

$G \curvearrowright X$ either set map $\alpha: G \times X \rightarrow X$ (satisfying 2 cond^{ns})

or gp hom $G \xrightarrow{\tau} \text{Aut}_{\text{Set}}(X)$

$x \in X$ $\begin{matrix} \nearrow \text{orbit } G \cdot x \\ \searrow \text{Stabilizer } \text{Stab}_G(x) = \{g \in G : g \cdot x = x\} \end{matrix}$

$G / \text{Stab}_G(x) \xleftrightarrow{\text{set bijection}} G \cdot x$

$\text{Stab}_G(x) \xrightarrow{\sim \text{gp iso}} \text{Stab}_G(\sigma \cdot x) \quad \forall \sigma \in G.$

$\begin{matrix} \wr \\ g \end{matrix} \mapsto \begin{matrix} \wr \\ \sigma g \sigma^{-1} \end{matrix}$

Ex: $S_n \curvearrowright X = \{1, \dots, n\}.$

$\begin{matrix} \wr \\ \sigma \end{matrix}$ in X , there is only one orbit, it has size n .

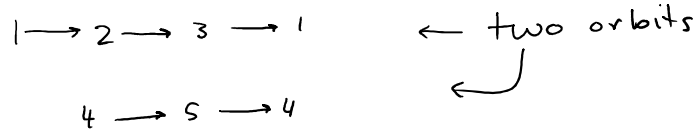
$\forall k \in \{1, \dots, n\}, \text{Stab}_{S_n}(k) = \{\sigma \in S_n \mid \sigma(k) = k\} \xrightarrow{\sim} S_{n-1} =: \text{Stab}_{S_n}(n).$

$\begin{matrix} \wr \\ w \end{matrix} \mapsto \begin{matrix} \wr \\ (kn) w (kn) \end{matrix}$

Pick $\pi \in S_n$ and let $H = \langle \pi \rangle \curvearrowright X.$

$\langle \pi \rangle$ -orbits in $X \longleftrightarrow$ cycles in π .

e.g. $n=5, \quad \pi = (123)(45)$



terms:

Transitive: $G \curvearrowright X$ is transitive if $\forall x, y \in X, \exists g \in G$ s.t. $g \cdot x = y$
(i.e. there is only one orbit)

Free: $G \curvearrowright X$ is free if $g \cdot x = x \implies g = e$

(i.e. $\text{Stab}_G(x) = \{e\} \forall x \in X$)

\iff every orbit has the same size, $|G|$.



Faithful: $G \longrightarrow \text{Aut}_{\text{set}}(X)$ is injective:

$$[g \cdot x = x \forall x \implies g = e]$$

$$\begin{array}{l} \{1, \dots, n\} \\ S_n \curvearrowright X \\ \begin{array}{ll} \text{Faithful} & \checkmark \\ \text{Free} & \times \\ \text{Transitive} & \checkmark \end{array} \end{array}$$

$$\begin{array}{l} D_{2n} \curvearrowright \mathbb{R}^2 \\ \begin{array}{ll} \text{Faithful} & \checkmark \\ \text{Free} & \times \\ \text{Transitive} & \times \end{array} \end{array}$$

$$S_n \xrightarrow{\text{sign}} \{\pm 1\}, \text{ so } S_n \curvearrowright \{\pm 1\} \text{ by } \alpha(\sigma, x) \xrightarrow{\text{not faithful}} \text{sign}(\sigma) \cdot x$$

$$\{\sigma \in S_n \mid \text{sign}(\sigma) = 1\} =: A_n \text{ (alternating group).}$$

Free \nearrow all orbits have same size

Transitive \longrightarrow exactly one orbit

Ex: Count # of ^{set} partitions $\{1, 2, \dots, 7\} = P_1 \cup P_2 \cup \dots \cup P_i$
 $|P_1| = 3, |P_2| = |P_3| = 2.$

$E =$ set of possible ways to break it like this
 $S_7 \curvearrowright \{1, 2, 3\} \sqcup \{4, 5\} \sqcup \{6, 7\} \xrightarrow{\sigma} \{\sigma(1), \sigma(2), \sigma(3)\} \sqcup \dots$

(1) $S_7 \curvearrowright E$ is transitive. if we pick a specific partition P

$$S_7 / \underset{S_7}{\text{Stab}(P)} \xrightarrow{\text{bijection}} E = \text{only one orbit}$$

$$\uparrow \\ S_3 \times S_2 \times S_2$$

$$\text{So } |E| = \frac{7!}{3! \cdot 2! \cdot 2!}$$

In general, we get the multinomial coefficient.

$$G \curvearrowright X, \quad X = \text{disjoint union of orbits}$$

$$|\mathcal{O}| = \frac{|G|}{|\text{Stab}_G(x)|}$$

\uparrow
 some $x \in \mathcal{O}$

So

$$|X| = \sum_{\mathcal{O} \in \text{Orbits}} \frac{|G|}{|\text{Stab}_G(x)|}$$

\uparrow
 $x \in \mathcal{O}$