Lec 10/18

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Hermitian Inner Product

$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ for $V/\mathbb{C}$

$\langle \alpha a + \beta b, c \rangle = \alpha \langle a, c \rangle + \beta \langle b, c \rangle$

$\langle c, \alpha a + \beta b \rangle = \bar{\alpha} \langle c, a \rangle + \bar{\beta} \langle c, b \rangle$

$\langle a, b \rangle = \overline{\langle b, a \rangle}$

$\|a\|^2 = \langle a, a \rangle$ is real, $\langle a, a \rangle \geq 0$, $\langle a, a \rangle = 0 \Leftrightarrow a = 0$

Existence of OB (V): $\{u_1, \ldots, u_n\}$ $\langle u_i, u_j \rangle = \delta_{ij}$ via gram-schmidt.

given an OB of V, $v = \sum_{j=1}^{n} \alpha_j u_j$, $\langle v, u_j \rangle = \alpha_j$

$\Rightarrow v = \sum_{j=1}^{n} \langle v, u_j \rangle u_j$, $w = \sum_{j=1}^{n} \overbrace{\langle w, u_j \rangle}^{\beta_j} u_j$

$\Rightarrow \langle v, w \rangle = \sum_{j=1}^{n} \langle v, u_j \rangle \overline{\langle w, u_j \rangle} = \sum_{j=1}^{n} \alpha_j \bar{\beta}_j$

Analogue of orthogonal Transformations.

Def. $U \in L(V,V)$ is Unitarie if $\|Uv\| = \|v\|$ $\forall v \in V$.

Thm: the following are equivalent

1) U is unitary

2) $\langle Uv, Uw \rangle = \langle v, w \rangle$

3) $U(OB_2) = OB_2$

4) $U \xrightarrow{OB} A \Rightarrow A\overline{A^T} = \overline{A^T}A = I.$

Proof. $1 \Rightarrow 2$: $\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle$

$= \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \langle w,v\rangle$ let $\langle v,w\rangle = \alpha + i\beta$

so $\langle w,v\rangle = \alpha - i\beta$

$\Rightarrow 2\alpha = \|v+w\|^2 - \|v\|^2 - \|w\|^2$

$-i\langle v,w\rangle = \beta - i\alpha$

$i\langle w,v\rangle = \beta + i\alpha$

and $\|v+iw\|^2 = \|v\|^2 + \|w\|^2 - i\langle v,w\rangle + i\langle w,v\rangle$

$\Rightarrow 2\beta = \|v+iw\|^2 - \|v\|^2 + \|w\|^2$

So inner product is expressible in terms of norms.

So $1 \Rightarrow 2$. $2 \Rightarrow 1$ trivially.

$2 \Rightarrow 3$ trivially

$3 \Rightarrow 4$: $U(u_i) = \sum_{i=1}^{n} \alpha_{ki} u_k$

so $\delta_{ij} = \sum_{k,l=1}^{n} \alpha_{ki}\overline{\alpha_{lj}} = \sum_{k=1}^{n} \alpha_{ki}\overline{\alpha_{kj}} = (\overline{A^T}A)_{ji}$

so A invertible, $A^{-1} = \overline{A^T}$.

and $4 \Rightarrow 3$ trivially

$3 \Rightarrow 2$: via algebra. So it is proved. □

$U \in \overbrace{\mathcal{U}(V)}^{\text{a group.}}$ notice $\|v\| = \|Iv\| = \|U(U^{-1}(v))\| = \|U^{-1}v\|$

If $W \subseteq V$ is an irreducible subsp. of U then $\dim W = 1$.

Since U has an eigenvalue over W, and

$S(\underset{\text{eigenvector}}{v}) \subseteq W$ is irreducible.

$V = W \oplus W^{\perp}$. all eigenvalues are on unit circle

since $\|Uv\| = \|\lambda v\| = |\lambda| \|v\| = \|v\|$

so U is diagonalizable wrt a OB.

$\operatorname{spec}(T) = \{ \lambda \in F : \exists v \neq 0 \text{ s.t. } Tv = \lambda v \}$.

$\operatorname{spec}(U) \subset S_1 = \{ z \in \mathbb{C} : \|z\| = 1 \text{ i.e. } z = \cos\theta + i\sin\theta \}$.