Lec 9/13

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Semi-direct Products

Thm Let $H \leq G$, $K \leq G$ s.t.

(i) $H \trianglelefteq G$, $K \trianglelefteq G$

(ii) $H \cap K = 1$

Then $HK \cong H \times K$ (so if $HK = G$, then G is isomorphic to $H \times K$)

Proof we have $HK \leq G$. Let $h \in H$, $k \in K$. Since $H \trianglelefteq G$,

$k^{-1}hk \in H$, so $h^{-1}k^{-1}hk \in H$. Similarly, $h^{-1}k^{-1}hk \in K$,

So it is 1. Thus $hk = kh$.

Let $\varphi : HK \to H \times K$, $hk \xmapsto{\varphi} (h, k)$.

φ is well-defined since every elt of HK

can be written uniquely in the form hk w/ $h \in H$, $k \in K$.

(If H and K are finite, $|HK| = \frac{|H||K|}{|H \cap K|} = |H||K|$, showing uniqueness; if H and K aren't assumed to be finite, exercise).

Note $\varphi((h_1k_1)(h_2k_2)) = \varphi(h_1h_2k_1k_2) = (h_1h_2, k_1k_2) = (h_1, k_1)\cdot(h_2, k_2) = \varphi(h_1k_1)\cdot\varphi(h_2k_2)$.

φ is a bijection by uniqueness of representation

So $HK \cong_{\varphi} H \times K$. $\square$

If H, K abelian, $H \times K$ abelian.

This is not necessarily the case for $\rtimes$.

<u>Thm</u> Let $H \leq G$, $K \leq G$ s.t.

(i) $H \trianglelefteq G$

(ii) $H \cap K = 1$

Let $\cdot$ denote the conjugation action of K on H, so $k \cdot h = khk^{-1} \in H$.

Let Q be the set of ordered pairs (h,k) with $h \in H$, $k \in K$.

Define the following op on Q:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1 k_2) \qquad (*)$$

(1) $(*)$ makes Q a group of size $|Q| = |H||K|$.

(2) $H \cong \tilde{H} = \{(h,1) \mid h \in H\} \leq Q$

$K \cong \tilde{K} = \{(1,k) \mid k \in K\} \leq Q$

(3) $\tilde{H} \trianglelefteq Q$ and $\tilde{H} \cap \tilde{K} = 1$

<u>Proof</u> Lets show $(*)$ is associative.

$$((h_1, k_1) \cdot (h_2, k_2)) \cdot (h_3, k_3) = (h_1(k_1 \cdot h_2), k_1 k_2) \cdot (h_3, k_3)$$

$$= (h_1 (k_1 \cdot h_2)(k_1 k_2 \cdot h_3), k_1 k_2 k_3)$$

$$= (h_1 (k_1 \cdot h_2)(k_1 \cdot (k_2 \cdot h_3)), k_1 k_2 k_3)$$

$$= (h_1 (k_1 \cdot (h_2 (k_2 \cdot h_3))), k_1 k_2 k_3)$$

$$= (h_1, k_1)(h_2(k_2 \cdot h_3), k_2 k_3)$$

$$= (h_1, k_1)((h_2, k_2)(h_3, k_3)) \quad \checkmark$$

Uses fact that conj action gives automorphism $K \longrightarrow \mathrm{Aut}(H)$

So $(*)$ is associative.

$(1,1) \in Q$ is the id. wrt $(*)$.

Lets verify that $(h,k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$

$$(h,k)(k^{-1}h^{-1}, k^{-1}) = (h\ k\cdot(k^{-1}\cdot h^{-1}), kk^{-1}) = (h\ 1\cdot h^{-1}, 1) = (1,1).$$

So Q is a group under $(*)$.

For (2), $(1, k_1)(1, k_2) = (1\ k_1\cdot 1,\ k_1k_2) = (1, k_1k_2) \in \tilde{K}$

$$(1,k)^{-1} = (k^{-1}\cdot 1, k^{-1}) = (1, k^{-1}) \in \hat{K}.$$

Similarly, $(h_1, 1)(h_2, 1) = (h_1\ 1\cdot h_2, 1) = (h_1h_2, 1) \in \tilde{H}$

$$(h,1)^{-1} = (1^{-1}\cdot h^{-1}, 1^{-1}) = (h^{-1}, 1) \in \tilde{H}$$

so $\hat{K}, \tilde{H} \leq Q$ (exercise: show details of isomorphisms $\tilde{H} \cong H$, $\tilde{K} \cong K$).

For (3), if $(h,1) = (1,k)$, $(1,1) = (h,1)^{-1}(1,k)$

$$= (h^{-1},1)(1,k)$$
$$= (h^{-1}\ 1\cdot 1,\ 1\cdot k)$$
$$= (h^{-1}, k)$$

So $h^{-1} = 1$, $k = 1$ so $h = 1$.

So $\tilde{H} \cap \tilde{K} = 1$.

Also, $(1,k)(h,1)(1,k)^{-1} = (k\cdot h, 1) \in \tilde{H}$

so $\tilde{K} \leq N_Q(\tilde{H})$.

Also, $\tilde{H} \leq N_Q(\tilde{H})$. So $\underbrace{\tilde{H}\tilde{K}}_{\substack{\| \\ Q}} \leq N_Q(\tilde{H})$. so $\tilde{H} \trianglelefteq Q$ $\square$

– recall exercise about size argument including when $\tilde{H}$ & $\tilde{K}$ are not finite

<u>Thm</u> let H and K be gps. Let φ be a homomorphism $K \longrightarrow \mathrm{Aut}(H)$.

Let $\cdot$ denote the action $k \cdot h = \varphi(k)(h)$. Let G be the set of pairs (h, k) where $h \in H$, $k \in K$. Define the multiplication

$$(h_1, k_1)(h_2, k_2) = (h_1\, k_1 \cdot h_2,\ k_1 k_2).$$

(1) this makes G a gp of order $|G| = |H||K|$, and this is called the semidirect product

$$G = H \rtimes_\varphi K.$$

(2) $\underset{h \leftrightarrow (h,1)}{H \cong \tilde{H}} \leq G, \quad \underset{k \leftrightarrow (1,k)}{K \cong \tilde{K}} \leq G$

(3) $\tilde{H} \trianglelefteq G$

Note identify H & K with $\hat{H}$ & $\tilde{K}$.

we have $k h k^{-1} = k \cdot h = \varphi(k)(h)$.

Note if φ is the trivial hom $K \longrightarrow Aut(H)$, $\varphi(k) = id$,

then $H \rtimes_\varphi K \cong H \times K$.

Ex let $H = \langle y \mid y^n = 1 \rangle$, $K = \langle x \mid x^2 = 1 \rangle$

$$\varphi : K \longrightarrow Aut(H)$$
$$x \longmapsto \sigma$$

where $\sigma(h) = h^{-1}$.

i.e. $\underbrace{x \cdot h = h^{-1}}_{\text{works bc } H \text{ is abelian.}}$

Then $H \rtimes_\varphi K \cong D_n$ (or D_{2n})