# Lec 10/30

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Ex: $R = \mathbb{Z}/6\mathbb{Z} \supset S = \{1, 2, 4\}$

$$S^{-1}R = \{0, \tfrac{1}{2}, \tfrac{1}{4}\}$$

$\frac{1}{4} = 1$

Basic operations w/ rings:

$$R_1 \times R_2 \xrightarrow{\pi_1} R_1, \quad R_1 \times R_2 \xrightarrow{\pi_2} R_2$$

$$R \xrightarrow{i} R[x],\ R[\![x]\!]$$

$R \xrightarrow{\pi} R/I \qquad I \subsetneq R$ proper ideal

$R \xrightarrow{j} S^{-1}R \qquad S \subset R$ mult. closed $(1 \in S,\ 0 \notin S,\ a, b \in S \Rightarrow ab \in S)$.

$\underset{\text{ideal}}{I} \subset R \rightsquigarrow S^{-1}I$ ideal $\subseteq S^{-1}R$

$\underset{\text{ideal}}{\tilde{I}} \subset S^{-1}R \rightsquigarrow j^{-1}(\tilde{I}) \subseteq R$ is an ideal

$\forall\, \tilde{I} \subset S^{-1}R,\quad S^{-1}(j^{-1}(\tilde{I})) = \tilde{I}$

but <u>not</u> the other way around.

$$j^{-1}(S^{-1}I) \supseteq I$$

not always $=$.

$j^{-1}(S^{-1}I) = I$ if $I \subseteq R$ is prime & $I \cap S = \emptyset$.

# Noetherian Rings

## Applications of Localization

$R$ : comm ring

$\mathcal{N}(R) = \{a \in R : a^n = 0 \text{ for some } n \geq 1\}$ is an ideal & is proper $(1 \notin \mathcal{N}(R))$.

If $P \subset R$ is any prime ideal & $a \in \mathcal{N}$ then $a^n \in P$ so $a \in P$.

$\Rightarrow \mathcal{N} \subset P \ \forall$ prime $P \subset R$.

$\Rightarrow \mathcal{N} \subset \bigcap\limits_{P \text{: prime ideal}} P$

**Lemma** $a \in P \ \forall$ prime $P \subset R \Longrightarrow a$ is nilpotent.

Pf assume $a \notin \mathcal{N}$. Then $\{1, a, a^2, \dots\} = S$ is multiplicatively closed.

$R \xrightarrow{j} S^{-1}R \ni \tilde{P}$ any prime ideal. $j^{-1}(\tilde{P})$ is a prime ideal.

$a \notin P$ since $P \cap S = \emptyset$. $\exists$ since every Ring has a max'l ideal & max'l is prime.

So $\mathcal{N} = \bigcap\limits_{\substack{P \subsetneq R \\ \text{prime}}} P$

II $R$ : comm ring

Jacobson Radical $\rightarrow J = \bigcap\limits_{\substack{M \subsetneq R \\ \text{is max'l}}} M \subsetneq R$ is a proper ideal.

$$J = \{ a \in R : 1 - ax \in R^\times \ \forall x \in R \} =: K$$

Since every Max'l ideal is prime, $J \supset \mathcal{N}$. $\mathcal{N} = J \cap \bigcap_{\substack{P \subsetneq R \\ \text{prime} \\ \text{not max'l}}} P$.

Proof for Local ring $(R, M)$:

In this case, $J = M$. To show: $1 - M \subset R^\times (= R \setminus M)$

If $1-a$ is not a unit then $a, 1-a \in M \implies 1 \in M$ $\times$

In general: $R$ : comm ring.

Assume $a \in R$ s.t. $1 - ax \notin R^\times$ for some $x \in R$.

then we will find a max'l ideal $M \subsetneq R$ s.t. $a \notin M$.

$b = 1 - ax \notin R^\times \iff (b) \subsetneq R$ is proper.

Every proper ideal is contained in some max'l ideal $M$.

but $a \notin M$ since $1 - ax \in M$ and $M$ is proper.

$J \subset K$

Suppose $a \in R$ and $a \notin M$ for some max'l ideal.

We'll show $\exists x \in R$ s.t. $1 - ax$ is not a unit.

"Localize at $M$" means take $S = R \setminus M$

$$R \xrightarrow{j} S^{-1}R$$

if $1 - ax$ were a unit $\forall x \in R$,

then $1 - ax'$ and $\frac{a}{1}$ are both units $\forall x' \in S^{-1}R$.

$\implies$ for $x' = \frac{1}{a}$, $1 - 1$ is a unit $\times$.

$J \supset K$