Lec 9/18

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Corollaries of MCT

automatic if $(X, \mathcal{M}, \mu)$ is complete.

(4) If $(f_n) \subset L^+$, $\underbrace{f \in L^+}$, and $f_n \nearrow f$ a.e., then $\int f = \lim \int f_n$.

pf Suppose $f_n \nearrow f$ on $E \in \mathcal{M}$ and E^c is μ-null.

Then $f - f\chi_E = 0$ a.e., $f_n - f_n\chi_E = 0$ a.e..

$$\int f = \underbrace{\int f\chi_E = \lim \int f_n \chi_E}_{\text{MCT.}} = \lim \int f_n . \qquad \square$$

(5) (Fatou's Lemma) If $(f_n) \subset L^+$,

$$\int \liminf f_n \leq \liminf \int f_n$$

pf $\forall j \geq k \in \mathbb{N}$, $\inf\limits_{n \geq k} f_n \leq f_j$, so

$$\int \inf_{n \geq k} f_n \leq \int f_j \quad \forall j \geq k.$$

Thus $\int \inf\limits_{n \geq k} f_n \leq \inf\limits_{j \geq k} \int f_j$. Letting $k \to \infty$ by MCT,

$$\int \liminf f_n = \lim_{k \to \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \to \infty} \inf_{n \geq k} \int f_n = \liminf \int f_n . \qquad \square$$

(6) $(f_n) \subset L^+$, $f \in L^+$, Suppose $f_n \to f$ a.e.

Then $\int f \le \liminf \int f_n$.

Pf Suppose $f_n \to f$ a.e. on E, E^c is μ-null.

Then $\int f_n = \int f \chi_E \overset{\text{fatou.}}{\le} \liminf \int f_n \chi_E = \liminf \int f_n$. $\square$

(7) If $f \in L^+$ and $\int f < \infty$, then $\{f = \infty\}$ is μ-null and $\{f > 0\}$ is σ-finite.

Pf Exercise (Doesn't need MCT).

Integration of $\overline{\mathbb{R}}$-valued fns

$(X, \mathcal{M}, \mu)$ measure space.

Def $f : X \to \overline{\mathbb{R}}$ $\mathcal{M}$-mble is called integrable if

$\int f_\pm < \infty$, where $f = f_+ - f_-$, $f_+ = f \vee 0$, $f_- = -(f \wedge 0)$.

observe f integrable $\Leftrightarrow \int |f| < \infty$.

Define $\int f := \int f_+ - \int f_-$.

$L^1(\mu, \overline{\mathbb{R}}) = \{\text{integrable fns } f : X \to \overline{\mathbb{R}}\}$

Prop: $L^1(\mu, \bar{\mathbb{R}})$ is a real v.s.

Moreover, $\int : L^1(\mu, \bar{\mathbb{R}}) \to \mathbb{R}$ is linear.

($ISF := SF \cap L^1(\mu, \bar{\mathbb{R}})$ is also a v.s.)

proof If $r \in \mathbb{R}$, $f, g \in L^1$, then

$|rf + g| \leq |r||f| + |g|$ integrable.

$\Rightarrow$ L^1 is an $\mathbb{R}$-v.s.

- If $r \in \mathbb{R}$ and $f \in L^1$, $\int rf = \int (rf)_+ - \int (rf)_-$

(if $r > 0$) $= \int rf_+ - \int rf_-$

(if $r < 0$, more cases). $= r\int f_+ - r\int f_-$

$= r\left(\int f_+ - \int f_-\right) = r\int f.$

- If $f, g \in L^1$, $(f+g)_+ - (f+g)_- = f_+ - f_- + g_+ - g_-$.

$\Rightarrow (f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+ \in L^+$

corollary 3 of MCT

$\Rightarrow \int (f+g)_+ + \int f_- + \int g_- = \int (f+g)_- + \int f_+ + \int g_+.$

rearrange.

$\square$

Ex: $c_k \neq 0$ measurable

① show $\psi = \underbrace{\sum c_k \chi_{E_k}}_{\substack{\text{not necessarily} \\ \text{in standard form.}}}$ integrable $\iff \mu(E_k) < \infty \ \forall k$.

② If ψ is integrable, $\int \psi = \sum c_k \mu(E_k)$

$\mathbb{C}$-valued fns:

Def $\mathcal{M}$-meas fn $f: X \longrightarrow \mathbb{C}$ is integrable if $\int |f| < \infty$.

Since $|f| \leq |\mathrm{Re} f| + |\mathrm{Im} f| \leq 2|f|$, f is integrable iff $\mathrm{Re} f$ & $\mathrm{Im} f$ are integrable.

(Recall f measurable iff $\mathrm{Re} f$ & $\mathrm{Im} f$ are mble)

Defn $\int f = \int \mathrm{Re} f + i \int \mathrm{Im} f$

$$= \int \mathrm{Re} f_+ - \int \mathrm{Re} f_- + i \int \mathrm{Im} f_+ - i \int \mathrm{Im} f_- .$$

$$\left(f = \mathrm{Re} f_+ - \mathrm{Re} f_- + i \,\mathrm{Im} f_+ - i \,\mathrm{Im} f_- \right).$$

$L^1 = L^1(\mu) = L^1(\mu, \mathbb{C}) = \{ \text{integrable } f: X \to \mathbb{C} \}$.

L^1 is a $\mathbb{C}$-v.s. and $\int$ is linear.

Prop $\forall f \in L^1$, $\left| \int f \right| \leq \int |f|$.

Pf **Step 1** if f is $\mathbb{R}$-valued,

$$\left|\int f\right| = \left|\int f_+ - \int f_-\right| \le \left|\int f_+\right| + \left|\int f_-\right| = \int f_+ + \int f_- = \int |f| .$$

Step 2 we may assume $\int f \neq 0$.

Trick Let $\displaystyle \operatorname{sign}\left(\int f\right) := \frac{\int f}{\left|\int f\right|} \in \mathbb{T} = U(1) = S^1$.

Then $\displaystyle \left|\int f\right| = \overline{\operatorname{sign}(\int f)} \int f = \int \overline{\operatorname{sign}(\int f)}\, f \in \mathbb{R}$.

Then
$$\begin{aligned}
\left|\int f\right| = \int \overline{\operatorname{sign}(\textstyle\int f)}\, f &= \operatorname{Re} \int \overline{\operatorname{sign}(\textstyle\int f)}\, f \\
&= \int \operatorname{Re}\left(\overline{\operatorname{sign}(\textstyle\int f)}\, f\right) \\
&\le \int \left|\operatorname{Re}\left(\overline{\operatorname{sign}(\textstyle\int f)}\, f\right)\right| \\
&\le \int \left|\overline{\operatorname{sign}(\textstyle\int f)}\, f\right| \\
&= \int |f| .
\end{aligned}$$

Step 1

$\square$