Lec 8/28

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<u>Outer measures</u>: $\mu^* : P(X) \longrightarrow [0, \infty]$

s.t. ⓪ $\mu^*(\emptyset) = 0$

① $E \subset F \Rightarrow \mu^*(E) \leq \mu^*(F)$

② $\mu^*(\bigcup E_n) \leq \sum \mu^*(E_n)$

<u>Prop</u>: if $\overset{\emptyset, X}{\overset{\cap}{\mathcal{E}}} \subset P(X)$ and $\rho : \mathcal{E} \longrightarrow [0, \infty]$, $\rho(\emptyset) = 0$,

Then $\mu^*(E) = \inf \left\{ \sum \rho(E_n) \mid E \subset \bigcup E_n,\ E_n \in \mathcal{E} \right\}$

is an outer measure.

outer measure $\overset{\text{on } P(X)}{\mu^*}$ $\underset{\text{today}}{\leadsto}$ measure μ. on M^*

today & friday

premeasure μ_0 on alg. $\mathcal{A}$ $\longrightarrow$ $\mathcal{A} \subset M^*$ and $\mu^*\big|_A = \mu_0$

(+ uniqueness)

use this to construct Lebesgue-Stieltjes measures on $\mathbb{R}$.

given an outer measure μ^*

Def: $\mathcal{M}^* = \left\{ E \subset X \;\middle|\; \begin{array}{c} \mu^*(E \cap F) + \mu^*(E^c \cap F) = \mu^*(F) \\ \forall F \subset X \end{array} \right\}$

These are the μ^*-measurable sets.

$E \in \mathcal{M}^* \iff E \cup E^c = X$ is a "good" partition wrt μ^*.

Prop: $\mathcal{M}^*$ is a σ-algebra. $\longleftarrow$ Save for later

Remarks: ① obviously $\mu^*(F) \leq \mu^*(E \cap F) + \mu^*(E^c \cap F)$

↳ $E \in \mathcal{M}^* \iff \mu^*(F) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F) \quad \forall F.$

② all μ^*-null sets are in $\mathcal{M}^*$.

pf if $F \subset X$, $\underbrace{\mu^*(F \cap N)}_{= 0} + \underbrace{\mu^*(F \setminus N)}_{\leq \mu^*(F)}$ (if N is μ^*-null).

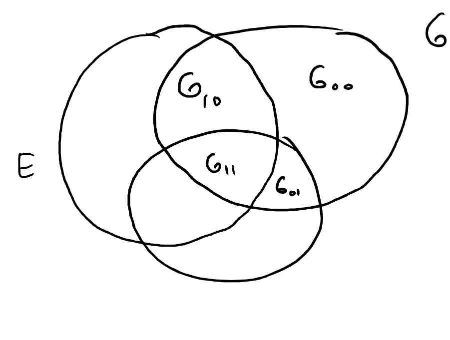
Lemma: Suppose $G \subset X$ and $E, F \in \mathcal{M}^*$.

Define $G_{00} = G \setminus (E \cup F)$

$G_{10} = G \cap (E \setminus F)$

$G_{01} = G \cap (F \setminus E)$

$G_{11} = G \cap E \cap F$



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Then $$\boxed{\mu^*(G) = \mu^*(G_{00}) + \mu^*(G_{01}) + \mu^*(G_{10}) + \mu^*(G_{11}).} \qquad (*)$$

Pf Since $E \in \mathcal{M}^*$, $\mu^*(G) = \mu^*(\underbrace{G \cap E}_{G_{10} \amalg G_{11}}) + \mu^*(\underbrace{G \cap E^c}_{G_{00} \amalg G_{01}})$

Since $F \in \mathcal{M}^*$,

$$\mu^*(G_{11} \amalg G_{10}) = \mu^*(\underbrace{(G_{11} \amalg G_{10}) \cap F}_{G_{11}}) + \mu^*(\underbrace{(G_{11} \amalg G_{10}) \cap F^c}_{G_{10}})$$

The other one is similar. $\square$

$$E \in \mathcal{M}^* \iff \forall F \subset X,\ \mu^*(F) = \mu^*(E \cap F) + \mu^*(E^c \cap F). \Big\}$$ Carathéodory criterion

Thm (Carathéodory):

① $\mathcal{M}^*$ is a σ-algebra

② $\mu^*|_{\mathcal{M}^*}$ is a complete measure

from remark, $\mathcal{M}^*$ contains μ^*-null sets

Pf Step 1: Show $\mathcal{M}^*$ is an algebra

⓪ $\emptyset \in \mathcal{M}^*$ since it is μ^*-null.

① If $E, F \in \mathcal{M}^*$, then $\forall G \subset X$, we have $(*)$

$$\mu^*((E \cup F) \cap G) = \mu^*(G_{10} \cup G_{01} \cup G_{11})$$

$$\underset{(*)}{=} \mu^*(G_{10}) + \mu^*(G_{01}) + \mu^*(G_{11})$$

$$\mu^*((E \cup F)^c \cap G) = \mu^*(G_{00})$$

So $\mu^*((E\cup F)\cap G) + \mu^*((E\cup F)^c \cap G) = \mu^*(G)$.

So $E\cup F$ $\mathcal{M}^*$. Induct.

② observe the carathéodory criterion is symmetric in E & E^c.

<u>Step 2</u>: $\mathcal{M}^*$ is a σ-algebra.

(an algebra is a σ-algebra iff it's closed under ctble disjoint unions).

pf Suppose (E_n) is a sequence of disjoint subsets, set $E := \coprod E_n$. by step 1, $\forall N\in\mathbb{N}$,

$$\coprod_1^N E_n \in \mathcal{M}^*.$$

Let $F\subset X$ and set $G := F\cap \coprod_1^N G_n$.

Since $E_N\in\mathcal{M}^*$,

$$\begin{aligned}\mu^*(F\cap \coprod_1^N G_n) &= \mu^*(G)\\ &= \mu^*(E_n^c\cap G) + \mu^*(E_n\cap G)\\ &= \mu^*(F\cap \coprod_1^{N-1} E_n) + \mu^*(F\cap E_n).\end{aligned}$$

by induction, $\mu^*(F\cap \coprod_1^N E_n) = \sum_1^N \mu^*(F\cap E_n)$.

Then $\forall N\in N$, $\mu^*(F) = \mu^*(F\cap \coprod_1^N E_n) + \mu^*(\underbrace{F\setminus \coprod_1^N E_n}_{\supset F\setminus E})$

$$\geq \sum_1^N \mu^*(F\cap E_n) + \mu^*(F\setminus E)$$

Taking $N\to\infty$.

$\mu^*(F) \geq \sum^{\infty} \mu^*(E\cap F) + \mu^*(F\ldots$

$$\left.\begin{aligned}\mu^*(F) &\geq \sum_1^\infty \mu^*(F\cap E_n) + \mu^*(F\setminus E) \\ &\geq \mu^*\left(\coprod (F\cap E_n)\right) + \mu^*(F\setminus E) \\ &= \mu^*(F\cap E) + \mu^*(F\setminus E)\end{aligned}\right\} \quad (**)$$

So $E \in \mathcal{M}^*$.

<u>Step 3</u> $\mu^*|_{\mathcal{M}^*}$ is a measure.

<u>Pf</u> Suppose $(E_n) \subset \mathcal{M}^*$ is disjoint.

take $F = E$ in $(**)$ above.

then $\mu^*(E) \geq \sum \mu^*(E_n) \geq \mu^*(\coprod E_n) = \mu^*(E)$.

So equality holds. $\square$

<u>Def</u>: Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra.

A fn $\mu_0 : \mathcal{A} \longrightarrow [0,\infty]$ is called a premeasure if

⓪ $\mu_0(\emptyset) = 0$

① $\forall$ sequence $(E_n) \subset \mathcal{A}$ of disjoint sets s.t. $\coprod E_n \in \mathcal{A}$, $\mu_0(\coprod E_n) = \sum \mu_0(E_n)$.

<u>Remarks</u>: premeasures are finitely additive.

Starting w/ a premeasure on $\mathcal{A}$ $\rightsquigarrow$ μ^* on $\mathcal{P}(X)$

via $\mu^*(E) = \inf\left\{ \sum \mu_0(E_n) \;\middle|\; E \subset \bigcup E_n,\ E_n \in \mathcal{A} \right\}$.

<u>Lemma</u>: $\mu^*|_{\mathcal{A}} = \mu_0$.

~~pf~~ Suppose $E \in \mathcal{A}$ w/ $E \subset \bigcup \overset{\in \mathcal{A}}{E_n}$ w/ $\sum \mu_0(E_n) < \mu^*(E) + \varepsilon$

let $\varepsilon > 0$

define $F_n := E \cap [E_n \setminus \bigcup_1^{n-1} E_i]$.

Then $F_n \in \mathcal{A}$ $\forall n$ and $\coprod F_n = E \in \mathcal{A}$.

So $\mu_0(E) = \sum \mu_0(F_n) \underbrace{\leq}_{F_n \subset E_n} \sum \mu_0(E_n) \leq \mu^*(E) + \varepsilon$.

$\varepsilon > 0$ was arbitrary, so $\mu_0(E) \leq \mu^*(E)$.

Conversely, $E \subset \bigcup G_n$ where $G_1 = E$, $G_n = \emptyset$ $\forall n > 1$.

So $\mu^*(E) \leq \sum \mu_0(G_n) = \mu_0(E)$. $\square$