Lec 9/6

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How many cycles are there of a given cycle type $C(\lambda)$

Cycle type: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$, $\sum \lambda_i = n$.

or $(\underbrace{1, 1, \ldots, 1}_{l_1}, \underbrace{2, \ldots, 2}_{l_2}, \ldots, \underbrace{n, \ldots, n}_{l_n})$

$$(1^{l_1}, 2^{l_2}, \ldots, n^{l_n}), \quad \sum i l_i = n$$

eg $n = 5$ $\quad (3,2)$ or $(1^0, 2^1, 3^1)$

$$\binom{5}{3} \times 2 = C((3,2)) = 20$$

# of ways to break $[n]$ into sets of size $\lambda_1, \ldots, \lambda_r$:

$$\binom{n}{\lambda} = \binom{n}{\lambda_1 \; \lambda_2 \cdots \lambda_r} = \frac{n!}{\lambda_1! \, \lambda_2! \cdots \lambda_r!}$$

# of ways to make a cycle from set of λ_i elts: $(\lambda_i - 1)!$

So total of $(\lambda_1 - 1)! \, (\lambda_2 - 1)! \cdots (\lambda_r - 1)!$

but overcounted by l_i if multiple of same length:

So $$\boxed{C(\lambda) = \frac{n!}{\lambda_1 \lambda_2 \cdots \lambda_r \cdot l_1! \, l_2! \cdots l_n!}}$$

$$l_i = \#\{\lambda_j = i\}$$

$$\begin{array}{ccc} G & \circlearrowright & X \\ \| & \uparrow & \| \\ S_n & \text{conj.} & S_n \end{array}$$

so $\sigma \cdot x = \sigma x \sigma^{-1} = \text{conj}_\sigma(x)$

Generalize: $G \circlearrowright G$ by conjugation.

$C(\lambda) = \#$ of elements in $\overset{\mathcal{O}_\lambda}{\vee}$ S_n-orbit (via conjugation)

of $\pi_\lambda = (1\ 2 \ldots \lambda_1)(\lambda_1+1 \ldots \lambda_1+\lambda_2)\ldots$

Pf $\sigma(x_1 \ldots x_\ell)\sigma^{-1} = (\sigma(x_1) \ldots \sigma(x_\ell))$

Orbits of $\underset{\text{by conj.}}{G \circlearrowright G}$ are called Conjugacy Classes.

$$|\text{Orbit}| = \frac{|\text{Group}|}{|\text{stabilizer}|}.$$

In our case $\underset{\text{conj}}{G \circlearrowright G}$, $\text{Stab}_G(x) = \{g \in G \mid gxg^{-1} = x\}$

= all elements which commute with x

$= Z_G(x) \leftarrow$ centralizer of x in G

Cor: # of elements of S_n commuting w/ w (which has type λ)

is $C(\lambda) = \dfrac{n!}{\lambda_1\lambda_2\cdots\lambda_r \cdot \ell_1!\ell_2!\cdots\ell_n!}$.

So far. ① $|\text{Orbit}| = \dfrac{|\text{Group}|}{|\text{stabilizer}|}$

② $$\frac{|X|}{|G|} = \sum_{\mathcal{O}:\text{orbit}} \frac{1}{|\mathrm{Stab}_G(x_{\mathcal{O}})|}$$

some elt. from $\mathcal{O}$.

e.g. $$1 = \sum_{\lambda} \frac{1}{z_\lambda}$$

$\hookrightarrow z_\lambda = \lambda_1 \lambda_2 \cdots \lambda_r \cdot l_1! \, l_2! \cdots l_n!$

eg $n=4$

λ	z_λ		
4	4	=	4
3+1	3	=	3
2+2	$4\cdot 2$	=	8
2+1+1	$2\cdot 2$	=	4
1+1+1+1	4!	=	24

$$\sum_\lambda \frac{1}{z_\lambda} = \frac{6+8+3+6+1}{24} = 1$$

Burnside's Theorem

$$|G\backslash X| = \frac{1}{|G|} \sum_{g\in G} |X^g|$$

Set of G-orbits of X, since each element is $g\cdot x$

Ex $G = S_n \curvearrowright X = \{1, \dots, n\}$. $\quad |G \backslash X| = 1$ (i.e. action is transitive)

$$1 = \frac{1}{n!} \sum_{\sigma \in S_n} |X^\sigma|$$

$= \ell_1$ in cycle type of σ

$\{ i \in [n] : \sigma(i) = i \}$

$$\boxed{1 = \frac{1}{n!} \sum_{\lambda : \text{cycle type}} c(\lambda)\, \ell_1}$$

So $\displaystyle \sum_\lambda \frac{1}{z_\lambda} = \sum_\lambda \frac{\ell_1}{z_\lambda}$ $\quad \left(\text{since } c(\lambda)\,\ell_1 = \frac{\ell_1}{z_1} n! \right)$

Proof of Burnside's Theorem

$$G \times X \supset F := \{ (g, x) \in G \times X \mid gx = x \}$$

$$\mathrm{Stab}_G(x) \cong \mathrm{Stab}_G(\sigma x)$$

$$g \longrightarrow \sigma g \sigma^{-1}$$

$$|F| = \sum_{g \in G} |X^g| = \sum_{x \in X} |\mathrm{Stab}_G(x)| = \sum_{\mathcal{O} : \text{orbit}} |\mathcal{O}|\, |\mathrm{Stab}_G(x_\mathcal{O})|$$

$$= \sum_{\mathcal{O} : \text{orbit}} |G| = |G| \cdot |G \backslash X| \qquad \square$$

idea:

$$\begin{tikzcd} X^g \subset & F \arrow[dl] \arrow[dr] & \supset \mathrm{Stab}_G(x) \\ g \in G & & X \ni x \end{tikzcd}$$

"fixed point formulae".

quiz: constructing group homomorphisms

Problem: prove that G has the following presentation $\mathcal{G} = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$.

method: ① map $\mathcal{G} \xrightarrow{\text{hom.}} G$

prescribe n elements $\{g_1, \dots, g_n\}$ in G which satisfy $r_1, \dots, r_m$.

② prove map is surjective: $\langle \{g_1, \dots, g_n\} \rangle = G$

③ prove map is injective: compare size (maybe)

or

write an inverse map which is a homomorphism.