Lec 10/31

Wednesday, October 31, 2018 11:34

Ex: R any comm ring:

$\partial : R \longrightarrow R$ hom. of ab gps s.t. $\partial(ab) = \partial(a) b + a\partial(b)$.

Then $\{ a \mid \partial(a) = 0 \}$ is a subring.

Noetherian Rings:

R is a comm ring. We say

R is Noetherian if $\underbrace{\text{ascending chain condition}}$ holds

$\boxed{\begin{array}{l}\text{Given any ascending chain of ideals } I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \\ \text{We can find } n \geq 1 \text{ s.t. } I_n = I_{n+1} = I_{n+2} = \dots\end{array}}$

Non-example: $R = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \text{ continuous} \}$

$[-1,1] \supset [-\frac{1}{2}, \frac{1}{2}] \supset [\frac{1}{3}, \frac{1}{3}] \supset \cdots$

Set $I_n = \{ f \in R : f([-\frac{1}{n}, \frac{1}{n}]) = \{0\} \}$

$I_1 \subset I_2 \subset I_3 \subset \cdots$ and all of these inclusions are proper (urysohn's lemma).

Prop: R comm ring. Then the following props are equivalent:

(1) R is noetherian

(2) $\forall$ non-empty set X of ideals in R,

$\exists\, I \in X$ s.t. $\begin{smallmatrix} I \subseteq J \\ J \in X \end{smallmatrix} \Rightarrow I = J$ (i.e. every non-empty set of ideals has a max'l ideal)

(3) every ideal is finitely generated.

<u>Cor</u>. Every $\underbrace{\text{principal ideal ring}}$ is Noetherian

eg: $\mathbb{Z}$, $K[x]$, $K[[x]]$, $\mathbb{Z}/n\mathbb{Z}$.

Pf (1) $\Rightarrow$ (2): Let X be a non-empty set of ideals in R.

Let $I_1 \in X$. if I_1 is maximal among all ideals in X, we are done. otherwise pick $I_2 \in X$ s.t. $I_2 \supsetneq I_1$.

Continue. (1) says this process must stop eventually.

(2) $\Rightarrow$ (3): Let $I \subseteq R$ be an ideal. Let X be the set of sub-ideals of I which are finitely generated.

$\exists$ a maximal one, and this must be I.

If not, I could make it bigger.

(3) $\Rightarrow$ (1): Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain in R. The union $I = \bigcup_{i=1}^{\infty} I_i$ is an ideal & so is finitely generated, $I = (a_1, a_2, \ldots, a_N)$ with $a_1 \in I_{k_1}$, $a_2 \in I_{k_2}$, $\ldots$, $a_N \in I_{k_N}$.

Let $M = \max\{k_1, k_2, \ldots, k_N\}$. Then $I_M = I_{M+1} = \cdots$

Our favorite operations on rings:

(1) $R_1 \times R_2$ is noetherian if R_1 & R_2 are noetherian

(2) R/I is noetherian if R is noetherian.

(3) $S^{-1}R$ is noetherian if R is noetherian

(4) Hilbert basis theorem: $R[x]$ is noetherian if R is noetherian

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(1) Ideals in $R_1 \times R_2$ look like $I_1 \times I_2$. (Ex fill in details)

(2) to prove: every $I' \subseteq R/I$ is finitely generated.

$$\begin{array}{ccc} R & \xrightarrow{\pi} & R/I \\ J & \longmapsto & I' \end{array}$$

$J \subseteq R$ ideal which contains I

$\Rightarrow$ J is finitely generated so I' is too.

(3) $R \xrightarrow{j} S^{-1}R$, $\quad \tilde{I} \subseteq S^{-1}R$

to prove: $\tilde{I}$ is f.g. but $\tilde{I} = S^{-1}I$,

and $I = (a_1, a_2, \ldots a_N)$ so $\tilde{I} = \left(\frac{a_1}{1}, \frac{a_2}{1}, \ldots, \frac{a_N}{1}\right)$

Another example of non-noetherian ring:

$R = K[\underbrace{X_1, X_2, \ldots}_{\infty\text{-many variables}}]$. This is still a domain

$\downarrow j$

$F(R) = (R\setminus\{0\})^{-1}R$ but fields are noetherian

So: a subring of a noetherian ring is not necessarily Noetherian.

Hilbert Basis Theorem

R: noetherian $\Longrightarrow$ $R[X]$ is noetherian.

$R \xhookrightarrow{i} R[x]$.

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$$\begin{array}{c} \rotatebox{90}{$\subseteq$} \\ \tilde{I} \end{array} \quad \text{ideal}$$

To prove: $\tilde{I}$ is finitely generated.

<u>Idea</u>: Define an ideal $LT(\tilde{I})$ leading term generated by leading coefficients a_N of polynomials $f(x) = a_0 + a_1 x + \cdots + a_N x^N \in \tilde{I}$.

Why is $LT(\tilde{I})$ an ideal?

$a_n = LT(f(x)) \implies r \cdot a_n = LT(r \cdot f(x))$.

$g(x) = 0 \Rightarrow LT(g(x)) = 0$ (convention)

$a, b \in LT(\tilde{I}) \implies \begin{array}{l} f(x) = a x^N + \cdots + a_0 \\ g(x) = b X^M + \cdots + b_0 \end{array} \in \tilde{I}$. Say $N \leq M$

then a is also leading coeff of $x^{M-N} f(x) = a x^M + \cdots + a_0 x^{M-N} \in \tilde{I}$

and $X^{M-N} f(x) \pm g(x)$ has leading coeff $a \pm b$.