Orthogonal Transformation

$T \in L(V,V)$, V/F , $F \subset \mathbb{C}$ subfield

Irreducible subspace for T.

$W \subseteq V$ (subspace) $T(W) \subseteq W$ (invariant) and there are no $S \subset W$ (proper subspace, nonzero) s.t. $T(S) \subset S$

So W is irreducible.

Counterex

$T^r = 0$, $r > 1$ minimal

$W = T^{r-1}(V) \subsetneq V$. if $T^{r-1}(V) = V$ then $T(V) = 0$ so $T = 0$ but then $T^{r-1} = 0$

V is not irreducible, since $\overset{0}{\neq}\, T^{r-1} \subsetneq V$ and $T(T^{r-1}(V)) = 0 \subset V$.

Prop1

Let W be an irreducible subspace for T.

1) If $F = \mathbb{C}$ Then $\dim W = 1$.

2) If $F = \mathbb{R}$ then $\dim W = 1$ or 2.

Proof

1) $T|_W$ (restricted to W, irreducible) has an eigenvalue $\lambda \in \mathbb{C}$ $\Rightarrow$ $\exists\, v \overset{0}{\underset{W}{\neq}}$ s.t. $T(v) = \lambda v$.

So $T(\mathbb{C}v) = \mathbb{C}v$ so $\mathbb{C}v$ is an invariant subsp. of W, must be $= W$ since W irred.

2) $T|_W$ has a minimal polynomial $m(x) = P_1(x)^{e_1} \cdots P_r(x)^{e_r}$ with P_i prime.

$W = W_1 \oplus \cdots \oplus W_r$ where $W_i = n(P_i(x)^{e_i})$. $T(W_i) \subset W_i$ since $v \in W_i$

implies $P_i(T)^{e_i} v = 0$ so $P_i(T)^{e_i} T(v) = T P_i(T)^{e_i}(v) = 0$ so $T(v) \in n(P_i(T)^{e_i}) = W_i$.

Irreducibility implies $r = 1$ so $m(x) = P(x)^e$. If $e > 1$ Then $0 \neq P(T)^{e-1} \subsetneq W$

(nonzero since m is minimal, not $= W$ since o.w. $P(T)W = P(T)P(T)^{e-1}W = P(T)^e W = 0 \Rightarrow e = 1$)

So $e = 1$, and $m(x) = P(x) = \begin{cases} x - \alpha & \text{for some } \alpha \in \mathbb{R} \\ x^2 + \alpha x + \ldots & \text{with } \alpha^2 \ldots \end{cases}$

So $e=1$, and $m(x) = P(x) = \begin{cases} x - \alpha & \text{for some } \alpha \in \mathbb{R} \\ x^2 + \alpha x + \beta & \text{with } \alpha^2 - 4\beta < 0. \end{cases}$

If in first case, α is an eigenvalue in W so $\exists\, v \in W$ s.t. $T(\mathbb{R}v) = \mathbb{R}v$

So $\dim W = 1$ (or it's not minimal).

In the second case, $T^2 + \alpha T + \beta I = 0$. Take $S(v, T(v)) \subseteq W$ for some $\overset{\overset{0}{\#}}{v} \in W$.

first, $T(v) = \lambda v$ implies T has a real eigenvalue so p is not prime.

$S(v, T(v))$ invariant under T since $T^2 = -\alpha T - \beta I$. but W irreducible

So $S(v, T(v)) = W$ so $\dim(W) = 2$.

Now let $F = \mathbb{R}$, V/F with inner product $\langle , \rangle$, and $T \in O(V)$

<u>Prop 2</u> If $W \subset V$ is invariant under $T \in O(V)$, then $W^{\perp} = \{ v \in V, \langle v, W \rangle = 0 \}$ is invariant and $V = W \oplus W^{\perp}$

<u>Proof</u> first, if $w \in W$, $w' \in W^{\perp}$ and $w + w' = 0$ then $\langle w, w + w' \rangle = 0 \Rightarrow |w|^2 = 0$ and $|w'|^2 = 0$.

take an orthonormal basis of W, extend it to one of V. $OB(V) = OB(W) \sqcup OB(W^{\perp})$.

Since any orthogonal vectors added will fall in $W^{\perp}$.

Let $w' \in W^{\perp}$, $v \in W$. $T|_W : W \to W$ is orthogonal and so has inverse T^{-1}, so $v = T(w)$

for some $w \in W$. then $\langle T(w'), v \rangle = \langle T(w'), T(w) \rangle = \langle w', w \rangle = 0$.

So $T(W^{\perp}) \subseteq W^{\perp}$ so $W^{\perp}$ is invariant.

Let $T \in O(\overset{\dim V = n.}{V})$. $\exists\, W_1$ irreducible subspace. $\dim W_1 = 1$ or 2. $V = W_1 \oplus W_1^{\perp}$

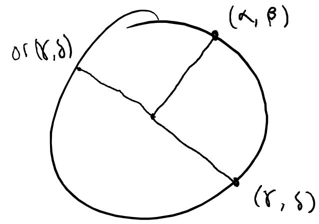
Inducting on $\dim V$, $W^{\perp}$ has dim $n-1$ or $n-2$. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ where

$\langle W_i, W_j \rangle = 0$ for $i \neq j$ and W_i irreducible.

1) if $\dim W_i = 1$ then $T(v) = \lambda v$ but $|Tv| = |v|$ so $|\lambda| = 1$ so $\lambda = \pm 1$

2) if $\dim W_i = 2$. let $\{u_1, u_2\}$ $OB(W_i)$. then $T(u_1) = \alpha u_1 + \beta u_2$, $T(u_2) = \gamma u_1 + \delta u_2$.

So $\alpha^2 + \beta^2 = 1$ and $\gamma^2 + \delta^2 = 1$, and $\alpha\gamma + \beta\delta = 0$.



so $T|_{W_i} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some θ.

or $T|_{W_i} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ but here we have $T^2 = I$

$T - I = 0$ or $T + I = 0$, contradiction.

so $T = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}$