Lec 9/30

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$(X, \mathcal{M})$, $(Y, \mathcal{N})$ mble spaces.

mble rectangles.

$\mathcal{M} \otimes \mathcal{N}$ = σ-algebra generated by $\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}$

Proposition: Suppose (X, ρ_X) and (Y, ρ_Y)

① $\mathcal{B}_X \otimes \mathcal{B}_Y$ is gen by $\{U \times Y \mid U \subseteq X \text{ open}\} \cup \{X \times V \mid V \subseteq Y \text{ open}\}$.

② $\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X \times Y}$ (use $\rho_X + \rho_Y$ or $\max\{\rho_X, \rho_Y\}$, they give same topology). check

③ If X, Y are separable, $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X \times Y}$

Pf ① follows from fact that $\mathcal{M} \otimes \mathcal{N}$ s.t. projections π_X, π_Y are mble.

② Since $U \times Y$ and $X \times V$ are open in $X \times Y$, $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X \times Y}$ by ①

③ Suppose $C \subset X$ and $D \subset Y$ are countable dense subsets. Let $\mathcal{E}_X$, $\mathcal{E}_Y$ be the collections of open balls centered at C, D resp, w/ rational radii. Then every open set in X or Y is a ctble union of sets in $\mathcal{E}_X$ or $\mathcal{E}_Y$ resp. Also, $C \times D$ is a countable dense subset of $X \times Y$ (Exercise). So the topology of $X \times Y$ is gen by $\mathcal{E}_X \times \mathcal{E}_Y \subset \mathcal{B}_X \otimes \mathcal{B}_Y$. Thus $\mathcal{B}_{X \times Y} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$. □

Recall for sets X & Y the product $X \times Y$ satisfies the universal property

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, dotted, "\exists!"] & \\ X & X \times Y \arrow[l, "\pi_x"] \arrow[r, "\pi_y"'] & Y \end{tikzcd}$$

$\forall\, Z,\ f: Z \longrightarrow X,\ g: Z \longrightarrow Y$

$\exists!$ fn $f \times g : Z \longrightarrow X \times Y$ s.t.

$\pi_X \circ (f \times g) = f$

$\pi_Y \circ (f \times g) = g$

<u>Exercise</u>: If X, Y, Z are topological / mble spaces, TFAE

① $f \times g$ is cts / mble.

② f & g are cts / mble.

<u>Prop</u> The following functions are continuous & thus mble

① $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ (can replace $\mathbb{R}$ with $[0, \infty]$ or $\mathbb{C}$)

② $\cdot : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ (can replace $\mathbb{R}$ with $\overline{\mathbb{R}}$ or $\mathbb{C}$)

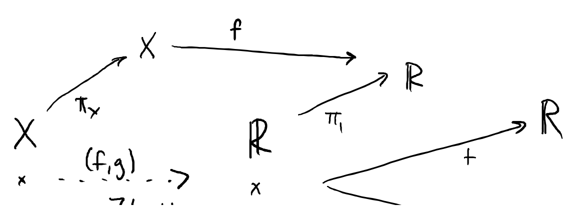
<u>Cor</u>: if $f : (X, \mathcal{M}) \longrightarrow \mathbb{R}$, $g : (Y, \mathcal{N}) \longrightarrow \mathbb{R}$ are mble, so are

$(x, y) \longmapsto f(x) + g(y)$

$(x, y) \longmapsto f(x)\, g(y)$

$: (X \times Y, \mathcal{M} \otimes \mathcal{N}) \longrightarrow \mathbb{R}$.

Pf



$$\begin{tikzcd}
X \times Y \arrow[r, dashed, "{(f,g)}", "\exists!\ \text{mble}"'] \arrow[d, "\pi_Y"'] & \mathbb{R} \times \mathbb{R} \arrow[r, "+"] \arrow[r, shift right=3, "\cdot"'] \arrow[d, "\pi_2"] & \mathbb{R} \\
Y \arrow[r, "g"'] & \mathbb{R} &
\end{tikzcd}$$

Product Measures: fix measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$.

Observe:

- $E_1 \times F_1 \cap E_2 \times F_2 = (E_1 \cap E_2) \times (F_1 \cap F_2)$
- $(E \times F)^c = (E^c \times F) \cup (E \times F^c) \cup (E^c \times F^c)$

$A := \{$finite disjoint unions of mble rectangles$\}$ is an algebra, it generates $\mathcal{M} \otimes \mathcal{N}$.

Prop: for $G = \coprod_1^n E_k \times F_k$, define

$$(\mu \times \nu)_0 (G) = \sum_1^n \mu(E_k)\nu(F_k).$$

Then $(\mu \times \nu)_0$ is a premeasure on A.

Pf: It suffices to show that if $E \times F = \coprod E_j \times F_j$ then

$$\mu(E)\nu(F) = \sum \mu(E_j)\nu(F_j)$$

Trick for every $x \in E$, $y \in F$, $\exists!\ j$ s.t. $(x,y) \in E_j \times F_j$.

Thus for $y \in F$, $E = \coprod_{j \text{ s.t. } y \in F_j} E_j$.

Now for $y \in F$,

$\underline{\infty}$,

Now for $y \in F$,

$$\mu(E) = \sum_{j \text{ s.t. } y \in F_j} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j) \chi_{F_j}(y)$$

Then $\mu(E)\chi_F(y) = \sum_{j=1}^{\infty} \mu(E_j) \chi_{F_j}(y)$

$$\int_Y \mu(E)\chi_F \, d\nu = \int_Y \sum \mu(E_j) \chi_{F_j} \, d\nu$$

$= \mu(E)\nu(F)$

$$= \sum \int_Y \mu(E_j) \chi_{F_j} \, d\nu$$

MCT

$$= \sum \mu(E_j) \int_Y \chi_{F_j} \, d\nu$$

$$= \sum \mu(E_j) \nu(F_j)$$

□

Use outer measure construction to get $(\mu \times \nu)^*$ on $P(X \times Y)$, restrict it to $(\mu \times \nu)^*$-mble sets $\supseteq \mathcal{M} \otimes \mathcal{N} \rightsquigarrow \mu \times \nu$ is a measure.