Lec 11/17

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Geometric series: $a + ar + ar^2 + \dots$ $\quad a, r \neq 0$

$$\sum_{j=1}^{\infty} ar^{j-1}$$

Theorem: Geometric series converge iff $|r|<1$. (to $\frac{a}{1-r}$).

Proof: $s_n = a + ar + \dots + ar^{n-1}$ (1)

$rs_n = ar + \dots + ar^{n-1} + ar^n$ (2)

$(1)-(2) = (1-r)s_n = a - ar^n$

$\Rightarrow \quad s_n = \dfrac{a-ar^n}{1-r} \quad r \neq 1$

If $|r|<1$ then $\lim_{n\to\infty} r^n = 0$ and $\lim_{n\to\infty} s_n = \frac{a}{1-r}$

If $|r| \geq 1$ then $\lim_{n\to\infty} ar^{n-1}$ either DNE or $= a$, so it $\neq 0$, so the series diverges. ∎

Geometric series useful for comparison w/ other series.

Notation: If $\sum_{j=1}^{\infty} a_j$ is a convergent infinite series, $R_n = \sum_{j=n+1}^{\infty} a_j$

This is the n^{th} remainder.

$$R_n = \sum_{j=1}^{\infty} a_j - s_n$$

Definition Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be convergent infinite series. We say that $\sum_{j=1}^{\infty} a_j$ converges faster than $\sum_{j=1}^{\infty} b_j$ if for some N we have that $|R_n^a| \leq |R_n^b|$ for all $n \geq N$.

Lemma for a convergent geometric series, $R_n = \frac{ar^n}{1-r}$ ← first neglected term

Proof: $R_n = \sum_{j=1}^{\infty} ar^{n-1} - S_n$

$$= \frac{a}{1-r} - \frac{a - ar^n}{1-r}$$

$$= \frac{ar^n}{1-r} \qquad \blacksquare$$

Proposition if $0<|r|<|s|<1$ then $\sum_{j=1}^{\infty} ar^{j-1}$ converges faster than $\sum_{j=1}^{\infty} bs^{j-1}$.

Proof wts that $\exists N$ s.t. $\forall n \geq N$ $\left|\frac{ar^n}{1-r}\right| \leq \left|\frac{bs^n}{1-s}\right|$

which is eq. to saying $\frac{|a||r|^n}{|1-r|} \leq \frac{|b||s|^n}{|1-s|}$

$$\Longleftrightarrow \quad \left(\frac{|r|}{|s|}\right)^n \leq \frac{|b|\,|1-r|}{|a|\,|1-s|} = \varepsilon, \text{ some positive num.}$$

$\left(\frac{|r|}{|s|}\right)^n$ tends to 0, so can find an N where this works. $\blacksquare$

Fundamental Convergence Thm: for any infinite series $\sum_{j=1}^{\infty} a_j$, there is an associated convergence parameter $q \in [0, \infty]$ such that

(1) if $q = 0$, then the series converges faster than any geometric series.

(2) if $0 < q < 1$ then the series converges faster than any geometric series $\sum_{j=1}^{\infty} bs^{j-1}$ with $s \in (q, 1)$

(3) if $q \in (1, \infty]$ then the series diverges.

(4) if $q = 1$, the series may or may not converge. If it converges, then (usually) it converges slower than any convergent geometric series.

$q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \sqrt[n]{a_n}$

or q is usually obtained from a refinement of the root test or the ratio test

Definition let $\{c_n\}$ be any sequence. we say that c is a cluster point of the sequence if some subsequence converges to c. ($c = \pm\infty$ is allowed).

Examples: (1) sequence converges $\Leftrightarrow$ it has a single finite cluster point.

(2) $\{(-1)^n\}_{n=1}^{\infty}$ has 2 cluster points, namely 1 and -1.

(3) $\{\sin(n)\}_{n=1}^{\infty}$ has the following set of cluster points: $[-1, 1]$.

$\Leftrightarrow$ π is irrational.

q = Largest cluster point of the sequence $\{|a_n|^{1/n}\}_{n=1}^{\infty}$

$= \limsup |a_n|^{1/n}$

Proposition: If $|a_n| < b_n$ for $n > N$ and $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges faster than $\sum_{j=1}^{\infty} b_j$. $|R_n^a| \le R_m^b$ for $n \geq N$ for some N.

Proof: Use the Cauchy Criterion to show that $\sum_{j=1}^{\infty} a_j$ converges.

If $n \geq m > N$, then $\left|\sum_{j=m}^{n} a_j\right| \le \sum_{j=m}^{n} |a_j| < \sum_{j=m}^{n} b_j < \varepsilon$ for $m, n > M$ for some M

Then for $n \geq m > \max(M, N)$, then $\left|\sum_{j=m}^{n} a_j\right| < \varepsilon$,

Hence $\{S_n\}$ are a Cauchy sequence so they converge.

for $m \geq N$, $|R_m^a| = \left|\sum_{j=m+1}^{\infty} a_j\right| = \lim_{n\to\infty}\left|\sum_{j=m+1}^{n} a_j\right| \le \lim_{n\to\infty}\sum_{j=m+1}^{n}|a_j| < \lim_{n\to\infty}\sum_{j=m+1}^{n} b_j = R_n^b$ ∎

Proof of Main Convergence theorem:

Suppose $q < 1$ and $q < s < 1$. Since q is the largest cluster point of $\{|a_n|^{1/n}\}$, we must have for some N that $|a_n|^{1/n} \le s$ for all $n > N$.

(if not we could find a subsequence $|a_{n_k}|^{1/n_k}$ which converges to something greater than $s > q$)

So $|a_n| \le s^n$ for $n > N$, and (by proposition), $\sum_{j=1}^{\infty} a_j$ converges faster than $\sum_{j=1}^{\infty} s^j$, a geometric sequence. This proves (1) and (2).