Lec 2/17

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Recall: H_0 null hypothesis, H_1 alternative hypothesis.

α = prob of type 1 error = $\mathbb{P}$(reject H_0, H_0 is true)

β = prob of type II error = $\mathbb{P}$(accept H_0, H_0 is false)

Remarks: ① as long as sample size is fixed, cannot decrease both α and β at once by changing critical region.

by collecting more data, can reduce α & β.

② The selection of the threshold (or in general, critical region) is based on application.

Ex: Blood Test for HIV.

H_0 = You don't have HIV. H_1 = you have HIV.

Let X be the value of HIV-Level. assume higher value of X, more likely to have HIV.

how to decide threshold t for acceptance region $x \leq t$?

Want to make t small; care more about β type II error rate (false negative).

Ex: Consider a RS $X_1, \ldots, X_n$ from a normal population with $\sigma^2 = 1$. Want to test hypotheses:

$H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$ where $\mu_1 > \mu_0$

find value K so that $\bar{X} > K$ provides a critical region for which $\alpha = 0.05$.

$\boxed{\text{Guess: } K = \mu_0 + Z_{\frac{0.05}{2}}}$ Note $\alpha = \mathbb{P}(\mu = \mu_0;\ \bar{X} > K)$.

under H_0, $\bar{X} \sim N(\mu_0, \frac{1}{n})$ so $\dfrac{\bar{X} - \mu_0}{\frac{1}{\sqrt{n}}} \sim N(0,1)$

so $\mathbb{P}\left(\dfrac{\bar{X} - \mu_0}{\frac{1}{\sqrt{n}}} > \dfrac{K - \mu_0}{\frac{1}{\sqrt{n}}}\right) = 0.5$ → so want $\dfrac{K - \mu_0}{\frac{1}{\sqrt{n}}} = Z_{0.05}$.

$$\Rightarrow K = \mu_0 + Z_{0.05} \frac{1}{\sqrt{n}}$$
$$= \mu_0 + 1.65 \frac{1}{\sqrt{n}}$$

Q: to test $H_0: \mu = 10$ vs $H_1: \mu = 11$, find minimal sample size needed in order to have $\beta \leq 0.06$ (w/ $\alpha = 0.05$ fixed)

$\beta = \mathbb{P}$(accept H_0; H_0 is false)

under H_1, this is $\mathbb{P}(\bar{X} < K, \mu = 11) = \mathbb{P}\left(\underbrace{\dfrac{\bar{X} - 11}{\frac{1}{\sqrt{n}}}}_{\sim N(0,1)} < \dfrac{K - 11}{\frac{1}{\sqrt{n}}}\right)$ → want this $\leq -Z_{0.06}$

$= \mathbb{P}(\bar{X} - 11 \quad 10 + 1.65 \frac{1}{\sqrt{n}} - 11$

want this $\le -Z_{0.06}$

$$= \mathbb{P}\left(\frac{\bar{X} - 11}{\frac{1}{\sqrt{n}}} < \frac{10 + 1.65\frac{1}{\sqrt{n}} - 11}{\frac{1}{\sqrt{n}}} \right) = 0.06$$

so want $1.65 - \sqrt{n} \le -Z_{0.06} = -1.56$, so want $n > (-1.56 - 1.65)^2 \approx 10.3$

so $n \geq 11$ is necessary.

<u>Remark</u>: in "classical" Hyp. testing, restrict ourselves to tests for which α is specified to be acceptably low.

generally, $\alpha \le 0.05$, but 0.01 and 0.1 are also used.

sometimes don't have cts dist so cannot pick $\alpha = 0.05$.

§ 12.4 : Neyman - Pearson lemma:

Consider fixed α and look for tests that minimize β (or maximize $1-\beta$) where $1-\beta$ is the power of the test. $\mathbb{P}$(reject H_0; H_0 is false)

We would like to find tests that maximize power for particular alternatives.

Let $f(x;\theta)$ be dist. of a pop.

Consider testing $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ (simple vs. simple)

would like to find test that has max power at $\theta = \theta_1$.

This is called the "best" or "most powerful" test, and corresponding critical region is called the "most powerful" critical region.

Given a sample $X_1, \dots, X_n$, to construct such a test, consider likelihood under two hypotheses.

$$L_0 = \prod_{i=1}^{n} f(x_i ; \theta_0), \quad L_1 = \prod_{i=1}^{n} f(x_i ; \theta_1).$$

Idea: Reject H_0 when L_0 small and L_1 big. one way is to look at ratio $\frac{L_0}{L_1}$ test statistic

Rejection / Critical region.

- for the values of sample points that fall inside critical region, $\frac{L_0}{L_1}$ should be small.
- " " outside " " large.

Define critical region to be $\left\{ \vec{x} : \frac{L_0(\vec{x})}{L_1(\vec{x})} \le K \right\}$