Lec 11/2

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Primary Ideal: $ab \in Q$, $a \notin Q \implies b^n \in Q$ for some $n \geq 1$.

[primes are primary]

$Q \subset R$ primary $\iff$ in R/Q zero divisors = nilpotents.

Ex: $R = \mathbb{Z} \supset (0)$ is primary. if $I \neq (0)$, $I \subsetneq \mathbb{Z}$ is primary

then $I = p^k\mathbb{Z}$ for some $k \geq 1$.

Pf $I = n\mathbb{Z}$. $n \geq 2$. $\mathbb{Z}/n\mathbb{Z}$: "zero divisor" $\iff$ "divides n"

nilpotent means $n \mid t^k$ for some $k \Rightarrow$

$\forall d$ s.t. $d \mid n$, $\exists k$ s.t. $n \mid d^k$. take $d = p$ which divides n.

so $p \mid n$, $n \mid p^k \Rightarrow n = p^\ell$.

Ex. $(4, t) \subset \mathbb{Z}[t]$. $\mathbb{Z}[t]/(4,t) \cong \mathbb{Z}/4\mathbb{Z}$ which has all zero divisors nilpotent.

In $\mathbb{Z}$, $n = p_1^{k_1} \cdots p_\ell^{k_\ell}$.

Want to say: in R noetherian, every ideal $I = \underbrace{Q_1 \cap Q_2 \cap \cdots \cap Q_\ell}_{\text{primary}}$

This representation is unique up to $k \leq \ell$.

Irreducible ideal: $I \subset R$ is irreducible if $I = I_1 \cap I_2 \Rightarrow I = I_1$ or $I = I_2$.

Lemma: In a noetherian ring, every ideal is a finite intersection of irreducible ideals.

Pf $\Sigma = \{ I \subsetneq R$ s.t. I cannot be written as a finite intersection of irred. ideals$\}$

If Σ is non-empty then Σ has a max'l element say $J \in \Sigma$.

$J = J \Rightarrow J$ is not irreducible.

$J = J_1 \cap J_2$ s.t. $J_1 \supsetneq J$, $J_2 \supsetneq J$. So $J_1, J_2 \notin \Sigma$.

$J_1 = K_1 \cap \dots \cap K_n$; $J_2 = L_1 \cap \dots \cap L_m$

$\Rightarrow J = K_1 \cap \dots \cap K_n \cap L_1 \cap \dots \cap L_m$ $\lightning$ to $J \in \Sigma$.

So $\Sigma = \phi$.

Lemma: R: Noetherian $I \subsetneq R$ irreducible $\Rightarrow$ I is primary.

Pf (Replace R by R/I)

Given: $(0) \subset R$ is irreducible. To prove: (0) is primary

$(ab = 0, a \neq 0 \Rightarrow b^n = 0)$

Form a chain of ideals: $Ann(b^i) = \{r \in R : rb^i = 0\}$.

$Ann(b) \subset Ann(b^2) \subset \dots \subset Ann(b^\ell) = Ann(b^{\ell+1})$

$\uparrow$ R noetherian.

Claim: $(0) = (a) \cap (b^\ell)$

$$\left.\begin{array}{l} \text{If } \xi \in (a) \Rightarrow b\xi = 0 \\ \xi \in (b^\ell) \Rightarrow \xi = cb^\ell \end{array}\right\} \Rightarrow cb^{\ell+1} = 0$$

$\Rightarrow c \in Ann(b^{\ell+1})$

$\Rightarrow c \in Ann(b^\ell)$

$\Rightarrow \xi = cb^\ell = 0$.

So $(a) \neq (0) \Rightarrow (b^\ell) = (0)$ so $b^\ell = 0$.

If $I \subset R$ is an ideal then $Rad(I) = \{x \in R : x^n \in I \text{ for some } n \geq 1\}$

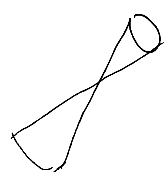
P_i

Uniqueness: $\mathrm{Rad}(Q_i)$ is prime & $\{P_1, \dots, P_\ell\}$ is uniquely determined by I.

$$\begin{tikzcd} R \arrow[r, "\pi"] & R/I & (I \subsetneq R) \\ \mathrm{Rad}(I) \arrow[u, phantom, "\cup" rotate=90] & \mathcal{N}(R/I) \arrow[l, "\pi^{-1}"'] \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

$$\begin{tikzcd} \text{Primary} \arrow[rr, bend left, "F"] \arrow[d, bend right, "F"'] & & \text{Power of prime} \arrow[ll, bend right, "F"'] \arrow[dll, bend right, "F"'] \\ \text{Irreducible} \arrow[u, "T"'] \arrow[urr, bend right, "F"'] & & \end{tikzcd}$$

$P^K = \{a_1 a_2 \cdots a_K : a_i \in P \ \forall i\}$

$K[x,y,z]/(z^2 - xy)$ 

$\cup$

$(x^2, xz, z^2) = (x,z)^2$ and (x,z) is prime.

$\uparrow$ not primary

$(4,t) \subset \mathbb{Z}[t]$ is primary, $\mathrm{Rad}((4,t)) = (2,t)$

but $(4,t)$ is not a power of a prime.

in $R = K[x,y]$: $(x^2, y) \cap (x, y^2) = (x^2, xy, y^2)$

$\uparrow$ primary