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$(M_\alpha)_{\alpha \in \Lambda}$ a family of R-modules

$\prod M_\alpha$ is the universal attracting object in category of modules N w/ homs $\varphi_\alpha : N \to M_\alpha$ $\forall \alpha$.

$\exists$ unique $\varphi : N \to \prod M_\alpha$ s.t.

$$\begin{tikzcd} & M_\alpha & \\ \prod M_\alpha \arrow[ur, "\pi_\alpha"] & & N \arrow[ul, "\varphi_\alpha"'] \arrow[ll, "\varphi"] \end{tikzcd}$$ commutes $\forall \alpha$

$\varphi = (\varphi_\alpha)_{\alpha \in \Lambda}$. $\longleftarrow$ defn.

$\bigoplus M_\alpha$ is universal repelling object in category of $(N, (\varphi_\alpha : M_\alpha \to N)_{\alpha \in \Lambda})$.

$\exists$ unique $\varphi : \bigoplus M_\alpha \to N$ s.t.

$$\begin{tikzcd} & M_\alpha \arrow[dl, "\eta"'] \arrow[dr, "\varphi_\alpha"] & \\ \bigoplus M_\alpha \arrow[rr, "\varphi"'] & & N \end{tikzcd}$$ commutes $\forall \alpha$

finite sum!

defn. $\longrightarrow$ $\varphi\left((u_\alpha)_{\alpha \in \Lambda}\right) = \sum_{\alpha \in \Lambda} \varphi_\alpha(u_\alpha)$

Given N, we have 1-1 correspondence

$$\{(\varphi_\alpha : N \to M_\alpha)_{\alpha \in \Lambda}\} \longleftrightarrow \{\varphi : N \to \prod M_\alpha\}$$

$$\prod_{\alpha \in \Lambda} \mathrm{Hom}(N, M_\alpha) \longleftrightarrow \mathrm{Hom}\left(N, \prod_{\alpha \in \Lambda} M_\alpha\right)$$

Claim: this is actually a module isomorphism.

Given N, we have 1-1 correspondence

$$\{(\varphi_\alpha : M_\alpha \to N)_{\alpha \in \Lambda}\} \longleftrightarrow \{\varphi : \oplus M_\alpha \to N\}$$

$$\prod_{\alpha \in \Lambda} \mathrm{Hom}(M_\alpha, N) \longleftrightarrow \mathrm{Hom}\left(\bigoplus_{\alpha \in \Lambda} M_\alpha, N\right)$$

claim: this is another module isomorphism.

$$\mathrm{Hom}(N, \prod M_\alpha) \cong \prod \mathrm{Hom}(N, M_\alpha)$$

$$\mathrm{Hom}(\oplus M_\alpha, N) \cong \prod \mathrm{Hom}(M_\alpha, N)$$

Chinese Remainder Theorem:

Let R be a commutative unital ring, M be an R-module, and $I_1, \ldots, I_n$ are pairwise comaximal (coprime) ideals in R. $(I_i + I_j = (1) = R) \; \forall i \neq j$

Then $I_1 M \cap \cdots \cap I_n M = (I_1 \cdots I_n) M$ and

$$M/(I_1 \cdots I_n) M \cong \bigoplus_{i=1}^{n} M/(I_i M)$$

<u>Proof</u> Let $n=2$. $I_1 + I_2 = (1)$. Find $\underset{\in I_1}{a_1} + \underset{\in I_2}{a_2} = 1$.

We have a hom-sm $\varphi : M \longrightarrow (M/I_1 M) \oplus (M/I_2 M)$

by $\varphi(u) = (u \bmod I_1 M,\ u \bmod I_2 M)$.

$\mathrm{Ker}(\varphi) = I_1 M \cap I_2 M$.

<u>Claim</u> φ is surjective.

<u>Proof</u> Take any $u_1, u_2 \in M$. Put $u = a_1 u_1 + a_2 u_2$

$$= (1-a_1) u_1 + a_1 u_2$$
$$= u_1 + a_1 (u_2 - u_1)$$
$$\equiv u_1 \bmod I_1$$

and similarly, $u \equiv u_2 \bmod I_2$,

So $\varphi(u) = (u_1 \bmod I_1,\ u_2 \bmod I_2)$.

<u>Claim 2</u>: $I_1 M \cap I_2 M = (I_1 I_2) M$.

<u>Proof</u>: clearly, $\supseteq$ holds.

Let $u \in I_1 M \cap I_2 M$. Then $u = 1u = \underset{\in I_1}{a_1} \underset{\in I_2 M}{u} + \underset{\in I_2}{a_2} \underset{\in I_1 M}{u} \in (I_1 I_2) M$.

for general n: use induction & the fact that if $I_1, \dots, I_n$ are pairwise comaximal then I_1 and $I_2 \cdots I_n$ are comaximal.

Proof: $\forall i = 2, \dots, n$, find $a_i \in I_1$, $b_i \in I_i$ s.t. $a_i + b_i = 1$.

then $1 = (a_2 + b_2) \cdots (a_n + b_n) = \underbrace{\sum \text{stuff with at least one } a}_{\in I_1} + \underbrace{b_2 \cdots b_n}_{\in I_2 \cdots I_n}$.

Free Modules:

Let R be a unital ring.

R^n is a free module of rank n.

Def M is a free R-module of rank n if $M \cong R^n$.

Def A set B is a basis of M if $\forall u \in M$, u is uniquely representable as a linear combination of elements of B: $u = a_1 b_1 + \dots + a_n b_n$, $a_i \in R$, $b_i \in B$.

Theorem: M is free of rank n iff M has a basis B with $|B| = n$.

Proof: R^n has a standard basis $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$.

if $M \xrightarrow[\varphi]{\cong} R^n$, take $b_i = \varphi^{-1}(e_i)$, then $\{b_1, \dots, b_n\}$ is a basis for M.

if $\{b_1, \dots, b_n\}$ is a basis, let $\varphi(u = \sum a_i b_i) = \sum a_i e_i \in R^n$.

this is an isomorphism. $a_1, \dots, a_n$ are called "coordinates".

" Free modules of rank $|\Lambda|$ is $\bigoplus_{\alpha \in \Lambda} R$ "

formal linear combinations

"free module generated by $S = \{s_\alpha : \alpha \in \Lambda\}$

is $\left\{ \sum_{i=1}^{n} a_i s_i : n \in \mathbb{N}, a_i \in R, s_i \in S \right\}$