Lec 12/4

Monday, December 4, 2017 14:22

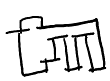
$\beta\mathbb{N}$ universal topological space (compactification of $\mathbb{N}$).

P.S. Aleksandroff : $\mathbb{R} \cup \{\pm\infty\}$, $\mathbb{R} \cup \{\infty\}$, $\mathbb{Z} \cup \{\pm\infty\}$, $\mathbb{Z} \cup \{\infty\}$

are compact.

Riemann sphere is a compactification

Ex: the riemann sphere correspondence is conformal (preserves angles)



$$\frac{az+b}{cz+d}$$

$\beta\mathbb{N}$: Stone – Čech compactification.

Hindman's Theorem: for any finite coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one C_i contains

$$FS((n_i)_{i=1}^{\infty}) = \{ n_{i_1} + \dots + n_{i_k} ; \; i_1 < \dots < i_k, \; k \in \mathbb{N} \}$$

Ex: Hindman's $\iff$ $\forall$ finite coloring $FS(n_i)_{i=1}^{\infty} = \bigcup_{i=1}^{r} C_i$, one C_i contains $FS(m_i)_{i=1}^{\infty}$

Color   density

VdW $\longleftrightarrow$ Szemeredi

Hindman $\longleftrightarrow$???

Schur $\longleftrightarrow$

$\downarrow$

one color contains $x, y, x+y$ $\Rightarrow$ $\forall n \in \mathbb{N}$, if $p \in P$ is large enough then $x^n + y^n \equiv z^n \mod p$ is nontrivially solvable

Exercise: $\forall A = FS(n_i)_{i=1}^{\infty}$, $A \cap k\mathbb{N} \neq \emptyset$ $\forall k$.

Consider $x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_{k+1}$. pigeonhole & subtract.

exercise: (Erdős's question) is it true that any $A \subseteq \mathbb{N}$ w/ $d(A) > 0$ contains a shift of $FS(n_i)_{i=1}^{\infty}$ ($\equiv$ IP-set)? No. Moreover, $\forall \varepsilon > 0$

Ernst Strauss

there is $A \subseteq \mathbb{N}$, $d(A) > 1 - \varepsilon$, s.t. A does not contain a shift of IP-set.

Hint: take large n. remove all multiples of $n!$ from $\mathbb{N}$. Then take a higher $k > n$, remove $(k! \pm 1)$ multiples (do it a little carefully)

exercise: Call $A \subseteq \mathbb{N}$ IP* set if $A \cap FS(n_i)_{i=1}^{\infty} \neq \emptyset$ for all $n_i \nearrow \infty$.

if A_1, A_2 are IP*, then $A_1 \cap A_2$ is also IP*.

Example of IP*: $B - B$ where $d^*(B) > 0$.

Facts: 1. $\beta\mathbb{N}$ is compact

2. $\mathbb{N} \subset \beta\mathbb{N}$ and $\overline{\mathbb{N}} = \beta\mathbb{N}$

3. $|\beta\mathbb{N}| = 2^{c}$

4. $\beta\mathbb{N}$ is not metrizable

Important: $\beta\mathbb{N}$ can be viewed as the set of all 0-1 valued finitely

Important: $\beta\mathbb{N}$ can be viewed as the set of all 0-1 valued finitely additive measures on $\mathcal{P}(\mathbb{N})$ (ultrafilters)

ex: $\mu(A) = \begin{cases} 1 & 17 \in A \\ 0 & 17 \notin A \end{cases}$

$A : d(A) = 1$

On $\beta\mathbb{N}$, extension of $+$ is heavily noncommutative.

$n\alpha \bmod 1$ is dense in $[0,1]$. take $\varepsilon = \frac{1}{10}$, take 11 element $\alpha, 2\alpha, \dots, 11\alpha$.

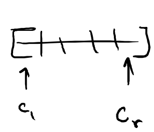
then $\underbrace{(i-j)}_{n_0}\alpha$ is ε-close to 0 or 1. then take $n_0\alpha, 2n_0\alpha, 3n_0\alpha, \dots$

What about $n^2\alpha \bmod 1$? Pigeonhole is not good enough.

Use 2-dim v.d.W. (equispaced grid)

$\forall$ finite coloring of $\mathbb{N}^2 = \bigcup_{i=1}^{r} C_i$

Exercise: verify that if $n^2\alpha \bmod 1$ gets arbitrarily close to 0 or 1 then $n^2\alpha$ is dense mod 1.

Now $mn\alpha \bmod 1 \longrightarrow$ 

finite coloring of $\mathbb{N}^2$

now we have $(n, m+d)$, $(n+d, m+d)$

in one color

(n,m) $(n+d,m)$

$(n+d)(m+d) - n(m+d) - m(n+d) - mn = d^2$, so we have a square $d^2\alpha$ ε-close to 0 or 1.

Sapienti sat.

(this d can be chosen from IP-set given in advance)

Exercise: use same reasoning to get $n^3\alpha$ is dense mod 1.

Cor. of proof (and of the appropriate version of vdW):

$\forall \varepsilon > 0$, $\{n\alpha : \|n^2\alpha\| < \varepsilon\}$ is IP* $\leftarrow$ Remember this for Final

$\uparrow$ distance to closest integer.

From well-distribution:

We know $\{n\alpha : \|n^2\alpha\| < \varepsilon\}$ is syndetic.

$$\frac{1}{N-M} \sum_{n=M}^{N-1} f(n^2\alpha) \longrightarrow \int_0^1 f$$

take $f = 1_{[0,\varepsilon] \cup [1-\varepsilon, 1]}$

Defs: U.D., W.D., vdC, multidimensional versions, Weyl criterion, Weyl's polynomial u.d., Fejer, normal #.

$$\frac{1}{N}\sum e^{2\pi i h x_n} \to 0 \quad \forall h$$

in multi-dim: $\frac{1}{N}\sum e^{2\pi i \vec{h}\cdot\vec{x}_n} \to 0 \quad \forall \vec{h} \in \mathbb{N}^2.$

$A\cdot A$ is $\mathbb{P}^*$

know $\mu, \Lambda, \sigma, \tau, \phi$

$\uparrow$

μ-inversion