Lec 1/12

Thursday, January 12, 2017 09:01

(X, d) metric space, $a \in S \subseteq X$

[$\underbrace{a \text{ is an interior pt}}$ if $\exists\, r_a > 0$ s.t. $B(r_a, a) \subseteq S$
S open if $\forall a \in S$, a]

[$\mathcal{T}$ is the set of all open sets in (X, d)
↳ topology associated to (X, d).] → [Metrics are equivalent if they determine the same topologies]

Notation: d_1, d_2 – metrics on X. $\mathcal{T}_i$ – topology associated w/ (X, d_i) $i = 1, 2$

$B_i(r, a) = \{x \in X \mid d_i(x, a) < r\}$ $i = 1, 2$ $id: X \to X$, $id(x) = x$

Theorem: The following are equivalent:

(1) $\mathcal{T}_2 \subseteq \mathcal{T}_1$

(2) $id: (X, d_1) \to (X, d_2)$ is continuous on X

(3) $\forall a \in X, \varepsilon > 0\ \exists\, \delta_{a,\varepsilon} > 0$ s.t. $B_1(\delta_{a,\varepsilon}, a) \subseteq B_2(\varepsilon, a)$

Proof: (2) $\Leftrightarrow$ (3):

$id: (X, d_1) \to (X, d_2)$ is cts at $a \in X$ means ($\Leftrightarrow$) $\forall \varepsilon > 0\ \exists\, \delta_{a,\varepsilon}$ s.t. $d_1(x, a) < \delta_{a,\varepsilon} \Rightarrow d_2(\overset{id(x)}{x}, \overset{id(a)}{a}) < \varepsilon$

which is equiv to saying $x \in B_1(\delta_{a,\varepsilon}, a) \Rightarrow x \in B_2(\varepsilon, a)$, so (2) $\Leftrightarrow$ (3).

(3) $\Rightarrow$ (1):

Suppose $U \in \mathcal{T}_2$. Then all points in U are interior points of U in (X, d_2)

Then $a \in U \Rightarrow B_2(\varepsilon_a, a) \subseteq U$ for some $\varepsilon_a > 0$.

By (3), we can find a δ_{a,ε_a} s.t. $B_1(\delta_{a,\varepsilon_a}, a) \subseteq B_2(\varepsilon_a, a) \subseteq U$,

now $U \subseteq \bigcup_{a \in U} B_1(\delta_{a,\varepsilon_a}, a) \subseteq \bigcup_{a \in U} B_2(\varepsilon_a, a) = U$

So U is a union of open balls in (X, d_1) so U is open in (X, d_1)

so $U \in \mathcal{T}_1$. This means $\mathcal{T}_2 \subseteq \mathcal{T}_1$

(1) $\Rightarrow$ (3):

$B_2(\varepsilon, a) \in \mathcal{T}_2 \Rightarrow B_2(\varepsilon, a) \in \mathcal{T}_1 \Rightarrow a$ is an interior point of $B_2(\varepsilon, a)$ in (X, d_1)

so for some $\delta_{a,\varepsilon}$, $B_1(\delta_{a,\varepsilon}, a) \subseteq B_2(\varepsilon, a)$. ∎

Corollary: The following are equivalent

(1) $\mathcal{T}_1 = \mathcal{T}_2 \Longleftrightarrow d_1, d_2$ are equivalent metrics

(2) $id: (X, d_1) \to (X, d_2)$ and $id: (X, d_2) \to (X, d_1)$ are continuous.

(3) $\forall a \in X, \varepsilon > 0, \exists \delta_{a,\varepsilon}, \delta'_{a,\varepsilon} > 0$ s.t. $B_1(\delta_{a,\varepsilon}, a) \subseteq B_2(\varepsilon, a)$
$B_2(\delta'_{a,\varepsilon}, a) \subseteq B_1(\varepsilon, a)$.

Definition we say that two metrics d_1, d_2 are strongly equivalent iff $\exists A, B \in \mathbb{R}^+$ s.t. $\forall x, y \in X$:

$$A d_1(x,y) \leq d_2(x,y) \leq B d_1(x,y)$$
$$\frac{1}{B} d_2(x,y) \leq d_1(x,y) \leq \frac{1}{A} d_2(x,y)$$

Theorem: Strongly equivalent metrics are equivalent.

Proof: $x \in B_2(A\varepsilon, a) \subseteq B_1(\varepsilon, a)$:

$$A d_1(x,a) \leq d_2(x,a) < A\varepsilon$$
$$\Rightarrow \quad d_1(x,a) < \varepsilon$$
$$\Rightarrow \quad x \in B_1(\varepsilon, a) \qquad (\text{i.e. take } \delta_{a,\varepsilon} = A\varepsilon)$$

$x \in B_1(\frac{\varepsilon}{B}, a) \subseteq B_2(\varepsilon, a)$:

$$d_1(x,a) < \frac{\varepsilon}{B}$$
$$\Rightarrow \quad d_2(x,a) \leq B d_1(x,a) < \varepsilon$$
$$\Rightarrow \quad x \in B_2(\varepsilon, a) \qquad (\text{i.e. take } \delta'_{a,\varepsilon} = \frac{1}{B}\varepsilon)$$

Note: if $X = \mathbb{R}^n$, $d_1(\vec{x}, \vec{y}) = \nu_1(\vec{x} - \vec{y})$

then d_1, d_2 strongly equiv $\Rightarrow A\nu_1(\vec{x}) \leq \nu_2(\vec{x}) \leq B\nu_1(\vec{x})$.

Exercise: if d_1, d_2 strongly equiv, d_2, d_3 s.e., then d_1, d_3 s.e.

Theorem: $|\,|_1$ = taxicab norm, $|\,|_2$ = euclidean norm, $|\,|_\infty$ = box norm are all strongly equivalent.

Proof: (I did this on the HW).

$$|\vec{x}|_\infty \leq |\vec{x}|_1 \leq n|\vec{x}|_\infty$$

$$\frac{1}{\sqrt{n}}|\vec{x}|_2 \le |\vec{x}|_\infty \le |\vec{x}|_2 = \sqrt{\sum_{i=1}^{n} |x_i|}$$

Definition: let $f: X \to Y$. Suppose $S \subseteq Y$. Then we define

$$f^{-1}(S) = \{x \in X \mid f(x) \in S\} \quad \text{(inverse image of } S \text{ under } f).$$

lemma: The inverse image has the following properties:

- $f^{-1}(Y) = X$, $f^{-1}(\emptyset) = \emptyset$
- $A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$
- $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$
- If $\{A_\alpha\}$ is an arbitrary collection of subsets of Y, then

$$f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha), \quad f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha)$$

Theorem: let $f: (X, d_1) \to (Y, d_2)$, $a \in X$.

(1) f is continuous at $a \iff$ for any neighborhood N of $f(a)$, $f^{-1}(N)$ is a neighborhood of a.

(2) f is continuous on $X \iff \left(V \subseteq Y \text{ open in } (Y, d_2) \Rightarrow f^{-1}(V) \text{ open in } (X, d_1)\right)$

(3) f is continuous on $X \iff \left(F \subseteq Y \text{ closed in } (Y, d_2) \Rightarrow f^{-1}(F) \text{ closed in } (X, d_1)\right)$

Proof: (1): $\Rightarrow$: Suppose f is continuous at a, N is a neighborhood of $f(a)$ in (Y, d_2).

Then $B_2(\varepsilon, f(a)) \subseteq N$ for some $\varepsilon > 0$.

So $\exists \delta_{f(a), \varepsilon} > 0$ s.t. $f(B_1(\delta_{f(a),\varepsilon}, a)) \subseteq B_2(\varepsilon, f(a)) \subseteq N$

$$\iff B_1(\delta, a) \subseteq f^{-1}(N).$$

$\Leftarrow$: suppose N is a neighborhood of $f(a)$ and $f^{-1}(N)$ is a neighborhood of a.

take $N = B_2(\varepsilon, f(a))$. Then for some $\delta > 0$, $B_1(\delta, a) \subseteq f^{-1}(N) = f^{-1}(B_2(\varepsilon, f(a)))$

$\iff d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.