Lec 11/15

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<u>Regular Singular Points</u>

$$L(y) = (x-x_0)^n y^{(n)} + a_1(x)(x-x_0)^{n-1} y^{(n-1)} + \cdots + a_n(x) y$$

$$\varphi(x) = (x-x_0)^r$$ (for convenience, take $x_0 = 0$.

so $$L(y) = x^n y^{(n)} + a_1(x) x^{n-1} y^{(n-1)} + \cdots + a_n(x) y$$

where $$a_j(x) = \sum_{k=0}^{\infty} \alpha_{jk} x^k$$

$$\varphi(x) = x^r$$

$$\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

$$\left.\begin{aligned} \varphi(0) &= c_0 \\ \varphi'(0) &= c_1 \\ &\vdots \\ \varphi^{(n-1)}(0) &= c_{n-1} \end{aligned}\right\}$$ gives n lin. indep. solns by taking $c_i = 1, c_{j \neq i} = 0$.

$$L(y) = x^2 y'' + a(x) x y' + b(x) y$$

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$$

$$b(x) = \sum_{k=0}^{\infty} \beta_k x^k$$

b(x)

$$\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{r+k}$$

$$\varphi'(x) = \sum^{\infty} c_k (r+k) x^{r+k}$$

$$\begin{array}{l|l} a(x) & x\,\varphi'(x) = \sum_{k=0}^{\infty} c_k (r+k)\, x^{r+k} \\ 1 & x^2\,\varphi''(x) = \sum_{k=0}^{\infty} c_k (r+k)(r+k-1)\, x^{r+k} \end{array}$$

$$a(x)\,x\,\varphi'(x) = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \alpha_\ell c_{n-\ell} (r+n-\ell) \right) x^{r+n}$$

$$b(x)\,\varphi(x) = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \beta_\ell c_{n-\ell} \right) x^{r+n}$$

So $$L(\varphi)(x) = \sum_{n=0}^{\infty} \left(c_n (r+n)(r+n-1) + \sum_{\ell=0}^{n} \alpha_\ell c_{n-\ell}(r+n-\ell) + \sum_{\ell=0}^{\infty} \beta_\ell c_{n-\ell} \right) x^{r+n}$$

$$= \sum_{n=0}^{\infty} \left[\left[(r+n)(r+n-1) + \alpha_0 (r+n) + \beta_0 \right] c_n + \underbrace{\sum_{\ell=1}^{n} \left[(r+n-\ell)\alpha_\ell + \beta_\ell \right] c_{n-\ell}}_{D_n(r)} \right] x^{r+n}$$

$$= q(r)\, c_0\, x^r + \sum_{n=1}^{\infty} \left[q(r+n)\, c_n + D_n(r) \right] x^{r+n}$$

Want this identically 0, but can't make $c_0 = 0$ so $q(r) = 0$.

so take $r =$ one of roots of $q(\xi) = 0$.

Also $q(r+1)\, c_1 + (\alpha_1 r + \beta_1)\, c_0 = 0$ must have $q(r_1 + 1) \neq 0$

or $q(r_2 + 1) \neq 0$.

I : $r_1 \neq r_2$, $\mathrm{Re}(r_1) < \mathrm{Re}(r_2)$

and $r_2 - r_1$ not an integer.

II :