Compactification

Def An embedding $e: X \longrightarrow Y$ is a cts injection which is a homeomorphism onto its image:

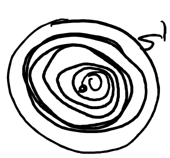
$$e^{-1}: e(X) \longrightarrow X \quad \text{is cts}$$

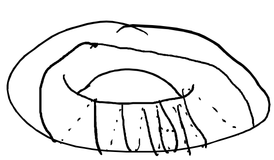
Def a compactification of X is a compact space K and an embedding $e: X \longrightarrow K$ s.t. $e(X)$ is dense in K.

Examples: $\mathbb{R}$ can be compactified:

1. $\overline{\mathbb{R}} = [-\infty, +\infty]$

2. Add one pt, ∞, and get S^1.

3. 

4. 

and X not cpt

If X is LCH, there is a one pt (Alexandroff) compactification.

Choose an object $\infty \notin X$. $X^\bullet = X \cup \{\infty\}$,

$\mathcal{U} \subset X^\bullet$ is open if $\mathcal{U} \subset X$ and $\mathcal{U}$ is open,

or $\infty \in \mathcal{U}$ and $\mathcal{U}^c \subset X$ is compact.

<u>Thm</u> $X^\bullet$ is compact Hausdorff, $X \hookrightarrow X^\bullet$ is an embedding

~~pf~~ cpt: if $\{\mathcal{U}_i\}$ is an open cover, $\exists\, \mathcal{U}_0$ s.t. $\infty \in \mathcal{U}_0$

so $\mathcal{U}_0^c$ is cpt. Thus $\{\mathcal{U}_i\}_{i \neq 0}$ covers $\mathcal{U}_0^c$

which is cpt.

<u>Hausdorff</u>: it suffices to separate ∞ from $x \in X$.

Since X is LCH, $\exists$ open $V \subset X$ s.t. $x \in V$, $\bar{V}$ cpt.

so $\infty \in \bar{V}^c$. □

<u>Def</u> A top sp is called <u>completely regular</u> if $\forall$ closed $F \subset X$

and $x \in F^c$, $\exists$ cts $f: X \to [0,1]$ s.t. $f(x) = 1$ and $f|_F = 0$.

Call X <u>Tychonoff</u> if X is completely regular & T_1.

Facts:

① Tychonoff $\Rightarrow$ Hausdorff

② Normal $\Rightarrow$ Tychonoff by Urysohn/Tietze

③ LCH $\Rightarrow$ Tychonoff

④ Any subspace of a Tychonoff sp is Tychonoff.

Embedding Lemma: Suppose $\Phi \subset C(X, [0,1])$.

Define $e : X \longrightarrow [0,1]^{\Phi}$ (cpt!) by

$$x \longmapsto (f(x))_{f \in \Phi}.$$

① e is cts

② e is injective iff Φ separates points

③ if Φ separates points from closed sets

($\forall F \subset X$ closed, $x \in F^c$, $\exists x \in \Phi$ s.t. $f(x) \notin \overline{f(F)}$),

e is open [when corestricted to $e(X)$].

④ If Φ separates pts And separates pts & closed sets, e is an embedding.

Pf ① Observe $\pi_f \circ e = f$ is cts $\forall f \in \Phi$.

② $e(x) \neq e(y) \Leftrightarrow \exists f \in \Phi$ s.t. $f(x) \neq f(y)$.

③ suppose Φ separates pts from closed sets.

Let $\mathcal{U} \subset X$ be open, $x \in \mathcal{U}$. Want to find open $V \subset [0,1]^{\Phi}$

s.t. $e(x) \in V \cap e(X) \subset e(\mathcal{U})$

Now $\exists f \in \Phi$ s.t. $f(x) \notin \overline{f(\mathcal{U}^c)}$. Then

$W := [0,1] \setminus \overline{f(\mathcal{U}^c)}$ is open and contains $f(x)$.

Thus $e(x) \in \pi_f^{-1}(W)$ open in $[0,1]^{\Phi}$.

Observe $e(y) \in \pi_f^{-1}(W) \cap e(X) \iff f(y) \notin \overline{f(\mathcal{U}^c)}$

$\implies y \in \mathcal{U}$.

so $e(x) \in \overset{V = \pi_f^{-1}(W)}{\overset{\downarrow}{V}} \cap e(X) \subset e(\mathcal{U})$.

④ by ① & ②, $e: X \hookrightarrow [0,1]^{\Phi}$ is a cts injection,

by ③, $e^{-1}: e(X) \to X$ is cts. $\square$

Cor: X is Tychonoff iff $\exists$ embedding $X \hookrightarrow [0,1]^I$.

pf $\Rightarrow$: take $I = \Phi = C(X, [0,1])$ and apply embedding lemma

$\Leftarrow$: $[0,1]^I$ is cpt hausdorff $\Rightarrow$ Normal $\Rightarrow$ tychonoff,

so X is a subspace of tychonoff & so is tychonoff. $\square$

Stone-Čech Compactification: Suppose X is tychonoff.

Let $\Phi = C(X, [0,1])$. Consider the embedding

$e: X \hookrightarrow [0,1]^{\Phi}$ and define $\beta X = \overline{e(X)}$.

Then $(\beta X, e)$ is a compactification of X.

<u>Theorem</u>: The compactification $(\beta X, e)$ satisfies:

① $\forall$ cpt Hausdorff Z and $f : X \longrightarrow Z$ cts, $\exists \tilde{f} : \beta X \longrightarrow Z$ cts s.t. $\tilde{f} \circ e = f$.

$$\begin{tikzcd} \beta X \arrow[dr, dotted, "\exists \tilde{f}"] & \\ X \arrow[u, hook, "e"] \arrow[r, "f"'] & X \end{tikzcd}$$

② The map $\tilde{f}$ in ① is unique.

③ βX is uniquely characterized by univ. prop. ①.

④ β is a functor $\{\text{Tychonoff Sp}\} \longrightarrow \{\text{Cpt Hausdorff sp}\}$.

<u>Prove in order</u>: ②, ③, ④, ①.

② Suppose $e' : X \longrightarrow K$ is a compactification.

And $f : X \longrightarrow Z$ is cts. There exists at most one cts $g : K \longrightarrow Z$ s.t. $g \circ e' = f$.

pf if $g_i \circ e = f$ for $i = 1, 2$, then $g_1 = g_2$ on $\underbrace{e'(X) \subset K. \text{ so } g_1 = g_2}_{\text{do this w/ nets}}$.

dense