Lec 10/31

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Uniform Continuity

vs. Continuity

Examples:

(1) $f(x) = \frac{1}{x}$ continuous $\forall x \neq 0$

$|x-a| < \min\left(\frac{|a|}{2}, \frac{\varepsilon |a|^2}{2}\right) = \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

(2) $f(x) = x^2$ continuous $\forall x$

$|x-a| < \min\left(1, \frac{\varepsilon}{2|a|+1}\right) = \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

δ depends on a.

Definition f is uniformly continuous on $A \subseteq \mathrm{dom}(f)$ (usually an interval)

if given $\varepsilon > 0$ you can find a $\delta > 0$ so that

$|x-a| < \delta$ & $x, a \in A \Rightarrow |f(x) - f(a)| < \varepsilon$

does not depend on a

Non-Examples:

(1) $f(x) = \frac{1}{x}$ is not uniformly cts on $(0,1)$.

Proof: take $\varepsilon = 1$. Suppose that there is a $\delta > 0$ s.t.

$|x-a| < \delta$ & $x, a \in (0,1) \Rightarrow \left|\frac{1}{x} - \frac{1}{a}\right| < 1$

Pick reciprocals of integers: $x = \frac{1}{m}$ $a = \frac{1}{n} \in (0, \delta)$, $m \neq n$

$\left|\frac{1}{x} - \frac{1}{a}\right| = |m-n| \geq 1$. Contradiction.

(2) $f(x) = x^2$ is not uniformly cts on $[1, \infty)$.

Proof: take $\varepsilon = 1$. Suppose that $\exists \delta > 0$ s.t.

$|x-a| < \delta$ & $x, a \in [1, \infty) \Rightarrow |x^2 - a^2| < 1$

Pick $a > \frac{1}{\delta}$ and $x = a + \frac{\delta}{2}$

$|x - a| = \frac{\delta}{2} < \delta$, $\quad x^2 - a^2 = (x+a)(x-a) = \left(2a + \frac{\delta}{2}\right)\frac{\delta}{2}$

$$= a\delta + \frac{\delta^2}{4} > a\delta > 1.$$

Example Suppose f' continuous on (c, d)

$\left(\lim_{x \to c^+} f'(x), \ \lim_{x \to d^-} f(x) \text{ exist}\right).$

then f is uniformly continuous on $[c, d]$

Proof: By EVT, $|f'|$ has a maximum M over $[c, d]$

given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M+1}$

b between $x, a \Rightarrow b \in (c, d)$

Suppose $|x - a| < \delta = \frac{\varepsilon}{M+1}$. Then $|f(x) - f(a)| = |f'(b)|\,|x - a|$

so $|f(x) - f(a)| < M \frac{\varepsilon}{M+1} < \varepsilon$.

Theorem If f is continuous on a closed finite interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof By contradiction. Suppose that f is not uniformly continuous on $[a, b]$. Then for some $\varepsilon > 0$, no $\delta > 0$ will work.

In particular, for any integer n, $\delta = \frac{1}{n}$ does not work.

So $\exists\, x_n, y_n \in [a, b]$ so that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

By BWP, we can find a subsequence $\{x_{n_k}\}$ which converges to some $c \in [a, b]$. Then $\{y_{n_k}\}$ converges to c as well since

$$0 \leq |y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \underset{\substack{\downarrow \\ 0}}{\frac{1}{n_k}} + \underset{\substack{\downarrow \\ 0}}{|x_{n_k} - c|}$$

so by Sq. thm,

$$\lim_{k \to \infty} |y_{n_k} - c| = 0 = \lim_{k \to \infty} y_{n_k} = c.$$

f cts.

$\lim f(x_{n_k}) = f(\lim x_{n_k}) = f(c)$

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c)$$

$$\lim_{k\to\infty} f(y_{n_k}) = f\left(\lim_{k\to\infty} y_{n_k}\right) = f(c)$$

then $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$

but, for all k, $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$

This is a contradiction. ∎

Integration

We can define $\int_a^b f$ for highly discontinuous functions,

why? multiple integration.

if $A = [a,b] \times [c,d]$

Fubini's theorem: $\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ for a continuous function f

for A not a rectangle, we have some options:

$$\int_a^b \left[\iint_{A \cap (\{x\}\times\mathbb{R})} f(x,y)\,dy\right] dx$$

for this, need to define $\int_S f(y)\,dy$ where $S \subseteq \mathbb{R}$ (arbitrary subset of $\mathbb{R}$), not necesarily an interval.

$\Rightarrow$ Lebesgue Integral
(better approach)

Other approach: enclose A in some rectangle $[a,b]\times[c,d]$
extend f to $[a,b]\times[c,d]$:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in A \\ 0 & (x,y) \notin A \end{cases}$$

So $\tilde{f}$ may be very discontinuous. → this is why.

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d \tilde{f}(x,y)\,dy \right] dx$$

This approach (Riemann integral)

requires $\int_a^b f$ where f is highly discontinuous.