Lec 2/24

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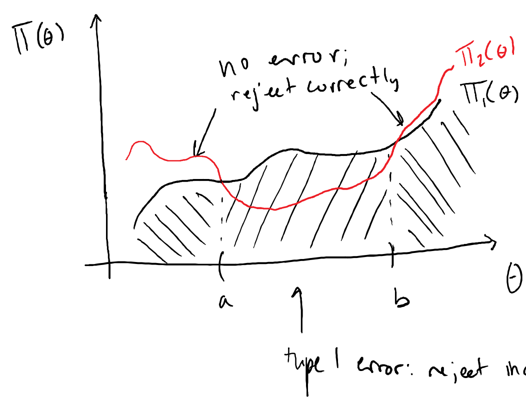
Power function $\pi(p) = \mathbb{P}(\text{reject } H_0 ; p)$

In simple vs. simple, $\alpha, 1-\beta$ are only values for type I error rate and power.

In composite vs composite, there are multiple values seen in power curve.

Recall: $\overset{\text{power function}}{\pi(\theta)} = \mathbb{P}(\text{reject } H_0 ; \theta) = \begin{cases} \alpha(\theta) & H_0 \text{ holds} \\ 1-\beta(\theta) & H_1 \text{ holds} \end{cases}$

e.g. $H_0 : \theta \in (a,b)$ vs $H_1 : \theta \notin (a,b)$

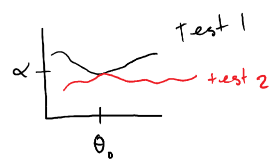


Power function is computed according to the "critical region" of the test.

Test 2 is a more powerful & better test.

Comparison of tests via power functions.

Ex: $H_0 : \theta = \theta_0$, $H_1 : \theta \neq \theta_1$



both test 1 & test 2 have same rate of type I error. (both of size α)

✲ Test 1 is uniformly more powerful than 2. So just compare powers, test 1 is better.

Def 12.6 There is a Uniformly Most Powerful (UMP) test.

Section 12.6 Likelihood Ratio tests (LRT).

Recall that NP lemma gives most powerful test when H_0, H_1 simple.

Not necessarily MP test when H_0 or H_1 composite.

let $X_1, \dots, X_n$ be a RS of size n from a density $f(x;\theta)$,

let Ω = all possible values of θ.

suppose want to test $H_0: \theta \in \omega$, $H_1: \theta \in \omega'$ where $\omega \subseteq \Omega$ and $\omega' = \Omega \setminus \omega$.

Previously, $|\omega| = |\omega'| = 1$ and looked at ratios of likelihoods, now use max val of Likelihoods.

Let $\max L_0 = \max_{\theta \in \omega} \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} f(x_i;\hat{\theta})$ where $\hat{\theta}$ is MLE of θ on ω

$\max L = \max_{\theta \in \Omega} \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} f(x_i;\hat{\hat{\theta}})$ where $\hat{\hat{\theta}}$ " on Ω

$$\Lambda = \frac{\max L_0}{\max L} = \text{likelihood ratio statistic (LRS)}, \quad 0 \le \Lambda \le 1.$$

under H_1 $\max L_0 \approx \max L \Rightarrow \Lambda \approx 1$

under H_0 $\max L_0 \ll \max L \Rightarrow \Lambda$ small

test: reject H_0 if Λ is too small (smaller than $k \in (0,1)$).

Question: why L instead of L_1? because then $k \in (0,1)$.

Q: how do we choose k?

If H_0 simple, pick so that CR is of size α

If H_0 composite, pick so that type I error $\le \alpha$ for all $\theta \in \omega$, with

- $*_1$ equality at more $\theta \in \omega$ to increase the power.
- $*_2$ equality at at least one θ if possible

Note: for testing simple vs simple case $H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$

$\max L_0 = L_0(\vec{x})$

$\max L = \max \begin{cases} L_0(\vec{x}) \\ L_1(\vec{x}) \end{cases}$ $\quad \omega = \{\theta_0\}$ $\quad \omega' = \{\theta_1\}$ $\quad \Omega = \{\theta_0, \theta_1\}$

So NP - test: $\frac{L_0(\vec{x})}{L_1(\vec{x})} \le K$, LRT: $\frac{L_0}{\max(L_0, L_1)} \le \overset{(0,1)}{\tilde{K}}$

the two tests are not equivalent: consider $K = 2$.

Ex: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \overset{\text{known}}{\sigma^2})$. find CR of LRT for $H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$.

sol. $\omega = \{\mu_0\}$ so $\hat{\mu} = \mu_0$, $\Omega = \mathbb{R}$ so $\hat{\hat{\mu}} = \bar{X}$.

$$\frac{\max L_0}{\max L} = \exp\left(\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left((X_i - \bar{X})^2 - (X_i - \mu_0)^2 \right) \right) = \exp\left(-\frac{n}{2\sigma^2} (\bar{X} - \mu_0)^2 \right)$$

LRT $\Rightarrow$ reject H_0 when $\exp\left(\frac{-n}{2\sigma^2} (\bar{X} - \mu_0)^2 \right) \le k$

$$\frac{-n}{2\sigma^2} (\bar{X} - \mu_0)^2 \le \log(k)$$

$$(\bar{X} - \mu_0)^2 \ge \frac{\log(k) 2\sigma^2}{-n}$$

$$\bar{X} \ge \underbrace{\sqrt{\frac{\log(k) 2\sigma^2}{-n}} + \mu_0}_{\tilde{K}}$$

$\tilde{K}$ determined s.t. CR has size α.