Lec 9/27

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Von Neumann's Thm:

if $(x_n) \subset [0,1]$ is dense then $\exists$ a rearrangement $(x_{\tilde{n}_i})$
s.t. $(x_{\tilde{n}_i})$ is u.d. mod 1.

Proof take $y_n = n\alpha \bmod 1$, $\alpha \notin \mathbb{Q}$ $\forall n\ \exists k_n$ s.t. $|y_n - x_{k_n}| < \frac{1}{2}$

$(\Rightarrow x_{k_n}$ u.d. mod 1$)$

If y_n u.d. and $|z_n - y_n| \xrightarrow{n\to\infty} 0$ then z_n u.d.

Claim: altering a u.d. sequence (x_n) at ns belonging to density 0 set does not change u.d. ness.

$$\frac{1}{N}\sum_{n=1}^{N} 1_A(n) = \frac{|A \cap \{1,\dots,N\}|}{N} \longrightarrow d(A)$$

What is the probability that $(n,m)=1$ (for some $n,m \in \mathbb{N}$)? $\frac{6}{\pi^2}$.

$d_{(\mathbb{N}^2)}(S) = \frac{6}{\pi^2}$ where $S = \{(n,m) \in \mathbb{N}^2 : (n,m)=1\}$.

$$\frac{1}{N^2}\sum_{n,m=1}^{N} 1_A(n,m) \longrightarrow d_{(\mathbb{N}^2)}(A) \quad \text{for } A \subset \mathbb{N}^2.$$

Options $(x_{n,m}) \subset [0,1]$ (1)

$(x_{n,m}) \subset [0,1]^2$ (2)

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$(x_n) \subset [0,1]^2$ (3)

example of (3).

$n^2\alpha \mod 1$ is u.d. (figure out why)

$((n\alpha, n\beta)) \subset [0,1]^2$? yes if $\alpha \neq p\beta$ for $p \in \mathbb{Q}$.

(iff α,β lin. indep. over $\mathbb{Q}$) (exercise) (both ways)

$((n\alpha, n^2\alpha)) \subset [0,1]^2$?

$(x_n) \subset [0,1]^2$ is u.d mod 1 if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_{(a,b]\times(c,d]}(x_n) = (b-a)(d-c)$$

(Lekkerkerker)

Van der Corput's Difference Theorem:

If $\forall h \in \mathbb{N}$, $(x_{n+h} - x_n), n \in \mathbb{N}$ is u.d. mod 1 then $(x_n), n \in \mathbb{N}$ is u.d. mod 1.

$x_n = n^2\alpha$, $x_{n+h} - x_n = (n^2 + 2nh + h^2)\alpha - n^2\alpha = n(2h\alpha) + \text{const}.$

Weyl handout Thm 3.2:

let $f(x) \in \mathbb{R}[x]$ and assume at least one coefficient of f (other than the constant term) is irrational.

then $(f(n))$ is u.d. mod 1. (exercise)

equivalent forms of u.d. mod 1^2.

1. $\forall f \in C([0,1]^2)$, $\frac{1}{N}\sum_{n=1}^{N} f(x_n, y_n) \longrightarrow \int_0^1\int_0^1 f(x,y)\,dx\,dy$ (also for $f \in R([0,1]^2)$).

2. Weyl's criterion: $\forall \vec{h} \in \mathbb{Z}^2 \setminus \{(0,0)\}$ $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \vec{h}\cdot(x_n, y_n)} \to 0$

$\Bigg(\Big(e^{2\pi i h x}$, $h \in \mathbb{Z}$ form a basis of $C(\mathbb{T})$ after closure of span $\Big)$ torus

$\Big(e^{2\pi i \vec{h}\cdot\vec{x}}$, $h \in \mathbb{Z}^2$ form a basis of $C(\mathbb{T}^2)$ after closure of span $\Big) \Bigg)$