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$$M = R^n / N$$

$$R^k = N \xrightarrow{\varphi} R^n \qquad \begin{pmatrix} | & | \end{pmatrix}$$

row ops change basis in R^n.

column ops change basis in N

$$M = R/(a), \qquad bM = b\left(R/(a)\right) = \left((b)+(a)\right)/(a)$$

$$c \mid b \Rightarrow cM / bM =$$

$\varphi : V \longrightarrow V$, $\varphi \in \mathrm{End}(V)$, $\dim(V) = n$,

$\exists$ basis s.t. $A_\varphi = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_m} \end{pmatrix}$ where $\forall i$, $A_i = \begin{pmatrix} 0 & & 0 & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \\ 0 & & 1 & -a_{d-1} \end{pmatrix}$

A_i is the companion matrix of the polynomial

$$x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = P_i$$

This is the rational normal form of φ and of the matrix of φ.

Note: $P_1 \mid P_2 \mid \cdots \mid P_m$ and P_i are uniquely defined

$\sum \deg p_i = n$. The rational normal form is unique.

$\forall$ $n \times n$ matrix A, $\exists$ invertible P s.t. PAP^{-1} has rat'l normal form.

2 matrices are conjugate <u>iff</u> they have the same rat'l normal form.

p_i are invariant factors of φ or of A.

Also, $\exists$ a form of matrix $\begin{pmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_k} \end{pmatrix}$ s.t.

B_i is companion matrix of $q_i^{r_i}$ where

q_i are irreducible.

This is uniquely defined up to swapping blocks.

p_m is the minimal polynomial of φ.

$V \cong F[X]^n / \text{Relns Module}$

Let $\{u_1, \dots, u_n\}$ be a basis in V.

it's also a set of generators of V

as an $F[X]$-module. But there are rel$^{\underline{ns}}$:

$$x u_1 = \varphi(u_1) = \begin{pmatrix} \vdots \end{pmatrix} = a_{11}u_1 + \cdots + a_{n1}u_n$$

The relation colum is $\begin{pmatrix} x - a_{11} \\ -a_{21} \\ \vdots \\ -a_{n1} \end{pmatrix}$

$$x u_2 = a_{12}u_1 + \cdots + a_{n2}u_n \rightsquigarrow \begin{pmatrix} -a_{12} \\ x - a_{22} \\ \vdots \\ -a_{n2} \end{pmatrix}$$

So we get a relation matrix

$$\begin{pmatrix} x - a_{11} & -a_{12} & & -a_{1n} \\ -a_{21} & x - a_{22} & \ddots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & & x - a_{nn} \end{pmatrix} = xI - A$$

matrix of φ in basis $\{u_1, \dots, u_n\}$.

Why is it complete rel$^{\underline{n}}$s matrix?

Let K be the submodule of $F[x]^n$ genrd by

$$\begin{pmatrix} x - a_{11} \\ \vdots \\ -a_{n1} \end{pmatrix} \cdots \begin{pmatrix} -a_{1n} \\ \vdots \\ x - a_{nn} \end{pmatrix}. \quad \text{Let } M = F[x]^n / K.$$

Then, Since all these relns hold in V, we have an epimorphism $M \to V \to 0$.

If $\dim_F(M) = n$, then this is an isomorphism

So $N = K$.

In M, $\begin{pmatrix} \bar{x} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \bar{x} \end{pmatrix} = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$

$$\begin{pmatrix} x^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = x\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} xa_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ xa_{n1} \end{pmatrix} = a_{11}\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + a_{n1}\begin{pmatrix} a_{1\cdot} \\ \vdots \\ a_{n1} \end{pmatrix}$$
$$= \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

So we can forget about x's,
all are equal to some column of
field elements,
so $\dim(M) \le n$, but this
means $M \cong V$.

So $xI - A$ is the complete rel^{n}s matrix of V
as an $F[x]$-module.

Reduce it to $\begin{pmatrix} P_1(x) & & 0 \\ & \ddots & \end{pmatrix}$ $P_i \in F[x]$, $P_1 | P_2 | \cdots | P_n$

Reduce it to $\begin{pmatrix} P_1(x) & & 0 \\ & \ddots & \\ 0 & & P_n(x) \end{pmatrix}$ $P_i \in F[x]$, $\underbrace{P_1 | P_2 | \cdots | P_n}_{\text{non-constant ones are invariant factors.}}$

V has no free component, so $P_i \neq 0 \ \forall i$

$$V = F[x]/(P_1) \oplus \cdots \oplus F[x]/(P_n).$$

$\begin{pmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_n \end{pmatrix}$ is the <u>Smith normal form</u> of φ.

$\det(xI - A) = \pm \det(\text{Smith normal form}) = \pm P_1(x) \cdots P_n(x).$

It is called the characteristic polynomial of φ.

$C_\varphi(x) = \prod$ inv factors P_i of φ.

In particular, $P_m = m_\varphi$ divides C_φ, so $C_\varphi(\varphi) = 0$.

(Hamilton-Cayley Theorem : $C_\varphi(\varphi) = 0$).