Lec 1/26

Friday, January 26, 2018 14:14

3 ice cream cone theorem: squeeze 3 balls (w. different radii) between 2 planes.

Propn

Let $u \in \mathbb{R}^3$ be a unit vector. Let $h \in [-1, 1]$. let α be a C^1 unit-speed curve in $\mathbb{R}^3$.

Then $\langle T, u \rangle \equiv h$ iff there is a curve β in a plane perpendicular to u such that $\forall s$, $\alpha(s) = \beta(s) + hsu$.

Remark Such an α is called a helix w/ axis u and pitch $\theta = \cos^{-1} h$ is the constant angle between T and u.

Proof of Propn:

$(\Longrightarrow)$ Suppose $\langle T, u \rangle \equiv h$. let $\beta(s) = \alpha(s) - hsu$ $\forall s$.

Fix $s_0 \in \text{domain}(\alpha)$ and let $x_0 = \beta(s_0)$. We'll show that

$\text{range}(\beta) \subseteq \Pi = \{x \in \mathbb{R}^3 : \langle x - x_0, u \rangle = 0\}$.

By the choice of x_0, $\langle \beta(s_0) - x_0, u \rangle = 0$.

now

$$\frac{d}{ds} \langle \beta(s) - x_0, u \rangle = \langle \beta'(s), u \rangle = \langle \alpha'(s) - hu, u \rangle$$

$$= \langle T(s), u \rangle - h\langle u, u \rangle = h - h = 0.$$

so $\langle \beta(s) - x_0, u \rangle = 0$ $\forall s$ and thus the result holds ■

$(\Longleftarrow)$ Suppose $\langle \beta(s) - x_0, u \rangle = 0$ $\forall s$. Then $\frac{d}{ds} \langle \beta(s) - x_0, u \rangle = \langle T(s), u \rangle - h = 0$

So $\langle T(s), u \rangle = h$ $\forall s$. ■

Remark: if α is unit speed, if $h = \pm 1$ then β is constant and so α is a straight line parallel to u.

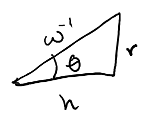
Thus if α is not a straight line, $h \in (-1, 1)$.

eg any planer curve is a helix w/ $u \perp$ the "planer plane" and $h=0$.
(i.e. $\theta = \frac{\pi}{2}$).

eg a unit-speed circular helix $\alpha(s) = (r\cos\omega s, r\sin\omega s, h\omega s)$,
where $\omega = \frac{1}{\sqrt{r^2+h^2}}$ and r is positive and $h \in \mathbb{R} \setminus \{0\}$,

with axis $(0,0,1)$ and pitch $\theta = \cos^{-1}\omega h$.

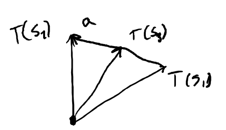
$\left(T(s) = \omega\,(-r\sin\omega s, r\cos\omega s, h)\right.$, so $\left.\langle T(s), u\rangle = \omega h\right)$.



Remark if α is a straight line, then T is constant, so $\forall\, u \in \mathbb{R}^3$,
$\langle T, u\rangle$ is constant, so the axis for α is arbitrary.

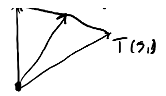
Propn Let $\alpha : I \to \mathbb{R}^3$ be a C^1 unit-speed in $\mathbb{R}^3$ which is a helix with nonzero curvature (not a straight line). Then the axis of α is unique up to a sign change.

Proof Note T is not constant. Let $E = \{v \in \mathbb{R}^3 : \langle T, v\rangle \text{ is constant}\}$.
E is a linear subspace of $\mathbb{R}^3$ clearly. In fact, $E = D^\perp$ where
$D = \{T(s_2) - T(s_1) : s_1, s_2 \in I\}$. Since T is not constant,
D is not $\{0\}$. ($\exists\, s_1, s_2 \in I$ s.t. $a := T(s_1) - T(s_2) \neq 0$). define
$f : I \to \mathbb{R}$ by $f(s) = \langle T(s) - T(s_1), a\rangle$. $f(s_1) = 0$. $f(s_2) = \langle a, a\rangle = \|a\|$.
Thus $\exists\, s_3$ between s_1, s_2 so that $f(s_3) = \frac{\|a\|}{2}$. Let $b = T(s_3) - T(s_1)$.
then $\langle b - \frac{1}{2}a, a\rangle = \langle b, a\rangle - \frac{1}{2}\langle a, a\rangle = 0$. so $b - \frac{1}{2}a \perp a$
and $b \neq \frac{1}{2}a$ since if it were then $|T(s_3)| = |\frac{1}{2}a + T(s_1)| < 1$ (see picture).



so $b - \frac{1}{2}a, a$ are linearly independent in D,

α is a helix



so $b - \frac{1}{2}a$, a are linearly independent in D,

α is a helix

So $\dim D \geq 2$. Also $\dim E \geq 1$ so $\dim D = 2$ and $\dim E = 1$.

And there are only 2 unit vectors (axes) in E, one the negative of the other.

Lancret's Theorem (1802, first proof given by De Saint Venant in 1845).

Let α be a C^3 unit-speed curve in $\mathbb{R}^3$ whose curvature κ is never 0. Then α is a helix iff its torsion τ satisfies $\tau = c\kappa$ for some constant c.

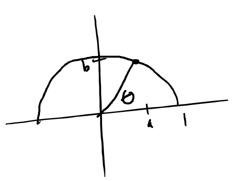
Proof

($\Rightarrow$) Suppose α is a helix. Then there is a unit vector u in $\mathbb{R}^3$ s.t. $\langle T, u \rangle \equiv a$ for some constant $a \in \mathbb{R}$. Since κ is never 0, α is not a straight line so $-1 < a < 1$. Now $N \perp u$ because $0 = a' = \langle T, u \rangle' = \langle T', u \rangle = \kappa \langle N, u \rangle$ and κ is never 0.

Hence $u = aT + bB$ where $b = \langle B, u \rangle$.

By the Pythagorean Theorem, $a^2 + b^2 = |u|^2 = 1$. So $b = \pm\sqrt{1 - a^2}$.

Hence b is constant because a is constant and b is cts.



Replacing u and a by $-u$ and $-a$ if necessary, we may suppose $b = \sqrt{1 - a^2}$. Then $\exists!\ \theta \in (0, \pi)$ s.t. $a = \cos\theta$, $b = \sin\theta$.

Then $u = T\cos\theta + B\sin\theta$ so

$$0 = u' = T'\cos\theta + B'\sin\theta = \kappa N\cos\theta - \tau N\sin\theta$$

$$\text{so } \tau = \kappa\cot\theta = \frac{\kappa}{\tan\theta}. \quad \text{So } c = \cot\theta.$$