Lec 8/23

Thursday, August 23, 2018 11:25

group

Defn: Order of an element. if $a \in G$,

$|a| = \text{ord}(a) =$ smallest positive integer ℓ s.t. $a^\ell = e$ (could be ∞).

if $\text{ord}(a) = \infty$ then $G \geq \langle a \rangle \cong \mathbb{Z}$.

Presentation of a group: $\langle \text{generators} \mid \text{relations} \rangle$

$$\langle a_1, a_2, a_3, \dots \mid r_1, r_2, r_3, \dots \rangle$$

- form words from alphabet $\{a_1, a_2, \dots\}$. eg. $a_3^{-7} a_2^2 a_3 a_1^4$
- $r_1, r_2, r_3, \dots$ are some such words.
- If $W = w_1 r_j w_2$ for some $j = 1, 2, \dots$, then $W = w_1 w_2$.

Ex: $\langle a, b \mid a^2, b^2 \rangle = \{a, b, ab, ba, abab\dots, bab\dots\}$

$$(aba)(aba) = e.$$

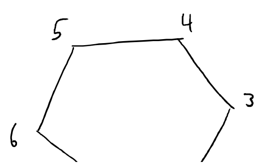
G is finitely generated, each generator is of finite order, but G is infinite.

$\text{ord}(ab) = \infty$, but $\text{ord}(a) = \text{ord}(b) = 2$

however if $ab = ba$ then $\text{ord}(ab) = \text{lcm}(\text{ord}(a), \text{ord}(b))$

Presentation of dihedral group D_{2n} = symmetries of regular n-gon.

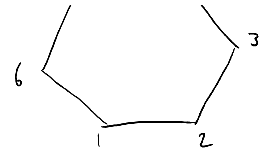
eg $n = 6$.



$D_{12} \ni \rho = (123456) =$ rotation by $60°$ about center

Ψ

$\sigma = (12)(36)(45) =$ flip across vertical axis



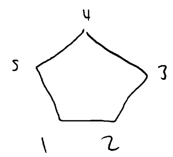
$\Downarrow$

$\sigma = (12)(36)(45) =$ flip across vertical axis

$\rho^6 = e = \sigma^2.$ $\quad \sigma\rho\sigma = (654321) = \rho^{-1}$

Side note: τ some permutation $\Rightarrow$ $\tau\overbrace{(n_1\, n_2 \cdots)}^{\text{some cycle}}\tau^{-1} = (\tau(n_1)\ \tau(n_2) \cdots)$

$n=5$

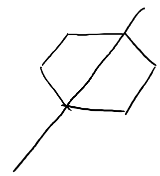


$\sigma = (12)(35)$
$\rho = (12345)$ $\quad \sigma\rho\sigma = (21543)$

$\boxed{\sigma\rho}\boxed{\sigma\rho} = e$ ← reflection

↳ another relation

if $n=6$, across



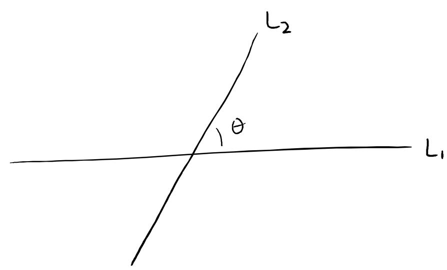
2n elements.

$$\langle s, r \mid s^2 = e = r^n,\ srs = r^{-1} \rangle = \{e, s, sr, sr^2, \ldots, sr^{n-1}, r, r^2, \ldots, r^{n-1}\}$$

So this is a presentation of D_{2n}

<u>Propn</u>: Product of 2 reflections is a rotation.

L_1, L_2 are 2 lines in $\mathbb{R}^2$. say L_1 is x-axis



claim: $S_2 S_1 = \rho_{2\theta}$

where S_i is reflection in L_i, and ρ_ψ is ccw rotation by ψ.

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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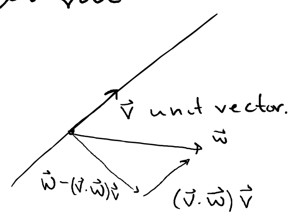
$$S_2 S_1 = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} = \rho_{2\theta}$$

in general:



so reflection of $\vec{w}$ in line spanned by $\vec{v}$ is $-\vec{w} + 2(\vec{v}\cdot\vec{w})\,\vec{v}$.

another presentation, since a rotation is a product of reflections:

$$D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle.$$

$\hookrightarrow$ example of groups generated by reflections (coxeter groups).

$$\left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{l} s_i^2 = e \ \forall i \\ (s_i s_j)^{m_{ij}} = e \end{array} \right\rangle$$

Parameter: square matrix (m_{ij}), $m_{ii} = 2$.

except for

Trouble with group presentations:

eg $\langle x, y \mid \underbrace{xy^2 = y^3 x}_{xy^2x^{-1} = y^3},\ \underbrace{yx^2 = x^3 y}_{yx^2y^{-1} = x^3} \rangle$

$xyx^{-3}y^2 = e$?   word problems are np-hard in general

in fact this group is $\{e\}$:

$$xy^2 = y^3x \Rightarrow x^2y^8x^{-2} = xy^{12}xx^{-2} = y^{18}x^2x^{-2} = y^{18}$$
$$x^3y^8x^{-3} = xy^{18}x^{-1} = (xy^2x^{-1})^9 = (y^3)^9 = y^{27}.$$

$$yx^2y^{-1} \Rightarrow y^{27} = x^3y^8x^{-3} = yx^2y^{-1}y^8yx^{-2}y^{-1} = yx^2y^8x^{-2}y^{-1} = yy^{18}y^{-1} = y^{18}.$$

so $y^9 = e$.

so $e = x^{-1}y^{9}x = (x^{-1}y^{3}x)^{3} = y^{6} \Rightarrow y^{3} = e \Rightarrow y^{2} = e \Rightarrow y = e \Rightarrow x = e.$