2/21

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Midterm - Monday March 4

$R$ - PID, $M$ finitely generated $R$-module.

Then $M \cong R^l \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$ s.t. $a_1 | a_2 | \cdots | a_m$.

$M$ is a direct sum of cyclic modules

$a_1, \ldots, a_m$ invariant factors

$M \cong R^l \oplus R/(p_1^{r_1}) \oplus \cdots \oplus R/(p_k^{r_k})$

$p_1^{r_1}, \ldots, p_k^{r_k}$ elementary divisors

Recall: $R/(a) \cong R/(p_1^{r_1}) \oplus \cdots \oplus R/(p_k^{r_k})$

if $a = p_1^{r_1} \cdots p_k^{r_k}$

Elem-ry divisors $\longrightarrow$ invariant factors

$p_1^{r_{1,1}}, \ldots, p_2^{r_{1,l_1}}, p_2^{r_{2,1}}, \ldots, p_2^{r_{2,l_2}}, \ldots, p_k^{r_{k,1}}, \ldots, p_k^{r_{k,l_k}}$

$r_{i,1} \geq \cdots \geq r_{i,l_i}$.

Put $a_m = p_1^{r_{1,1}} \cdot p_2^{r_{2,1}} \cdot \ldots \cdot p_k^{r_{k,1}}$ $m = \max\{l_i\}$.

$a_{m-1} = p_1^{r_{1,2}} \cdots p_k^{r_{k,2}}$ we can take some $r = 0$.

$\vdots$

then $a_1 | a_2 | \cdots | a_m$.

$$R/(a_m) \cong R/(p_1^{r_{1,1}}) \oplus \cdots \oplus R/(p_k^{r_{k,1}})$$

$\vdots$

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$$\bigoplus R/(p_i^{r_{i,j}}) = \bigoplus R/(a_i) .$$

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Uniqueness: $R/(p^n) \not\cong R/(p^m)$ if $n \neq m$.

~~pf~~ assume $n > m$. Then $(p^n) \subseteq (p^m)$

$$R/(p^{n+1}) \Big/ R/(p^n) \cong R/(p) \quad \text{— a field.}$$

$$(p^n) = p^n \cdot R .$$

$$p^m \cdot R / p^{m+1} \cdot R \cong R/p = F$$

$$p^{m+1} \cdot R / p^{m+2} \cdot R \cong R/p$$

$p^m \cdot R$ — tower of modules $p^m R \supseteq p^{m+1} R \supseteq \cdots \supseteq p^n R$,

and $\forall i$, $p^i R / p^{i+1} R = F$.

$I \subsetneq J \Rightarrow R/J \subsetneq R/I$

$R/(p^{n_1}) \oplus R/(p^{n_2}) \qquad R/(p^{m_1}) \oplus R/(p^{m_2})$

M - finitely generated module.

- invariant factors?

$M = K/N$ where $M$ is generated by $u_1, \dots, u_n$

$K$ is freely generated by $u_1, \dots, u_n$.

$N$ is generated by some $\left\{ \begin{array}{l} a_{11}u_1 + \dots + a_{n1}u_n = v_1 \\ \vdots \\ a_{1m}u_1 + \dots + a_{nm}u_n = v_m \end{array} \right\}$

$N = \mathrm{Ker}: K \to M$

$v_1, \dots, v_m = 0$ in $M$.

the equalities $\begin{array}{l} v_1 = 0 \\ \vdots \\ v_m = 0 \end{array}$ are relns in $M$,

and all other relns are their linear combonations.

Relations Matrix of $M$ is $\left( v_1 \mid \dots \mid v_m \right) = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$

eg $G = \left\langle u_1, u_2, u_3 \;\middle|\; \begin{array}{l} 2u_1 - 2u_2 + 3 = 0 \\ u_1 + 3u_3 = 0 \end{array} \right\rangle$

So relⁿ matrix is $\begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 1 & 3 \end{pmatrix}$

After we reduce the relⁿ matrix to

$\left(\begin{array}{c|c} \begin{matrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$, the relations become

$a_1\tilde{u}_1 = 0, \dots, a_k\hat{u}_k = 0$ when $\{\tilde{u}_1, \dots, \tilde{u}_n\}$ is a new basis in $K$.

$$K/N = M = R/(a_1) \oplus \cdots \oplus R/(a_k) \oplus R^{n-k}$$

Example: Let $G = \langle u_1, u_2, u_3 \mid u_i u_j = u_j u_i, 2u_1 + 2u_2 - u_3 = 0, u_1 + 3u_3 = 0 \rangle$.

Matrix is $\begin{pmatrix} 2 & 1 \\ 2 & 0 \\ -1 & 3 \end{pmatrix} \longmapsto \begin{pmatrix} -1 & 3 \\ 0 & 7 \\ 0 & 6 \end{pmatrix} \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 7 \\ 0 & 6 \end{pmatrix} \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 6 \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

So $G = \mathbb{Z}/(-1) \oplus \mathbb{Z}/(1) \oplus \mathbb{Z} = \mathbb{Z}$.