# Lec 12/2

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<u>Last time</u>: Lebesgue-Radon-Nikodym thm

$\mu$ $\sigma$-finite pos measure, $\nu$ $\sigma$-finite signed measure.

$\Rightarrow$ $\exists!$ $\sigma$-finite $\lambda, \rho$ s.t. $\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \lambda + \rho$. $\exists!$ $\overset{\mu-}{\text{a.e.}}$ extended $\mu$-$\int$ble $f$ s.t. $d\rho = f d\mu$. If $\nu$ is pos/finite, so is $\lambda/\rho$.

<u>Remark</u>: If $\nu \ll \mu$, $\exists!$ $f$ s.t. $d\nu = f d\mu$.

call $f$ the Radon-Nikodym derivative of $\nu$ wrt $\mu$:

$$f = \frac{d\nu}{d\mu}.$$

<u>Exercise</u>: if $\mu$ $\sigma$-finite signed meas,

$$\left| \frac{d\mu}{d|\mu|} \right| = 1 \quad \mu\text{-a.e.} \quad \leftarrow \quad \underline{\text{know how to do this}}$$

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<u>Def</u>: let $X$ be LCH. A signed Borel measure $\mu$ on $X$ is a <u>signed Radon measure</u> if $\mu_\pm$ are Radon.

(Here $\mu = \mu_+ - \mu_-$ is the ! Jordan decomp).

let $RM \subseteq \underset{\sim}{M}$ be the subspace of finite signed Radon measures.

Banach space of finite signed measures, $\|\mu\| := |\mu|(X)$.

Exercise: If $\mu$ is a positive Radon measure on $X$ (LCH),

Then $C_c(X)$ is dense in $L^1(\mu)$.

Lusin's Thm: If $\mu$ is a positive Radon meas on $X$ (LCH),

and $f: X \to \mathbb{C}$ is mble & vanishes outside a set of finite measure, then $\forall \varepsilon > 0$ $\exists g \in C_c(X)$ s.t. $g = f$ except on a set of measure $< \varepsilon$.

If $\|f\|_\infty < \infty$, we can pick $g$ s.t. $\|g\|_\infty < \|f\|_\infty$.

If $\mathrm{Im}(f) \subset \mathbb{R}$, we can pick $g$ s.t. $\mathrm{Im}(g) \subset \mathbb{R}$.

Theorem (Riesz Rep'n): Suppose $X$ is LCH, define $\varphi: RM \longrightarrow C_0(X, \mathbb{R})^*$ by $\mu \mapsto \varphi_\mu$ where $\varphi_\mu(f) = \int f\, d\mu$. Then $\varphi$ is an iso. iso.

($\Rightarrow RM \subset M$ is closed subspace $\leftarrow$ know how to prove this directly).

Pf we already showed $\varphi$ is surjective. If $\mu \in RM$, then

$$\left|\int f\, d\mu\right| = \left|\int f d\mu_+ - \int f d\mu_-\right| \le \left|\int f d\mu_+\right| + \left|\int f\, d\mu_-\right| \le \int |f| d\mu_+ + \int |f| d\mu_- = \int |f| d|\mu| \le \|f\|_\infty \|\mu\|.$$

So $\|\varphi_\mu\| \le \|\mu\|$ (LHS is $|\varphi_\mu(f)|$). Moreover, $\left|\frac{d\mu}{d|\mu|}\right| = 1$ $|\mu|$-a.e.

Let $\varepsilon > 0$. By Lusin's thm, $\exists$ $f \in C_c(X, \mathbb{R})$ s.t. $\|f\|_\infty = 1$ and

$f = \overline{\frac{d\mu}{d|\mu|}}$ except on $E \in \mathcal{B}_X$ w/ $|\mu|(E) < \frac{\varepsilon}{2}$. Then

$f = \overline{\frac{d\mu}{d|\mu|}}$ except on $E \in \mathcal{B}_X$ w/ $|\mu|(E) < \frac{\varepsilon}{2}$. Then

$$\|\mu\| = \int d|\mu| = \int \left|\frac{d\mu}{d|\mu|}\right|^2 d|\mu| = \int \overline{\frac{d\mu}{d|\mu|}} \frac{d\mu}{d|\mu|} d|\mu| = \int \overline{\frac{d\mu}{d|\mu|}} d\mu$$

$$\leq \left|\int f\,d\mu\right| + \left|\int \underbrace{\left(f - \overline{\frac{d\mu}{d|\mu|}}\right)}_{\|f - \overline{\frac{d\mu}{d|\mu|}}\|_\infty \leq 2} d\mu\right| \leq \|\varphi_\mu\| \cdot \underbrace{\|f\|_\infty}_{1} + 2|\mu|(E) = \|\varphi_\mu\| + \varepsilon$$

so $\|\mu\| \leq \|\varphi_\mu\|$, and so $\|\varphi_\mu\| = \|\mu\|$. So $\varphi$ is iso iso. $\square$

<u>Complex measures</u>: Let $(X, \mathcal{M})$ be mble sp. a fn $\nu : \mathcal{M} \to \mathbb{C}$ is a complex measure if

- $\nu(\emptyset) = 0$, and
- $\forall$ disjoint seq $(E_n) \subset \mathcal{M}$, $\nu\left(\bigsqcup E_n\right) = \underbrace{\sum \nu(E_n)}_{\text{absolutely convergent.}}$

<u>Exercise</u>: If $\nu$ complex meas on $(X, \mathcal{M})$, $\mathrm{Re}(\nu)$, $\mathrm{Im}(\nu)$ are <u>finite</u> signed measures, & $\nu = \mathrm{Re}(\nu) + \mathrm{Im}(\nu)$.

<u>Examples</u>:

① If $\mu_0, \mu_1, \mu_2, \mu_3$ are finite (positive) measures on $(X, \mathcal{M})$, $\sum_{k=0}^{3} i^k \mu_k$ is a complex meas.

② for $g \in L^1(\underset{\text{pos}}{\mu}, \mathbb{C})$, $\nu(E) := \int_E f\,d\mu$ is complex.

By Jordan decomp, we get

<u>Cor</u>: If $\nu$ is a complex measure on $(X, \mathcal{M})$, $\exists!$ pairs of mutually singular finite pos measures $Re(\nu)_\pm$, $Im(\nu)_\pm$ s.t. $\nu = Re(\nu)_+ - Re(\nu)_- + i[Im(\nu)_+ - Im(\nu)_-]$.

<u>Def</u>: If $\nu$ is a complex measure, $\mu$ is a positive measure on $(X, \mathcal{M})$, then $\nu \perp \mu$ if $Re\,\nu \perp \mu$, $Im\,\nu \perp \mu$, same thing for abs. continuity.

<u>Thm</u> If $\nu$ complex, $\mu$ $\sigma$-finite positive on $(X, \mathcal{M})$, $\exists!$ complex meas $\lambda$ and $f \in L^1(\mu)$ s.t. $\lambda \perp \mu$ & $d\nu = d\lambda + f\,d\mu$.

[If $\lambda' \perp \mu$ & $f' \in L^1(\mu)$ s.t. $d\nu = d\lambda' + f'd\mu$ then $\lambda = \lambda'$ & $f = f'$ in $L^1(\mu)$]

Pf apply LRN to $Re(\nu)$, $Im(\nu)$, recombine. □

<u>Lemma</u> Suppose $\nu$ is a complex measure.

$\exists!$ pos measure $|\nu|$ satisfying:

- $\forall$ pos meas $\mu$ & $f \in L^1(\mu)$ s.t. $d\nu = f\,d\mu$, $d|\nu| = |f|d\mu$.

Call this $|\nu|$ the <u>total variation</u> of $\nu$.

Pf first, consider $\mu = |Re(\nu)| + |Im(\nu)|$. By LRN, $\exists f \in L^1(\mu)$ s.t. $d\nu = f\,d\mu$. Define $d|\nu| = |f|d\mu$, we claim that this $|\nu|$ satisfies the above universal property.

If $d\nu = g\,d\rho$ for another pos meas $\rho$ & $g \in \mathcal{L}^1(\rho)$, consider $\mu+\rho$ on $(X,\mathcal{M})$. Observe: $\nu \ll \mu, \rho \ll \mu+\rho$.

So $d\mu = \frac{d\mu}{d(\mu+\rho)}\,d(\mu+\rho)$, $d\rho = \frac{d\rho}{d(\mu+\rho)}\,d(\mu+\rho)$.

Exercise/discussion: If $\nu \ll \mu \ll \lambda$ w/ $\mu,\lambda$ $\sigma$-finite pos meas, & $\nu$ is either $\sigma$-finite signed or complex, then

① $\forall f \in \mathcal{L}^1(\nu)$, $f\frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu)$, $\int f\,d\nu = \int f \frac{d\nu}{d\mu}\,d\mu$.

② $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ ($\lambda$-a.e.)

Chain Rule

Since $f\frac{d\mu}{d(\mu+\rho)}d(\mu+\rho) = f\,d\mu = d\nu = g\,d\rho = g\frac{d\rho}{d(\mu+\rho)}d(\mu+\rho)$,

we have $f\frac{d\mu}{d(\mu+\rho)} = g\frac{d\rho}{d(\mu+\rho)}$, so

$$|f|\frac{d\mu}{d(\mu+\rho)} = \left|f\frac{d\mu}{d(\mu+\rho)}\right| = \left|g\frac{d\mu}{d(\mu+\rho)}\right| = |g|\frac{d\rho}{d(\mu+\rho)}$$

$\mu+\rho$ a.e.

So $|f|\,d\mu = |g|\,d\rho$, so $|\nu|$ is indep. of choice. $\square$