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(if this is well-defined).

<u>Def</u> The rank of a module is the cardinality of its maximal linearly independent subset.

<u>eg</u> $M = \mathbb{Q},\ R = \mathbb{Z}$ any single element is a max'l linearly indep subset.

$$cb\frac{a}{b} - ad\frac{c}{d} = 0.$$

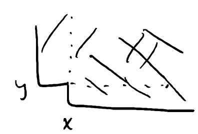
And $\mathbb{Q}/\mathbb{Z}$ is a torsion module: $\mathbb{Q}/\mathbb{Z} \cong$ roots of unity

<u>eg</u> $R = \overset{\text{field}}{F}[x,y],\quad M = (x,y)$. $\quad \overset{R}{x}\cdot\overset{M}{y} - \overset{R}{y}\cdot\overset{M}{x} = 0$ so $\{x,y\}$ are linearly dependent

$\{x\}$ is a max'l linearly indep. subset, so is $\{y\}$.

$$M/R\{x\} = \{ b_1 y + \cdots + b_n y^n : b_i \in R \}$$ is a torsion module.

Rank $M = 1$. but $M \ncong R$.



<u>Theorem</u>: any vector space is a free module (has a basis B s.t. $V = \bigoplus_{b \in B} F$).

<u>Note</u>: $\underset{\text{many copies}}{\prod} F \cong \underset{\text{many more copies}}{\bigoplus} F$

<u>Proof</u>: Vector spaces have no torsion elements.

Let B be a max'l linearly independent subset in V.

Then B generate V. Indeed, let $u \in V$.

Then $\{u\} \cup B$ is a linearly dependent set so

$\exists\, a, a_1, \dots, a_n \in F,\ v, v_1, \dots, v_n \in V$ s.t. $au + a_1 v_1 + \cdots + a_n v_n = 0$

& not all $a, a_i = 0$. $a \neq 0$ otherwise B is lin. dep.

So $u = -a^{-1}(a_1 v_1 + \cdots + a_n v_n)$.

If C is a linearly indp subset in a module M,

$\exists$ maxl'l lin indp. subset of M containing C.

for v.s., any lin. indp set can be extended to a basis.

Theorem: if V is a vector space & W is a subspace,

then W is a direct summand of V. $\exists U \subset V$ s.t. $V = W \oplus U$.

Proof: find a basis C of V. extend it to B. let $U = \mathrm{Span}(B \setminus C) = F \cdot (B \setminus C)$.

then $V = W \oplus U$.

proof 2:
$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] & V \arrow[r] & V/W \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

$\hookrightarrow$ Vector Space, so is a free module.

Split the sequence as follows: Send basis to any Preimages of Basis.

Since the free module is a universal object here.

So $V = W \oplus U$ where $U \cong V/W$.

Lemma: If N is a free module and $\varphi : M \to N$ is surjective, then φ has a section $\sigma : N \to M$ s.t. $\varphi \circ \sigma = \mathrm{id}_N$.

Proof: let B be a basis in N. $\forall u \in B$, choose $v_u \in \varphi^{-1}(u) \subseteq M$,

and let $\sigma\left(\sum_{b \in B}^{\text{finite}} a_b \cdot b\right) = \sum_{b \in B}^{\text{finite}} a_b \cdot v_b$.

Theorem: dim = rank of a vector space is well-defined. Any two bases have the same cardinality.

Proof: Finite case:
Let $B = \{u_1, \dots, u_n\}$, $C = \{v_1, \dots, v_m\}$ be two Bases.

$v_1 = a_1 u_1 + \cdots + a_n u_n$, let $a_1 \neq 0$. Then $u_1 = a_1^{-1}(v_1 - a_2 u_2 - \cdots - a_n u_n)$.

Put $B_1 = \{v_1, u_2, \dots, u_n\}$. Claim: B_1 is still a basis of V.

Indeed, $u_1 \in Span(B_1)$ so $Span(B_1) = V$. if The vectors were dependent, $c_1 v_1 + c_2 u_2 + \cdots + c_n u_n = 0$, then

$c_1 a_1 u_1 + (c_1 a_2 + c_2) u_2 + \cdots + (c_1 a_n + c_n) u_n = 0$ and $c_1 a_1 \neq 0$ $\lightning$.

Replace u_2 by one of remaining v_i. At some point, $B_n = \{v_1, \dots, v_n\}$ will be a basis. So if $n < m$, Then C is linearly dependent.

for infinite case, we need Zorn Lemma to make it rigorous.

B, C bases. Consider the family $\mathcal{A}$ of sets of the form (B', C') s.t. $B' \subseteq B$, $C' \subseteq C$, and $B' \cup C'$ is a basis in V, and $\exists$ a bijection $B \setminus B' \longleftrightarrow C'$.

Order on $\mathcal{A}$ is defined by $(B', C') < (B'', C'')$ if $B'' \subset B'$, $C'' \supset C'$.

Max'l elt will be some $(\emptyset, C')$