Lec 8/24

Thursday, August 24, 2017 10:19

removing field axioms.

- don't require $1 \neq 0$, allow 0^{-1}! then $F = \{0\}$.

Vector Spaces:

$\mathbb{R}^2$ is a vector space on $\mathbb{R}$.

so is $\mathbb{R}^n$.

$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad (a, b) \longmapsto a+b$

$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad (\lambda, a) \longmapsto \lambda a$

can replace $\mathbb{R}$ by any field F.

- $\cdot$ and $\times$ are attributes of specific vector spaces, not all of them.

Vector Space rules:

$(a+b)+c = a+(b+c)$

$a+b = b+a$

$\exists 0$ s.t. $a+0 = a$

$\exists (-a)$ s.t. $a+(-a) = 0$

$\lambda(a+b) = \lambda a + \lambda b$

$(\lambda + \delta) a = \lambda a + \delta a$

$\lambda(\delta a) = (\lambda \delta) a$

$1a = a$

V is a vector space over the field F if $V \neq \emptyset$ is endowed with operations:

$V \times V \xrightarrow{+} V$

$F \times V \xrightarrow{\cdot} V$

satisfying these rules

Examples:

-

Polynomials with real coeffs $\mathbb{R}[x]$.

$$P = \alpha_0 X^n + \alpha_1 X^{n-1} + \cdots + \alpha_{n-1} X + \alpha_n \quad \text{where } \alpha_i \in \mathbb{R}$$

$\mathbb{R}_n[X]$ is polynomials of degree at most n, bij w/ $\mathbb{R}^{n+1}$

$S \neq \emptyset$ $\quad \mathcal{F}(S, F) = \{ f : S \to F \}$ where F is a fixed field.

$$(f + g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$
$x \in S$, $\lambda \in F$

$\mathcal{F}(S, F)$ is a vector space over F.

This statement includes all V.S. mentioned thus far:

$\mathcal{F}(\{1, \dots, n\}, F)$ bij F^n

$\mathcal{F}(\mathbb{N}, F)$ bij $F[x]$

Now:

$C(\mathbb{R})$ cont. fns from $\mathbb{R} \to \mathbb{R}$ is a v. space.

$C^1(\mathbb{R}) \subset C(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$. $\quad C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

$\cup$

$\vdots$

$C^k(\mathbb{R})$

$\cup$

$\vdots$

infinite

etc.

$$\begin{array}{c} C^k(\mathbb{R}) \\ \cup \\ \vdots \\ C^\infty(\mathbb{R}) \\ \cup \\ \mathbb{R}[x] \end{array}$$

infinite dimension

let $y \in C^n(\mathbb{R})$ satisfying:

$$(*)(y) := \quad \alpha_0 y^{(n)} + \alpha_1 y^{(n-1)} + \cdots + \alpha_{n-1} y' + \alpha_n y = 0$$

the $\underbrace{\text{set of such functions}}$ is a subspace of $C^n(\mathbb{R})$

$\{ y \in C^n(\mathbb{R}) \mid (*)(y) \}$ (it's a vector space over $\mathbb{R}$).

but it is finite-dimensional (we'll show this in ODE).

$\underset{F}{0} \, \underset{V}{x} = \underset{V}{0}$. This is b.c. $0x = (0+0)x = 0x + 0x \Rightarrow 0 = 0x$

$-x \overset{?}{=} (-1)x$ Yes, $x + (-1)x = 1x + (-1)x = (1-1)x = 0x = 0$.