Lec 9/21

Thursday, September 21, 2017 10:26

Alternating multilinear forms on V/F, $\dim V = n$.

$$\Phi : V^n \to \underset{\cap}{F} \quad \mathbb{C}$$

multilinear.

$$\Phi(u_1, \dots, u_{i-1}, \alpha a + \beta b, u_{i+1}, \dots, u_n) =$$

$$\alpha \Phi(u_1, \dots, u_{i-1}, a, u_{i+1}, \dots, u_n) + \beta \Phi(u_1, \dots, u_{i-1}, b, u_{i+1}, \dots, u_n)$$

$$\Phi(u_{\sigma(1)}, \dots, u_{\sigma(n)}) = \varepsilon(\sigma) \Phi(u_1, \dots, u_n) \quad \sigma \in S_n \quad \leftarrow \text{alternating.}$$

Note: $\Phi(v_1, \dots, v, \dots, v, \dots, v_n) = 0$.

In particular, exchanging two inputs negates the value.

It's enough to require $\Phi(v_1, \dots, v_n) = 0$ if $v_i = v_j$ for some $i \neq j$.

Existence:

Fix any basis of V: $\{u_1, \dots, u_n\}$.

It's enough to define Φ on $u_1, \dots, u_n$.

$\Phi(v_1, \dots, v_n) =$ some sum of multiples of $\Phi(u_1, \dots, u_n)$:

$$\Phi\left(\sum \alpha_{i_1 1} u_{i_1}, \dots, \sum \alpha_{i_n n} u_{i_n}\right) = \sum_{\underbrace{i_1, \dots, i_n}_{\text{Pairwise distinct}} = 1}^{n} \alpha_{i_1 1} \alpha_{i_2 2} \cdots \alpha_{i_n n} \Phi(u_{i_1}, u_{i_2}, \dots, u_{i_n})$$

$$= \Phi(u_1, \dots, u_n) \sum_{\sigma \in S_n} \varepsilon(\sigma) \alpha_{\sigma(1) 1} \alpha_{\sigma(2) 2} \cdots \alpha_{\sigma(n) n}$$

alternating multilinear forms are unique up to scaler multiplication.

$\underbrace{\text{Alternating } n\text{-linear forms}}_{\overset{\shortparallel}{\Lambda^n V}}$ are a vector space over F.

$\dim \Lambda^n V = 1.$

Let $V = F^n$, $A \in M_{n,n}(F)$, $A = (\alpha_{ij})$.

$= S(\{e_1, \dots, e_n\})$, $\{Ae_1, \dots, Ae_n\}$, $Ae_j = \sum_{i=1}^{n} \alpha_{ij} e_i$

Take $\Phi_{st} \in \Lambda^n F^n$ to be determined by $\Phi_{st}(e_1, \dots, e_n) = 1$.

<u>Def:</u> $\det(A) = \Phi_{st}(Ae_1, \dots, Ae_n)$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \alpha_{\sigma(1)1} \alpha_{\sigma(2)2} \cdots \alpha_{\sigma(n)n}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) \alpha_{1\sigma^{-1}(1)} \alpha_{2\sigma^{-1}(2)} \cdots \alpha_{n\sigma^{-1}(n)}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \alpha_{1\sigma(1)} \alpha_{2\sigma(2)} \cdots \alpha_{n\sigma(n)}$$

$\det(A) = \det(A^T)$

Hardest property to check:

$$\det(AB) = \det(A)\det(B)$$

$$\Phi_{st}(ABe_1, \ldots, ABe_n)$$

$$\in \Lambda^n F^n$$

Def $\Psi_A(v_1, \ldots, v_n) = \Phi_{st}(Av_1, \ldots, Av_n)$

$$= \lambda\, \Phi_{st}(v_1, \ldots, v_n)$$

$$\lambda \Phi(e_1, \ldots, e_n) = \Psi_A(e_1, \ldots, e_n) = \Phi_{st}(Ae_1, \ldots, Ae_n)$$

$$\lambda = \det(A)$$

now $\Psi_A(Be_1, \ldots, Be_n) = \det(A)\, \Phi_A(Be_1, \ldots, Be_n)$

$$\|$$

$$\Phi_{st}(ABe_1, \ldots, ABe_n) \quad = \det(A)\det(B)$$

$$= \det(AB)$$