Lec 11/3

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Koopmanism

if $T: X \to X$ is a measure-preserving invertible transformation on $(X, \mathcal{B}, \mu)$

then $Uf(x) := f(Tx)$ is a unitary operator.

$U: \mathbb{C}^n \to \mathbb{C}^n$ is unitary if $\|Ux\| = \|x\| \quad \forall x \in \mathbb{C}^n$.

$$\Updownarrow$$

$\langle Ux, Uy \rangle = \langle x, y \rangle$ (exercise: show this equivalence)

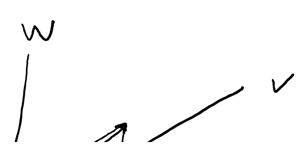
In $\mathbb{C}$	In invertible linear operators on $\mathbb{C}^n$.
$\lvert\lambda\rvert = 1$	Unitary $\to$ Appropos all have unimodular eigenvalues.
real: $\lambda = \bar{\lambda}$.	Self-adjoint $A = \overset{A^*}{\overset{\parallel}{A^t}}$. $\to$ real eigenvalues
Positive real	positive eigenvalues.
$a + bi = \rho(\cos\varphi + i\sin\varphi)$	$A = PU = \tilde{U}\tilde{P}$
	Projection $\to P^2 = P$, eigenvalues are 0 or 1.

Halmos : finite-dim vector spaces

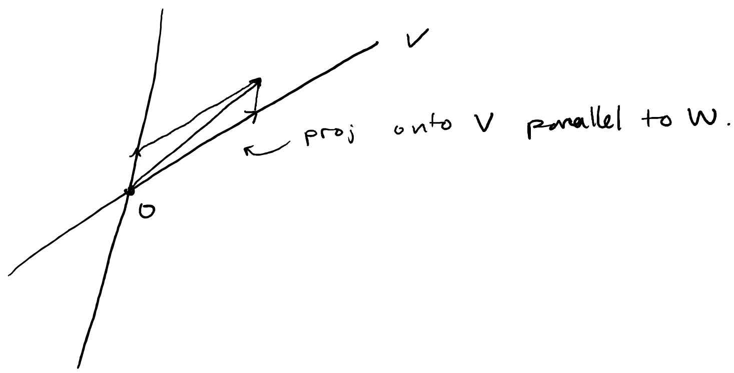
Theorem: $\forall$ unitary operator $U: \mathbb{C}^n \to \mathbb{C}^n$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} U^k = P$$ (projection operator on subspace of U-invariant vectors)

Projection



Projection



Proof Assume first that n (in $\mathbb{C}^n$) is 1. then

$$\frac{1}{N}\sum_{k=0}^{N-1}\lambda^k \longrightarrow \begin{cases} 1 & \lambda = 1 \\ 0 & \lambda \neq 1 \end{cases}$$

in abs value $= \left|\frac{1}{N}\frac{\lambda^N - 1}{\lambda - 1}\right|$ if $\lambda \neq 1$

$$\leq \frac{2}{N|\lambda - 1|} \to 0$$

if $n \neq 1$ then

$$AUA^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and so} \quad AU^kA^{-1} = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

So the $n=1$ case proves it.

$$\mathbb{C}^n = \{x : Ux = x\} \overset{\perp}{\oplus} \underbrace{\operatorname{span}\{x - Ux,\ x \in \mathbb{C}^n\}}$$ (exercise: prove this)

$\Rightarrow$ It is enough to prove $\frac{1}{N}\sum U^n y \to 0$ $\forall y \in \underbrace{}_{\circledast}$

$$y = y_1 + y_2$$

$$U^n y = U^n y_1 + U^n y_2$$

$$\| \qquad \downarrow$$

$$U y = U y_1 + U y_2$$
$$\quad\ \ \| \qquad\ \ \downarrow$$
$$\quad\ y_1 \qquad\ 0$$

part 1:

$z \in U$ invariant, $\langle z, x - Ux \rangle = \langle z, x \rangle - \langle z, Ux \rangle$

$= \langle z, x \rangle - \langle U^* z, x \rangle$

$= \langle z, x \rangle - \langle z, x \rangle$

$= 0.$

if $Ux = x$, then $U^* x = x$
since $U^* = U^{-1}$

it is enough to prove ⊛ for $y = x - Ux$ for all $x \in \mathbb{C}^n$.

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n y \right\| = \left\| \frac{(x - Ux) + U(x - Ux) + U^2(x - Ux) + \cdots + U^{N-1}(x - Ux)}{N} \right\|$$

$$= \left\| \frac{x - U^N x}{N} \right\| \leq \frac{2\|x\|}{N}$$

Also true:

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{k=M}^{N-1} U^k = P$$

Theorem of Von Neumann

Proof above is by F. Riesz

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(U_t x)\, dt \xrightarrow[T_2 - T_1 \to \infty]{\|\ \|} P_{inv} f = \int_X f\, d\mu \quad \text{if } U_t \text{ is ergodic.}$$

converges in norm

U_t is a m.p. transformation $\forall t$, measurable in t. (or cts, $U_t f \to f$ as $t \to 0$)

$\downarrow$

$f^* = f^*(U_t x) \quad \forall t.$

$$(**) \qquad \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \xrightarrow[N\to\infty]{a.e.} \int f \qquad \forall f \text{ iff } T \text{ is ergodic.}$$

multiply by g and integrate:

$$\frac{1}{N}\sum_{n=0}^{N-1}\int g(x)\, f(T^n x)\, dx = \int\left(g \int f\right) = \int f \int g$$

Now let $f = 1_A$, $g = 1_B$.

thus
$$\underbrace{\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}B) = \mu(A)\,\mu(B).}$$

$\mu(A\cap B) = \mu(A)\,\mu(B)$ independence.

$$\mu(A\cap T^{-n}B) = \mu(A)\,\mu(T^{-n}B)$$
$$= \mu(A)\,\mu(B)$$

direct independence, not always true.

Exercise: these two are equivalent

But ergodicity gives "on average" independence for all sets.

take $A = B$ so

$(*)$

Take $A = B$ so

$$\circledast \quad \frac{1}{N} \sum \mu(A \cap T^{-n} A) \longrightarrow \mu^2(A)$$

exercise : $\varphi(n) = \langle U^n x, x \rangle$ is p.d.

equiv. to T ergodic, sufficient to check for the generator sets of σ-algebra $\mathcal{B}$.

exercise: by checking $\circledast$ for (preimages of) intervals check that $x \mapsto 2x \mod 1$ is ergodic.

exercise : derive from ergodicity of $x \longmapsto 2x \mod 1$ and $(\ast\ast)$ that A.E. binary number is normal.

Hint/start Take $Tx = 2x \mod 1$, take $f = 1_{[0,\frac{1}{2}]}$

$$\frac{1}{N} \sum 1_{[0,\frac{1}{2}]}(T^n x) \xrightarrow{a.e.} \frac{1}{2}$$

$$\|$$

$$\frac{1}{N} \sum 1_{[0,1/2]}(2^n x \mod 1) \xrightarrow{a.e.} \int 1_{[0,1/2]} = \frac{1}{2}$$

So a.e. x is simply normal.