Lec 10/6

Thursday, October 6, 2016 9:09 AM

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Interpretations of Derivative

* slope of tangent line

  Precisely: the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is the value of the derivative at a.

Ex: $f(x) = x^{3/2}$ $x > 0$

$$f'(a) = \frac{3}{2} a^{1/2}$$

eqn of tangent line at $(a, f(a))$:

$$y - f(a) = \frac{3}{2} a^{1/2} (x - a)$$

Avoid this mistake:

$$y - f(a) = \frac{3}{2} x^{1/2} (x - a)$$

* Instantaneous velocity

  $x = f(t)$: position of an object moving along a straight line

$$\frac{f(t) - f(a)}{t - a} \quad : \text{ avg velocity over } [a, t]$$

$$f'(a) = \lim_{t \to a} \left(\text{avg velocity over } [a, t] \right)$$

If we let a vary, we get a new function

$v(t) = f'(t)$ (instantaneous velocity).

$$\frac{v(t) - v(b)}{t - b} \quad : \text{ avg acceleration over } [b, t]$$

$$f''(b) = \lim_{t \to b} \left(\text{avg acceleration over } [b, t] \right)$$

$a(t) = f''(t)$ (instantaneous acceleration)

$F = ma$

Leibnitz notation

$$\left.\frac{\partial f}{\partial x}\right|_{x=a} = f'(a) \qquad \frac{\partial f}{\partial x} \text{ is a function.}$$

$$\frac{\partial}{\partial x} : \text{functions} \longrightarrow \text{functions}$$

$$\frac{\partial}{\partial x}(x^{3/2}) = \frac{3}{2}x^{1/2}$$

$$f'' = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$$

$$f^{(n)} = \frac{\partial^n f}{\partial x^n}$$

Example: $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Case 1: $a > 0$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x - a}{x - a} \qquad \text{(localization principle over } (0, a) \cup (a, \infty))$$

$$= 1$$

Case 2: $a < 0$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim \frac{-x - (-a)}{} \qquad \text{(localization principle on } (-\infty, a) \cup (a, 0))$$

$$\ldots \lim_{x\to a} \frac{\ldots}{x-a}$$

$$= \lim_{x\to a} \frac{-x-(-a)}{x-a} \qquad \text{(localization principle on } (-\infty,a)\cup(a,0)\text{)}$$

$$= -1$$

Case 3: $a = 0$

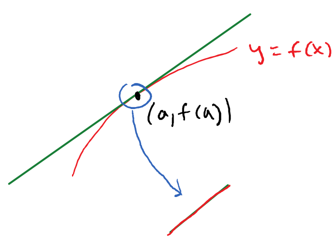
$$f'(0) = \lim_{x\to 0} \frac{|x|-0}{x-0}$$

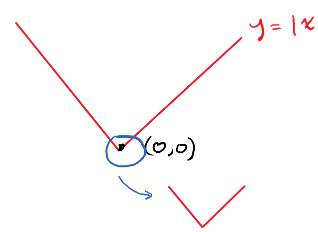
$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \qquad LP(-\infty, 0)$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1 \qquad LP(0,\infty)$$

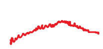
$1 \neq -1$ so limit DNE and f has no derivative at 0.

graphical interpretation of derivative

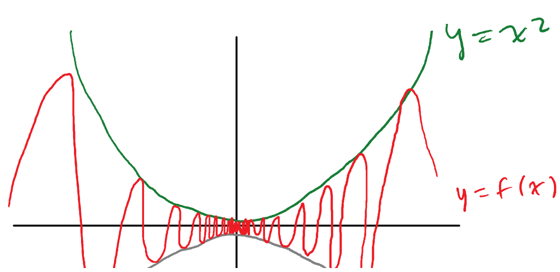


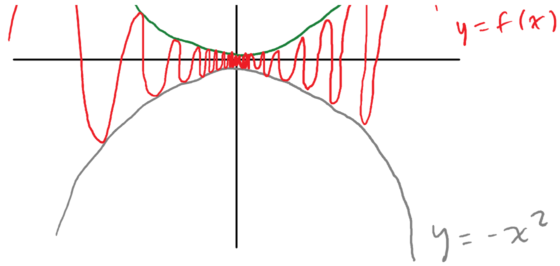


functions exist that are continuous everywhere but diffable nowhere.

 fractals.

Example: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$





$$f'(x) = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x}\left(\frac{-1}{x^2}\right) \qquad \text{(by product \& chain rules)}$$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

This is valid for $x \neq 0$ by using Localization principle (implicitly)

To compute $f'(0)$, need to use definition.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} \qquad LP\ (-\infty, 0) \cup (0, \infty)$$

$$= \lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

$$= 0$$

by squeeze theorem. $\underset{\downarrow\atop 0}{-|x|} \le \underset{\downarrow\atop 0}{x\sin\frac{1}{x}} \le \underset{\downarrow\atop 0}{|x|}$

However $\lim_{x\to 0} f'(x) \neq 0$ (the limit DNE)

So derivative need not be continuous. ✱

Differentiation Rules

(1) $(f+g)'(x) = f'(x) + g'(x)$ (provided these exist)

(2) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

(3) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

(4) $(f \circ g)'(x) = f'(g(x))\,g'(x)$

(5) $(x^n)' = nx^{n-1}$ (for all numbers n, provided this makes sense.)