# Lec 9/25

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$M_n(R)$ is the ring of $n \times n$ matrices over $R$.

When $R \neq 0$ and $n > 1$, $M_n(R)$ is not commutative.

Pf let $e_{ij}$ be the matrix w/ 1 in the $(i,j)$-position and 0 elsewhere.

Then $e_{ij} e_{kl} = \delta_{jk} e_{il}$.

$$\begin{array}{ccc} R & \hookrightarrow & M_n(R) \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ a & \longmapsto & \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix} \end{array}$$ is a ring hom.

we can define scalar multiplication with this identification.

**Def** suppose $R$ is commutative. if $\overset{[a_{ij}]}{\overset{\shortparallel}{A}} \in M_n(R)$,

$$\det(A) := \sum_{\pi \in S_n} (\operatorname{sgn} \pi)\, a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

**Fact** $\det(1) = 1$ and $\det(AB) = \det(A)\det(B)$.

**Def** The gp of units in $M_n(R)$ is called the general linear group, denoted $GL_n(R)$.

**Thm** If $R$ is commutative, $A \in M_n(R)$ is invertible iff $\det(A)$ is invertible in $R$.

i.e $GL_n(R) = \{A \in M_n(R) \mid \det A \in R^\times\}$.

**Corollary** if $R$ is a division ring, $GL_n(R) = \{A \in M_n(R) \mid \det A \neq 0\}$.

To prove the theorem, we give a formula for $A^{-1}$ when it exists.

**Def** let $A = [a_{ij}] \in M_n(R)$. The cofactor of $a_{ij}$ in $A$ is

$$A_{ij} = (-1)^{i+j} \det \tilde{A}_{ij} \quad \text{where}$$

$\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i^{th}$ row & $j^{th}$ column.

**prop** If $R$ is commutative, we have the "orthogonality relations"

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn} = \delta_{ij} \det A.$$

$$a_{1j} A_{1i} + a_{2j} A_{2i} + \cdots + a_{nj} A_{ni} = \delta_{ij} \det A.$$

**Def** The adjoint matrix is $\operatorname{adj} A = [A_{ji}] = [A_{ij}]^T$.

in particular, $A(\operatorname{adj} A) = \underbrace{\det A}_{\in M_n(R)} = (\operatorname{adj} A) A$.

Thus, if $\det A \in R^\times$, $A^{-1} = (\det A)^{-1}(\operatorname{adj} A)$.

Also, if $A^{-1}$ exists, $\det(A)^{-1} = \det(A^{-1})$. $\square$

## Quaternions

Recall $\mathbb{C} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$.

Now $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}$.

$\mathbb{H}$ is a division ring.