Dual Modules: R: unital, commutative.
M: R-module.

Def: dual module $M^* = \mathrm{Hom}_R(M,R)$
Elements of M^* are called linear forms on M, or linear functions on M, or covectors, or linear functionals $f: M \to R$.

Properties: $R^* \cong R$, $(M_1 \oplus M_2)^* \cong M_1^* \oplus M_2^*$

$\mathrm{Hom}(M_1 \oplus M_2, R) \cong \mathrm{Hom}(M_1, R) \oplus \mathrm{Hom}(M_2, R)$

$(R^n)^* \cong (R^*)^n \cong R^n$. $(M^n)^* \cong (M^*)^n$ by induction.

$$\left(\bigoplus_{\alpha \in \Lambda} M_\alpha\right)^* \cong \prod_{\alpha \in \Lambda} M_\alpha^*$$

Bc $(u_1, u_2, \dots, 0, \dots) \xrightarrow{f} R$

$(f_1(u_1), f_2 u_2), \dots 0, \dots) = \sum^{\text{finite}} f_i(u_i)$

So $\left(\bigoplus M_i\right)^*$ are sequences $(f_i) \in \prod M_i^*$

$u \in M$, $f \in M^* \Rightarrow f(u) \in R$

$$\begin{aligned} M \times M^* &\longrightarrow R \\ (u, f) &\longmapsto f(u) \end{aligned}$$ is bilinear.

So we have a homomorphism

$$\begin{aligned} M \otimes M^* &\longrightarrow R \\ u \otimes f &\longmapsto f(u) \end{aligned} \qquad \sum a_i u_i \otimes f_i \overset{\text{contraction}}{\longmapsto} \sum a_i f_i(u_i)$$

This mapping is called contraction of (1,1)-tensors.

Define $u(f) = f(u)$ - this is a linear form on M^*
that is, an element in $(M^*)^*$.

So we have a hom $M \longrightarrow M^{**}$
$u \longmapsto$ form defined by $u(f) = f(u)$.

Proposition: $\forall M, N,\ (M \otimes N)^* \cong \mathrm{Hom}(M, N^*)$.

proof: $\mathrm{Hom}(M \otimes N, R) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, R))$. (we had this for $R = K$, general).

$f \in (M \otimes N)^* \Rightarrow \forall u \in M,\ f(u \otimes \cdot) : N \longrightarrow R$.

If $\varphi : M \to N$ is a hom-sm, then we have a dual hom-sm $\varphi^* : N^* \to M^*$ defined by

$\varphi^*(f) = f \circ \varphi \quad \forall f \in N^*$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\varphi^*(f)"'] & N \arrow[d, "f"] \\ & R \end{tikzcd}$$

$\forall \quad A \xrightarrow{\varphi} B \xrightarrow{\psi} C, \quad (\psi \circ \varphi)^* = \varphi^* \circ \psi^*$

$* = \mathrm{Hom}(\cdot, R)$ – a contravariant functor from R-Mod to R-Mod.

$*$ is a left-exact functor, $*$ is exact iff R is injective as an R-module.

(but $\mathbb{Z}$ is not injective).

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0 \quad \text{exact}$$

$$\Downarrow$$

$$0 \longrightarrow C^* \xrightarrow{\psi^*} B^* \xrightarrow{\varphi^*} A^* \quad \text{exact}$$

$$A \subseteq B \qquad \mathrm{Ker}\,\varphi^* = \overset{\mathrm{Ann}(A)}{\overset{\shortparallel}{\{f \in B^* : f|_A = 0\}}}$$

$= \{f \in B^*$ that define elements of $C^*\}$

↳ since $C = B/A$ so if $\mathrm{Ker}(f) \supseteq A$ then f is on B/A.

$= \psi^*(C^*)$

If $A \subseteq B$, then $\mathrm{Ann}(A) = \{f \in B^* : f|_A = 0\}$

$= \mathrm{Ker}(B^* \longrightarrow A^*)$

$$\underbrace{M \otimes \cdots \otimes M}_{k} \otimes \underbrace{M^* \otimes \cdots \otimes M^*}_{\ell}$$

elements here are called (k, ℓ)-tensors, k-times contravariant, ℓ-times covariant.

Let R be an integral domain

M is divisible if $\forall\, u \in M$, $\forall\, a \in R \setminus \{0\}$,

$u = av$ for some $v \in M$.

this means the mapping $\begin{array}{l} M \longrightarrow M \\ v \longmapsto av \end{array}$ is surjective.

Theorem: If a module M is injective then it is divisible.

(note: $\mathbb{Z}$ - not injective but $\mathbb{Q}$ is injective, so is $\mathbb{Q}/\mathbb{Z}$ (as $\mathbb{Z}$-modules)).

If R is a PID then this is a criterion for injectivity.

Proof:
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\varphi"] \arrow[dr, "\alpha"'] & B \arrow[d, "\beta"] \\ & & M \end{tikzcd}$$
$\forall \alpha\ \exists \beta$ s.t. diagram commutes (if M is injective).

Consider
$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook, "\varphi"', "b \mapsto ab"] \arrow[dr, "\alpha"'] & R \arrow[d, "\beta"] \\ & & M \end{tikzcd}$$
injective since R is ID.

let $u \in M$. define $\alpha : R \longrightarrow M$ by $\alpha(1) = u$.

Let $\beta : R \longrightarrow M$ be such that $\alpha = \beta \circ \varphi$

Let $v = \beta(1)$. then $u = \alpha(1) = \beta(\varphi(1)) = \beta(a) = av$.

Midterm: R - unital & commutative.

① Def of Modules

Submodules, generators

Quotient modules

Torsion elements, Torsion submodule.

(2) Homomorphisms of Modules, Ker, Im

Isomorphism Theorems

Module $H(M,N)$, Algebra $End(M)$

Commutative diagrams, exact sequences

(3) Direct products & direct Sums

$M_1, M_2 \Longrightarrow M_1 \oplus M_2$

or: $M_1, M_2 \subseteq M \overset{?}{\Rightarrow} M = M_1 \oplus M_2$

Universal Properties

(4) Free modules, bases, max lin indp subsets, Rank

(5) Tensor Products

(6) Tor_0, $Hom(\cdot, K)$, $Hom(K, \cdot)$, flat, injective, projective modules.