$M = R^n$ (R : ID)

$\varphi, \psi : M \to N$ s.t. $\psi \circ \varphi = Id_M$. Then $\varphi \circ \psi = Id_M$

Pf $0 \to M \underset{\psi}{\overset{\varphi}{\rightleftarrows}} M \to M/\varphi(M) \to 0$ splits,

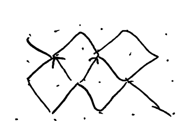
so $M = \varphi(M) \oplus M/\varphi(M)$, but M has same rank as $\varphi(M)$ so $M/\varphi(M) = 0$.

Finitely generated Modules over PID: (PID: $\mathbb{Z}$, $F[x]$, $\mathbb{Z}[i]$)

Theorem: Let R be a PID, let $M \cong R^m$, and let N be a submodule of M. Then $N \cong R^n$ for some $n \leq m$. Moreover, $\exists$ a basis $\{u_1, \dots, u_m\}$ of M & elements $a_1, \dots, a_n \in R$ s.t. $\{a_1 u_1, \dots, a_n u_n\}$ is a basis of N and $a_1 | a_2 | \cdots | a_k$.

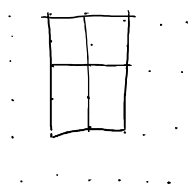
Note: $I = (x, y)$ in $F[x,y] = R$ is a submodule of R, but I isn't free.

Example: $M = \mathbb{Z}^2$, $N = \mathbb{Z} \cdot \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



put $u_1 = \binom{1}{1}$, $u_2 = \binom{0}{1}$. Then $\{u_1, u_2\}$ is a basis in M,

and $\{u_1, 2u_2\}$ is a basis in N.



Proof. Let $n = \operatorname{rank} M$, $k = \operatorname{rank} N$.

R is a Noetherian ring if every ideal in R is finitely generated, OR
Any system of ideals in R has a maximal element I_0 s.t. there is no other ideal I s.t. $I_0 \subsetneq I$.

Any PID is Noetherian.

Induction on n. If $n = 1$, $M \cong R$, $N \cong$ an ideal in R.
So $N = (a)$. $\{1\}$ - basis in M, $\{a\}$ - basis in N.

$\forall f \in M^*$, let $I_f = f(N) = \{f(u) : u \in N\}$.
Then I_f is an ideal in R. Let $h \in M^*$ be s.t. I_h is max'l in $\{I_f : f \in M^*\}$. Let $a_1 \in R$ s.t. $I_h = (a_1)$.

$a_1 \neq 0$ because: let $u = \overbrace{c_1 u_1 + \cdots + c_n u_n}^{\text{some basis}}$. Let $\{f_1, \dots, f_n\}$ be the dual basis. Then $f_1(u) = c_1 \neq 0$.

$a_1 \in I_h$ so $\exists\, v_1 \in N$ s.t. $a_1 = h(v_1)$.

<u>claim</u>: $a_1 \mid f(v_1)\ \forall f \in M^*$.

Indeed, let $I = v_1(M^*) = \{f(v_1) : f \in M^*\}$.

Then $I \overset{\text{ideal}}{\subseteq} R$ so $I = (b)$. Then $a_1 = h(v_1) \in I$ so $b \mid a_1$.

But $b = f(v_1)$ for some f, so $b \in I_f$.

So $I_h = (a_1) \subseteq (b) = I \subseteq I_f$ so $I_h = I_f = I$.

(in fact, this shows $a_1 \mid f(v)\ \forall f \in M^*,\ v \in N$).

So v_1, as an element of M^{**} is div. by a_1.

($v_1(f) = f(v_1)$ - div. by a_1).

So $\exists\, u_1 \in M^{**} = M$ s.t. $v_1 = a_1 u_1$.

(or, coordinates of v_1 are all div. by a_1, put $u_1 = v_1/a_1$).

let $K = \operatorname{Ker} h$. ($h(v_1) = a_1$, $h(u_1) = 1$).

$\forall u \in M$, $u = \underset{\substack{\cap \\ Ru_1}}{h(u)u_1} + \underset{\substack{\cap \\ K}}{(u - h(u)u_1)}$. $\rightsquigarrow$ since $h(u - h(u)u_1) = h(u) - h(u)\cdot 1 = 0$

So $M = Ru_1 \oplus K$

If $u \in N$, $h(u) = \frac{h(u)}{a_1} \cdot a_1$, so $u = \underset{\substack{Rv_1 \\ \cup}}{c \cdot v_1} + \underset{\substack{K \cap N \\ \cup}}{(u - h(u)u_1)}$

So also $N = Rv_1 \oplus (K \cap N)$.

Induction on $k = \operatorname{rank} N$:

$\operatorname{rank}(K \cap N) = k-1$ so by induction, it is free, so N is free

We proved that every submodule of M is free.

So K is free. Then by induction on $n = \text{rank } M$,
$\exists$ a basis $\{u_2, \dots, u_n\}$ of K and #s $a_2, \dots, a_k \in R$
s.t. $\{a_2 u_2, \dots, a_k u_k\}$ is a basis in $K \cap N$,
and $a_2 | a_3 | \cdots | a_k$.

So $\{u_1, \dots, u_n\}$ is a basis of M, and $\{a_1 u_1, \dots, a_k u_k\}$ is one for N.

Now to prove: $a_1 | a_2$.

define $f(\underbrace{x_1 u_1 + \cdots + x_n u_n}_{\text{chosen basis}}) = x_1 + x_2$, $f \in M^*$.

then $f(a_1 u_1) = a_1$

So $a_1 \in f(N)$, so $(a_1) = I_h \subseteq I_f \Rightarrow I_h = I_f$

And $f(a_2 u_2) = a_2$, so $a_2 \in I_f = (a_1)$ so $a_1 | a_2$.