Lec 1/13

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Next week: Jennings 155.

Recall: $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta) = \frac{1}{2}(1+\theta x)$ $\quad (x, \theta) \in (-1,1)^2$

trying to estimate

is $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ an unbiased estimator of θ?

$E(\bar{X}) = E(x) = \frac{1}{2}\int_{-1}^{1} x(1+\theta x)\,dx = \frac{\theta}{3}$ so No, bias $= -\frac{2\theta}{3}$.

$E(X) - \theta$

So $\hat{\theta} = 3\bar{X}$ is an unbiased estimator of θ.

Ex: for a population w/ mean μ and variance σ^2, what is an unbiased estimator for μ^2? Is $\bar{X}^2$ one?

In general, no.

$E(\bar{X}^2) = E(\bar{X})^2 + Var(\bar{X})$, so unless $Var(\bar{X}) = 0$ $\bar{X}^2$ is biased.

$\mu^2 + \frac{\sigma^2}{n} \neq \mu^2$. Bias $= \frac{\sigma^2}{n}$

so $\bar{X}^2 - \frac{S^2}{n}$ is an unbiased estimator for μ^2.

Remarks:

1: $\hat{\theta}$ u.b.e. of $\theta \not\Rightarrow f(\hat{\theta})$ u.b.e. of $f(\theta)$.

2: If $\lim_{n\to\infty} Bias(\hat{\theta}) = 0$ then $\hat{\theta}$ is asymptotically unbiased.

3: for any distribution w/ finite variance σ^2, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is u.b.e. for σ^2.

(see thm 10.1 & Proof is analogous to poisson example).

10.3 Efficiency (another desirable property of an estimator).

we might also care about Variance of estimator.

find unbiased estimator w/ minimum variance.

☆ Cramer-Rao Inequality (a lower bound on variance of an unbiased estimator).

$$Var(\hat{\theta}) \geq \frac{1}{\underbrace{n E\left[\left(\frac{\partial \log(f(x))}{\partial \theta}\right)^2\right]}}, \quad \text{where} \quad f(x) = f(x;\theta) \quad (\text{a pdf/pmf of } X_i\text{'s})$$

unbiased estimator

↳ the denom. is also called fisher information about θ.

more (larger) information $\Rightarrow$ smaller variance.

Ex: (Poisson dist. $f(x) = f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$

Find Cramer-Rao bound.

$$\log(f(x)) = \log\left(\frac{\lambda^x e^{-\lambda}}{x!}\right) = \log(\lambda^x) + \log(e^{-\lambda}) - \log(x!)$$

$$= x\log(\lambda) - \lambda - \log(x!)$$

$$\left(\frac{\partial \log(f(x))}{\partial \lambda}\right)^2 = \left(\frac{x}{\lambda} - 1\right)^2 = \frac{x^2}{\lambda^2} - \frac{2x}{\lambda} + 1$$

$$E(\;\;) = E\left(\frac{x^2}{\lambda^2}\right) - \underbrace{2\underbrace{E\left(\frac{x}{\lambda}\right)}_{=1} + 1}_{=-1}$$

$$= \frac{1}{\lambda^2}E(x^2) - 1$$

$$= \frac{1}{\lambda^2}\left(Var(X) + E(x)^2\right) - 1$$

$$= \frac{1}{\lambda^2}(\lambda + \lambda^2) - 1$$

$$= \frac{1}{\lambda} + 1 - 1 = \frac{1}{\lambda}.$$

So C-R bound is $\frac{\lambda}{n}$.

last time, showed that $\bar{X}$ & S^2 are unbiased for λ.

$Var(\bar{X}) = \frac{\lambda}{n}$ (best possible).

Not easy to compute $Var(S^2)$, but can show that it is $> \frac{\lambda}{n}$.

So the estimator $\bar{X}$ is preferred.

Definition: If $\hat{\theta}$ is an unbiased estimator of θ and $Var(\hat{\theta})$ = CR lower bound, then $\hat{\theta}$ is a minimum variance unbiased estimator of θ. (MVUE)

If there is no MVUE, we can compare variances of proposed estimators.

Definition Relative efficiency - efficiency of $\hat{\theta}_2$ relative to $\hat{\theta}_1 = \dfrac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)}$

if $RE > 1$, $\hat{\theta}_2$ is better.