Lec 1/8

Wednesday, January 8, 2020 11:30

Monic: Left cancellation.

$$C \overset{g_1}{\underset{g_2}{\rightrightarrows}} A \xrightarrow{f} B$$

$$fg_1 = fg_2 \Rightarrow g_1 = g_2$$

Epic: Right Cancellation

$$A \xrightarrow{f} B \overset{h_1}{\underset{h_2}{\rightrightarrows}} C$$

$$h_1 f = h_2 f \Longrightarrow h_1 = h_2$$

<u>Basic Facts</u>

$$A \xrightarrow{f} B \xrightarrow{g} C$$

f, g monic (epic) $\Rightarrow$ gf monic (epic),

gf monic (epic) $\Rightarrow$ f monic (g epic).

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

if $gf = 1$ then f is monic & g is epic.

<u>Example</u>: in <u>Set</u>, <u>R-mod</u>, <u>Grp</u>, monic $\Leftrightarrow$ injective & epic $\Leftrightarrow$ surjective.

<u>pf ($\Rightarrow$)</u> (R-mod) • Suppose $f : A \to B$ is not injective. Let $C = \ker(f)$. $C \neq 0$.

Let $g : C \hookrightarrow A$. Then $f0 = fg$ but $0 \neq g$.

(R-mod) Let $g : C \hookrightarrow A$. Then $f0 = fg$ but $0 \neq g$.

- Suppose $f_{:A \to B}$ is not surjective. Let $I = \mathrm{Im}(f)$. Let

  $g : B \longrightarrow B/_{\mathrm{Im}(f)}$. Then $gf = 0f$ but $g \neq 0$ since $\mathrm{Im}(f) \neq B$.

  $\qquad\qquad\quad \| \\ \qquad\qquad \mathrm{coker}(f).$

For groups, the first argument works but the second doesnt (coker is not necessarily in the category).

<u>Example</u>: in <u>Ring</u> monic $\Leftrightarrow$ injective and epic $\Leftarrow$ surjective but epic $\not\Rightarrow$ surjective.

<u>pf</u> $(\Rightarrow)$ Suppose $f_{:A \to B}$ not injective. We form the product ring $A \times A$ and consider $K = \{(a_1, a_2) \in A \times A \mid f(a_1) = f(a_2)\}$.

$$K \supsetneq D = \{(a,a) \mid a \in A\}.$$

maps $g_1, g_2 : K \longrightarrow A$ are projections onto 1^{st} & 2^{nd} factors.

$$\begin{tikzcd} & K \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ A \arrow[d, "f"] & & A \arrow[d, "f"'] \\ B & & B \end{tikzcd}$$

Then $fg_1 = fg_2$ but $g_1 \neq g_2$

since $\exists (a_1, a_2) \in K \setminus D$.

$$g_1 \downarrow a_1 \qquad g_2 \downarrow a_2$$

<u>Counterexample</u>: $\mathbb{Z} \longrightarrow \mathbb{Q}$ is epic.

<u>slight generalization</u>: R - any comm ring

$\cup$

S - any mult. closed subset

$R \hookrightarrow S^{-1}R$ is epic but not necessarily surj.