Lec 8/26

Friday, August 26, 2016 9:07 AM

Tentative room assignment for Sunday 3-5: CH 218 → Cockins Hall

Definition if $a \in \mathbb{F}$, an ordered field

then $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$

triangle inequality $\forall a, b \in \mathbb{F}, \quad |a+b| \leq |a| + |b|$

brute force by cases $\left(\begin{matrix} a \geq 0 \\ b \geq 0 \end{matrix} \middle| \begin{matrix} a \geq 0 \\ b < 0 \end{matrix} \middle| \begin{matrix} a < 0 \\ b \geq 0 \end{matrix} \middle| \begin{matrix} a < 0 \\ b < 0 \end{matrix} \right)$

more elegant proof:

Lemma 1: $\forall a \in \mathbb{F}, \quad a \leq |a|$

Proof: if $a \geq 0$, $|a| = a$ so $a \leq |a|$

if $a < 0$, $a < 0 < |a|$ so $a \leq |a|$

Proposition: if $0 \leq a$, $0 \leq b$, and $a^2 = b^2$ then $a = b$

Proof: $0 = (a-b)(a+b)$

if $a > 0$ or $b > 0$ then

$0 = 0\,(a+b)$ so $0 = (a-b)$ so $a = b$

if $a = b = 0$ then $a = b$

what

Lemma 2: $|a|^2 = a^2$

Proof: if $a \geq 0$ then $|a| = a \Rightarrow a^2 = |a|^2$

if $a < 0$ then $|a|^2 = (-a)(-a) = a^2$

Lemma 3: $|ab| = |a||b|$

proof:

$ab^2 = a^2b^2$ P8, P5

$|ab|^2 = |a|^2|b|^2$ Lemma 2

$|ab|^2 = (|a||b|)^2$ P8, P5

$|ab| = |a||b|$ by proposition

Prove triangle inequality: (backwards)

$$|a+b|^2 \leq (|a|+|b|)^2$$

$$(a+b)^2 \leq |a|^2 + 2|a||b| + |b|^2 \qquad \text{by Lemma 2}$$

$$\cancel{a^2}+2ab+\cancel{b^2} \leq \cancel{a^2} + |2ab| + \cancel{b^2}$$

$$2ab \leq |2ab|$$

Actual proof:

$2ab \leq \lvert 2ab \rvert$	Lemma 1
$2ab \leq 2\lvert a\rvert\lvert b\rvert$	Lemma 3
$a^2+2ab+b^2 \leq a^2 + 2\lvert a\rvert\lvert b\rvert + b^2$	Addition defined
$a^2+2ab+b^2 \leq \lvert a\rvert^2 + 2\lvert a\rvert\lvert b\rvert + \lvert b\rvert^2$	Lemma 2
$(a+b)^2 \leq (\lvert a\rvert + \lvert b\rvert)^2$	P9, P4, P8, P1
$\lvert a+b\rvert^2 \leq (\lvert a\rvert+\lvert b\rvert)^2$	Lemma 2
$\lvert a+b\rvert \leq \lvert a\rvert+\lvert b\rvert$	Proposition above.

So far:

P1 - P9 field axioms

P10 - P12 ordering axioms

$\mathbb{Q}$ satisfies P1 - P12

$\mathbb{R}$ satisfies P1 - P12

So P1-P12 insufficient to distinguish $\mathbb{Q}$ and $\mathbb{R}$

$$\mathbb{Q} \neq \mathbb{R}$$

Theorem: the equation $x^2 = 2$ has no solutions in $\mathbb{Q}$

Proof: by contradiction.

Suppose $\pm x = \frac{p}{q}$ are rational solutions

Assume $x > 0$ so $p > 0$, $q > 0$

Assume positive integers have unique prime factorizations

Reduce $\frac{p}{q}$ by cancelling common prime factors in p and q

So can assume p, q have no common factors.

$\frac{2}{1} = 2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} \Rightarrow p^2, q^2$ have no common prime factors

hence q^2 has no prime factors $\Rightarrow$ q has no prime factors

because 2 is an integer

$\Rightarrow$ q is 1

hence q^2 has no prime factors $\Rightarrow$ q has no prime factors
$\Rightarrow q$ is 1 — because 2 is an integer

hence $p^2 = 2$ $\quad p$ positive integer

Case 1: $p = 1 \Rightarrow 1^2 = 2$ contradiction
Case 2: $p \geq 2 \Rightarrow p^2 \geq 4 > 2$ contradiction

So assuming that $x^2 = 2$ has a rational solution leads to contradiction

Need another axiom so $x^2 = 2$ has solution in $\mathbb{R}$:

Dedekind cuts

Defn: for $\mathbb{F}$ an ordered field a dedekind cut of $\mathbb{F}$ is a pair of subsets (A, B)

$A \subseteq \mathbb{F}$, $B \subseteq \mathbb{F}$ satisfying

(1) $\mathbb{F} = A \cup B$
(2) $A \neq \emptyset$, $B \neq \emptyset$
(3) $a \in A$, $b \in B \Rightarrow a < b$

interval notation

Let $\mathbb{F}$ be an ordered field (P1-P12)

Definitions:

$[a, b]_{\mathbb{F}} = \{x \in \mathbb{F} \mid a \leq x \leq b\}$
$[a, b)_{\mathbb{F}} = \{x \in \mathbb{F} \mid a \leq x < b\}$
$(a, \infty)_{\mathbb{F}} = \{x \in \mathbb{F} \mid a < x\}$
$\vdots$
etc.

Note: $\pm\infty$ is just an abstract symbol, not an element

otherwise $\infty + 1 = \infty$

P13: the completion axiom:

The only dedekind cuts of $\mathbb{R}$ are of the form

$((-\infty, c), [c, \infty))$ and $((-\infty, c], (c, \infty))$

for some $c \in \mathbb{R}$ (c called the cut point of A and B)

(so one of A or B must be closed at c)