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Let K/F be an extension, $\alpha \in K$.

β is conjugate of α iff $\exists$ embedding $F(\alpha) \longrightarrow K$
s.t. $\alpha \longmapsto \beta$.

<u>Defn</u> Let L/F be a subextension of K/F.

Then L'/F is conjugate to L/F if

$\exists$ an embedding $\varphi : L/F \longrightarrow K/F$

s.t. $\varphi(L) = L'$.

$$\begin{tikzcd}
K \arrow[d, dash] \arrow[dr, dash] & \\
L \arrow[r, "\varphi"'] \arrow[d, dash] & L' \\
F \arrow[ur, dash] &
\end{tikzcd}$$

<u>Defn</u> An algebraic extension K/F is normal

if $\forall \alpha \in K$, $m_{\alpha,F}$ splits completely in K.

(all conjugates of α are in K).

($\forall$ extension E/K, any conjugate of α in E is in K.)

Equivalently, if $f \in F[x]$ is irreducible & has a root in K, then f splits completely in K.

<u>Theorem</u> if K/F is normal, then $\forall$ subextension L/F,

then K/L is normal.

<u>Theorem</u> If $L_1, L_2 \subseteq K$, L_1/F, L_2/F are normal,

then $(L_1 \cap L_2)/F$ is normal.

<u>Theorem</u> (Finite) K/F is normal iff $\forall$ extension E/K, any

embedding $K/F \xrightarrow{\varphi} E/F$ is an automorphism of K $(\varphi(K)=K)$.

<u>Proof</u> Let K/F be normal. $\forall \alpha \in K$, $\forall \varphi : K/F \to E/F$,

$\varphi(\alpha)$ is a conjugate of α so $\varphi(\alpha) \in K$.

so $\varphi(K) \subseteq K$, so $\varphi(K) = K$, $\varphi \in \mathrm{Aut}(K/F)$.

Now assume $\alpha \in K$, $\alpha' \in E$, α' is conjugate of α, $\alpha' \notin K$.

Let $K = F(\alpha, \alpha_1, \alpha_2, \dots, \alpha_k)$. Let $f = \prod m_{\alpha_i, F}$.

Let $\tilde{E}$ be the splitting field of f over F.

$$\begin{array}{ccc}
F(\alpha, \alpha_2, \dots, \alpha_k) & \xrightarrow{\varphi_k} & \tilde{E} \\
| & & | \\
\vdots & & \vdots \\
| & & | \\
F(\alpha_1, \alpha_2) & \xrightarrow{\varphi_1} & F(\alpha_1', \alpha_1') \\
\vdots & \vdots & \vdots
\end{array}$$

$m_{\alpha_1, F(\alpha)} = f_1$, $f_1' = \varphi(f_1)$.

Let α_1' be a root of f_1'

$\exists\, \varphi_1 : F(\alpha, \alpha_1) \longrightarrow \hat{E}$

$\alpha_1 \longmapsto \alpha_1'$

$$\begin{tikzcd}
F(\alpha_1, \alpha_2) \arrow[r, "\varphi_1"] \arrow[d, dash] \arrow[dr, phantom, "\rotatebox{90}{$\subseteq$}"] & F(\alpha_1', \alpha_1') \arrow[d, dash] \\
\varphi : F(\alpha) \arrow[r] & F(\alpha')
\end{tikzcd}$$

$\alpha_1 \longmapsto \alpha_1'$

$\varphi|_{F(\alpha)} = \varphi.$

If $f = m_{\alpha_1, F}$ then $f_1 | f$ and

so $\underset{\substack{\| \\ f_1'}}{\varphi(f_1)} \,\Big|\, \varphi(f) = f$

so f_1' splits in $\tilde{E}$

so $\alpha_1' \in \tilde{E}$.

by induction, $\exists$ embedding $\varphi_k : K \longrightarrow \tilde{E}$

Such that $\varphi_k(\alpha) = \alpha' \notin K$, so

$\varphi_k \notin \operatorname{Aut}(K)$. $\square$

<u>Corollary</u> Let $K = F(\alpha_1, \dots, \alpha_k)$ and assume that $\forall i$, all conjugates of α_i are in K. then K/F is normal.

<u>proof</u> $\forall \varphi : K/F \longrightarrow E/F$, $\forall i$, $\varphi(\alpha_i) \in K$,

So $\varphi(K) \subseteq K$, so $\varphi(K) = K$.

So K is normal.

So a finite extension is normal iff it is a splitting field of some polynomial.

If $K = F(\alpha_1, \ldots, \alpha_n)$ and K/F is normal, then K is the splitting field of $f = \prod m_{\alpha_i, F}$.

If K is the splitting field of F, let $\alpha_1, \ldots, \alpha_k$ be the roots of f in K,

then $K = F(\alpha_1, \ldots, \alpha_k)$ and all conjugates of each α_i are in K.

<u>Corollary</u> If $L_1, L_2 \subseteq K$, L_1/F, L_2/F are normal, then $L_1 L_2/F$ is normal.

~~proof~~ $L_1 = F(\alpha_1, \ldots, \alpha_k)$, $L_2 = F(\beta_1, \ldots, \beta_\ell)$.

Then $L_1 L_2 = F(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$

α_i and β_i are all "good" (their conjugates are in $L_1 L_2$)

So $L_1 L_2$ is normal.

<u>Def</u> if L/F is algebraic, a normal closure of L/F is the minimal normal extension K/F with $L \subseteq K$

It exists & is unique up to isomorphism:

Proof assuming L is finite $\longrightarrow$ K is the splitting field of $\prod_{i=1}^{k} m_{\alpha_i, F}$ where $L = F(\alpha_1, \dots, \alpha_k)$.

<u>Theorem</u> if K/F be normal, L/F is a subextension of K/F, and $\varphi : L/F \longrightarrow K/F$ be an embedding. Then $\exists\, \psi \in Aut(K/F)$ s.t. $\psi|_L = \varphi$.

~~pf~~ (assume K is finite)

$$\begin{array}{ccc} K & \xrightarrow{\psi} & K \\ | & \cup & | \\ \vdots & & \vdots \\ | & \cup & | \\ L(\alpha_1) & \longrightarrow & L'(\alpha_1') \\ | & \cup & | \\ \varphi : L & \xrightarrow[\varphi]{} & L' \end{array}$$

let $K = L(\alpha_1, \dots, \alpha_k)$.

let $f_1 = m_{\alpha_1, L}$, $f_1' = \varphi(f_1)$

then $\underbrace{f_1' \mid m_{\alpha, F}}$ so f_1' has a root $\alpha_1' \in K$.

$f_1' = \varphi(f_1) \mid \varphi(m_{\alpha_1, F})$ since $f_1 = m_{\alpha_1, L} \mid m_{\alpha_1, F}$.

$\vdots$

et cetera

<u>Defn</u> A finite normal separable extension is called

a <u>Galois Extension</u>

If K/F is Galois, $\mathrm{Aut}(K/F)$ is called the <u>Galois group</u> of K/F, denoted $\mathrm{Gal}(K/F)$.

$$|\mathrm{Gal}(K/F)| = [K:F].$$