Lec 8/28

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Order change : 5 → 7 → 6

Basis

$S \subseteq V/F$ is finitely generated if $S = S(\overbrace{a_1, \dots, a_n}^{\text{set of generators}})$ for $a_i \in V$.

Thm 1 Assume $S = S(a_1, \dots, a_n) \subseteq V$.

Let $b_1, b_2, \dots, b_m \in S$. If $m > n$, $\{b_1, \dots, b_m\}$ is linearly dependent.

Proof By induction on n.

$n = 1$, $S = S(a_1)$, $b_1 = \beta_1 a_1$, $b_2 = \beta_2 a_1, \dots$, $b_m = \beta_m a_1$, $m > 1$

if any $\beta_i = 0$ we are done, else $b_2 = \frac{\beta_2}{\beta_1} b_1$ so we are done.

assume for $1, 2, \dots, n$. Let $S = S(a_1, \dots, a_{n+1})$ and $m > n+1$. $b_i = \beta_{i1} a_1 + \dots + \beta_{i\,n+1} a_{n+1}$ for $i \in [m]$.

Assume not all b_{1j} are 0, say $b_{11} \neq 0$, then $\frac{1}{\beta_{11}} b_1 = a_1 + \lambda_{12} a_2 + \dots + \lambda_{1\,n+1} a_{n+1}$.

subtract $\frac{1}{\beta_{11}} b_1$ from all others, multiplied by β_{i1}.

so we get a set of $\overset{m-1}{\vee}$ vectors $(b_i - \frac{\beta_{i1}}{\beta_{11}} b_1)$ which are liner combos of $\overbrace{\{a_2, \dots, a_{n+1}\}}^{n \text{ vectors}}$.

but these are now linerly dependent by induction hypothesis, so

$$\underbrace{0 b_1 + \lambda_2 (b_2 - \frac{\beta_{21}}{\beta_{11}} b_1) + \dots + \lambda_m (b_m - \frac{\beta_{m1}}{\beta_{11}} b_1)}_{\text{liner combo of } b_i\text{s, so } b_i\text{s are lin. dep.}} = 0$$

$\frac{7}{13} = \frac{1}{1 + \frac{1}{\ }}$

for a rational number $\frac{p}{q}$ in least terms, $\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$ where $r_{-1} = \frac{p}{q}$

$a_i = \lfloor \frac{1}{r_{i-1}} \rfloor$, $r_i = \frac{1}{r_{i-1}} - \lfloor \frac{1}{r_{i-1}} \rfloor$

the r_is are rational and if $r_i = \frac{p_i}{q_i}$ then $r_{i+1} = \frac{q_i - n p_i}{p_i}$ so both p_i and q_i are strictly decreasing seqs. of integers meaning one will be 1 at some point so that r_j is an integer so $r_k = 0$, $k > j$.

Thm 1': If $\{b_1, \dots, b_m\}$ is lin. indp. Then $m \leq n$.

Cor 1: Assume $S = S(a_1, \dots, a_n) = S(b_1, \dots, b_m) \subseteq V/F$. If both $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ are lin. indp. then $n = m$.

Pf: $m \leq n$ and $n \leq m$.

Def ... erators of S

so $r_k = 0$, $k > j$.

Def: Assume $\{a_1, \ldots, a_n\}$ are $\begin{cases} \text{generators of } S \\ \text{lin. indp} \end{cases}$ then $\dim(S) = n$.

$\dim(\mathbb{R}^2) = 2$ since $\{e_1, e_2\}$ satisfy these.

In fact, any 2 noncolliner vectors generate $\mathbb{R}^2$.

$\mathcal{P}(\mathbb{R}) = \mathbb{R}[X]$ = polynomials w/ real coeffs. Not finitely generated since any finite set of polys has a higest degree.

$\mathcal{P}_n(\mathbb{R})$ is finitely generated with dimension n, generated by $\{1, x, x^2, \ldots, x^n\}$.

A basis of V is a generating set of size $\dim(V)$ if this is finite.

Let V/F have a basis $\{a_1, \ldots, a_n\}$

Then $\forall a \in V$, $\exists!\, \alpha_1, \ldots, a_n \in F$ s.t. $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$.

existence comes immediately, uniqueness comes via lin. indp.

Assume can write a as $a = \beta_1 a_1 + \cdots + \beta_n a_n$

So $(\alpha_1 - \beta_1) a_1 + \cdots + (a_n - \beta_n) a_n = 0 \Rightarrow \alpha_i - \beta = 0 \Rightarrow \alpha_i = \beta_i$.