Lec 8/22

Wednesday, August 22, 2018 11:31

Last time: G a group. examples: $\underset{}{S_n}^{\,n!}$, $D_{2n}^{\,2n}$, $GL_2(\mathbb{R})$

Lemma: If $a \in G$ and $b, b' \in G$ are inverses of a, then $b = b'$.

pf: $b = b1 = b(ab') = (ba)b' = 1b' = b'$. $\square$

Properties of inverse: $(ab)^{-1} = b^{-1}a^{-1}$. $(a^{-1})^{-1} = a$.

Cancellation prop.: $ab = ac \Rightarrow b = c \Leftarrow ba = ca$

Defn: Subgroup. nonempty $H \subseteq G$ closed under $\_\cdot\_$ and $\_^{-1}$. $H \leq G$ ($H < G$ if proper)

Lemma: $\emptyset \neq H \subseteq G$, H subgroup of $G \iff xy^{-1} \in H \ \forall x, y \in H$.

Pf $\Rightarrow$ $x, y \in H \Rightarrow x, y^{-1} \in H \Rightarrow xy^{-1} \in H$.

$\Leftarrow$ $H \neq \emptyset \Rightarrow \exists a \in H \Rightarrow aa^{-1} = e \in H$

$x, e \in H \Rightarrow ex^{-1} = x^{-1} \in H$

$x, y \in H \Rightarrow x, y^{-1} \in H \Rightarrow x(y^{-1})^{-1} = xy \in H$. $\square$

Subgroup generated by a subset.

$A \subset G \rightsquigarrow \langle A \rangle =$ smallest subgroup containing A.

$= \bigcap_{\substack{H \leq G \\ H \supset A}} H \longrightarrow$ ex: intersection of subgroups is a subgroup.

Convention: $\langle \emptyset \rangle = \{e\}$

We say G is generated by $A \subset G$ if $G = \langle A \rangle$. → or A is a set of generators for A

Ex: $G = S_4$. Check: $\overset{A_1}{\{(12), (1234)\}}$, $\overset{A_2}{\{(13), (23), (34)\}}$ are both generating sets for S_4.

G is finitely generated if $G = \langle A \rangle$ w/ $|A| < \infty$.

Ex: F_2 = free group on 2 letters. finitely generated infinite group.

notation.

Cyclic Group: a group generated by one element $G = \langle \{a\} \rangle = \langle a \rangle$ for some $a \in G$

$$G = \{\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots\}$$

all cyclic groups

option A: if G is <u>infinite</u> then $\underset{a^n}{\overset{G}{\cup}} \cong \underset{n}{\overset{\mathbb{Z}}{\cup}}$ [isomorphism].

$a^n \leftrightarrow n$

$a^n a^m = a^{n+m} \leftrightarrow nm = n+m$

option B: if G is <u>finite</u> then for some k, $a^k \in \{e, a, a^2, \ldots, a^{k-1}\}$.

let k be the smallest. Claim: $a^k = e$.

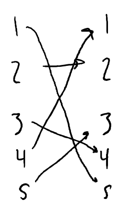
we know $a^k = a^l \Rightarrow a^{k-l} \in \{e, a, a^2, \ldots, a^{k-1}\}$

so if $l \neq 0$, k isn't minimal. Thus $G \cong \mathbb{Z}/_k\mathbb{Z}$.

Cyclic Groups: $\langle a \mid \underbrace{a^k = e}_{\text{here or not}} \rangle$ "free"

Example of S_n.

$(1\ 5\ 3\ 4)$ means



$(\quad)(\quad)$

$\xleftarrow{\text{R to L}}$

So one set of generators is all cycles $(i_1, i_2, \ldots, i_l)$

$(12)(34) = (34)(12)$ (disjoint)

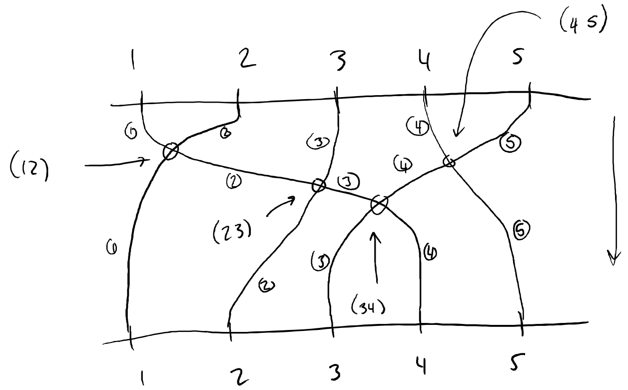
but $(1534) = (15)(53)(34)$,

so we have a better set of generators:

transpositions.

cycles of length 2. $S_n = \{(i,j) \mid 1 \le i < j \le n\}$

Lemma: $\{ \sigma_{i,i+1} \mid 1 \le i < n \}$ generate S_n.



so $(14532) = (34)(23)(12)(45)$

Puzzle: soul switching: 5 jumbled souls, how many extra people do you need to return them.