Connected Sums, Genus, Factorization   Recall: # is a well-defined op on equivalence classes of knots.

Corollary of well-definedness lemma: $[K_1]\#[K_2] = [K_2]\#[K_1]$. ← Knot monoid is commutative

(Note that it is a monoid! it's associative and O is the identity).

Note: $\overline{K_1}\#\overline{K_2} = \overline{K_1\#K_2}$. However $\overline{K_1}\#K_2 \neq K_1\#K_2$ in general (granny vs. square).

$K_1^{-1}\#K_2^{-1} = (K_1\#K_2)^{-1}$. But $K_1^{-1}\#K_2 \neq K_1\#K_2$ in general (must use (large) non-invertible knots)
↳ $K_1 = P(3,5,7)$, $K_2 = P(3,5,9)$ is smallest example.

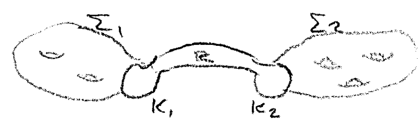
A knot is prime if $K = A\#B \Rightarrow A$ or B is unknot.

Genus: $g(K) = \min\{g \mid K$ has a seifert surface of genus $g\}$. $g(K)=0 \Rightarrow K$ is unknot.

Thm: $g(K_1\#K_2) = g(K_1)+g(K_2)$ (g : Knot Monoid $\to \mathbb{N}\cup\{0\}$ is a homomorphism)

Pf Suppose Σ_i minimal seifert surface for K_i. Suppose $K_1\#K_2$ is connected by ribbon R.

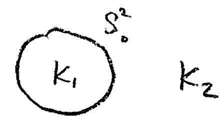
Then $\Sigma_1 \sqcup R \sqcup \Sigma_2$ is a seifert surface for $K_1\#K_2$.



So $g(K_1\#K_2) \leq g(K_1)+g(K_2)$.

Conversely, given minimal genus Seifert surface Σ of $K_1\#K_2$,

and $K_1\#K_2 \cap S^2_\circ = 2$ points. So $\Sigma \pitchfork S^2_\circ$ is a 1-mfd in $S^2_\circ$

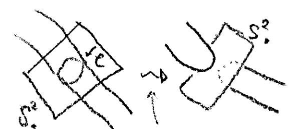


boundary ∂Q of $Q = \Sigma\cap S^2_\circ$ is 2 points (it is $K_1\#K_2 \cap S^2_\circ$)

So $Q = J \cup S^1 \cup \cdots \cup S^1$ where J is an interval. $S^2_\circ - J = \mathbb{R}^2$.

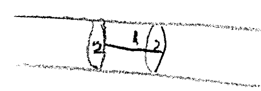
Pick innermost conn. comp $\mathcal{C}$ in Q in $S^2_\circ$. $\mathcal{C}$ bounds a disc in $S^2_\circ$

Do surgery to eliminate $\mathcal{C}$ from Q. We get a new surface Σ'.



surgery along disc.

Change in χ = Euler char. under surgery: $\chi(\Sigma') = \chi(\Sigma)+2$



Cases Σ' is connected $= \Sigma_{h,1}$. $\chi(\Sigma') = 1-2h$, $\chi(\Sigma) = 1-2g \Rightarrow h = g-1$ contradiction.

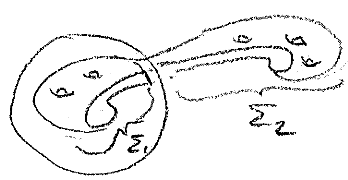
so Σ' is disconnected: $\Sigma' = \Sigma_{p,1} \sqcup \Sigma_{q,0}$. $\chi(\Sigma') = (1-2p)+(2-2q) = 1-2g+2 \Rightarrow g = p+q$.

by minimality, $p \geq g$, so $p = g$, $q = 0$. i.e. $\Sigma' \approx \tilde{\Sigma} \cup S^2$, and $\tilde{\Sigma}$ is a Seifert surface for $K_1\#K_2$ of minimal genus still.

After removing all circles, $Q = J$.

Split $\tilde{\tilde{\Sigma}}$ into two along $S^2_\circ$ and add in J to both to get

Σ_1 & Σ_2 w/ $g(\Sigma_1)+g(\Sigma_2) = g(\tilde{\tilde{\Sigma}})$, and Σ_i seifert surface for K_i □



Cor $g(K) = 1 \Rightarrow K$ is prime. $K\#R = R \Rightarrow K$ is unknot.

Also, Wild knots don't fit into this theory: $T = \boxed{K}-\boxed{K}-\boxed{K}\cdots \Rightarrow T = K\#T$.

Prime decomposition of K: $K = K_1\#K_2\#\cdots\#K_m$.

Theorem Every knot has a prime decomposition, and this decomposition is unique.

proof Existence by induction on genus.

K prime ✓ $g(K)=1$

$K = A\#B$, $A \neq O \neq B \Rightarrow g(A), g(B) < g(K)$.   (uniqueness next time)