Rec 10/24

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Filters: X any set. $\mathcal{P}(X)$: power set.

Filter: $\mathcal{F} \subseteq \mathcal{P}(X)$: family of subsets s.t.

① $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$

② if $A \in \mathcal{F}$ and $B \supset A$, then $B \in \mathcal{F}$

③ $\underbrace{\text{if } A, B \in \mathcal{F}, \text{ then } A \cap B \in \mathcal{F}}_{\text{This together w/ ① means } A \cap B \neq \emptyset}$.

Examples: $X = \mathbb{N}$, $\mathcal{F} = \{A \mid 1 \in A\}$

$(\mathcal{F}_x = \{A \mid x \in A\})$ ← principal filter

Cofinite filter: $A \in \mathcal{F}$ if $X \setminus A$ is finite. (X must be infinite)

Ultrafilter $\mathcal{F}$ has the property that $\forall A \subset X$,

either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Example any principal filter.

Non-example cofinite filter.

But,

Thm Every filter is contained in an ultrafilter.

Use for filters:

Consider the sequence $x_1, x_2, \ldots$

$x : \mathbb{N} \longrightarrow \mathbb{R}$.

$\lim_{x \to \infty} x_n = x_0$ means "for every open $\mathcal{U} \ni x_0$, all but finitely many x_n are in $\mathcal{U}$"

$\Leftrightarrow$ "$x^{-1}(\mathcal{U})$ is in the co-finite filter on $\mathbb{N}$" ($\forall$ open $\mathcal{U} \ni x_0$)

Define $\mathcal{U}\mathbb{N}$ = set of ultrafilters on $\mathbb{N}$

eg $\mathcal{U}\mathbb{N}$ contains $\mathcal{F}_j$ (principal ultrafilter) $\forall j$.

#53 Define $[S] = \{ \mathcal{F} \in \mathcal{U}\mathbb{N} \mid S \in \mathcal{F} \}$

Ex $S = \{1\}$. Then $\mathcal{F}_1 \in [S]$, $\mathcal{F}_j \notin [S]$ $\forall j \neq 1$.

$A \subseteq \mathcal{P}(X)$, A nonempty, A has FIP.

$\rightsquigarrow$ A generates a filter.

eg $A = \{\{1,2\}, \{1,3\}\}$.

Show $[\emptyset] = \emptyset$