$\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ is free w/ basis $\{1\otimes 1, 1\otimes x, \dots\}$

So $1\otimes x - 1\otimes x \neq 0$ (unique repns in this basis)

Let $\beta(p(x), q(x)) = p(x)q(x)$, this is bilinear so it extends to a homomorphism $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]$ that maps all nonzero simple tensors to nonzero elements.

⑤ $L \subseteq M$ but $M \neq L \oplus N$

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0$$

$2\mathbb{Z} \subseteq \mathbb{Z}$ but $\nexists\ \sigma : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

Torsion-free module which is not free.

$\mathbb{Q}$ over $\mathbb{Z}$: rank 1 bc any two elements are linearly indep, but no single element generates $\mathbb{Q}$.

($\mathbb{Q}$ is infinitely generated.)

(x,y) in $F[x,y]$: generated by no less than $\{x,y\}$ but has rank 1: $xy - yx = 0$.

R - commutative

I - max'l ideal

M/IM is an R/I-vector space

$M/IM \cong (R/I) \otimes M$ (extension of scalars)

If $M = R^n$, then $(R/I) \otimes R^n = (R/I)^n$ - n.dim v.s.

Since dim is uniquely defined, so is rank.

R - integral domain

F - its field of fractions.

rank $M = n$ if $0 \longrightarrow R^n \longrightarrow M \longrightarrow \underbrace{M/R^n}_{\text{torsion module}} \longrightarrow 0$ is exact

then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^n \otimes F & \longrightarrow & M \otimes F & \longrightarrow & M/R^n \otimes F & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & F^n & \longrightarrow & M \otimes F & \longrightarrow & 0 & & \end{array}$$

is exact.

So $M \otimes F \cong F^n$

$$\left(\oplus M_\alpha\right) \otimes N \cong \oplus \left(M_\alpha \otimes N\right)$$

$$((u_\alpha, \alpha \in \Lambda), v) \longmapsto (u_\alpha \otimes v, \alpha \in \Lambda)$$

homsm : $(u_\alpha)_\alpha \otimes v \longmapsto (u_\alpha \otimes v)_\alpha$

$$\begin{tikzcd}
\varphi_\alpha : M_\alpha \otimes N \arrow[r] \arrow[d, Rightarrow] & K \quad \forall \alpha \\
\varphi : \bigoplus (M_\alpha \otimes N) \arrow[ur] &
\end{tikzcd}$$

where $\varphi(u_\alpha \otimes v_\alpha,\ \alpha \in \Lambda) = \sum \varphi_\alpha(u_\alpha \otimes v_\alpha)$

$$\varphi_\alpha : M_\alpha \otimes N \longrightarrow \left(\bigoplus M_\alpha\right) \otimes N$$

$$u_\alpha \otimes v \longmapsto (u_\alpha \otimes v,\ 0 \text{ for } \beta \neq \alpha)$$

$$\Downarrow$$

$$\varphi : \bigoplus (M_\alpha \otimes N) \longrightarrow \left(\bigoplus M_\alpha\right) \otimes N,$$

This is inverse of other map

But $\left(\prod M_\alpha\right) \otimes N \neq \prod (M_\alpha \otimes N)$