Lec 1/8

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<u>Last time</u>: Hölder inequality.

if $p, q \in (1,\infty)$ w/ $\frac{1}{p} + \frac{1}{q} = 1$ (conjugate exponents)

and $f \in L^p$, $g \in L^q$, then $fg \in L^1$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q .$$

<u>Minkowski ineq</u>: for $1 \le p < \infty$, $f, g \in L^p$, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

<u>Pf</u> for $p > 1$. $|f+g|^p \le (|f| + |g|)\, |f+g|^{p-1}$

$$\rightsquigarrow \|f+g\|_p^p \le \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1}$$

$$\le \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \, \| |f+g|^{p-1} \|_q$$

$$= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^{q(p-1)} \right)^{1/q}$$

$$= \left(\|f\|_p + \|g\|_p\right) \|f+g\|_p^{p/q}$$

$$\rightsquigarrow \|f+g\|_p \le \|f\|_p + \|g\|_p \qquad \square$$

Note:

$$\frac{p+q}{pq} = 1$$

$$p+q = pq$$

$$q(p-1) = p$$

and

$$p - p/q = p\left(1 - \frac{1}{q}\right)$$

$$= p \cdot \frac{1}{p} = 1.$$

<u>Ex</u>: When does equality hold in Minkowski?

<u>Examples</u> of L^p spaces.

1) $X = \mathbb{R}^d$ w/ Lebesgue measure.

$L^p(\mathbb{R}^d)$ ← Wheeder-Zygmund.

2) $X = \mathbb{N}$ w/ counting measure

$\ell^p(\mathbb{N}) =: \ell^p$

Theorem 1: for $1 < p < \infty$, L^p is a Banach space.

We need:

- A normed space X is complete iff absolutely convergent series converge.
- Monotone Convergence thm

  (f_n) in L^+ nondecreasing sequence $\Rightarrow \lim \int f_n = \int \lim f_n$.

- Dominated convergence thm.

  (f_n) in L^+ s.t. $f_n \to f$ a.e. & $\exists g \in L^1$ s.t $|f_n| \leq g$ $\forall n$ a.e. $\Rightarrow \lim \int f_n = \int \lim f_n$.

Proof of theorem 1: Let $\sum_1^\infty f_k$ be an absolutely convergent series (i.e. $\sum_1^\infty \|f_k\|_p =: B < \infty$).

Partial sums $S_n = \sum_1^n |f_k|$, $S = \sum_1^\infty |f_k|$. $\|S_n\|_p \leq \sum_1^n \|f_n\|_p \leq B$ $\forall n$.

By MCT, $\int \lim S_n^p = \lim \int S_n^p \Rightarrow \int S^p = \lim \int S_n^p \leq B^p < \infty$,

So $S \in L^p$ so $S < \infty$ a.e. so $\sum_1^\infty f_k$ converges a.e. (to F, say).

$\left| F - \sum_1^n f_k \right|^p \leq (2S)^p \in L^1$.

$\underset{\text{DCT}}{\Rightarrow} \lim \int \left| F - \sum_1^n f_k \right|^p = \int \lim \left| F - \sum_1^n f_k \right|^p = \int 0 = 0$

$\rightsquigarrow \sum_1^\infty f_k$ converges in L^p. $\square$

useful: (but not here) ??

$$\left\{ \begin{aligned} |f+g|^p &\leq (2 \max(|f|, |g|))^p \\ &\leq 2^p (|f|^p + |g|^p) \end{aligned} \right.$$

Examples of L^p functions:

Consider $L^p(\mathbb{R}^d)$. Let

$$f_\alpha(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise: $f_\alpha \in L^p$ iff $p\alpha < d$.

Prop^n if X has positive finite measure & $p \le q$ then $L^p \supset L^q$

Proof direct application of Hölder:

suppose $f \in L^q$. let $r = \frac{q}{p}$, s = conjugate of r.

$$\|f\|_p^p = \int |f|^p = \int |f|^p \cdot 1 \le \|f^p\|_r \cdot \|1\|_s = \left(\int (|f|^p)^{q/p}\right)^{p/q} \cdot \mu(X)^{1/s} = \|f\|_q^p \cdot \mu(X)^{1/s}$$

$$\rightsquigarrow \|f\|_p \le \|f\|_q \cdot \mu(X)^{\frac{1}{ps}} = \|f\|_q \cdot \mu(X)^{\frac{1}{p} - \frac{1}{q}}.$$

since $\frac{1}{s} = 1 - \frac{p}{q} \rightsquigarrow \frac{1}{ps} = \frac{1}{p} - \frac{1}{q}$

$\square$

Prop^n Simple functions are dense in L^p for $1 \le p < \infty$.