Lec 9/30

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Applications of IVT & EVT to polynomials.

Theorem: Any odd-degree polynomial has at least 1 real root.

$$p(x) = \sum_{j=0}^{n} a_j x^j \quad (n \text{ is odd}, \ a_n \neq 0)$$

By dividing out by a_n we may replace $p(x)$ by an equivalent polynomial with $a_n = 1$. So, wolog, we can assume

$$p(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$$

$$= x^n \left(1 + \sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}}\right).$$

Lemma: if $|x| \geq 1$ then $\left|\sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}}\right| \leq \frac{\sum_{j=0}^{n-1} |a_j|}{|x|}$

Proof: $\left|\sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}}\right| \leq \sum_{j=0}^{n-1} \frac{|a_j|}{|x|^{n-j}} \leq \sum_{j=0}^{n-1} \frac{|a_j|}{|x|}$ (since $|x| \geq 1$) ∎

Corollary: if $|x| \geq \max(1, 2\sum_{j=0}^{n-1} |a_j|)$, then $1 + \sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}} \geq \frac{1}{2}$

Proof: by lemma: $\left|\sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}}\right| \leq \frac{\sum_{j=0}^{n-1} |a_j|}{|x|} \leq \frac{\sum_{j=0}^{n-1} |a_j|}{2\sum_{j=0}^{n-1} |a_j|} = \frac{1}{2}$

$\Rightarrow 1 + \sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}} \in \left[\frac{1}{2}, \frac{3}{2}\right]$ so it's greater than $\frac{1}{2}$.

Proof of Theorem: if $|x| \geq \underbrace{\max(1, 2\sum_{j=0}^{n-1} |a_j|)}_{M}$, then

$p(x) = x^n \cdot$ (some positive number)

so $p(x)$ has the same sign as x^n.

so $p(M) > 0$, $p(-M) < 0$

so by IVT, $p(x) = 0$ for some $x \in (-M, M)$. ∎

Definition: if $f: (-\infty, \infty) \to \mathbb{R}$ has a maximum over $(-\infty, \infty)$ at x_{max}, we say f has a 'global maximum' at x_{max}.

'global minimum' is defined similarly.

Theorem 2: Let $p(x)$ be a polynomial of even degree. Then p has a global minimum if a_n is positive and a global maximum if a_n is negative.

Remark: it suffices to consider ol nomia's with $a_n = 1$. If $p(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$ has a global minimum then $A\,p(x)$ has a global min if $A > 0$ and a global max if $A < 0$.

Proof: by same argument as theorem 1,

$p(x)$ is $x^n\left(1 + \sum_{j=0}^{n-1} \frac{a_j}{x^{n-j}}\right) \geq \underbrace{\tfrac{1}{2}x^n}_{\text{positive.}}$ if $|x| \geq M_1$ (M in theorem 1)

Let $M = \max\left(\sqrt[n]{2|p(0)|},\ M_1\right)$. ($n^{th}$ roots exist: $f(x) = x^n - a \Rightarrow f(0) < 0$, $f(x) > 0$ for x large)

Then if $|x| > M$, $p(x) \geq \frac{1}{2}x^n = \frac{1}{2}|x|^n > \frac{1}{2}2|p(0)| \geq p(0)$.

Then EVT $\Rightarrow$ p has a minimum over $[-M, M]$ ($x_{min} \in [-M, M]$)

so $p(x) \geq p(x_{min})$ $\forall x$.

if $x \in [-M, M]$ true by def.

if $x \notin [-M, M]$ then $|x| > M$

$\Rightarrow p(x) > p(0) \geq p(x_{min})$ ∎