Lec 11/1

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Hilbert basis thm: R noetherian $\Rightarrow$ $R[x]$ noetherian.

$L(f) =$ Leading coeff of f.

Ex: Let $J \subset R[x]$ be a subgroup s.t. $RJ \subset J$.

then $C_k(J) = \{\text{Coeff of } x^k \text{ in } f(x) : f(x) \in J\}$ is an ideal in R.

$\tilde{I} \overset{\text{ideal}}{\subset} R[x] \rightsquigarrow I = L(\tilde{I}) \subset R$ is an ideal (proved yesterday)

R is noetherian so $I = (a_1, \dots, a_N)$ in R.

We get $g_1(x), g_2(x), \dots, g_N(x) \in \tilde{I}$ s.t. $L(g_i) = a_i$, $d_i = \deg(g_i)$

Division: $\forall g(x) \in \tilde{I}$ $\exists \bar{g}(x) \in \tilde{I}$ s.t. $g(x) = \bar{g}(x) \mod (g_1(x), \dots, g_N(x))$

and $\deg(\bar{g}) < D = \max\{d_1, \dots, d_N\}$

Pf $g(x) = \gamma x^M + \cdots$. $M < D \Rightarrow$ nothing to do.

$M \geq D \Rightarrow \gamma \in L(\tilde{I}) \Rightarrow \gamma = r_1 a_1 + \cdots + r_N a_N$ for some $r_1, \dots, r_N \in R$

$\Rightarrow \tilde{g}(x) = g(x) - r_1 g_1(x) x^{M-d_1} - \cdots - r_N g_N(x) x^{M-d_N}$

has degree less than g. continue this process.

Propn R Noetherian, $D \in \mathbb{Z}_{\geq 1} \Rightarrow R[x]/(x^D)$ is noetherian

More generally, if $J \subseteq R[x]/(x^D)$ is an abelian group

(*) s.t. $R \cdot J \subset J$, then $\exists f_1(x), \dots, f_\ell(x) \in J$ s.t.

$J = R \cdot f_1(x) + R \cdot f_2(x) + \cdots + R \cdot f_\ell(x)$.

Using proposition, we finish pf of HBT:

$\tilde{I} \subset R[x]$ an ideal. We found $g_1(x), \dots, g_N(x) \in \tilde{I}$

s.t. $\forall g \in \tilde{I}$, $\exists \bar{g} \in \tilde{I}$ s.t. $g \equiv \bar{g} \mod (g_1, \dots, g_N)$ and $\deg(\bar{g}) < D$.

$$\tilde{I}_{(<D)} = \{ f(x) \in \tilde{I} : \deg(f) < D \}$$

$\tilde{I}_{(<D)}$ is an abelian subgr & $R\,\tilde{I}_{(<D)} \subset \tilde{I}_{(<D)}$.

As we will see in the proof of ⊛, (eg take

$J = \pi(\tilde{I}_{(<D)}) \subset R[x]/(x^D)$) we will get $f_1(x), \dots, f_\ell(x) \in \tilde{I}_{(<D)}$

s.t. $\tilde{I}_{<D} = R \cdot f_1(x) + \dots + R \cdot f_\ell(x)$

$\Rightarrow \tilde{I} = (g_1, \dots, g_N, f_1, \dots, f_\ell)$ is finitely generated. □

Pf of ⊛: For each $k \in \{0, \dots, D-1\}$, define $C_k(J) = \{ a \in R : \exists f(x) = a x^k + * x^{k+1} + \dots + * x^{D-1} \in J \}$

Claim: $C_k(J) \subset R$ is an ideal. (Ex)

$$C_k(J) = (\alpha_1^{(k)}, \dots, \alpha_{m_k}^{(k)}) \text{ for some } \alpha_i^{(j)} \in \mathcal{O}.$$

$k=0$ $C_0(J) = (\alpha_1^{(0)}, \dots, \alpha_{m_0}^{(0)})$

We get:

$$J \ni f_1^{(0)}(x) = \alpha_1^{(0)} + \boxed{\text{higher terms}}$$

$$\vdots$$

$$J \ni f_{m_0}^{(0)}(x) = \alpha_{m_0}^{(0)} + \boxed{\text{higher terms}}$$

$$J \ni f_i^{(k)} = \alpha_i^{(k)} x^k + \boxed{\text{higher terms}}$$

$\forall\, k \in \{0, \dots, D-1\}$, $i \in \{1, \dots, m_k\}$.

Claim every element in J is a linear combination (coefficients from R) of

$$\{f_1^{(0)}, \dots, f_{m_0}^{(0)}, f_1^{(1)}, \dots, f_{m_1}^{(1)}, \ddots, f_1^{(D-1)}, \dots, f_{m_{D-1}}^{(D-1)}\}$$

Pf Let $g(x) = \gamma \cdot x^{\ell} + \boxed{\text{higher terms}} \in J$.

then $\tilde{g}(x) = g(x) - \sum_{j=1}^{m_\ell} r_j f_j^{(\ell)}(x)$ has higher degree

(where $\gamma = \sum r_j \alpha_j^{(\ell)}$) $\square$

Note: $R[\![x]\!]$ is noetherian if R is.

<u>Cor</u>: $R[x_1, \dots, x_N]$ is Noetherian

noetherian ring.

Hilbert's original statement:

every $I \subset K[x_1, \dots, x_N]$ is finitely generated

field.

<u>Decomposition Theorem</u>: $n = p_1^{k_1} \cdots p_\ell^{k_\ell}$ in integers uniquely.

Any ideal $I \subset R$ (where R is noetherian)

Can be written as $I = Q_1 \cap Q_2 \cap \dots \cap Q_\ell$

where $Q_1, \dots, Q_\ell \subset R$ are <u>primary</u> ideals

(uniqueness only up to $K \leq \ell$)

Primary ideal: $\underset{\text{ideal}}{Q \subsetneq R}$ is primary if $ab \in Q, a \notin Q, \Rightarrow b^n \in Q$ for some $n \geq 1$.

Quiz tomorrow: Ideals in $S^{-1}R$.