Lec 11/29

Tuesday, November 29, 2016 9:04 AM

Power series:

$\sum_{n=0}^{\infty} c_n (x-a)^n = f(x)$ $\quad q = cp(\sum c_n)$ $\quad q_x = "q|x-a|"$ $\quad R = "\frac{1}{q}"$

need to check endpoints of interval. endpoints could have conditional convergence.

Integration & Differentiation

Generally, cannot integrate/Differentiate $\sum_{n=0}^{\infty} f_n(x) = f(x)$.

Simple counterexample: $\sum_{n=1}^{\infty} \left(\frac{\sin(n^2 x)}{n} - \frac{\sin((n+1)^2 x)}{n+1} \right)$, $S_n = \sin(x) - \frac{\sin((n+1)^2 x)}{n+1}$

$\lim_{n\to\infty} S_n = \sin(x)$ for all x.

but $\cos(x) \neq \sum_{n=1}^{\infty} \left(\frac{\cos(n^2 x) n^2}{n} - \frac{\cos((n+1)^2 x)(n+1)^2}{n+1} \right)$

$\lim_{n\to\infty} \cos(x) - \frac{(n+1)^2 \cos((n+1)^2 x)}{n+1}$ DNE.

Theorem (1) $\sum_{n=0}^{\infty} c_n (x-a)^n$, (2) $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$, and (3) $\sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ have same radius of convergence.

$\Leftrightarrow$ The convergence parameters of series of coefficients are the same. $(R = \frac{1}{q})$

Proposition Suppose $\lim_{n\to\infty} a_n = L \neq 0$, and $\{b_n\}$ is any sequence.

Then cluster points of $\{a_n \cdot b_n\}$ are L times the cluster points of $\{b_n\}$.

Proof: If c is a cluster point of $\{b_n\}$ then $\lim_{j\to\infty} b_{n_j} = c$. for some subsequence $\{b_{n_j}\}$.

then $\lim_{j\to\infty} a_{n_j} b_{n_j} = L \cdot C$ since $\lim_{j\to\infty} a_{n_j} = \lim_{n\to\infty} a_n$.

Conversely, if $\hat{c}$ is a cluster point of $\{a_n b_n\}$ then $\lim_{j\to\infty} a_{n_j} b_{n_j} = \hat{c}$ so

$\lim_{j\to\infty} b_{n_j} = \lim_{j\to\infty} (a_{n_j})^{-1} \cdot a_{n_j} b_{n_j} = L^{-1} \hat{c}$. ∎

Proof of theorem: let $q =$ largest cluster point of $\{|c_n|^{1/n}\}$.

$q_2 =$ " " $\{|n c_n|^{1/n-1}\}$.

$q_3 =$ " " $\{|\frac{c_n}{n+1}|^{1/n+1}\}$.

for q_2 take $a_n = n^{1/n-1}$ $\quad b_n = |c_n|^{1/n-1}$

for q_3 take $a_n' = \left(\frac{1}{n+1}\right)^{1/n+1}$ $\quad b_n' = |c_n|^{1/n+1}$

$\lim_{n\to\infty} a_n = e^{\lim_{n\to\infty} \log(n)/n-1 \to 0}] \to 1$

$L = 1$.

$\lim a_n' = 1$ as well.

$\lim_{n\to\infty} a_n = e^{\cdots} \quad \cdots \to 1$

$L = 1.$

$\lim_{n\to\infty} a_n' = 1$ as well.

now need to show that cluster points of $\{|c_n|^{1/n+1}\}$ and $\{|c_n|^{1/n-1}\}$ are the same as those of $\{|c_n|^{1/n}\}$.

Partial proof: suppose c is a cluster point of $\{|c_n|^{1/n}\}$.

Then $c = \lim_{j\to\infty} |c_{n_j}|^{1/n_j} = \lim_{j\to\infty} e^{\log|c_{n_j}|/n_j} \iff \lim_{j\to\infty} \frac{\log|c_{n_j}|}{n_j} = \log c.$

$\iff \lim_{j\to\infty} \frac{\log|c_{n_j}|}{n_j \pm 1} = \log c.$

The trouble w/ int/dif term by term is from fact that cant interchange limits & derivatives in general: $\left(\lim_{n\to\infty} f_n(x)\right)' \neq \lim_{n\to\infty} f_n'(x)$

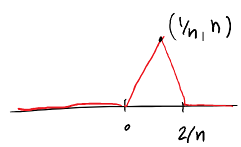
$$\left(\lim_{n\to\infty} \frac{\sin(n^2 x)}{n} = 0\right)' = 0$$

$$\lim_{n\to\infty}\left(\frac{\sin(n^2x)'}{n}\right) = \lim_{n\to\infty} \cos(n^2 x)\, n \quad \text{DNE.}$$

integrals are a bit better, since if $f(x) \le g(x)$ $\nearrow$ for $x \in I$ then $\int_I f \le \int_I g$

whereas $f \le g \not\Rightarrow f' \le g'$

even integrals have issues: $f_n(x) =$



$\lim_{n\to\infty} f_n(x) = 0$ for all x (triangle squeezes to the left).

so $\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = 0$ but $\lim_{n\to\infty} \int_0^1 (f_n(x))\, dx = \lim_{n\to\infty} \left[\text{area of triangle} = \tfrac{1}{2}\left(\tfrac{2}{n}\cdot n\right)\right] = 1.$

Theorem Suppose $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of integrable functions over some interval $[a,b]$.

suppose there is a sequence of positive constants $\{M_n\}$ such that

(1) $|f_n(x)| \le M_n$ for all $x \in [a,b]$.

(2) $\lim_{n\to\infty} M_n = 0$

Then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = 0 = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx.$

Proof: $-M_n \le f_n(x) \le M_n \Rightarrow \lim_{n\to\infty} f_n(x) = 0$ for $x \in [a,b]$ by sq. Thm.

$\Rightarrow \int^b (\lim f_n(x))\, dx = 0$

Proof: $-M_n \le f_n(x) \le M_n \Rightarrow \lim_{n\to\infty} f_n(x) = 0$ for $x \in [a,b]$ by sq. Thm.

$$\Rightarrow \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx = 0$$

$$\underset{\substack{\downarrow \\ 0}}{-M_n(b-a)} = \int_a^b -M_n\, dx \le \underset{\substack{\downarrow \\ 0 \\ \text{by sq. thm.}}}{\int_a^b f_n(x)\, dx} \le \int_a^b M_n\, dx = \underset{\substack{\downarrow \\ 0}}{M_n(b-a)}$$

$\blacksquare$

(*) Apply this theorem to remainders of power series.