Lec 12/2

Friday, December 2, 2016 9:08 AM

Definition f is analytic at a if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a|<R$ for power series.

If f is analytic at a, $c_n = \frac{f^{(n)}(a)}{n!}$

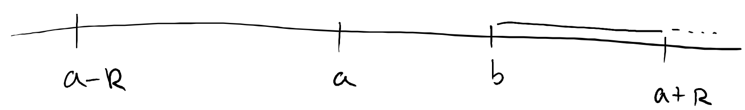
If $|b-a|<R$, then we can form the taylor series for f about b.

$$f^{(m)}(x) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (x-a)^n \quad \text{for all } x \in (a-R, a+R).$$

We can plug in $x=b$ and form the power series that way:

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(b)}{m!} (x-b)^m$$

Theorem (recentering) if f is analytic at a and Taylor series at a has radius R (>0) and $|b-a|<R$, then f is analytic at b and the Taylor series at b has a radius of convergence $R_b \geq R - |b-a|$



Remark $R_b \leq R + |a-b|$ since exchanging a and b would get another recentering theorem thing.

Examples of Recentering:

(1) Geom. series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad a=0$.

let $|b|<1$ $\quad u = x-b$, $\quad x = u+b$, $\quad \frac{1}{1-x} = \frac{1}{1-b-u} = \frac{1}{1-b} \frac{1}{1-\frac{u}{1-b}}$

Provided $|u| < 1-b$, then this $= \frac{1}{1-b} \sum_{n=0}^{\infty} \left(\frac{u}{1-b}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{1-b}\right)^{n+1} (x-b)^n$

for $|x-b| < 1-b = R_b$ so $R_b \geq 1-|b|$

(this holds $\forall b \neq 1$, better than recentering thm).

(2) exponential series: $e^x = e^b e^{x-b} = e^b \sum_{n=0}^{\infty} \frac{(x-b)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^b}{n!} (x-b)^n$

(3) Trigonometric series: $\cos(x) = \cos(b + x - b)$

$$= \cos(b)\cos(x-b) - \sin(b)\sin(x-b)$$

$$= \cos(b) \sum_{n=0}^{\infty} (-1)^n \frac{(x-b)^{2n}}{(2n)!} - \sin(b) \sum_{n=0}^{\infty} (-1)^n \frac{(x-b)^{2n+1}}{(2n+1)!}$$

$$C_m = \begin{cases} \frac{(-1)^n \cos(b)}{(2n)!} & \text{if } m = 2n \\ \frac{(-1)^{n+1} \sin(b)}{(2n+1)!} & \text{if } m = 2n+1 \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cos(b)}{(2n)!} (x-b)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sin(b)}{(2n-1)!} (x-b)^{2n+1}$$

$$= \sum_{m=0}^{\infty} C_m (x-b)^m$$

Similarly, can do something like this for $\sin(x)$.

We needed to know that $\sum_{n=1}^{\infty} c_n (x-a)^n$ was continuous within interval of convergence.

General context for proving this: notion of uniform convergence:

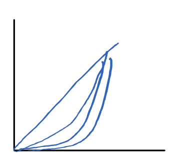
Definition Let $A \subseteq \mathbb{R}$ and suppose that $f_n : A \to \mathbb{R}$ $n = 0, 1, 2, \ldots$ and $f : A \to \mathbb{R}$. We say that $\{f_n\}$ converges uniformly to f on A if $\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N, x \in A, |f(x) - f_n(x)| < \varepsilon$.

Contrast this with pointwise convergence: we say that f_n converges pointwise to f on A if $\forall x \in A, \forall \varepsilon > 0, \exists N$ s.t. $\forall n > N$ $|f(x) - f_n(x)| < \varepsilon$.

Example of pointwise convergence which is not uniform:

Let $A = [0,1]$, $f_n(x) = x^n$, $f(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$

$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = f(x)$ $\forall x$.



Proposition: Let $\sum_{m=0}^{\infty} c_m (x-a)^m$ be a power series w. $R > 0$.

Then for any closed finite interval $[b, c] \subseteq (a-R, a+R)$

The partial sums $\left\{ S_n(x) = \sum_{m=1}^{n} c_m (x-a)^m \right\}$ converge uniformly to $\sum_{m=0}^{\infty} c_m (x-a)^m$ on $[b, c]$.

Proof: Let d be the furthest from a of b and c.

$$\left| \sum_{m=0}^{\infty} c_m (x-a)^m - S_n(x) \right| = \left| \sum_{m=n+1}^{\infty} c_m (x-a)^m \right| \le \sum_{m=n+1}^{\infty} |c_m| |x-a|^m \le \sum_{m=n+1}^{\infty} |c_m| |d-a|^m < \varepsilon$$

for n large enough since the power series converges absolutely for $x = d$, $q_d < 1$. $(|c-a| < R,\ |b-a| < R)$.

Proposition is a special case of:

Theorem (Weierstrass M-test): If $|f_m(x)| \le M_m$ for all $x \in A$ and $\sum_{m=0}^{\infty} M_m$ converges (absolutely), then $\left\{ \sum_{m=0}^{n} f_m(x) \right\}$ converges uniformly to $\sum_{m=0}^{\infty} f_m(x)$.

Proof: $\left| \sum_{m=0}^{\infty} f_m(x) - \sum_{m=0}^{n} f_m(x) \right| = \left| \sum_{m=n+1}^{\infty} f_m(x) \right| \le \sum_{m=n+1}^{\infty} M_m < \varepsilon$ for n large enough.

(in previous proof $|c_m| |d-a|^m$ stands in for M_m).

Theorem: If $\{f_n\}$ converges uniformly to f on A and f_n's are continuous on A, then f is continuous on A.

Proof: Let $\varepsilon > 0$ be given. Take N s.t. $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ for $n > N$.

Pick $n_0 > N$. Since f_{n_0} is continuous at any $a \in A$, can find a $\delta > 0$ s.t. $|f_{n_0}(x) - f_{n_0}(a)| < \frac{\varepsilon}{3}$ if $|x-a| < \delta$ and $x \in A$.

Then for $|x-a| < \delta$, $x \in A$, we have that

$$|f(x) - f(a)| \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(a)| + |f_{n_0}(a) - f(a)| < \frac{\varepsilon}{3} \times 3 = \varepsilon. \blacksquare$$