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$\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{(n,m)}$.

Pf: We have a hom $\varphi: \mathbb{Z} \to \mathbb{Z}_n \otimes \mathbb{Z}_m$, $k \xmapsto{\varphi} k \otimes 1$ surjective.
$(d) \subset \operatorname{Ker} \varphi$ so φ reduces to a hom-sm $\mathbb{Z}_d \to \mathbb{Z}_n \otimes \mathbb{Z}_n$.

Consider $\beta: \mathbb{Z}_n \times \mathbb{Z}_m \longrightarrow \mathbb{Z}_d$ defined by $\beta(k,\ell) = k\ell \mod d$.
β is bilinear so it induces a hom-sm $\psi: \mathbb{Z}_n \otimes \mathbb{Z}_m \to \mathbb{Z}_d$.
Claim: $\psi = \varphi^{-1}$.

Pf $\varphi(\psi(k \otimes \ell)) = \varphi(k\ell) = k\ell \otimes 1 = k \otimes \ell$
$\psi(\varphi(k)) = \psi(k \otimes 1) = k$

Theorems: $M_1 \otimes M_2 \cong M_2 \otimes M_1$
①: $u_1 \otimes u_2 \longleftrightarrow u_2 \otimes u_1$

proof: $\beta: M_1 \times M_2 \longrightarrow M_2 \otimes M_1$
$\beta(u_1, u_2) = u_2 \otimes u_1$ is bilinear

So, induces $\varphi: M_1 \otimes M_2 \longrightarrow M_2 \otimes M_1$
$u_1 \otimes u_2 \longmapsto u_2 \otimes u_1$,

Same way the other way, and $\psi = \varphi^{-1}$.

In $M \otimes M$, $u_1 \otimes u_2 \neq u_2 \otimes u_1$. Steiner's Lemma

②: $\forall M_1, M_2, M_3$, $M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3$

Good

$$u_1 \otimes (u_2 \otimes u_3) \longleftrightarrow (u_1 \otimes u_2) \otimes u_3$$

3: $(M_1 \oplus M_2) \otimes M_3 = (M_1 \otimes M_3) \oplus (M_2 \otimes M_3)$

$$(u_1, u_2) \otimes u_3 \longleftrightarrow (u_1 \otimes u_3, u_2 \otimes u_3)$$

proof: Consider $\beta : (M_1 \oplus M_2) \times M_3 \longrightarrow (M_1 \otimes M_3) \oplus (M_2 \otimes M_3)$

$$((u_1, u_2), u_3) \longmapsto (u_1 \otimes u_3, \; u_2 \otimes u_3)$$

So β induces a hom-sm $\varphi : (M_1 \oplus M_2) \otimes M_3 \longrightarrow (M_1 \otimes M_3) \oplus (M_2 \otimes M_3)$

s.t. $\varphi((u_1, u_2) \otimes u_3) = \beta((u_1, u_2), u_3)$.

define $\varphi_1 : M_1 \otimes M_3 \longrightarrow (M_1 \oplus M_2) \otimes M_3$ by $\varphi_1(u_1 \otimes u_3) = (u_1, 0) \otimes u_3$.

$\beta : M_1 \times M_3 \nearrow$ by $\beta(u_1, u_3) = (u_1, 0) \otimes u_3$ is bilinear.

$\varphi_2 : M_2 \otimes M_3 \longrightarrow (M_1 \oplus M_2) \otimes M_3$ by $\varphi_2(u_2 \otimes u_3) = (0, u_2) \otimes u_3$

By universal property of $\oplus$, $\exists$ unique

$$\varphi : (M_1 \otimes M_3) \oplus (M_2 \otimes M_3) \longrightarrow (M_1 \oplus M_2) \otimes M_3$$

which agrees with φ_1 & φ_2.

$$\begin{aligned}
\varphi((u_1 \otimes u_3, u_2 \otimes u_3)) &= (\varphi_1(u_1 \otimes u_3), \; \varphi_2(u_2 \otimes u_3)) \\
&= (u_1, 0) \otimes u_3 + (0, u_2) \otimes u_3 \\
&= (u_1 + u_2) \otimes u_3 .
\end{aligned}$$

define $M^n = \bigoplus_n M$.

$$\underset{\parallel}{M_1^{n_1}} \otimes \underset{\parallel}{M_2^{n_2}} = (M_1 \otimes M_2)^{n_1 n_2}$$

$\forall M_1, M_2 \quad \forall n_1, n_2 \in \mathbb{Z} \quad (\bigoplus M_1) \otimes (\ldots) \ldots \bigoplus \ldots$

$$\forall M_1, M_2, \ \forall n_1, n_2 \in \mathbb{Z}, \ \left(\bigoplus_{n_1} M_1\right) \otimes \left(\bigoplus_{n_2} M_2\right) = \bigoplus_{n_1 \cdot n_2} (M_1 \otimes M_2)$$

$$R^n \otimes M = \bigoplus_n (R \otimes M) = \bigoplus_n M = M^n$$

$$R^n \otimes R^m = R^{nm}$$

<u>Extension of Scalars</u>: Let S be an R-algebra.

Let M be an R-module.

$S \otimes_R M$ - R-module

It has an S-module structure by

$\alpha(\beta \otimes u) = (\alpha\beta) \otimes u$. $\quad$ Note: $(\beta \otimes u \neq \beta \cdot (1 \otimes u))$

<u>Check</u>: $\alpha_1(\alpha_2(\beta \otimes u)) = (\alpha_1\alpha_2\beta) \otimes u = (\alpha_1\alpha_2)(\beta \otimes u)$

$(\alpha_1 + \alpha_2)(\beta \otimes u) = \alpha_1(\beta \otimes u) = \alpha_2(\beta \otimes u)$

$\alpha(\omega_1 + \omega_2) = \alpha\omega_1 + \alpha\omega_2$? $\quad$ (where $\omega_1, \omega_2 \in S \otimes M$).