Rec 11/7

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① **Def** R is Noetherian if every ideal is finitely generated

— iff R satisfies ascending chain condition (ACC)

② R is Artinian if R satisfies Descending Chain Condition (DCC)

Non-Noetherian example: $R[x_1, x_2, x_3, \ldots]$.

$$\text{Int}(K) = \{ f \in K(x) \mid f(n) \in \mathbb{Z} \ \forall n \in \mathbb{Z} \}$$

(whenever $K \supset \mathbb{Z}$ is a field)

eg $f(x) = \frac{x(x+1)}{2} \in \text{Int}(K)$.

If R is Noetherian, then

- $R[x]$ is Noetherian,
- R_s is Noetherian (Localization)
- R/I is Noetherian
- R has only finitely many minimal prime ideals

If R is comm & Artinian then R is Noetherian.

(1888)

<u>Hilbert Basis Thm</u>: R Noetherian $\Rightarrow R[x]$ Noetherian.

pf Let $I \subset R[x]$ be an ideal, L be the set of leading coefficients of elements of I. Then L is an ideal of R.

<u>why</u>: • $0 \in L$ since $0 \in I$.

• $ra - b \in L$ $\forall a, b \in L$, $r \in R$ since if $f = ax^n + \cdots$, $g = bx^m + \cdots$

then $rfx^m - gx^n \in I$.

So L is finitely gen, say $L = (a_1, \ldots, a_n)$.

For each i, let $f_i \in I$ be of minimal degree w/ leading coefficient a_i. Let $N = \max\{e_i := \deg f_i\}$.

For each $d \in \{0, \ldots, N-1\}$, let L_d be the set of (0 and the) leading coefficients of polynomials in I w/ degree d

Each L_d is an ideal of R, so each L_d is f.g.

$L_d = (b_{d,1}, b_{d,2}, \ldots, b_{d,n_d})$. Let $f_{d,i} \in I$ w/ degree d and leading coefficient $b_{d,i}$.

<u>Claim</u>: $I = \left(\{f_1, \ldots, f_n\} \cup \{f_{d,i} \mid 0 \le d \le N-1, 1 \le i \le n_d\}\right) =: I'$.

<u>why</u>: $I' \subset I$ by construction. Assume $\exists f \in I \setminus I'$ w/ minimal degree d and leading coefficient a.

<u>case 1</u>: $d < N$. Then $a \in L_d$ so $a = r_1 b_{d,1} + r_2 b_{d,2} + \cdots + r_{n_d} b_{d,n_d}$.

let $g = r_1 f_{d,1} + \cdots + r_{n_d} f_{d,n_d}$. Then $g \in I'$ w/ same degree as f and same leading coeff.

Thus $f-g \notin I \setminus I'$ has degree $< d$. Contradiction

<u>Case 2</u>: $d \geq N$. something similar.

$\square$