Lec 9/9

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Group Actions 1

eg $G \circlearrowright$ space of right cosets of H: $g \cdot (Hx) = Hxg^{-1}$.

Q: what is the kernel?

def Two actions of G on S & S' are equivalent if there is a bijection $\alpha : S \to S'$ s.t. $\alpha(gx) = g\alpha(x)$.

So if T, T' are the associated homomorphisms of the actions, then $\alpha T(g) = T'(g)\alpha \quad \forall g \in G$.

So
$$\begin{tikzcd} S \arrow[r, "T(g)"] \arrow[d, "\alpha"'] & S \arrow[d, "\alpha"] \\ S' \arrow[r, "T'(g)"'] & S' \end{tikzcd}$$
commutes.

eg If $\underbrace{S = G}_{\text{left-mult}}$ and $\underbrace{S' = G}_{\text{right-mult}}$

Then $\alpha : S \to S'$, $x \mapsto x^{-1}$ is an equivalence of actions:

$$\alpha(g \overset{}{\underset{\text{left}\,\cdot}{\cdot}} x) = x^{-1} g^{-1} = g \overset{\text{right}\,\cdot}{\cdot} \alpha(x)$$

Group Actions 2

$G \circlearrowright S$. take $x, y \in S$. we write $x \sim_G y$ to mean $y = gx$ for some $g \in G$.

The G-orbit of x, $G \cdot x = \{gx \mid g \in G\}$, is an eq. class under $\sim_G$.

G acts transitively if there is one orbit.

eg let $G \circlearrowright G$ by conjugation: $\overset{g_x}{\overbrace{g \cdot x}} = gxg^{-1}$

orbits are called "conjugacy classes"

Theorem Let $G \circlearrowright S$ transitively. For $x \in S$, let $H = \mathrm{Stab}\, x = \{g \in G \mid gx = x\}$. Then $G \circlearrowright S$ is equivalent to $G \circlearrowright G/H$ by left multiplication.

notes: $\mathrm{Stab}\, x \leq G$. $\mathrm{Stab}\, gx = g \cdot \mathrm{Stab}\, x \cdot g^{-1}$

e.g. $\mathrm{Stab}\, x$ under $G \circlearrowright G$ by conjugation is $C(x)$.

Q: What is $\bigcap_{x \in S} \mathrm{Stab}\, x = ?$ A: Kernel of action.

proof Fix $x \in S$. Since g acts trans, $Gx = S$.

Let $\bar{g} = \{a \in G \mid ax = gx\} = \{a \in G \mid g^{-1}ax = x\}$

$= \{a \in G \mid g^{-1}a \in \mathrm{Stab}\, x\}$

$= g \cdot \mathrm{Stab}\, x$

So $G = \bigcup_{g \in G} \bar{g} = \bigcup_{g \in G} g \cdot \mathrm{Stab}\, x \qquad (*)$

We claim $\alpha : G/_{\operatorname{Stab} x} \longrightarrow S$, $g \cdot \operatorname{Stab} x \longrightarrow gx$

is an equivalence of actions.

by $(*)$, α is surjective (by transitivity)

It's clear that α is injective by definition of $\bar{g}$.

So α is a bijection. Also

$$\alpha(g \cdot \bar{g'}) = \alpha(\overline{gg'}) = gg'x = g(g'x) = g \cdot \alpha(\bar{g'}) \quad \forall g, g' \in G. \quad \square$$

<u>Corollary</u> If finite G acts on S transitively, then

$$\underbrace{|S| = [G : \operatorname{Stab} x]}_{\text{orbit-stabilizer formula}} = \frac{|G|}{|\operatorname{Stab} x|} \qquad \text{for all } x \in S.$$

<u>Corollary</u> If finite $G \circlearrowright S$, then $|G \cdot x| = \dfrac{|G|}{|\operatorname{Stab} x|}$ $\forall x$,

so $$|S| = \sum_{\text{orbit } O} \frac{|G|}{|\operatorname{Stab} x_0|} \quad \text{where } x_0 \in O \ \forall \text{ orbit } O$$

$$= \sum_{\text{orbit } O} [G : \operatorname{Stab} x_0]$$

<u>proof</u> $\forall$ orbit O, $G \circlearrowright O$ transitively,

and $S = \coprod_{\text{orbit } O} O$.

Class Equation for Finite Groups

Thm Let G be finite. Then

$$|G| = |Z(G)| + \sum_{y_i} [G : C(y_i)]$$

$\uparrow$ y_i runs over conjugacy classes with more than one element

$\nearrow p^r$

Thm If $G \neq 1$ and $|G|$ is a prime power, then $Z(G) \neq 1$.

pf
$$|Z(G)| = |G| - \sum_{y_i} [G : C(y_i)].$$

$\nearrow$ y_i doesn't include trivial conjugacy classes.

everything in RHS is div. by P, so $|Z(G)|$ is too.

Burnside's Formula Let finite G act on S. Suppose the G-action has r orbits. Then

$$r = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}_g(S)|$$

where
$$\underbrace{\mathrm{Fix}_g(S) = \{ x \in S \mid gx = x \}}.$$

$$\bigcap_{g \in G} \mathrm{Fix}_g(S) \subseteq S$$

is the set of fixed points of the action.