Lec 3/30

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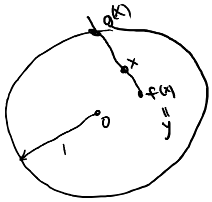

$$\underbrace{|y+t(x-y)|^2 - 1}_{h(x,y,t)} = 0$$

$$h : G \longrightarrow \mathbb{R}$$

$G = \{(x,y,t) : x, y \in \mathbb{R}^n,\ x \neq y,\ t > 0\}$, open in $\mathbb{R}^{2n+1}$

Solve for t in terms of x and y, and then

$$g(x) = f(x) + t(x,y)\,(x - f(x)).$$

Show $\dfrac{\partial h}{\partial t} \neq 0$, then apply implicit f_n theorem.

$\chi : \mathcal{U}$ open $\subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V$ open $\subseteq M$, χ a C^2 coord. patch.

<u>Propn 4.2</u>

(a) (Gauss's Formulae)

$$\chi_{ij} = L_{ij}\, n + \sum_k \Gamma_{ij}^k \chi_k$$

Where $L_{ij} = \langle \chi_{ij} | n \rangle$ and $\Gamma_{ij}^k = \sum_\ell \langle \chi_{ij} | \chi_\ell \rangle g^{\ell k}$

(b) For any C^2 unit speed curve $s \longmapsto \gamma(s) = \chi(\gamma^1(s), \gamma^2(s))$ in V,

We have $K_n = \sum_{i,j} L_{ij} \dfrac{d\gamma^i}{ds} \dfrac{d\gamma^j}{ds}$

and $K_g S = \sum_k \left[\dfrac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \dfrac{d\gamma^i}{ds} \dfrac{d\gamma^j}{ds} \right] \chi_k$

<u>Pf</u> (a) we did last time. for (b):

$$\kappa N = \frac{dT}{ds} = K_n n + K_g S. \text{ but } \frac{dT}{ds} = \frac{d^2\gamma}{ds^2} = \frac{d}{ds}\frac{d\gamma}{ds} = \frac{d}{ds}\left[\chi_1 \frac{d\gamma^1}{ds} + \chi_2 \frac{d\gamma^2}{ds}\right]$$

$$= \sum_i \left[\left(\frac{d}{ds} \frac{\partial \chi}{\partial u^i} \right) \frac{d\gamma^i}{ds} + \frac{\partial \chi}{\partial u^i} \frac{d^2 \gamma^i}{ds} \right]$$

$$= \sum_i \left[\left(\frac{d}{ds} \frac{\partial x}{\partial u^i} \right) \frac{d\gamma^i}{ds} + \frac{\partial x}{\partial u^i} \frac{d^2 \gamma^i}{ds^2} \right]$$

$$= \sum_i \left[\left(\sum_j \frac{\partial^2 x}{\partial u^i \partial u^j} \frac{d\gamma^j}{ds} \right) \frac{d\gamma^i}{ds} + \frac{\partial x}{\partial u^i} \frac{d^2\gamma^i}{ds^2} \right]$$

$$= \sum_i \sum_j \left(x_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} \right) + \sum_i x_i \frac{d^2\gamma^i}{ds^2}$$

$$= \sum_i \sum_j \left[\left(L_{ij}\, n + \sum_k \Gamma_{ij}^k x_k \right) \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} \right] + \sum_k x_k \frac{d^2\gamma^k}{ds^2}$$

$$= \underbrace{\left(\sum_{i,j} L_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} \right) n}_{\substack{\text{normal to surface.} \\ \text{hence } \kappa_n n}} + \underbrace{\sum_k \left(\frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} \right) x_k}_{\substack{\text{tangential to surface.} \\ \text{hence } \kappa_g S}}$$

Which proves the proposition $\square$

eg Define $x : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $x(u^1, u^2) = (u^1, u^2, u^1 u^2)$. x is C^∞.

$x_1 = (1, 0, u^2)$, $x_2 = (0, 1, u^1)$, $x_{12} = x_{21} = (0, 0, 1)$, $x_{11} = x_{22} = (0, 0, 0)$

$x_1 \times x_2 = (-u^2, -u^1, 1)$, which is never 0, so x is an immersion

$g = \det(g_{ij}) = |x_1 \times x_2|^2 = (u^2)^2 + (u^1)^2 + 1$ so

$$n = \frac{x_1 \times x_2}{\sqrt{g}} = \frac{(-u^2, -u^1, 1)}{\sqrt{(u^2)^2 + (u^1)^2 + 1}}$$

where $(g_{ij}) = (\langle x_i | x_j \rangle) = \begin{pmatrix} 1 + (u^2)^2 & u^1 u^2 \\ u^1 u^2 & 1 + (u^1)^2 \end{pmatrix}$

so $(L_{ij}) = (\langle x_{ij} | n \rangle) = \frac{1}{\sqrt{g}} (\langle x_{ij} | x_1 \times x_2 \rangle) = \frac{1}{\sqrt{g}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $M = \{ x(u) : u \in \mathbb{R}^2 \}$. Then $\forall\, v = (v^1, v^2, v^3) \in M$ we have

$x^{-1}(v^1, v^2, v^3) = (v^1, v^2)$. Thus x^{-1} is continuous

So x is a global C^∞ coordinate patch on M so M is a C^∞ surface in $\mathbb{R}^3$.

Consider a unit speed curve $s \longmapsto \gamma(s) = x(\gamma^1(s), \gamma^2(s))$ which passes through the point $(0,0,0)$ when $s=0$.

let $a = \left(\frac{d\gamma^1}{ds}, \frac{d\gamma^2}{ds} \right)\Big|_{s=0}$. Then $\frac{d\gamma}{ds} = \frac{dx}{du^1}\frac{d\gamma^1}{ds} + \frac{\partial x}{\partial u^2}\frac{d\gamma^2}{ds} = (1,0,\gamma^2(s))\frac{d\gamma^1}{ds} + (0,1,\gamma^1(s))\frac{d\gamma^2}{ds}$

So $\frac{d\gamma}{ds}\Big|_{s=0} = (a^1, a^2, 0)$, so $T(0) = (a^1, a^2, 0)$.

(this is because $T_{(0,0,0)}M$ is horizontal)

Since $T(0) = (a^1, a^2, 0)$ and $|T(0)| = 1$, $(a^1)^2 + (a^2)^2 = 1$.

Now $K_n(0) = \left(\sum_{i,j} L_{ij} \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} \right)_{s=0} = \sum_{i,j} L_{ij}(0,0)\, a^i a^j$

$$= \frac{0 + a^1a^2 + a^2a^1 + 0}{\sqrt{g(0,0)}} = 2a^1a^2$$

for such a curve γ, the maximum possible value of $K_n(0)$ is called
$K_1(0,0) = 2\cdot\left(\frac{1}{\sqrt{2}}\right)^2 = 1$. The minimum possible value is
$K_2(0,0) = 2\cdot\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -1$.

$K_1(0,0)$, $K_2(0,0)$ are called the 'principal curvatures' of x at $(0,0)$ or of M at $(0,0,0) = x(0,0)$

The Gaussian Curvature of x at $(0,0)$ (or M at $(0,0,0) = x(0,0)$) is

$$K(0,0) = K_1(0,0)\,K_2(0,0) = 1\cdot -1 = -1.$$

(more on principal + Gaussian curvature in §4-8).