Lec 9/16

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Short exact sequences

Def a seq. of gps and homomorphisms

$$H \xrightarrow{\alpha} G \xrightarrow{\beta} K$$

is called <u>exact</u> if $\text{im}\,\alpha = \ker \beta$

eg To say $1 \to G \xrightarrow{f} K$ is exact at G means f is injective.

To say $H \xrightarrow{f} G \to 1$ is exact at G means f is surjective.

Def A short exact seq is a seq

$$1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$$

that is exact at H, G, and K.

so α is injective, β is surjective, and $\ker\beta = \text{im}\,\alpha$.

eg when $N \trianglelefteq G$, $1 \longrightarrow N \hookrightarrow G \twoheadrightarrow G/N \longrightarrow 1$.

eg $1 \to H \hookrightarrow H \times K \twoheadrightarrow K \to 1$ (special case of above).

eg $1 \to H \hookrightarrow H \rtimes K \twoheadrightarrow K \to 1$

Def two SES's $1 \to H_1 \xrightarrow{\alpha_1} G_1 \xrightarrow{\beta_1} K_1 \to 1$

and $1 \to H_2 \xrightarrow{\alpha_2} G_2 \xrightarrow{\beta_2} K_2 \to 1$

are equivalent if they fit into a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & H_1 \arrow[r] \arrow[d, "\cong"'] & G_1 \arrow[r] \arrow[d, "\cong"'] & K_1 \arrow[r] \arrow[d, "\cong"'] & 1 \\
1 \arrow[r] & H_2 \arrow[r] & G_2 \arrow[r] & K_2 \arrow[r] & 1
\end{tikzcd}$$

$(N \trianglelefteq G)$

eg every SES is equivalent to one of the form $1 \to N \to G \to G/N \to 1$.

pf Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a SES. Then

α restricts to an isomorphism $H \cong \operatorname{Im}\alpha$ since α is injective

β induces an isomorphism $\bar{\beta} : G/_{\operatorname{Im}\alpha} = G/_{\ker\beta} \cong K$.

So
$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & G \arrow[r, "\beta"] \arrow[d, "id"] & K \arrow[r] \arrow[d, "\bar{\beta}"] & 1 \\
1 \arrow[r] & \operatorname{Im}\alpha \arrow[r, hook] & G \arrow[r, "\pi"] & G/_{\ker\beta} \arrow[r] & 1
\end{tikzcd}$$
. $\square$

Note equivalence of SES is an equivalence relation.

Thm Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a SES. TFAE.

(1) $\exists$ a homomorphism $\alpha' : G \to H$ s.t. $\alpha'(\alpha(h)) = h \ \forall h \in H$.

(2) $\exists$ an isomorphism $\theta : G \to H \times K$ s.t. the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "\theta"] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r, hook] & H \times K \arrow[r, "\pi"] & K \arrow[r] & 1
\end{tikzcd}$$

Since α' & β are.

proof (1) $\Rightarrow$ (2):

Define $\theta : G \to H \times K$ by $\theta(g) = (\alpha'(g), \beta(g))$. θ is a homomorphism!

Suppose $\theta(g) = (1,1)$. Then $\alpha'(g) = 1$ and $\beta(g) = 1$. So $g = \alpha(h)$ for some h (by exactness at G).

Thus $\alpha'(\alpha(h)) = 1$ So $h = 1$ so $g = 1$. So θ is injective.

Suppose $(h,k) \in H \times K$. By exactness at G, $\exists\ g \in G$ w/ $\beta(g) = k$.

by exactness at G, $k = \beta(g\alpha(x))$ for all $x \in H$. We want x s.t. $\alpha'(g\,\alpha(x)) = h$.

Equiv, we want x s.t. $\alpha'(g)\,\alpha'(\alpha(x)) = h \Leftrightarrow \alpha'(g)\,x = h$. Take $x = (\alpha'(g))^{-1}h$.

So $\theta(g\alpha(x)) = (h,k)$. So θ is surjective.

To show the diagram is commutative, the first square is

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, equal] & G \arrow[d, "\theta"] \\ H \arrow[r, hook] & H \times K \end{tikzcd},$$

and $\theta(\alpha(h)) = (\alpha'(\alpha(h)), \beta(\alpha(h))) = (h, 1) = \iota(id(h))$.

Similarly, the second square commutes.

$(2) \Rightarrow (1)$: Suppose $\exists$ isomorphism $\theta : G \cong H \times K$ s.t.

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, equal] & G \arrow[r, "\beta"] \arrow[d, "\theta"] & K \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H \arrow[r, hook] & H \times K \arrow[r, "\pi"] & K \arrow[r] & 1 \end{tikzcd}$$

is commutative. we have $\theta(g) = (\alpha'(g), \beta(g))$ definition of α'.

α' is a function $G \to H$. it is also a hom: $\alpha' = \pi_H \circ \theta$.

and $\forall\ h \in H$, $\alpha'(\alpha(h)) = h$ by commutativity. $\square$

Thm Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a SES. TFAE:

(1) $\exists$ a homomorphism $\beta' : K \to G$ s.t. $\beta(\beta'(k)) = k$ $\forall k \in K$.

(2) $\exists$ a homomorphism $\varphi : K \longrightarrow \mathrm{Aut}(H)$ and an isomorphism $\theta : G \longrightarrow H \rtimes_\varphi K$ s.t.

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, equal] & G \arrow[r, "\beta"] \arrow[d, "\theta"] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r, hook] & H \rtimes_\varphi K \arrow[r, "\pi"] & K \arrow[r] & 1
\end{tikzcd}$$

commutes. (it is easier to construct θ^{-1}).

<u>Def</u> a SES $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ is said to <u>split</u> if $\exists$ hom $\beta' : K \longrightarrow G$ s.t $\beta(\beta'(k)) = k \ \forall k \in K$.