Lec 10/4

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Sequential formulations of CA.

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

Let $\{n_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of indices (in $\mathbb{N}^+$).

we then form the sequence $\{a_{n_j}\}_{j=1}^{\infty}$ (Composition of functions)

we call $\{a_{n_j}\}_{j=1}^{\infty}$ a subsequence of $\{a_n\}_{n=1}^{\infty}$

and we say that $\{a_n\}_{n=1}^{\infty}$ contains the subsequence $\{a_{n_j}\}_{j=1}^{\infty}$

Examples: $\{a_{2n}\}$: subsequence of even terms

$\{a_{2n-1}\}$: subsequence of odd terms.

Observation/Lemma: $n_j \geq j$

Proposition: If $\lim_{n\to\infty} a_n = L$, then $\lim_{j\to\infty} a_{n_j} = L$ for any subsequence.

Proof: Let $\varepsilon > 0$ be given. then for some N

$n > N \Rightarrow |a_n - L| < \varepsilon$

so $j > N \Rightarrow |a_{n_j} - L| < \varepsilon$

because $n_j \geq j$. ■

Monotone Convergence property (MCP)

1) every bounded increasing sequence has a limit

2) " " decreasing " "

(1) $\Leftrightarrow$ (2) because $\{a_n\}$ increasing $\Leftrightarrow$ $\{-a_n\}$ decreasing

and $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} -a_n = -L$.

Theorem 1 MCP $\Leftrightarrow$ CA

Proof: $\Leftarrow$: Since CA $\Leftrightarrow$ LUBP

it suffices to show that LUBP $\Rightarrow$ MCP

Let $\{a_n\}$ be bounded and increasing.

Consider the set $S = \{a_n : n \geq 1\}$

S is bounded above. so $\sup S = L$ exists by LUBP

* we show that $\lim_{n\to\infty} a_n = L$

Let $\varepsilon > 0$ be given.

Then $L - \varepsilon$ is not an upper bound for S. so

$L \geq a_N > L - \varepsilon$ for some N.

Then for $n > N$, $L \geq a_n > a_N > L - \varepsilon$

so $|a_n - L| < \varepsilon$.

$\Rightarrow$: Since CA $\Leftrightarrow$ NIP & $\mathbb{R}$ has no infinitesimals.

* first show MCP $\Rightarrow$ $\mathbb{R}$ has no infinitesimals $\Leftrightarrow$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing and bounded

so MCP implies $\lim_{n\to\infty} \frac{1}{n} = L$ exists.

$\{\frac{1}{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{\frac{1}{n}\}_{n=1}^{\infty}$

$\Rightarrow$ $\lim_{n\to\infty} \frac{1}{n+1} = L$.

because this is also a subsequence

$$0 = L - L = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \left(\frac{1}{n(n+1)}\right) = L$$

so $L = 0$ and $\mathbb{R}$ has no infinitesimals.

* Now show MCP $\Rightarrow$ NIP

Let $I_n = [a_n, b_n]$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

and $\lim_{n\to\infty} (b_n - a_n) = 0$

Then $\{a_n\}$ is a bounded increasing sequence

So MCP says $\lim_{n\to\infty} a_n = L$ exists.

$a_n \leq L \leq b_n$ for all n

So $L \in \bigcap_{n=1}^{\infty} I_n$

condition that $(b_n - a_n) \to 0$ implies $\bigcap_{n=1}^{\infty} I_n$ can't contain two distinct elements.

So $\bigcap_{n=1}^{\infty} I_n = \{L\}$ ∎

Bolzano-Weierstrauss Property (BWP)

Every bounded sequence contains a subseqence that has a limit

Theorem 2: CA $\Leftrightarrow$ BWP

Lemma: any sequence contains a monotonic subsequence

Definition: we say that an index N is a peak point of $\{a_n\}_{n=1}^{\infty}$ if $a_N > a_n$ for all $n > N$

Proof of Lemma: If there are infinitely many peak points, then there is a strictly decreasing subsequence (take the successive peak points).

Otherwise, there is a (nonstrictly) increasing subsequence (go to last peak point & ...). ∎

Proof of Thm 2: (CA $\Leftrightarrow$ MCP, so it suffices to show MCP $\Leftrightarrow$ BWP.

$\Rightarrow$: If $\{a_n\}$ is a bounded sequence, it has a monotonic subsequence (which is also bounded).

So by MCP, this subsequence has a limit.

$\Leftarrow$: If $\{a_n\}$ is bounded monotonically, then BWP implies $\lim_{j\to\infty} a_{n_j} = L$ for some subsequence $\{a_{n_j}\}$

So $\lim_{n\to\infty} a_n = L$ ∎

Cauchy Completeness Property (CCP)

Every cauchy sequence converges.

Definition: a sequence $\{a_n\}$ is a cauchy sequence if $\forall \varepsilon > 0 \ \exists N$ s.t. $|a_n - a_m| < \varepsilon$ for all $n, m > N$.

Intuitively: sequence bunches up.

Theorem 3: $CA \Leftrightarrow CCP$ and $\mathbb{R}$ has no infinitesimals.

Proof: $\Rightarrow$: $CA \Leftrightarrow BWP$ so it suffices to show that $BWP \Rightarrow CCP$

Let $\{a_n\}$ be a cauchy sequence.

* We will show $\{a_n\}$ is bounded first.

Let $\varepsilon = 1$. then find an index N s.t.

$$|a_m - a_n| < 1 \quad \forall_{n,m} \geq N.$$

Let $B_1 = \max(a_1, a_2, \ldots, a_N)$

$B = B_1 + 1$

Then $|a_m| \leq B_1 < B$ for $m < N$

$$|a_m| \leq |a_N| + |a_m - a_N| \leq B_1 + 1 = B \quad \text{for } m \geq N.$$

So $\{a_n\}$ is bounded.

So by BWP, $\lim_{j \to \infty} a_{n_j} = L$ exists for some subsequence.

Now we will show $\lim_{n \to \infty} a_n = L$.

Let $\varepsilon > 0$ be given. Then for some J,

$$|a_{n_j} - a_{n_J}| < \frac{\varepsilon}{2} \quad \text{for } j \geq J.$$

and for some N,

$$|a_m - a_n| < \frac{\varepsilon}{2} \quad \text{for } m, n \geq N.$$

let $M = \max\{N, n_J\}$

Suppose $n \geq M$. Then

$$|a_n - L| \leq |a_n - a_{n_M}| + |a_{n_M} - L|$$

$|a_n - a_{n_M}| < \frac{\varepsilon}{2}$ since $n \geq M$, $n_M \geq M$

$|a_{n_M} - L| < \frac{\varepsilon}{2}$ since $M \geq n_J \geq J$

so $|a_n - L| < \varepsilon$

* and we already know CA $\Rightarrow \mathbb{R}$ has no infinitesimals.

$\Leftarrow$: Use NIP.