Lec 3/26

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Arc length on a Surface

Let M be a C^1 surface in $\mathbb{R}^3$.

Let $x : \mathcal{U} \text{ open} \subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V \overset{\text{open}}{\subseteq} M$ be a C^1 patch in M.

Let $\alpha : (a,b) \longrightarrow V$ be a C^1 curve.

Then $x^{-1} \circ \alpha$ is a C^1 curve in $\mathcal{U}$ with coordinates /components α^1 and α^2. $x^{-1}(\alpha(t)) = (\alpha^1(t), \alpha^2(t))$.

So $\alpha(t) = x(\alpha^1(t), \alpha^2(t))$, so $\frac{d\alpha}{dt} = \frac{\partial x}{\partial u^1}\frac{d\alpha^1}{dt} + \frac{\partial x}{\partial u^2}\frac{d\alpha^2}{dt} = x_1 \frac{d\alpha^1}{dt} + x_2 \frac{d\alpha^2}{dt}$.

So
$$\underbrace{\left|\frac{d\alpha}{dt}\right|^2}_{\left(\frac{ds}{dt}\right)^2} = \left\langle x_1 \frac{d\alpha^1}{dt} + x_2 \frac{d\alpha^2}{dt} \,\middle|\, x_1 \frac{d\alpha^1}{dt} + x_2 \frac{d\alpha^2}{dt} \right\rangle$$
$$= \left\langle \sum_i x_i \frac{d\alpha^i}{dt} \,\middle|\, \sum_j x_j \frac{d\alpha^j}{dt} \right\rangle$$
$$= \sum_i \sum_j \langle x_i \mid x_j \rangle \frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}$$
$$= \sum_{i,j} g_{ij} \frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}$$

Thus $(ds)^2 = \sum_{i,j} g_{ij}\, du^i\, du^j$

The Weingarten Map

Let $x, \mathcal{U}, V, M$ be as before but with M, x being C^2.

The unit normal vector field is $n = \frac{x_1 \times x_2}{|x_1 \times x_2|}$

The Weingarten Map is the map $p \longmapsto L_p$ on V defined by

$$L_p(X) = -n'(p)(X) \qquad \forall\, X \in T_pM \qquad (n'(p) : T_pM \longrightarrow T_{n(p)}S^2)$$

if $x(u) = p$, then $L_p(x_i(u)) = -\frac{\partial n}{\partial u^i}(u)$.

Note that
$$T_{n(p)} S^2 \overset{\substack{\text{special property} \\ \text{of } S^2}}{=} \{n(p)\}^\perp = T_pM$$

So $L_p : T_pM \xrightarrow{\text{linear}} T_pM$

The second fundamental form

This is the function $p \longmapsto \mathbb{I}_p$ on M (or just on V) defined by

$$\mathbb{I}_p(X, Y) = \langle L_p(X) \mid Y \rangle \quad \forall\, X, Y \in T_pM$$

As we'll see, we also have $\mathbb{I}_p(X, Y) = \langle X \mid L_p(Y) \rangle$ so L_p is self-adjoint.

Let $X, Y \in T_pM$ with $X = \sum_i X^i x_i(u)$, $Y = \sum_j Y^j x_j(u)$

Then $\mathbb{I}_p(X, Y) = \underbrace{\langle L_p(X) \mid Y \rangle}_{\text{linear in } X \text{ and in } Y} = \sum_{i,j} L_{ij}(u) X^i Y^j$

where $L_{ij}(u) = \mathbb{I}_p(x_i(u), x_j(u)) = \langle L_p(x_i(u)) \mid x_j(u) \rangle = \langle -n'(p)(\overset{\text{really } n \circ x}{x_i(u)}) \mid x_j(u) \rangle$

$$= \left\langle -\frac{\partial n}{\partial u^i}(u) \;\middle|\; x_j(u) \right\rangle = \left\langle -\frac{\partial}{\partial u^i}\left(\frac{x_1 \times x_2}{\sqrt{g}}\right) \;\middle|\; x_j \right\rangle \quad \text{(drop the } u\text{)}$$

$$= -\left\langle \frac{1}{\sqrt{g}}(x_{1i} \times x_2 + x_1 \times x_{2i}) + \overbrace{(x_1 \times x_2)}^{\text{orthogonal to } x_j} \frac{\partial}{\partial u^i}\frac{1}{\sqrt{g}} \;\middle|\; x_j \right\rangle$$

$$= -\frac{1}{\sqrt{g}} \langle x_{1i} \times x_2 + x_1 \times x_{2i} \mid x_j \rangle$$

Thus $L_{ij} = -\frac{1}{\sqrt{g}} \left(\langle x_{1i} \mid x_2 \times x_j \rangle + \langle x_{2i} \mid x_j \times x_1 \rangle \right)$ (cyclically permuting determinant columns).

$$= \left\langle x_{1i} \;\middle|\; \frac{x_j \times x_2}{\sqrt{g}} \right\rangle + \left\langle x_{2i} \;\middle|\; \frac{x_1 \times x_j}{\sqrt{g}} \right\rangle$$

$$= \left\langle x_{i1} \;\middle|\; \frac{x_j \times x_2}{\sqrt{g}} \right\rangle + \left\langle x_{i2} \;\middle|\; \frac{x_1 \times x_j}{\sqrt{g}} \right\rangle$$

$$= \langle x_{ij} \mid n \rangle \quad \text{(consider cases } j=1 \text{ and } j=2\text{).}$$

Well $x_{ij} = x_{ji}$ so $L_{ij} = L_{ji}$. So (L_{ij}) is symmetric.

This is why $\langle L_p(X) \mid Y \rangle = \langle X \mid L_p(Y) \rangle$.

4-4 Normal Curvature, Geodesic curvature, and Gauss's Formulae.

Let M be a C^2 surface in $\mathbb{R}^3$. Let $x: U \overset{\text{open}}{\subseteq} \mathbb{R}^2 \longrightarrow V \overset{\text{open}}{\subseteq} M$ be a C^2 patch.

Let $\gamma: (a,b) \longrightarrow V$ be a C^2 unit speed curve with unit tangent vector field T.

Then the intrinsic normal for γ is $S = n \times T$. $S(s) = n(\gamma(s)) \times T(s)$

S is a unit vector normal to γ and tangent to M. (Also, T tangent to M).