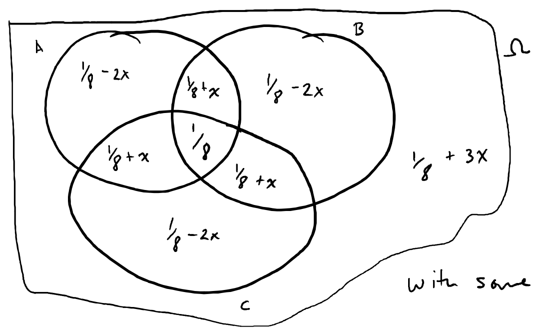
Independence: $P(AB) = P(A)P(B)$

$$P\Big(\bigcap_{\alpha\in I} A_\alpha\Big) = \prod_{\alpha\in J} P(A_\alpha) \qquad \forall\ \overset{\emptyset}{\overset{\neq}{J}} \subseteq I.$$

eg $A = \{\text{heads on } 1^{st} \text{ toss}\}$ $B = \{\text{heads on } 2^{nd} \text{ toss}\}$ $C = \{\text{same result on both coins}\}$

A, B, C are pairwise independent but not independent.

eg



has $P(ABC) = P(A)P(B)P(C)$

but no pair of events is independent.

with some $x \in \left[-\frac{1}{24}, \frac{1}{16}\right] \setminus \{0\}$.

Propn Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

Then for all $A, B, C_1, C_2, C_3, \ldots \in \mathcal{F}$,

(a) if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(b) if $A \subseteq B$ and $\mu(A) < \infty$, then $\mu(B) - \mu(A) = \mu(B \setminus A)$.

Pf: $B = A \sqcup B\setminus A$, so $\mu(B) = \mu(A) + \mu(B\setminus A)$

(c) if $A \subseteq \bigcup_n C_n$ then $\mu(A) \leq \sum_n \mu(C_n)$

Pf: Let $A_n = A \cap C_n$. $\mu(A_n) \leq \mu(C_n)$.

$\bigcup_n A_n = A$ since $A \subseteq \bigcup_n C_n$

Let $D_n = A_n \setminus \bigcup_{m<n} A_m$. Then $D_1, D_2, \ldots$ are disjoint & $\dot{\bigcup_n} D_n = A$.

hence $\mu(A) = \sum_n \mu(D_n) \leq \sum_n \mu(A_n) \leq \sum_n \mu(C_n)$.

Property (c) is called "Countable subadditivity."

(d) If $C_n \uparrow A$ (i.e. $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ and $\bigcup_n C_n = A$), then $\mu(C_n) \longrightarrow \mu(A)$.

Pf Let $C_0 = \emptyset$. Then $\dot{\bigcup}_{n=1}^{\infty} (C_n \setminus C_{n-1}) = A$, so $\sum_{n=1}^{\infty} (\mu(C_n \setminus C_{n-1})) = \mu(A)$.

and $\dot{\bigcup}_{k=1}^{n} (C_k \setminus C_{k-1}) = C_n$ so $\mu(C_n) = \sum_{k=1}^{n} (\mu(C_k \setminus C_{k-1})) \longrightarrow \mu(A)$ as $n \to \infty$.

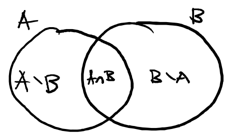
(c) If $C_k \downarrow A$ (i.e. $C_1 \supseteq C_2 \supseteq \cdots$ and $\bigcap_k C_k = A$), and $\mu(C_1) < \infty$, then $\mu(C_n) \longrightarrow \mu(A)$.

Pf Let $C_k' = C_1 \setminus C_k$. Then $C_k' \uparrow C_1 \setminus A$, so $\mu(C_k') \longrightarrow \mu(C_1) - \mu(A)$.

but $\mu(C_k') = \mu(C_1) - \mu(C_k)$ so $\mu(C_k) \longrightarrow \mu(A)$ as $k \to \infty$.

Propn Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $A, B \in \mathcal{F}$.

Then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.

Pf 

$$\mu(A \cup B) = \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A).$$
$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B).$$
$$\mu(B) = \mu(A \setminus B) + \mu(A \cap B).$$

Propn Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $A_1, A_2, \ldots, A_n \in \mathcal{F}$.

Then (a) $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 A_2)$.

(b) $\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) - \mu(A_1 A_2) - \mu(A_1 A_3) - \mu(A_2 A_3) + \mu(A_1 A_2 A_3)$.

(c) $\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|-1} \mu\left(\bigcap_{i \in I} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{i_1 < \cdots < i_k} \mu\left(\bigcap_{j=1}^{k} A_{i_j}\right)\right)$.