Lec 9/6

Tuesday, September 6, 2016 9:08 AM

more on functions

Operations on functions

1) addition, subtraction, mult & div

$f+g$ $\quad (f+g)(x) = f(x)+g(x) \quad$ $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$f-g$ $\quad (f-g)(x) = f(x)-g(x) \quad$ $\mathrm{dom}(f-g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$f \cdot g$ $\quad (f \cdot g)(x) = f(x) \cdot g(x) \quad$ $\mathrm{dom}(f \cdot g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

f/g $\quad (f/g)(x) = \dfrac{f(x)}{g(x)} \quad$ $\mathrm{dom}(f/g) = \mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) : g(x) \neq 0\}$

$f \circ g$ $\quad (f \circ g)(x) = f(g(x)) \quad$ $\mathrm{dom}(f \circ g) = \{x \in \mathrm{dom}(g) : g(x) \in \mathrm{dom}(f)\}$

↳ associative but not commutative

$f \circ (g \circ h) = (f \circ g) \circ h \qquad f(g(h(x)))$ for both

$f \circ g \neq g \circ f$ in general:

$f(x) := x^2 \quad g(x) := x+1. \qquad (f \circ g)(x) = f(x+1) = x^2+2x+1$ ≠

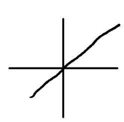
$\qquad (g \circ f)(x) = g(x^2) = x^2+1$

More operations:

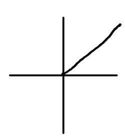
Restriction Suppose that $S \subseteq \mathrm{dom}(f)$

$$f|_S = \{(x, f(x)) : x \in S\}$$

Ex: $f(x) = x$:



$f|_{[0,\infty)}$:



Splicing if f and g are functions then $f \cup g$ is a function iff $f(x) = g(x) \ \forall x \in dom(f) \cap dom(g)$

iff $f|_{dom(f) \cap dom(g)} = g|_{dom(f) \cap dom(g)}$

Ex: $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$ note: $I(x) = x$

$f = I|_{[0,\infty)} \cup -I|_{(-\infty,0)} = \text{sqrt} \circ \text{square}$

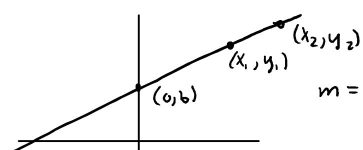
§4: More general graphs

Given an equation in two variables, x and y, its graph is the set of points in the plane w/ cartesian coordinates (x,y) satisfying the equation.

A function f is a graph of the equation $y = f(x)$

linear functions:

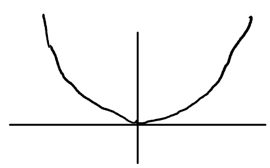
$y = mx + b$



$m = \frac{y_1 - y_2}{x_1 - x_2}$

quadratic graphs:

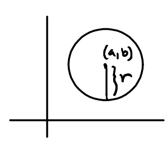
$y = x^2$



parabola

Circles:

$(x-a)^2 + (y-b)^2 = r$



more generally, graphs of non-degenerate quadratic equations can be classified as one of 3 geometric types:

1) parabolas ?

as one of 3 geometric types:

1) parabolas
2) ellipses
3) hyperbolas

$\left.\right\}$ Conic sections

Deriving the eqn of a hyperbola

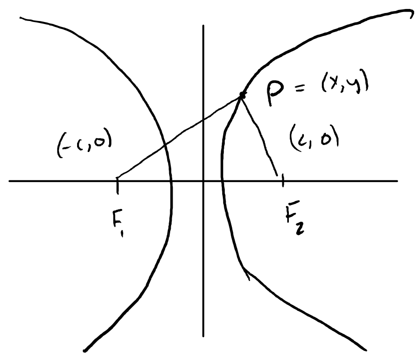
Geometric definition of a hyperbola:

set of points in the plane such that

$|$ difference of distances to two fixed points F_1 and $F_2| = \alpha$

where α is a fixed positive number

and F_1, F_2 are two fixed points (the foci)



$$|PF_1 - PF_2| = 2a$$

We will assume that $0 < a < c$

$$PF_1 = \sqrt{(x+c)^2 + (y-0)^2}$$

$$PF_2 = \sqrt{(x-c)^2 + (y-0)^2}$$

(1) $\left| \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} \right| = 2a$

(2) $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$

(3) $\sqrt{(x+c)^2 + y^2} = \pm 2a + \sqrt{(x-c)^2 + y^2}$

(4) $(x+c)^2+y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$

(5) $\cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2} + \cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2}$

(6) $\cancel{4}cx - \cancel{4}a^2 = \pm \cancel{4}a\sqrt{(x-c)^2+y^2}$

(7) $(cx-a^2)^2 = a^2\left((x-c)^2+y^2\right)$

(8) $c^2x^2 - \cancel{2a^2cx} + a^4 = a^2x^2 - \cancel{2a^2xc} + a^2c^2 + a^2y^2$

(9) $(c^2-a^2)x^2 - a^2y^2 = a^2c^2 - a^4 = a^2(c^2-a^2)$

(10) $b^2x^2 - a^2y^2 = a^2b^2$

(11) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\left\{ \begin{array}{l} c^2-a^2>0 \\ \text{let } b^2 = c^2-a^2 \end{array} \right.$

for homework:

$\left\{ \begin{array}{l} c^2-a^2=0 \\ \text{so } a^2y^2=0 \\ \text{so } y^2=0 \\ \text{so } y=0 \end{array} \right.$

$\left\{ \begin{array}{l} c^2-a^2<0 \\ \text{let } b^2=-(c^2-a^2) \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \end{array} \right.$

but are these steps reversible?

we have shown that (1) $\Rightarrow$ (11)

but does (11) $\Rightarrow$ (1)?

$A^2=B^2 \not\Rightarrow A=B$

we squared both sides at (7) and (4) (7) $\Rightarrow$ (6) ok because we have $\pm$

now need to show (4) $\Rightarrow$ (3) in order to show that (11) $\Rightarrow$ (1)

ie rule out:

(3') $\sqrt{(x+c)^2+y^2} = -\left(\pm 2a + \sqrt{(x-c)^2+y^2}\right)$

<u>Case 1</u>: $\pm 2a = 2a$

$\sqrt{(x+c)^2+y^2} = -\left(2a + \sqrt{(x-c)^2+y^2}\right)$

impossible because left ≥ 0, right < 0.

<u>Case 2</u>: $\pm 2a = -2a$

$\sqrt{(x+c)^2+y^2} = -\left(-2a + \sqrt{(x-c)^2+y^2}\right)$

$= 2a - \sqrt{(x-c)^2+y^2}$

$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

Why is this impossible?

Hint: triangle inequality in the plane (-a)