Lec 9/12

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4.1 #5 (a) $S_3 \curvearrowright X$ = set of ordered triples $\{(i,j,k) : 1 \le i,j,k \le 3\}$.

$$(x,y,z) \longmapsto (\sigma(x), \sigma(y), \sigma(z))$$

$\{(i,i,i) : 1 \le i \le 3\}$ $\quad \mathcal{O}_1 \quad$ 3

$(1,1,2) \xrightarrow{\text{any}} \underset{i \neq j}{(i,i,j)}$ so $\{(i,i,j) : i \neq j\}$ $\quad \mathcal{O}_2 \quad$ 6

(i,j,i) $\quad \mathcal{O}_3 \quad$ 6

(j,i,i) $\quad \mathcal{O}_4 \quad$ 6

$\{(1,2,3), (2,1,3), \dots\}$ $\quad \mathcal{O}_6 \quad$ 6

$\sigma = (12) \in S_3$: on $\mathcal{O}_1$: $(1,1,1) \rightleftarrows (2,2,2)$

Sylow Theorems: (1) Sylow p-subgroups exist,

(2) are unique up to conjugation,

(3) and # Sylow p-subgps $\equiv 1 \pmod p$.

$|G| = 45 \Rightarrow G \cong \begin{pmatrix}\text{group w/} \\ 9 \text{ elts}\end{pmatrix} \times \begin{pmatrix}\text{group w/} \\ 5 \text{ elts}\end{pmatrix}$.

Lemma: Let G be a group and $N_1, N_2 \trianglelefteq G$. Assume $\overset{(1)}{G = \langle N_1, N_2 \rangle}$ & $\overset{(2)}{N_1 \cap N_2 = 1}$.

Then $N_1 \times N_2 \xrightarrow[f]{\cong} G$ gp. iso.

$$(x_1, x_2) \longmapsto x_1 x_2$$

Proof: N_1, N_2 normal & $N_1 \cap N_2 = \{e\} \Longrightarrow ab = ba \;\; \forall a \in N_1, b \in N_2$.

$\langle N_1, N_2 \rangle$.

Proof: N_1, N_2 normal & $N_1 \cap N_2 = \{e\} \Longrightarrow ab = ba \quad \forall a \in N_1, b \in N_2$.

So every element in $\underset{\langle N_1, N_2 \rangle}{G}$ can be written as xy for some $x \in N_1, y \in N_2$

So f is surjective. It's also a gp. hom: $N_1 \times N_2 = \{(x_1, x_2) : x_i \in N_i\}$,

Componentwise multiplication. So $f(x_1y_1, x_2y_2) = x_1y_1x_2y_2 = x_1x_2y_1y_2 = f(x_1x_2)f(y_1y_2)$.

$Ker(f) \ni (x_1, x_2)$ means $x_1x_2 = e$, $N_1 \ni x_1 = x_2^{-1} \in N_2 \Rightarrow x_1 = e = x_2$.

So f is injective: any gp-hom is 1-1 iff $Ker(f) = \{e\}$. $\square$

$G \times H$ is called a direct product

Actually, if $N_1, N_2, \ldots, N_k \trianglelefteq G$ and $N_i \cap N_j = \{e\} \ \forall i \neq j$ and $G = \langle \{N_k : k\} \rangle$

then $G \cong N_1 \times N_2 \times \cdots \times N_k$

Cor. So if $|G| = n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ and G is abelian then

$G \cong P_1 \times P_2 \times \cdots \times P_t$ where P_i is a ("the" since G is abelian.) Sylow p_i-subgp

Pf (2) $P_i \cap P_j = \{e\} \ \forall i \neq j$, since $\sigma \in P_i \cap P_j \Longrightarrow ord(\sigma) \mid p_i^{a_i}$ and $ord(\sigma) \mid p_j^{a_j}$.

(1) $\langle P_1, P_2, \ldots, P_t \rangle =: H \Longrightarrow |H|$ is div by $p_i^{a_i} \ \forall i \Rightarrow n \mid |H| \leq n = |G| \Rightarrow G = H$.

(0) P_i are all normal since G is abelian.

Theorem: Let G be an abelian p-group (i.e. $|G| = p^r$).

Read Notes for Proof!

Then $G \cong \mathbb{Z}/_{p^{a_1}}\mathbb{Z} \times \cdots \times \mathbb{Z}/_{p^{a_k}}\mathbb{Z}$ where $\sum a_i = r$.

if we assert that $a_1 \leq a_2 \leq \cdots \leq a_k$, then $K, a_1, a_2, \ldots, a_k$ are unique (determined by G.)

Lemma: Let us assume we have an abelian group H of order p^ℓ.

Assume $\begin{cases} H_1 \cong \mathbb{Z}/_{p^{\ell_1}}\mathbb{Z} \trianglelefteq H & \text{where } \ell_1 = \max\{r : p^r \text{ is order of some } x \in H\}. \\ H_2 := H/H_1 = \cong \mathbb{Z}/_{p^{\ell_2}}\mathbb{Z} & (\ell_1 + \ell_2 = \ell) \end{cases}$

. .

Then $\forall$ generator $y \in H_2$, $\exists\, \sigma_2 \in H$ s.t. $y = \sigma_2 \cdot H_1$, $\operatorname{ord}(\sigma_2) = p^{l_2}$.

Pf (For convenience, additive notation : $e = 0$, $\cdot = +$, $x + y = y + x$, $\operatorname{ord}(x)\, x = 0$)

$$\begin{array}{lcl} H_1 \trianglelefteq H \,; & \qquad & H/H_1 =: H_2 \\ \| \wr & & \| \wr \\ \mathbb{Z}/p^{l_1}\mathbb{Z} \ni \sigma_1 \text{ a generator.} & & \mathbb{Z}/p^{l_2}\mathbb{Z} \\ & & \quad \Psi \\ & & y \text{ a generator} \\ & & \text{s.t. } y = \underset{\substack{\Psi \\ H}}{\sigma} + H_1 . \end{array}$$

$p^{l_2} y = 0$ means $p^{l_2}\sigma \in H_1 = \{ j \cdot \sigma_1 : 0 \le j \le p^{l_1} - 1 \}$

$$(*) \quad p^{l_2} \cdot \sigma = p^{s} \underbrace{m\sigma_1}_{\text{has order } p^{l_1}} \quad \text{where} \quad s \ge 0, \quad (p, m) = 1 .$$

claim: $s \ge l_2$ so $p^{l_2}(\underbrace{\sigma - p^{s - l_2} m \sigma_1}_{\text{choose this to be } \sigma_2 .}) = 0$

pf of claim: $\operatorname{Ord}(\sigma) = p^{l_1 + l_2 - s}$ (multiply both sides of $(*)$ by $p^{l_1 - s}$).

so $l_1 + l_2 - s \le l_1$ by defn $\Longrightarrow s \ge l_2$.