Lec 2/1

Wednesday, February 1, 2017 15:00

$(1-\alpha)\times 100\%$ CI for mean: $\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ (where σ^2 is known)

Population is normal $\Leftrightarrow$ CI Exact

Changing n (sample size) affects the width of CI inversely.

" level of confidence " " in the same direction (lower conf, lower width)

What if we don't know σ^2?

$S^2 \longrightarrow \sigma^2$ as $n \to \infty$

(S^2: Sample variance)

for large n, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \overset{\text{Approx}}{\sim} N(0,1) \Rightarrow \frac{\bar{X}-\mu}{S/\sqrt{n}} \overset{\text{Approx}}{\sim} N(0,1)$

$\Rightarrow \bar{X} \pm Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$ is an approx $(1-\alpha)\times 100\%$ CI for μ.

how big should n be? it depends on the application. (for now use $n \geq 30$).

What if n is not big enough? $n < 30$ and σ^2 unknown?

Review: the t distribution:

Thm. 8.12 If Y, Z are indep. RVs and $Y \sim \chi^2_k$ $Z \sim N(0,1)$ then

$$T = \frac{Z}{\sqrt{Y/k}} \sim f(t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k}\;\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{t^2}{k}\right)^{-\frac{k+1}{2}}$$

t distribution w/ k degrees of freedom.

Thm 8.13 $\bar{X}, S^2$ sample mean & var from RS of size n of normal pop:

$$T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$$ has t distribution w/ $n-1$ degrees of freedom.

Proof: Thm 8.11 $\Rightarrow$ $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ and $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

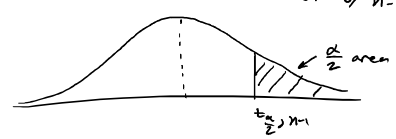
$$\text{Thm 8.12} \Rightarrow \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \Big/ \sqrt{\frac{(n-1)S^2}{\sigma^2}\Big/(n-1)} = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S} = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

Thm 11.3 $\overline{X}, S$ sample mean & SD of size n RS from normal population, then

$$\overline{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$$

is a $(1-\alpha) \times 100\%$ CI for μ.

t dist w/ n-1 degrees of freedom.



Properties of t distribution:

1: Symmetric about origin

2: More tail than normal dist

3: $t_n \longrightarrow N(0,1)$ as $n \to \infty$

so $t_{\frac{\alpha}{2}, n} \longrightarrow z_{\frac{\alpha}{2}}$ as $n \to \infty$

Ex: a machine produces metal rods. a RS of 15 rods' diameters.

$\overline{X} = 8.234$ $S^2 = 0.00064$.

Assume the pop is nearly normal find 95% CI for μ

$$\overline{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}} = 8.234 \pm \underbrace{2.145}_{= \, t_{0.025, 14}} \frac{\sqrt{0.00064}}{\sqrt{14}} = (8.22, 8.25) \text{ mm.}$$

Remarks:

①

	σ^2 known	σ^2 unknown, $n \geq 30$	σ^2 unknown, $n < 30$
Normal pop	$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ (exact)	$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$ (approx)	$\overline{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$ (exact)
non-normal pop	$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ (approx	$\overline{X} \pm z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$ (approx)	hopeless.