Lec 8/23

Wednesday, August 23, 2017 14:18

- Systems of Diophantine Eqns.

(polynomial eqns w/ rational coeffs)

$x^2 - Dy^2 = 1$ Pell's eqn. where $D \in \mathbb{N}$ is not a square.

Know: group, field, ring

$\downarrow$

e.g. S_n symmetric group order n, set of mappings $\varphi : \{1, \dots, n\} \circlearrowleft$ under $\circ$

Problem: Give an example of an infinite (countable) Abelian group which doesn't have nontrivial infinite subgroups.

Analytic Number Theory: (maybe)

$P = \{2, 3, 5, \dots\}$

$|P| = \infty$.

$\frac{a}{b} = \sqrt{2} \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2 \Longrightarrow 4\tilde{a}^2 = 2b^2 \Rightarrow 2\tilde{a}^2 = b^2$ What happened?

$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ Basel Problem – Euler

Infinitude of primes:

$2 \le n$. $n, n+1$ are relatively prime $(n, n+1) = 1$

(in general, $(a,b)=1$ if a,b are coprime, (a,b) is GCD).

So $n(n+1)$ has at least 2 prime factors.

$(n(n+1),\ n(n+1)+1) = 1$ so $n(n+1)(n(n+1)+1)$ has at least 3 prime factors,

etc.

$\underbrace{2^{2^n}+1 \text{ are coprime.}}_{\text{Goldbach}}$ if ∞ coprime #s, ∞ primes.

Goldbach (Exercise)

$$\left.\left(\frac{\sqrt{2}}{2}\right)^n \longrightarrow 0\right] \Rightarrow \sqrt{2} \notin \mathbb{Q} \qquad \text{(Exercise)}$$

Euler: $\sum_{p \in P} \frac{1}{p} = \infty$

$$P_n \sim n \log n \qquad \left(\frac{P_n}{n \log n} \to 1\right)$$

Ex: Suppose you know that $\exists\, c_1, c_2 > 0$ s.t.

$$c_1 < \frac{P_n}{n \log n} < c_2$$

$\forall\, n > 1$, prove $\sum \frac{1}{p} = \infty$

$\pi(n) :=$ number of primes $\le n$

$$\pi(n) \sim \frac{n}{\log n} \iff P_n \sim n \log n \qquad \text{(exercise)}$$

$$\pi(n) \sim \frac{n}{\log n} \iff P_n \sim n \log n$$ (exercise)

Algebraic Number theory:

$\Gamma = \{ a+bi \mid a,b \in \mathbb{Z} \}$ Gaussian Integers.

An analog of $\mathbb{Z}$.

$$(2+i)(3+i) = 5+5i$$

get ± 1, $\pm i$ 'for free' like 1 in $\mathbb{Z}$.

Algebraic number theory: algebraic number

$f \in \mathbb{Q}[t]$. If $f(t_0) = 0$, $t_0 \in \mathbb{A}$.

Liouville

$\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is non algebraic (so called L. number)

Cantor: algebraic #s are countable (easy proof now, exercise)

Geometry of Numbers.

Number theory ~ counting integer points in complex bodies.

<u>Theorem</u> If $p = 4n+1$ then $p = n^2 + m^2$

Definition: Upper density of $A \subseteq \mathbb{N}$ is $\bar{\delta}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$

AP-Rich: has arbitrarily long arithmetic progressions.