Lec 9/29

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Extreme Value Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$
then f attains a maximal and minimal
value on $[a,b]$

Extreme Value $\Rightarrow$ mean value $\Rightarrow$ Fundamental theorem of Calculus.

Definition: Suppose $f:S\to\mathbb{R}$. We say that f attains a maximal value on S at $x_{max}\in S$ if $f(x)\le f(x_{max})$ for all $x\in S$.

Definition: Suppose $f:S\to\mathbb{R}$. We say that f attains a minimal value on S at $x_{min}\in S$ if $f(x)\ge f(x_{min})$ for all $x\in S$.

First prove a weak version of EVT - that f does not blow up if it is continuous.
To make this precise, we need another definition.

Definition: Suppose $f:S\to\mathbb{R}$ we say that f is bounded on S if there is some number $B>0$ s.t. $|f(x)|\le B$ for all $x\in S$.

Remark: If f is bounded on S and on T, then it is bounded on $S\cup T$.
If $|f(x)|\le B_S$ $\forall x\in S$ and $|f(x)|\le B_T$ $\forall x\in T$,
Then $|f(x)|\le \max(B_S,B_T)$ $\forall x\in S\cup T$. ∎

Lemma: If f is continuous at $c\in \mathrm{dom}(f)$, then for some $\delta>0$,

f is bounded on $(c-\delta, c+\delta) \cap dom(f)$.

Proof: Pick $\varepsilon = 1$, then find $\delta > 0$ s.t.

$$|x-c| < \delta \ \& \ x \in dom(f) \Rightarrow |f(x) - f(c)| < 1$$
$$\Rightarrow |f(x)| < \underbrace{|f(c)| + 1}_{\text{bound.}}$$

∎

Weak EVT: if $f: [a,b] \to \mathbb{R}$ is continuous, it is bounded on $[a,b]$.

Proof: by Contradiction. Suppose that f is not bounded on $[a,b]$.

Recursively define a sequence of nested intervals $[a_n, b_n]$ on which f is not bounded.

Let $a_0 = a$, $b_0 = b$. having defined a_n, b_n. Let $c_n = \frac{a_n + b_n}{2}$.

$[a_n, b_n] = [a_n, c_n] \cup [c_n, b_n]$. By the remark above, it cannot be the case that f is bounded on both subintervals.

if f is not bounded on $[a_n, c_n]$, Let $a_{n+1} = a_n$, $b_{n+1} = c_n$

if f is bounded on $[a_n, c_n]$, it must be unbounded on $[c_n, b_n]$ so let $a_{n+1} = c_n$ and $b_{n+1} = b_n$.

note that $b_n - a_n = \frac{b_0 - a_0}{2^n} \to 0$. (no infinitessimals in $\mathbb{R}$)

and $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$

So the NIP applies to $\{[a_n, b_n]\}_{n=1}^{\infty}$

So $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{c\}$ for some $c \in [a,b]$.

By lemma, we can find a $\delta > 0$ such that f is bounded on $(c-\delta, c+\delta) \cap dom(f)$.

Otoh, by NIP, we can find an N s.t. $[a_n, b_n] \subseteq (c-\delta, c+\delta) \cap [a,b]$ for $n > N$.

But we've constructed this interval so f is unbounded on $[a_n, b_n]$. This is a contradiction. ∎

Weak EVT $\Rightarrow$ EVT (strong)

Proof: by weak EVT, $f([a,b]) \subseteq [-B, B]$ for some $B > 0$

hence $f([a,b])$ is bounded above. Let $u = \sup(f([a,b]))$.

Then $f(x) \le u$ for all $x \in [a,b]$.

it suffices to show that $f(x) = u$ for some $x \in [a,b]$.

* We prove this by contradiction.

Suppose $f(x) < u$ for all $x \in [a,b]$. $\rightarrow u - f(x) > 0$.

Now consider $g(x) = \frac{1}{u-f(x)}$. Then g is continuous

on $[a,b]$ because its denominator never $= 0$ and f continuous.

By weak EVT, g is then bounded on $[a,b]$.

i.e. $0 < g(x) \le \overline{B}$ for some $\overline{B}$. and all $x \in [a,b]$

Then $\frac{1}{u-f(x)} \le \overline{B}$

$$u - f(x) \geq \frac{1}{\overline{B}}$$

$$u - \frac{1}{\overline{B}} \geq f(x) \quad \forall x \in [a,b].$$

so $u - \frac{1}{\overline{B}}$ is an upper bound for $f([a,b])$

but u was the least upper bound.

so we have a contradiction.

So $f(x) = u$ for some x.

So f attains a maximal value on $[a,b]$. ∎

a similar argument shows f attains a minimal value on $[a,b]$.

A quicker way: f attains min. val. $\Leftrightarrow$ $-f$ attains max val. ∎

For HW: Do the thing with $\lim_{x\to\infty} f(x) = \infty$ by setting $N = f(0)$!

Consequences of the IVT and EVT for polynomials.

<u>Theorem</u>: any polynomial of odd degree has at least one real root.

<u>Remark</u>: $P(x) = \sum_{j=0}^{n} a_j x^j \qquad n \text{ odd}, \; a_n \neq 0.$

$$P(x) = 0 \iff \frac{1}{a_n} P(x) = 0.$$

wolog, assume $a_n = 1$.

and $p(x) = x^n + \sum_{j=0}^{n-1} a_j x^j = x^n \left(1 + \sum_{j=0}^{n-1} a_j \frac{1}{x^{n-j}}\right)$

<u>Lemma</u>: for some $M > 0$, $|x| \geq M \Rightarrow 1 + \sum_{j=0}^{n-1} a_j \frac{1}{x^{n-j}} \geq \frac{1}{2}$

<u>Proof</u>: assume $|x| \geq 1$. Then $\frac{1}{|x|^{n-j}} \leq \frac{1}{|x|}$

So, $$\left|\sum_{j=0}^{n-1} a_j \frac{1}{x^{n-j}}\right| \leq \sum_{j=0}^{n-1} a_{n-j} \frac{1}{|x|^{n-j}}$$

$\vdots$

$$1 + \sum_{j=0}^{n-1} a_j \frac{1}{x^{n-j}} \leq \frac{1}{2} \quad ?$$

if $|x| \geq M = \max\left(1, 2\sum_{j=0}^{n-1} |a_j|\right)$. ∎