# Lec 8/30

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$f: G_1 \longrightarrow G_2$ gp hom.

$$\begin{array}{ccc} G_1 & \longrightarrow & G_2 \\ \cup & & \cup \\ \mathrm{Ker}(f) & & \mathrm{Im}(f) \\ \| & & \| \\ \{x \in G_1 : f(x) = e_2\} & & \{y \in G_2 : \exists x \in G_1 \ni f(x) = y\}. \end{array}$$

**Thm:** $G_1/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$

$$\begin{tikzcd} G_1/\mathrm{Ker}(f) \arrow[r, bend right, "\bar{f}"'] & \mathrm{Im}(f) \end{tikzcd}$$

**Lemma:** $f: G_1 \longrightarrow G_2$ surjective gp hom and $H \trianglelefteq G_1$.

$H = \mathrm{Ker}(f) \iff G_1/H \cong G_2$ induced from $f$.

**Pf** $(\Longrightarrow)$: $1^{st}$ Iso Thm.

$(\Longleftarrow)$: Let

$$\begin{tikzcd} G_1 \arrow[r, "\pi"] \arrow[dr, "f"'] & G_1/H \arrow[d, phantom, "\cong"] \\ & G_2 \end{tikzcd}$$

natural projection

So $\mathrm{Ker}(f) = \mathrm{Ker}(\pi) = H$.

**Ex:** $G_1 = \mathrm{Free}(2) \xrightarrow{\quad p \quad} G_2 = \mathbb{Z}^2$

$\geq$

$H = \langle \{\alpha\beta\alpha^{-1}\beta^{-1} : \alpha, \beta \in \mathrm{Free}(2)\} \rangle.$

$H \subset \mathrm{Ker}(p).$

Claim: $H$ is normal. Pf later

then

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] & \mathbb{Z}^2 \\ G_1/H \arrow[ur, phantom, "\cong"] & \end{tikzcd}$$

$a^m b^n \longleftrightarrow (m,n)$

$\|$

$\langle a, b \mid ab = ba \rangle$

Commutator subgroup

**Lemma** Let $G$ be a group & $H \leq G$ generated by $\{\underbrace{xyx^{-1}y^{-1}}_{\text{commutator of } x \text{ and } y.} : x, y \in G\}$ then $H \trianglelefteq G$.

to Prove: $aha^{-1} \in H \ \forall h \in H, a \in G$

Reduction 1: enough to show for generators: $a h_1 h_2 \cdots h_\ell a^{-1} = ah_1a^{-1}ah_2a^{-1} \cdots ah_\ell a^{-1}$.

Pf. let $a \in G$, $h = xyx^{-1}y^{-1}$. $a(xyx^{-1}y^{-1})a^{-1} = (axa^{-1})(aya^{-1})(axa^{-1})^{-1}(aya^{-1})^{-1} \in H$. $\square$

Reduction 2: we can also assume $a \in$ some set of generators of $G$ (associativity).

**Definition**: Conjugation by $a \in G$ is $\begin{matrix} G & \longrightarrow & G \\ \rotatebox{90}{$\in$} & & \\ x & \longmapsto & axa^{-1} \end{matrix}$.

(1): Conj(a) is a group homomorphism.

(2): $\mathrm{Conj}(a) \circ \mathrm{Conj}(b) = \mathrm{Conj}(ab)$

(3): $\mathrm{Conj}(e) = \mathrm{Identity}_G$

Hence $\underset{\substack{\uparrow \\ \text{inverse of a bijection.}}}{\mathrm{Conj}(a)^{-1}} = \mathrm{Conj}(a^{-1})$ so Conj(a) is an isomorphism

Precise defn. of a group given by generators & relns.

$A$ : a set (generators)

Free$(A) \ni w$ can be uniquely written as $x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}$ where $x_i \in A$, $n_i \in \mathbb{Z}_{\neq 0}$, $x_{i-1} \neq x_i$.

$R \subset \mathrm{Free}(A)$ (relations).

$$\langle A \mid R \rangle := \mathrm{Free}(A) \Big/ \begin{array}{l}\text{Smallest} \\ \text{Normal subgroup of Free(A)} \\ \text{containing } R.\end{array}$$

$$\| $$

$$\bigcap_{H \trianglelefteq \mathrm{Free}(2)} H \longrightarrow \text{Ex: } N_1 \cap N_2 \trianglelefteq G \text{ if } N_1, N_2 \trianglelefteq G$$

**Q**: how to define a group homomorphism $\mathrm{Free}(A) \longrightarrow H$? ← some group

**A**: Just specify where to send each $x \in A$.

Then $\underset{\mathrm{Free}(A)}{W} = x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell} \xmapsto{f} f(x_1)^{n_1} f(x_2)^{n_2} \cdots f(x_\ell)^{n_\ell}$.
(not ambiguous)

$$\left\{ \begin{matrix} \text{Group hom}^s \\ \mathrm{Free}(A) \longrightarrow H \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{set maps} \\ A \longrightarrow H \end{matrix} \right\}$$

**Q**: how to define a group hom. $G \longrightarrow H$ where $G = \langle A \mid R \rangle$?

**A**: (1) Specify where each $x \in A$ goes $\left( \rightsquigarrow \tilde{f} : \mathrm{Free}(A) \xrightarrow{\text{hom}} H \right)$.
(2) Make sure $\tilde{f}(r) = e \quad \forall\, r \in R$

**Ex**: $D_{2n} :=$ symmetries of regular $n$-gon. $|D_{2n}| = 2n$.

$\quad \cup$

$s, r =$ rotation by $\frac{2\pi}{n}$.

$(s^2 = e,\ r^n = e,\ srs = r^{-1})$.

**Define** $G = \langle \sigma, \rho \mid \sigma^2, \rho^n, \sigma\rho\sigma\rho \rangle$

$$\begin{array}{c} G \\ {\scriptstyle \text{hom}} \big\downarrow \quad \sigma \mapsto s,\ \rho \mapsto r \\ D_{2n} \end{array}$$

- Check that this map is onto
- check that $|G| = 2n$.

**Ex**: there exists a unique gp hom $D_{2n} \longrightarrow \{\pm 1\}$ ← gp w/ 2 elts.

s.t. $\left\{ \begin{matrix} s_1 \xmapsto{\text{sign}} -1 \\ s_2 \xmapsto{\text{sign}} -1 \end{matrix} \right.$

**Pf** we just check $\mathrm{sign}(s_1)^2 = \mathrm{sign}(s_2)^2 = (\mathrm{sign}(s_1)\mathrm{sign}(s_2))^n$. $\square$

**Note**: Can't do $\begin{matrix} s_1 \longmapsto -1 \\ s_2 \longmapsto 1 \end{matrix}$ if $n$ is odd.

Let's believe that

$$S_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = e \\ s_i s_j = s_j s_i \ \text{ if } |i-j| \geq 2 \end{array} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \right\rangle$$

$$\begin{array}{cc} S_n & \sigma_i \\ \downarrow & \downarrow \\ \{\pm 1\} & -1 \end{array}$$