2/12

Tuesday, February 12, 2019 11:31

Free modules of finite rank over integral domains.

R^n - standard free module of rank n.

M is a free module of rank n if $M \cong R^n$, isomorphism is defined by a basis in M.

$$\mathrm{Hom}(R^n, R^m) \cong R^{nm}$$

$e_1 = (1, 0, \ldots, 0)$ $\quad$ $e_1' = (1, 0, \ldots, 0)$

$\vdots$ $\quad$ $\vdots$ $\quad$ bases for R^n & R^m

$e_n = (0, \ldots, 0, 1)$ $\quad$ $e_m' = (0, \ldots, 0, 1)$

$$\mathrm{Hom}(R^n, R^m) \cong \bigoplus_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \mathrm{Hom}(R, R), \qquad \mathrm{Hom}(R, R) \cong R$$

$$\varphi \longmapsto \varphi(1)$$

basis in $\mathrm{Hom}(R^n, R^m)$ is $\{\varphi_{ij} : \varphi_{ij}(e_i) = e_j, \ \varphi_{ij}(e_k) = 0 \text{ for } k \neq i\}$

$\forall \varphi \in \mathrm{Hom}(R^n, R^m)$, $\quad \varphi = \sum a_{ij} \varphi_{ij}$

$$\varphi(b_1, \ldots, b_n) = \sum a_{ij} \varphi_{ij}(b_1, \ldots, b_n) = \sum a_{ij} b_i e_j'$$

$$= \left(\sum a_{i1} b_i, \ldots, \sum a_{im} b_i\right).$$

the coordinates a_{ij} of φ ...

the coordinates a_{ij} of φ form an $m \times n$ matrix $\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{pmatrix}$

it's called the matrix of φ, A_{φ}, and

$$\varphi(b) = A_{\varphi} \cdot b$$

$\varphi : R^n \longrightarrow R^m$, $\psi : R^m \longrightarrow R^k$.

$\psi \circ \varphi$. A_{φ} - matrix of φ A_{ψ} - matrix of ψ.

$$A_{\psi \circ \varphi} = A_{\psi} \cdot A_{\varphi}$$

$$\forall_i \quad \begin{pmatrix} a_{i1} \\ \vdots \\ a_{im} \end{pmatrix} = \varphi(e_i)$$

Let M be a free module of rank n.

Let $\{u_1, \ldots, u_n\}$ be a basis in M.

Then $\varphi_1 : R^n \longrightarrow M$, $e_i \longmapsto u_i$ is an isomorphism.

Let $\{v_1, \ldots, v_n\}$ be another basis in M.

then $\varphi_2 : R^n \longrightarrow M$, $e_i \longmapsto v_i$

$$\begin{tikzcd} & R^n \arrow[dl, "\varphi_1"', "\sim"] \arrow[dd, "\varphi_2^{-1}\circ\varphi_1"] \\ M & \\ & R^n \arrow[ul, "\varphi_2", "\sim"'] \end{tikzcd}$$

$\varphi_2^{-1}\circ\varphi_1$ is an isomorphism, the "change of basis" isomorphism

Let $P = A_{\varphi_2^{-1}\circ\varphi_1}$.

$$(\varphi_2^{-1}\circ\varphi_1)(\underset{\substack{\uparrow \\ \text{old coordinates}}}{a_1,\dots,a_n}) = (\underset{\substack{\uparrow \\ \text{new coordinates}}}{b_1,\dots,b_n})$$

So $\underset{\substack{\uparrow \\ \text{transition matrix}}}{P}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

transition matrix for the change of basis

$\{u_1,\dots,u_n\} \longrightarrow \{v_1,\dots,v_n\}$

$$\mathrm{Mat}_{n\times n}(R) \cong \mathrm{End}_R(R^n)$$

A_φ invertible $\Leftrightarrow$ φ invertible.

P is invertible, $P^{-1} = A_{\varphi_1^{-1}\circ\varphi_2}$, the transition matrix

$\{v_1,\dots,v_n\} \longmapsto \{u_1,\dots,u_n\}$.

Let $\varphi : M \longrightarrow N$, bases $\{u_1, \dots, u_n\}$ in M, $\{v_1, \dots, v_m\}$ in N

$$\begin{array}{ccc} M & \longrightarrow & N \\ \wr\wr & & \wr\wr \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

$\Rightarrow A_\varphi$

$\{u_1', \dots, u_n'\}$ in M, $\{v_1', \dots, v_m'\}$ in N

$\Rightarrow A'_\varphi$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\eta^{-1}\circ\varphi\circ\tau"] \arrow[d, "\{u_1,\dots,u_n\} - \tau"'] & \mathbb{R}^m \arrow[d, "\eta - \{v_1,\dots,v_m\}"] \\
M \arrow[r, "\varphi"'] & N
\end{tikzcd}$$

The matrix $A_{\eta^{-1}\circ\varphi\circ\tau}$ is the matrix A_φ wrt the bases $\{u_1, \dots, u_n\}$ & $\{v_1, \dots, v_m\}$

the i^{th} column is the coordinates of $\varphi(u_i)$ in basis $\{v_1, \dots, v_m\}$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A_\varphi"] \arrow[d, "\tau"'] \arrow[dd, bend right=60, "P"'] & \mathbb{R}^m \arrow[d, "\eta"] \arrow[dd, bend left=60, "Q"] \\
M \arrow[r, "\varphi"] & N \\
\mathbb{R}^n \arrow[u, "\tau'"] \arrow[r, "A'_\varphi"'] & \mathbb{R}^m \arrow[u, "\eta'"']
\end{tikzcd}$$

P & Q transition matrices

$$A'_\varphi = Q A_\varphi P^{-1}$$

if $N=M$, $Q=P$ and $A'_\varphi = PA_\varphi P^{-1}$

If $\underset{n\times n}{B} = P\underset{n\times n}{A}P^{-1}$ for an invertible P, A & B are called conjugate.

(square matrices)

if B & A are conjugate, $B = PAP^{-1}$,

Take any basis $\{u_1, \dots, u_n\}$ in $\mathbb{R}^n$.

Define $\{Pu_1, \dots, Pu_n\}$ as a new basis in $\mathbb{R}^n$.

Then the transition matrix $\{u_1, \dots, u_n\} \longmapsto \{Pu_1, \dots, Pu_n\}$

Let φ be the the transfn $\varphi(u) = A(u)$ in basis $\{u_1, \dots, u_n\}$.

then B is the matrix of φ in basis $\{Pu_1, \dots, Pu_n\}$.

$\varphi: V \longrightarrow W$ finite dim. vector spaces

$\Rightarrow$ $V = \operatorname{Ker}\varphi \oplus V_2$, $\varphi\big|_{V_2}$ = isomorphism $V_2 \cong \varphi(V) \subseteq W$.

$\varphi(V) \oplus W_2 = W$

Choose $\{u_1, \dots, u_n\}$ s.t. $\{u_1, \dots, u_k\}$ is basis in V_2.

Then $\{\varphi(u_1), \dots, \varphi(u_k)\}$ is a basis in $\varphi(V)$.

Let $v_{k+1}, \dots, v_m$ be a basis in W_2.

Then in $\{u_1, \dots, u_n\}$, $\{\varphi(u_1), \dots, \varphi(u_k), v_{k+1}, \dots, v_m\}$,

$$A_\varphi = \begin{array}{r} k\left\{ \vphantom{\begin{matrix} 1 \\ \ddots \\ 1 \end{matrix}} \right. \\ \vphantom{0} \\ \vphantom{0} \end{array} \begin{array}{c} \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{k} & 0 \\ \hline 0 & 0 \end{array} \right) \\ \begin{array}{cc} V_2 & \operatorname{Ker}(\varphi) \end{array} \end{array} \begin{array}{l} \varphi(V) \\ W_2 \\ \vphantom{0} \end{array}$$