Lec 10/25

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$a_n b_m = 0$

$a_{n-1} b_m + a_n b_{m-1} = 0$

$\Rightarrow \quad a_n^2 b_{m-1} = 0$

induction $\Rightarrow \quad a_n^{k+1} b_{m-k}$

$\rightarrow \quad a_n^{m+1} b_0 = 0 \Rightarrow a_n$ is nilpotent

induction: $p(x) - a_n x^n$ is still a unit.

comm ring w/ unique max'l ideal M

Local ring (R, M)

in this case $M = R \setminus R^\times$.

set of nilpotent elements in R.

$$(R[x])^\times = R^\times + x\, \mathcal{N}[x]$$

[Optional] $\mathbb{C} \ni z_0 \qquad \mathcal{O}_{z_0} = \{ f : U \longrightarrow \mathbb{C}$ holomorphic, U some domain containing $z_0 \}$

$f = g$ if they agree on some domain containing z_0.

Local ring w/ $M = \{ f \in \mathcal{O}_{z_0} : f(z_0) = 0 \}$

Our examples:

$\mathbb{C}[\![x]\!]$, $\mathbb{Z}_p$ $\leftarrow \{ \frac{a}{b} :$ p does not divide b, $(a,b) = 1 \}$

$\hookrightarrow M = (p)$

How to localize?

- by non-vanishing
- by completion! $\longrightarrow$ maybe not in this course

Definition: $S \subset R$ is multiplicatively closed if

(i) $1 \in S$, $0 \notin S$.

(ii) $a, b \in S \Rightarrow ab \in S$.

Construction of ring of fractions

$\rightsquigarrow$ new ring $S^{-1}R \overset{\text{as a set}}{=} \dfrac{\{(r,s) : r \in R,\ s \in S\}}{\sim}$

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists\, t \in S \text{ s.t. } t(s_1 r_2 - s_2 r_1) = 0$$

claim: $\sim$ is an equiv. reln.

(i) reflexive $rs - sr = 0$ ✓

(ii) symmetric $-0 = 0$ ✓

(iii) transitive
$$t(s_2 t')(s_1 r_3) = tt'(s_1 s_3 r_2)$$
$$= t' t(s_3 s_2 r_1)$$
$$\Rightarrow \underbrace{tt's_2}_{\in S}(s_1 r_3 - s_3 r_1) = 0 \ ✓$$

$\dfrac{r}{s}$ = equivalence class of (r, s).

$\dfrac{r_1}{s_1} = \dfrac{r_2}{s_2}$ means $t(r_1 s_2 - r_2 s_1) = 0$ for some $t \in S$.

addition $\dfrac{r_1}{s_1} + \dfrac{r_2}{s_2} = \dfrac{r_1 s_2 + r_2 s_1}{s_1 s_2}$

multiplication $\dfrac{r_1}{s_1} \cdot \dfrac{r_2}{s_2} = \dfrac{r_1 r_2}{s_1 s_2}$

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We have to make sure the ops are well-defined.

then $0_{S^{-1}R} = \frac{0}{1}$, $1_{S^{-1}R} = \frac{1}{1}$

Addition:

$$\frac{r_1}{s_1} = \frac{r_1'}{s_1'} , \quad \frac{r_2}{s_2} = \frac{r_2'}{s_2'} \implies \frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1'}{s_1'} + \frac{r_2'}{s_2'}$$

to show
↓
$\exists t$ s.t. $t\left(s_1's_2'(r_1s_2 + r_2s_1) - s_1s_2(r_1's_2' + r_2's_1')\right) = 0$

given $\exists t_1, t_2$ s.t. $t_1(r_1s_1' - r_1's_1) = 0$
$t_2(r_2s_2' - r_2's_2) = 0$

it all works out. use $t = t_1t_2s_1s_2s_1's_2'$

R comm ring
$\cup$
S mult. closed $\rightsquigarrow$ $S^{-1}R$ ring of fractions

(ore domains)

Eg $R = \mathbb{Z} \supset S_1 = \mathbb{Z} \setminus \{0\}$, $S_1^{-1}R = \mathbb{Q}$ is a field

$R \setminus \{0\} \subset R$ is mult-closed iff R is an integral domain

Eg $R = \mathbb{Z} \supset S_2 = \mathbb{Z} \setminus p\mathbb{Z}$, $S_2^{-1}R = \mathbb{Z}_p = \{ \frac{a}{b} : \substack{p \nmid b \\ (a,b)=1} \}$

Eg $R = \mathbb{Z} \supset S_3 = \{1, p, p^2, \dots\}$, $S_3^{-1}R = \{ \frac{a}{p^e} : a \in \mathbb{Z}, e \in \mathbb{Z}_{\geq 0} \}$

Observation: R comm ring
$\cup\!|$
P ideal

P prime $\iff$ $R \setminus P$ is multiplicatively closed

($\Rightarrow$) Since $ab \notin S \Rightarrow ab \in P \Rightarrow$ one of $a, b \in P \Rightarrow$ one of $a, b \notin S$.

So $a, b \in S \Rightarrow ab \in S$.

Also $1 \notin P$, $0 \in P$.

($\Leftarrow$) $a \notin P$, $b \notin P \Rightarrow ab \notin P$

Notation: $R_P = (R \setminus P)^{-1} R$

"R localized at P" $\leftarrow$ tomorrow:

Maxl ideal

$M = \left\{ \frac{r}{s} : \begin{array}{l} r \in P \\ s \in R \setminus P \end{array} \right\}$