$$\begin{tikzcd}[column sep=small] & K = L_1 L_2 \arrow[dl, dash] \arrow[dr, dash] & \\ L_1 \arrow[dr, equal] & & L_2 \arrow[dl, dash] \\ & F = L_1 \cap L_2 & \end{tikzcd}$$

$$Gal(K/F) \overset{?}{=} Gal(\quad) \rtimes Gal(\quad)$$

$$\begin{matrix} K & & 1 \\ | & & | \\ L_1 & & H_1 \\ \| & & \| \\ F & & G \end{matrix}$$

Sequence splits

$$\begin{tikzcd} & & & H_1 \arrow[dl, bend right] \arrow[d, equal] & \\ 1 \arrow[r] & H_1 \arrow[r] & G \arrow[r] & G/H_1 \arrow[r] & 1 \end{tikzcd}$$

$$1 \to Gal(K/L_1) \to Gal(K/F) \to Gal(L_1/F) \to 1$$

If $\exists L_2$ s.t. $L_1 \cap L_2 = F$, $L_1 L_2 = K$,

then the sequence splits

& $Gal(K/F) = Gal(K/L_1) \rtimes \underset{\substack{\cong \\ H_2}}{Gal(L_1/F)}$

K - splitting field of $x^8 - 2$, $K = \mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\alpha, i)$,

$\alpha = \sqrt[8]{2}$, $\omega = e^{2\pi i/8} = \frac{i+1}{\sqrt{2}}$.

$$G = \langle \varphi, \psi \mid \varphi^8 = \psi^2 = 1, \ \psi\varphi\psi^{-1} = \varphi^3 \rangle$$

$$\varphi : \begin{array}{l} \alpha \longmapsto \alpha\omega \\ i \longmapsto i \end{array} \qquad \psi : \begin{array}{l} \alpha \longmapsto \alpha \\ i \longmapsto -i \end{array}$$

$$H_1 = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{2})) = \{\eta \in G : \eta(\sqrt{2}) = \sqrt{2}\}$$

$$\begin{array}{c} K \\ \big| \, 8 \\ \mathbb{Q}(\sqrt{2}) \\ \big| \, 2 \\ \mathbb{Q} \end{array} \qquad \begin{array}{c} 1 \\ \big| \, 8 \\ H_1 \\ \big| \, 2 \\ G \end{array}$$

$\varphi(\sqrt{2}) = -\sqrt{2}$, $\quad \varphi \notin H_1$

$\varphi^2(\sqrt{2}) = \sqrt{2}$, $\quad \varphi \in H_1$

$\psi(\sqrt{2}) = \sqrt{2}$, $\quad \psi \in H_1$

So $H_1 = \{1, \varphi^2, \varphi^4, \varphi^6, \psi, \psi\varphi^2, \psi\varphi^4, \psi\varphi^6\} = \langle \varphi^2, \psi \rangle.$

$$H_1 = \langle \tilde{\varphi}, \psi \mid \tilde{\varphi}^4 = \psi^2 = 1, \ \psi\tilde{\varphi}\psi = \tilde{\varphi}^{-1} \rangle \cong D_8$$

$H_2 = \mathrm{Gal}(K/\mathbb{Q}(i\sqrt{2}))$.

$\varphi(i\sqrt{2}) = -i\sqrt{2}$, $\varphi \notin H_2$ but $\overset{a}{\overset{\shortparallel}{\varphi^2}} \in H_2$

$\psi(i\sqrt{2}) = -i\sqrt{2}$, $\psi \notin H_2$ but $\underset{b}{\underset{\shortparallel}{\psi\varphi}} \in H_2$

$$H_2 = \langle a, b \mid a^4 = b^4 = 1,\ b^2 = a^2,\ ab = ba^{-1} \rangle \cong Q_8$$

$$Q_8 = \{\pm 1, \pm j, \pm i, \pm k\}$$

$$b^2 = \psi\varphi\psi\varphi = \varphi^4 = a^2$$

$$ab = \varphi^2 \cdot \psi\varphi = \psi\varphi^7 = ba^3$$

where $a \longleftrightarrow i$

$b \longleftrightarrow j$

$a^2 = b^2 \longleftrightarrow -1$

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[8]{3}) \arrow[dr, dash] & & \mathbb{Q}(\omega) \arrow[dl, dash] \\
& \mathbb{Q} &
\end{tikzcd}$$

$x^8 - 3$ gives a different situation to $x^8 - 2$.

---

char $F \neq 2$,

$$K = F(\sqrt{D_1}, \sqrt{D_2}), \qquad Gal(K/F) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

as long as

$$\begin{tikzcd}
& K & \\
F(\sqrt{D_1}) \arrow[ur, dash, "2"] & & F(\sqrt{D_2}) \arrow[ul, dash, "2"'] \\
& F \arrow[ul, dash, "2"] \arrow[ur, dash, "2"'] &
\end{tikzcd}$$

i.e. $\sqrt{D_1}, \sqrt{D_2} \notin F$,

and $F(\sqrt{D_1}) \neq F(\sqrt{D_2})$.

---

and $F(\sqrt{D_1}) \neq F(\sqrt{D_2})$.

That is, $\sqrt{\frac{D_1}{D_2}} \notin F \iff \sqrt{D_1 D_2} \notin F$.

Conversely, if K/F is Galois and $Gal(K/F) = \mathbb{Z}_2^2 = V_4$,

then $K = F(\sqrt{D_1}, \sqrt{D_2})$.

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & & & & K \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
\mathbb{Z}_2 \arrow[dr, dash, "2"'] & & \mathbb{Z}_2 \arrow[dl, dash, "2"] \arrow[rr] & & L_1 \arrow[dr, dash, "2"'] & & L_2 \arrow[dl, dash, "2"] \\
 & G & & & & F &
\end{tikzcd}$$

$\Rightarrow L_1 = F(\sqrt{D_1}),\ L_2 = F(\sqrt{D_2})$,

since L_i are extensions of degree 2.

$x^4 - 2x^2 - 2$ $\qquad$ K : splitting field.

$\pm \alpha_1 = \pm \sqrt{1+\sqrt{3}}$,

$\pm \alpha_2 = \pm \sqrt{1-\sqrt{3}}$

$K_1 = \mathbb{Q}(\alpha_1), \qquad K_2 = \mathbb{Q}(\alpha_2),$

$G = \mathrm{Gal}(K/\mathbb{Q})$

$K_1 \cap K_2 = F = \mathbb{Q}(\sqrt{3})$

$$\begin{tikzcd}[column sep=small] & K = K_1 K_2 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ K_1 \arrow[dr, dash, "2"'] & & K_2 \arrow[dl, dash, "2"] \\ & F \arrow[d, dash, "2"] & \\ & \mathbb{Q} & \end{tikzcd}$$

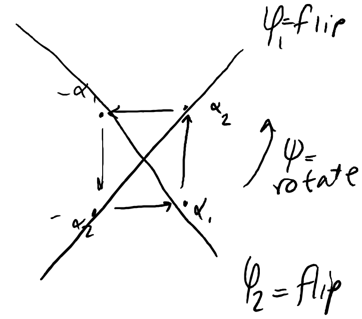
$\mathrm{Gal}(K/F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \trianglelefteq G$

$\varphi_1 : \begin{array}{l} \alpha_1 \longmapsto -\alpha_1 \\ \alpha_2 \longmapsto \alpha_2 \\ \sqrt{3} \longmapsto \sqrt{3} \end{array}$

$\varphi_2 : \begin{array}{l} \alpha_1 \longmapsto \alpha_1 \\ \alpha_2 \longmapsto -\alpha_2 \\ \sqrt{3} \longmapsto \sqrt{3} \end{array}$

$\psi : \sqrt{3} \longmapsto -\sqrt{3}$

$\alpha_1 \longmapsto \alpha_2 \longmapsto -\alpha_1 \longmapsto -\alpha_2 \longmapsto \alpha_1$



$\psi^4 = 1$

$\varphi_1^2 = \varphi_2^2 = 1$

$\varphi_1 \psi \varphi_1^{-1} = \psi^3$

$$\begin{tikzcd} \langle \psi \rangle \arrow[d, equal] \\ G \end{tikzcd}$$