Lec 4/11

Wednesday, April 11, 2018 12:38

Sack Splitting Theorem: Let A be r.e. but not recursive. Then A can be partitioned into two incomparable r.e. sets.

Proof: Let $a_1, a_2, \ldots$ be an enumeration of A.

At stage s, we'll put a_s in either B or C.

Let $R_{e,B}: \chi_A \neq \Phi_e^B$ and $R_{e,C}: \chi_A \neq \Phi_e^C$.

[these requirements suffice because collectively they say $A \not\leq_T B$ and $A \not\leq_T C$, and so if $B \leq_T C$ then, for any x, to see if $x \in A$ we can use our oracle for C to see if $x \in C$ and if $x \in B$. So $A \leq_T C$, a contradiction so $B \not\leq_T C$ (and similarly $C \not\leq_T B$).

Consider $R_{1,B}$. We will use a length of agreement argument.

DFNS: 1. $\varphi_e^B(K)[S] = 1 +$ the rightmost position of the oracle head during the computation $\Phi_e^B(k)[S]$.

length of agreement $\rightarrow$ 2. $\ell(e,B,s) = \max\{n : (\forall k < n)\ \chi_A(k)[S] = \Phi_e^B(k)[S]\}$ where $A_s = \{a_0, \ldots, a_s\}$.

high-water mark $\rightarrow$ 3. $L(e,B,s) = \max\limits_{t \leq s} \ell(e,B,t)$

4. $u(e,B,s) = \max\limits_{K < L(e,B,s)} \varphi_e^B(K)[S]$

(and all four for C instead of B too).

We prioritize the requirements as follows:

$$R_{1,B} < R_{1,C} < R_{2,B} < R_{2,C} < \cdots$$

Here's the construction:

Stage 0:

$B \leftarrow \emptyset$ $\qquad L(e,B,0) \leftarrow 0$

$C \leftarrow \emptyset$ $\qquad L(e,C,0) \leftarrow 0$

Stage $s+1$:

<u>If</u> there's an $e \le s$ s.t. $a_s < u(e,B,s)$ or $a_s < u(e,C,s)$

<u>then</u> $e \leftarrow$ a least such #

If $a_s < u(e,B,s)$ <u>then</u> $C \leftarrow C \cup \{e\}$

<u>else</u> $B \leftarrow B \cup \{e\}$

<u>else</u> $B \leftarrow B \cup \{a_s\}$ (or $C \leftarrow C \cup \{a_s\}$ it doesn't matter).

Update the u function on each weaker priority than $u(s,C,s+1)$.

We'll first argue that $R_{1,B}$ is met.

<u>Lemma 1</u>: $(\exists m)(\forall s)[\ell(1,B,s) < m]$ (i.e. $R_{1,B}$ has a high water mark).

<u>Pf</u> Assume the contrary, i.e. that $\lim_{s\to\infty} L(1,B,s) = \infty$. Then $\forall k$, $\exists$ inf. many stages s such that $\ell(1,B,s) > k$. For each such stage,

$$\Phi_1^B(k) \underset{(1)}{=} \Phi_1^B(k)[s] \underset{(2)}{=} A_s(k) \underset{(3)}{=} A(k).$$

(1) is true because elements less than $u(1,B,s)$ are prevented from entering B after stage s.

(2) is true because the length of agreement is $> k$.

(3) is true because, if $k \in A_s \subset A$ then $k \in A$.
if $k \notin A$ then by (1) and (2), $\Phi_1^B(k) = 0$ and since $\ell(1,B,s)$ is unbounded we have $k \notin A$

So we can decide if $k \in A$ by running construction until $L(1,B,s) > k$, so A is recursive, $\otimes$

ill be fine on my own she said

Let $m = \lim_{s\to\infty} L(1,B,s)$.

<u>Lemma 2</u> at least one number $y \le m$ is a permanent witness

to $R_{1,B}$.

<u>Proof</u> Let s be the first stage at which the high water mark is hit, i.e. $m = L(1, B, s)$. If $\exists y < m$ that enters A at stage $\tilde{s} > s$, then y is a permanent witness for $R_{1,B}$, in particular $(\forall t \geq \tilde{s})\ [A_t(y) = 1 \neq 0 = \Phi_e^B(y)[t]]$, because $\Phi_e^B(y)[\tilde{s}]\downarrow = 0$, and we forever prevent elements below $u(1, B, s)$ from entering B.