Lec 9/30

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<u>Ideals</u> Let I & J be ideals.

$$I+J = (I \cup J).$$

$$IJ = (\{ab \mid a \in I, b \in J\}).$$

As sets, $I+J = \{a+b \mid a \in I, b \in J\}$

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \;\middle|\; n \in \mathbb{N}, a_i \in I, b_i \in J \right\}.$$

<u>Ex</u> let I, J, K be ideals. Then

$$I(J+K) = \{a_1(b_1+c_1) + \cdots + a_m(b_m+c_m) \mid a_i \in I, b_i \in J, c_i \in K, m \in \mathbb{N}\}$$

$$= \{(a_1b_1 + \cdots + a_mb_m) + (a_1c_1 + \cdots + a_mc_m) \mid \text{"} \quad \text{"}\}$$

$$\subseteq IJ + IK.$$

Conversely $IJ \subseteq I(J+K)$ and $IK \subseteq I(J+K)$

so $IJ + IK \subseteq I(J+K)$. So $I(J+K) = IJ + IK$.

<u>Def</u> Ring homomorphisms$^{\eta}$ are homomorphisms of rings.

(Note that $\underbrace{1 \mapsto 1}$).

this condition is superfluous if η is surj.

since $\eta(1)\eta(x) = \eta(x) = \eta(x)\eta(1)$, and 1 is unique.

Ex Let I be an ideal in R. $\pi : R \longrightarrow R/I$ is the natural projection homomorphism. it's surjective.

Ex Let $\eta : R \longrightarrow R'$ be a ring hom. The kernel of η is $I = \eta^{-1}(0)$, which is an ideal of R.

This induces an injective ring hom
$$R/I \xrightarrow{\bar{\eta}} R', \quad a + I \longmapsto \eta(a).$$

Fundamental thm of Ring homs:

Let $\eta : R \to R'$ be a ring hom & let $I = \ker(\eta)$. then the following diagram commutes.

$$\begin{tikzcd}
R \arrow[r, "\eta"] \arrow[d, "\pi"'] & R' \\
R/I \arrow[ur, dashed, "\exists!\,\bar{\eta}"'] &
\end{tikzcd}$$

Corollary: $R/I \cong \eta(R)$

Thm The 1-1 correspondence of the set of subgps of $(R, +, 0)$

containing an ideal $I \subseteq R$ with the set of subgps of $(R/I, +, 0+I)$ pairs subrings of R containing I with subrings of R/I, and pairs ideals $J \supset I$ with ideals of R/I.

Moreover, if J is an ideal containing I, then $R/J \cong {}^{R/I}\big/_{J/I}$ via the map $a + J \longmapsto \pi(a) + J/I$.

<u>Ex</u> a <u>maximal ideal</u> of R is a proper ideal J of R s.t. there is no ideal $I \subsetneq R$ with $J \subsetneq I$.

Suppose R is a commutative ring and I is a maximal ideal of R. Then R/I is a field since the only ideals of R/I are 0 and R/I.

Converselly, if R/I is a field then I is maximal.

<u>Ex</u> A <u>prime ideal</u> of a commutative ring is a proper ideal I s.t. if $ab \in I$ then $a \in I$ or $b \in I$.

<u>Prop</u> Let R be commutative. If $I \subseteq R$ is a maximal ideal, I is prime.

Pf Suppose not. Then $\exists\, a,b \in R$ s.t. $ab \in I$ but neither of a,b is in I.
So looking at R/I, $\overline{ab} = 0$ but $\bar{a} \neq 0$, $\bar{b} \neq 0$. but this contradicts the fact that a field has no zero divisors.

Ex Let $R = \mathbb{Z}/(6)$. ideals of $R = \{(\bar{0}), (\bar{2}), (\bar{3}), \underset{\overset{\|}{R}}{(\bar{1})}\}$

Prime ideals $= \{(\bar{2}), (\bar{3})\}$

max'l ideals $=$ same

subrings $= \{(\bar{0}), R = (\bar{1})\}$

Note: a subring must use the same multiplicative identity as the ring.

Thm Let R be a ring, S a subring, I an ideal in R.
Then $S + I = \{s + a \mid s \in S, a \in I\}$ is a subring of R;
I is an ideal of $S+I$, $S \cap I$ is an ideal of S,
and $(S+I)/I \cong S/S\cap I$ via $s + I \longmapsto s + (S \cap I)$.

$s + (S \cap I) \longmapsto s + I$

Let e denote the mult. identity of R.

The map $\mathbb{Z} \longrightarrow R$ given by $n \longmapsto ne$ is a ring homomorphism.

So the image of $\mathbb{Z}$ is a subring of R. $\mathbb{Z}e$

Moreover, any nonzero subring of R contains $\mathbb{Z}e$ since it contains e.

$\mathbb{Z}e$ is called the prime ring of R.

By the fund thm, $\mathbb{Z}e = \mathbb{Z}/K$ where K is kernel of map.

$K = (k)$ for some $k \geq 0$, by classification of ideals in $\mathbb{Z}$.

so $\mathbb{Z}e = \mathbb{Z}/(k)$.