Lec 9/14

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Inner product:

$(a,b) \longmapsto \langle a,b \rangle$

$V \times V \longrightarrow \mathbb{R}$ where $V/\mathbb{R}$

Properties:

Bilinear: $\langle \lambda a+\mu b, c\rangle = \lambda\langle a,c\rangle + \mu\langle b,c\rangle$

$\langle a, \lambda b+\mu c\rangle = \lambda\langle a,b\rangle + \mu\langle a,c\rangle$

Symmetric: $\langle a,b\rangle = \langle b,a\rangle$

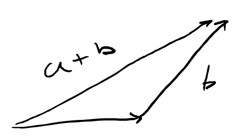
Positive Definite: $\langle a,a\rangle \geq 0$ with $=$ iff $a=0$.

Examples: Dot product $\langle a,b\rangle = a \cdot b$ for $\mathbb{R}^n/\mathbb{R}$

$\langle f,g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ for $f, g \in C([-1,1])/\mathbb{R}$

Define 'length' to be $|a| = \sqrt{\langle a,a\rangle}$ (also called norm)

<u>Triangle Ineq:</u> $|a+b| \leq |a| + |b|$ (Theorem)



Note: for $\mathbb{R}/\mathbb{R}$, $\langle a,b\rangle = ab$, $|a+b| \leq |a|+|b|$

<u>Lemma</u>: $|\langle a,b\rangle| \leq |a||b|$ (cauchy-schwarz inequality)

<u>Proof</u>: wlog assume $a \neq 0 \neq b$.

Lemma: $|\langle a, b\rangle| \le |a||b|$ (cauchy-schwarz inequality)

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So lemma is equiv. to $\left|\left\langle \frac{a}{|a|}, \frac{b}{|b|}\right\rangle\right| \le 1$.

$|u| = \left|\frac{a}{|a|}\right| = 1$, $|v| = \left|\frac{b}{|b|}\right| = 1$.

$$|u+v|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$
$$= |u|^2 + 2\langle u,v\rangle + |v|^2$$
$$= 2(1+\langle u,v\rangle)$$

so $(1+\langle u,v\rangle)$ is positive,
$\langle u,v\rangle \ge -1$.

by checking $\langle u-v, u-v\rangle$ we get $|u-v|^2 = 2(1-\langle u,v\rangle)$
so $\langle u,v\rangle \le 1$.

so $\left|\left\langle \frac{a}{|a|} \frac{b}{|b|}\right\rangle\right| \le 1$ so the Lemma holds. ∎

$\checkmark$

$$\left|\int_{-1}^{1} f(x)g(x)\,dx\right| \le \left(\int_{-1}^{1} f(x)^2 dx\right)^{1/2}\left(\int_{-1}^{1} g(x)^2 dx\right)^{1/2}$$

(cauchy-schwarz)

Proof of Thm:

$$|a+b|^2 = \langle a+b, a+b\rangle = |a|^2 + 2\langle a,b\rangle + |b|^2$$
$$\le |a|^2 + 2|a||b| + |b|^2$$
$$= (|a|+|b|)^2$$

So $$\left(\int_{-1}^{1} (f(x)+g(x))^2 dx\right)^{1/2} \le \left(\int_{-1}^{1} f(x)^2\, dx\right)^{1/2} + \left(\int_{-1}^{1} g(x)^2\, dx\right)^{1/2}$$