Lec 11/28

Tuesday, November 28, 2017 10:47

Picard's Approximation Method: (to solve integral eqn $\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt$)

$$\varphi_0(x) = y_0$$

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$\varphi_2(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_1(t))\, dt$$

$$\vdots$$

$$\varphi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_n(t))\, dt$$

Problem: $\varphi_1(t)$ may not be in $\overset{[x_0-a,\, x_0+a] \times [y_0-b,\, y_0+b]}{\overset{\shortparallel}{D}}$ (where $f: D \to \mathbb{R}$.)

$$M = \sup_{(x,y)\in D} |f(x,y)|$$

but $\left|\varphi_1(t) - y_0\right| \le \left|\int_{x_0}^{y} |f(t, y_0)|\,dt\right| \le M|x-x_0| \le b.$

So we must take a new domain:

$$\left\{ |x-x_0| \le \min\left\{a, \tfrac{b}{M}\right\},\ |y-y_0| \le b \right\}$$

by induction, $\forall n$, $|\varphi_n - y_0| \le M|x-x_0|$.

Suppose true for φ_{n-1}.

Then same argument goes thru

$$\left|\varphi_{n+1}(x) - \varphi_n(x)\right| \le \int^{x} |f(t, \varphi_n(t)) - f(t, \varphi_{n-1}(t))|\, dt$$

$$\left| \varphi_{n+1}(x) - \varphi_n(x) \right| \leq \int_{x_0}^{x} \left| f(t, \varphi_n(t)) - f(t, \varphi_{n-1}(t)) \right| dt$$

Lipschitz Condition on f:

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

$$\forall \; x, y_1, y_2$$

Sufficient to check

$$\sup_{(x,y) \in D} \left| \frac{\partial f}{\partial y}(x,y) \right| \leq K \neq \infty$$

(if f has Lipschitz)

$$\leq K \int_{x_0}^{x} |\varphi_n(t) - \varphi_{n-1}(t)| \, dt$$

$$\leq MK \int_{x_0}^{x} |t - x_0| \, dt$$

$$= MK \left| \int_0^{x-x_0} u \, du \right|$$

$$= \frac{MK |x - x_0|^2}{2}$$

So guess:

$$|\varphi_n(x) - \varphi_{n-1}(x)| \leq M \frac{K^{n-1} |x - x_0|^n}{n!} \longrightarrow 0 \text{ as } n \to \infty$$

fast

so $\varphi_n \to \varphi$.

Since $\varphi_0 + (\varphi_1(x) - \varphi_0(x)) + (\varphi_2(x) - \varphi_1(x)) + \cdots$

$\longrightarrow$ limit if it exists of φ_n

and it exists by comparison test,