Flat modules:

it may be that

$$N \subseteq M \text{ but } N \otimes K \subsetneq M \otimes K$$

$$\mathbb{Z} \subseteq \mathbb{Q} \text{ but } \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \mathbb{Z}_2 \text{ and } \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_2 = 0.$$

$$0 \longrightarrow N \xrightarrow{\varphi} M \qquad \text{exact}$$

$$\Downarrow$$

$$0 \longrightarrow N \otimes K \xrightarrow{\varphi \otimes 1_K} M \otimes K \qquad \text{may not be exact.}$$

$$u \otimes v \longmapsto \varphi(u) \otimes v$$

The module K is called <u>flat</u> if

$$0 \longrightarrow N \otimes K \longrightarrow M \otimes K \quad \text{is exact whenever}$$

$$0 \longrightarrow N \longrightarrow M \quad \text{is exact.}$$

①: R is a flat module since

$$\begin{array}{ccccc} 0 \longrightarrow & N \otimes R & \longrightarrow & M \otimes R \\ & \wr\| & & \wr\| \\ 0 \longrightarrow & N & \longrightarrow & M \end{array},$$

② : If K_1, K_2 are flat then $K_1 \oplus K_2$ is flat.

Proof: Let $0 \longrightarrow N \longrightarrow M$ be exact. Then

$0 \longrightarrow N \otimes K_1 \longrightarrow M \otimes K_1$ & $0 \longrightarrow N \otimes K_2 \longrightarrow M \otimes K_2$

are exact. then

$$\begin{array}{ccccc} 0 & \longrightarrow & N \otimes (K_1 \oplus K_2) & \longrightarrow & M \otimes (K_1 \otimes K_2) \\ & & \parallel & & \parallel \\ & & (N \otimes K_1) \oplus (N \otimes K_2) & & (M \otimes K_1) \oplus (M \otimes K_2) \end{array}$$ is exact.

③ conversely, if $K_1 \oplus K_2$ is flat then K_1, K_2 are flat.

④ $\forall n$, R^n is flat. Also, any direct summand of R is flat.

⑤ if K_1, K_2 are flat then $K_1 \otimes K_2$ is flat.

$$B \xrightarrow{\psi} C \longrightarrow 0 \quad \text{exact}$$

$$\Downarrow$$

$$B \otimes K \xrightarrow{\psi \otimes 1_K} C \otimes K \longrightarrow 0 \quad \text{exact ?}$$

Yes. proof: $\forall\, u \otimes v \in C \otimes K$, $u = \psi(w)$ for some w,

so $\psi \otimes 1_K (w \otimes v) = u \otimes v$.

So $u \otimes v \in$ img of $\psi \otimes 1_K$ so $\psi \otimes 1_K$ is surj.

<u>Theorem</u>: Let $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ be exact.

Then $A \otimes K \xrightarrow{\varphi \otimes 1_K} B \otimes K \xrightarrow{\psi \otimes 1_K} C \longrightarrow 0$ is exact $\forall K$.

<u>Proof</u> $\forall u \in A$, $v \in K$, $\psi \otimes 1_K(\varphi \otimes 1_K(u \otimes v)) = \psi(\varphi(u)) \otimes v = 0$

so $\operatorname{Im}(\varphi \otimes 1_K) \subseteq \operatorname{Ker}(\psi \otimes 1_K)$.

So we have a surj. hom'sm

$$(B \otimes K) / \varphi \otimes 1_K (A \otimes K) \longrightarrow C \otimes K$$

Construct an inverse: Let $u \otimes v \in C \otimes K$.

Find $w \in B$ s.t. $\psi(w) = u$.

define $\xi(u \otimes v) = w \otimes v \mod \operatorname{Im}(\varphi \otimes 1_K) \in (B \otimes K) / \operatorname{Im}(\varphi \otimes 1_K) = \varphi(A) \otimes K$

ξ is well-defined: if $\psi(w') = \psi(w) = u$ then

$w' = w \mod \operatorname{Im}(\varphi)$. Then $w' \otimes v = w \otimes v \mod \varphi(A) \otimes K$.

Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two categories.

A functor $F : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ is "a mapping" s.t.

$\forall$ object A in $\mathcal{K}_1$ you have an object $F(A)$ in $\mathcal{K}_2$.

$\forall$ morphism $A \xrightarrow{\varphi} B$ in $\mathcal{K}_1$ you have a morphism $F(A) \xrightarrow{F(\varphi)} F(B)$ in $\mathcal{K}_2$.

s.t. $\forall$ φ, ψ, $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$

$\mathrm{Tor}_0(\cdot, k) : R\text{-Mod} \longrightarrow R\text{-Mod}$ functor.

$$A \longmapsto A \otimes K$$

$$\varphi \longmapsto \varphi \otimes 1_k$$

$$\begin{matrix} \because & & \because \\ A \to B & & A \otimes K \to B \otimes k. \end{matrix}$$

<u>Def</u>: A functor F is called <u>right exact</u> if

$\forall$ exact $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$

the sequence $F(A) \to F(B) \to F(C) \longrightarrow 0$ is exact.

($\mathrm{Tor}_0(\cdot, k)$ is right exact.)

F is left exact if

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \quad \text{is exact.}$$

F is exact if

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \quad \text{is exact.}$$

K is flat $\Longleftrightarrow$ $\mathrm{Tor}_0(\cdot, K)$ is an exact functor.

K: module:

Functors: $\mathrm{Hom}(K, \cdot) \;:\; A \longmapsto \mathrm{Hom}(K, A)$.

$$(\varphi : A \to B) \longmapsto \big((\psi : K \to A) \longmapsto (\varphi \circ \psi : K \to B)\big)$$

Left-exact.

Functor $\mathrm{Hom}(\cdot, K) \;:\; A \longmapsto \mathrm{Hom}(A, K)$

$$(\varphi : A \to B) \longmapsto \big((\psi : B \to K) \longmapsto (\psi \circ \varphi : A \to K)\big)$$

Bad!

$A \to B$ goes to $\mathrm{Hom}(B, K) \to \mathrm{Hom}(A, K)$!

This is a <u>contravariant</u> functor

(the earlier ones were <u>covariant</u>).

for contravariant F:

$$F(\varphi \circ \psi) = F(\psi) \circ F(\varphi).$$

$\mathrm{Hom}(\cdot, K)$ is contravariant functor.

Let R be an integral domain.

<u>Thm</u> If K is flat then K is torsion-free.

Proof Let $v \in K$ be a torsion elt: $av = 0$, $\overset{\in R}{a} \neq 0$.

Let $F = (R \setminus \{0\})^{-1} R$ (field of fractions of R).

Consider $0 \to R \to F$ and $0 \to \underset{\underset{K}{\cong}}{R \otimes K} \to F \otimes K$.

This is not injection since $\overset{0 \neq v}{1 \otimes v} \mapsto 1 \otimes v = \frac{1}{a} \otimes av = 0$

Criterion: K is flat iff $\forall$ ideal I in R,

$I \otimes K \longrightarrow R \otimes K \cong K$ is an injection.

$\Downarrow$

if R is PID then K is flat $\Longleftrightarrow$ K is torsion free.

<u>Theorem</u> if R is an integral domain

then its field of fractions is flat.