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Thursday, April 18, 2019 11:30

Cubics & Quartics

$f(x) = x^3 + ax^2 + bx + c = y^3 + py + q \in F[x]$, char $F \neq 2,3$.

$\sqrt[3]{1} = \frac{1+\sqrt{-3}}{2}$

f is irreducible over F.

assume $\sqrt{-3} \in F$.

D - discriminant of f,

$D = -4p^3 - 27q^2$.

Let $G = Gal(f/F)$. $\alpha_1, \alpha_2, \alpha_3$ be roots of f.

$G \leq S_3$ since G permutes $\{\alpha_1, \alpha_2, \alpha_3\}$.

G acts transitively on the roots, so the only possible subgroups are $G \cong \mathbb{Z}_3$ or $G \cong S_3$.

$$\mathbb{Z}_3 = A_3$$

$G \cong A_3$ iff $\sqrt{D} \in F$.

for $A_3$: for $S_3$:

$1$ $\quad K = F(\alpha_1, \alpha_2, \alpha_3)$ $\quad 1$

$\|$ $\quad \|$ $\quad \|$

$$\begin{array}{ccc} 1 & K = F(\alpha_1, \alpha_2, \alpha_3) & 1 \\ \| \, 3 & \| & \| \\ A_3 & L = F(\sqrt{D}) & A_3 \\ \| \, 2 \text{ or } 1 & \| & \| \\ G \cap S_3 & F & S_3 \end{array}$$

$\alpha_1 \in K, \quad \varphi : \alpha_1 \longmapsto \alpha_2 \longmapsto \alpha_3$

$$(\alpha_1, \omega) = \alpha_1 + \omega \varphi(\alpha_1) + \omega^2 \varphi^2(\alpha_1)$$

$$= \alpha_1 + \omega \alpha_2 + \omega^2 \alpha_3$$

and $(\alpha_1, \omega)^3 \in F(\sqrt{D})$

let $b = (\alpha_1, \omega)^3, \quad (\alpha, \omega) = \sqrt[3]{b}$ .

Note: $b_1 = -\frac{27}{q} + \frac{3}{2}\sqrt{-3D}$

Recall: $f(x) = x^3 + px + q$

$b_2 = -\frac{27}{q} - \frac{3}{2}\sqrt{-3D}$

Cardano formulas:

$$\alpha_1 = \frac{1}{3}(b_1 + b_2)$$

$$\alpha_2 = \frac{1}{3}(\omega b_1 + \omega^2 b_2)$$

$$\alpha_3 = \frac{1}{3}(\omega^2 b_1 + \omega b_2)$$

obtained by solving $\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3 = \sqrt[3]{b_1}$

$$\alpha_2 + \omega\alpha_3 + \omega^2\alpha_1 = \sqrt[3]{b_2}$$

since they add up to coeff of $x^2$. $\longrightarrow$ $\alpha_1 + \alpha_2 + \alpha_3 = 0$

$G \cong S_3$ if $\sqrt{D} \notin F$

$G \cong A_3$ if $\sqrt{D} \in F$

$A_3 \cong \mathbb{Z}_3$

Over $\mathbb{Q}$ now

If $f$ has 2 non-real roots,

complex conjugation $\in G$,

so $G \cong S_3$.

If all roots are real,

then $G \cong \mathbb{Z}_3$ or $G \cong S_3$.

<u>Lemma</u>: $f$ has 3 real roots iff $D > 0$.

pf $D = \prod_{i>j} (\alpha_i - \alpha_j)^2 > 0$ if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

If $D>0$, then $\sqrt{D}\in\mathbb{R}$, so $\mathbb{Q}(\sqrt{D})\subseteq\mathbb{R}$,

and $(K=\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3))$ $K/\mathbb{Q}(\sqrt{D}) \cong \mathbb{Z}_3$

is abelian. If $\alpha_1$ is real, then

$K=\mathbb{Q}(\sqrt{D})(\alpha_1)$, so $\alpha_2,\alpha_3$ are also real.

## Casus Irreducibilis

If cubic $f$ is irreducible & $\alpha_1,\alpha_2,\alpha_3$ are real, There is no tower of real radical extensions containing $\alpha_1,\alpha_2,\alpha_3$.

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## Quartics

$f(x)=x^4+px^2+qx+r \in F[x]$, char $F\neq 2,3$,

$\sqrt{-3}\in F$.

$f$ is irr-ble, $\alpha_1,\dots,\alpha_4$ - roots,

$K=F(\alpha_1,\dots,\alpha_4)$,

$$D=-27p^4-108p^3q-162p^2q^2-108pq^3-27q^4+256r^3.$$

$G = \mathrm{Gal}(f/F)$, $4 \mid |G|$ since $|G| = [K:F]$.

$G \leq S_4$

$G$ acts transitively on $\{\alpha_1, \dots, \alpha_4\}$.

<u>Possibilities for $G$</u>

transitive subgroups of $S_4$:

$$\underset{24}{S_4}, \quad \underset{12}{A_4}, \quad H_1 = \langle (1\,3\,2\,4), (1\,2) \rangle,$$

$$H_1 \cong D_8 \cong H_2, \quad D_8 \cong H_3$$

$$H_2 = \langle (1\,2\,3\,4), (1\,3) \rangle, \quad H_3 = \langle (1\,2\,4\,3), (1\,4) \rangle,$$

$$V = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \cong \mathbb{Z}_2^2$$

normal in $S_4$

$$C_1 = \langle (1\,2\,3\,4) \rangle \cong C_2 = \langle (1\,3\,2\,4) \rangle \cong C_3 = \langle (1\,2\,4\,3) \rangle \cong \mathbb{Z}_4$$

all 3 are conjugate.

$$\begin{array}{c} 1 \\ 4 \,|\, \\ V \\ 3 \,\|\, \\ A_4 \\ 2 \,|\, \\ S_4 \end{array} \quad \Longrightarrow \quad \begin{array}{c} 1 \\ \| \; 1, 2, \text{ or } 4 \\ V \cap G \\ \| \; 1 \text{ or } 3 \\ A_4 \cap G \\ \| \; 1 \text{ or } 2 \\ G \end{array} \quad \longleftrightarrow \quad \begin{array}{c} K \\ \| \; 1, 2, \text{ or } 4 \\ L \\ \| \; 1 \text{ or } 3 \\ F(\sqrt{D}) \\ \| \; 1 \text{ or } 2 \\ F \end{array}$$

fixed by $V$ $\Big\{$ $\theta_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$

another choice

$\theta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$

fixed by V $\left\{ \begin{aligned} \theta_1 &= (\alpha_1+\alpha_2)(\alpha_3+\alpha_4) \\ \theta_2 &= (\alpha_1+\alpha_3)(\alpha_2+\alpha_4) \\ \theta_3 &= (\alpha_1+\alpha_4)(\alpha_2+\alpha_3) \end{aligned} \right.$

$\theta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$

$\vdots$

They are roots of $R(x)$,

$$R(x) = (x-\theta_1)(x-\theta_2)(x-\theta_3)$$
$$= x^3 - 2px^2 + (p^2-4r)x + q^2$$

(cubic resolvent of $f$).

**Lemma** Discriminant of $R$ = Discriminant of $f$

roots of $f$ are:

$$\alpha_1 = \frac{1}{2}\left(\sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3}\right), \quad \alpha_2 = \frac{1}{2}\left(\sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{-\theta_3}\right)$$

$$\alpha_3 = \frac{1}{2}\left(-\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{-\theta_3}\right), \quad \alpha_4 = \frac{1}{2}\left(-\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{-\theta_3}\right)$$

If $R$ is irr-le & $\sqrt{D} \notin F$, $G = S_4$

If $R$ is irr-le & $\sqrt{D} \in F$, $G = A_4$

$\vdots$