$\varphi \in End(V)$, $\dim_F V = n$

$\Rightarrow$ $p_1 \mid p_2 \mid \cdots \mid p_n$ - invariant factors of φ

P_m - minimal polynomial

Characteristic polynomial $C_\varphi = P_1 P_2 \cdots P_m$.

(all polynomials are assumed to be monic).

$C_\varphi(\varphi) = 0$ since $P_m \mid C_\varphi$.

on the other hand, $C_\varphi \mid P_m \cdot P_m \cdot \dots \cdot P_m = P_m^m$

So C_φ and $P_m = m_\varphi$ have same irreducible factors.

$n \times n$ matrix $A \Rightarrow$ we have $p_1 \mid P_2 \mid \cdots \mid P_m$, ...

A & B are conjugate iff they have the same invariant factors.

So they have the same rational normal form.

$xI - A \longmapsto \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & P_m \end{pmatrix}$ - remove units, get inv. factors.

Problem.

$A, B, C \qquad c_A = c_B = c_C = (x-2)^2(x-3).$

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$$

Elementary divisors:

Two options: $x-2,\ x-2,\ x-3$

$(x-2)^2,\ x-3$

$\Rightarrow$

Invariant factors

$\underbrace{(x-2)(x-3)}_{m_A = P_2},\ \underbrace{x-2}_{P_1} \Rightarrow P_1 \mid P_2 \qquad (*)$

$(x-2)^2(x-3) = m_A = c_A \qquad (**)$

Rational Normal forms:

for $(*)$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$

for $(**)$ $\begin{pmatrix} 0 & 0 & 12 \\ 1 & 0 & -16 \\ 0 & 1 & 7 \end{pmatrix}$

"Elementary divisors" form

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}$

up to permuting blocks

12.2 (2) if $A = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_k} \end{pmatrix}$ then $m_A = \operatorname{lcm}(m_{A_i})$

$C_A = \prod_{i=1}^{k} C_{A_i}$ ← obvious

$V = V_1 \oplus \cdots \oplus V_k$

to kill A completely, you must kill each component.

$P(A) = 0$ iff $m_i \mid P \ \forall i.$

③ non-scalar 2x2 matrices are conjugate iff they have the same characteristic polynomial

Invariant factors:

P_1, P_2 of degree 1, $\underline{P_1 \mid P_2 \Rightarrow P_1 = P_2}$

Scaler case $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

or P_1 of degree 2.

in 2^{nd} case 2 matrices are conj iff they

have the same $P_1 = C_\varphi$.

④ 3x3 matrices are similar iff they have the

same minimal & char. polynomial.

$$C_A = C_B, \quad m_A = m_B.$$

Cases: (1) P, P, P $\deg P = 1$ then $\begin{pmatrix} a & & 0 \\ & a & \\ 0 & & a \end{pmatrix}$ $m_A = x - a$

$C_A = (x-a)^3$

(2) P_1, P_2 $\deg P_1 = 1, \deg P_2 = 2$ $C_A = P_1 P_2$ $m_A = P_2$

(3) $P = C_A = m_A$ $\deg P = 3$.

for 4x4 we can have $m_A = m_B$, $C_A = C_B$ but $A \nsim B$.

$m_A = (x-2)^2$

2 classes: $x-2, x-2, (x-2)^2$ or $(x-2)^2, (x-2)^2$.

⑧ A is companion of $p \Rightarrow C_A = p$.

⑩ Similarity Classes of 6x6 matrices over $\mathbb{Q}$ w/ minimal polynomial $(x+2)^2(x-1)$

There are (probably) 6 of them

⑫ Similarity classes of 3x3 matrices A satisfying $A^6 = 1 \Rightarrow m_A \mid x^6 - 1$.

$x^6 - 1 = (x-1)(x^2+x+1)(x+1)(x^2-x+1)$ over $\mathbb{F}_2 = \mathbb{Z}_2$.