# Lec 2/2

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2.2 #6 Slick solution.

$$\sum a_j \nabla f_j(\vec{a}) \cdot \vec{h}$$

matrix product $= \underset{1\times n}{(a_1, \dots, a_n)} \underset{n \times n}{\left(\partial_k f_j(\vec{a})\right)} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = 0$

$$\Updownarrow$$

$$(a_1, \dots, a_n)\left(\partial_k f_j(\vec{a})\right) = \vec{0}$$

$\left(\partial_k f_j(\vec{a})\right) = DF(\vec{a})$ Jacobian where $\vec{F} = (f_1, \dots, f_n)$

$$\vec{F}(\vec{x}) = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \dots, \frac{x_n}{|x|}\right) = \frac{\vec{x}}{|\vec{x}|}$$

Jacobian? of some function $\vec{g}: \mathbb{R}^n \to \mathbb{R}$ $\nabla \vec{g}(\vec{a}) = (a_1, \dots, a_n)$

$\frac{\partial g}{\partial x_i} = x_i$ $\vec{g}(\vec{x}) = \frac{1}{2}(x_1^2 + x_2^2 + \dots x_n^2)$.

so $g(\vec{x}) = \frac{1}{2}|\vec{x}|^2$

Which means $(a_1, \dots, a_n)\left(\partial_k f_j(\vec{a})\right)$

$(g \circ \vec{F})(\vec{x}) = \frac{1}{2}$ constant

So $D(g \circ \vec{F})(\vec{x}) = Dg(\vec{F}(\vec{a})) \cdot D\vec{F}(\vec{a}) = 0$

$$Dg(\vec{F}(\vec{a})) \cdot D\vec{F}(\vec{a}) = \vec{F}(\vec{a}) \cdot \left(\partial_k f_j(\vec{a})\right) = 0$$

$$\vec{F}(\vec{a}) \cdot \left(\partial_k f_j(\vec{a})\right) = \frac{1}{|\vec{a}|}\vec{a} \cdot \left(\partial_k f_j(\vec{a})\right) = 0$$

$$\vec{a} \cdot \left(\partial_k f_j(\vec{a})\right) = 0. \checkmark$$

**Corollary 7:** Let $U \subseteq \mathbb{R}^n$ be open, $f: U \to \mathbb{R}$ and for some pair of indices $1 \le i < j \le n$ $\partial_i f$, $\partial_j f$ defined on all of $U$, and one of the mixed partials $\partial_i \partial_j f$ or

(6 is this but with $\mathbb{R}^2$ only)

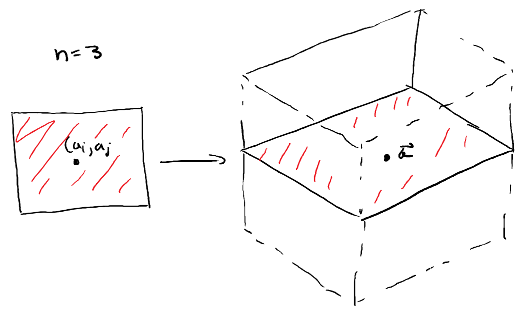
$\partial_j \partial_i f$ is defined & continuous on $\mathcal{U}$. Then the other mixed partial is also defined & cts on $\mathcal{U}$ and they are both equal.

Proof: Pick $\vec{a} \in \mathcal{U}$, choose $r > 0$ so that $B_\infty(r, \vec{a}) \subseteq \mathcal{U}$. Then $B_\infty(r, (a_i, a_j)) \subseteq \mathbb{R}^2$

Let $\vec{\lambda} : B_\infty(r, (a_i, a_j)) \longrightarrow B_\infty(r, \vec{a})$

$\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$

$$\lambda_k(x, y) = \begin{cases} x & k = i \\ y & k = j \\ a_k & k \neq i,\ k \neq j \end{cases}$$




$$\partial_1(f \circ \vec{\lambda})(x, y) = \partial_i f(\vec{\lambda}(x, y))$$
$$\partial_2(f \circ \vec{\lambda})(x, y) = \partial_j f(\vec{\lambda}(x, y))$$
$$\partial_1 \partial_2(f \circ \vec{\lambda})(x, y) = \partial_i \partial_j f(\vec{\lambda}(x, y))$$
$$\partial_2 \partial_1(f \circ \vec{\lambda})(x, y) = \partial_j \partial_i f(\vec{\lambda}(x, y))$$

Corr 6

$\Longrightarrow \partial_i \partial_j f(\vec{a}) = \partial_j \partial_i f(\vec{a})$

but $\vec{a}$ was arbitrary.

∎

Multi-indices: $f : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}$

k-th order $\quad I = (i_1, \dots, i_k) \quad 1 \leq i_j \leq n \quad j = 1, \dots, k \quad$ (indices can repeat).

$$\partial_I f = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f \qquad \text{allow } I = \emptyset. \quad \partial_\emptyset f = f$$

entropy of $I$ is total number of deranged pairs in $I$.

i.e. $p < q$ but $i_p > i_q$

$I$ has entropy $0 \Leftrightarrow i_1 \leq i_2 \leq \cdots \leq i_k$

$I = (2, 3, 1, 3, 3, 2)$

$(2,1), (3,1), (3,2), (3,2), (3,2) \Rightarrow$ entropy $= 5$

$I * J =$ concatenation of $I$ & $J$.

$= (i_1, i_2, \dots, i_k, j_1, \dots, j_l)$

Theorem 10: Let $\mathcal{U} \subseteq \mathbb{R}^n$ open, $f : \mathcal{U} \to \mathbb{R}$. Let $I = (i_1, \dots, i_k)$ be a multi-index

$\mathcal{A} = \{ I$ and all rearrangements of $I \}$

Suppose that $\partial_J f$ is defined & continuous on $U$ $\forall J \in \mathcal{A}$.

Then $\partial_J f = \partial_I f$.

<u>Proof</u>: Wolog, assume entropy of $I$ is 0. We will show by induction on entropy that $\partial_J f = \partial_I f$.

entropy $J = 0 \Rightarrow J = I$ so trivially true.

Suppose $\partial_{J'} f = \partial_I f$ $\forall J'$ with entropy $< r$.

let $J$ have entropy $r > 0$.

Hence for some pair of adjacent indices $(i_p, i_{p+1})$ we have $i_{p+1} < i_p$.

$J = J_1 * (i_p, i_{p+1}) * J_2$

let $g = \partial_{J_2} f$.

then $\partial_{i_p} g$, $\partial_{i_{p+1}} g$, $\partial_{i_p} \partial_{i_{p+1}} g$, $\partial_{p+1} \partial_p g$ are all defined and continuous on $U$.

(since this holds for all $\partial_{J'} f$)

Then by Corr 7.1 $\partial_{i_p} \partial_{i_{p+1}} g = \partial_{i_{p+1}} \partial_{i_p} g$

So $\partial_J f = \partial_{J_1} \partial_{i_p} \partial_{i_{p+1}} g = \partial_{J'} f$

but entropy $J' < r$ so $\partial_{J'} f = \partial_I f$.