Lec 4/4

Wednesday, April 4, 2018 14:23

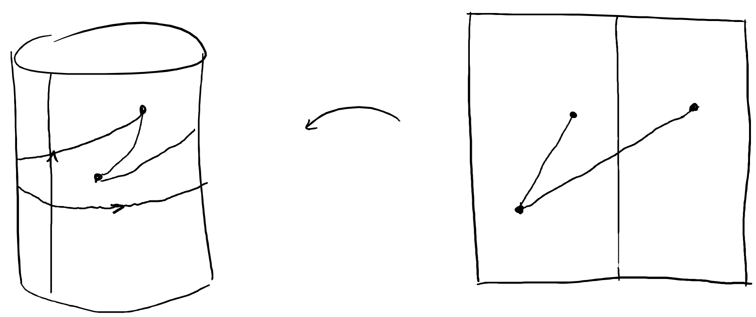
Geodesics on a Sphere

- Great circles

$\frac{dT}{ds}$ is normal to sphere, so points towards the center of the sphere.

For a circle of latitude, $\frac{dT}{ds}$ points towards center of circle, so a circle of latitude is not a geodesic unless it is a great circle.

Geodesics on a Cylinder

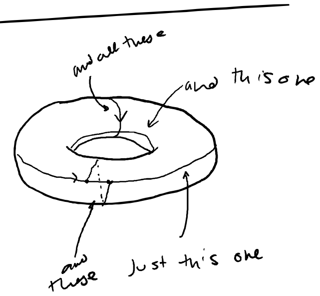


"drawing a string taut between two pts"

$x(\theta, z) = (\cos\theta, \sin\theta, z)$

straight lines $\longmapsto$ geodesics

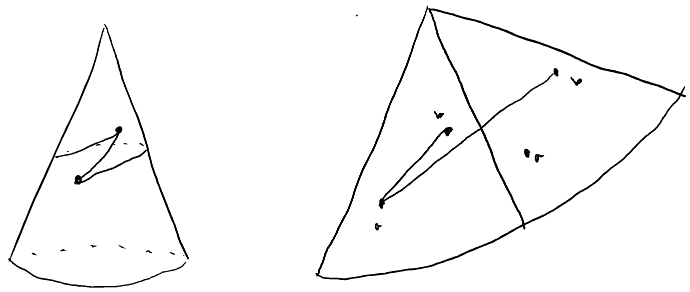
Geodesics on a Torus



meridians, inner & outer eqns.

also, certain curves that spiral around

Geodesics on a cone



Remark:

Suppose $x : U \text{ open} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an immersion.

Then x_{ij} can still be written as a tangential part and a normal part. The tangential part is still of the form $\sum_k \Gamma_{ij}^k x_k$ and the Γ_{ij}^k's are still intrinsic.

But the dimension of the Normal space is $n-m$, not necessarily 1.

Defns Let $\gamma : (a,b) \longrightarrow M$ be a C^1 curve on a C^2 surface M in $\mathbb{R}^3$. A C^1 vector field on M along γ is a C^1 function $X : (a,b) \longrightarrow \mathbb{R}^3$ s.t. $\forall t \in (a,b)$, $X(t) \in T_{\gamma(t)} M$

- The covariant derivative $\nabla_{\dot\gamma} X$ of X along γ is the component of $\frac{d}{dt} X(t)$ tangent to M at $\gamma(t)$.

- To say X is parallel along γ (rel. to M) means $\nabla_{\dot\gamma} X \equiv 0$.

Propn Let $x : U \text{ open} \subseteq \mathbb{R}^2 \xrightarrow{\text{onto}} V \text{ open} \subseteq M$ be a C^2 patch on M. Let $\gamma : (a,b) \longrightarrow V$ be a C^1 curve and let $X : (a,b) \longrightarrow \mathbb{R}^3$ be a C^1 v.f. on M along γ.
Then $\nabla_{\dot\gamma} X = \sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k X^i \frac{d\gamma^j}{dt} \right) x_k$ where $\gamma(t) = x(\gamma^1(t), \gamma^2(t))$

and $X(t) = \sum_k X^k(t)\, x_k(\gamma^1(t), \gamma^2(t))$ for all $t \in (a,b)$.

Thus $\nabla_{\dot\gamma} X$ is intrinsic and also X is parallel along γ iff $\forall k$,

$$\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k X^i \frac{d\gamma^j}{dt} \equiv 0.$$

Pf

$$\frac{dX}{dt} = \frac{d}{dt}\left(\sum_k X^k x_k(\gamma^1(t), \gamma^2(t))\right) = \sum_k \left(\frac{dX^k}{dt} x_k + X^k \frac{dx_k}{dt}\right)$$

$$= \left(\sum_k \frac{dX^k}{dt} x_k\right) + \left(\sum_i X^i \frac{dx_i}{dt}\right)$$

$$= \left(\text{—''—}\right) + \left(\sum_i X^i \left(\sum_j x_{ij} \frac{d\gamma^j}{dt}\right)\right)$$

$$= \left(\text{—''—}\right) + \sum_{i,j}\left(L_{ij}\, n + \sum_k \Gamma_{ij}^k x_k\right) X^i \frac{d\gamma^j}{dt}$$

Thus the tangential part of $\frac{dX}{dt}$ is

$$\sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k X^i \frac{d\gamma^j}{dt}\right) x_k \quad \text{as required.} \qquad \square$$

<u>Propn</u> Let X, Y be C^1 vector fields on M along γ.

Then $\frac{d}{dt}\langle X, Y\rangle = \langle \nabla_{\dot\gamma} X, Y\rangle + \langle X, \nabla_{\dot\gamma} Y\rangle$.

<u>Corollary</u> If X, Y are C^1 vector fields on M along γ and each is parallel along γ then $t \mapsto \langle X(t), Y(t)\rangle$ is constant.

<u>Corollary</u> If X is a C^1 vector field on M along γ and X is parallel along γ then $t \longmapsto |X(t)|$ is constant.