Lec 8/30

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$A \subset P(X)$ algebra.

$\mu_0 : A \longrightarrow [0, \infty]$ is a premeasure if

⓪ $\mu_0(\emptyset) = 0$

① If $\coprod E_n \in A$ then $\mu_0(\coprod E_n) = \sum \mu_0(E_n)$

Properties: ⓪ finite additivity

$\Downarrow$

① monotonicity

② Countable subadditivity: $\mu_0(\bigcup E_n) \leq \sum \mu_0(E_n)$

if $\bigcup E_n \in A$.

②' if $E \in A$ and $(E_n) \subset A$ s.t. $E \subset \bigcup E_n$ then $\mu_0(E) \leq \sum \mu_0(E_n)$.

μ_0 on A $\rightsquigarrow$ μ^* on $P(A)$ by

$$\mu^*(E) = \inf\{\sum \mu_0(E_n) \mid \bigcup E_n \supset E,\ E_n \in A\}$$

Lemma: $\mu_* \big|_A = \mu_0$ ✓

Lemma: $A \subset M^*$

~~Pf~~ Suppose $E \in A$ and $F \subset X$.

WTS: $\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \setminus E)$.

Let $\varepsilon > 0$. Pick $(F_n) \subset A$ s.t. $F \subset \bigcup F_n$ and

$\sum \mu_0(F_n) \le \mu^*(F) + \varepsilon$. Since μ_0 is additive on A,

$$\mu^*(F) + \varepsilon \geq \sum \mu_0(F_n) = \sum \mu_0(F_n \cap E) + \mu_0(F_n \setminus E)$$

$$\geq \mu^*(F \cap E) + \mu^*(F \setminus E) \quad \square$$

$\varepsilon > 0$ was arbitrary.

Get a (complete) measure $\mu := \mu^*|_{M^*}$ on M^*.

If $M = M(A)$ then $M \subset M^*$ and $\mu|_M$ is a measure s.t. $\mu|_A = \mu_0$.

<u>Theorem</u> If ν is a measure on $M(A)$ s.t. $\nu|_A = \mu_0$, then $\nu(E) \le \mu(E)$ $\forall E \in M$ with equality when $\mu(E) < \infty$.

Pf Suppose $E \in M$ and $E \subset \bigcup E_n$ where $E_n \in A$ & $\sum \mu_0(E_n) \le \mu^*(E) + \varepsilon$.

then $\nu(E) \le \sum \mu(E_n) = \sum \mu_0(E_n) \le \mu^*(E) + \varepsilon = \mu(E) + \varepsilon$.

So $\underbrace{\nu(E) \le \mu(E)}_{(0)}$ as ε was arbitrary.

If $\mu(E)$ is finite, then

① $\mu((\bigcup E_n) \setminus E) \le \varepsilon$. $\quad \bigcup E_n = (\bigcup E_n \setminus E) \amalg E$.

② by continuity from below for ν & μ,

$$\mu(\bigcup E_n) = \lim_N \mu(\bigcup_1^N E_n)$$

$$= \lim_N \mu_0(\bigcup_1^N E_n)$$

$$= \lim_N \nu\left(\bigcup_1^N E_n\right)$$

$$= \nu\left(\bigcup E_n\right).$$

So $\mu(E) \le \mu(\bigcup E_n) = \nu(\bigcup E_n) = \nu(E) + \nu((\bigcup E_n) \setminus E)$

$\le \nu(E) + \mu((\bigcup E_n) \setminus E)$ by (0)

$\le \nu(E) + \varepsilon$ by (1)

So $\mu(E) \le \nu(E)$. $\square$

__Cor__: If μ_0 is σ-finite $[X = \coprod X_n$ w/ $X_n \in A$ & $\mu_0(X_n) < \infty \ \forall n]$.

Then μ is the unique extension of μ_0 to $M(A)$.

~~Pf~~ For any other ν extending μ_0 and $E \in M(A)$,

$$\nu(E) = \nu(E \cap X) = \nu(E \cap \coprod X_n) = \nu(\coprod(X_n \cap E))$$

$$= \sum \nu(E \cap X_n) = \sum \mu(E \cap X_n) = \cdots = \mu(E). \quad \square$$

Construction of Lebesgue-Stieltjes measures in $\mathbb{R}$.

__Def__: $\mathcal{H} = \{\emptyset\} \cup \{(a,b] \mid -\infty \le a < b < \infty\} \cup \{(a,\infty) \mid -\infty \le a < \infty\}$.

are called h-intervals.

$A = \{$ finite disjoint unions of elts of $\mathcal{H}\}$

By __HW3__, A is an algebra.

$M(A) = B_{\mathbb{R}}$.

$F : \mathbb{R} \to \mathbb{R}$ non-decreasing $(s \le t \Rightarrow F(s) \le F(t))$

and right-continuous $(a_n \searrow a \Rightarrow F(a_n) \to F(a))$

Extend F to $F : [-\infty, \infty] \longrightarrow [-\infty, \infty]$ by

$$F(-\infty) = \lim_{a \to -\infty} F(a), \quad F(\infty) = \lim_{a \to \infty} F(a).$$

$$\underset{[-\infty,\infty)}{\pitchfork} \qquad\qquad \underset{(-\infty,\infty]}{\pitchfork}$$

<u>Define</u>: $\mu_0 : \mathcal{H} \longrightarrow [0, \infty]$ by

- $\mu_0(\emptyset) = 0$
- $\mu_0((a,b]) = F(b) - F(a)$
- $\mu_0((a,\infty)) = F(\infty) - F(a)$

<u>Goal</u>: Extend μ_0 to A, show it's a premeasure.

<u>Step 1</u>: If $(a,b] = \coprod_1^n (a_j, b_j]$ then $\mu_0((a,b]) = \sum_1^n \mu_0(a_j, b_j]$

<u>Pf</u> after reindexing, we may assume

$$a = a_1 < b_1 = a_2 < b_2 = \cdots < b_n = b.$$

Then
$$\begin{aligned}\mu_0(a,b] &= F(b) - F(a) \\ &= \sum_1^n F(b_j) - F(a_j) \\ &= \sum_1^n \mu_0(a_j, b_j] \qquad \square\end{aligned}$$

This doesn't work if $n = \infty$.

Step 2: If $(a,\infty) = (a_0,\infty) \cup \coprod_1^n (a_j, b_j]$, then

$$\mu_0(a,\infty) = \mu_0(a_0,\infty) + \sum_1^n \mu_0(a_j, b_j].$$

Pf similar to step 1. □

Step 3: If $E_1, \dots, E_n \in \mathcal{H}$ are disjoint and

$F \in \mathcal{H}$ s.t. $F \subset \coprod_1^n E_i$, then

$$\mu_0(F) = \sum_1^n \mu_0(E_i \cap F)$$

Pf We may remove E_i if $E_i \cap F = \emptyset$.

So we may assume $\underbrace{E_i \cap F}_{\in \mathcal{H}}$ $\forall i$.

Then $F = \coprod_1^n E_i \cap F$. use step 1 & 2. □

Step 4: If $(E_i)_1^m$ & $(F_j)_1^n$ are two sets of disjoint h-intervals

s.t. $\coprod_1^m E_i = \coprod_1^n F_j$, Then

$$\sum_1^m \mu_0(E_i) = \sum_1^n \mu_0(F_j).$$

hence μ_0 extends to a well-defined function

from $A \longrightarrow [0,\infty]$ by $\mu_0(\coprod_1^m \underbrace{E_i}_{\in \mathcal{H}}) = \sum_1^m \mu_0(E_i)$.

Pf by step 3,

$$\sum_1^m \mu_0(E_i) = \sum_1^m \sum_1^n \mu_0(E_i \cap F_j) = \sum_1^n \mu_0(F_j).$$ □