Lec 9/1

Thursday, September 1, 2016 9:06 AM

Definition: $S \subseteq \mathbb{R}$ is dense if any open interval contains an element of S.

Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$

Proof: Let (a, b) be an open interval. Want to find $r \in (a,b)$, $r \in \mathbb{Q} \Leftrightarrow a<r<b$

1) it suffices to prove this for $0 \le a < b$

   if $a<0$ and $b>0$ then $0 \in (a, b)$

   if $b < a \le 0$ then $0 \le -a < b$

   so $r \in (-a, -b) \Rightarrow -r \in (b, a)$

2) find a positive integer q s.t. $\frac{1}{q} < b-a$

   We can do this because P13 implies no infinitesimals.

3) Use the well ordering principle for $\mathbb{N}$. Let p be the least element of $T = \{n \in \mathbb{N} : \frac{n}{q} \geq b\}$

   First need to verify that $T \neq \emptyset$

   $T = \{n \in \mathbb{N} : n \geq bq\}$.

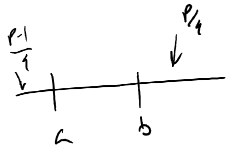
   so $T \neq \emptyset$ otherwise bq would be pseudo-infinite which would contradict P13

4) $\frac{p-1}{q} \in (a, b)$ i.e. $a < \frac{p-1}{q} < b$

These are not true:

$$\begin{cases} \frac{p-1}{q} \geq b \Rightarrow p-1 \in T & \text{(contradiction, p should be least elem. of T, so } \frac{p-1}{q} < b\text{)} \\ \frac{p-1}{q} \leq a < b \leq \frac{p}{q} \Rightarrow \frac{p}{q} - \frac{p-1}{q} \geq b-a & \text{(contradiction of Part 2)} \end{cases}$$

$\blacksquare$



Theorem: the irrational numbers are dense in $\mathbb{R}$

Proof: Suppose that (a, b) is an open interval. Want to find an irrational in (a,b) by previous theorem, we can find a rational $c \in (a,b)$ and another rational $d \in (c,b)$. so $(c,d) \subseteq (a,b)$

find a rational $r \in \left(\frac{c}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)$ so $\frac{c}{\sqrt{2}} < r < \frac{d}{\sqrt{2}}$ so $c < r\sqrt{2} < d$

so $r\sqrt{2}$ irrational and $r\sqrt{2} \in (a,b)$

$\downarrow$

if $r\sqrt{2}$ was rat'l, then $\sqrt{2} = \frac{\overset{\text{rat'l}}{r\sqrt{2}}}{\underset{\text{rat'l}}{r}} \in \mathbb{Q}$

* Could have just used a,b instead of c,d for this proof.

Last problem on HW 2:

$\mathbb{L} = \{a + b\sqrt{2} : a,b \in \mathbb{Q}\}$

Theorem: $\mathbb{L}$ is a field

Proof 1: Assuming we know $\mathbb{R}$ is a field (and it exists)

$\mathbb{L} \subseteq \mathbb{R}$. $\mathbb{R}$ has $+$ and $\cdot$, so apply those to $\mathbb{L}$

To show that $\mathbb{L}$ has $+$ and $\cdot$, we need to show that sums & products of elements of $\mathbb{L}$ are in $\mathbb{L}$.

$$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \underbrace{(a_1 + a_2)} + \underbrace{(b_1 + b_2)}\sqrt{2} \in \mathbb{L}$$

sum of rat'ls is rat'l.

$$(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + b_1b_2(\sqrt{2})^2$$

$$= \underbrace{(a_1a_2 + 2b_1b_2)} + \underbrace{(a_1b_2 + a_2b_1)}\sqrt{2} \in \mathbb{L}$$

product of rat'ls is rat'l.

$0 = 0 + 0\sqrt{2} \in \mathbb{L}$

$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} \in \mathbb{L}$

$1 = 1 + 0\sqrt{2} \in \mathbb{L}$

hence $\mathbb{L}$ satisfies P1-P4, P5, P6, P8, P9

To verify P7 (existence of inverses), we need to show that $a + b\sqrt{2} \neq 0 + 0\sqrt{2} \Rightarrow (a + b\sqrt{2})^{-1} \in \mathbb{L}$

$$\frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - b^2\sqrt{2}^2} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$$

We need to check that $a + b\sqrt{2} \neq 0 \Rightarrow a^2 - 2b^2 \neq 0$

assume $a^2 - 2b^2 = 0$ i.e. $a^2 = 2b^2$ so if $b \neq 0$ then $\left(\frac{a^2}{b^2}\right) = \left(\frac{a}{b}\right)^2 = 2$

$a+b\sqrt{2} \neq 0 \Rightarrow a^2-2b^2 \neq 0$

assume $a^2-2b^2=0$ ie $a^2=2b^2$ so if $b\neq 0$ Then $\left(\frac{a^2}{b^2}\right)=\left(\frac{a}{b}\right)^2=2$

but there are no rational solutions to that, so $a^2-2b^2\neq 0$

but if $b^2=0$ then $a^2=0$ so no inverse.

Proof 2: don't assume $\mathbb{R}$ exists.

We assume $\mathbb{Q}$ exists Add imaginary element, j, to $\mathbb{Q}$ satisfying $j^2=2$

Let $\mathbb{L}=\{a+bj : a,b\in\mathbb{Q}\}$

$(a_1+b_1j)+(a_2+b_2j)=(a_1+a_2)+(b_1+b_2)j \in \mathbb{L}$

$(a_1+b_1j)(a_2+b_2j) = a_1a_2+a_1b_2j+b_1a_2j+b_1b_2j^2 = (a_1a_2+2b_1b_2)+(b_1a_2+b_2a_1)j \in \mathbb{L}$

$$\frac{1}{a+bj}=\frac{1}{a+bj}\,\frac{a-bj}{a-bj}=\frac{a-bj}{a^2-2b^2}=\frac{a}{a^2-2b^2}-\frac{b}{a^2-2b^2}j$$

$a^2-2b^2=0 \Leftrightarrow a=0=b$

note:

$\mathbb{Q}\subseteq\mathbb{L}$

$a\in\mathbb{Q} = (a,0)\in\mathbb{L}$

Formal definition of $\mathbb{L}$

$$\mathbb{L}=\{\underset{a+bj}{\underset{\parallel}{(a,b)}} : a,b\in\mathbb{Q}\}$$

So $(a_1,b_1)+(a_2,b_2)=(a_1+a_2,\ b_1+b_2)$

$(a_1,b_1)\cdot(a_2,b_2)=(a_1a_2+2b_1b_2,\ a_1b_2+a_2b_1)$

$(a,b)^{-1}=\left(\frac{a}{a^2-2b^2},\ \frac{b}{a^2-2b^2}\right)$