Lec 8/21

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Informal discussion on measures:

X - nonempty set.

A <u>measure</u> on X is a fn $\mu : \mathcal{M} \longrightarrow [0,\infty]$

$\mathcal{M} \subseteq \mathcal{P}(X)$

such that

(0): $\mu(\emptyset) = 0$

(1): $\mu(\coprod E_n) = \sum \mu(E_n)$ whenever (E_n) is a (countable) sequence of disjoint sets

disjoint union

Call μ finite if $\mu(X) < \infty$

What properties should $\mathcal{M}$ satisfy?

- $\emptyset, X \in \mathcal{M}$
- closed under countable disjoint unions

<u>Examples</u>:

countable = uncountable = ∞

(0) $\mathcal{M} = \mathcal{P}(X)$, $\mu(E) = |E|$

counting measure

(1) There is a measure λ on $\mathcal{M} \subset \mathcal{P}(\mathbb{R})$

s.t. all intervals are in $\mathcal{M}$, and

$$\lambda([0,1)) = 1$$

$$\lambda(E+r) = \lambda(E) \quad \forall\, r \in \mathbb{R},\ E \in \mathcal{M}$$

Under basic axioms of set theory (A.C),

$\mathcal{M}$ cannot be $\mathcal{P}(\mathbb{R})$!

- work mod 1 for convenience

$$[\quad) = S^1$$

Define an eq. rel'n on S^1 by

$$x \sim y \iff x - y \in \mathbb{Q}.$$

Using A.C, pick one representative

from each equivalence class; call this set E.

for $q \in \mathbb{Q} \cap S^1$, set $E_q := E + q$.

If $\mathcal{P}(X) = \mathcal{M}$ then $E_q \in \mathcal{M}$ $\forall q$.

Notice $[0,1) = \coprod_{q \in \mathbb{Q} \cap S^1} E_q$

$$1 = \lambda([0,1)) = \lambda\left(\coprod_q E_q\right) = \sum_q \lambda(E_q)$$

$$= \sum_q \lambda(E) = \lambda(E) \cdot \underbrace{\sum_q 1}_{=\infty} .$$

contradiction (nothing $\cdot \infty = 1$).

formal discussion:

X a non-empty set

"measurable sets"

Defn: call $\mathcal{M} \subset \mathcal{P}(X)$ an algebra if

⓪ $\mathcal{M} \neq \emptyset$

① $\mathcal{M}$ is closed under finite unions

② $\mathcal{M}$ is closed under complements $(E^c = X \setminus E)$.

Observe: Every algebra $\mathcal{M}$

- contains $X = E \amalg E^c$
- contains $\emptyset = X^c$
- is closed under finite intersections:

is true

$$\bigcap_1^n E_i = \Big[\bigcap_1^n E_i\Big]^{cc} = \Big[\bigcup_1^n E_i^c\Big]^c$$

If in addition $\mathcal{M}$ is closed under <u>countable</u> unions, call $\mathcal{M}$ a σ-algebra

<u>Exs</u>

⓪: $\{\emptyset, X\}$ trivial σ-algebra

①: $\mathcal{P}(X)$ discrete σ-algebra

②: if X is uncountable, $\mathcal{M} = \{E \mid E \text{ or } E^c \text{ is ctble}\}$ is a σ-algebra.

<u>Exercises</u>:

- [Disjointification] If $(E_n) \subset \mathcal{M}$ is a sequence of subsets of X, let $F_1 = E_1$, $F_n = E_n \setminus \bigcup_1^{n-1} E_i$. $\quad (A \setminus B = A \cap B^c)$

  Then $(F_n) \subset \mathcal{M}$ and $\bigcup E_n = \coprod F_n$.