Lec 10/19

Wednesday, October 19, 2016 9:06 AM

2nd Derivative test

Suppose $f'(c) = 0$ and $f''(c)$ is defined.

(1) if $f''(c) > 0$, f has a local min at c

(2) " $f''(c) < 0$, " " max "

(3) If $f''(c) = 0$, the test is inconclusive.

Ex: (3) $f(x) = x^3$ $f'(0) = f''(0) = 0$ Neither local min or max

$f(x) = x^4$ $f'(0) = f''(0) = 0$ local min

$f(x) = -x^4$ $f'(0) = f''(0) = 0$ local max

Lemma If $g'(c) > 0$, then there is an interval (a,b) containing c so that $g(x) < g(c) < g(y)$ $\forall$ $x < c < y \in (a,b)$.

Note: g not necessarily increasing on (a,b)

Proof of 2D Test:

Suppose $f''(c) > 0$. Take $g = f'$ in lemma. For some open interval (a,b) containing c, we have

(a) $f'(x) < f'(c) = 0$ for $x \in (a,c)$

(b) $f'(x) > f'(c) = 0$ for $x \in (c,b)$

(a) $\Rightarrow$ f is decreasing on $(a,c]$

$\Rightarrow$ $f(x) > f(c)$ $\forall x \in (a,c)$

(b) $\Rightarrow$ f increasing on $[c,b)$

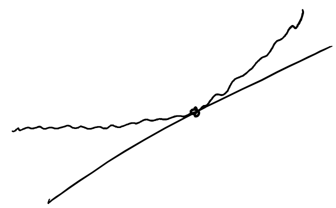
$\Rightarrow$ $f(x) > f(c)$ $\forall x \in (c,b)$

Therefore, f has a local min.

Case 2: by case 1, $-f$ has local min $\Rightarrow$ f has local max. $\blacksquare$

Corollary (1) Suppose $f''(c) > 0$. Then for some open interval $(a,b) \ni c$, the graph of f lies above the tangent line at c except at c.

$f(x) > f'(c)(x-c) + f(c) \quad \forall x \in (a,b) \setminus \{c\}$.



(2) Similar but $f''(c) < 0$, f above tangent line.

Proof: Let $g(x) = f(x) - f'(c)(x-c) - f(c)$.

Then $g'(x) = f'(x) - f'(c)$

$g''(c) = f''(c)$

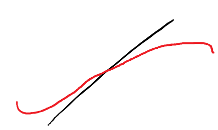
(1) $g'(c) = 0$, $g''(c) > 0 \Rightarrow g$ has a local min at c.

$\Rightarrow g(x) > g(c) = 0$ for $x \in (a,c) \cup (c,b)$ $\blacksquare$?

Definition We say that f has an inflection point at c if f has a tangent line at c which cuts through the graph at c. More precisely, for some open interval $(a,b) \ni c$, we have either

(1) $f(x) > f'(c)(x-c) + f(c) \quad \forall x \in (a,c)$

$f(x) < f'(c)(x-c) + f(c) \quad \forall x \in (c,b)$



(2) $f(x) < f'(c)(x-c) + f(c) \quad \forall x \in (a,c)$

$$f(x) > f'(c)(x-c) + f(c) \quad \forall x \in (c,b)$$

Corollary 2: if f has an inflection point at c & $f''(c)$ is defined, then $f''(c) = 0$.

Remark: $f''(c) = 0$ does not imply that f has an inflection point at c.
consider $f(x) = x^4$. $f''(0) = 0$ but min at 0, not inf. pt.

Theorem if f'' is defined on an open interval $(a,b) \ni c$, and $f''(c) = 0$, and $f''(x)$ has opposite signs on (a,c) and (c,b), Then f has an inflection point at c.

Proof: Suppose $f''(x) > 0$ on (a,c) and $f''(x) < 0$ on (c,b).

$$\left.\begin{aligned} &\text{Let } g(x) = f(x) - f'(c)(x-c) - f(c). \\ &\text{Then } g'(x) = f'(x) - f'(c) \\ &\qquad g''(x) = f''(x) \end{aligned}\right\} x \in (a,b)$$

$g''(x) = f''(x) > 0$ on $(a,c) \Rightarrow g'$ increasing on (a,c)
$\Rightarrow g'(x) < g'(c) = 0 \ \forall x \in (a,c)$
$\Rightarrow g$ is decreasing on (a,c)
$\Rightarrow g(x) > g(c) = 0 \ \forall x \in (a,c)$

$g''(x) = f''(x) < 0$ on $(c,b) \Rightarrow g'$ decreasing on (c,b)
$\Rightarrow g'(x) > g'(c) = 0 \ \forall x \in (c,b)$
$\Rightarrow g$ decreasing on (c,b)
$\Rightarrow g(x) < g(c) = 0 \ \forall x \in (c,b)$

So tangent line passes through graph at c. ∎

Inverse functions

Inverse functions

Definition We say that f^{-1} is the inverse function of f if $f^{-1} = \{(x,y) : (y,x) \in f\}$ is a function.

- This is equivalent to saying that for any $x \in \mathrm{dom}(f^{-1})$ there is a unique y s.t. $f(y) = x$.
- This is equivalent to saying $y_1 \neq y_2 \in \mathrm{dom}(f) \Rightarrow f(y_1) \neq f(y_2)$ $\Leftrightarrow$ $y_1 < y_2 \in \mathrm{dom}(f) \Rightarrow f(y_1) \neq f(y_2)$

Definition: f is 1-1 or injective if the above condition holds.

Theorem f is 1-1, $\mathrm{dom}(f)$ is an interval I, f continuous on I $\Leftrightarrow$ f is increasing on I or f is decreasing on I.

Proof: it is obvious. ∎