Lec 9/19

Monday, September 19, 2016 9:10 AM

ε-δ definition of continuity & limits

$(-\infty,0)\cup(0,\infty)$

Ex: use ε-δ ~~that~~ $f(x)=\frac{1}{x}$ is continuous throughout its domain.

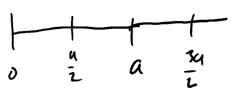
Solution: first assume $a>0$

Let $\varepsilon>0$ be arbitrary. Want to find δ so that

$$|x-a|<\delta \text{ and } x\neq 0 \Rightarrow |f(x)-f(a)|<\varepsilon$$

$$|f(x)-f(a)| = \left|\frac{1}{x}-\frac{1}{a}\right| = \left|\frac{a-x}{xa}\right| = \frac{|a-x|}{a|x|}$$

Can't allow x to come close to 0



if $|x-a|<\frac{a}{2}$ then $x\neq 0$

$x\in\left(\frac{a}{2},\frac{3a}{2}\right) \Rightarrow x>\frac{a}{2}>0 \Rightarrow \frac{1}{x}<\frac{2}{a}$

so $\dfrac{|a-x|}{a|x|} \le \dfrac{|x-a|}{a}\dfrac{2}{a} = |x-a|\dfrac{2}{a^2}$

we want this $<\varepsilon$

so $|x-a|<\dfrac{a^2\varepsilon}{2}$ so take $\delta=\min\left(\frac{a}{2},\frac{a^2\varepsilon}{2}\right)$

Then $|x-a|<\delta \Rightarrow |f(x)-f(a)|<$

$\Downarrow$

$x>0$

Might be on midterm!

$\downarrow$ (prove this works for $a<0$)

if $a<0$ then $\delta=\min\left(-\frac{a}{2},\frac{a^2\varepsilon}{2}\right)$ will work. ☆

$\forall a\in \text{dom}(f)$, $\delta=\min\left(\frac{|a|}{2},\frac{a^2\varepsilon}{2}\right)$ will work.

Definition: We say that $\lim\limits_{x\to a} f(x)=l$ if $\forall\varepsilon>0, \exists\delta>0$ s.t.

$$0<|x-a|<\delta \Rightarrow x\in\text{dom}(f) \text{ and } |f(x)-l|<\varepsilon$$

Definition: f is continuous at a point $a\in\text{dom}(f)$ if $\forall\varepsilon>0, \exists\delta>0$ s.t.

$$|x-a|<\delta \text{ and } x\in\text{dom}(f) \Rightarrow |f(x)-f(a)|<\varepsilon$$

Compare:

difference (1) " $0<|x-a|$ " in limit definition

(2) position of " $x \in \mathrm{dom}(f)$ "

(1) often when taking $\lim_{x\to a} f(x) = L$, $f(a)$ is undefined so if $|x-a|=0$ then $f(x)=f(a)$ which is indeterminate. So we avoid this possibility. even if $f(a)$ is defined, we ignore the case where $x=a$ because of hole discontinuities.

This can lead to mistakes:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{u\to b} g(u) = a \quad \not\Rightarrow \quad \lim_{u\to b} f(g(u)) = L$$

take $f(x) = \begin{cases} x^2 & x\neq 0 \\ -1 & x=0 \end{cases}$ and $g(u)=0$

then $f(g(u)) = f(0) = -1 \quad \forall u$

so $\lim_{u\to b} f(g(u)) = -1 \neq 0 = \lim_{x\to 0} f(x)$

(2) According to spivak, f is continuous at $a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$ (defn)

Not according to our definitions. $\sqrt{x}$ is continuous at 0.

but $\lim_{x\to 0} \sqrt{x}$ does not exist.

__Theorem:__ $\lim_{x\to a} f(x) = f(a) \iff f$ is continuous at a <u>and</u> $(c,d) \subseteq \mathrm{dom}(f)$ for some $c<a<d$

__Proof:__ $\Rightarrow$. Given $\varepsilon>0$ we can find $\delta>0$ s.t.

$0<|x-a|<\delta \Rightarrow x\in\mathrm{dom}(f)$ and $|f(x)-f(a)|<\varepsilon$

$\Rightarrow \underbrace{(a-\delta}_{c}, \underbrace{a+\delta)}_{d} \subseteq \mathrm{dom}(f)$

and $|f(x)-f(a)|<\varepsilon$ for $0<|x-a|<\varepsilon$.

for $x = a$ trivially since $f(a) - f(a) = 0 < \varepsilon$
so for $|x-a| < \varepsilon$

$\Leftarrow$. Given $\varepsilon > 0$ we can find $\delta_1 > 0$ s.t.

$|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon$

Also $a \in (c,d) \subseteq dom(f)$ for some $c < a < d$

let $\delta = \min(\delta_1, a-c, d-a)$

Then $0 < |x-a| < \delta \Rightarrow x \in (c,a) \cup (a,d) \subseteq dom(f)$

and $|f(x) - f(a)| < \varepsilon$

So it goes both ways ■

Confusing the position of "$x \in dom(f)$" in definitions of limit & continuity can lead to paradoxial results.

$f : [0, \infty) \to \mathbb{R}$ $\quad f(x) = \sqrt{x}$

$g : (-\infty, 0] \to \mathbb{R}$ $\quad g(x) = \sqrt{-x}$

f & g are continuous at 0.

$\overset{\text{spirak}}{\Rightarrow}$ $\lim_{x \to 0} f(x) = 0$ $\quad \lim_{x \to 0} g(x) = 0$

$\Rightarrow \lim_{x \to 0} (f(x) + g(x)) = 0$

$dom(f+g) = \{0\}$ but in taking limit we should ignore val. at 0.

Theorem1: if f and g are continuous at a, so are $\overset{(i)}{f+g}$ and $\overset{(ii)}{f \cdot g}$

Theorem2: if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

then $\overset{(i)}{\lim_{x \to a} (f(x) + g(x)) = l + m}$ and $\overset{(ii)}{\lim_{x \to a} (f(x)g(x)) = lm}$

Proof of 1(i): Let $\varepsilon > 0$ be arbitrary. by continuity of f and g at a,

find $\delta_1, \delta_2 > 0$ so that:

$|x-a| < \delta_1$ and $x \in \mathrm{dom}(f) \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2}$

$|x-a| < \delta_2$ and $x \in \mathrm{dom}(g) \Rightarrow |g(x) - g(a)| < \frac{\varepsilon}{2}$

let $\delta = \min(\delta_1, \delta_2)$. Then:

$|x-a| < \delta$ and $x \in \mathrm{dom}(f+g) \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2}$

and $|g(x) - g(a)| < \frac{\varepsilon}{2}$

$\Downarrow$:

$$\left|(f+g)(x) - (f+g)(a)\right| < \varepsilon \qquad \blacksquare$$

<u>Proof of 2 (i)</u>: Given $\varepsilon > 0$, find δ_1, δ_2

$0 < |x-a| < \delta_1 \Rightarrow x \in \mathrm{dom}(f)$ and $|f(x) - l| < \frac{\varepsilon}{2}$

$0 < |x-a| < \delta_2 \Rightarrow x \in \mathrm{dom}(g)$ and $|g(x) - m| < \frac{\varepsilon}{2}$

let $\delta = \min(\delta_1, \delta_2)$ Then

$0 < |x-a| < \delta \Rightarrow x \in \mathrm{dom}(f)$ and $|f(x) - l| < \frac{\varepsilon}{2}$

and $x \in \mathrm{dom}(g)$ and $|g(x) - m| < \frac{\varepsilon}{2}$

$\Rightarrow x \in \mathrm{dom}(f+g)$ and something

$\Downarrow$:

$$|(f+g)(x) - (l+m)| < \varepsilon \qquad \blacksquare$$