Lec 10/18

Tuesday, October 18, 2016 9:09 AM

Def: f convex over I, if $\forall a<b \in I$, $f(x) < \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$

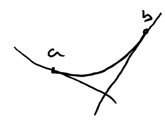
Thm 1: f convex over I, $a<b\in I$, $f'(a), f'(b)$ defined, then $f'(a) < f'(b)$

Thm 2: f' increasing over $I \Rightarrow f$ convex over I

Thm 3: Spoze f is diffable on I & $\forall a \in I$, The graph of f lies above the tangent line over $I \setminus \{a\}$. then f convex over I.

Proof: We will show f' increasing over I.

Let $a<b\in I$. show $f'(a) < f'(b)$.



Since $f(a) > f'(b)(a-b)+f(b)$

$f(b) > f'(a)(b-a)+f(a)$

$f(a)-f(b) > f'(b)(a-b)$

so $f'(b) > \frac{f(a)-f(b)}{a-b}$ because $a-b<0$

$f(b)-f(a) > f'(a)(b-a)$

so $f'(a) < \frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(b)}{a-b}$

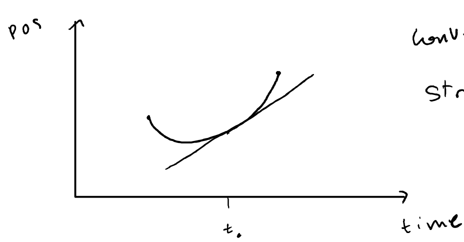
so $f'(b) > f'(a)$, so f' is increasing.

Therefore, by Thm 2, f is convex. ∎

Converse to Thm 3: If f is convex on an open interval I, then the graph of f over I lies above the tangent line at any point $a \in I$ over $I \setminus \{a\}$.

↳ thm 3'

physics interpretation:



convex graph represents an accelerating object (car A)

straight line represents constant speed. (car B)

If line of Car B tangent to graph of Car A at time t_0

At t_0, they are next to each other, at same speed.

B catches up to A for an instant.

Proof of Thm 3': use MVT. Let $a \in I$ and $x \in I$ s.t. $x < a$. Then

$f(x) - f(a) = f'(c)(x-a)$ for some $c \in (x, a)$. but

$f'(c) < f'(a)$ since f is convex & $c < a$. (Thm 1).

So $f(x) - f(a) > f'(a)(x-a)$ ← since $x - a < 0$

hence $f(x) > f'(a)(x-a) + f(a)$

So $f(x)$ lies above the tangent line of $(a, f(a))$.

Similar argument on the other side:

Let $a < x \in I$. then $f(x) - f(a) = f'(c)(x-a)$ for some $c \in (a, x)$.

$f'(c) > f'(a) \Rightarrow f(x) - f(a) > f'(a)(x-a)$

$\Rightarrow f(x) > f'(a)(x-a) + f(a)$ ■

Applications to Maxima/minima (2nd derivative test)

Spose that f' exists on $(a-\delta, a+\delta)$, a is a critical pt; $f'(a) = 0$, and $f''(a)$ exists

Then if $f''(a) > 0$, then f has a local minimum at a.

$f''(a) < 0$, " " maximum "

$f''(a) = 0$, The test is inconclusive.

Proof 1: Assuming f'' is defined on an open interval I around a and f'' is continuous at a. Then

(1) if $f''(a) > 0$ then $f''(x) > 0$ on some open interval J around a.

$\Rightarrow$ f' increasing on $J \Rightarrow f$ Convex over J

$\Rightarrow$ graph of f over J above horizontal tangent line at $(a, f(a))$.

$\Rightarrow$ f has a local min at a.

(2) replace f by $-f$ $\Rightarrow$ $-f$ has a local min $\Rightarrow$ f has a local max.

<u>Proof 2</u> (without extra hypotheses): Recall that if $g'(a) > 0$ then in some open interval J around a, $g(x) < g(a) < g(y)$ for $x < a < y \in J$.

Nevermind. Might need extra hypotheses.

<u>Theorem 4</u> if f' is defined on an interval I, and intersects any of its tangent lines just once on I, then f is either convex or concave.

<u>Lemma:</u> with hypothesis of Thm 4, any straight line can only intersect the graph of f over I once or twice.

<u>Proof of Lemma:</u> Assume for contradiction that a line intersects the graph of f over I 3 times. at $a < c < b$. Then $\dfrac{f(b) - f(a)}{b-a} = \dfrac{f(c) - f(a)}{c-a}$. $(*)$

Let $g(x) = \dfrac{f(x) - f(a)}{x-a}$, $x \in [c, b]$. $g(b) = g(c)$ by $(*)$

So by rolle's thm, $g'(d) = 0$ for some $d \in (c, b)$.

By quotient rule, we have that $g'(x) = \dfrac{f'(x)(x-a) - (f(x) - f(a))}{(x-a)^2}$.

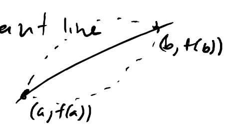
$g'(d) = 0 \Rightarrow 0 = f'(d)(d-a) - f(d) + f(a)$.

$\Rightarrow f(a) = f(d) + f'(d)(a-d)$

so the tangent line at d passes through $(a, f(a))$.

So that contradicts hypothesis. ∎

<u>Proof of Theorem:</u> Suppose $a < b \in I$. Then secant line cannot intersect graph anywhere in (a, b). So the graph must be above or below it on (a, b).



We must show that if $f(x)$ is below (or above) secant line thru $(a, f(a))$ $(b, f(b))$

then the same holds true for any other pair.

Suppose that for one pair $a<b \in I$, $f(x) <$ secant line

and for $a'<b' \in I$, $f(x) >$ secant line.

Choose $c \in (a,b)$ and $c' \in (a',b')$.

for $t \in [1,0]$ let $a_t = (1-t)a + ta'$

$$b_t = (1-t)b + tb'$$

$$c_t = (1-t)c + tc'$$

Then $a_t < c_t < b_t \in I$. Then $g(t) = f(c_t) =$ secant line thru $(a_t, f(a_t)), (b_t, f(b_t))$.

$g(0) > 0$, $g(1) < 0$ so $g(t) = 0$ for some $t \in (0,1)$

Then t-secant line intersects graph in 3 pts. $\times$.

So f is either convex or concave. ∎