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$\alpha = \sqrt{2} + \sqrt{3}$, $\deg_{\mathbb{Q}}(\alpha) = 4 \iff \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$

~~Pf~~ $\sqrt{3} - \sqrt{2} = \dfrac{3-2}{\sqrt{3}+\sqrt{2}} = \dfrac{1}{\alpha}$,

So $\sqrt{3} - \sqrt{2} \in \mathbb{Q}(\alpha)$, so $\sqrt{3}, \sqrt{2} \in \mathbb{Q}(\alpha)$, and $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

$L_1, L_2 \subset K$, L_1/F , L_2/F finite extensions.

$$\begin{array}{c} L_1L_2/F \\ | \\ \text{composite} \end{array} \quad \stackrel{?}{=} \left\{ \frac{\sum \alpha_i \beta_i}{\sum \gamma_j \delta_j} : \alpha_i, \gamma_j \in L_1, \beta_i, \delta_j \in L_2 \right\}$$

but if L_1/F, L_2/F are finite, we don't need to divide.

$$\begin{array}{c} L_1 L_2 \\ | \\ L_1 \\ | \\ F \end{array}$$

Theorem if L_1/F, L_2/F are finite, $L_1, L_2 \subseteq K$,

$\{\alpha_1, \dots, \alpha_n\}$ generates L_1 over F.

$\{\beta_1, \dots, \beta_m\}$ generates L_2 over F,

Then $\{\alpha_i \beta_j : 1 \le i \le n, \ 1 \le j \le m\}$ generates $L_1 L_2$ over F.

So $[L_1 L_2 : F] \le [L_1 : F] \cdot [L_2 : F]$.

<u>Proof</u> $L_1 L_2 = F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$.

$= F[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m]$

So any $\gamma \in L_1 L_2 = \sum u_k v_k$, $u_k \in L_1$, $v_k \in L_2$.

$$\begin{tikzcd} & L_1 L_2 \arrow[dl, dash, "n_2'"'] \arrow[dr, dash, "n_1'"] \arrow[dd, dash, "n"] & \\ L_1 \arrow[dr, dash, "n_1"'] & & L_2 \arrow[dl, dash, "n_2"] \\ & F & \end{tikzcd}$$

$\Longrightarrow$ $n_2' \le n_2$, $n_1' \le n_1$

also $n_1 \mid n$, and $n_2 \mid n$,

So $\operatorname{lcm}(n_1, n_2) \mid n$

So $\operatorname{lcm}(n_1, n_2) \le n \le n_1 \cdot n_2$

<u>Corollary</u>: if $(n_1, n_2) = 1$, $n = n_1 \cdot n_2$

$L_1 \otimes L_2 \longrightarrow L_1 L_2$ is surjective if L_1/F, L_2/F are finite.

$(\alpha, \beta) \longmapsto \alpha\beta$

if $[L_1L_2 : F] = [L_1 : F] \cdot [L_2 : F]$, it's an isomorphism.

K/F is algebraic if every $\alpha \in K$ is algebraic over F.

K/F is transcendental otherwise.

any finite extension is algebraic

<u>Theorem</u>: an algebraic extension is finite iff it is finitely generated, $K = F(\alpha_1, \dots, \alpha_n)$.

~~pf~~ Finite $\Rightarrow$ choose basis $\{\alpha_1, \dots, \alpha_n\}$

F.G. $\Rightarrow$ top of a finite tower of simple algebraic extensions

$F(\alpha_1, \dots, \alpha_n) / F(\alpha_1, \dots, \alpha_{n-1}) / \cdots / F(\alpha_1) / F$.

<u>Theorem</u>: if L_1/F, L_2/F are algebraic, then L_1L_2/F is algebraic.

If K/L and L/F are algebraic, then

K/F is algebraic

Pf Let $\alpha \in L_1 L_2$, then $\alpha = \dfrac{\sum \beta_i \gamma_j}{\sum \delta_i \tau_j}$ $\quad \beta_i, \delta_i \in L_1$, $\gamma_j, \tau_j \in L_2$.

So $\alpha \in F(\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_m, \tau_1, \dots, \tau_m)$ - finite.

Let $\alpha \in K$. Then α is a root of $f(x) = x^n + \beta_{n-1} x^{n-1} + \dots + \beta_1 x + \beta_0$, with $\beta_i \in L$. Then α is algebraic over $F(\beta_0, \dots, \beta_{n-1})$.

So $F(\alpha, \beta_0, \dots, \beta_{n-1}) / F(\beta_0, \dots, \beta_{n-1})$ is finite, and $F(\beta_0, \dots, \beta_{n-1})/F$ is finite, so $F(\alpha, \beta_0, \dots, \beta_{n-1})/F$ is finite & so algebraic, so α is algebraic over F. So K/F is algebraic.

Corollary: α, β are algebraic over F

$\Rightarrow \alpha \pm \beta, \alpha \cdot \beta, \frac{\alpha}{\beta}$ are algebraic.

So if K/F is any extension, then

$\{\alpha \in K : \alpha \text{ is algebraic over } F\} / F$ is a subextension.

(Max'l algebraic subfield of K).

Quadratic extension: $[K:F]=2$. (Char $F \neq 2$)

$\alpha \in K \setminus F$, $\deg_F m_\alpha = 2$. $m_\alpha = x^2+ax+b$.

So $\alpha = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4}-b}$

Let $\delta = \sqrt{\frac{a^2}{4}-b}$. Then $\delta \notin F$,

So $K = F(\delta)$.

$D = \delta^2 \in F$, so $K = F(\sqrt{D})$

Char $F \neq 2$

(8) Biquadratic extension $K = F(\sqrt{D_1}, \sqrt{D_2})$

s.t. $[K:F]=4$.

this is so iff $\sqrt{D_1}, \sqrt{D_2}, \sqrt{D_1D_2} \notin F$.

Pf if $\sqrt{D_1} \in F$, Then $K = F(\sqrt{D_2})$ and $[K:F] \leq 2$

if $\sqrt{D_1 D_2} \in F$, then $\sqrt{D_2} = \frac{a}{\sqrt{D_1}} \in F(\sqrt{D_1})$ so $K = F(\sqrt{D_1})$.

If $\sqrt{D_1}, \sqrt{D_2}, \sqrt{D_1D_2} \notin F$, then

$$\begin{array}{c} K \\ \big|\,2 \\ F(\sqrt{D_1}) \\ \big|\,2 \end{array} \Rightarrow \begin{array}{c} K \\ \big|\,4 \\ F \end{array}.$$

$$\begin{array}{c} F(\sqrt{D_1}) \\ \big|\,{\scriptstyle 2} \\ F \end{array} \quad \Rightarrow \quad \begin{array}{c} \big|\,{\scriptstyle 4}. \\ F \end{array}$$