Lec 1/19

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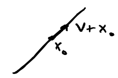
Some examples of curves:

eg Let $x_0, \overset{\circ}{\underset{\shortparallel}{}} v \in \mathbb{R}^d$. define $\alpha : \mathbb{R} \to \mathbb{R}^d$ by $\alpha(t) = x_0 + tv$.

This is a C^∞-regular curve. $\alpha'(t) = v \neq 0 \quad \forall t$.

The range of α is $\{x_0 + tv : t \in \mathbb{R}\} =$ 

The velocity is v, the speed is $|v|$. The line tangent to α is the range of α.

Now define $\beta : \mathbb{R} \to \mathbb{R}^d$ by $\beta(t) = \alpha(t^3)$. $t \mapsto t^3$ is strictly increasing map $\mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$.

so α, β have same range but β is not regular since $\beta'(t) = 3t^2 v$, which is 0 when $t = 0$.

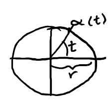
Define $\gamma : \mathbb{R} \to \mathbb{R}^d$ by $\gamma(t) = \alpha(t + t^3)$ so $\gamma'(t) = (1 + 3t^2)v \neq 0 \quad \forall t$ so γ is regular.

γ has the same range as α since $t \mapsto t + t^3 \nearrow \mathbb{R} \to \mathbb{R}$ onto.

eg Let $g : \mathbb{R} \to \mathbb{R}$ be C^k. Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t, g(t))$.

This is the graph of g. $\alpha'(t) = (1, g'(t)) \neq 0$ so α is regular. α is C^k as well.

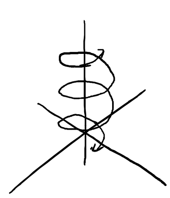
Let $r \in (0, \infty)$. Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (r\cos t, r\sin t)$. This is C^∞, regular. Circle.



Not 1-1. $|\alpha'(t)| = \overset{\circ}{\underset{\shortparallel}{}} r$ so α is regular. $\alpha'(t) = (-r\sin t, r\cos t)$.

$\alpha''(t) = -\alpha(t)$

Eg Let $r, h \in (0, \infty)$. define $\alpha : \mathbb{R} \to \mathbb{R}^3$ by $\alpha(t) = (r\cos t, r\sin t, ht)$.



helix

$\alpha'(t) = (-r\sin t, r\cos t, h)$

$|\alpha'(t)| = \sqrt{r^2 + h^2} > 0$, α regular

$\alpha''(t) = (-r\cos t, -r\sin t, 0)$

↑ particle accelerates towards the z-axis.

Reparameterizations:

Let $\alpha : (a,b) \longrightarrow \mathbb{R}^n$ be a C^k-regular curve.

Let $g : (c,d) \longrightarrow (a,b)$ be a 1-1, onto, C^k function s.t. $g^{-1} : (a,b) \longrightarrow (c,d)$ is also C^k.

Let $\beta = \alpha \circ g$. $\beta : (c,d) \longrightarrow \mathbb{R}^n$ is also C^k and is regular?

$\beta'(t) = \alpha'(g(t))\, g'(t)$ and $g'(t) \neq 0$ since g^{-1} is C^k.

$$\left(\frac{d}{dt} g^{-1}(t) = \frac{1}{g'(g^{-1}(t))} \right) \qquad g(t) = u, \quad \frac{du}{dt} = \frac{1}{dt/du}$$

$$1 = \frac{dt}{dt} = \frac{d}{dt} g(g^{-1}(t)) = g'(g^{-1}(t))(g^{-1})'(t) \qquad \text{etc.}$$

Counterexample from earlier.

Define $g : \mathbb{R} \longrightarrow \mathbb{R}$ by $g(u) = u^3$

$g^{-1}(t) = t^{1/3}$ which is not diffable at 0.

2-2 arc length

Let $\alpha : [a,b] \longrightarrow \mathbb{R}^n$ be a C^k curve. the length of α is

$$L = \int_a^b \left| \frac{d\alpha}{dt} \right| dt$$

Let $S(t_0) = \displaystyle\int_a^{t_0} \left| \frac{d\alpha}{dt} \right| dt \qquad S : [a,b] \longrightarrow [0,L]. \quad \frac{ds}{dt} = \left| \frac{d\alpha}{dt} \right|.$

and C^k.

a

Suppose α is regular. then $\frac{ds}{dt} > 0$ $\forall$ input vals, so s is H & s^{-1} is C^k. (and C^k.)

Let $s \longmapsto t(s)$ be the inverse function of $t \longmapsto s(t)$

then $\beta = \alpha \circ t$ has $|\beta'(s)| = |\alpha'(t(s))||t'(s)| = \left|\frac{d\alpha}{dt}(t(s))\right|\left|\frac{dt}{ds}\right| = 1.$

$\beta : [0, L] \longrightarrow \mathbb{R}^n$

2-3 Curvature & the Frenet-Serret apparatus

Consider a C^2 unit-speed ($\left|\frac{d\alpha}{ds}\right| = 1$ at all points in (a,b).) curve. $\alpha : (a,b) \longrightarrow \mathbb{R}^2.$

$s \longmapsto \alpha(s)$

$\alpha'(s) = T(s)$

The curvature K of α is defined by $K(s) = |T'(s)|$ for $s \in (a,b)$.

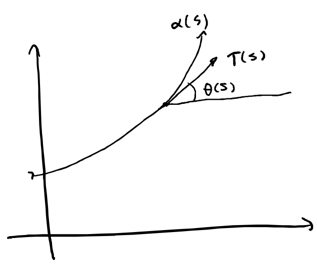
Define $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ Let $r \in (0, \infty)$.

$s \longrightarrow (r\cos\frac{s}{r}, r\sin\frac{s}{r})$.

$|\alpha'(s)| = |(-\sin\frac{s}{r}, \cos\frac{s}{r})| = 1.$

$K(s) = |T'(s)| = \frac{1}{r}|(-\cos\frac{s}{r}, -\sin\frac{s}{r})| = \frac{1}{r}.$

More generally, let $\alpha : (a,b) \longrightarrow \mathbb{R}^2$ be any C^2 unit-speed curve, let θ be as depicted next.



θ is at least C^1 since T is.

$T(s) = (\cos(\theta(s)), \sin(\theta(s)))$

$T'(s) = \theta'(s)\,(-\sin(\theta(s)),\, \cos(\theta(s)))$

$|T'(s)| = |\theta'(s)| = \kappa(s)$.

<u>Define</u> Let α be a C^3 unit-speed curve in $\mathbb{R}^3$.

$T(s) = \alpha'(s) \qquad |T(s)| = 1$

$\kappa(s) = |T'(s)|$.

$N(s) = \dfrac{T'(s)}{\kappa(s)}$, $|N(s)| = 1$ This is the principal normal vector to the curve.

$B(s) = T(s) \times N(s)$

↳ is the "Binormal"

$\tau(s) = -\langle B'(s), N(s) \rangle$ is the "torsion"

(κ, τ, T, N, B) is the Frenet-Serret Apparatus for α.