Lec 9/4

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Important Stuff:

(1) Definitions:

group, subgroup, cyclic groups, order: element/set. Cosets: left (G/H), right $(H \backslash G)$

generated by a subset

Normal subgroups, quotient group homs, iso-s, presentations

(2) Abstract Results:

cyclic grps: $\{ \mathbb{Z}/k\mathbb{Z} \; : \; k = 0,1,2,\dots \}$

$$\left| G/H \right| = |G|/|H| \Rightarrow \text{ord}(a) \mid \text{divides } |G| \Longrightarrow \left(|G| = p \Rightarrow \text{any } a \in G \text{ generates } G \right)$$

1^{st} iso $\left[G/\text{Ker}(f) \cong \text{Im}(f) \right.$

2^{nd} iso

$$N \trianglelefteq G \rightsquigarrow G \xrightarrow{\pi} G/N$$

natural projection?

$$\left\{ \begin{matrix} \text{(normal)} \\ \text{subgps of } G \\ \text{containing } N \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{(normal)} \\ \text{Subgroups of} \\ G/N \end{matrix} \right\}$$

$$G \trianglerighteq H \supset N \Longrightarrow G/H \cong (G/N)/(H/N)$$

(3) concrete examples:

D_{2n} $\quad |D_{2n}| = 2n, \; D_{2n} = \langle r, s \; : \; r^n = s^2 = (rs)^2 = e \rangle$

Free(2) $\quad$ smallest normal subgr containing $aba^{-1}b^{-1}$ is same as $\text{Ker}(p)$ $\quad p: \text{Free}(2) \to \mathbb{Z}^2$

S_n $\quad$ disjoint cycles, order of elts, presentation, $|S_n| = n!$

Group Actions (on sets)

Definition A group G acts on a set X if we have a set map $G \times X \xrightarrow{\alpha} X$, $(g, x) \longmapsto \alpha(g,x) = g \cdot x$

satisfying $(\forall x \in X, \; g_1, g_2 \in G)$ $\underset{(1)}{\alpha(e,x) = x}$ and $\underset{(2)}{\alpha(g_1 g_2, x) = \alpha(g_1, \alpha(g_2, x))}$.

Notation: $G \circlearrowright X$

$$\left\{ \begin{matrix} \alpha : G \times X \longrightarrow X \\ \text{satisfying} \\ (1) \ \& \ (2) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{group hom} \\ G \longrightarrow S_X \end{matrix} \right\}$$

Automorphisms: Isomorphism $X \longrightarrow X$

$$\alpha \rightsquigarrow \begin{matrix} G \xrightarrow{\tau} \mathrm{Aut}_{set}(X) \\ g \longmapsto (x \longmapsto \alpha(g, x)) \end{matrix}$$

set automorphism (as opposed to group/vector space/etc).

$\mathrm{Aut}_{set}(\mathbb{R}^2) \neq \mathrm{Aut}_{\mathbb{R}\text{-vs}}(\mathbb{R}^2) = GL_2(\mathbb{R})$.

$$\tau(e) = \mathrm{Id}_X . \quad \tau(g_1 g_2) = \tau(g_1) \circ \tau(g_2) .$$

$G \circlearrowright X$ words

- Orbit of $x \in X$ is $G \cdot x = \{ g \cdot x \mid g \in G \} \subseteq X$
- Stabilizer of x in G is $\mathrm{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \} \leq G$ ← check this
- fixed points of $g \in G$ is $X^g := \{ x \in X \mid g \cdot x = x \} \subseteq X$

Examples:

$D_{2n} \circlearrowright \mathbb{R}^2$. $D_{2n} \xrightarrow{\text{gp-hom}} GL_2(\mathbb{R})$

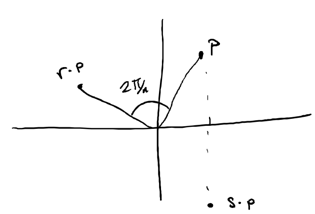
$D_{2n} \longrightarrow \mathrm{Aut}_{set}(\mathbb{R}^2)$, $GL_2(\mathbb{R}) = \mathrm{Aut}_{set}(\mathbb{R}^2)$

$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in GL_2(\mathbb{R})$$

$$r \longmapsto \begin{bmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{bmatrix}$$

Let $X = \mathbb{R}^2 \setminus \{0\}$ $\longrightarrow$ $\mathrm{Stab}_{D_{2n}}(0) = G$, $D_{2n} \cdot 0 = \{0\}$.

$\circlearrowright D_{2n}$



for $p \in X$, $D_{2n} p \supset \{ p, r \cdot p, r^2 \cdot p, \dots, r^{n-1} \cdot p \}$

$D_{2n} \cdot p$ has $2n$ elements iff $\overset{\#}{s \cdot p}$ iff not p_0 is on x-axis.

or $p_1 =$ scalar multiple of $\begin{bmatrix} \cos(\pi/n) \\ \pm \sin(\pi/n) \end{bmatrix}$

$\lceil \cos(2\pi/n) \rceil$

$_{2n}$ ... x-axis.

or p_1 = scalar multiple of $\begin{bmatrix} \cos(\pi/n) \\ \pm\sin(\pi/n) \end{bmatrix}$

or p_2 = scalar multiple of $\begin{bmatrix} \cos(2\pi/n) \\ \pm\sin(2\pi/n) \end{bmatrix}$

$\vdots$

or p_k = scalar multiple of $\begin{bmatrix} \cos(k\pi/n) \\ \pm\sin(k\pi/n) \end{bmatrix}$

$\vdots$

$\mathrm{Stab}_{D_{2n}}(p) = \{e\}$ unless p is one of these p_i's.

$\mathrm{Stab}_{D_{2n}}(p_j) = \{e, sr^j\} \cong \mathbb{Z}/2\mathbb{Z}$.

note: $\mathrm{Stab}_G(x)$ is <u>not</u> always a normal subgp.

<u>Lemma</u>: $G/\mathrm{Stab}_G(x) \xrightarrow[\text{bijection}]{\sim} G \cdot x$

<u>Proof</u> fix x. define $\begin{array}{ccc} G & \xrightarrow{\text{surj.}} & G \cdot x \\ g & \longmapsto & g \cdot x \end{array}$. $g_1 x = g_2 x \iff g_1^{-1} g_2 \in \mathrm{Stab}_G(x)$ map is injective

i.e. $g_1 \mathrm{Stab}_G(x) = g_2 \mathrm{Stab}_G(x)$

$\Rightarrow G/\mathrm{Stab}_G(x) \longrightarrow G \cdot x$ bijection

<u>Ex.</u> $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ acts by rotation of $\mathrm{Arg}(z)$. $\mathrm{Stab}_{S^1}(p) = \{e\}$. $S^1 \cdot p =$ circle w/radius $|p|$.

$\mathbb{R}^2 \setminus \{0\}$

So set of orbits is $\mathbb{R}_{>0}$.