$(1+x)^n \geq 1+nx$

$\frac{1+x}{n\sqrt{n}} \geq \frac{1+x}{n} \geq \sqrt[n]{1+x}$

$\Rightarrow \quad 1+x \geq \sqrt[n]{n+nx}$

$\Rightarrow \quad (1+x)^n \geq n+nx \geq 1+nx$

Likelihood Ratio test

$\Lambda = \frac{\max L_0}{\max L}, \quad 0 \leq \Lambda \leq 1.$

test: reject H_0 if $\Lambda \leq K$ for some $K \in (0,1)$.

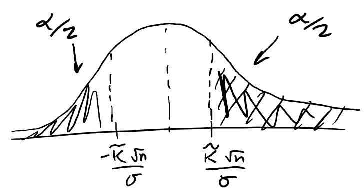
↳ simplify ratio, get equivalent eqn.

Q: how to find $\tilde{K}$? (refer back to last time).

$\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$ since under H_0, $\mu = \mu_0$ and we're trying to find α.

$\Rightarrow \quad \mathbb{P}(|\bar{X} - \mu_0| > \tilde{K}) = \alpha.$

$\Rightarrow \quad \mathbb{P}\left(\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > \frac{\tilde{K}}{\sigma/\sqrt{n}}\right) = \alpha$



$z_{\frac{\alpha}{2}} = \frac{\tilde{K}\sqrt{n}}{\sigma}$

$\Rightarrow \quad \hat{K} = \frac{z_{\frac{\alpha}{2}}\,\sigma}{\sqrt{n}}$

So reject H_0 if

$|\bar{X} - \mu_0| \geq \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}$. (two-sided test).

Q: In this example, RS is from a normal distribution. So we can get distribution of $\bar{X}$. What if we don't have a normal dist?

Try to prove this theorem to test math skillz.

Thm 12.2 (Wilk's Thm).

For large n, the distribution of $-2\log(\Lambda)$ approaches χ^2_1 distribution under general conditions.

LRT Approximation: Reject H_0 if $-2\log(\Lambda) \geq \chi^2_{\alpha,1}$

Check with our example: $-2\log(\Lambda) = -2\left(\frac{n}{2\sigma^2}(\bar{X}-\mu_0)^2\right) = \frac{n}{\sigma^2}(\bar{X}-\mu_0)^2 = \left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right)^2$

and $(N(0,1))^2 = \chi^2$.

$$\ldots = \frac{n}{\sigma^2}(\bar{X}-\mu_0)^2 = \left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right)^2$$

and $(N(0,1))^2 = \chi^2_1$.

<u>Problem 12.22</u>: Let X_i's be iid $N(\mu, \sigma^2)$. Both μ and σ^2 unknown.

Find LRT of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ with level α.

<u>Sol</u>. $\Omega = \mathbb{R} \times (0, \infty)$. $\omega = \{(\mu_0, \sigma^2) : \sigma^2 > 0\}$.

(1) find MLEs:

under Ω, $\hat{\mu} = \bar{X}$ $\quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$

under ω, $\hat{\mu} = \mu_0$ $\quad \hat{\hat{\sigma}}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2$

$$\Lambda = \frac{\max L_0}{\max L} = \frac{\left(\frac{1}{\hat{\hat{\sigma}}^2}\right)^{n/2} \exp\left\{\frac{-1}{2\hat{\hat{\sigma}}^2}\sum_{i=1}^n (X_i-\mu_0)^2\right\}}{\left(\frac{1}{\hat{\sigma}^2}\right)^{n/2} \exp\left\{\frac{-1}{2\hat{\sigma}^2}\sum_{i=1}^n (X_i-\bar{X})^2\right\}} \leq K.$$

$$\left(\frac{\hat{\hat{\sigma}}^2}{\hat{\sigma}^2}\right)^{-n/2} = \left(\frac{\sum_{i=1}^n (X_i-\mu_0)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}\right)^{-n/2}$$

$$\sum_{i=1}^n (X_i-\mu_0)^2 = \sum_{i=1}^n \left((X_i-\bar{X}) + (\bar{X}-\mu_0)\right)^2 = \sum_{i=1}^n (X_i-\bar{X})^2 + 2\Big(\underbrace{\sum_{i=1}^n (X_i-\bar{X})}_{0}\Big)\cdot(\bar{X}-\mu_0) + \sum_{i=1}^n (\bar{X}-\mu_0)^2$$

$$= \sum_{i=1}^n (X_i-\bar{X})^2 + n(\bar{X}-\mu_0)^2$$

and the $\frac{\exp\{\cdots\}}{\exp\{\cdots\}}$ part is 1 since there is cancellation.

So
$$\Lambda < K \Leftrightarrow \left[\frac{\sum_{i=1}^n (X_i-\bar{X})^2 + n(\bar{X}-\mu_0)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}\right]^{-n/2} \leq K.$$

$$\Leftrightarrow 1 + \frac{n(\bar{X}-\mu_0)^2}{\sum_{i=1}^n (X_i-\bar{X})^2} \geq K^{-2/n}$$

$$\Leftrightarrow \quad \frac{n(\bar{x}-\mu_0)^2}{\sum (x_i - \bar{x})^2} \geq K^{-2/n} - 1$$

$$\Leftrightarrow \quad \frac{(\bar{x}-\mu_0)^2}{\frac{1}{n}\left(\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right)} \geq (n-1)\left(K^{-2/n} - 1\right)$$

$$\Leftrightarrow \quad \underbrace{\left| \frac{\bar{x}-\mu_0}{S/\sqrt{n}} \right|}_{t_{n-1}} \geq \underbrace{\sqrt{(n-1)\left(K^{-2/n} - 1\right)}}_{\tilde{K}}$$

(2) find K So $\tilde{K}$ should be $t_{n-1, \frac{\alpha}{2}}$.

So reject H_0 when $|\bar{X} - \mu_0| \geq t_{n-1, \frac{\alpha}{2}} \frac{S}{\sqrt{n}}$