Lec 10/25

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Let G_i be the set of #s representable as product of $0 \le i \le \infty$ distinct prime powers. Show that $\delta(G_i) = 0$. (exercise)

Convex functions: $\forall x_1, x_2$, $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$

(midpoint convexity, adequate for continuous fns).

Does it imply $f(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f(x_1) + \alpha_2 f(x_2)$ (exercise)

$(\forall\ \alpha_1 + \alpha_2 = 1,\ \alpha_i > 0)$

exercise give an example of a discontinuous yet midpoint convex function which is not convex in the sense above.

(hint: use Hamel Basis)

Convex combination of $x_1, \dots, x_k$ is any $\sum_{i=1}^{k} \alpha_i x_i$ where $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \ge 0$.

$$\frac{a_1 + a_2}{2} \ge \sqrt{a_1 a_2} \qquad \forall \alpha_i \ge 0.$$

$$\alpha_1 x_1 + \alpha_2 x_2 \ge x_1^{\alpha_1} x_2^{\alpha_2} \qquad \forall \alpha_i \ge 0,\ \alpha_1 + \alpha_2 = 1.$$ (exercise)

$$\frac{x_1 + \cdots + x_k}{k} \ge \sqrt[k]{x_1 \cdots x_k}$$ (Lagrange multipliers)

$$\alpha_1 x_1 + \cdots + \alpha_k x_k \ge x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$ (exercise)

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \le \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

ex: show this is true $\forall$

$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \leq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$

$\alpha_i \geq 0, \quad \sum \alpha_i = 1$ (Jensen) Ex: show this is true $\forall$ convex f.

$x^2 + 1 \equiv 0 \mod p.$ Use: $x^2 \equiv -1 \mod 4$, Legendre symbols.

$x^2 + y^2 + 1 \equiv 0 \mod p$

Measures:

$\mu(A) \geq 0, \quad \mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for measurable sets.

σ - additivity.

$\mu(\emptyset) = 0.$

If $d(A)$, $d(B)$ exist, does $d(A \cup B)$ exist? No. (exercise)

or $A \cap B$

$\downarrow$ $\downarrow$

$2\mathbb{N}$ intermittent $2\mathbb{N}, 2\mathbb{N}+1$

$\bar{d}(A \cap (A - n^2)) > 0 \qquad \mu(A \cap T^{-n^2} A^2) \qquad x \in T^{-1}(A) \Leftrightarrow Tx \in A.$

$\bar{d}(A - 17) = \bar{d}(A) \quad \forall 17.$

Measure spaces $(X, \mathcal{B}, \mu)$

set, nice subsets of X, measure on $\mathcal{B}$.

If $\mu(X)=1$ (we will usually assume this) then sometimes $(X, \mathcal{B}, \mu)$ called probability space.

$\mu(A_i) \geq 0$, $\mu(\cup A_i) = \sum \mu(A_i)$, $\mu(\emptyset) = 0$.

$X = \{0,1\}^{\mathbb{N}}$ $\qquad \overbrace{A_{1,0} = \{x \in \{0,1\}^{\mathbb{N}} : x_1 = 0\} \quad A_{11} = \{x \in \{0,1\}^{\mathbb{N}} : x_1 = 1\}}^{\text{Cylinder sets.}}$.

$\mu(A_0) = p \quad \mu(A_1) = q \quad A_0, A_1 =$ clopen, $A_0 \cup A_1 = X$

$\rho(x,y) = \sum \frac{|x_i - y_i|}{2^i}$ induces topology.

$\mu(A_{17,0}) = p$, $\mu(A_{17,1}) = q$

$\mu(\underset{0}{\times} \underset{1}{\times} \underset{1}{\times}) = pq^2$

p-q product measure (assign ps, qs to cylinders).

$\mu(A \cap B) = \mu(A)\mu(B)$ (independence)

$T: X \to X$ is called measure preserving if

$\forall A \in \mathcal{B}$, $\mu(T^{-1}A) = \mu(A)$ $\quad (x \in T^{-1}A \Leftrightarrow Tx \in A)$

$f: \mathbb{R} \to \mathbb{R}$ is cont. iff $\forall$ open U, $f^{-1}(U)$ is open.

exercise: prove equivalence to ε-δ

$x \mapsto 2x \bmod 1$ measure preserving

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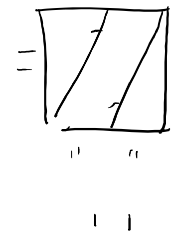
(but not under 'Dan's defn')



Theorem: let T preserve measure on $(X, \mathcal{B}, \mu)$.

then $\forall A \in \mathcal{B}$ w $\mu(A) > 0$, $\exists n \in \mathbb{N}$ s.t.

$$\mu(A \cap T^{-n} A) > 0 \quad \text{(Poincare recurrence thm)}$$

Proof: Consider $A, T^{-1}A, \ldots, T^{-i}A, \ldots, T^{-j}A, \ldots$

$$0 < \mu(T^{-i}A \cap T^{-j}A) = \mu(A \cap T^{-(j-i)}A)$$

use: $T^{-i}A \cap T^{-j}A \equiv T^{-i}(A \cap T^{-(j-i)}A)$ (check why, google use preimage)