Lec 12/4

Wednesday, December 4, 2019 10:21

$\nu: \mathcal{M} \to \mathbb{C}$ cplx measure.

<u>Lem</u>: $\exists!$ pos measure $|\nu|$ s.t.

$*$ $\forall$ pos μ & $f \in \mathcal{L}^1(\mu)$ s.t. $d\nu = f\,d\mu$, $d|\nu| = |f|\,d\mu$.

call $|\nu|$ the total variation

<u>Last time</u>: Existence. $\mu = |Re(\nu)| + |Im(\nu)|$, $\overbrace{d|\nu| = \left|\frac{d\nu}{d\mu}\right| d\mu}^{(**)}$ works.

We showed: if $\exists$ pos meas ρ & $g \in \mathcal{L}^1(\rho)$ s.t. $d\nu = g\,d\rho$,

then $|f|\,d\mu = |g|\,d\rho$. (this shows $*$ holds). ✓

<u>This time</u>: Uniqueness. Suppose ρ is another measure satisfying $*$.

Then $|\nu|$ is a pos. meas. & $\frac{d\nu}{d|\nu|} \in \mathcal{L}^1(|\nu|)$ s.t. $d\nu = \frac{d\nu}{d|\nu|} d|\nu|$.

Moreover, $\left|\frac{d\nu}{d|\nu|}\right| = 1$ a.e. so $d\rho = \left|\frac{d\nu}{d|\nu|}\right| d|\nu| = d|\nu|$. ✓

<u>Note</u>: we need to show that $\frac{d\nu}{d|\nu|}$ exists ($\nu \ll |\nu|$).

Suppose $|\nu|(E) = 0$. Then $\left|\frac{d\nu}{d\mu}\right| = 0$ μ-a.e. on E, so $\frac{d\nu}{d|\nu|} = 0$ μ-a.e. on E.

so $\nu(E) = \int_E \frac{d\nu}{d\mu}\,d\mu = 0$.

<u>Guess</u>: "$|\nu| = \left[|Re(\nu)|^2 + |Im(\nu)|^2\right]^{1/2}$"

actually, $$\frac{d|\nu|}{d\mu} = \left[\left(\frac{d|Re(\nu)|}{d\mu}\right)^2 + \left(\frac{d|Im(\nu)|}{d\mu}\right)^2\right]^{1/2}$$

Observe: if ν is finite & signed then

$$d\nu = (\overbrace{\chi_P - \chi_{P^c}}^{f})\, d|\nu| \qquad \text{where } X = P \sqcup P^c \text{ is Hahn decomp.}$$

$$d|\nu| = \underbrace{1}_{|f|}\, d|\nu|$$

$\Rightarrow$ $|\nu|$ agrees w/ old defn.

<u>Defn</u> $M_{\mathbb{C}} := M(X, \mathcal{M}, \mathbb{C}) := \{\text{complex measures on } (X, \mathcal{M})\}$

$$M = M(X, \mathcal{M}, \mathbb{R}) \oplus i\, M(X, \mathcal{M}, \mathbb{R})$$

$$\|\nu\|_{M_{\mathbb{C}}} := |\nu|(X).$$

<u>claim</u>: $(M_{\mathbb{C}}, \|\cdot\|)$ is Banach

pf $\|\nu\|_{\mathbb{C}} = \left[\|\mathrm{Re}(\nu)\|_{\mathbb{R}}^2 + \|\mathrm{Im}(\nu)\|_{\mathbb{R}}^2\right]^{1/2}$

is this true?? No, see below.

When X is LCH, $RM_{\mathbb{C}} \subset M_{\mathbb{C}}$ is the subspace of Radon complex measures (i.e. Re & Im are Radon)

Thm (Riesz Repn): If X is LCH, define $\varphi : RM_{\mathbb{C}} \to C_0(X, \mathbb{C})^*$ by $\varphi_\nu(f) := \int f \, d\nu$. Then φ is an isometric isomorphism.

$$X = \{ \underset{i}{\cdot} \quad \underset{1}{\cdot} \}$$

$\|\nu\| = 2$, but the other thing is $\sqrt{2}$.

Note: $\|\nu\|$ is also not the 1-norm, etc.