Solvable Groups

Def a seq of subgps $G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_s \trianglerighteq G_{s+1} = 1$ $(*)$

is called a normal series for the gp G.

eg If G is abelian, any descending seq. of subgps that terminates in 1 is a normal series for G.

Def to say a gp G is solvable means that G has a normal series $(*)$ whose factors G_i/G_{i+1} are all abelian.

eg any abelian gp is solvable: $G \trianglerighteq 1$.

Thm Any p-group is solvable.

pf Let G be a p-group. So $|G| = p^n$, $n \geq 1$, p prime.

Since G is a p-gp, $Z(G) \neq 1$. Put $Z_1 = Z(G)$.

If $Z_1 = G$, G is abelian & hence solvable.

So suppose $Z_1 \neq G$. Then G/Z_1 is a nontrivial p-group,

so $Z(G/Z_1) \neq 1$, and has the form Z_2/Z_1, and

$Z_2 \trianglelefteq G$ (since Z_2/Z_1 is normal in G/Z_1).

If $Z_2 = G$, we are done: $G = Z_2 \trianglerighteq Z_1 \trianglerighteq 1$.

Suppose $Z_2 \neq G$, so $Z(G/Z_2) \neq 1$, and has the form Z_3/Z_2.

Again $Z_3 \trianglelefteq G$, et cetera.

obtain a sequence of normal subgps

$1 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq Z_3 \trianglelefteq \ldots$

such that $1 \subsetneq Z_1 \subsetneq Z_2 \subsetneq Z_3 \subsetneq \ldots$

Since G is finite, this must stop at

Some point, $Z_k = G$ for some k. $\square$

<u>Def</u> If $g, h \in G$, the commutator $[g,h] = g^{-1}h^{-1}gh$.

eg $gh = hg[g,h]$. $[g,h]^{-1} = [h,g]$.

<u>Def</u> The derived (or commutator) subgroup G' of G is the subgp generated by all commutators:

$$G' = [G,G] = \{[g,h] \mid g,h \in G\}.$$

eg the elts of G' are the products of the form

$[g_1,h_1][g_2,h_2]\cdots[g_k,h_k]$ where all $g_i, h_i \in G$.

<u>Prop</u>: Let $\eta: G \longrightarrow \tilde{G}$ be a gp homomorphism.

Then $\eta(G') \subseteq \eta(\tilde{G}')$ with equality if η is surjective.

<u>Pf</u> for the first part, note that $\eta([g,h]) = [\eta(g), \eta(h)]$.

for the second part, note that $\{[\tilde{g},\tilde{h}] \mid \tilde{g},\tilde{h} \in G\} \subseteq \eta(G')$. □

<u>Corollary</u> Let $\eta : G \to G$. Then $\eta(G') \subseteq G'$. if η is surjective, $\eta(G') = G'$.

<u>Prop</u> if $K \trianglelefteq G$, then $K' \trianglelefteq G$. in particular $G' \trianglelefteq G$.

<u>proof</u> Any inner automorphism $I_a : x \mapsto axa^{-1}$ of G induces an endomorphism of K. Hence, by corollary, $I_a(K') \subseteq K'$ $\forall a \in G$. So $K' \trianglelefteq G$.

<u>Def</u> the k^{th} derived group of G is $G^{(k)} = (G^{(k-1)})'$, $G^{(0)} = G$.

<u>eg</u> $G^{(2)} = (G')' = G''$.

<u>Prop</u> $G^{(k)} \trianglelefteq G$ for all $k \geq 0$.

<u>pf</u> $G \trianglelefteq G \Rightarrow G' \trianglelefteq G \Rightarrow G'' \trianglelefteq G \Rightarrow \cdots$

by prop above.

<u>Thm</u> A gp G is solvable iff $G^{(k)} = 1$ for some $k \geq 1$.

<u>pf</u> assume $G^{(k)} = 1$ for some $k \geq 1$. Then

$$G \trianglerighteq G' \trianglerighteq G'' \trianglerighteq \cdots \trianglerighteq G^{(k)} = 1$$ is a normal series for G.

Moreover, every $G^{(i)}/G^{(i+1)}$ is abelian $\Big[H/H'$ is always abelian since $[uH', vH'] = [u,v]H' = H'$ for all $u, v \in H$. Conversely, If H/K is abelian then $K \supseteq H' \Big]$. So G is solvable.

Conversely, suppose G is solvable so we have a series

$$G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_s \trianglerighteq G_{s+1} = 1$$

where G_i/G_{i+1} is abelian $\forall i$.

We claim that $G_i \supseteq G^{(i)}$ for $i = 1, \ldots, s+1$.

If $i=1$, $G_i = G \supseteq G' = G^{(1)}$.

by converse in [] above.

Assume $G_i \supseteq G^{(i)}$. Then $G_i \supseteq G_{i+1} \supseteq G_i' \supseteq (G^{(i)})' = G^{(i+1)}$.

So $G_{s+1} = 1 \implies G^{(s+1)} = 1$ so commutator series terminates. □

<u>Thm</u> any subgp of a solvable group is solvable,
any homomorphic image of a solvable group is solvable.

Pf
- $H \leq G \implies H^{(i)} \leq G^{(i)}$.
- Suppose $\eta : G \to H$ surjective. $\eta(G') = H'$, $\eta(G^{(i)}) = H^{(i)}$. □

<u>Thm</u> Suppose $K \trianglelefteq G$ and G/K are both solvable. Then G is solvable.

Pf $\pi : G \to G/K$, $\pi(G^{(i)}) = (G/K)^{(i)}$, so $G^{(\ell)}$ will eventually $= K$.
And $K^{(s)} = 1$ for some s, so $G^{(\ell+s)} = 1$. □

<u>Thm</u> S_n is not solvable if $n \geq 5$.

<u>proof</u> Suppose to the contrary that S_n is solvable.
Then A_n is solvable. But by the theorem below,
A_n is simple so $A_n' \trianglelefteq A_n$ is either 1 or A_n.

$A_n' \neq 1$ since A_n is not abelian. $A_n' = A_n$ contradicts the fact that

<u>Thm</u> A_n is simple if $n \geqslant 5$.