Lec 11/15

Friday, November 15, 2019 10:25

X LCH Space

Goal: Compute $C_0(X)^*$

Recall: a Borel measure μ on X is called

outer regular on E if $\mu(E) = \inf\{\mu(U) \mid E \subset U \text{ open}\}$ and

inner regular on E if $\mu(E) = \sup\{\mu(K) \mid E \supset K \text{ cpt}\}$.

If μ is inner & outer regular on all Borel E, μ is regular

Eg: Lebesgue-Stieltjes measures.

Def A Radon measure on X is a Borel measure which is

- finite on compact sets
- outer regular on Borel sets
- inner regular on open sets

($\Rightarrow$ inner reg on σ-finite sets so X σ-finite $\Rightarrow$ μ regular. Btw, X σ-cpt $\Rightarrow$ X σ-finite.)

Consider $C_c(X)$ cts fns of cpt supp.

A Radon integral on X is a positive linear functional

$$\varphi : C_c(X) \longrightarrow \mathbb{C}, \quad \text{i.e. } \varphi(f) \geq 0 \text{ if } f \geq 0.$$

Lemma Radon integrals are bdd on compact subsets of X.

i.e. Let $K \subset X$ cpt. Show $\exists\, C_K > 0$ s.t. $\forall f \in C_c(X)$ w/ $\operatorname{supp}(f) \subset K$, $|\varphi(f)| \le C_K \|f\|_\infty$.

Pf By taking Re + Im parts, we may assume f is $\mathbb{R}$-valued.

Choose $g \in C_c(X)$ s.t. $g = 1$ on K and $g \geq 0$ by Urysohn's Lemma

If $\operatorname{supp}(f) \subset K$, $|f| \le \|f\|_\infty \cdot g$. Then $\|f\|_\infty \cdot g - |f| \geq 0$, and so

$\|f\|_\infty \cdot g \pm f \geq 0$. So $\|f\|_\infty \varphi(g) \pm \varphi(f) \geq 0$, so $|\varphi(f)| \le \varphi(g)\, \|f\|_\infty$. $\square$

Properties of Radon Measures: X LCH, μ Radon.

① If $E \subset X$ is σ-finite, then μ is inner regular on E.

$\Rightarrow$ Every σ-finite Radon meas is regular

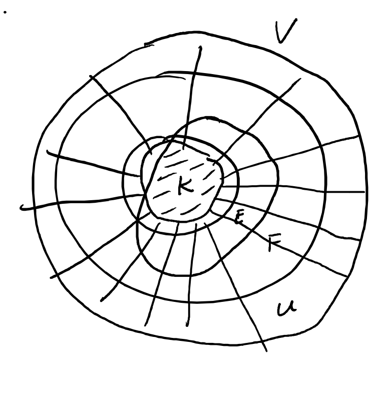
$\Rightarrow$ X σ-cpt $\Rightarrow$

Pf If $\mu(E) < \infty$, then let $\varepsilon > 0$. Pick $U \supset E$ open s.t. $\mu(U) < \mu(E) + \frac{\varepsilon}{2}$.

And pick a cpt $F \subset U$ s.t. $\mu(F) > \mu(U) - \frac{\varepsilon}{2}$.

Then $\mu(U \setminus E) < \frac{\varepsilon}{2}$, $\exists$ open $V \supset U \setminus E$ w/ $\mu(V) < \frac{\varepsilon}{2}$.

Let $K = F \setminus V \subset E$. K is cpt.



$$\begin{aligned}\mu(K) &= \mu(F) - \mu(F \cap V) \\ &> \mu(U) - \tfrac{\varepsilon}{2} - \mu(V) \\ &> \mu(E) - \tfrac{\varepsilon}{2} - \mu(V) \\ &> \mu(E) - \varepsilon.\end{aligned}$$

$\Rightarrow \mu$ inner regular on E.

If $\mu(E)=\infty$, $E=\bigcup E_j$ s.t. $E_j \subset E_{j+1}$ & $\mu(E_j)<\infty$.

So $\forall n \in \mathbb{N}$, $\exists j$ s.t. $\mu(E_j) > N$ and $\exists$ cpt $K \subset E_j \subset E$ s.t. $\mu(K) > N$. So μ inner regular on E. $\square$

② If μ is σ-finite on X and $E \subset X$ is Borel, then $\forall \varepsilon > 0$, $\exists F \subset E \subset \mathcal{U}$ w/ F closed, $\mathcal{U}$ open, & $\mu(\mathcal{U} \setminus F) < \varepsilon$.

③ Suppose X is LCH s.t. every open set is σ-cpt. [Eg X is 2^{nd} ctble]. Then every Borel measure which is finite on cpt sets is regular (and so Radon).

pf: Suppose μ is finite on cpt sets. Then $C_c(X) \subset L^1(\mu)$. So...

Riesz Representation Thm: $\forall$ Radon integral φ on X (LCH), $\exists!$ Radon measure μ_φ on X s.t. $\varphi(f) = \int f \, d\mu_\varphi$ $\forall f \in C_c(X)$. Moreover,

ⓐ $\mu_\varphi(\mathcal{U}) = \sup\{\varphi(f) \mid f \in C_c(X) \text{ w/ } f \prec \mathcal{U}\}$ $\forall \mathcal{U}$ open

$0 \le f \le 1$, $\mathrm{supp}(f) \subseteq \mathcal{U}$

ⓑ $\mu_\varphi(K) = \inf\{\varphi(f) \mid f \in C_c(X) \text{ w/ } f \ge \chi_K\}$ $\forall$ K cpt.

pf of uniqueness If μ Radon s.t. $\varphi(f) = \int f \, d\mu$ $\forall f \in C_c(X)$ and $\mathcal{U} \subset X$ is open, then $\varphi(f) \le \mu(\mathcal{U})$ $\forall f \prec \mathcal{U}$.

If $K \subset \mathcal{U}$ is cpt, by LCH Urysohn, $\exists f \in C_c(X)$ w/ $f \prec \mathcal{U}$ and $f|_K = 1$. So $\mu(K) \le \int f \, d\mu = \varphi(f)$.

Since μ is inner regular on $\mathcal{U}$, (a) is satisfied. So μ is determined by φ on open sets & thus on all Borel sets by outer regularity. $\square$

So μ is determined by ψ on open sets & thus on all Borel sets by outer regularity. $\square$

Back to pf of ③: Let ν be the unique Radon measure on X s.t. $\int f d\nu = \underbrace{\int f d\mu}_{\text{Radon Integral.}}$ $\forall f \in C_c(X)$. Show μ is Radon so $\mu = \nu$.

For $\mathcal{U} \subseteq X$ open, write $\mathcal{U} = \bigcup K_n$ w/ K_n cpt. Inductively find $f_n \in C_c(X)$ s.t. $f_n \prec \mathcal{U}$, $f_n = 1$ on $\bigcup^n K_j$ and 1 on cpt set $\bigcup^{n-1} \overline{\mathrm{supp}(f_j)}$. So $f_n \nearrow \chi_{\mathcal{U}}$ pointwise. So, by MCT,

$$\mu(\mathcal{U}) = \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int f_n d\nu = \nu(\mathcal{U}).$$

If E is Borel & $\varepsilon > 0$, take $\overset{\text{closed}}{F} \subset E \subset \overset{\text{open}}{\mathcal{U}}$ s.t. $\mu(\mathcal{U} \setminus F) = \nu(\mathcal{U} \setminus F) < \varepsilon$. Then

$\underbrace{\mu(\mathcal{U}) - \varepsilon \leq \mu(E)}_{\text{Outer regular}} \leq \cancel{\mu(F) + \varepsilon}$. So $\nu(E) = \mu(E) \Rightarrow \mu = \nu$. $\square$