Lec 9/20

Wednesday, September 20, 2017 10:45

group of permutations $\xrightarrow[\text{Homomorphism}]{}$ group $\{\pm 1\}$

via $\varepsilon(\sigma) =$ sign of σ

$$= \frac{\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j}(x_i - x_j)} \quad \text{for } x_i \neq x_j \text{ for } i \neq j.$$

$\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$

Now even permutations form a subgroup since their sign is 1.

$\tau_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ 1 & 2 & \cdots & j & \cdots & i & \cdots & n \end{pmatrix}$ is a transposition.

$\varepsilon(\tau_{ij}) = -1.$ since only one change is made.

Note $\tau_{ij}^{-1} = \tau_{ij}$. Identity is ι (iota)

<u>Thm</u> (1) every permutation $\sigma \in S(n)$ is a product of transpositions

(2) This product of factors is not unique, but the parity of the product is unique.

<u>Proof</u> (2): suppose $\sigma = \tau_1 \tau_2 \cdots \tau_p$

then $\varepsilon(\sigma) = \varepsilon(\tau_1)\varepsilon(\tau_2)\cdots\varepsilon(\tau_p) = (-1)^p$

which is unique, so p is either even or odd for σ.

(1): by induction on n (as in Δn)

base case: Δ_1 trivial, Δ_2, $S_2 = \{\iota, \tau_{12}\}$ so it works.

Assume it's true for S_n.

take $\sigma \in S_{n+1}$. if $\sigma(n+1) = n+1$, then σ acting on $\{1, \dots, n\}$ is a permutation of S_n so it's a product of transpositions in S_n.

Otoh, if $\sigma(n+1) = i \le n$ then $\tau_{i\,n+1}\sigma(n+1) = n+1$ is a product of transpositions in S_n, and $\sigma = \tau_{i\,n+1}\tau_{i\,n+1}\sigma$ so it's also a product of transpositions $\blacksquare$