Lec 10/12

Wednesday, October 12, 2016 9:08 AM

Cauchy Mean Value Theorem

Suppose f and g are continuous on $[a,b]$ and $f'(x)$ and $g'(x)$ exist $\forall x \in (a,b)$. Then $\exists c \in (a,b)$ s.t. $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$

hence if $g'(c) \neq 0$ and $g(b)-g(a) \neq 0$ then $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

(if $g(x) = x$ we have MVT).

Proof: Let $h(x) = (f(x)-f(a))(g(b)-g(a)) - (g(x)-g(a))(f(b)-f(a))$

h is cts on $[a,b]$ and $h'(x) = f'(x)(g(b)-g(a)) - g'(x)(f(b)-f(a))$

and $h'(x)$ is defined $\forall x \in (a,b)$.

$h(a) = 0$ and $h(b) = 0$ so by Rolle's Thm, $\exists c \in (a,b)$ s.t.

$h'(c) = 0$. ∎

L'Hôpital's Rule (proved by Bernoulli)

Suppose that f, g are continuous on $(a-\delta, a) \cup (a, a+\delta)$ for some δ.

and $\lim_{x\to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$ also suppose $f'(x)$ and $g'(x)$ are defined on $(a-\delta, a) \cup (a, a+\delta)$.

and $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L$. (obvious modifications for one-sided limits).

Bonus problem for second midterm:

Disprove L'Hopital's Rule:

Let $f(x) = \dfrac{x}{e^{\sin(\frac{1}{x})}(2 + x\sin(\frac{2}{x}))}$

$g(x) = \dfrac{x}{2 + x\sin(\frac{2}{x})}$

Then $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ does not exist.

Hint: transform limits at 0 to limits at ∞.

Bonus problem and $f(x) = \dfrac{1}{e^{\sin(\frac{1}{x})}} g(x)$

After some gruesome calculations, we can show that

$$\left.\begin{aligned} f'(x) &= \alpha(x)\cos\left(\tfrac{1}{x}\right) \\ g'(x) &= \beta(x)\cos\left(\tfrac{1}{x}\right) \end{aligned}\right\} \text{ for } x \neq 0$$

$\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)}$ "=" $\lim\limits_{x\to 0} \frac{\alpha(x)}{\beta(x)} = 0$, so L'Hopital's rule is wrong.

In fact, $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |x| < \delta \Rightarrow \overbrace{x \in \operatorname{dom}\left(\frac{f'(x)}{g'(x)} = \frac{\alpha(x)\cos(\frac{1}{x})}{\beta(x)\cos(\frac{1}{x})}\right)}^{①}$ & $\left|\frac{f'(x)}{g'(x)}\right| < \varepsilon$

but "$\lim\limits_{x\to 0}$" $\frac{f'(x)}{g'(x)}$ (where ① is on the left) exists and $= 0$.

<u>Proof of L'Hopital's rule</u> (we will prove the right-handed version)

Hypothesis of RH-version: f and g are cts on $(a, a+\delta)$ for some $\delta > 0$.

and $f'(x)$ and $g'(x)$ are defined $\forall x \in (a, a+\delta)$. and $\lim\limits_{x\to a^+} f(x) = 0 = \lim\limits_{x\to a^+} g(x)$

and $\lim\limits_{x\to a^+} \frac{f'(x)}{g'(x)} = L$. Then $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = L$ too.

We know: for some $0 < \delta' \leq \delta$, $g'(x) \neq 0$ $\forall x \in (a, a+\delta')$

(Re)define if necessary, $f(a) = g(a) = 0$.

Then f and g are cts on $[a, a+\delta)$.

Also $g(x) \neq 0$ for $x \in (a, a+\delta')$. If $g(x_0) = 0$ for $x_0 \in (a, a+\delta')$,

then by Rolle's thm, $g'(c) = 0$ for some $c \in (a, x_0) \subseteq (a, a+\delta)$ (contradiction).

By CMVT, for $x \in (a, a+\delta')$,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{for some } c \in (a, x_0) \subseteq (a, a+\delta')$$

As $x \to a^+$, $c \to a^+$. so $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = \lim\limits_{c\to a^+} \frac{f'(c)}{g'(c)} = L$

ε-λ justification:

Given $\varepsilon > 0$ we can find λ, $0 < \lambda < \delta$ s.t.

$0 < c - a < \lambda \Rightarrow c \in \operatorname{dom}\left(\frac{f'}{g'}\right)$ and $\left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon$

So

$$0 < x - a < \lambda \Rightarrow 0 < c - a < \delta \Rightarrow \left| \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} - L \right| < \varepsilon$$