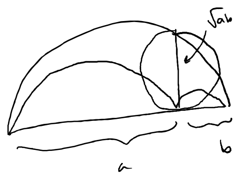


$A_{circ} = \frac{ab}{4}\pi$

$A_{moon} = \frac{1}{2}\left(\frac{a+b}{2}\right)^2\pi - \frac{1}{2}\left(\frac{a}{2}\right)^2\pi - \frac{1}{2}\left(\frac{b}{2}\right)^2\pi = \frac{ab}{4}\pi$

Eqns 9-5 can be rewritten as

$$R_{i\ jk}^{\ l} = L_{ik}L_j^l - L_{ij}L_k^l \qquad \text{(Gauss's eqns, 1827).}$$

where $R_{ijk}^{l} = \frac{\partial \Gamma_{ik}^l}{\partial u^j} - \frac{\partial \Gamma_{ij}^l}{\partial u^k} + \sum_p \left(\Gamma_{ik}^p \Gamma_{pj}^l - \Gamma_{ij}^p \Gamma_{pk}^l\right)$.

Although the L's are not intrinsic, the R's <u>are</u>.

The functions $R_{i\ jk}^{\ l}$ are called the components of the Riemann curvature tensor (of type (1,3)).

The functions $R_{imjk} = \sum_l R_{i\ jk}^{\ l} g_{lm}$ are called the components of the Riemann curvature tensor (of type (0,4)).

<u>Theorem 9.2</u> (Gauss's Theorema Egregium):

The Gaussian Curvature K of a C^3 surface M in $\mathbb{R}^3$ is intrinsic.

<u>Pf</u> By Gauss's equations, $R_{i\ jk}^{\ l} = L_{ik}L_j^l - L_{ij}L_k^l$, so, lowering the index l, we get

$$R_{imjk} = \sum_l R_{i\ jk}^{\ l} g_{lm} = L_{ik}\sum_l L_j^l g_{lm} - L_{ij}\sum_l L_k^l g_{lm}$$

$$= L_{ik}L_{mj} - L_{ij}L_{mk}$$

In particular, when $i=k=1$ and $j=m=2$ we get

$$R_{1221} = L_{11}L_{22} - L_{12}L_{21} = \det(L_{ij}) = \det\left((g_{ik})(L_j^k)\right) = \det(g_{ik})\det(L_j^k)$$

$$= g K_1 K_2 = gK.$$

$\square$

So $K = \frac{R_{1221}}{g}$ is intrinsic.

<u>Remark</u> It can be shown that $R_{imjk} = -R_{mijk}$, $R_{imjk} = R_{jkim}$.

hence if $i=m$ or if $j=k$ then $R_{imjk}=0$.

Thus for a surface the only possibly nonzero components of (R_{imjk}) are

$$R_{1221} = -R_{2121} = R_{2112} = -R_{1212} = \underbrace{gK}_{\text{gaussian Kurvature}}.$$

<u>Remark</u> (about the curvature tensor for higher dimensional manifolds)

Since K is intrinsic, it can be defined for surfaces in $\mathbb{R}^n$ where $n \geq 2$.

Let M be a C^3 manifold of dimension ≥ 2 in $\mathbb{R}^n$ (this memo $n \geq 2$ as well).

Let $p \in M$. Let V be a 2 dimensional linear subspace of T_pM

Let X, Y be an orthonormal basis for V, $X = \sum_i X^i x_i$ and $Y = \sum_j Y^j x_j$.

The collection of constant speed geodesics γ in M with $\gamma(0)=p$ and $\gamma'(0) \in V$

fills out a surface $S \subseteq M$.

The Gaussian Curvature of S at p is given by

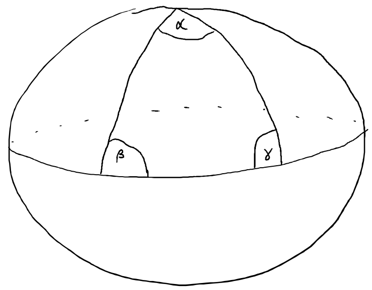
$$K_p(S) = R(X,Y,Y,X)(p) \stackrel{\text{def}}{=} \sum_{i,m,j,k} R_{imjk}(p)\, X^i Y^m Y^j X^k$$

<u>Special case</u>: suppose M is a surface in $\mathbb{R}^3$. Then

$$\begin{aligned}
\sum_{i,m,j,k} R_{imjk} X^i Y^m Y^j X^k &= gK\left(X^1Y^2Y^2X^1 - X^2Y^1Y^2X^1 + X^2Y^1Y^1X^2 - X^1Y^2Y^1X^2\right) \\
&= gK\left((X^1)^2(Y^2)^2 - 2X^1X^2Y^1Y^2 + (X^2)^2(Y^1)^2\right) \\
&= gK\left(X^1Y^2 - X^2Y^1\right)^2 \\
&= K
\end{aligned}$$

Since $1 = |X \times Y|^2 = \left\| \begin{matrix} x_1 & x_2 & x_1 \times x_2 \\ X^1 & X^2 & 0 \\ Y^1 & Y^2 & 0 \end{matrix} \right\|^2 = (X^1Y^2 - X^2Y^1)^2\, |x_1 \times x_2|^2 = (X^1Y^2 - X^2Y^1)^2\, g$.

<u>The sum of the angles in a spherical triangle.</u>



Unit sphere $K = 1$ everywhere.

in the picture, $\beta = \gamma = \frac{\pi}{2}$

and $0 < \alpha < 2\pi$,

The indicated triangle has area $A = \alpha$, so sum of angles is $\pi + A$.

If sphere has radius R, $K = \frac{1}{R^2}$, and $A = \alpha R^2$ so sum of angles is $\overbrace{\pi + AK}^{(*)}$

It turns out that for any geodesic triangle on the sphere, the formula

$\alpha + \beta + \gamma = \pi + KA$ still holds.

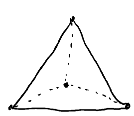
Actually, $(*)$ holds for a geodesic triangle on any surface of constant Gaussian Curvature K.

In fact, Gauss (1827) showed that on any (smooth enough) surface, the sum of the interior angles $\beta_1, \beta_2, \beta_3$ in a geodesic triangle Δ is

$$\beta_1 + \beta_2 + \beta_3 = \pi + \int_{\Delta} K \, dA .$$

Bonnet (1848) extended this result to non-geodesic triangles by adding an "$\int_{\partial\Delta} k_g \, ds$" term to account for the geodesic curvature of the sides (generalized turning tangents theorem or Local Gauss-Bonnet theorem).

<u>Regular tetrahedron M</u>



4 faces, each is an equilateral triangle.

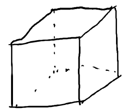
6 edges, 4 vertices.

$\chi = V - E + F = 4 - 6 + 4 = 2$.

At each corner, the sum of the angles is $\pi/3 + \pi/3 + \pi/3 = \pi = 2\pi - \pi$.

$$4\pi = 2\pi\chi = \int_M K \, dA = \pi + \pi + \pi + \pi$$

<u>Cube M</u>

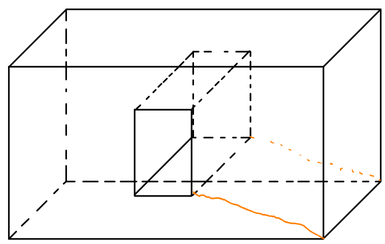


$\chi = V - E + F = 8 - 12 + 6 = 2.$

at each corner, the sum of the angles is $\frac{3\pi}{2} = 2\pi - \frac{\pi}{2}$.

$$\int_M K\,dA = \frac{\pi}{2} \cdot 8 = 4\pi = 2\pi\chi.$$

Rectangular Torus M



cuts since we require "face" $\sim \Delta$ homeomorphic

$F = 10$, $E = 9 + 9 + 4 + 4 = 26$, $V = 16$, $\chi = V - E + F = 0$

at each "outside" vertex, the sum of the angles is $2\pi - \frac{\pi}{2}$.

at each "inside" vertex, the sum of the angles is $\frac{\pi}{2} + \frac{\pi}{2} + \frac{3\pi}{2} = 2\pi + \frac{\pi}{2}$

So $\int K\,dA = 0.$