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Friday, February 22, 2019 10:35

An abstract simplicial complex

is $X = (V, \Sigma)$ s.t.

$V$ is finite,

$(\{\cdot\} \; \langle\$\rangle \; V) \subseteq \Sigma$

$\Sigma \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and $\forall \sigma \in \Sigma$ if $\emptyset \neq \tau \subseteq \sigma$, $\tau \in \Sigma$.

The Standard abstract $n$-simplex is

$\Delta^n = (V, \Sigma)$ where $V = \{0, 1, \dots, n\}$ and $\Sigma = \mathcal{P}(V) \setminus \{0\}$.

$$\begin{tikzcd} \text{geometric} \arrow[r, shift left, "\text{easy}"] & \text{abstract} \arrow[l, shift left, bend left, "\underline{\text{propn}}."] \end{tikzcd} \text{ simplicial complexes.}$$

~~pf~~ let $X = (V, \Sigma)$ with $|V| = n+1$.

embed $V$ into $\mathbb{R}^{n+1}$ as vertices of the

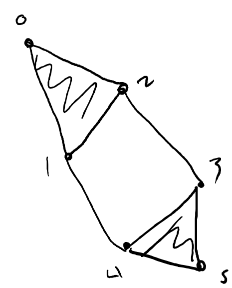
Standard geometric $n$-dim'l sp cpx.

Build $\mathcal{X}$ by including $\mathrm{Conv}(\sigma)$ $\forall \sigma \in \Sigma$. $\square$

<u>Ex</u>: $X = (V, \Sigma)$ where

$V = \{0, 1, 2, 3, 4, 5, 6\}$

$$\Sigma = \left\{ \begin{array}{l} 0, 1, 2, 3, 4, 5, \\ 01, 02, 12, 34, 35, 45, 23, 14, \\ 345, 012 \end{array} \right\}$$




<u>Def</u> Let $X$ & $Y$ be abstract simplicial complexes.

A simplicial map is a function

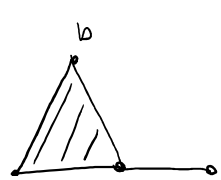
$$f : V(X) \longrightarrow V(Y)$$

s.t. $\forall \sigma \in \Sigma(X)$, $f(\sigma) \in \Sigma(Y)$.

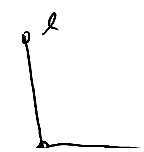
If $f$ is a bijective map & $f^{-1}$ is also simplicial, then $f$ is a simplicial isomorphism.

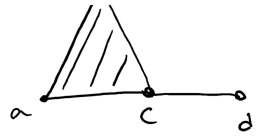
<u>Ex</u>:

$X$




$Y$



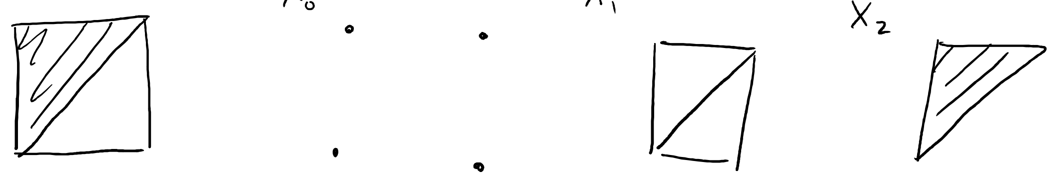




$$f = \begin{cases} a \mapsto l \\ b \mapsto l \\ c \mapsto m \\ d \mapsto n \end{cases}$$ is a simplicial map.

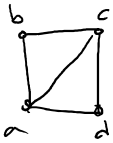
**Def**: for a geometric (resp abstract) simplicial complex $\mathcal{X}$ $(X)$, the k-skeleton of $\mathcal{X}$ (or $X$) is the collection of all k-simplices. denoted $\mathcal{X}_k$ or $X_k$.

eg $X$    $X_0$    $X_1$    $X_2$



**Goal** develop an algorithm to compute topological invariants of sp. cpx's

more precisely, define vector spcs $H_k(X)$ for $k = 0, 1, 2, \ldots$
s.t. if $H_k(X) \neq H_k(Y)$ for some $k$, then $X \approx Y$.

**Ex**: $\mathcal{X} =$ 


$$X = \begin{Bmatrix} a & b & c & d \\ ab & ac & ad & bc & cd \end{Bmatrix}$$

Consider 2 vector spaces:

$$C_0 = \mathrm{Span}_{\mathbb{F}_2}\{a, b, c, d\}$$

$$C_1 = \mathrm{Span}_{\mathbb{F}_2}\{ab, ac, ad, bc, cd\}$$

free vector spaces

$$C_n = 0 \quad \text{for} \quad n > 1.$$

<u>define</u> $\partial_1 : C_1 \longrightarrow C_0$

$$\partial_1(xy) = x + y$$

Then $\partial_1$ has matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} ab & ac & ad & bc & cd \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

find $\mathrm{Ker}\, \partial_1$ by row-reducing.

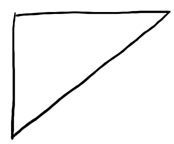
$$\downarrow$$

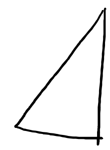
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$x_1 = x_2 + x_3 = x_4 + x_5 + x_5 = x_4$$
$$x_2 = x_3 + x_4 = x_5 + x_4$$
$$x_3 = x_5$$

$$\text{So } \operatorname{Ker} \partial_1 = \left\{ x_4 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_4, x_5 \in \mathbb{F}_2 \right\}$$

$ab + ac + bc$ $\qquad$ $ac + ad + cd$





$\operatorname{Ker} \partial_1$ is picking out loops in $X$.