Lec 8/25

Thursday, August 25, 2016 9:02 AM

$$(a+b)+((-a)+(-b)) = (a+(-a))+(b+(-b)) \qquad P1 \,\&\, P4$$
$$= 0+0 \qquad P3$$
$$= 0 \qquad P2$$

$$-(a+b)+((a+b)+((-a)+(-b))) = -(a+b)+0 \qquad \text{addition is unambiguous}$$
$$(-(a+b)+(a+b))+(-a)+(-b) = -(a+b) \qquad P1, P2$$
$$0+(-a)+(-b) = -(a+b) \qquad P3$$
$$(-a)+(-b) = -(a+b) \qquad P2$$

$\mathbb{F}$ field P1 - P9

$\mathbb{F}$ ordered field if $\exists P \subseteq \mathbb{F}\setminus\{0\}$ so P satisfies P10 - P12

Definition: $a<b \Leftrightarrow b-a \in P \quad \forall a,b \in \mathbb{F}$ (ordered)

$a \le b \Leftrightarrow b-a \in P \cup \{0\}$

Trichotomy: $\forall a,b \in \mathbb{F}$, exactly one of the following holds.

(1) $a=b$
(2) $a<b$
(3) $a>b$

$a \in P \Leftrightarrow 0<a \Leftrightarrow a>0$

<u>Proposition</u> If $x \in \mathbb{F}$, then $x^2 \geq 0$, if $x \neq 0$ then $x^2>0$

<u>proof</u>

<u>Lemma</u> (i) if $x=0$ then $x^2=0$ so $x^2 \geq 0$

<u>Lemma</u> (ii) if $x>0$ then $x^2>0$ so $x^2 \geq 0$ by P12

<u>Lemma</u> (iii) if $x<0$ then $x^2>0$

<u>Proof</u> $x<0 \Rightarrow -x>0$ by P10

$(-x)(-x)>0$ P12

$(-x)(-x) = x^2 > 0$ Proved in class

Using P11,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$$

$$\| $$

$$0 \Leftrightarrow x_1 = x_2 = \cdots = x_n = 0$$

any ordered field $\mathbb{F}$ contains all the rational numbers $\mathbb{Q}$.

any rational number can be written as $\pm \frac{n}{m}$, $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+ \cup \{0\}$

$$\pm n/m = \pm(\underbrace{1 + \cdots + 1}_{n})(\underbrace{1 + \cdots + 1}_{m})^{-1}$$

Handout problem $a > 0 \Rightarrow a^{-1} > 0$

Lemma 1 $a > 0 \Longrightarrow a^{-1} \neq 0$ proved yesterday

Lemma 2 $a > 0 \Longrightarrow a^{-1} \not< 0$

	Justification
Proof assume $a^{-1} < 0$	
$-a^{-1} > 0$	P10
$a(-a^{-1}) > 0$	P12
$-(a \cdot a^{-1}) > 0$	proved
$-1 > 0$	P7
$-1^2 > 0$	
Contradiction, so $a^{-1} \not< 0$	

$a^{-1} > 0$, $a^{-1} < 0$, or $a^{-1} = 0$ P10

$a^{-1} > 0$ exclusion by lemmas 1 & 2

Corollary: $0 < a < b \Rightarrow b^{-1} < a^{-1}$

discover the proof.

want $0 < a^{-1} - a^{-b}$

$0 < (a^{-1} - b^{-1})(ab)$

$0 < (a^{-1})ab - ab(b^{-1})$ P6

$0 < b - a \rightarrow$ true

actual proof goes down to up.

Proof:

$0 < b - a$ (hypothesis)

$0 < a^{-1}b^{-1}(b-a)$ handout problem, $a^{-1}, b^{-1} > 0$ bc $a, b > 0$, P12

$0 < a^{-1}b^{-1}b - a^{-1}b^{-1}a$ P9

$0 < a^{-1} + (-b^{-1}) = a^{-1} - b^{-1}$ P8, P4

so $b^{-1} < a^{-1}$

Basic properties of $<$

Prop 1 $a < b$ & $b < c \Rightarrow a < c$

Proof $c - a = \underbrace{(b-a)}_{>0} + \underbrace{(c-b)}_{>0} > 0$ P4, P1, P11, P3

Prop 2 $a < b \Rightarrow a + c < b + c$

Proof $(b+c) - (a+c) = b - a > 0$

Prop 3 $a < b$ and $c > 0 \Rightarrow ac < bc$

$a < b$ and $c < 0 \Rightarrow ac > bc$

Proof $bc - ac = c(b-a) > 0$ P9, P12

Proposition $a < b \Leftrightarrow a^3 < b^3$

Proof $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$ P9, P1, P4, P8, P5 +

$= (b-a)(b^2 + ab + \frac{a^2}{4} + 3a^2/4)$

$= (b-a)\underbrace{((b + \frac{a}{2})^2 + \frac{3}{4}a^2)}_{>0 \text{ P11}} > 0$ so $b^3 > a^3$

$(b+\frac{a}{2})^2 > 0$, $\frac{3}{4}a^2 > 0$ P12 P12

Proposition $a < b$, $a \geq 0$, $b \geq 0 \Rightarrow a^2 < b^2$

$a^2 < b^2$, $a \geq 0$, $b \geq 0 \Rightarrow a < b$

Proof $b^2 - a^2 = (b-a)(a+b) > 0$

Proof
sketch

$$b^2 - a^2 = \underset{\substack{\vee \\ 0}}{(b-a)}\,\underset{\substack{\vee \\ 0}}{(a+b)} > 0$$