$f, g \in F[x]$ are relatively prime if $(f, g) = 1$

iff f & g have no common divisor.

Also, iff f & g have no common root in any extension of F.

Proof if p is an irreducible common divisor of f & g,
let α be a root of p. Then α is a common root of f & g in $F(\alpha)$.
If f & g have a common root α in some extension K/F,
then $m_{\alpha,F} \mid f$ & $m_{\alpha,F} \mid g$ so f & g are not rel. prime.

Defn K is algebraically closed if $\forall f \in K[x]$,
f has a root in K (then f splits completely in K by induction on $\deg f$).

Note K is algebraically closed if it has no nontrivial algebraic extensions.
All irred. pol-ls in $K[x]$ are linear.

Example : $\mathbb{C}$.

Defn Let F be a field. K/F is an algebraic closure of F if

$\forall f \in F[x]$, f splits completely in K, and K is a minimal field w/ this property.

Other Defn: K/F is algebraic and any $f \in F[x]$ splits completely in K

Theorem If K is an algebraic closure of F, then K is alg. closed.

Proof Let α be algebraic over K. $K(\alpha)/K/F$ implies $K(\alpha)/F$ is algebraic, so α is algebraic over F. but $m_{\alpha,F} \in F[x]$ splits in K so $\alpha \in K$.

Theorem $\forall$ field F, an algebraic closure of F exists & is unique up to isomorphism which is identical on F.

Moreover, if L/F is algebraic, $\exists$ an algebraic closure K of F s.t. $L \subseteq K$.

Or: $\forall$ algebraic closure K of F, $L \cong$ subfield of K.

Algebraic Closure of $\mathbb{Q}$ is the field of all algebraic numbers (roots of polynomials from $\mathbb{Q}[x]$).

(it's denoted $\overline{F}$ sometimes).

$$|\overline{F}| \geq |F|, \quad \text{and} \quad |\overline{F}| \leq \sum_{\substack{\text{irreducible} \\ \text{pol.} \ f \in F[x]}} \deg f$$

$$\Big|\underbrace{\{\text{pol's of degree } n \text{ in } F\}}_{F_n}\Big| = |F^{n+1}| = |F| \quad \text{if} \quad F \text{ is infinite.}$$

$$|F[x]| = \Big|\bigcup_n F_n\Big| = |F| \quad \text{if } F \text{ is infinite}$$

$$|\bar{F}| \leq \sum_n n|F_n| = \sum_n |F| = |F| \quad \text{if } F \text{ is infinite}$$

So $|\bar{F}| = |F|$ if F is infinite.

Algebraic ext-ns L/F.

$L_1 \geq L_2$ if L_1/L_2.

$\forall$ chain $\{L_\alpha\}$, $\bigcup_\alpha L_\alpha$ - algebraic extension of F.

By Zorn, $\exists$ maximal algebraic extension K of F.

If $\exists f \in F[x]$ that doesn't split in K, adjoin a root $\alpha \notin K$ of f to K. Then $K(\alpha) > K$, contradiction.

Gap: There is no set of algebraic extensions of F.

Another Proof: Let $R = F[x_f : f \in F[x], f \text{ is monic}]$.

Let $I = (f(x_f) : f \in F[x])$.

(in R/I, $f(x_f) = 0 \ \forall f$)

Claim 1: $I \neq R$. Indeed, assume $I = R$. Then

$$1 = \sum_i^n g_i \cdot f_i(x_{f_i}), \quad g_i \in R.$$

Let $\alpha_1, \dots, \alpha_n$ be roots of $f_1, \dots, f_n$ in some field K.

Consider $R \longrightarrow K$; $x_{f_i} \longmapsto \alpha_i$ for $i = 1, \dots, n$, $x_h \mapsto 0 \ \forall$ other h.

Then $1 = \varphi(1) = \varphi(\Sigma) = 0$, contradiction.

Let M be a max'l ideal of R that contains I.

(this exists by Zorn Lemma).

Let $K_1 = R/M$, K_1 is a field, and $\forall f \in F[x]$,

f has a root in K_1, namely x_f mod M.

Now construct K_2 from K_1, same way as K_1, from F.

get $F \subseteq K_1 \subseteq K_2 \subseteq \dots$.

Put $\tilde{K} = \bigcup_n K_n$.

Claim: $\tilde{K}$ is algebraically closed.

Pf: if $f \in \tilde{K}[x]$ then $f \in K_n[x]$ for some n,

So f has a root in K_{n+1}.

Let K be the set of elements of $\tilde{K}$ algebraic over F

= max'l algebraic over F subextension of F.

Then any $f \in F[x]$ splits in $\tilde{K}$, so it splits in K.

Thus K is the alg. closure of F.