Lec 9/5

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$G \circlearrowright X$ · either set map $\alpha : G \times X \longrightarrow X$ (satisfying 2 cond$^{\underline{n}}$s)

or gp hom $G \xrightarrow{\tau} \mathrm{Aut}_{Set}(X)$

$x \in X$ ⇝ orbit $G \cdot x$

$x \in X$ ⇝ Stabilizer $\mathrm{Stab}_G(x) = \{ g \in G : g \cdot x = x \}$ $\leq G$

$$G / \mathrm{Stab}_G(x) \xleftrightarrow{\text{set bijection}} G \cdot x$$

$$\mathrm{Stab}_G(x) \xrightarrow[\text{gp iso}]{\sim} \mathrm{Stab}_G(\sigma \cdot x) \qquad \forall \sigma \in G.$$

$$g \longmapsto \sigma g \sigma^{-1}$$

<u>Ex</u>: $S_n \circlearrowright X = \{1, \ldots, n\}$.

$\sigma \in S_n$ in X, there is only one orbit, it has size n.

$$\forall k \in \{1, \ldots, n\}, \quad \mathrm{Stab}_{S_n}(k) = \{ \sigma \in S_n \mid \sigma(k) = k \} \xrightarrow{\sim} S_{n-1} =: \mathrm{Stab}_{S_n}(n).$$

$$w \longmapsto (k\,n)\, w\, (k\,n)$$

Pick $\pi \in S_n$ and let $H = \langle \pi \rangle \circlearrowright X$.

$\langle \pi \rangle$-orbits in $X \longleftrightarrow$ cycles in π.

e.g. $n = 5$, $\quad \pi = (123)(45)$

$1 \to 2 \to 3 \to 1$ $\quad\leftarrow$ two orbits

$4 \to 5 \to 4$

terms:

Transitive: $\quad G \circlearrowright X$ is transitive if $\forall x, y \in X$, $\exists g \in G$ s.t. $g \cdot x = y$

(i.e. there is only one orbit)

Free: $\quad G \circlearrowright X$ is free if $g \cdot x = x \implies g = e$

(i.e. $\mathrm{Stab}_G(x) = \{e\}$ $\forall x \in X$)

$\iff$ every orbit has the same size, $|G|$.

$\Downarrow$

Faithful: $G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X)$ is injective:

$[g \cdot x = x \;\forall x \implies g = e]$

$S_n \circlearrowright X = \{1, \dots, n\}$

Faithful ✓

Free ×

Transitive ✓

$D_{2n} \circlearrowright \mathbb{R}^2$

Faithful ✓

Free ×

Transitive ×

$S_n \xrightarrow{\text{sign}} \{\pm 1\}$, so $S_n \circlearrowright \{\pm 1\}$ by $\alpha(\sigma, x) \longrightarrow \mathrm{sign}(\sigma) \cdot x$ ← not faithful n

$\{\sigma \in S_n \mid \text{sign}(\sigma) = 1\} =: A_n$ (alternating group).

Free $\longrightarrow$ all orbits have same size

Transitive $\longrightarrow$ exactly one orbit

<u>Ex:</u> Count # of set partitions $\{1, 2, \dots, 7\} = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_i$

$$\underbrace{|P_1| = 3, \quad |P_2| = |P_3| = 2.}$$

$E =$ set of possible ways to break it like this

$S_7 \circlearrowright E$

$$\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\} \xrightarrow{\sigma} \{\sigma(1), \sigma(2), \sigma(3)\} \sqcup \cdots$$

(1) $S_7 \curvearrowright E$ is <u>transitive</u>. if we pick a specific partition P

$$S_7 / \text{Stab}_{S_7}(P) \xrightarrow{\text{bijection}} E$$ = only one orbit

$S_3 \times S_2 \times S_2$

So $|E| = \dfrac{7!}{3! \cdot 2! \cdot 2!}$

In general, we get the multinomial coefficient.

$G \curvearrowright X$, $X =$ disjoint union of orbits

$$|\mathcal{O}| = \frac{|G|}{|\mathrm{Stab}_G(x)|}$$

some $x \in \mathcal{O}$

So $$|X| = \sum_{\mathcal{O} \in \text{Orbits}} \frac{|G|}{|\mathrm{Stab}_G(x)|}$$

$x \in \mathcal{O}$