I Group Theory

1. Let $f: G_1 \longrightarrow G_2$ be a gp hom. Prove that $G_1/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$

2. There is a bijection $(\forall N \trianglelefteq G)$ $\left\{ \begin{matrix} H \leq G \text{ s.t.} \\ N \subset H \end{matrix} \right\} \longleftrightarrow \{ \overline{H} \leq G/N \}$

   Similarly, $\left\{ \begin{matrix} H \trianglelefteq G \text{ s.t.} \\ N \subset H \end{matrix} \right\} \longleftrightarrow \{ \overline{H} \trianglelefteq G/N \}$.

   So $G/H \cong (G/N)/(H/N)$

3. $G \circlearrowright X$: Prove $|G \cdot x| = \dfrac{|G|}{|\mathrm{Stab}_G(x)|}$

$$|X| = \sum_{\mathcal{O} \text{ : orbit}} |\mathcal{O}| = \sum_{i=1}^{n} \frac{|G|}{|\mathrm{Stab}_G(x_i)|} \quad \left(\text{where } \{G \cdot x_1, \dots, G \cdot x_n\} = {}_G\backslash^X \right)$$

   (a) $|G/H| = |G|/|H|$

   (b) $\mathrm{Order}_G(g) \mid |G|$

(we never saw an application)

   (c) write all this for $G \circlearrowright G$ by conj : $g \cdot x = gxg^{-1}$.

4[†]: $|{}_G\backslash^X| = \dfrac{1}{|G|} \sum_{g \in G} |X^g|$ (Burnside's Counting Lemma)

5: $|G| = p^n$ ($n \geq 1$, p prime), $G \circlearrowright X \implies |X| \equiv |X^G| \bmod p$

(proof won't appear on exam)

6**: Sylow Theorems:

   (1) Sylow p-subgroups exist

   (2) $Q \leq G$, $Q \in \mathrm{Syl}_p(G)$, $H \leq G$, $|H| = p^k$

   $\implies gHg^{-1} \leq Q$ for some $g \in G$.

(3) $n_p \equiv 1 \mod p$; $n_p \mid |G|$

7: Semi-direct products:

(1) $G \cong N \rtimes_\alpha H$ (for some $\alpha : H \to \mathrm{Aut}_{gp}(N)$)

(2) $H \leq G$, $N \trianglelefteq G$, $H \cap N = \{e\}$, $G = H \cdot N$ $(= N \cdot H)$

Prove (1) $\iff$ (2).

8: Commutator Subgroups

(1) $A, B \trianglelefteq G \implies [A;B] \trianglelefteq G$

(2) $[G;G]$ is smallest $N \trianglelefteq G$ s.t. G/N is abelian.

II Rings (commutative and nontrivial: $0 \neq 1 \in R$)

9: Isomorphism Theorems: Let $f : R_1 \to R_2$ be a ring hom.

then $R_1/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ (Note that hom means $\begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 1 \end{matrix}$)

10: $\left\{ \begin{matrix} \tilde{I} \overset{\text{ideal}}{\subseteq} R \text{ s.t.} \\ I \subseteq \tilde{I} \end{matrix} \right\} \longleftrightarrow \left\{ \text{Ideals of } R/I \right\}$

$$R/\tilde{I} \cong (R/I)\Big/(\tilde{I}/I)$$

11**: Maximal ideals exist

12: Maximal ideals are prime

13: Coprime ideals $(I_1 + I_2 = R)$: $I_1 \cdot I_2 = I_1 \cap I_2$

and $R/I_1 \cdot I_2 \cong R/I_1 \times R/I_2$ (Chinese remainder thm).

14: I_1, I_2 coprime $\Longrightarrow$ $I_1^{n_1}, I_2^{n_2}$ coprime $(\forall\, n_1, n_2 \geq 1)$

15: If $S \subsetneq R$ is multiplicatively closed then

$$j : \begin{array}{ccc} R & \longrightarrow & S^{-1}R \\ r & \longmapsto & \frac{r}{1} \end{array}$$ is a ring hom,

and $j(t) \in (S^{-1}R)^{\times}$ $\forall\, t \in S$.

16: Ideals in $S^{-1}R = \{ S^{-1}I \mid I \subset R \text{ is an ideal} \}$ (Note: it is possible that $S^{-1}I_1 = S^{-1}I_2$ when $I_1 \neq I_2$)

17: $S^{-1}I = R \Longleftrightarrow I \cap S \neq \emptyset$

18: $P_1, P_2 \subsetneq R$ distinct prime ideals s.t. $P_1 \cap S = P_2 \cap S = \emptyset$

$\Longrightarrow$ $S^{-1}P_1 \neq S^{-1}P_2$

19: R: domain; $S = R \setminus \{0\} \Longrightarrow S^{-1}R$ is a field.

20: R: any ring, $P \subsetneq R$ prime ideal, $S = R \setminus P \Rightarrow S^{-1}R$ is a local ring.

21: $\mathcal{N}(R) :=$ set of nilpotent elements. Then $\mathcal{N}(R) \subsetneq R$ is an ideal,

and $\mathcal{N}(R) = \bigcap_{P \subsetneq R \text{ prime ideal}} P$ (nilradical)

22: R: PID $\Longrightarrow$ Max'l ideals = nonzero prime ideals

23: (1) every ideal in R is finitely generated

(2) Ascending Chain Condition

(3) every non-empty set of ideals has a max'l element

are three <u>equivalent</u> definitions of "R is a Noetherian Ring"

↑ prove this

24: Prove that Euclidean domains are PIDs.

25†: Hilbert Basis Theorem

26: (1) Q : primary ideal $\Longrightarrow$ $\mathrm{Rad}(Q)$ is prime

(2) $\mathrm{Rad}(I)$ maximal $\Longrightarrow$ I is primary

(3) $\mathrm{Rad}(I_1 \cap I_2) = \mathrm{Rad}(I_1) \cap \mathrm{Rad}(I_2)$

(4) (Noetherian) $\mathrm{Rad}(I)^N \subset I$ for some $N \geq 1$.

(5) (PID) Primary ideal $=$ (Prime ideal)^N

27†: PID $\Longrightarrow$ UFD (UFD $\not\Longrightarrow$ PID could be on exam)

28: $R = K[x]$ where K is a field.

(1) $K[x]$ is a euclidean domain with norm = degree

(2) $\{\text{Max'l in } K[x]\} = \{(f) \text{ where } f \text{ is non-zero irreducible}\}$

(3) $f(x) = (x-\alpha)^{N+1} g(x)$ for some $g(x)$ s.t. $g(\alpha) \neq 0$

iff $f(\alpha) = f'(\alpha) = \dots = f^{(N)}(\alpha) = 0$ & $f^{(N+1)}(\alpha) \neq 0$.

[in words: f has a root in $K \Rightarrow f$ is not irreducible]

29: Characteristic of a field K is p : prime or 0.

In the first case, $\mathbb{F}_p \subset K$.

In the second case, $\mathbb{Q} \subset K$

30: $\mathrm{Char}(K) = p$: prime implies $\begin{array}{ccc} K & \xrightarrow{\sigma_p} & K \\ x & \longmapsto & x^p \end{array}$ is a ring hom

(this is the Frobenius homomorphism)

(as K is a field, σ_p is injective and $\sigma_p(a) = a \ \forall a \in \mathbb{F}_p$).