Lec 9/27

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Topics for quiz: $\mathrm{Aut}_{gp}(W) = ?$ Semidirect products.

Definition: a composition series Σ of G is a descending sequence of normal subgroups:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

(G_2 not necessarily normal in G_0).

e.g. $\mathbb{Z}/4\mathbb{Z} \trianglerighteq \mathbb{Z}/2\mathbb{Z} \trianglerighteq \{0\}$ is a composition series for $\mathbb{Z}/4\mathbb{Z}$.

$$\begin{array}{ccc} \mathbb{Z}/6\mathbb{Z} & \trianglerighteq & \mathbb{Z}/3\mathbb{Z} \\ \triangledown\!\!| & & \triangledown\!\!| \\ \mathbb{Z}/2\mathbb{Z} & \trianglerighteq & \{0\} \end{array}$$

not unique.

Associated graded pieces:

$$G_0/G_1 \; ; \; G_1/G_2 \; ; \; \cdots \; ; \; G_{n-1}/G_n$$

$$gr_i^{\Sigma}(G) = G_i/G_{i+1}$$

in ex.1: 2 copies of $\mathbb{Z}/2\mathbb{Z}$.

in ex 2: $\mathbb{Z}/3\mathbb{Z}$; $\mathbb{Z}/2\mathbb{Z}$ in both orders.

<u>Defn</u> two composition series Σ_1 & Σ_2 are equivalent if, up to permutation, associated graded pieces are the same.

Σ_1 : $G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$

Σ_2 : $H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$

Σ_1 equiv to Σ_2 :

(1) $n = m$

(2) $\exists$ a bijection $\sigma : \{0, 1, \ldots, n-1\} \circlearrowleft$

s.t. $gr_i^{\Sigma_1}(G_0) \cong gr_{\sigma(i)}^{\Sigma_2}(H_0) \quad \forall\, i.$

eg. $(\mathbb{Z}/2\mathbb{Z})^2 \trianglerighteq \mathbb{Z}/2\mathbb{Z} \trianglerighteq \{0\}$

and $\mathbb{Z}/4\mathbb{Z} \trianglerighteq \mathbb{Z}/2\mathbb{Z} \trianglerighteq \{0\}$

are equiv.

More examples:

can be refined if n is not prime

$$D_{2n} \trianglerighteq \underset{\mathbb{Z}/n\mathbb{Z}}{\underset{\cong}{\langle r \rangle}} \trianglerighteq \{0\}$$

$S_n \trianglerighteq A_n \trianglerighteq \{e\}$ A_n is simple if $n \geq 5$ so we can't keep going.

<u>Defn</u> if Σ and Σ' are two composition series of the same group, we say Σ' is finer than Σ if

Σ can be obtained from Σ' by omitting a few terms.

2 elements of order 4.

Example: $G = D_8$ composition series

$D_8 \trianglerighteq \mathbb{Z}/4 \trianglerighteq \mathbb{Z}/2 \trianglerighteq \{e\}$

So $D_8 \not\cong Q$

6 elements of order 4.

Defn $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. $i^2 = j^2 = k^2 = -1$

quaternion gr

$ij = k, \quad jk = i, \quad ki = j.$

$$\begin{array}{ccccccc} Q_8 & \trianglerighteq & \mathbb{Z}/4 & \trianglerighteq & \mathbb{Z}/2 & \trianglerighteq & \{0\} \\ & & \| & & \| & & \\ & & \{\pm 1, \pm i\} & & \{\pm 1\} & & \end{array}$$

Same graded pieces as that of D_8.

Defn a composition series Σ is strict if all its graded pieces are nontrivial.

$\Sigma: G = G_0 \ntrianglerighteq G_1 \ntrianglerighteq \cdots \ntrianglerighteq G_n = \{e\}$

Jordan-Hölder: a maximal among all strict composition series

Σ is Jordan-Hölder iff $\forall\, \Sigma'$ finer than Σ, Σ' contains repeats (of the things in Σ).

Non-example: $\mathbb{Z} \triangleright 2\mathbb{Z} \triangleright 2^2\mathbb{Z} \triangleright \cdots \triangleright 2^N\mathbb{Z} \triangleright \{0\}$

no JH-series by)

no JH-series by

We will prove: if G admits a Jordan-Hölder series, then any strict composition series can be refined to a Jordan-Hölder series

$G \rightsquigarrow \ell(G) \in \mathbb{N}$.

length of group

For Finite groups:

(i) J-H series exists

(ii) Any two J-H series are equivalent. (in particular, same length)

Proof of (i): Let $G (= G_0)$ be a finite group.

find largest proper $(\neq G)$ normal subgp.

If there aren't any, G is simple so

J-H series is $G \triangleright \{e\}$.

Otherwise, $G_0 \underset{\neq}{\triangleright} G_1 \underset{\neq}{\triangleright} \{e\}$

G_0/G_1 is simple since

$\{\text{normal subgps in } G/N\} \sim \{\text{normal subgps in } G \text{ containing } N\}$

And proceed inductively since $|G_1| < |G_0|$. $\square$

Nice thing: Σ is J-H iff G_i/G_{i+1} is simple $\forall i$.

($\{e\}$ is not simple).

length 1: Simple groups.

once we prove uniqueness, we will have:

Fix a simple gp S. define $[G:S]$ = # of times S appears as associated graded piece

multiplicity of S in G

= # { . . . }

multiplicity of S in G $= \#\{i \mid G_i/G_{i+1} \cong S\}$

as associated graded piece

(invariant of J-H Σ).

How to prove uniqueness?

Given two composition series Σ_1 & Σ_2 of a group G

We can refine Σ_1 using Σ_2 & vice versa

$$\begin{array}{ccc} \Sigma_1 & & \Sigma_2 \\ \rightsquigarrow & & \rightsquigarrow \\ \Sigma_1' & \sim & \Sigma_2' \end{array}$$

Replace G_i in Σ_1 by $(K_0 \cap G_i)H_{i+1} \trianglerighteq (K_1 \cap G_i)H_{i+1} \trianglerighteq \dots \trianglerighteq (K_m \cap G_i)H_{i+1}$