Lec 11/8

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Facts about $\langle \cdot , \cdot \rangle$ sesq. form on $H / \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

- Polarization (c last time)

suppose $\langle \cdot , \cdot \rangle$ positive, self-adjoint. Then $\left(\|x\| = \sqrt{\langle x, x \rangle} \right)$

⓪: if $\mathbb{K} = \mathbb{R}$, $4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2$

①: Pythagorean thm

②: $x \perp y \iff \forall \alpha \in \mathbb{C}$ or $\mathbb{R}$, $\|x\|^2 \leq \|x + \alpha y\|^2$

③: $\|x\|^2 = 0 \iff \langle x, y \rangle = 0 \ \forall y$

so Definite $\iff$ nondegenerate

④: (Cauchy-Schwarz) $\forall x, y \in H$, $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.

⑤: if $\langle \cdot , \cdot \rangle$ definite, $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ iff $\{x, y\}$ lin. dep.

⑥: $\|\cdot\| : H \longrightarrow [0, \infty)$ is a seminorm. it's a norm exactly when $\langle \cdot , \cdot \rangle$ definite.

Exercise: A norm $\|\cdot\|$ on a $\mathbb{C}$ vs comes from an inner product iff $\|\cdot\|$ satisfies:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2 \right] \qquad \forall x, y$$

(parallelogram rule)

A Hilbert space is an inner product space which is complete wrt $\|\|$ $(= \|\cdot\|_2)$.

Examples:

① $\ell^2 = \{(x_n) \mid \sum |x_n|^2 < \infty\}$ $\qquad \langle x, y\rangle = \sum x_e \bar{y}_e$

② $\ell^2(E) = \{f : E \to \mathbb{C} \mid \sum_{e \in E} |f(e)|^2 < \infty\}$

③ $L^2(X, \mu) = \{\text{mble } f : X \to \mathbb{C} \mid \int |f|^2 < \infty\}/\sim$ $\qquad \langle f, g\rangle = \int f \bar{g}$

From now on, H is a Hilbert space $(/\mathbb{C})$.

Thm Suppose $C \subset H$ is a closed convex set and $z \notin C$.
$\exists!\, x \in C$ s.t. $\|z - x\| = \inf_{y \in C} \|z - y\|$

pf By translation, assume $z = 0 \notin C$. Suppose $(x_n) \subset C$ st $\|x_n\| \to r := \inf_{y \in C} \|y\|$. By the parallelogram law,

$$\left\| \frac{x_n - x_m}{2} \right\|^2 + \left\| \underbrace{\frac{x_m + x_n}{2}}_{\in C} \right\|^2 = 2\left[\left\| \frac{x_m}{2} \right\|^2 + \left\| \frac{x_n}{2} \right\|^2 \right]$$

$$= \frac{1}{2} \underbrace{\|\underset{\in C}{x_m}\|^2}_{\to r^2} + \frac{1}{2} \underbrace{\|\underset{\in C}{x_n}\|^2}_{\to r^2}$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - 4\left\| \frac{x_m - x_n}{2} \right\|^2$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\underbrace{\|x_m\|^2}_{\to r^2} + 2\underbrace{\|x_n\|^2}_{\to r^2} - 4\underbrace{\left\|\frac{x_m - x_n}{2}\right\|^2}_{\geq r^2}$$

$\rightsquigarrow \limsup_{n,m} \|x_n - x_m\|^2 = 0$, so (x_n) Cauchy.

Since H complete, $\exists\, x$ s.t. $x_n \to x$.

Since $\|\cdot\|$ cts, $\|x\| = r$, and $x \in C$ closed.

If $x' \in C$ also has $\|x'\| = r$, then

$x, x', x, x', \ldots$ is a seq in C s.t. every term has norm r, so by the preceding argument, it is Cauchy! so $x' = x$. $\square$

For $S \subset H$, let $S^\perp := \{x \in H \mid \langle x, s\rangle = 0 \ \forall s \in S\}$.

<u>Observe</u>: $S^\perp$ is a closed subspace

since $|\langle x, y\rangle| \leq \|x\|\,\|y\| \Rightarrow \langle x, \cdot\rangle$ cts.

Let $M \subseteq H$ be a subspace

① $M \cap M^\perp = \{0\}$.

② $(M^\perp)^\perp = \overline{M}$.

Pf If $x \in \overline{M}$, let $(x_n) \subset M$ w/ $x_n \to x$.

Then $\forall y \in M^\perp$, $\langle x, y\rangle = \lim_n \langle x_n, y\rangle = 0$.

Conversely, if $x \in M^\perp$, $\forall y \in M^\perp$, $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$,

so $\overline{M} \subset (M^\perp)^\perp$ (again!)

really conversely, suppose $x \in (M^\perp)^\perp$. Then $\langle x, y\rangle = 0 \ \forall \ y \in M^\perp$.

$\overline{M}$ is a closed convex set, so if $z \in (M^\perp)^\perp \setminus \overline{M}$, $\exists! \ x \in \overline{M}$

minimizing dist to z ...

③ $H = M \oplus M^\perp$.