Lec 11/27

Monday, November 27, 2017 10:27

$$y' = f(x,y) \qquad f: D \longrightarrow \mathbb{R}$$
$$y(x_0) = y_0 \qquad D \subseteq \mathbb{R}^2, \quad (x_0, y_0) \in D.$$

Case: Separable (separated variables)

$$f(x,y) = \frac{g(x)}{h(y)}$$

So eqn is $h(y)\,y' = g(x)$.

if $\varphi(x)$ is a soln, then $\varphi'(x) = \dfrac{g(x)}{h(\varphi(x))}$

$$\iff h(\varphi(x))\,\varphi'(x) = g(x)$$

$$H(y) = \int_{y_0}^{y} h(s)\,ds, \qquad G(x) = \int_{x_0}^{x} g(t)\,dt$$

$$\frac{d}{dx} H(\varphi(x)) = h(\varphi(x))\,\varphi'(x) \qquad G'(x) = g(x)$$

$$\text{so } H(\varphi(x)) = G(x) + C$$

$$\text{so } \underbrace{H(y_0)}_{=0} = \underbrace{G(x_0)}_{=0} + C$$

so $C = 0$. (if no IVP, leave as $+C$)

general soln is (implicitly) $H(y) = G(x) + c$

example: $y' = x^2 y^2 - x^2$

so $\dfrac{y'}{1-y^2} = -x^2$

$$\int \frac{y'}{1-y^2} = -\int x^2$$

$$\operatorname{arctanh}(y) = -\frac{x^3}{3} + c$$

$$y = \tanh\left(c - \frac{x^3}{3}\right)$$

if $y(x_0) = x_0$ then $y_0 = \tanh\left(c - \frac{x_0^3}{3}\right)$

so $c = \operatorname{arctanh}(y_0) + \dfrac{x_0^3}{3}$

So soln is $y = \tanh\left(\operatorname{arctanh}(y_0) + \frac{x_0^3}{3} - \frac{x^3}{3}\right)$

example

$$y' = \frac{x^2 + xy + y^2}{x^2 + y^2} = \frac{1 + \frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{y^2}{x^2}}$$

let $u = \frac{y}{x}$. then $y = ux$ so $y' = xu' + u$

so $\quad x u' + u = \dfrac{1 + u + u^2}{1 + u^2}$

So $\quad x u' = \dfrac{1 + u^2 - u^3}{1 + u^2}$

so $\quad \dfrac{1 + u^2}{1 + u^2 - u^3} u' = \dfrac{1}{x}$

$f(tx, ty) = f(x, y)$

So taking $t = \frac{1}{x}$ gives $\quad f(x, y) = f(1, \frac{y}{x})$

exact equations

$f(x,y) = -\dfrac{M(x,y)}{N(x,y)}$ is exact if $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ("curl" is 0)

$M, N :$ Rectangle $\longrightarrow \mathbb{R}$

so $\exists F :$ Rectangle $\longrightarrow \mathbb{R}$ s.t. $\quad M = \dfrac{\partial F}{\partial x}$ and $N = \dfrac{\partial F}{\partial y}$

(potential function)

Now $\quad M(x,y)\,dx + N(x,y)\,dy = 0$

if $\varphi(x)$ solves,
$$\underbrace{M(x, \varphi(x)) + N(x, \varphi(x))\,\varphi'(x)}_{\frac{d}{dx} F(x, \varphi(x))} = 0$$

Suppose we found F. Then $\frac{d}{dx} F(x, \varphi(x)) = 0$

So the solution renders $F(x, \varphi(x))$ constant.

So $F(x,y) = c$ is implicit equation giving solution.

to find F:

$$F(x,y) - F(x_0, y) = \int_{x_0}^{x} M(s,y)\,ds$$

and $$F(x,y) - F(x, y_0) = \int_{y_0}^{y} N(x,t)\,dt$$

So $$F(x_0, y) - F(x_0, y_0) = \int_{y_0}^{y} N(x_0,t)\,dt$$

So $$F(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt + F(x_0, y_0)$$

or $$F(x,y) = \int M(s,y)\,ds + \int N(x,t)\,dt + C$$

or $$F(x,y) = \int_{\mathcal{P}} \left(M(x,y)\,dx + N(x,y)\,dy\right)$$ where $\mathcal{P}$ connects (x_0, y_0)

or $F(x,y) = \int_{P} \left(M(x,y)\,\partial x + N(x,y)\,\partial y \right)$ where P connects (x_0, y_0) to (x,y)