Lec 9/22

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Let $(x_n) \subset \{0,1\}^{\mathbb{N}}$ be normal.

is (x_{an+b}), $n \in \mathbb{N}$ normal? (Exercise: yes)

what about $(x_{f(n)})$ where $d(f(\mathbb{N})) > 0$? (for $d(f(\mathbb{N})) = 0$, clearly no)

f should be so 'deterministic'.

$$\underbrace{a\mathbb{Z}+b}_{\omega} \in \{0,1\}^{\mathbb{Z}}$$

$$\sigma(\omega(n)) = \omega(n+1)$$

left shift of seq ω.

$\{\sigma^k \omega : k \in \mathbb{Z}\}$ orbit of ω.

Closure of this is in $\{0,1\}^{\mathbb{Z}}$

this is finite if $\omega(n) = 1$ if $n \in a\mathbb{Z}+b$

$$d(x,y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$$

$2x \bmod 1$ = left shift

$X \cong b_1 b_2 \cdots$ is normal if $2^n x \bmod 1$ is u.d. mod 1.

base 2

$\overline{\{\sigma^k \omega : k \in \mathbb{Z}\}} = \{0,1\}^{\mathbb{Z}}$ iff ω is weakly normal

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(Exercise: use d as above)

$\nu(n) = \#$ of distinct words of length n in ω

'complexity' of ω is how fast ν grows.

$$\frac{\log \nu(n)}{n} \longrightarrow \text{'entropy' of } \omega$$

<u>Theorem.</u> Let (x_n) be normal.

$(x_{f(n)})$ is normal iff $(f(n))$ is 'deterministic' sequence.

(X, d) compact metric space.

$T : X \to X$ homeomorphism

$x \in \underset{\text{open}}{U} \subset X$

$A = \{n : T^n x \in U\}$

for decent U, certain T, $\delta(A) > 0$ and A is syndetic.

Let $S \subset P$, and let $A_s = \{x : x = \prod_{p \in S} p^{\alpha}\}$. iff is harder

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T/F? $d(A_S) > 0$ if $\sum_{p \in P \setminus S} \frac{1}{p} < \infty$ (exercise)

(think intersections of progressions)

i.e. $\sum_{p \in P \setminus S} \frac{1}{p} = \infty \iff d(A_S) = 0$

Let $P_1 \cup P_2 = P$, $\sum_{p \in P_1} \frac{1}{p} = \sum_{p \in P_2} \frac{1}{p} = \infty$.

T/F? $d(A_1) = d(A_2) = 0$? $\left(A_i = \{n : n = \prod_{p \in P_i} p^{\alpha}\}\right)$ (exercise)

Let C be the classical Cantor set (middle thirds)

Then $C + C = [0,2]$, $C - C = [-1,1]$. (exercise)

Is it true for other cantor sets $\subset [0,1]$ (exercise: no)

Is there a cantor set $K \subset [0,1]$ where $\text{interior}(K - K) = \emptyset$

homeomorphic to C.

$\text{int}(S)$ = all points x in S s.t. there's a neighborhood around x in S

Claim: The Classical Cantor set contains a Hamel basis.

$\mathbb{Q}(C+C) = \mathbb{R}$ so C spans $\mathbb{R}/\mathbb{Q}$

(exercise) $\{0,1\}^{\mathbb{N}} \approx \{0,1\}^{\mathbb{Z}} \approx \{0,1,\ldots,17\}^{\mathbb{N}} \approx \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \approx \cdots \approx C$

(exercise) $\{0,1\}^{\mathbb{N}} \underset{\substack{\uparrow \\ \text{homeomorphism}}}{\approx} \{0,1\} \approx \underbrace{\{0,1,\ldots,17\}}_{\sum \frac{|x_i-y_i|}{18^i}} \approx \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \approx \cdots \approx C$

Hardy: $SL(2,\mathbb{Z})$ p. 184.

$$x^2 - Dy^2 = 1$$