Monday, August 26, 2019 10:18

Bosic Properties of masures: (X, M, M) musure space.

- (both in M) = M(F) & M(F)
- ② (En) cm = M(UEn) ≤ ZM(En)
- 3 E, CE2 CE3 CON = (UEn) = (IM M(En) (all in m)
- (continuity from a bove) EDEDEZOEZO. (all in M), W M(E,) < 00 => u (NEn) = lim u(En)

f Let Fn=E, En. Then since E, sEz>Ez>..., F. C.F. C.F. C.F. Moreover, $\mu(E_i) = \mu(E_p) + \mu(F_p)$

Then UFn = U(E, En) = E, (n En). So $\lim_{n} \mu(F_n) = \mu(UF_n) = \mu(E_1) - \mu(nE_n)$ Everything is $< \infty$ so so subtraction is $\lim_{n} \mu(E_1) - \mu(E_n) = E_1 - \lim_{n} \mu(E_n)$ ok. So M(NEn) = lim M(En).

 $\underline{\text{Cor}}$: If $E \in M$ s.t. $\mu(E) = 0$, and $F \in M$ s.t. $f \in E$, M(F)=0 too.

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Thum (completion): Spose
$$(X, M, M)$$
 is a measure space. define $\overline{M} = \{ E \cup F \mid E \in M \text{ and } \exists N \in M \text{ s.e. } \mu(N) = 0 \text{ and } F \subset N \}.$
Then $O \in M$ is a σ -algebra $W \in M$

(1) Suppose
$$(E_n \cup F_n)$$
 is a sequence in \overline{m} . Then $U(\overline{t}_n \cup F_n) = (UE_n) \cup (UF_n)$. For each n , choose \overline{m} .

$$N_n \in \mathbb{M}$$
 s.t. $\mu(N_n) = 0$ and $F_n \subset N_n$. Then $UF_n \subset UN_n$ and $\mu(UN_n) \leq \sum \mu(N_n) = 0$. So $U(E_n \cup F_n) \in \overline{\mathbb{M}}$.

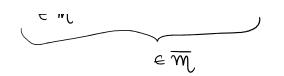
$$\frac{t_{\text{rick}}: X = N \coprod N^{c}}{(E_{\text{U}}F)^{c}} = E^{c} \cap F^{c} \cap X$$

$$= E^{c} \cap F^{c} \cap [N \coprod N^{c}]$$

$$= (E^{c} \cap F^{c} \cap N^{c}) \coprod (E^{c} \cap F^{c} \cap N)$$

$$= N^{c}$$

$$= M$$



Hence m is a o-algebra, m c m.

IT: Suppose Then Y EUF C TO with F C N nonelly

 $\mu(E) = \overline{\mu}(E) \leq \overline{\mu}(E \cup F) \leq \overline{\mu}(E) + \overline{\mu}(F) \leq \overline{\mu}(E) + \overline{\mu}(N) = \mu(E) + \mu(N) = \mu(E).$

3 so μ(E υ F) = μ(E). Let μ(E υ F):=μ(E)

This is well-defined:

if E, UF, = EZUFZ, Then

E C E UF = EZUFZ C EZUNZ

so $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2).$

Similarly M(Ez) = M(E,) So

I is ind. of choice of representative EVF.

- $0 \ \overline{\mu}(\phi) = \mu(\phi) = 0.$
- o suppose (En UFn) c m ore disjoint.

Then so are (En).

 $\overline{\mu} \left(\underline{\prod} (E_n \cup F_n) \right) = \overline{\mu} \left(\underline{H} E_n \cup \underline{\Pi} F_n \right) \\
= \underline{\mu} (\underline{H} E_n) \\
= \underline{\mu} (E_n) \\
= \underline{\mu} (E_n \cup F_n) .$

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Outer Measures:

$$\mu^*: P(X) \longrightarrow [0,\infty]$$
 s.t.

Strategy:

Suppose
$$\rho: \mathcal{E} \longrightarrow (0, \infty)$$
 s.t. $\rho(\emptyset) = 0$.

Define
$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \middle| E_n \in E, E \in UE_n \right\}$$

They is an outer mensure.

$$H = 0$$
 Taking $f_n = \emptyset + 1$, $u^*(\emptyset) = 0$.

②
$$T_{\underline{Y},\underline{C},\underline{k}}: \varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$
.

Suppose (En) is a seq. of sets. let E>O.

For each
$$n_j$$
 $\exists (F_j^n)_{j=1}^\infty$ st. $E_n \in \bigcup_j F_j^n$

and
$$\sum_{j} \rho(F_{j}^{n}) \leq \mu^{*}(E_{n}) + \frac{\varepsilon}{2^{n}}$$
.

Thun
$$VE_n = \bigcup_{n \in J} F_j^n$$
. So

$$\mu^{*} (UE_{n}) \leq \sum_{n} \sum_{j} \rho(F_{j}^{n})$$

$$\leq \sum_{n} (\mu^{*}(E_{n}) + \frac{\varepsilon}{2n})$$

$$= \sum_{n} \mu^{*}(E_{n}) + \sum_{n} \frac{\varepsilon}{2n}$$

$$= \sum_{n} \mu^{*}(E_{n}) + \varepsilon.$$

Since E>0 was orbitrary, M*(UEn) & Zu*(En). "