Lec 9/11

Monday, September 11, 2017 14:20

if $P = 4k+1$ then $\sum_{i=1}^{k} [\sqrt{ip}] = \frac{p^2-1}{12}$ Try to do it by Wednesday.

Upper Banach density

$f(x) \in C[0,1]$, $f \to f(\frac{1}{17})$

"functional"

for $A \subseteq \mathbb{N}$, let $d^*(A) = \limsup_{N-M \to \infty} \frac{|A \cap \{M+1, M+2, \dots, N\}|}{N-M}$

recall $\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{1, \dots, N\}|}{N}$

Note: understand limsup.

natural density, $\lim_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N}$

Example: $A = \bigcup_{n=1}^{\infty} [2^n, 2^n+n]$ $\bar{d}(A) = d(A) = 0$ but $d^*(A) = 1$.

Definition 2

If $d^*(A) = \alpha$ then $\exists \{I_n\} = \{\{M_n+1, \dots, N_n\}\}$ with $|I_n| \to \infty$

s.t. $d^*(A) = \alpha = \lim_{n\to\infty} \frac{|A \cap I_n|}{|I_n|}$

and for any other $\{J_n\}$, $\limsup_{n\to\infty} \frac{|A \cap J_n|}{|J_n|} \le \alpha$.

$d^*(A) \geq \bar{d}(A)$ trivially.

Szemerédi in yet another equiv. form:

if $A \subseteq \mathbb{N}$, $d^*(A) > 0$, A AP-rich

$\Updownarrow$

finitistic version

Hardest / strongest Sz:

$\bar{d}^*(A) > 0 \Rightarrow \bar{d}^*(A \cap (A-d) \cap \dots \cap (A - \ell d)) > 0 \ \forall \ell$ for some d.

$d^*(A \cup B) \leq d^*(A) + d^*(B)$? (exercise)

Can it happen that $d^*(A) = 1$, $d^*(B) = 1$, but $A \cap B = \emptyset$?

Yes, $\bigcup_{n=100}^{\infty} [2^n, 2^n + n]$ and $\bigcup_{n=100}^{\infty} [2^n + n + 1, 2^n + 2n]$.

how about $\bar{d}(A) = 1$, $\bar{d}(B) = 1$, but $A \cap B = \emptyset$? (exercise: yes)

$\bar{d}(P) = 0$ (exercise) Hint: for sequence to have positive density, it must grow linearly.

maybe try Sárközy.

Note that $d^*(P) = 0$ but this is not an exercise.

$\delta(S) = \frac{6}{\pi^2}$. how about $\delta^*(S)$?

$\square$-free

Exercise: $\delta^*(S) < 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{pmatrix}.$$

$u_0 = 0$
$u_1 = 1$
$u_2 = 1$
$u_3 = 2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$u_n \sim \left(\frac{1+\sqrt{5}}{2}\right)^n$$

eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$

so $\lambda = \varphi$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = T^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} T$$

Exercise: figure out formula for nth fibonacci number.

Exercise: if (for $c>1$), $a_n = \lfloor 2^c \rfloor^n$, $b_n = \lfloor 2^{cn} \rfloor$.

Prove that $\delta^*(\{a_n\})$ and $\delta^*(\{b_n\})$ are 0.

H. Weyl, 1916

Defn Sequence $(x_n) \subset [0,1]$ is uniformly distributed

(*) if $\forall\, 0 \le a < b \le 1$, $\displaystyle\lim_{N\to\infty} \frac{\#\{1 \le n \le N : x_n \in (a,b)\}}{N} = b - a$

Example: $x_n = n\alpha \bmod 1$

equivalent: $\forall\, 0 \le a < b \le 1$, $\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{(a,b)}(x_n) \longrightarrow \int_0^1 \mathbb{1}_{(a,b)}(x)\, dx$

$\forall f \in C[0,1]$

$(**)$ $\frac{1}{N} \sum_{n=1}^{N} f(x_n) \longrightarrow \int_0^1 f(x)\, dx$

exercise: $(*) \Leftrightarrow (**)$

(more generally equiv to $(**)$ with f Riemann integrable)

Note: polynomials are dense in $C[0,1]$