Lec 4/23

Monday, April 23, 2018 14:22

Temporarily jump ahead to § 6-7.

Propn 7.2: Let V and W be vector fields on a cpt orientable surface M, each having just finitely many zeroes. Then $I(V) = I(W)$. i.e. $\sum_{p:V(p)=0} i_p(V) = \sum_{p:W(p)=0} i_p(W)$.

Proof: Break M up into finitely many "triangles" γ_i so each triangle is contained in a simply connected coordinate patch, each triangle has at most one zero of V and one zero of W in its interior, and each zero of V or of W is in the interior of some triangle.

For each i, let u_i be a continuous field of unit vectors on M in the patch containing γ_i. Then $I(V) = \frac{1}{2\pi}\sum_i \delta\sphericalangle(u_i, V, \gamma_i)$, and $I(W) = \frac{1}{2\pi}\sum_i \delta\sphericalangle(u_i, W, \gamma_i)$. Hence:

$$I(V) - I(W) = \frac{1}{2\pi}\sum_i \left[\delta\sphericalangle(u_i, V, \gamma_i) - \delta\sphericalangle(u_i, W, \gamma_i)\right]$$

$$= \frac{1}{2\pi}\sum_i \delta\sphericalangle(W, V, \gamma_i)$$

$$= \frac{1}{2\pi}\sum_{\text{edges}} \delta\sphericalangle(W, V) = 0.$$

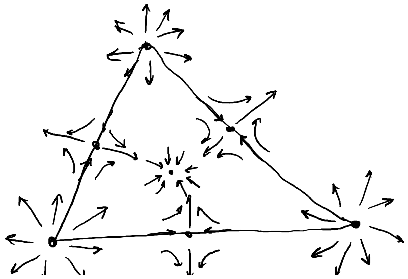
□

Thm 7.3 (Poincaré – Brouwer):

Let M be a compact orientable surface. Let V be a C^∞ vector field on M having just finitely many zeroes. Then $I(V) = \chi(M)$.

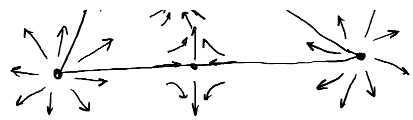
Pf: by the proposition, it suffices to exhibit a C^∞ vector field W on M w/ just finitely many zeroes s.t. $I(W) = \chi(M)$.

Triangulate M into finitely many positively oriented triangles γ_i.



$V - E + F$

□



$\square$

<u>Corollary</u> Let M be a compact orientable surface w/ $\chi(M) \neq 0$.

(i.e. M is not homeomorphic to a torus).

Let V be a cts vector field on M having just finitely many zeroes. Then V has at least one zero.

<u>eg</u> Let M be a torus, so $\chi(M)=0$. Then there is a cts v. field V on M s.t. V has no zeroes.

<u>Back to Section 6-4, The Gauss-Bonnet Formula</u>

<u>(or the local Gauss-Bonnet Theorem).</u>

<u>The Generalized rotation index theorem</u>

Let $x: \mathcal{U} \subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} \mathcal{R} \subseteq M$ be a positively oriented C^1 patch and let γ be a C^1-regular simple loop in $\mathcal{R}$. Let $\tilde{\gamma}(t) = \gamma(e^{it})$ for $t \in [0, 2\pi]$. Let $\tilde{T}$ be the unit tangent field for $\tilde{\gamma}$.
$\tilde{T}(t) = T(e^{it})$.

Then $\int \measuredangle (x_1, \tilde{T}, \tilde{\gamma}) = \pm 2\pi$ (determined by γ's orientation).

<u>Pf</u> $\int \measuredangle (x_1, T, \tilde{\gamma}) = 2\pi n$ where n is the winding # of the loop

$\alpha_0 = \langle\langle x_1, T, \tilde{\gamma} \rangle\rangle$ wrt the origin in $\mathbb{R}^2$.

Let $x_1 = V^1 x_1 + V^2 x_2$ $(V^1 = 1, V^2 = 0)$. Let $T = W^1 x_1 + W^2 x_2$.

Then $\alpha_0 = \left(\sum_{i,j} g_{ij} V^i W^j \, , \, (V^1 W^2 - V^2 W^1)\sqrt{g} \right)$. Let $\alpha_1 = T = (W^1, W^2)$.

The winding # of α_1 is ± 1 by the rotation index theorem.

For $0 \leq \lambda \leq 1$, let $g_{ij}^{\lambda} = (1-\lambda) g_{ij} + \lambda g_{ij}$.

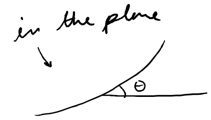
let $\alpha_\lambda = \left(\sum_{i,j} g^\lambda_{ij} V^i W^j, \;\; (V^1W^2 - V^2W^1)\sqrt{g^\lambda} \right)$.

(g^λ_{ij}) is strictly positive definite (Convex combination of spds).

So α_λ is a loop in $\mathbb{R}^2 \setminus \{(0,0)\}$.

$\lambda \longmapsto \alpha_\lambda$ is a cts deformation of α_0 to α_1 in $\mathbb{R}^2 \setminus \{(0,0)\}$,

so they have the same winding # wrt the origin. $\square$



$\dot\theta = \kappa$.

Geodesic Curvature
as the rate of change of an oriented angle

Let (M, ν) be an oriented C^2 surface in $\mathbb{R}^3$. Let γ be a C^2 unit-speed curve in M with tangent field T. Let V be a C^1 field of unit vectors that is parallel along γ on M. Let $\theta \in \measuredangle(V, T; \gamma)$. Then $\frac{d\theta}{ds} = \kappa_g$.

Pf let $W(s) = \nu(\gamma(s)) \times V(s)$. Then W is also parallel along γ on M

(let $s_0 \in \mathrm{dom}(\gamma)$ and let $\tilde W$ be parallel along γ on M and satisfy $\tilde W(s_0) = W(s_0)$.

Then $|\tilde W| = 1$, $\langle \tilde W | V\rangle = 0$, $|W| = 1$, and $\langle W | V \rangle = 0$.

Hence $\tilde W = \pm W$ at each pt, and the $\pm$ can't change (by continuity) and at s_0 it's $+$,

so $\tilde W = W$, so W is parallel along γ on M).

Now $\kappa_g S$ = tangential part of T' = tangential part of $\frac{d}{ds}(V\cos\theta + W\sin\theta)$

$= (-V\sin\theta + W\cos\theta)\frac{d\theta}{ds}$ (since V, W are parallel so $\frac{dV}{ds}, \frac{dW}{ds}$ have 0 tangential part).

But $S = \nu \times T = \nu \times (V\cos\theta + W\sin\theta) = W\cos\theta - V\sin\theta$,

so $\kappa_g S = S\frac{d\theta}{ds}$ so $\kappa_g = \frac{d\theta}{ds}$. $\square$

Thm 6-4 (The Local Gauss-Bonnet Theorem)

Let (M, ν) be an oriented C^3 surface in $\mathbb{R}^3$. Let R be a curvilinear n-gonal region on M with piecewise C^2 boundary ∂R.

Let $\beta_1, \dots, \beta_n \in (0, 2\pi)$ be the interior angles at the vertices of R.

Assume $R \subseteq V$ where $x : U \subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V \subseteq M$ is a C^3 patch in M. Then

Assume $R \subseteq V$ where $x : U \subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V \subseteq M$ is a C^∞ patch in M. Then

$$\iint_R K\,dA + \oint_{\partial R} k_g\,ds + \sum_{i=1}^{n} (\pi - \beta_i) = 2\pi .$$

Corollary (The Gauss-Bonnet Thm)

Let M be a compact orientable surface. Then

$$\iint_M K\,dA = 2\pi\,\chi(M).$$