Lec 9/26

Wednesday, September 26, 2018 11:31

$N \rtimes_\alpha H \leadsto (H, N, \alpha)$ — what can we love here & get the same answer?

α can be changed to $T \alpha T^{-1}$ for some $T \in Aut_{gp}(N)$.

i.e. $h \longmapsto T \alpha(h) T^{-1}$

$H \longrightarrow Aut_{gp}(N)$.

Lemma: If $\alpha_j(h)(n) = j(h)\, n\, j(h)^{-1}$ for some gp hom $j : H \longrightarrow N$

then $N \rtimes_{\alpha_j} H \cong N \times H$.

Pf $(n, h) \xmapsto{f} (n \cdot j(h), h)$

f is a gr hom & it clearly has an inverse. $\square$

$$(n_1, h_1) \cdot_{\alpha_j} (n_2, h_2) = (n_1 \alpha_j(h_1)(n_2), h_1 h_2)$$
$$= (n_1\, j(h_1)\, n_2\, j(h_1)^{-1}, h_1 h_2)$$

$n_1 \cdots$ it works out.

Group Cohomology*.

Inner automorphism of W: conjugation by an element of W

Hölder's Program: classify all simple groups, See how they hit together.

Defn: A group G is called simple if it has no nontrivial proper normal subgroups.

Ex: If G is abelian & simple then $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p.

(Schur-type)
Lemma: G_1, G_2 are two groups, G_1 is simple, let $f: G_1 \xrightarrow[\text{hom}]{\text{gp}} G_2$.

then $\text{Ker}(f) = \{e\}$ or G_1

$\{e\}$: nobody goes to identity

G_1: everybody does

Convention: $\{e\}$ is not considered simple (just like 1 is not prime).

Today: A_n is simple for $n \geq 5$

S_n: Symmetric group on n symbols

There is sign homomorphism $\varepsilon: S_n \longrightarrow \{\pm 1\}$ s.t. $\varepsilon(2\text{-cycle}) = -1$.

transposition

every $\pi \in S_n$ is a product of transpositions.

and $A_n \stackrel{\text{def}}{=} \text{Ker}(\varepsilon) \trianglelefteq S_n$.

index $2 = |S_n/A_n| = |\{\pm 1\}|$.

$\varepsilon(\ell\text{-cycle}) = (-1)^{\ell-1}$

Recall we proved ε is a gr hom by showing

$S \quad \langle \quad \mid s_i^2 = e \quad \rangle$

Recall we proved ε is a group homomorphism

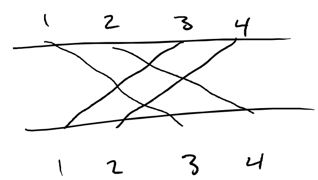
$$S_n = \left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{l} s_i^2 = e \\ s_i s_j = s_j s_i \text{ if } |i-j| \geq 2 \\ (s_i s_{i+1})^3 = e \end{array} \right\rangle$$

$$\Big\downarrow \varepsilon$$

$$H$$

but now we give a direct pf.

Idea: $\varepsilon(\pi)$



nontransversally
monotonically.

count crossings
(even or odd)

$\downarrow \qquad\qquad \downarrow$

$\varepsilon(\pi) = +1 \qquad \varepsilon(\pi) = -1$

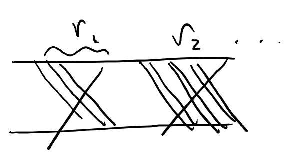
how many disjoint cycles

$$N(\pi) = (r_1 - 1) + (r_2 - 1) + \dots + (r_\ell - 1) = n - \ell$$

$\downarrow$

$\pi = \pi_1 \pi_2 \cdots \pi_\ell$, π_i disjoint cycles of length r_i.

$r_1 \geq r_2 \geq \dots \geq r_\ell$, $\sum r_i = n$.



Lemma: If $\pi = \tau_1 \cdots \tau_q$, $q \equiv N(\pi) \pmod 2$.

Proof

$$\underset{\ell+k+2}{(a\, c_1 \cdots c_k\, b\, d_1 \cdots d_\ell)} = (a b)\, \underset{\ell+1}{(b\, d_1 \cdots d_\ell)}\, \underset{k+1}{(a\, c_1 \cdots c_k)}.$$

(easy check of this identity in S_n).

So we get a nice inductive formula:

$$N((ab)\pi) = N(\pi) + 1 \text{ or } N(\pi) - 1$$

$\equiv N(\pi) + 1 \pmod 2$

rest is induction on l $\square$

Examples of A_n:

$A_2 = \{e\}$, $A_3 = \ker(S_3 \xrightarrow{\varepsilon} \{\pm 1\})$
$\cong \mathbb{Z}/3\mathbb{Z}$

4 elements to choose not in cycle ×2 directions = 8.

$$A_4 = \{e, \text{ all 3-cycles}, \underset{\substack{\parallel \\ x}}{(12)(34)}, \underset{\substack{\parallel \\ y}}{(13)(24)}, \underset{\substack{\parallel \\ z}}{(14)(23)}\}$$
$\trianglerighteq$
$\{e, x, y, z\}$.

So A_4 is not simple.

Theorem A_n is simple for $n \geq 5$.

Steps: (i) A_n is generated by 3-cycles. $(n \geq 3)$

(ii) any two 3-cycles are conjugate to each other in A_n. $(n \geq 5)$

(iii) if $K \trianglelefteq A_n$ is nontrivial, it must contain a 3-cycle. $(n \geq 5)$

Pf (i) Every element in A_n can be written as an even # of transpositions.

$(a\,b)(a\,b) = e$

$(a\,b)(a\,c) = (a\,c\,b)$

$(a\,b)(c\,d) = (a\,b\,c)(b\,c\,d)$ $\square$

(ii) $(a_1\,b_1\,c_1) \stackrel{?}{=} \gamma (a_2\,b_2\,c_2) \gamma^{-1}$ with $\gamma \in A_n$.

$\exists \sigma \in S_n$ s.t. $(a_1\,b_1\,c_2) = \sigma (a_2\,b_2\,c_2) \sigma^{-1}$

$\varepsilon(\sigma) = +1 \Rightarrow$ we are done

$\varepsilon(\sigma) = -1 \Rightarrow$ we are not done.

find $x \neq y$, $x, y \notin \{a_1, b_1, c_1\}$

and change σ to $(x\ y) \cdot \sigma = \gamma$.

(iii) Given $K \trianglelefteq A_n$, $K \neq \{e\}$, choose $\sigma \in K \setminus \{e\}$ which maximizes # of fixed points $= |X^\sigma| =$ # of 1-cycles.

- $\sigma = (a_1\ a_2\ a_3 \cdots) \cdots$

one cycle of length ≥ 3.

1: if $\sigma \neq (a_1\ a_2\ a_3)$, pick a_4, a_5 s.t. $\sigma(a_4) \neq a_4$, $\sigma(a_5) = a_5$

let $\tau = (a_3\ a_4\ a_5)$.

$\rightarrow$ Ex: $\tau \sigma \tau^{-1} \sigma^{-1}$ has more fixed points than σ.

So we have a contradiction.

2: $\sigma =$ product of disjoint transpositions... also get a contradiction if ℓ is too big.