Lec 3/21

Wednesday, March 21, 2018 14:22

Let $U \subseteq \mathbb{R}^m$ be open. to say x is a C^k immersion from U into $\mathbb{R}^n$ means x is C^k from U into $\mathbb{R}^n$ and for each $p \in U$, $\underbrace{x'(p)}_{n \times m}$ has rank m.

eg a C^k-regular curve $\gamma : I \longrightarrow \mathbb{R}^n$

Remark Let $x : U (\overset{\text{open}}{\subseteq} \mathbb{R}^m) \longrightarrow \mathbb{R}^n$ be a C^k immersion. Let $p \in U$. Then $\exists V \overset{\text{open}}{\subseteq} \mathbb{R}^m$ with $p \in V \subseteq U$ s.t. $x|_V$ is a homeomorphism from V onto $x[V]$.

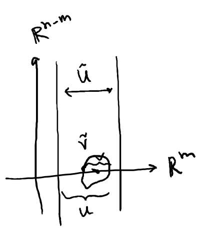
Pf We'll use the inverse function theorem. Since $\text{Rank}(\overbrace{x'(p)}^{n \times m}) = m$, $m \leq n$.

Think of $\mathbb{R}^m$ as $\{(u_1, \dots, u_n) \in \mathbb{R}^n : u_{m+1} = u_{m+2} = \dots = u_n = 0\}$. Let $v_1, \dots, v_m$ be the columns of $x'(p)$. These are linearly independent $\underset{\text{in } \mathbb{R}^m}{}$ since $\text{Rank}(x'(p)) = m$.

We can thus choose $v_{m+1}, \dots, v_n \in \mathbb{R}^n$ s.t. $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$.

Define $\tilde{x} : U \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$ by $\tilde{x}(u_1, \dots, u_m, u_{m+1}, \dots, u_n) = x(u_1, \dots, u_m) + u_{m+1} v_{m+1} + \dots + u_n v_n$.

$\tilde{x}' = \left(x' \,\middle|\, v_{m+1} \cdots v_n \right)$ so $\tilde{x}'(\tilde{p})$ is invertible where $\tilde{p} = (p, 0) \in \mathbb{R}^n$.

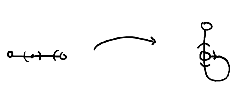


So apply the inverse function theorem to obtain an open set $\tilde{V} \subseteq U \times \mathbb{R}^{n-m}$ s.t. $\tilde{x}|_{\tilde{V}}$ is a 1-1 mapping from $\tilde{V}$ onto $\tilde{W} \subseteq \mathbb{R}^n$ whose inverse is a C^k fn from $\tilde{W}$ onto $\tilde{V}$. In particular, $\tilde{x}|_{\tilde{V}}$ is a homeomorphism from $\tilde{V}$ onto $\tilde{W}$.

let $V = \{(u_1, \dots, u_n) \in \tilde{V} : u_{m+1} = u_{m+2} = \dots = u_n = 0\} = \tilde{V} \cap \mathbb{R}^m$. Then V is open in $\mathbb{R}^m$, $p \in V \subseteq U$, and $x|_V$ is a homeomorphism from V onto $x[V]$. $\square$

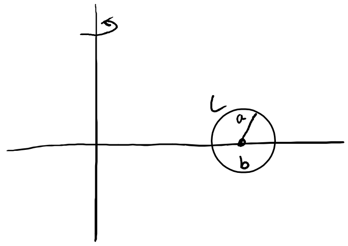
Note: the proof also shows that $(x|_V)^{-1}$ is C^k in the sense that $(x|_V)^{-1}$ can be extended to a C^k map on an open subset of $\mathbb{R}^n$ containing $x[V]$.

Q Is a one-to-one C^k immersion a homeomorphism?

A  (No)

eg (torus) Let $0 < a < b < \infty$.

off
off
off
off
Swervin
Like a
Donut



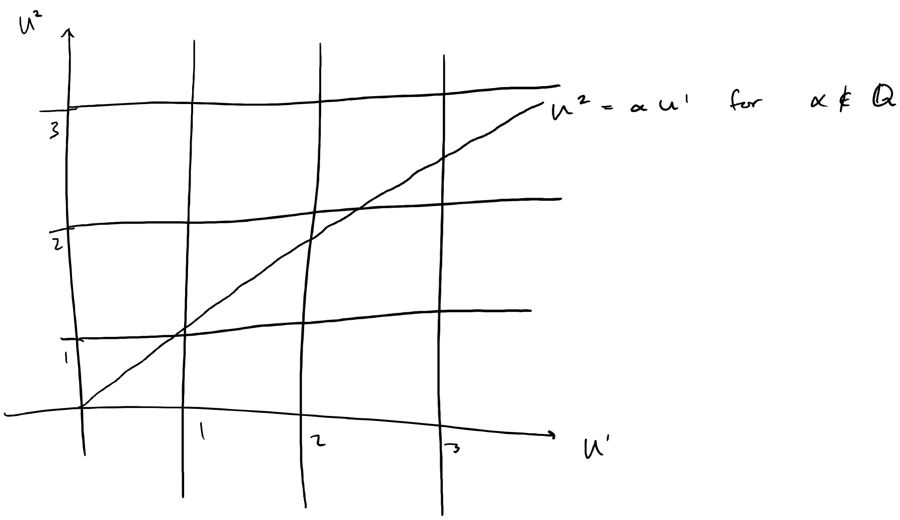
Define $\xi : \mathbb{R} \longrightarrow \mathbb{R}^3$ by $\xi(u^1) = (\cos 2\pi u^1, \sin 2\pi u^1, 0)$. Let $\eta = (0,0,1)$.

Define $x : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $x(u^1, u^2) = b\,\xi(u^1) + a\left[(\cos 2\pi u^2)\,\xi(u^1) + (\sin 2\pi u^2)\,\eta\right]$

$x[[0,1)\times[0,1)]$ is the torus we get by revolving C around vertical axis.

(so is $x[\mathbb{R}^2]$)

x is an immersion from $\mathbb{R}^2$ into $\mathbb{R}^3$.



$u^1 \longmapsto x(u^1, \alpha u^1)$ is a one-to-one immersion from $\mathbb{R}$ into $\mathbb{R}^3$.

$\{x(u^1, \alpha u^1) : u^1 \in \mathbb{R}\}$ is dense in $x[\mathbb{R}^2]$ (the torus).

<u>Defn</u> To say that x is a C^k <u>coordinate patch</u> (or <u>C^k simple surface</u>) in $\mathbb{R}^3$ means that x maps some open subset of $\mathbb{R}^2$ into $\mathbb{R}^3$ and is a C^k-immersion.

$\left(\frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2} \neq 0\right.$ at each point in U). Also it's a homeomorphism on its range.