16.1 5 M: R-module
I: left ideal in R.

Then $IM = \{a_1u_1 + \cdots + a_nu_n : n \in \mathbb{N}, a_i \in I, u_i \in M\}$.

<u>Claim</u>: IM is a submodule of M.

<u>Example</u>: A: abelian group (a $\mathbb{Z}$-module)

then $2A$ is a subgroup

7 If $N_1 \subseteq N_2 \subseteq \cdots$: submodules of M

then $\bigcup_{i=1}^{\infty} N_i$ is a submodule of M.

$\mathrm{Tor}(M) \neq 0$

8c If R has zero divisors then any R-module M has nonzero torsion.

pick $u \in M \setminus 0$.

$ab = 0 \implies a \cdot (bu) = 0$. $bu \in \mathrm{Tor}(M)$. if $bu = 0$, $u \in \mathrm{Tor}(M)$.

18 Let $V = \mathbb{R}^2$. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. V is an $\mathbb{R}[x]$-module by $xu = T(u)$.

prove V is a simple module.

$\forall\ u \in V \setminus 0$, $\mathrm{Span}\{u, T(u)\} = V$ so V & 0 are the only T-invariant subspaces.

19 $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$. V is not simple. 0, V, $\left\{\begin{pmatrix} 0 \\ y \end{pmatrix}\right\}$, and $\left\{\begin{pmatrix} x \\ 0 \end{pmatrix}\right\}$ are all $\mathbb{R}[x]$-submodules.

(20) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$. Any $\mathbb{R}$-subspace is an $\mathbb{R}[x]$-submodule.

because action of T is multiplication by a scaler.

(22) R: comm-ring

<u>Def 1</u> A is an R-algebra if A is an R-module & a ring

& $a(\alpha\beta) = \alpha(a\beta) = (a\alpha)\beta \quad \forall a \in R, \ \alpha, \beta \in A$.

<u>Def 2</u> A is an R-algebra if A is a ring, $1_R \in R$, $1_A \in A$, and

a ring hom-sm $f: R \to A$ is defined s.t. $f(1_R) = 1_A$ and

$f(R) \subseteq \text{Center}(A)$.

Let $1_R \in R$, $1_A \in A$

<u>Def 2 ⇒ Def 1</u>: Define $a \cdot \alpha = f(a)\alpha$.

Then $a(\alpha\beta) = f(a)(\alpha\beta) = (f(a)\alpha)\beta = (\alpha f(a))\beta$ since $f(a) \in \text{Center}(A)$.

<u>Def 1 ⇒ Def 2</u>: Define $f: R \to A$ by $f(a) = a \cdot 1_A$ for $a \in R$.

Then f is a ring hom & $f(1_R) = 1_A$.

And $\forall a \in R$, any $\alpha, \beta \in A$, $(a \cdot 1_A)\alpha = a \cdot (1_A \alpha) = a \cdot (\alpha 1_A) = \alpha(a \cdot 1_A)$.

So $f(a) \in \text{Center}(A)$.

<u>10.2.8</u> $\varphi: M \to N$: R-module hom-sm $(\varphi \in \text{Hom}_R(M, N))$

Then $\varphi(\text{Tor}(M)) \subseteq \text{Tor}(N)$.

<u>proof</u> Let $u \in \text{Tor}(M)$. Let $a \in R \setminus 0$ s.t. $a \cdot u = 0$.

$a\varphi(u) = \varphi(au) = \varphi(0) = 0$.

<u>10.2.7</u> $z \in$ Center of R. M: R-module.

R-module

then $\varphi_z : M \to M$ defined by $\varphi_z(u) = zu$ is an endomorphism of M.

Proof $z(u+v) = zu + zv$. $z(au) = a(zu)$.

Let R be commutative. Then $z \longmapsto \varphi_z$ is a ring hom-sm $R \to End_R(M)$.

$$\varphi_{z_1 z_2} = \varphi_{z_1} \circ \varphi_{z_2}.$$

10.2.11 Let $A_1, \dots, A_k$ be R-modules. $\forall i$ let B_i be a submodule of A_i.

Then $B_1 \times \dots \times B_k$ is a submodule of $A_1 \times \dots \times A_k$, and

$$(A_1 \times \dots \times A_k) / (B_1 \times \dots \times B_k) \cong A_1/B_1 \times \dots \times A_k/B_k.$$

Pf define $\varphi : A_1 \times \dots \times A_k \longrightarrow A_1/B_1 \times \dots \times A_k/B_k$

$$(a_1, \dots, a_k) \longrightarrow (a_1 \bmod B_1, \dots, a_k \bmod B_k).$$

it's a hom, it's surjective & $Ker(\varphi) = B_1 \times \dots \times B_k$.

⋮

(12) I = ideal in R. Then $M^n/(IM^n) \cong (M/(IM))^n$.

$$IM^n = (IM)^n.$$

10.3.5 $M = Tor(M)$ & M is finitely generated & R integral domain $\Rightarrow Ann(M) \neq 0$.

Let $M = R\{u_1, \dots, u_n\}$, let $a_i u_i = 0$ with $a_i \neq 0$.

$Ann(M) \ni a_1 \cdots a_n$.