Lec 11/4

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<u>Thm</u> (open Mapping): if X, Y Banach & $T \in \mathcal{L}(X,Y)$ is surj, T is open.

<u>cor</u>: If T is bijective, T is bdd.

<u>cor</u>: if X Banach under $\|\cdot\|_1$ & $\|\cdot\|_2$ and $\exists c>0$ s.t. $\|x\|_1 \le c\|x\|_2$ $\forall x \in X$,

Then $\|\cdot\|_1$ & $\|\cdot\|_2$ are equiv.

<u>Exercise</u>: suppose $\|\cdot\|_1$ & $\|\cdot\|_2$ are norms on X ~~Banach~~ normed which induce the same topology. Then $\|\cdot\|_1 \sim \|\cdot\|_2$.

<u>Def</u>: X, Y normed spaces & $T: X \to Y$ linear.

The graph of T is $\Gamma(T) = \{(x,y) \mid Tx = y\} \subset X \times Y$ subspace

T is <u>closed</u> if $\Gamma(T)$ is closed. $\|\cdot\|_{X\times Y} = \max\{\|x\|, \|y\|\}$

<u>Thm</u> (Closed graph): $T: X \to Y$ linear, Closed $\Rightarrow$ bdd.

<u>Thm</u> (Banach-Steinhaus/Uniform bddness Principle):

Suppose X, Y normed sp & $\mathcal{S} \subset \mathcal{L}(X,Y)$.

① if $\sup_{T\in\mathcal{S}} \|Tx\| < \infty$ $\forall x$ in a non-meager subset of X,

Then $\sup_{T\in\mathcal{S}} \|T\| < \infty$

② If X banach & $\sup_{T\in\mathcal{S}} \|Tx\| < \infty$ $\forall x \in X$, $\sup_{T\in\mathcal{S}} \|T\| < \infty$.

$^{i \to \infty}$

#① Define $E_n = \{x \in X \mid \sup_{T \in \mathcal{S}} \|Tx\| \le n\}$

$$= \bigcap_{T \in \mathcal{S}} \underbrace{\{x \mid \|Tx\| \le n\}}_{\text{Closed}}$$

$\Rightarrow E_n$ is closed. $\cup E_n$ nonmeager so some E not nowhere dense.

By assumption, $\exists x_0 \in X$, $r > 0$, $n > 0$ s.t.

$\overline{B_r(x_0)} \subset E_n$

$\overline{B_r(0)} \subset E_{2n}$:

$$\|Tx\| \le \|T(x - x_0)\| + \|Tx_0\| \le 2n \text{ when } \|x\| \le r$$

Thus $\forall\, T \in \mathcal{S}$, $\forall\, x$ w/ $\|x\| \le r$, $\|Tx\| \le 2n$.

$\Rightarrow \|T\| \le \frac{2n}{r}\ \forall\, T \in \mathcal{S}$

$\Rightarrow \sup_{T \in \mathcal{S}} \|T\| \le \frac{2n}{r}$.

② If X is banach, the sets E_n in ①
cannot all be meager by BCT
$(X = \cup E_n)$. $\square$

$\|(y_T)\| = \sup_{T \in \mathcal{S}} \|y_T\|$

<u>Using closed graph Thm:</u> $X \xrightarrow{S} \prod_{T \in \mathcal{S}} Y$ ← Y banach

X banach

X banach $\quad T \in \mathcal{S}$

$$x \longmapsto (Tx)_{T\in\mathcal{S}} \qquad \text{linear op.}$$

Claim: $S : X \longrightarrow \prod_{T\in\mathcal{S}} Y$ has closed graph,

so $\|S\| < \infty$

pf suppose $x_\lambda \longrightarrow x$ in X

$Sx_\lambda \longrightarrow y$ in $\prod Y$

Show $Sx = y$.

$\|y_T - Tx_\lambda\| \xrightarrow{\lambda} 0 \quad \forall\, T$

$y_T = Tx \quad \forall\, T \in \mathcal{S}$

Exercise: flesh this out.

Reference: Pedersen: Analysis Now.

Topological Vector Spaces: A v.s. X over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) equipped w/ a topology is a TVS if

$+ : X \times X \longrightarrow X$

$\cdot : \mathbb{K} \times X \longrightarrow X$

are cts.

- A subset $C \subseteq X$ is called convex if $\forall\, x, y \in C$, $\forall\, t \in [0,1]$, $tx + (1-t)y \in C$ as well.

- We'll focus on <u>locally convex</u> TVS's

$\forall$ open $U \subset X$, $x \in U$, $\exists$ open convex $V \subset U$ s.t $x \in V$.

- Let X be a $\mathbb{K}$-vs and $\{p_i\}_{i\in I}$ a family of <u>seminorms</u> on X.

for $x \in X$, $i \in I$, $\varepsilon > 0$, define

$$U_{x,i,\varepsilon} = \{ y \in X \mid p_i(x-y) < \varepsilon \}.$$

Let τ be the topology gen by $U_{x i \varepsilon}$'s.

<u>Facts</u>:

⓪ τ is weakest top s.t. $p_i : X \longrightarrow [0,\infty)$ is cts $\forall i$.

① $\forall x \in X$, finite intersections $\{ y \in X \mid p_i(x-y) < \varepsilon \ \ \forall i \in \{i_1, \dots, i_k\} \}$

$$= \bigcap_{j=1}^{k} U_{x\, i_j\, \varepsilon} \longleftarrow \text{Basic open sets.}$$

constant

form a nhd base at x.

Pf τ consists of arbitrary unions of finite intersections of $U_{xi\varepsilon}$'s.

Suppose $x \in \overset{\text{open}}{U} \subset X$ Then $\exists \begin{matrix} i_1, \dots, i_n \\ x_1, \dots, x_n \\ \varepsilon_1, \dots, \varepsilon_n \end{matrix}$ s.t. $x \in \bigcap_{k=1}^{n} U_{x_k i_k \varepsilon_k}$.

Define $\varepsilon = \min_k \Big[\underbrace{\varepsilon_k - p_{i_k}(x - x_k)}_{>0 \ \forall k} \Big]$

Then $\forall y \in \bigcap_{k=1}^{n} U_{x i_k \varepsilon}$, $\quad p_{i_k}(x-y) \le p_{i_k}(x-x_k) + p_{i_k}(x_k - y)$

$$\le \varepsilon_k - \varepsilon \quad + \varepsilon$$

$$= \varepsilon_k$$

$\rightsquigarrow y \in \bigcap_{k=1}^{n} U_{x_k i_k \varepsilon_k}$.

② If $(x_\lambda) \subset X$ is a net, $x_\lambda \to x$ iff $p_i(x_\lambda - x) \to 0 \quad \forall i \in I$.

(possible Notation change: $S = \{p_i\}_{i \in I}$, so $\{p_i \mid i \in I\} = \{p \mid p \in S\}$)

pf observe $x_\lambda \to x$ iff x_λ eventually in $U_{x i \varepsilon}$ $\forall$ $\varepsilon > 0$, $i \in I$

iff $p_i(x - x_\lambda) \to 0 \ \forall i$

③ (X, τ) is a TVS:

$+$: $x_\lambda \to x$, $y_\lambda \to y$. then $\forall$ $i \in I$,

$$p_i(x + y - x_\lambda - y_\lambda) \le p_i(x - x_\lambda) + p_i(y - y_\lambda) \longrightarrow 0$$

$\cdot$: if $\alpha_\lambda \to \alpha$, $x_\lambda \to x$, then $\forall i \in I$,

$$p_i(\alpha x - \alpha_\lambda x_\lambda) \le p_i(\alpha_\lambda x_\lambda - \alpha_\lambda x) + p_i(\alpha_\lambda x - \alpha x)$$

$$\le \alpha_\lambda p_i(x_\lambda - x) + (\alpha_\lambda - \alpha) p_i(x) \longrightarrow 0$$

④ (X, τ) is Locally convex

pf observe $U_{x i \varepsilon}$ is convex:

if $y, z \in U_{x i \varepsilon}$ then $\forall t \in [0,1]$,

$$p_i(x-(ty+(1-t)z)) \le p_i(tx-ty) + p_i((1-t)x-(1-t)z)$$
$$= t\,p_i(x-y) + (1-t)\,p_i(x-z)$$
$$< t\varepsilon + (1-t)\varepsilon = \varepsilon.$$

Exercises:

(5) X is Hausdorff iff $\{p_i\}_{i\in I}$ separates pts

iff $\forall x \in X$, $\exists i \in I$ s.t. $p_i(x) \neq 0$.

(6) If X Hausdorff & I is ctble, $\exists$ a translation invariant metric $\rho : X\times X \to [0,\infty)$ s.t.

$\rho(x+z, y+z) = \rho(x,y)$ $\forall z \in X$ which induces τ

$$\rho(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^i}\, p_i(x-y).$$