Lec 1/18

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<u>Ex:</u> Suppose 2 ind. RS' of size n_1, n_2. are taken from a population w/ mean μ and variance σ^2.

Let $\bar{X}_1$ and $\bar{X}_2$ be corresponding sample means.

Define $\hat{\mu} = w\bar{X}_1 + (1-w)\bar{X}_2$

is $\hat{\mu}$ an unbiased estimator of μ? Yes, $E(\hat{\mu}) = wE(\bar{X}_1) + (1-w)E(\bar{X}_2)$
$= w\mu + (1-w)\mu = \mu$.

Q_2: find w to minimize Variance of $\hat{\mu}$.

$$Var(\hat{\mu}) = w^2 Var(\bar{X}_1) + (1-w)^2 Var(\bar{X}_2) = w^2\frac{\sigma^2}{n_1} + (1-w)^2\frac{\sigma^2}{n_2}$$

$$\frac{\partial Var(\hat{\mu})}{\partial w} = 2w\frac{\sigma^2}{n_1} - 2(1-w)\frac{\sigma^2}{n_2} = \cancel{2}\left(\frac{\cancel{\sigma^2}}{n_1} + \frac{\cancel{\sigma^2}}{n_2}\right)w - \cancel{2}\frac{\cancel{\sigma^2}}{n_2} = 0$$

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right)w = \frac{1}{n_2} \Rightarrow w = \frac{\frac{1}{n_2}}{\frac{1}{n_1} + \frac{1}{n_2}} = \frac{n_1}{n_1+n_2}$$

Check 2nd derivative: $2\frac{\sigma^2}{} + 2\frac{\sigma^2}{n^2} > 0$ so $\nearrow$ is a minimum.

so $w = \frac{n_1}{n_1+n_2}$ will minimize variance of $\hat{\mu}$.

Note: In this section, compared variances of <u>unbiased</u> estimators but sometimes we want to use a biased estimator. w/ small |bias| & small variance. (when unbiased & large variance is other option).

In this case, comparing bias & variance together, use <u>Mean-squared-error</u> (MSE).

estimator of θ.

$$MSE(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta}-\theta)^2\right] = \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta\right)^2\right]$$

$$= \underbrace{\mathbb{E}\left[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2\right]} + 2\,\mathbb{E}\Big[\underbrace{(\hat{\theta} - \mathbb{E}(\hat{\theta}))}_{\mathbb{E}(\hat{\theta}) - \mathbb{E}\hat{\theta} = 0}\underbrace{(\mathbb{E}(\hat{\theta}) - \theta)}_{\text{non-random}}\Big] + \underbrace{\mathbb{E}\left[(\mathbb{E}(\hat{\theta}) - \theta)^2\right]}_{=}$$

$$= Var(\hat{\theta}) + 0 + [Bias(\hat{\theta})]^2$$

$$= Var(\hat{\theta}) + (Bias(\hat{\theta}))^2$$

Hence, Small MSE $\Rightarrow$ small variance & small $Bias^2$.

§ 10.4 Consistency (another desirable property of an estimator).

Idea: as more data is collected we expect estimate to get better.

Def 10.2: $\hat{\theta}$ is a consistent estimator of θ iff $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}-\theta| < \varepsilon) = 1. \qquad (n \text{ is sample size}).$$

AKA convergence in probability $(\hat{\theta} \xrightarrow{\mathbb{P}} \theta)$

Recall: Chebyshev's Thm (4.8). μ = mean, σ^2 = Variance of X. $k > 0$

$$\mathbb{P}(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

A consequence of the inequality is:

Thm 10.3: if $\hat{\theta}$ is an unbiased estimator of θ, and $Var(\hat{\theta}) \to 0$ as $n \to \infty$,

then $\hat{\theta}$ is a consistent estimator of θ.

Ex: Let $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$.

Consider sample mean $\bar{X}_n = \bar{X}$ (depends on n).

(1) $\bar{X}_n$ is unbiased: $E(\bar{X}_n) = \theta$.

(2) $Var\, \bar{X}_n = \frac{1}{n} \to 0$ as $n \to \infty$, so $\bar{X}_n$ is a consistent estimator of θ By 10.3.

Method 2: (by the definition) $\mathbb{P}(|\bar{X}_n - \theta| < \varepsilon) = \mathbb{P}\left(-\varepsilon < \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta) < \varepsilon\right)$

$$= \mathbb{P}\Bigg(-\varepsilon\sqrt{n} < \underbrace{\frac{\sum_{i=1}^{n}(x_i-\theta)}{\sqrt{n}}}_{N(0,1)} < \varepsilon\sqrt{n}\Bigg) \longrightarrow 1.$$