Lec 8/31

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See lecture notes online: N = smallest normal subgp containing $aba^{-1}b^{-1}$ = Ker(p).
& do some reading too.

Back to S_n:

Facts: (1) $|S_n| = n!$

(2) if $s_i = (i \ \ i+1)$ then $\{s_1, \dots, s_{n-1}\}$ generates S_n.

(3) $s_i^2 = e$, $s_i s_j = s_j s_i$ if $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ $\forall 1 \leq i \leq n-2$

This implies that if $\mathcal{G}_n = \left\{ \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i^2 = e \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |j-i| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\}$

Then there is a surjective gr hom

$$\begin{array}{ccc} \mathcal{G}_n & \xrightarrow{\pi_n} & S_n \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma_i & \longmapsto & s_i \end{array} \Rightarrow |\mathcal{G}_n| \geq n!$$

We prove $\mathcal{G}_n \cong S_n$ by induction on n.

Base case $n=2$: $\mathcal{G}_2 = \langle \sigma \mid \sigma^2 = e \rangle \cong \mathbb{Z}/2\mathbb{Z} \cong S_2$

Induction hypothesis: $\pi_1, \dots, \pi_n$ are all <u>iso</u>'s

Induction step: to prove: π_{n+1} is an <u>iso</u> $\Leftrightarrow |\mathcal{G}_{n+1}| \leq (n+1)!$.

$$\left.\begin{array}{l} \{\sigma_1, \dots, \sigma_n\} \longleftarrow \text{generators of } \mathcal{G}_{n+1} \\ \{\sigma_1, \dots, \sigma_{n-1}\} \text{ satisfy relns of } \mathcal{G}_n \end{array}\right\} \begin{array}{c} i_n : \mathcal{G}_n \longrightarrow \mathcal{G}_{n+1} \\ \text{gr hom.} \end{array}$$

$$G_n = \text{Image}(i_n) \leq \mathcal{G}_{n+1} \Rightarrow |G_n| \leq |\mathcal{G}_n| = n!$$

Claim: $\left| \mathcal{G}_{n+1} / G_n \right| \le n+1 \Rightarrow \left| \mathcal{G}_{n+1} \right| \le |G_n| (n+1) \le (n+1)!$

Pf of Claim. idea: $\left[\begin{array}{l} S_N / S_{N-1} \xleftrightarrow{bij} \{1, \dots, N\} \quad \text{since } S_N = S_{N-1} \sqcup (1N) S_{N-1} \sqcup (2N) S_{N-1} \sqcup \cdots \\ (1N) = \underbrace{s_1 s_2 \cdots s_{N-2}}_{\substack{\cup \\ S_{N-1}}} s_{N-1} \underbrace{s_{N-2} \cdots s_2 s_1}_{\substack{\cup \\ S_{N-1}}} \end{array} \right.$

set $H_\ell = \sigma_\ell \sigma_{\ell+1} \cdots \sigma_n G_n$ for $\ell = 1, \dots, n$. $H_{n+1} = G_n$.

$H_\ell = \pi_{n+1}^{-1} \left(\{ \tau \in S_{n+1} \mid \tau(n+1) = \ell \} \right)$

Claim: $\{H_1, H_2, \dots, H_{n+1}\} = \mathcal{G}_{n+1} / G_n$

$\left[\begin{array}{l} \text{Fun fact: If } \overset{\emptyset}{\underset{\#}{}} X \subset G/H \text{ s.t. } gX \subset X \ \forall g \in G \text{ then } X = G/H. \\ \uparrow \\ \text{If } X \ne \emptyset \ \& \ gH \in X \text{ then } hH = (hg^{-1}) gH. \end{array} \right.$

$\sigma_k H_\ell = ?$ if $\ell = n+1$ $H_{n+1} = G_n$, $\sigma_k H_{n+1} = H_{n+1}$ if $1 \le k \le n-1$, $\sigma_n H_{n+1} = H_n$.

otherwise: $\sigma_k (\underbrace{\sigma_\ell \sigma_{\ell+1} \cdots \sigma_n G_n}_{H_\ell}) = H_\ell$ if $k \le \ell - 2$.

$\sigma_{\ell-1} H_\ell = H_{\ell-1}$. $\sigma_\ell H_\ell = H_{\ell+1}$,

if $k > \ell$, $\sigma_k (\sigma_\ell \sigma_{\ell+1} \cdots \sigma_k \cdots \sigma_{n-1} \sigma_n G_n)$

$= \sigma_\ell \sigma_{\ell+1} \cdots (\sigma_k \sigma_{k-1} \sigma_k) \cdots \sigma_{n-1} \sigma_n G_n$

$= \sigma_\ell \sigma_{\ell+1} \cdots \sigma_{k-1} \sigma_k \sigma_{k-1} \cdots \sigma_{n-1} \sigma_n G_n$

$= H_\ell$

gets absorbed

Thus $\left| \mathcal{G}_{n+1} / G_n \right| = n+1 \Rightarrow \pi_{n+1}$ is an iso. $\square$

Conclusion: S_n is isomorphic to $\mathcal{G}_n$. it is presented by those generators & rel^{n}s.

Circular proof of $\begin{array}{l} S_n \longrightarrow \{\pm 1\} \\ s_i \longrightarrow -1 \end{array}$ is a gp-hom.

"Proof": $\begin{tikzcd} S_n \arrow[r, "\text{gp hom}"] \arrow[rr, bend left, "\text{sign}"] & GL_n(\mathbb{R}) \arrow[r, "\det"] & \mathbb{R}_{\neq 0} \end{tikzcd}$,

$\sigma \longmapsto X_\sigma$, $(X_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$

but to show det is a gp hom we use the fact that sign is a gp hom.

Theorem (Artin): $\left\langle T_1, \dots, T_{n-1} \;\middle|\; \begin{array}{ll} T_i T_j = T_j T_i & \text{if } |i,j| \geq 2 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & 1 \leq i \leq n-2 \end{array} \right\rangle$

$\parallel$

fundamental group of configuration space of n points in the plane.