations, the IVP $(\dagger)$ has a unique solution (a, u_1, u_2, u_3).

Now if α is a C^3 unit-speed curve in $\mathbb{R}^3$ satisfying $(*)$ then $(\alpha, T_\alpha, N_\alpha, B_\alpha)$ satisfies $(\dagger)$ so it is <u>the</u> solution of $(\dagger)$. This proves uniqueness for $(*)$.

To prove existence, define α to be a, and let us show that α is a C^3 unit-speed curve in $\mathbb{R}^3$ with

$T_\alpha = u_1, \quad N_\alpha = u_2, \quad B_\alpha = u_3, \quad \kappa_\alpha = \kappa, \quad \tau_\alpha = \tau.$

Define $U : I \to \mathbb{R}^{3\times 3}$ by $U(s) = (u_1(s), u_2(s), u_3(s))$.

$$A : I \to \mathbb{R}^{3\times 3} \text{ by } A(s) = \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}.$$

Note that $A(s)$ is skew symmetric $\forall s$, and

$U' = UA$ by $(\dagger)$, and $U(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal.

Thus $U(s)$ is orthogonal $\forall s \in I$.

$\det(U(s)) = \pm 1$ and $\det(U(0)) = 1$ so $\det(U(s)) = 1$ by IVT.

Thus $\forall s \in I$, $u_1(s), u_2(s), u_3(s)$ is a positively oriented ONB for $\mathbb{R}^3$, so $u_1(s) \times u_2(s) = u_3(s)$. Now $|\alpha'| = |a'| = |u_1| = 1$ so α is unit-speed. $T_\alpha = \alpha' = u_1$.

Next, $\alpha'' = T_\alpha' = u_1' = \kappa u_2$, so $\kappa_\alpha = |T_\alpha'| = \kappa |u_2| = \kappa$.

And $N_\alpha = \dfrac{T_\alpha'}{\kappa} = u_2$. And $B_\alpha = T_\alpha \times N_\alpha = u_1 \times u_2 = u_3$.

Also $B_\alpha' = u_3' = -\tau u_2 = -\tau N_\alpha$ so $\tau_\alpha = \tau$.

$\alpha''' = (\kappa u_2)' = \kappa' u_2 + \kappa u_2'$ which exists.

So α is C^3, and thus existence is proved. $\square$

Review: Picard's Theorem:

Let I be an interval in $\mathbb{R}$ with non-zero length. Let $t_0 \in I$. Let $f: I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous and satisfy $|f(t,x) - f(t,y)| \leq K(t)\,|x-y|$ for all $t \in I$ where $K: I \longrightarrow [0,\infty)$ such that $\forall\, a,b \in I$ with $a<b$,

$$\int_a^b K(t)\,dt < \infty .$$

(for simplicity, K continuous would suffice).

Let $y_0 \in \mathbb{R}^n$. Then the IVP $\overbrace{y'(t) = f(t, y(t)), \quad y(0) = y_0}^{(*)}$ has a unique solution.

Proof Summary: Let $y: I \to \mathbb{R}^n$ be cts. Then y is differentiable and satisfies $(*)$ iff $\forall t \in I$, $\underbrace{y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds}_{(**)}$.

Let $\overset{\nearrow \text{any old function}}{y_1}: I \longrightarrow \mathbb{R}^n$ which is continuous. If $y_1, \dots, y_k$ have already been defined, let $y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\,ds$ $\quad (y_{k+1}: I \longrightarrow \mathbb{R}^n)$.

Then (y_k) converges uniformly on each compact subinterval of I. Thus its limit y is continuous & satisfies $(**)$, so y satisfies $(*)$.