

Free modules of finite rank over integral domains.

R^n - standard free module of rank n .

M is a free module of rank n if $M \cong R^n$, isomorphism is defined by a basis in M .

$$\text{Hom}(R^n, R^m) \cong R^{nm}$$

$$\begin{array}{ll} e_1 = (1, 0, \dots, 0) & e'_1 = (1, 0, \dots, 0) \\ \vdots & \vdots \\ e_n = (0, \dots, 0, 1) & e'_m = (0, \dots, 0, 1) \end{array} \quad \text{bases for } R^n \text{ \& } R^m$$

$$\text{Hom}(R^n, R^m) \cong \bigoplus_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \text{Hom}(R, R), \quad \text{Hom}(R, R) \cong R$$

$$\varphi \longmapsto \varphi(1)$$

basis in $\text{Hom}(R^n, R^m)$ is $\{\varphi_{ij} : \varphi_{ij}(e_i) = e_j, \varphi_{ij}(e_k) = 0 \text{ for } k \neq i\}$

$$\forall \varphi \in \text{Hom}(R^n, R^m), \quad \varphi = \sum a_{ij} \varphi_{ij}$$

$$\begin{aligned} \varphi(b_1, \dots, b_n) &= \sum a_{ij} \varphi_{ij}(b_1, \dots, b_n) = \sum a_{ij} b_i e'_j \\ &= (\sum a_{i1} b_i, \dots, \sum a_{im} b_i). \end{aligned}$$

the coordinates a_{ij} of φ are a_{11}, \dots, a_{nm} .

the coordinates a_{ij} of φ form an $m \times n$ matrix $\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

it's called the matrix of φ , A_φ , and

$$\varphi(b) = A_\varphi \cdot b$$

$$\varphi: R^n \rightarrow R^m, \quad \psi: R^m \rightarrow R^k.$$

$\psi \circ \varphi$. A_φ - matrix of φ A_ψ - matrix of ψ .

$$A_{\psi \circ \varphi} = A_\psi \cdot A_\varphi$$

$$\forall i \quad \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = \varphi(e_i)$$

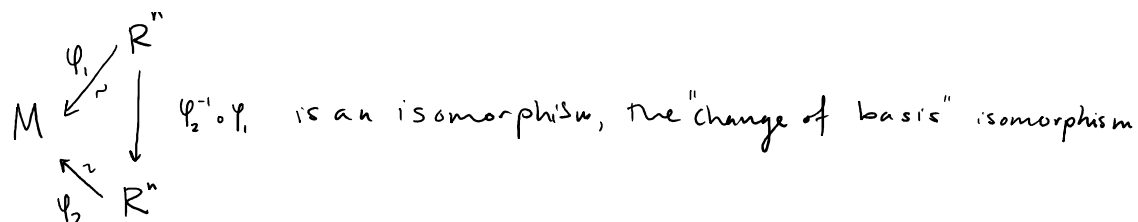
Let M be a free module of rank n .

Let $\{u_1, \dots, u_n\}$ be a basis in M .

then $\varphi_1: R^n \rightarrow M$ is an isomorphism.
 $e_i \mapsto u_i$

Let $\{v_1, \dots, v_n\}$ be another basis in M .

then $\varphi_2: R^n \rightarrow M$
 $e_i \mapsto v_i$



Let $P = A_{\varphi_2^{-1} \circ \varphi_1}$.

$$(\varphi_2^{-1} \circ \varphi_1)(a_1, \dots, a_n) = (b_1, \dots, b_n)$$

\uparrow old coordinates \uparrow new coordinates

So $P \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

\uparrow
 transition matrix for the change of basis
 $\{u_1, \dots, u_n\} \rightarrow \{v_1, \dots, v_n\}$

$$\text{Mat}_{n \times n}(R) \cong \text{End}_R(R^n)$$

A_φ invertible $\Leftrightarrow \varphi$ invertible.

P is invertible, $P^{-1} = A_{\varphi_1^{-1} \circ \varphi_2}$, the transition matrix
 $\{v_1, \dots, v_n\} \mapsto \{u_1, \dots, u_n\}$.

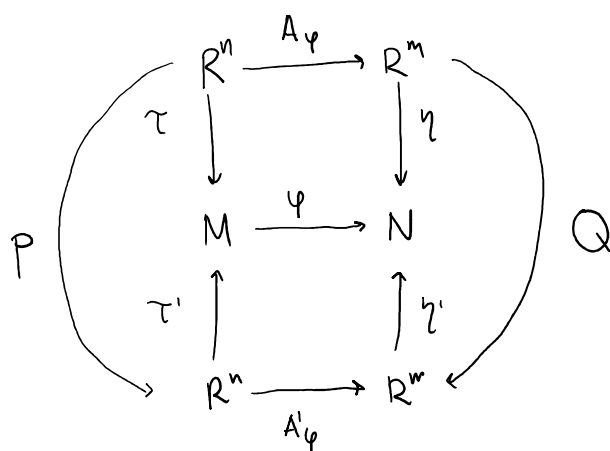
Let $\varphi: M \rightarrow N$, bases $\{u_1, \dots, u_n\}$ in M , $\{v_1, \dots, v_m\}$ in N
 $\parallel \begin{matrix} \mathbb{R}^n \\ \mathbb{R}^m \end{matrix} \Rightarrow A_\varphi$

$\{u'_1, \dots, u'_n\}$ in M , $\{v'_1, \dots, v'_m\}$ in N
 $\Rightarrow A'_\varphi$

$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tau^{-1} \circ \varphi \circ \tau} & \mathbb{R}^m \\ \downarrow \tau & & \downarrow \eta \\ M & \xrightarrow{\varphi} & N \end{array}$ The matrix $A_{\tau^{-1} \circ \varphi \circ \tau}$ is the
 matrix A_φ wrt the bases

$\{u_1, \dots, u_n\}$ & $\{v_1, \dots, v_m\}$

the i^{th} column is the coordinates of $\varphi(u_i)$ in basis $\{v_1, \dots, v_m\}$.



P & Q transition
 matrices

$$A'_\varphi = Q A_\varphi P^{-1}$$

if $N=M$, $Q=P$ and $A'_\varphi = P A_\varphi P^{-1}$

if $B = P A P^{-1}$ for an invertible P , A & B are called
(square matrices) conjugate.

if B & A are conjugate, $B = P A P^{-1}$,

Take any basis $\{u_1, \dots, u_n\}$ in \mathbb{R}^n .

Define $\{P u_1, \dots, P u_n\}$ as a new basis in \mathbb{R}^n .

Then the transition matrix $\{u_1, \dots, u_n\} \mapsto \{P u_1, \dots, P u_n\}$

Let φ be the transform $\varphi(u) = A(u)$ in basis $\{u_1, \dots, u_n\}$.

then B is the matrix of φ in basis $\{P u_1, \dots, P u_n\}$.

$\varphi: V \rightarrow W$ finite dim. vector spaces

$\Rightarrow V = \text{Ker } \varphi \oplus V_2$, $\varphi|_{V_2} = \text{isomorphism } V_2 \cong \varphi(V) \subseteq W$.

$$\varphi(V) \oplus W_2 = W$$

Choose $\{u_1, \dots, u_n\}$ s.t. $\{u_1, \dots, u_k\}$ is basis in V_2 .

Then $\{\varphi(u_1), \dots, \varphi(u_k)\}$ is a basis in $\varphi(V)$.

Let v_{k+1}, \dots, v_m be a basis in W_2 .

Then in $\{u_1, \dots, u_n\}, \{\varphi(u_1), \dots, \varphi(u_k), v_{k+1}, \dots, v_m\}$,

$$A_\varphi = \begin{matrix} & \overset{k}{\text{---}} \\ \begin{pmatrix} \begin{array}{c|c} \begin{matrix} \ddots & \ddots \\ 0 & 0 \end{matrix} & \begin{matrix} \circ \\ \circ \end{matrix} \\ \hline \begin{matrix} \circ \\ \circ \end{matrix} & \begin{matrix} \circ \\ \circ \end{matrix} \end{array} & \begin{matrix} \varphi(v) \\ W_2 \end{matrix} \\ \begin{matrix} V_2 & \text{Ker}(\varphi) \end{matrix} \end{matrix}$$