Lec 11/1

Tuesday, November 1, 2016 9:10 AM

Integration

$$\int_a^b f = \int_a^b f(x)\,dx$$

Definition: A partition P of a finite closed interval $[a,b]$ is a finite subset of $[a,b]$ which includes a, b. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

Definition: Suppose f is bounded on $[a,b]$ ($|f(x)| \le B$ for some B and all $x \in [a,b]$) and let P be a partition of $[a,b]$. We define
$$U(f,P) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$
$$L(f,P) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$
where $M_i = \sup f([x_{i-1}, x_i])$ and $m_i = \inf f([x_{i-1}, x_i])$.

Note: P not required to be regular (equal width).
and $L(f,P) \le U(f,P)$ since $m_i \le M_i$

Lemma: If $P \subseteq Q$, then $L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$.

Proof: It suffices to prove this when $Q = P \cup \{z\}$ with $z \in (x_{i-1}, x_i)$

$U(f,Q)$ has the same terms as $U(f,P)$ except for $M_i(x_{i-1} - x_i)$ is replaced by $\underbrace{M_i'(z - x_{i-1}) + M_i''(x_i - z)}$. $M_i' = \sup f([x_{i-1}, z])$, $M_i'' = \sup f([z, x_i])$

$M_i' \le M$, $M_i'' \le M_i$, so this $< M_i(z - x_{i-1}) + M_i(x_i - z) = M_i(x_i - x_{i-1})$.

Similar reasoning for L. ∎

Corollary: For any two partitions, P, Q, of $[a,b]$, $L(f,P) \le U(f,Q)$

Proof: $L(f,P) \le L(f, P\cup Q) \le U(f, P \cup Q) \le U(f,Q)$ ∎

Definition:
$$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ a partition of } [a,b]\} \quad \text{upper integral}$$
$$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \quad \text{lower integral.}$$

$\underline{\int_a^b} f = \sup\{L(f,P) : P$ a partition of $[a,b]\}$ lower integral.

Note: $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

f is integrable over $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f =: \int_a^b f$

Example of a non-integrable function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$$0 = \underline{\int_0^1} f < \overline{\int_0^1} f = 1$$

Proof: Let P be any partition of $[0,1]$. $M_i = 1$ for any $[x_{i-1}, x_i]$, $m_i = 0$.

Lemma (useful): f is integrable over $[a,b]$ iff for any $\varepsilon > 0$ we can find a partition P so that $U(f,P) - L(f,P) < \varepsilon$.

Proof: From the definition. $\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$

Since $\varepsilon > 0$ is arbitrary, $\Rightarrow$ $\overline{\int_a^b} f = \underline{\int_a^b} f$.

Conversely, if $\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f$ then we can find a partition P so that $\int_a^b f - L(f,P) < \varepsilon/2$ and another partition Q so that $U(f,Q) - \int_a^b f < \varepsilon/2$

so $U(f,Q) - L(f,P) < \varepsilon$ so $U(f, \underset{\overset{\shortparallel}{P_\varepsilon}}{P \cup Q}) - L(f, \underset{\overset{\shortparallel}{P_\varepsilon}}{P \cup Q}) < \varepsilon$ too. ∎

Theorem: If f is continuous over $[a,b]$ then f is integrable over $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Note that f is uniformly continuous. Hence there is a $\delta > 0$ s.t. $\forall x, y \in [a,b]$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$.

Pick a partition P so that $x_i - x_{i-1} < \delta$ for all i. (regular partition where $n > \frac{b-a}{\delta}$).

So $x_i = a + i\frac{b-a}{n}$. It follows that $M_i - m_i \leq \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a}$

So $U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \cdot \frac{b-a}{n} = \varepsilon$ ∎

So $U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \cdot \frac{b-a}{\delta} = \varepsilon$ ∎

Definition A subset $A \subseteq \mathbb{R}$ has content 0 if $\forall \varepsilon > 0$, $\exists$ a finite set of open intervals (may overlap) $\{(u_i, v_i)\}_{i=1}^{n}$ so that $A \subseteq \bigcup_{i=1}^{n} (a_i, b_i)$ and $\sum_{i=1}^{n} (b_i - a_i) < \varepsilon$.

Examples:

(1) any finite set. (enclose each point in a small interval)

(2) $A = \{\frac{1}{n}\}_{n=1}^{\infty}$. given $\varepsilon > 0$, find N s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$. Then part of $A \subseteq (0, \frac{1}{N})$, the rest is a finite set.

Theorem if the set of discontinuities of f over $[a,b]$ has content 0, f is integrable over $[a,b]$.

Proof sketch. Enclose the set D of discontinuities in a union of open intervals of total length $\frac{\varepsilon}{4B(b-a)}$, $|f(x)| \leq B$ $\forall x \in [a,b]$.

After consolidating these open intervals which overlap or have common endpoints, the intervals have length $< \frac{\varepsilon}{4B(b-a)}$

Then $[a,b] \setminus \bigcup_{i=1}^{n} (u_i, v_i) = \bigcup_{j=1}^{m} ([a_j, b_j])$, f is continuous on.

f is unif. cts. on that too.

Then can find a partition P_j of each interval so $U(f,P_j) - L(f,P_j) < \frac{\varepsilon}{2M}$

let $P = \bigcup_j P_j$, so $U(f,P) - L(f,P)$