Lec 2/7

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Subspaces of Topological spaces

<u>Propn</u> Let (X, ρ) be a pseudometric space. Let $X_0 \subseteq X$ and let $\rho_0 = \rho|_{X_0 \times X_0}$

Let $\mathcal{G} = \{G \subseteq X : G \text{ is } \rho\text{-open}\}$. Let $\mathcal{G}_0 = \{G \subseteq X_0 : G \text{ is } \rho_0\text{-open}\}$.

Then $\mathcal{G}_0 = \{G \cap X_0 : G \in \mathcal{G}\}$.

<u>Pf</u> Let $G \in \mathcal{G}$. Let $p \in G \cap X_0$. Then $\exists\, r > 0$ s.t. $\underbrace{\{x \in X : \rho(x, p) < r\}}_{B_X(p, r)} \subseteq G$

Then $B_{X_0}(p, r) = \{x \in X_0 : \rho_0(p, x) < r\} \subseteq \{x \in X : \rho(x, p) < r\} \subseteq G$

And $B_{X_0}(p, r) \subseteq X_0$ so $B_{X_0}(p, r) \subseteq G \cap X_0$, so $G \cap X_0$ is ρ_0-open in X_0.

Conversely, suppose G_0 is any ρ_0-open subset of X_0.

Let $S = \{(x, r) : x \in G_0,\ r > 0, \text{ and } B_{X_0}(x, r) \subseteq G_0\}$.

Let $G = \bigcup_{(x,r) \in S} B_X(x, r)$. Then G is ρ-open in X since open balls are open.

Now $G \cap X_0 = \bigcup_{(x,r) \in S} (\underbrace{B_X(x, r) \cap X_0}_{B_{X_0}(x, r)}) = G_0$.

Thus $\mathcal{G}_0 = \{G \cap X_0 : G \in \mathcal{G}\}$ $\square$

<u>Propn</u>: Let $(X, \mathcal{G})$ be a topological space. Let $X_0 \subseteq X$, $\mathcal{G}_0 = \{G \cap X_0 : G \in \mathcal{G}\}$

Then $(X_0, \mathcal{G}_0)$ is a topological space. ($\mathcal{G}_0$ is called the subspace topology that X_0 inherits from X).

<u>Propn</u>: Let X be a topological space. Let $X_0 \subseteq X$. Then the subspace topology X_0 inherits from X is the unique topology on X_0 with the property that for each topological space W and each $f: W \to X_0$, f is continuous from $W \to X_0$ iff f is continuous from W to X.

Pf You do it. (Hint make particular choices for W and f in order to prove that $\mathcal{G}_0$ is the only topology that works) $\square$

Back to connectedness.

Thm Let X be a topological space. Then TFAE:

a) X is connected

b) $\forall$ continuous $f : X \longrightarrow \mathbb{R}$, $f(X)$ is an interval.

Pf We already did (a) $\Rightarrow$ (b). Suppose (b) holds. Let U be a clopen $\subseteq X$. We wish to show that $U = \emptyset$ or $U = X$. Let $f = \mathbb{1}_U$. Then f is continuous because U and $X \setminus U$ are open. Thus $f[X]$ is an interval subset of $\{0, 1\}$. The only possible "intervals" are $\{0\}$, $\{1\}$, and $\emptyset$. If $f[X] = \{0\}$ then $U = \emptyset$, if $f[X] = \{1\}$ then $U = X$. If $f[X] = \emptyset$, $X = U = \emptyset$. So X is connected. $\square$

Corollary: Let I be a connected subset of $\mathbb{R}$. Then I is an interval. $\square$

Pf Apply the theorem with $X = I$ and $f : X \longrightarrow \mathbb{R}$ defined by $f(x) = x$.

Theorem Intervals are connected.

Lemma: Let $A \subseteq [0,1]$ s.t. (a) $0 \in A$, (b) $\forall a \in [0,1)$, if $[0,a] \subseteq A$, $\exists b \in (a,1]$, $[0,b] \subseteq A$, and (c) $\forall a \in (0,1]$, if $[0,a) \subseteq A$ then $[0,a] \subseteq A$. Then $A = [0,1]$

Pf Let $E = \{v \in [0,1] : [0,v] \subseteq A\}$. $0 \in E$ by (a). Let $a = \sup E$. Then $a \in [0,1]$. In fact, $a \in (0,1]$ by (b). Let $x \in [0,a)$. Then $\exists v \in E$ s.t. $x < v$ since x is not an upper bound for E (it is less than $\sup E$). Since $v \in E$, $[0,v] \subseteq A$ so $x \in A$, meaning $[0,a) \in A$ by (c). Thus $[0,a] \in A$. Thus if $a < 1$ then by (b) $\exists b > a$ s.t. $[0,b] \subseteq A$ so $a \neq \sup E$. Thus $a = 1$ and so $[0,1] \subseteq A$ so $A = [0,1]$ $\square$

Corollary Let $A \subseteq [0,1]$ s.t. $\overset{(a)}{0 \in A}$, $\overset{(b)}{A \text{ is open in } [0,1] \text{ for the right topology}}$, $\overset{(c)}{A \text{ is closed in } [0,1] \text{ for the left topology}}$.
Then $A = [0,1]$.

Proof by (b), the condition (b) of the lemma holds. by (c) the condition (c) of the lemma holds. thus the result follows. □

Right topology on $\mathbb{R}$ is the collection of sets $G \subseteq \mathbb{R}$ s.t. $\forall x \in G$ $\exists \varepsilon > 0$ s.t. $[x, x+\varepsilon) \subseteq G$.

Left topology on $\mathbb{R}$ is $\{G \subseteq \mathbb{R} : \forall x \in G, (x-\varepsilon, x] \subseteq G\}$.

f on $\mathbb{R}$ is right continuous iff f is cts wrt right topology.

Corollary let $A \subseteq [0,1]$ s.t. $0 \in A$, A is open in $[0,1]$, A is closed in $[0,1]$ (or in $\mathbb{R}$). Then $A = [0,1]$

pf A open $\Rightarrow$ A open wrt right topology. A closed $\Rightarrow$ A closed wrt left topology. □

Corollary $[0,1]$ is connected.