$M = R^n \implies M^{**} = M$ even if R is not an ID.

$M = R^n$. $\quad \mathrm{Bil}(M, R) \cong M^* \otimes M^*$ $\quad \beta \leftrightarrow \omega$

$\cong$

$(M \otimes M)^*$

universal property of Tensor Product

β is symmetric means $\beta(u,v) = \beta(v,u)$ $\forall u, v \in M$.

$\omega = \sum a_{ij} f_i \otimes f_j$

$\omega(u,v) = \sum a_{ij} f_i(u) f_j(v)$

If β is symmetric, then for corr-ing ω,

$\sum a_{ij} f_i(u) f_j(v) = \sum a_{ij} f_i(v) f_j(u) \quad \forall u, v.$

So pick $u = u_i$, $v = u_j$

So $a_{ij} = a_{ji}$ so ω is symmetric.

Alternatively, let $\tau : u \otimes v \longmapsto v \otimes u$

$f \otimes g \longmapsto g \otimes f$

then $\omega(\tau(\alpha)) = \tau(\omega)(\alpha)$ where $\alpha \in M \otimes M$
$\omega \in M^* \otimes M^*$

ω is a symmetric bilinear form if $\omega(\tau(\alpha)) = \omega(\alpha)$
for all $\alpha \in M \otimes M$ $(\beta(u,v) = \beta(v,u)\ \forall u,v \in M)$.

ω is a symmetric tensor if $\tau(\omega) = \omega$.

since $\omega(\tau(\alpha)) = \tau(\omega)(\alpha)$.

12: $M, N \cong R^n$, $\varphi : M \to N$.

if φ is inj is φ surj?

if φ is surj is φ inj?

(a) false $\mathbb{Z} \longrightarrow \mathbb{Z}$ $n \longmapsto 2n$. (inj. but not surj).

(b) true because one is true.

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$
$$\quad\quad 0 \quad\ n \quad\ n$$

$(K = \ker(\varphi)$ and rank is additive$)$

rank K = rank M - rank N = 0 but K is free and torsion so $K=0$ so φ is injective.

(4) $\{u_1, u_2\}$, $\{f_1, f_2\}$ dual basis. $f = 4f_1 + 5f_2$.

$v_1 = 2u_1$, $v_2 = 3u_2$. (assuming 2, 3 are units).

$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$ - transition matrix $\{u_1, u_2\} \longrightarrow \{v_1, v_2\}$.

$\{g_1, g_2\}$ dual of $\{v_1, v_2\}$.

$f(v_1) = f(2u_1) = 8$, $f(v_2) = f(3u_2) = 15$

so $f = 8g_1 + 15g_2$.

$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ - transition matrix $\{f_1, f_2\} \longrightarrow \{g_1, g_2\}$.

P: transition matrix in M,

then transition matrix in M^* is $(P^{-1})^*$

$$M \xrightarrow{\varphi_1} R^n \text{ - old}, \quad R^n \xrightarrow{\psi} \quad\quad M^* \xleftarrow{\varphi_1^*} R^n \text{ - old}, \quad R^n \xrightarrow{\tilde{\psi}} \quad\quad \begin{array}{ccc} M & \longrightarrow & R^n \\ & & \downarrow \\ (R^n)^* & \longrightarrow & M^* \end{array}$$

$$\begin{tikzcd} M \arrow[dr, "\varphi_2"'] & \\ & \mathbb{R}^n\text{-new} \end{tikzcd} \quad \downarrow \psi \qquad \begin{tikzcd} M^* & \\ & \mathbb{R}^n\text{-new} \arrow[ul, "\varphi_2^*"'] \end{tikzcd} \quad \downarrow \tilde{\psi}$$

$$\tilde{\psi} = (\varphi_2^*)^{-1} \circ \varphi_1^* = (\psi^{-1})^*$$

$$\psi = \varphi_2 \circ \varphi_1^{-1}$$

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③ M, N, K free of finite rank.

$$\varphi : M \to N \quad \overset{\text{tensor from}}{\longleftrightarrow} \quad N \otimes M^*$$

$$\psi : N \to K \quad \longleftrightarrow \quad K \otimes N^*$$

$$\psi \circ \varphi : M \to K \quad \longleftrightarrow \quad K \otimes N^*$$

$$(v \otimes f)(u) = f(u)\, v$$

$v \otimes f \in N \otimes M^*$

$w \otimes g \in K \otimes N^*$

$$u \in M, \quad \big((w \otimes g) \circ (v \otimes f)\big)(u)$$
$$= (w \otimes g)(f(u) v)$$
$$= g(f(u)\, v)\, w$$
$$= f(u)\, g(v)\, w = g(v)\,(f(u)\, w)$$

and $g(v) \xleftarrow{\text{contract}} g \otimes v$ so Statement is true for simple tensors.

$$(\mathrm{Hom}(M,N),\ \mathrm{Hom}(N,K)) \longrightarrow \mathrm{Hom}(M,K)$$

$$\rightsquigarrow$$

$$\begin{array}{ccc} \mathrm{Hom}(M,N) \otimes \mathrm{Hom}(N,K) & \longrightarrow & \mathrm{Hom}(M,K) \\ \cong \quad\quad \cong & & \cong \\ N\otimes M^* \otimes K\otimes N^* & & K\otimes M^* \\ \cong & & \\ K\otimes N^* \otimes N \otimes M^* & \nearrow & \end{array}$$

contraction on simple tensors so contraction everywhere.

$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$ Solve for x!

if $n=2$, quadratic eqn

$x^2 = 2$. $x=\sqrt{2}$ — number α s.t. $\alpha^2 = 2$.

$x^3 + ax^2 + bx + c = 0 \implies x^3 + px + q = 0$

$x \mapsto x + a$

1486 – Valmes : cubic $\Rightarrow$ quartic

Cardano /
ferrari formula: roots of cubic are:

$$\underbrace{\sqrt[3]{\frac{p}{q} + \sqrt{\frac{q^2}{4} + \frac{p^2}{27}}}}_{\alpha} + \underbrace{\sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}_{\beta} \qquad \text{s.t. } \alpha\beta = -p/3$$

Can we doit for quintic?

1802–1829 $\longrightarrow$ Abel found a polynomial in degree 5 for which there is no "solution in radicals."

1811 – 1832 $\longrightarrow$ Evariste Galois