$$D_{2n} = \langle s, r \mid s^2 = r^n = (sr)^2 = e \rangle = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle$$

<u>Definition</u>. free group on a set A. $\langle A \mid \text{no relations} \rangle$

Remark: every $w \in \text{Free}(A)$ can be written <u>uniquely</u> as $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ $\begin{cases} x_1, x_2, \ldots, x_n \in A \\ m_1, m_2, \ldots, m_n \in \mathbb{Z} \\ x_1 \neq x_2,\ x_2 \neq x_3, \ldots \end{cases}$

not true in non-free groups

<u>eg</u> $D_8 = \langle s_1, s_2 \mid s_1^2 = e = s_2^2 = (s_1 s_2)^4 \rangle$, $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

$\text{Free}(n) = \text{Free}(\{1, \ldots, n\}) \cong \text{Free}(A)$ if $|A| = n$.

$\text{Free}(1) \cong \mathbb{Z}$. $\text{Free}(2) \leftarrow$ gives counterexample to "Subgroups of finitely generated groups are finitely generated"

"Commutator Subgroup of Free(2)"

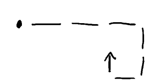
$\text{Free}(2) \geq H$ = subgroup generated by $\{xyx^{-1}y^{-1} \mid x, y \in \text{Free}(2)\}$.

<u>Propn</u>: H is not finitely generated

<u>Pf</u> $w \in \text{Free}(2)$.

$\downarrow$

$\gamma(w)$ path in $\mathbb{R}^2$ $\begin{cases} a & \text{means move horizontally,} \\ b & \text{means move vertically} \end{cases}$

i.e. $a^3 b^{-2} a^{-1} b \rightsquigarrow$ 

$\gamma(w)$ is a loop $\forall w \in H$ (easy fact).

also easy fact

If $d(w) = \max\{|(0,0) - p| : p \text{ on path } \gamma(w)\}$, then $\forall w_1, w_2 \in H$ $d(w_1 w_2) = \max\{d(w_1), d(w_2)\}$.

If H was finitely generated (by $w_1, w_2, \ldots, w_n$) then

$d(w) \le \max\{d(w_1), \dots, d(w_n)\}$ $\forall w \in H$. but $d(w)$ is clearly unbounded. $\square$

Presentation of Symmetric Group S_n $(n \geq 3)$.

$\sigma_{ij} = (i\,j)$ $\quad (1 \le i < j \le n)$ $\qquad \binom{n}{2}$

$s_i = (i\ \ i{+}1)$ $\quad (1 \le i \le n{-}1)$ $\qquad n-1$

<u>Prop.</u> Every permutation can be written as a product of s_j's.

<u>Proof</u>. 1. enough to prove it for cycles

2. $(k_1 k_2 \cdots k_\ell) = (k_1 k_2)(k_2 k_3) \cdots (k_{\ell-1} k_\ell)$ so enough to prove for σ_{ij}'s.

3. $(1\ n) = (n{-}1\ n)\ (1\ n{-}1)\ (n{-}1\ n)$

$\vdots$

$= s_{n-1} s_{n-2} \cdots s_2 s_1 s_2 \cdots s_{n-2} s_{n-1}$ $\square$

List of relations

$s_i^2 = e$. $\quad s_i s_j = s_j s_i$ if $\{i, i{+}1\} \cap \{j, j{+}1\} = \emptyset$

Assume $j = i+1$. We know $(s_i s_{i+1})^k = e$ for some k.

$s_i s_{i+1} = (i\ i{+}1)(i{+}1\ i{+}2) = (i\ i{+}1\ i{+}2)$ so $k = 3$.

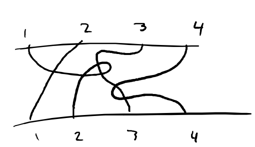
$$P_n = \langle s_1, s_2, \dots, s_{n-1} \mid s_i^2 = e\ \forall i,\ s_i s_j = s_j s_i \text{ if } |i-j| \ge 2,\ (s_i s_{i+1})^3 = e \text{ for } 1 \le i \le n-2 \rangle$$

Claim: $P_n \cong S_n$. (next week)

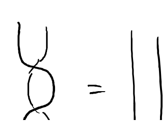
Recall: $\sigma (x_1\ x_2 \dots x_\ell)\,\sigma^{-1} = (\sigma(x_1)\ \sigma(x_2) \cdots \sigma(x_\ell))$ if $\sigma \in S_n$.

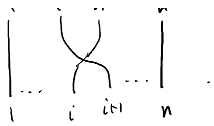
<u>Pf</u> LHS maps $\sigma(x_k) \longrightarrow x_k \longrightarrow x_{k+1} \longrightarrow \sigma(x_{k+1})$.

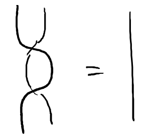
Use the relations to untangle "braid representation".

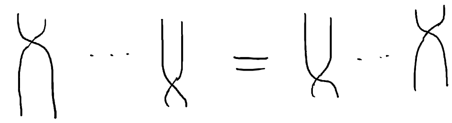


$s_i \longleftrightarrow$ 

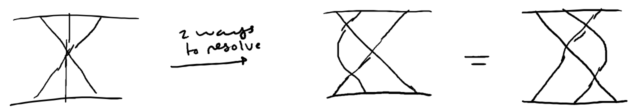
$s_i^2 = e \longleftrightarrow$ 

$s_i \longleftrightarrow$ 

$s_i^2 = e \longleftrightarrow$ 

$s_i s_j = s_j s_i \longleftrightarrow$ 

yang-baxter eq^{n}.

$(s_i s_{i+1})^3 = e \longleftrightarrow$ 

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

removing $s_i^2 = e$ gives braid group.