Lec 10/9

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Jordan Decomposition Theorem

Note: If $AB = BA$, $(A+B)^2 = A^2 + 2AB + B^2$

so $(A+B)^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i}$ as usual.

So N, N_1 both nilpotent $\overset{\text{w/ } r, r_1}{\Rightarrow}$ $(N - N_1)^{r+r_1} = \sum_{k=0}^{r+r_1} \binom{r+r_1}{k} N^k (-N_1)^{r+r_1-k} = 0$

So their difference is Nilpotent

$\underbrace{D - D_1}_{\text{diagonalizable}} = \underbrace{N - N_1}_{\text{Nilpotent}}$ $\quad$ 0 is only nilpotent diagonal matrix.

$(\mu_i - \lambda_i)^{r+r_1} = 0 \quad \forall i$ so $\mu_i - \lambda_i = 0$.

So JD is unique. (Note that $DN = ND$ is a requirement).

$(\text{since } m(x) \mid x^r \text{ and } r \leq n)$.

$T \sim \underbrace{A}_{n \times n}$, $A = D + N$, $N^r = 0$, $ND = DN$, $r \leq n$ $\quad$ What is $A^{1000000} = D^{1000000} + \sum_{i=1}^{r-1} \binom{1000000}{i} D^{1000000-i} N^i$

And D^m is easy to compute.

$T \in L(V, V)$, $h(x) = \det(xI - T)$

$h(x) = \det(xI - A)$ where A is triangular form.

So $h(x) = \prod_{i=1}^{n} (x - \lambda_i)$ where $\lambda_i = D_{ii}$. (not unique necessarily)

$$h(A) = 0 \quad \text{so} \quad m(x) \mid h(x)$$

$$\underbrace{h(A) = (A - \lambda_1 I)^{d_1} (A - \lambda_2 I)^{d_2} \cdots (A - \lambda_r I)^{d_r} = 0}$$

$V = V_1 \oplus \cdots \oplus V_r$ since $v = v_1 + \cdots + v_r$ and $(A - \lambda_i I)^{d_r}$ can commute to the front.

so $m \mid h$ so $\deg m \le \deg h$.

$h(T) = 0$ (Cayley - Hamilton)

Subbing T for X in $\det(XI - T)$ doesn't prove it.

Problems

$F \subseteq \mathbb{C}$ (subfield.)

(4) $T \in L(V,V)$, all eigenvalues $= 0$. Show T nilpotent.

$$m(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r} = x^e \Rightarrow T^e = 0.$$

(5) $T^2 = T$. is there a basis of eigenvectors for T?

$m \mid x^2 - x = x(x-1)$

1) $m = x \Rightarrow T = 0$

2) $m = x - 1 \Rightarrow T = 1$

3) $m = x(x-1) \Rightarrow V = V_1 \oplus V_2$, $V_1 = n(T)$, $V_2 = \underbrace{n(T - I)}_{\text{Range}(T) = T(V)}$.

v_1: $T(v_1) = 0$; v_2: $T(v_2) = v_2$

So $\exists$ basis of eigenvectors.

⑥ $T^r = I$ is there a basis of eigenvectors

$$m \mid x^r - 1 = (x-1)(x^{r-1} + x^{r-2} + \cdots + x + 1)$$

$$x^3 - 1 = (x-1)\,(x^2 + x + 1)$$

$$\frac{1}{(n+a-1)(n+1)}$$

Diagonalizabe over $\mathbb{C}$, eigenvalues are roots of unity.