Lec 9/4

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Quotient groups:

$K \leq G$, G/K is the set of cosets.

this is a gp if $K \trianglelefteq G$:

$(*)$ $CC' = C \cdot C' = \{ab \mid a \in C,\ b \in C'\}$

Since K is normal, $(*)$ is a binary op:

Say $C = xK$, $C' = x'K$.

Then $CC' = (xK)(x'K) = x(Kx')K = x(x'K)K$

$= xx'K$.

The coset $1K = K$ is identity in G/K.

$C = xK$, $C^{-1} = (xK)^{-1} = K^{-1}x^{-1} = x^{-1}K$.

We denote xK by $\bar{x}$.

So $(G/K, \cdot, \bar{1})$ is called the quotient gp of G by K.

$\bar{x}\bar{y} = \overline{xy}$, $\bar{x} = \bar{y} \iff xy^{-1} \in K$.

Prop $K \trianglelefteq G$ iff $K = \underbrace{\ker\varphi}_{}$ for some homomorphism φ on G. $= \{g \in G \mid \varphi(g) = 1\}$.

Note: if $\varphi : G \xrightarrow{\text{hom}} G'$ then $\varphi(1) = 1'$, $\varphi(g^{-1}) = \varphi(g)^{-1}$.

if $K = \ker\varphi$, then $g^{-1}Kg \subset K$ since $\varphi(g^{-1}kg) = \varphi(g)^{-1}\varphi(g) = 1$ (hence $K \trianglelefteq G$).

if K is normal, the projection

$$\pi : G \longmapsto G/K \qquad \text{has kernel } K.$$
$$x \longmapsto xK$$

Thm Let $\varphi : G \to G'$ be a homomorphism, Then

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\pi"', "\text{(surjective)}"'] & & G' \\ & G/\ker\varphi \arrow[ur, "\overline{\varphi}"', "\text{(injective)}"] & \end{tikzcd}$$

commutes, where $\overline{\varphi}(x\ker\varphi) = \varphi(x)$.

The map $\overline{\varphi}$ is "induced by φ on $G/\ker\varphi$".

i.e $\varphi = \pi \cdot \varphi$.

Thm If $\varphi: G \longrightarrow G'$ is a homomorphism

then $\overline{\varphi}$ is an isomorphism

$$G/_{\operatorname{Ker}\varphi} \cong_{\overline{\varphi}} \operatorname{Im}\varphi \; (:= \varphi(G))$$

(this is the 1st isomorphism thm).

Thm Let $K \trianglelefteq G$, $H \leq G$, suppose $K \leq H$.

Then $H/_K \leq G/_K$, and the map

$$H \longleftrightarrow \underset{\overline{H}}{\overset{\|}{H/K}} \quad \text{is a bijection}$$

$$\left\{ \begin{matrix} H \leq G \\ H \text{ contains } K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{subgps of} \\ G/K \end{matrix} \right\},$$

moreover $H \trianglelefteq G$ iff $H/_K \trianglelefteq G/_K$.

union of cosets in $\overline{H}$.

pf For $\overline{H} \leq \overline{G}$, $\overline{H}$ is the image of $\underbrace{HK}_{\text{will prove this is a subgp soon}} \leq G$ under π

Let $H_1, H_2 \leq G$ containing K,

If $\overline{H_1} = \overline{H_2}$, then

$\forall\, h_1 \in H_1$, $h_1 K = h_2 K$ for some $h_2 \in H_2$.

So $h_2^{-1} h_1 \in K$, so $h_2^{-1} h_1 = k$ for some $k \in K$.

So $h_1 = h_2 k$, so $h_1 \in H_2$. So $H_1 \subset H_2$.

Similarly, $H_2 \subset H_1$. So $H_1 = H_2$.

other part is an exercise. $\square$

Thm let H, K be normal subgps of G, with $K \leq H$. then $G/H \cong \underbrace{G/K}_{\overline{G}} / \underbrace{H/K}_{\overline{H}}$.

proof Consider the composition $\varphi\pi : G \longrightarrow \overline{G}/\overline{H}$ of $\pi : G \longrightarrow \overline{G}$ and

$\varphi : \overline{G} \longrightarrow \overline{G}/\overline{H}$, $\quad \varphi : \overline{g} \longmapsto \overline{g}\,\overline{H}$.

Note $\varphi\pi$ is surjective.

$$\text{Ker}(\varphi\pi) = \{g \in G \mid \overline{g} \in \overline{H}\} = \{g \in G \mid gK = hK \text{ for some } h \in H\}$$

$$= \{g \in G \mid g^{-1}h \in K \subset H\} = \{g \in G \mid g \in H\} = H.$$

So use 1^{st} isomorphism thm. $\square$

Thm let $H \leq G$ and $K \trianglelefteq G$. Then

$\underbrace{HK}_{\{hk \,:\, h \in H,\, k \in K\}} \leq G$, $H \cap K \trianglelefteq H$, and

$$HK/K \cong H/(H \cap K).$$

$\bigcup_{h \in H} hK \qquad \bigcup_{h \in H} Kh$

proof Since $K \trianglelefteq G$, $HK = KH$, and so $(HK)^{-1} = K^{-1}H^{-1} = KH = HK$.
Also, $(HK)^2 = H^2K^2 = HK$ so HK is closed under multiplication & inverses. So $HK \leq G$.

For the last part, consider the restriction hom. $\pi' = \pi\big|_H$ where $\pi : G \to G/K$ is natural proj.

Then $\operatorname{im}\pi' = HK/K$ and $\operatorname{Ker}\pi' = \operatorname{Ker}\pi \cap H = K \cap H$.

apply 1st isomorphism thm. □

Def a group G is simple means only normal subgroups of G are G & 1.

eg if G is simple & $\varphi : G \to G'$ then φ is injective or trivial.

Counting lemma: If $H, K \leq G$, H, K finite, then

$$\frac{|HK|}{|K|} = \frac{|H|}{|H \cap K|}.$$

proof $HK = \bigcup_{h \in H} hK$. Now $h_1K = h_2K$ iff $h_2^{-1}h_1 \in K$

iff $h_2^{-1}h_1 \in H \cap K$.

iff $h_1(H \cap K) = h_2(H \cap K)$.

So the $\overbrace{\text{\# of distinct cosets } hK}^{\frac{|HK|}{|K|}}$ as $h \in H$ is equal to the $\underbrace{\text{\# of distinct cosets } h(H \cap K) \text{ as } h \in H}_{\frac{|H|}{|H \cap K|}}$. $\square$