Lec 9/6

Wednesday, September 6, 2017 14:24

① $x - y = n^2$ solvable in any $A \subseteq \mathbb{N}$ with $\bar{d}(A) > 0$.

$\Updownarrow$

② $\bar{d}(A \cap (A - n^2)) > 0$. ← Sárközy

If $\bar{d}(A) > 0$ is there $d \in \mathbb{N}$ s.t. A contains arbitrarily long APs w/ difference d. (*)

Ch 7 reading: beginning thru 112. (Wolstenholme & von Staudt optional)

Chebyshev's bias:

$4n+1$ and $4n+3$ seem to be unequal.

but:

Dirichlet: If $(a,b) = 1$ then $\{an+b : n \in \mathbb{N}\}$ contains infinitely many primes.

Sharper form of Dirichlet's Theorem:

$$\frac{|P \cap \{4n+1,\ n \leq N\}|}{|P \cap \{4n+3,\ n \leq N\}|} \longrightarrow 1$$

(can generalize this to $an+b$).

Littlewood: Chebyshev bias "switches" infinitely many times.

$\sum \frac{1}{n} \to \infty$

is γ irrational?

$\sum_{k=1}^{n} \frac{1}{k} \sim \log n$ stronger than $\left(\sum_{k=1}^{n} \frac{1}{k} - \log n \to \gamma \right)$

$\frac{n^2+n}{n^2} \to 1$ but this is weaker equivalence. $n^2+n - n^2 \to \infty$.

$p_n \sim n \log n$

Speed of: $\boxed{\sum_{k=1}^{n} \frac{1}{p_k} \sim \log(\log n)}$ Exercise: is this true?

is it true that $\sum_{k=2}^{n} \frac{1}{k \log k} \sim \log(\log n)$? (exercise)

$\sum \frac{1}{q_n} < \infty$ where q_n is a "leading twin prime"

(Viggo Brun)

Szemerédi: ① $\bar{d}(A) > 0 \Rightarrow A$ is AP-rich

$\Updownarrow$

② $\bar{d}(A) > 0 \Rightarrow \forall n \, \exists k$ s.t. $\bar{d}(A \cap (A-n) \cap \cdots \cap (A-kn)) > 0$

If $\{n : A \cap (A-n) \cap \underline{(A-2n)} \neq \emptyset\}$ is finite

[if $\bar{d}(A) = a$, then $\bar{d}(A \cap (N, \infty)) = a$

$\exists n$ s.t. A contains infinitely many

$\boxed{\exists n \text{ s.t. } A \text{ contains infinitely many progressions of length 3 and difference } n.}$

follows from

③ (finitistic version of Szemerédi theorem)

$\forall \varepsilon > 0$, $\forall \ell \in \mathbb{N}$, $\exists C = C(\varepsilon, \ell)$ s.t. if

I is an interval of length $\geq C$ and

$A \subset I$ and $\dfrac{|A|}{|I|} \geq \varepsilon$, Then

A contains AP of length ℓ.

(Exercise): show (3) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2)

(use equal-length intervals)

(Exercise) formulate finitistic version of Sárközy.

Hint for exercise (☆): What are 'bad sets'

$\{ n : \ n\alpha \bmod 1 < \varepsilon \}$

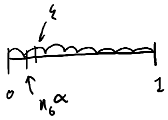
Claim: $\forall \alpha \notin \mathbb{Q}$, the sequence $n\alpha \bmod 1 = \{n\alpha\} = n\alpha - [n\alpha]$ $n \in \mathbb{N}$.

is dense in $[0,1]$.

Proof: it is enough to show $\forall \varepsilon > 0$ $\exists n$ s.t. $n\alpha \bmod 1 < \varepsilon$.

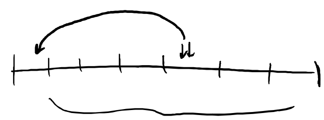
| Reason: it is enough to show $S = \{ n\alpha \bmod 1;\ n \in \mathbb{N} \}$

ε



(Reason: it is enough to show $S = \{n\alpha \bmod 1;\ n \in \mathbb{N}\}$ is ε-dense in $[0,1]$. Let $n_0\alpha \bmod 1 < \varepsilon$.
$n \cdot n_0\alpha$ fills gaps)

Take $\varepsilon = \frac{1}{10}$. Consider $\alpha, 2\alpha, \dots, 10\alpha, 11\alpha$



10 subintervals, 11 numbers, 2 end up in same interval. their positive difference is $< \frac{1}{10}$.

$= A,\ \bar{d}(A) \geq \varepsilon > 0.$

$\downarrow$

infinite by above argument.