# Lec 11/1

Wednesday, November 1, 2017 14:19

disregard: [A measure $\mu$ is "complete" if $\forall A$ with $\mu(A)=0$, one has $\mu(B)=0$ $\forall B \subseteq A$.

$\mu(A) > \mu(B)$ if $A \supset B$ and both are measurable.

Take $C$, classical cantor set.

Clearly $\mu(C)=0$.

the $\sigma$-algebra of Borel sets is the $\sigma$-algebra generated by intervals.

[A family of sets is called an algebra of sets if it is closed under finite set theoretical operations $\cup, \cap, []^c$

a $\sigma$-algebra is an algebra which is closed under countable operations.

First "stage": $G_\delta$ and $F_\sigma$ sets.

second stage: $G_{\delta\sigma}$, $F_{\sigma\delta}$

So the cardinality ~~of~~ of $\sigma$-algebra of Borel sets, $\mathcal{B}$, in $\mathbb{R}$ has

$$|\mathcal{B}| = |\mathbb{R}|$$

Since $|C| = |\mathbb{R}|$ $|P(C)| = 2^{|\mathbb{R}|}$ so some subsets of $C$ are not in $B$ (not Borel measurable)

Completing $B$ gives lebesgue - measurable set

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## What is on the midterm:

**H & W:**

9: Nim, Integers w/ missing digits

Cantor sets, measure zero

almost all #s are normal from ergodic thm

10: Formulas (165, 166). Thm 150, 157

Equivalent #s (orbit of action of $SL(2,\mathbb{Z})$) Thm 175.

Theorem 177 & its converse (know proof)

Hurwitz theorem / Dirichlet (10.15), Pell eqn

"fundamental pell eqn theorem" Thm 181, 182

Google: Pisot-Vijayaraghavan numbers.

11: Dirichlet Thm (Thm 185 / §11.3)

two proofs that transcendental #s exist

(cardinality + Liouville number) Thm 191.

Thm 196. Transcendence of e.

Irrationality of $e^r$ for $r \in \mathbb{Q}$.

12: Primes in $k(i)$ (§12.7). Sums of two squares (handout) NZM

Thm 252 ↔ quadratic integers handout

13: know proof in 13.4, $x^3 + y^3 = z^3$. (see quadratic integers handout)

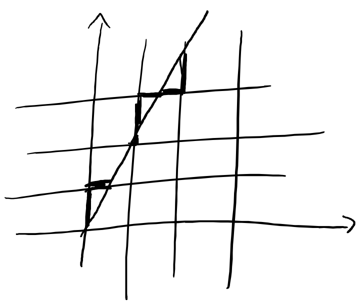
(find a shorter proof, at most 2 pages, or maybe not).

know proof in 13.3 $x^4 + y^4 = z^4$

14: know Algebraic numbers, Thm 236, 237, 238

Examples of fields which do not obey fundamental thm. (14.6)

$$z \longrightarrow \frac{az+b}{cz+d} \quad \text{if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$



Where is $SL(2,\mathbb{Z})$?

$$\xi = \underset{\text{expression in 10.11}}{\varphi(\eta)}$$

$$\eta \rightarrow \varphi(\eta) \longrightarrow \psi(\varphi(\eta)) = (\psi \circ \varphi)(\eta)$$

Group $G$ acts by $(T_g)_{g \in G}$ on a space $X$ if $\forall g \in G$, $T_g : X \to X$ is a bijection and

$$T_{g_1}(T_{g_2}(x)) = T_{g_1 g_2}(x) \quad \forall x \in X$$

and $T_e(x) = x \quad \forall x \in X$.

T/F: if $x \notin \mathbb{N}$, $x > 1$, $\{x^n \bmod 1\}$ has at least 17 accumulation points

For almost every $x$, $\{x^n \bmod 1\}$ $\underbrace{\text{is u.d. mod 1.}}_{\text{Koksma}}$

(unknown for $(3/2)^n \bmod 1$)

Examples of fields which do not obey fundamental thm. (14.6)

15: know a little bit about this section for T/F

(unknown for $(3/2)^n \bmod 1$) Koksma

google:

Pisot-Vijayaraghavan

Pell eqn Theorem $x^2 - Dy^2 = 1$, $D \in \mathbb{Z}$ nonsquare,

There are infinitely many solutions which can be obtained from the minimal one by $\begin{vmatrix} x & Dy \\ y & x \end{vmatrix}^n$

for which $|x \pm \sqrt{D}y|$ is minimal.

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$(X, \mathcal{B}, \mu, T)$ is ergodic iff $\forall$ nice $f$ ($f = 1_A$ for $\mu$-able $A$).

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^{-n}x) \longrightarrow \int_X f \, d\mu$$

iff

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}B) \longrightarrow \mu(A)\mu(B) \qquad \forall A, B \in \mathcal{B}$$

iff

$$\frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A) \longrightarrow \mu(A)^2 \qquad \forall A \in \mathcal{B}$$

Def: $T$ (on $(X, \mathcal{B}, \mu, T)$) ergodic if it moves every nontrivial set.

$\Leftrightarrow$ if there are no nice non-constant $T$-invariant functions

$T$-invariance: $f(Tx) = f(x)$ for a.e. $x$

General Thm: $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \longrightarrow f^*(x)$ a.e. $(*)$

assume $T$ ergodic, and prove that $f^* = \int f$ a.e.

1. first note that $f^*$ is $T$-invariant, hence constant if $T$ ergodic
2. Integrate both sides in $(*)$ to find value.

$\frac{1}{N} \sum$