Lec 11/3

Thursday, November 3, 2016 9:09 AM

Definition If $f: I \to \mathbb{R}$, we say that $F: I \to \mathbb{R}$ is an antiderivative of f (interval)

if F is continuous on I and $F'(x) = f(x)$ $\forall x \in$ interior of I.

Notation: $F = \int f$ unique up to constant.

FTCv2: If F is an antiderivative of a function f integrable over $[a,b]$,

then $\int_a^b f = F(b) - F(a)$.

Example / Misapplication of FTC:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} = \frac{-1}{x}$$

so $$\int_{-1}^{1} \frac{1}{x^2}\,dx = \frac{-1}{1} - \frac{-1}{-1} = -2$$

but $\frac{1}{x^2}$ not defined at $0 \Rightarrow \frac{1}{x}$ <u>not</u> an antiderivative of $\frac{1}{x^2}$ on $[-1,1]$

Substitution Rule for Antiderivatives: Reversing the chain rule.

$$\left(F(g(x))\right)' = f(g(x))\,g'(x) \qquad (\text{where } F \text{ is an antiderivative for } f)$$

so $$\int f(g(x))\,g'(x)\,dx = F(g(x))$$

given $\int h(x)\,dx$, find a factorization $h(x) = f(g(x))g'(x)$

Substitution rule = algebraic formalism for easing this process.

guess $u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \underset{?}{\Rightarrow} \frac{du}{g'(x)} = dx$

$$\int \tilde{h}(x,u)\,\frac{du}{g(x)} = \int f(u)\,du = F(u) = F(g(x))$$

Sometimes is tricky:

$$\int x\sqrt{2x+1}\,dx \qquad u = 2x+1 \qquad \frac{du}{dx} = 2 \qquad \frac{du}{2} = dx$$

$$= \int x \sqrt{u} \, \frac{du}{2} \qquad x = \frac{u-1}{2}$$

$$= \int \frac{u-1}{2} \sqrt{u} \, \frac{du}{2}$$

$$= \int \frac{u^{3/2} - u^{1/2}}{4} \, du$$

$$\int \sqrt{x^2 + x^4} \, dx \qquad u = 1 + x^2, \quad \frac{du}{2x} = dx$$

$$= \int |x| \sqrt{1 + x^2} \, dx$$

$$= \begin{cases} \int x \sqrt{1+x^2} \, dx & \text{if } x \geq 0 \quad = \frac{1}{2} \int \sqrt{u} \, du \quad = \frac{1}{3}(1+x^2)^{3/2} \\ -\int x \sqrt{1+x^2} \, dx & \text{if } x < 0 \quad = -\frac{1}{2} \int \sqrt{u} \, du \end{cases}$$

$$\text{so} \quad \int \sqrt{x^2 + x^4} \, dx = \begin{cases} \frac{1}{3}(1+x^2)^{3/2} + C & \text{if } x \geq 0 \\ -\frac{1}{3}(1+x^2)^{3/2} + C & \text{if } x < 0 \end{cases}$$

Not an antiderivative on $\mathbb{R}$: discontinuous at 0.

$$F(x) = \int \sqrt{x^2 + x^4} \, dx = \begin{cases} \frac{1}{3}(1+x^2)^{3/2} + C & \text{if } x \geq 0 \\ -\frac{1}{3}(1+x^2)^{3/2} + \frac{2}{3} + C & \text{if } x < 0 \end{cases}$$

To check that this is really the antiderivative on $(-\infty, \infty)$,

it's clear that it's an antiderivative on $(-\infty, 0)$, $(0, \infty)$

Check at 0.

$$\lim_{x \to 0^-} F'(x) = \lim_{x \to 0^-} \left(-\frac{1}{3}(1+x^2)^{3/2} + \frac{2}{3} + C \right)' = \lim_{x \to 0^-} \left(-\frac{1}{2}(1+x^2)^{1/2} 2x \right) = 0$$

$$\lim_{x \to 0^+} F'(x) = 0 \quad \text{too.} \Rightarrow \lim_{x \to 0} F'(x) = 0$$

So F is an antiderivative

$$\left(F'(0) = \lim_{x\to 0} \frac{F(x) - \frac{1}{3}}{x} \overset{L'H}{=} \lim_{x\to 0} F'(x) = 0 = f(0) \right)$$

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \sqrt{1+u^8}\, du \right) = \frac{d}{dx}\left(F(u)\Big|_{u=x^2}^{u=x^3} \right) = \frac{d}{dx}\left(F(x^3) - F(x^2) \right)$$

Let $F(x) = \int \sqrt{1+x^8}\, dx$

$$= F'(x^3)\, 3x^2 - F'(x^2)\, 2x$$

$$= \sqrt{1+x^{3\cdot 8}}\; 3x^2 - \sqrt{1+x^{2\cdot 8}}\; 2x$$

$$= 3x^2\sqrt{1+x^{24}} - 2x\sqrt{1+x^{16}}$$

Ch 18 Logs & Exps

$x^a \neq a^x$

if $x = \frac{p}{q}$, define $a^x = \left(\sqrt[q]{a}\right)^p$

if $a < 0$ then $a^{p/q}$ is not well defined $\forall q$.

$$(-8)^{1/3} = \sqrt[3]{-8} = -2$$

$$= (-8)^{2/6} = \left(\sqrt[6]{-8}\right)^2 \text{ DNE}$$

we exclude $a < 0$. $a = 0 \Rightarrow a^r = 0$ if $r > 0$

undef. if $r \leq 0$

$a = 1 \Rightarrow a^r = 1$

Hence we only consider bases $a \in (0,1) \cup (1,\infty)$

how to define a^x if x irrational?

obvious soln: $a^x = \text{"}\lim_{r \to x}\text{"}\, a^r$ for $r \in \mathbb{Q}$ but gruesome to work with.

To get around these, define inverse: the logarithm:

we say that $y = \log_a x$ if $a^y = x$

History: $\log_2 3 \iff$ solve $2^y = 3$

$1.1^7 \approx 2$ $\Rightarrow 3 = 2^y \approx (1.1^7)^y \approx 1.1^{11}$

$1.1^{11} \approx 3$

$7y \approx 11$

$y \approx \frac{11}{7}$

$1.001^{693} \approx 2$

$1.001^{1009} \approx 3$ $\Rightarrow$ $y \approx \frac{1009}{693}$

look: $\ln 2 \approx 0.693$

$\ln 3 \approx 1.009$

$\log_e 2$ $(e \approx 2.7)$

$1.001^{1000} \approx e$

$y \approx \frac{693}{1000}$

$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ $n = 1000 \Rightarrow 1.001^{1000} \approx e$