Lec 2/2

Friday, February 2, 2018 14:22

Non-unit-speed curves

$t \longmapsto \beta(t)$ a regular curve in $\mathbb{R}^3$. (β is C^3)

$I \longrightarrow \mathbb{R}^3$

$t_0 \in I$

Define s on I by $s(t) = \int_{t_0}^{t} |\beta'(\tilde{t})|\, d\tilde{t}$, the arclength from t_0 to t

$\beta(t) = \alpha(s(t))$ where α is β reparametrized by arclength.

Notation $\dot\beta = \frac{d\beta}{dt}$, $\ddot\beta = \frac{d^2\beta}{dt^2}$, $\dddot\beta = \frac{d^3\beta}{dt^3}$.

Propn 6.1 Let $t \longmapsto \beta(t)$ be a regular curve in $\mathbb{R}^3$. Then:

(a) $T = \frac{\dot\beta}{|\dot\beta|}$. (b) $B = \frac{\dot\beta \times \ddot\beta}{|\dot\beta \times \ddot\beta|}$. (c) $N = B \times T$.

(d) $\kappa = \frac{|\dot\beta \times \ddot\beta|}{|\dot\beta|^3}$ (e) $\tau = \frac{\det(\dot\beta, \ddot\beta, \dddot\beta)}{|\dot\beta \times \ddot\beta|^2}$

In (b), (c), (e), we need $|\dot\beta \times \ddot\beta| \neq 0 \iff \kappa \neq 0$.

Pf Let s be arclength along β, measured from some t_0.

Let α be β reparameterized in terms of s, s.t. $\forall t$, $\beta(t) = \alpha(s(t))$.

(a) $\dot\beta = \frac{d\beta}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = \alpha' \overbrace{|\dot\beta|}^{\dot s} = T|\dot\beta| \Rightarrow T = \frac{\dot\beta}{|\dot\beta|}$.

(b) $\ddot\beta = \frac{d}{dt}(\dot s T) = \ddot s T + \dot s \frac{dT}{dt} = \ddot s T + \dot s \frac{dT}{ds}\frac{ds}{dt} = \ddot s T + (\dot s)^2 \kappa N$

so $\dot\beta \times \ddot\beta = \dot s T \times (\ddot s T + (\dot s)^2 \kappa N) = (\dot s)^3 \kappa B$, so $B = \frac{\dot\beta \times \ddot\beta}{|\dot\beta \times \ddot\beta|}$

(c) $N = B \times T$ since TNB is $+$-oriented ONB for $\mathbb{R}^3$

(d) $|\dot\beta \times \ddot\beta| = (\dot s)^3 \kappa$ so $\kappa = \dfrac{|\dot\beta \times \ddot\beta|}{|\dot\beta|^3}$.

(e) $\dddot\beta = \dfrac{d}{dt}\left(\ddot s T + (\dot s)^2 \kappa N\right) = \dddot s T + \ddot s \dfrac{dT}{ds}\dfrac{ds}{dt} + 2 \dot s \ddot s \kappa N + (\dot s)^2 \dot\kappa N + (\dot s)^2 \kappa \dfrac{dN}{ds}\dfrac{ds}{dt}$

$= \dddot s T + 3 \ddot s \dot s \kappa N + (\dot s)^2 \dot\kappa N + (\dot s)^3 \kappa \left(-\kappa T + \tau B\right)$

$= \overbrace{\left(\dddot s - (\dot s)^3 \kappa^2\right)}^{*_1} T + \overbrace{\left(3 \dot s \ddot s \kappa\right)}^{*_2} N + \left((\dot s)^3 \kappa \tau\right) B$.

so $\det(\dot\beta, \ddot\beta, \dddot\beta) = \begin{vmatrix} \dot s & \ddot s & *_1 \\ 0 & (\dot s)^2 \kappa & *_2 \\ 0 & 0 & (\dot s)^3 \kappa \tau \end{vmatrix} = \left((\dot s)^3 \kappa\right)^2 \tau$

and $|\dot\beta \times \ddot\beta| = (\dot s)^3 \kappa$ so $\tau = \dfrac{\det(\dot\beta, \ddot\beta, \dddot\beta)}{|\dot\beta \times \ddot\beta|^2}$

$\det(\dot\beta, \ddot\beta, \dddot\beta) = \langle \dot\beta \times \ddot\beta, \dddot\beta \rangle = \langle (\dot s)^3 \kappa B, (\dot s)^3 \kappa \tau B \rangle = \left((\dot s)^3 \kappa\right)^2 \tau$. $\square$

eg 6.2 define $\beta : \mathbb{R} \to \mathbb{R}^3$ by $\beta(t) = (1+t^2, t, t^3)$. β is C^∞.

$\dot\beta(t) = (2t, 1, 3t^2)$, $|\dot\beta(t)| = \sqrt{1 + 4t^2 + 9t^4} \geq 1$ so β is regular.

$\ddot\beta(t) = (2, 0, 6t)$, $\dddot\beta(t) = (0, 0, 6)$. $\dot\beta \times \ddot\beta = (6t, -6t^2, -2)$.

$|\dot\beta \times \ddot\beta| = \sqrt{4 + 36t^2 + 36t^4}$. $\det(\dot\beta, \ddot\beta, \dddot\beta) = \langle \dot\beta \times \ddot\beta, \dddot\beta \rangle = -12$.

$\kappa = \dfrac{|\dot\beta \times \ddot\beta|}{|\dot\beta|^3} = \dfrac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \neq 0$.

$\tau = \dfrac{-12}{4 + 36t^2 + 36t^4} = \dfrac{-3}{1 + 9t^2 + 9t^4}$.

$T = \dfrac{(2t, 1, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}}$.

$B = \dfrac{(6t, -6t^2, -2)}{}$

$$B = \frac{(6t, -6t^2, -2)}{2\sqrt{1+9t^2+9t^4}}$$

$$N = B \times T = \frac{(6t, -6t^2, 2) \times (2t, 1, 3t^2)}{2\left((1+4t^2+9t^4)(1+9t^2+9t^4)\right)^{1/2}} = \frac{(-18t^4+2, -4-18t^3, 6t+12t^3)}{2\left((1+4t^2+9t^4)(1+9t^2+9t^4)\right)^{1/2}}$$

eg Define $\beta : (0,\infty) \longrightarrow \mathbb{R}^3$ by $\beta(t) = (t, 1+t^{-1}, t^{-1}-t)$.

$\dot\beta = (1, -t^{-2}, -t^{-2}-1)$, $\ddot\beta = (0, 2t^{-3}, 2t^{-3})$.

$$B = \frac{\dot\beta \times \ddot\beta}{|\dot\beta \times \ddot\beta|} = \frac{(2t^{-3}, -2t^{-3}, 2t^{-3})}{2t^{-3}\sqrt{3}} = \left(\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$$ which is constant

so $\tau = 0$, meaning β is planar.

<u>Propn</u> Let β be a regular curve in $\mathbb{R}^3$. Let $v(t) = |\dot\beta(t)|$. Then:

(a) $\dot T = \kappa v N$

(b) $\dot N = -\kappa v T + \tau v B$

(c) $\dot B = -\tau v N$

<u>Pf</u> Problem 6.5. $\square$