Lec 9/7

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S = subsp. of $\mathbb{R}^3$ generated by

$(2, 1, -3), \; (1, -1, 0), \; (1, 3, -4)$

find minimal set of eqns which determine S as solution space

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -4 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -4 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $S = S(\overbrace{(1, -1, 0), (0, 1, -1)}^{\text{basis}})$

$y_1 - y_2 = 0$ $\qquad$ $y_1 = y_2 = y_3$

$y_2 - y_3 = 0$ $\qquad$ so $\mathbb{R}(1, 1, 1)$ is a general solution

$x_1 + x_2 + x_3 = 0$ is the system we're looking for.

linear mfd $M \parallel S$

$V = (1, 2, 3) \in M$

$M = v + S$

system of $n-h$ lin eqn determining M:

$x_1 + x_2 + x_3 = 6$.

Linear Transf'mation

V/F finitely generated w/ basis $\{v_1, v_2, \dots, v_n\}$

If $a \in V$, $b \in V$ then $a = \alpha_1 v_1 + \dots + \alpha_n v_n$, $b = \beta_1 v_1 + \dots + \beta_n v_n$ uniquely.

$T: V \longrightarrow F^n$

$$T(a) = (\alpha_1, \dots, \alpha_n)$$

$$T(b) = (\beta_1, \dots, \beta_n)$$

$$\lambda a + \mu b = (\lambda \alpha_1 + \mu \beta_1) v_1 + \dots + (\lambda \alpha_n + \mu \beta_n) v_n$$

$$T(\lambda a + \mu b) = (\lambda \alpha_1 + \mu \beta_1, \dots, \lambda \alpha_n + \mu \beta_n)$$

$$= \lambda T(a) + \mu T(b)$$

this makes T linear.

$f(a+b) = f(a) + f(b) \Rightarrow f(na) = n f(a)$, $n \in \mathbb{Z}$ by induction.

$$f\left(\tfrac{p}{q} a\right) = \tfrac{p}{q}\tfrac{q}{p} f\left(\tfrac{p}{q} a\right) = \tfrac{p}{q}\tfrac{1}{p} f(pa) = \tfrac{p}{q} f(a)$$

$f(xa) = x f(a)$ by completeness.

Def: $T: V/F \longrightarrow W/F$ is linear if:

$$T(\lambda a) = \lambda T(a)$$

$$T(a+b) = T(a) + T(b)$$

$$D : C^1(\mathbb{R}) \longrightarrow C(\mathbb{R})$$

$$D(f) = f'$$

$$D(\lambda f + \mu g) = \lambda D(f) + \mu D(g)$$

D linear but not invertible.

$$I_\alpha^\beta : C(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$I_a^\beta(f) = \int_\alpha^\beta f$$

is also linear but not invertible.

$V, W / F$. $\quad L(V,W) = \{T : V \to W,\ T \text{ is a linear transformation}\}$

$$(T_1 + T_2)(v) = T_1(v) + T_2(v) \qquad \forall v \in V$$

$T_1 + T_2 \in L(V, W)$ since it's Lineer.

$(\lambda T)(v) = \lambda T(v)$ is also linear.

so $L(V, W)$ is a vector space over F.

Conjecture: $\dim L(V,W) = \dim V \cdot \dim W$ if these dimensions exist.

Proof: $\{v_1, \dots, v_n\}$ = basis V, $\{w_1, \dots, w_m\}$ = basis W

$$\left\{ t_{ij} = \begin{matrix} v_i \mapsto w_j \\ v_{k \neq i} \mapsto 0 \end{matrix} : 1 \le i \le n,\ 1 \le j \le m \right\} = \text{basis } L(V,W)$$

Let $T(v_i) = \alpha_{i1} w_1 + \cdots + \alpha_{im} w_m \quad \forall i.$

then $T = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} t_{ij}$

and if $\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} t_{ij} = 0$ then $\alpha_{ij} = 0$ so the set is lin. indp too. ∎

when $L(V,V) \equiv L(V)$, we also have composition:

$$(T_1 \circ T_2)(v) = T_1(T_2(v)) \in L(V) \quad \text{when } T_1, T_2 \in L(V)$$

this is also liner.

$$T_3 \circ (T_2 \circ T_1) = (T_3 \circ T_2) \circ T_1$$

$$T_1 \circ (T_2 + T_3) = T_1 \circ T_2 + T_1 \circ T_3$$

$$(T_2 + T_3) \circ T_1 = T_2 \circ T_1 + T_3 \circ T_1$$

$I : V \to V \quad I(v) = v$ is neutral for $\circ$.

$T \circ I = I \circ T = T \qquad O : V \to V \quad O(v) = 0$ is neutral for $+$.

$L(V) \binom{+}{\circ}$ is a ring (non-commutative)

$L(V)$ is an algebra over a field F (Ring & V.space)