Lec 9/28

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Composition Series

$\Sigma: G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$

n ← length of Σ

Strict if $G_i \supsetneq G_{i+1}$

$gr_i^{\Sigma}(G) = G_i / G_{i+1}$ graded pieces.

$\underset{\text{equivalent}}{\Sigma \sim \Sigma'}$ if lengths are $=$ & graded pieces are $=$ up to permutation.

Σ' is finer than Σ if $\Sigma \leftarrow \Sigma'$ by removing some elts.

Theorem: $\underbrace{\text{Jordan – Hölder}}_{\text{maximal strict sequences}}$ series exist for finite groups

Σ is J-H iff $gr_i^{\Sigma}(G)$ is simple $\forall i$.

uses: $\pi : H \xrightarrow[\text{projection}]{\text{natural}} H/N$ sets up a bijection

$$\left\{ \begin{array}{c} \text{Normal subgps of } H \\ \text{containing } N \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Normal subgps} \\ \text{of } H/N \end{array} \right\}.$$

Theorem (Schrier): Given two composition series Σ_1 & Σ_2 of G, there exist Σ_1', Σ_2' s.t.

(i) Σ_1' is finer than Σ_1

(ii) Σ_2' is finer than Σ_2

(iii) $\Sigma_1' \sim \Sigma_2'$.

This implies uniqueness of Jordan-Hölder series:

If Σ_1 & Σ_2 are two J-H series,

$\exists$ Σ_1' finer than Σ_1 (Σ_i' contain repeats)
s Σ_2' finer than Σ_2

but nontrivial graded pieces of Σ_i' are exactly the gp of Σ_i

so $\Sigma_1 \sim \Sigma_2$. $\square$

Pf of Schrier: define $\mathcal{L}_{ij} = (H_i \cap K_j) \cdot H_{i+1}$

to show $\mathcal{L}_{i,j} \trianglerighteq \mathcal{L}_{i,j+1}$:

$$\begin{array}{ccc} & H_i & \trianglerighteq \; H_{i+1} \\ & \rotatebox{-45}{$\geq$} & \\ & H_i \cap K_j & \\ \text{Since } K_j \trianglerighteq K_{j+1} \longrightarrow & \rotatebox{-90}{$\trianglerighteq$} & \\ & H_i \cap K_{j+1} & \end{array}$$

Ex: $\mathcal{G} \geq H_1 \trianglerighteq H_2 \implies H_2 N \trianglelefteq H_1 N$
$\rotatebox{-90}{$\trianglerighteq$}$
N

So $\mathcal{L}_{i,j} \trianglerighteq \mathcal{L}_{i,j+1}$ ✓

$\mathcal{L}$	$\mathcal{R}$
$(H_i \cap K_j) \cdot H_{i+1}$	$(H_i \cap K_j) \cdot K_{j+1}$
$\trianglerighteq$	$\trianglerighteq$
$(H_i \cap K_{j+1}) \cdot H_{i+1}$	$(H_{i+1} \cap K_j) \cdot K_{j+1}$

Claim: Corresponding quotients are isomorphic.

Zassenhaus Lemma:

<u>if</u> H, K two subgroups of G.

$H' \trianglelefteq H, \ K' \trianglelefteq K$

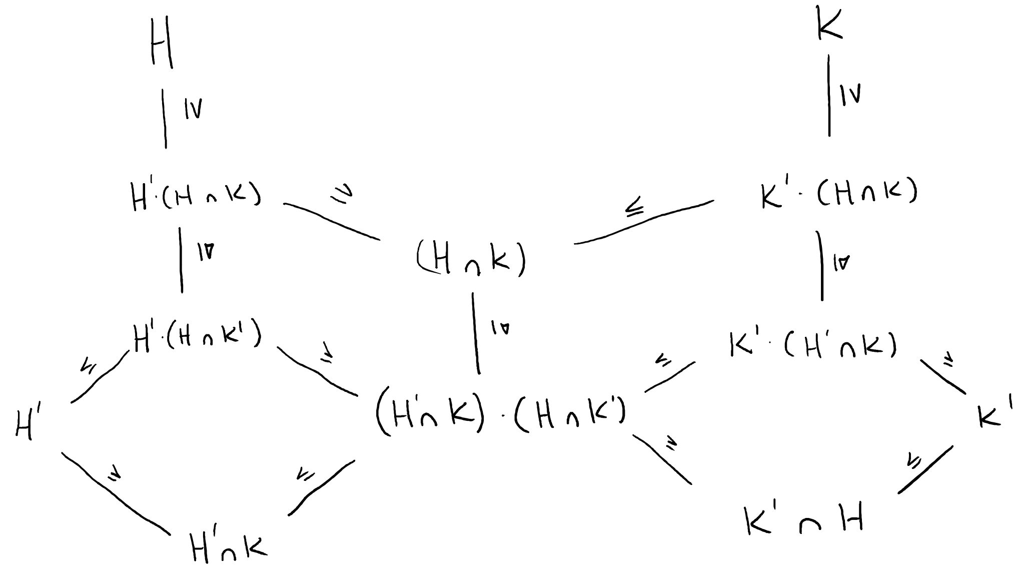
<u>then</u> (i) $(H \cap K) \cdot H' \trianglerighteq (H \cap K') \cdot H'$

$(H \cap K) \cdot K' \trianglerighteq (H' \cap K) \cdot K'$ $\Big\}$ already proved.

(ii) $(H \cap K) \cdot H' \big/ (H \cap K') \cdot H' \cong (H \cap K) \cdot K' \big/ (H' \cap K) \cdot K'$

<u>pf idea</u>: $\text{LHS} \cong \underbrace{H \cap K \big/ (H' \cap K) \cdot (H \cap K')}$ $\cong \text{RHS}$

symmetric in H & K



$$\underbrace{\text{Subgp}/_{\text{subgp} \cap \text{normal}} \cong \text{subgp} \cdot \text{normal}/_{\text{normal}}}_{3^{rd} \text{ isomorphism theorem.}}$$

$$(H \cap K) \cdot H'$$

$$(H \cap K') \cdot H' \trianglelefteq \qquad \geq H \cap K$$

So
$$\frac{(H \cap K) \cdot H'}{((H \cap K') \cdot H' \cap H \cap K)} \cong \frac{(H \cap K) \cdot H' \cdot (H \cap K') \cdot H'}{(H \cap K') \cdot H'}$$

$$\frac{H \cap K}{(H' \cap K) \cdot (H \cap K')} \cong \frac{(H \cap K) \cdot H'}{(H \cap K') \cdot H'}$$

↑ using lemma that denominators are equal

Pf of lemma.

$H' \cap K \subset H' \subset (H \cap K') \cdot H'$ $\Rightarrow$ LHS $\supset H \cap K'$

$H' \cap K \subset H \cap K$

similarly, LHS $\supset K' \cap H$

$\Rightarrow$ RHS $\subset$ LHS.

to prove LHS $\subset$ RHS, let $x \in$ LHS.

$\underset{H \cap K}{\overset{\cap}{x}} = ab$ where $a \in \underset{H \cap K}{\overset{\cap}{H \cap K'}}$ and $b \in H'$

so $a^{-1}x = b \in H \cap K \Rightarrow b \in H' \cap K$

so $x \in$ RHS. $\square$