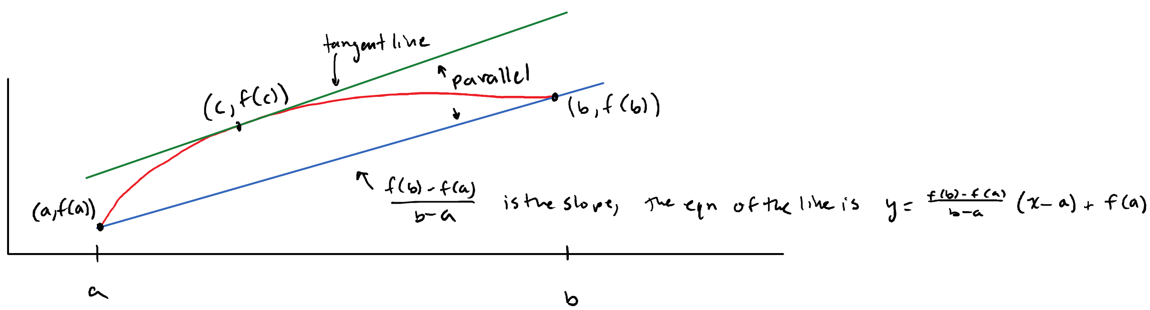
Lec 10/11

Tuesday, October 11, 2016 9:05 AM

Mean Value Theorem and Applications. (MVT)

Theorem: Suppose that $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $f'(x)$ is defined $\forall x \in (a,b)$. Then for some $c \in (a,b)$, (not necessarily unique) we have $f'(c) = \frac{f(b)-f(a)}{b-a} \iff f(b)-f(a) = f(c)(b-a)$



* MVT is important for proving FTC. It shows that antiderivatives are unique up to a constant.

Corollary of MVT: If $f'(x) = 0$ for all x in the interior of an interval I and f is continuous on I then f is constant on I.

Proof: Choose any $a < b \in I$. Then f is continuous on $[a,b]$ and $f'(x) = 0$ on (a,b) so it follows from MVT that $f(b) - f(a) = 0 \Rightarrow f(b) = f(a)$ for all $a < b \in I$. ■

Physical Interpretation: if velocity is always 0, then you don't move.

Corollary² of MVT: Antiderivatives over an interval are unique up to a constant.

Proof: suppose $F(x), G(x)$ are both antiderivatives for f defined on an interval I. Then $F'(x) = f(x) = G'(x)$ $\forall x \in I$, so $(F-G)'(x) = 0$ $\forall x \in I$ so $F - G$ is constant. ■

Special Case of MVT: Rolle's Theorem: Suppose the hypothesis of MVT and that $f(a)=f(b)$. Then $f'(c)=0$ for some $c \in (a,b)$.

Proof: By EVT, f has a max & min on $[a,b]$. Each max or min must occur at either (1) a critical point so $f'(c)=0$ or (2) a singular point so $f'(c)$ DNE (ruled out by hypothesis) or (3) an endpoint a or b. If both max and min occur at endpoints, then the function is constant bc. $f(a) \le f(x) \le f(b) = f(a)$. So, $f'(c)=0$ for any $c \in (a,b)$. ∎

Proof of MVT: Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$ the function of the secant line.

Because f is cts on $[a,b]$ and a linear function is cts on $\mathbb{R}$. g is cts on $[a,b]$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, defined $\forall x \in [a,b]$.

$g(a) = f(a) - f(a) - 0 = 0$. $g(b) = f(b) - f(a) - [f(b) - f(a)] = 0$.

So by rolle's theorem, $g'(c) = 0$ for some c. Therefore,

$f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$ ∎

Applications of MVT to graphing functions

Definition: We say that f defined on some interval I is increasing on I if $\forall x_1 < x_2 \in I$, $f(x_1) < f(x_2)$. Similar definition for decreasing.

Corollary of MVT: if f is continuous on $[a,b]$ and $f'(x) > 0$ $\forall x \in (a,b)$ then f is increasing on $[a,b]$. If $f'(x) < 0$ $\forall x \in (a,b)$ then f is decreasing on $[a,b]$.

Proof: Let $x_1 < x_2$ be in $[a,b]$. Then by MVT on $[x_1, x_2]$, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. If $f'(c) > 0$, then $f(x_2) > f(x_1)$ and if $f'(c) < 0$ then $f(x_2) < f(x_1)$. ∎

Definition: We

Definition: We say f has a local/relative maximum at a if for some $\delta > 0$, f has a maximum over $(a-\delta, a+\delta)$ at a. Similarly for minimum.

Corollary^2 of MVT (first derivative test): If for some $\delta > 0$, $f'(x) > 0$ for $x \in (a-\delta, a)$ and $f'(x) < 0$ for $x \in (a, a+\delta)$ (derivative needn't be defined at a) Then f has a local max at a. Similarly for minimum.

Recall that for $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, $f'(0) = 0$ but $\lim_{x \to 0} f'(x)$ dne. However:

Corollary of MVT If $\lim_{x \to a} f'(x)$ exists and f cts at a, then $\lim_{x \to a} f'(x) = f'(a)$

Proof: $\lim_{x \to a} f'(x)$ exists & f cts at a $\Rightarrow$ f is cts on $[a-\delta, a+\delta]$ for some $\delta > 0$. and $f'(x)$ is defined on $(a-\delta) \cup (a+\delta)$. Let $x \in (a, a+\delta)$. Then $\frac{f(x) - f(a)}{x-a} = f'(c)$ for some $c \in (a, x) \subseteq (a, a+\delta)$. So $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{c \to a^+} f'(c)$ which exists.

Similar argument shows $\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{c \to a^-} f'(c) \ldots$