Lec 10/27

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$L: V = C^2(\mathbb{R}) \to C(\mathbb{R})$

$L(y) = y'' + a_1 y' + a_2 y$

$\dim(\mathcal{N}(L)) = 2$

$\varphi_1, \varphi_2 \in C^2(\mathbb{R})$ lin indp.

$$c_1\varphi_1 + c_2\varphi_2 = 0 \Rightarrow c_1 = c_2 = 0$$
$$\Downarrow$$
$$c_1\varphi_1' + c_2\varphi_2' = 0$$

Wronskian

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} \neq 0 \Rightarrow \varphi_1, \varphi_2 \text{ lin indp}$$

if $\varphi_1, \varphi_2 \in \mathcal{N}(L)$ then φ_1, φ_2 lin indp $\Rightarrow W(x) \neq 0$ for all x.

Suppose $W(x_0) = 0$.

then $\exists$ c_1, c_2 nonzero s.t. $c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = 0$

$c_1\varphi_1'(x_0) + c_2\varphi_2(x_0) = 0$

let $\varphi(x) = c_1\varphi_1(x) + c_2\varphi_2(x)$

so $\varphi(x_0) = 0$, $\varphi'(x_0) = 0$

so $\varphi(x) = 0$ $\forall$ o. (by uniqueness of soln for $L(y) = 0$)

so φ_1, φ_2 lin dep.

Thus <u>Thm 1</u>: suppose $\varphi_1, \varphi_2 \in N(L)$. The following are equivalent:

$\{\varphi_1, \varphi_2\}$ lin. indp.

$W_{\varphi_1\varphi_2}(x) \neq 0 \quad \forall x \in \mathbb{R}$

$W_{\varphi_1\varphi_2}(x) \neq 0$ for some $x_0 \in \mathbb{R}$.

$a_1, a_2 \in \mathbb{R}$

$p(r) = r^2 + a_1 r + a_2, \quad r_1, r_2 \in \mathbb{C} \Rightarrow r_1 = \alpha + i\beta, \ r_2 = \alpha - i\beta$

then $e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$

$e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

Now $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are lin indp & in $\mathcal{N}(L)$

So these form a basis as well.
of $\mathcal{N}(L)$

$$W(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix}. \qquad W'(x) = \begin{vmatrix} \varphi_1'(x) & \varphi_2'(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} + \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1''(x) & \varphi_2'(x) \end{vmatrix}$$

take $W(x) = \begin{vmatrix} \varphi_1(x) & \cdots & \varphi_n(x) \\ \varphi_1'(x) & \cdots & \varphi_n'(x) \\ \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{vmatrix}$, $\quad W'(x) = \begin{vmatrix} \varphi_1'(x) & \cdots & \varphi_n'(x) \\ \varphi_1'(x) & \cdots & \varphi_n'(x) \\ \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_1(x) & \cdots & \varphi_n(x) \\ \vdots & \ddots & \vdots \\ \varphi_1^{(n-2)}(x) & \cdots & \varphi_n^{(n-2)}(x) \\ \varphi_1^{(n)}(x) & \cdots & \varphi_n^{(n)}(x) \end{vmatrix}$

$$\det A(x) = \sum_{\sigma \in S} \varepsilon(\sigma)\, a_{1\sigma(1)}(x) \cdots a_{n\sigma(n)}(x)$$

$(\det A)'(x) = \quad , \ldots$

So $W' = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1'' & \varphi_2'' \end{vmatrix} = \begin{vmatrix} \varphi_1 & \varphi_2 \\ -a_1\varphi_1' - a_2\varphi_1 & -a_1\varphi_2' - a_2\varphi_2 \end{vmatrix} = -\begin{vmatrix} \varphi_1 & \varphi_2 \\ a_1\varphi_1' & a_1\varphi_2' \end{vmatrix}$

$$\begin{vmatrix} \varphi_1'' & \varphi_2'' \end{vmatrix} - \begin{vmatrix} -a_1\varphi_1' - a_2\varphi_1 & -a_1\varphi_2' - a_2\varphi_2 \end{vmatrix} \quad \begin{vmatrix} a_1\varphi_1' & a_1\varphi_2' \end{vmatrix}$$

$$\Rightarrow \quad W' = -a_1 W.$$

$$\Rightarrow \quad W(x) = c\, e^{-a_1 x} \quad \text{where } c = e^{a_1 x_0} W(x_0)$$

So $W(\varphi_1, \varphi_2)(x) = e^{-a_1(x - x_0)} W(\varphi_1, \varphi_2)(x_0)$

So $W(x) \neq 0 \Leftrightarrow W(x_0) \neq 0$

1b. $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

$r^2 - 2r - 3$, roots are $1 \pm \sqrt{1+3} = 1 \pm 2$.

$\varphi_1(x) = e^{3x} \quad \varphi_2(x) = e^{-x}$

$$\begin{aligned} \varphi(x) &= c_1 e^{3x} + c_2 e^{-x} \\ \varphi'(x) &= 3c_1 e^{3x} - c_2 e^{-x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 - c_2 &= 1 \end{aligned}$$

$$\Rightarrow \quad c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{4}.$$

$y'' + (4i+1)y' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$r^2 + (4i+1)r + 1$, roots: $\dfrac{-(4i+1) \pm \sqrt{-16 + 8i - 4 + 1}}{2} = \dfrac{-(4i+1) + \sqrt{8i - 19}}{2}$

$(\alpha + i\beta)^2 = \alpha^2 - \beta^2 + 2i\alpha\beta = -19 + 8i$

$$\Rightarrow \quad \alpha^2 - \beta^2 = -19$$

$$\alpha\beta = 4$$

$$\Rightarrow \quad \alpha^2 - \frac{16}{\alpha^2} + 19 = 0$$

$$\Rightarrow \quad \alpha^4 + 19\alpha^2 - 16 = 0 \quad \Rightarrow \quad \alpha^2 = \frac{-19 \pm \sqrt{19^2 + 64}}{2}$$

$$\Rightarrow \quad \beta = \sqrt{\frac{32}{-19 \pm \sqrt{19^2 + 64}}}$$

$$\alpha + i\beta = \sqrt{\alpha^2 + \beta^2}\,(\cos\theta + i\sin\theta) \qquad \text{where } \theta = \arctan\left(\frac{\beta}{\alpha}\right).$$