Lec 9/15

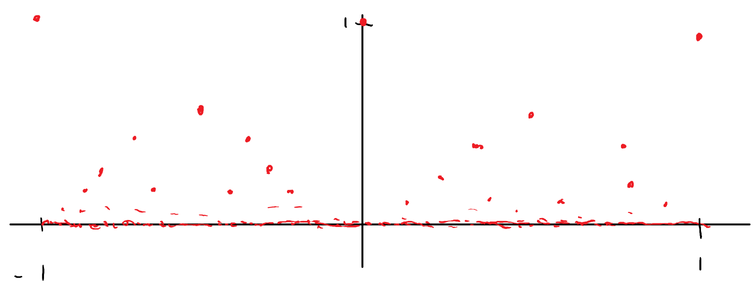
Thursday, September 15, 2016 9:09 AM

Midterm 1 - next friday

Covers: Ch 1-6 + Carmen lecture notes

Popcorn function:

$$pop(x) = \begin{cases} 1/q & \text{if } x \in \mathbb{Q},\ x = \frac{p}{q} \text{ in lowest terms} \quad (p, q \text{ coprime integers, } q > 0) \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$



Show that pop(x) is continuous at $x \notin \mathbb{Q}$ and discontinuous at $x \in \mathbb{Q}$

Informal argument: if $a \in \mathbb{Q}$, $a = \frac{p}{q}$, then there are irrational nums arbitrarily close to a, //so $J_{pop, a} = [0, \frac{1}{q}]$

So for any δ $\exists x \notin \mathbb{Q}$ so $|x-a| < \delta$ but $|pop(x) - pop(a)| = |0 - \frac{1}{q}| = \frac{1}{q} > \varepsilon = \frac{1}{q+1}$

so pop is discontinuous at rational nums.

OtOh, if $a \notin \mathbb{Q}$ and $|x-a| < \delta$ very small then either $x \notin \mathbb{Q}$ $(|x - pop(a)| = 0 < \varepsilon)$

or $x \in \mathbb{Q}$, $x = \frac{p}{q}$, $|pop(x) - pop(a)| = \frac{1}{q} < \varepsilon$

large — can make $\frac{1}{q} < \varepsilon$ for any ε.

Lemma 1: Suppose $a = \frac{p}{q}$, p, q coprime integers, $q > 0$ and $x = \frac{m}{n}$ m, n coprime, $0 < n < q$

Then $|x - a| > \frac{1}{q^2}$.

Proof: $|x - a| = \left|\frac{m}{n} - \frac{p}{q}\right| = \frac{|mq - pn|}{nq}$

$|mq-pn|$ a positive integer since $x \neq a$, so $1 \leq |mq-pn|$

otoh, $nq < q^2$ because $0 < n < q$

so $\dfrac{|mq-pn|}{nq} > \dfrac{1}{q^2}$ ∎

Lemma 2: Suppose $a \notin \mathbb{Q}$ and $x = \frac{m}{n}$ m, n coprime, $0 < n \leq q$ where q a fixed positive integer

Then there is some positive real number $\lambda_q > 0$ such that $|x - a| > \frac{\lambda_q}{q!}$

only depends on q

Proof: let $\lambda_q =$ the distance between $q!a$ and the closest integer.

there is a closest integer b/c $n!a$ irrational. (well-ordering principle).

$q!x = q!\frac{m}{n}$ is an integer since n is a factor of $q!$

$|q!x - q!a| \geqslant \lambda_q \Rightarrow |x - a| \geqslant \frac{\lambda_q}{q!}$ ∎

Theorem 1: If $a = \frac{p}{q}$ (usual business) and $0 < \delta \leq \frac{1}{q^2}$ then $J_{pop, a, \delta} = [0, \frac{1}{q}]$.

Consequently, the jump interval $J_{pop, a} = [0, \frac{1}{q}]$. Hence pop is discontinuous at a.

proof: $J_{pop, a, \delta} =$ the smallest closed interval containing $\{pop(x) : x \in [a-\delta, a+\delta]\}$.

first: $[a-\delta, a+\delta]$ contains a and it contains some irrational x_0, Hence $J_{pop, a, \delta}$ must contain $pop(x_0) = 0$ and $pop(a) = \frac{1}{q} \Rightarrow [0, \frac{1}{q}] \subseteq J_{pop, a, \delta}$

Conversely, $J_{pop, a, \delta}$ does not contain $pop(\frac{m}{n})$ with $0 < n < q$

since, by Lemma 1, $|\frac{m}{n} - \frac{p}{q}| > \frac{1}{q^2} \geqslant \delta$

$\Rightarrow J_{pop, a, \delta} = [0, \frac{1}{q}]$

Theorem 2: If $a \notin \mathbb{Q}$ and $0 < \delta < \frac{\lambda_q}{q!}$ (q is a fixed positive num.) Then $J_{pop, a, \delta} \subseteq [0, \frac{1}{q}]$

so $J_{pop, a} = \{0\}$ and pop is continuous at a.

proof: If $x = \frac{m}{n} \in [a-\delta, a+\delta]$ then $n \geqslant q$. for, by Lemma 2, otherwise $|x - a| > \frac{\lambda_q}{q!} > \delta$. $\Rightarrow J_{pop, a, \delta} \subseteq [0, \frac{1}{q}]$

$J_{pop, a} = \bigcap_{\delta} J_{pop, a, \delta} = \bigcap [0, \frac{1}{q}] = \{0\}$

$\cdot \qquad \tau>0$

so $p \circ p$ is continuous at a.