Lec 11/25

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<u>Def</u> Let ν be a signed meas on $(X, \mathcal{M})$, μ a pos meas on $(X, \mathcal{M})$.

Say ν is absolutely cts wrt μ ($\nu \ll \mu$) wrt μ if

$\mu(E) = 0 \Rightarrow \nu(E) = 0$. ($E$ μ-null $\to$ E ν-null).

<u>Eg</u> $f \in \mathcal{L}^1(X, \mathbb{R}, \mu)$, $\nu(E) := \int_E f \, d\mu$ $\quad (d\nu = f\,d\mu)$. $\quad \nu \ll \mu$.

(or f ext μ-$\int$ble).

<u>Exercise:</u> ① TFAE:

ⓐ $\nu \ll \mu$

ⓑ $\nu_\pm \ll \mu$

ⓒ $|\nu| \ll \mu$

② $\nu \ll \mu$ & $\nu \perp \mu \Rightarrow \nu = 0$.

<u>Prop:</u> Suppose ν is a finite signed measure on $(X, \mathcal{M})$ and μ is a pos meas. TFAE:

① $\nu \ll \mu$

② $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

<u>Pf</u> Since $\nu \ll \mu$ iff $|\nu| \ll \mu$, and since $|\nu(E)| \leq |\nu|(E)$, we may assume ν positive.

Clearly ② $\Rightarrow$ ①. And $\neg$② $\Rightarrow$ $\neg$①: suppose $\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$ $\exists E_n \in \mathcal{M}$ w/ $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$. Set $F = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n$. Since $\mu(\bigcup_{n=k}^\infty E_n) \leq \sum_{n=k}^\infty 2^{-n} = 2^{-k+1}$, $\mu(F) = 0$.

but since ν is finite, $\nu(F) = \lim_{k \to \infty} \nu(\bigcup_{n=k}^\infty E_n) \geq \lim_{k \to \infty} \nu(E_n) \geq \varepsilon$. $\square$

<u>Example:</u> $(X, \mathcal{M}) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))$. $\mu(E) = \sum_{n \in E} \frac{1}{2^n}$, $\nu(E) = \sum_{n \in E} 2^n$.

Then $\mu \ll \nu$ and $\nu \ll \mu$. But ② fails since $\nu(\mathbb{N}) = \infty$.

Lemma: Suppose μ, ν finite pos meas on $(X, \mathcal{M})$.

either $\nu \perp \mu$ or $\exists\, \varepsilon > 0$ & $E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and

$\nu \geq \varepsilon\mu$ on E (E is positive for $\nu - \varepsilon\mu$, the signed measure).

pf Let $X = P_n \amalg P_n^c$ be a Hahn decomp for $\nu - \frac{1}{n}\mu$.

Set $P = \bigcup P_n$ and observe $P^c = \bigcap P_n^c$. Then P^c is negative

for $\nu - \frac{1}{n}\mu$ $\forall n \in \mathbb{N}$. So $0 \leq \nu(P^c) \leq \frac{1}{n}\mu(P^c)$ $\forall n \in \mathbb{N}$.

So $\nu(P^c) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$.

If $\mu(P) > 0$, Then $\mu(P_n) > 0$ for some n, and

P_n is positive for $\nu - \frac{1}{n}\mu$. $\square$

Thm (Lebesgue - Radon - Nikodym): Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. $\exists!$ σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ s.t.

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, $\exists!$ (up to μ-a.e.) extended μ-$\int$ble function f s.t. $d\rho = f\,d\mu$.

f is called the Radon-Nikodym derivative.

- If ν is positive or finite, so are λ & ρ, and $f \in \mathcal{L}^+$ or $\mathcal{L}^1$

pf (uniqueness): Suppose λ, λ' are σ-finite signed measures s.t. $\lambda, \lambda' \perp \mu$ and $\underbrace{f, f' \in \mathcal{L}^1}_{\text{hole}}$ are extended μ-$\int$ble s.t.

$$d\nu = d\lambda + f\,d\mu = d\lambda' + f'\,d\mu.$$

Then $d(\lambda-\lambda') = (f'-f)\, d\mu$ as signed measures. Also $\lambda-\lambda' \perp \mu$.

And $f'-f\, d\mu \ll d\mu$, so both sides are zero. Conclude $\lambda=\lambda'$, $f=f' \in \mathcal{L}'$.

(Existence): Case 1; ν & μ are finite positive measures.

Let $A = \{f \in \mathcal{L}^1(X,\mu,[0,\infty]) \mid \int_E f\,d\mu \le \nu(E)\ \forall\ E \in \mathcal{M}\}$. $0 \in A$.

And if $f, g \in A$ then $f \vee g \in A$ [set $G = \{g > f\}$. Then $\forall\ E \in \mathcal{M}$,

$$\int_E f \vee g\, d\mu = \int_{E\cap G} g\, d\mu + \int_{E \setminus G} f\, d\mu \le \nu(E \cap G) + \nu(E \setminus G) = \nu(E).]$$

Set $m = \sup\{\int f d\mu \mid f \in A\}$. $m \le \nu(X) < \infty$.

Choose $(f_n) \subset A$ s.t. $\int f_n \nearrow m$. Set $g_n = \max\{f_1, \dots, f_n\} \in A$.

And let $f = \sup g_n$. Then $\int f_n\, d\mu \le \int g_n d\mu \nearrow m$.

Since $g_n \nearrow f$ ptwise, by MCT, $\int_E f d\mu = \lim_{n\to\infty} \int_E g_n\, d\mu \le \nu(E)$ $\forall\ E \in \mathcal{M}$.

So $f \in A$, and $\int f d\mu = m$.

<u>Claim</u>: $\lambda(E) := \nu(E) - \overbrace{\int_E f d\mu}^{\rho(E)} \ge 0$ is singular wrt μ.

So $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$, $d\nu = d\lambda + f\, d\mu$

<u>pf of claim</u>: If not, by the lemma above, $\exists\ E \in \mathcal{M}$ & $\varepsilon > 0$ s.t. $\mu(E) > 0$ & $\lambda \ge \varepsilon\mu$ on E. But then, $\forall F \in \mathcal{M}$,

$$\int_F f + \chi_E\, d\mu = \int_F f\, d\mu + \varepsilon\mu(E \cap F) \le \int_F f\, d\mu + \lambda(E \cap F)$$

$$= \int_F f d\mu + \nu(E \cap F) - \int_{E \cap F} f\, d\mu$$

$$= \int_{F \setminus E} f d\mu + \nu(E \cap F)$$

$$\le \nu(F \setminus E) + \nu(E \cap F) = \nu(F).$$

Hence $f + \varepsilon\chi_E \in A$. but $\int (f + \varepsilon\chi_E) d\mu = m + \varepsilon\mu(E) > m$ ↯

<u>Case 2</u>: Suppose μ & ν are σ-finite measures.

write $X = \sqcup X_n$ s.t. $\mu(X_n), \nu(X_n) < \infty$. Set $\mu_n(E) = \mu(E \cap X_n)$, $\nu_n(E) = \nu(E \cap X_n)$.

By case 1, $\exists$ pos meas $\lambda_n \perp \mu_n$ & $f_n \in \mathcal{L}^1_+(X_n, \mu_n)$ s.t. $d\nu_n = d\lambda_n + f_n d\mu_n$.

Since $\mu_n(X_n^c) = \nu_n(X_n^c) = 0$, $\lambda(X_n^c) = \nu_n(X_n^c) - \int_{X_n^c} f_n \, d\mu_n = 0$.

So we may extend f_n to X by $f|_{X_n^c} = 0$.

Let $\lambda = \sum \lambda_n$ and $f = \sum f_n \in \mathcal{L}^+$.

Then $\lambda \perp \mu$, $d\lambda$ & $f\,d\mu$ are σ-finite, and $d\nu = d\lambda + f\,d\mu$.

(Uniqueness is a little trickier here: have to restrict to X_n's ...)

Case 3: μ σ-finite & positive, ν σ-finite & signed.

Use Jordan decomp to write $\nu = \nu_+ - \nu_-$ w $\nu_+ \perp \nu_-$.

Then $\underbrace{\nu_+, \nu_- \text{ are } \sigma\text{-finite}}_{\boxed{\text{Know how to prove this for final!}}}$. Apply case 2 to ν_+, ν_- seperately, subtract results. $\square$