2/28

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$\omega \in \mathcal{T}^n(M^*)$, $\quad \omega : \mathcal{T}^n(M) \longrightarrow R$

$\mathcal{T}^n(M^*) \times \mathcal{T}^n(M) \qquad \omega = f_1 \otimes \cdots \otimes f_n$

$$\omega(u_1 \otimes \cdots \otimes u_n) = f_1(u_1) \cdot \cdots \cdot f_n(u_n).$$

$$M^* \times \cdots \times M^* \times M \times \cdots \times M \longrightarrow R$$

$$(f_1, \ldots, f_n, u_1, \ldots, u_n) \longmapsto \prod f_i(u_i)$$

$\omega \in \mathcal{T}^n(M^*)$ is symmetric, then $\omega \in (S^n(M))^*$. $\omega : S^n(M) \longrightarrow R$.

$$S^n(M) = \mathcal{T}^n(M) / C^n(M)$$

$$\begin{tikzcd} \mathcal{T}^n(M) \arrow[r, "\omega"] \arrow[d] & R \\ S^n(M) \arrow[ur] & \end{tikzcd}$$

$C^n(M)$ generated by

$u_1 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$

ω must be 0 on $C^n(M)$.

i.e. $\omega(u_1 \otimes \cdots \otimes u_n) = \omega(u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)})$.

if $\omega = f_1 \otimes \cdots \otimes f_n$ then $\omega(u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}) = \prod f_i(u_{\sigma(i)}) = \prod f_{\sigma^{-1}(i)}(u_i)$.

$= \sigma^{-1}(\omega)(u)$

$u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$

So if $\sigma(\omega) = \omega \ \forall \sigma$, then $\omega(\sigma(u)) = \omega(u)$ ✓

① $M = R^n$. $\{u_1, \ldots, u_n\}$ bases in M, then $\{u_1 \wedge \cdots \wedge u_n\}$

(9) $M = R^n$, $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ bases in M, then $\{u_1 \wedge \cdots \wedge u_n\}$, $\{v_1 \wedge \cdots \wedge v_n\}$ are bases in $\Lambda^n M = R$

So $u_1 \wedge \cdots \wedge u_n = c\, v_1 \wedge \cdots \wedge v_n$, but $b u_1 \wedge \cdots \wedge u_n = v_1 \wedge \cdots \wedge v_n$ so $c \in R^\times$.

(5) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$ Jordan Normal Form

Companion of $x^2 - 3x + 2 = \underline{(x-1)(x-2)}$

min pol so jordan Normal form is

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

So it's

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(18) Find all JNF for $C_p(x) = (x-2)^3 (x-3)^2$

Elem divisors: $(x-2)^3, \ (x-3)^2 \longrightarrow \begin{pmatrix} 2 & 1 & & & 0 \\ & 2 & 1 & & \\ & & 2 & 0 & \\ & & & 3 & 1 \\ 0 & & & & 3 \end{pmatrix}$

or $(x-2)^2, (x-2), (x-3)^2 \longrightarrow \begin{pmatrix} 2 & 1 & & & 0 \\ & 2 & 0 & & \\ & & 2 & 0 & \\ & & & 3 & 1 \\ 0 & & & & 3 \end{pmatrix}$

or $(x-2), (x-2), (x-3)^2 \longrightarrow \begin{pmatrix} 2 & 0 & & & 0 \\ & 2 & 0 & & \\ & & 2 & 0 & \\ & & & 3 & 1 \\ 0 & & & & 3 \end{pmatrix}$

or ――――――――

" " " $(x-3), (x-3) \to$ " but w/ $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ instead.

(14) Smith $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x+2 & 0 \\ 0 & 0 & \cdots & \end{pmatrix}$ rat'l form, char poly.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & x+2 & 0 \\ 0 & 0 & 0 & (x+2)(x^2+3) \end{pmatrix}$$ rat'l form, char poly.

$C_\varphi = (x+2)^2(x^2+3)$

$$\begin{array}{c} x+2 \end{array} \left(\begin{array}{c|ccc} -2 & & 0 & \\ \hline & 0 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ & 0 & 1 & -2 \end{array}\right)$$

$\underbrace{(x+2)(x^2+3)}_{\text{minimal polynomial.}} = x^3+2x^2+3x+6$

(20) $A = \begin{pmatrix} 0 & 0 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ & & \ddots & & \vdots \\ 0 & & & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \\ 0 & & & & 1\,1 \\ & & & & 1 \end{pmatrix}$ $p \times p$ matrices over $F_p = \mathbb{Z}_p$.

A, B are similar.

$C_A = m_A =$ companion of $A = x^p - 1 = (x-1)^p$ in F_p

$(x-1)^p$ is only elem. div. so JNF of A is B.

1 Jordan cell

(21) if $A^2 = A$ then A is similar to $\begin{pmatrix} \begin{matrix} 1 & & 0 \\ 0 & \ddots & \\ & & 1 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$

A satisfies $x^2 - x = 0 \Rightarrow m_A(x) \mid x(x-1)$.

So $m_A(x) = x$ (i.e. $A = 0$)

or $m_A(x) = x-1$ (i.e. $A = I$)

or $m_A(x) = x^2 - x$

Then elem. div's of A are $\underbrace{x, \dots, x}_{n \text{ times}}, \underbrace{x-1, \dots, x-1}_{m \text{ times}}$

So in the 2 blocks of A, one has $A = 0$, one has $A = I$.

(23) A is 2×2 matrix over $\mathbb{Q}$, $A^3 = I$, $A \neq I$.

write rat'l normal form and Jordan form over $\mathbb{C}$.

$$x^3 - 1 = \underset{\text{doesn't satisfy this}}{(x-1)} \underbrace{(x^2+x+1)}_{m_A = c_A}$$

Note: rat'l normal form doesn't depend on field

$\Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ rat'l normal form

$$x^2 + x + 1 = (x - \lambda_1)(x - \lambda_2), \quad \lambda_i = \frac{-1 \pm \sqrt{-3}}{2}$$

$$JNF = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$F_1 \subseteq F_2$ fields

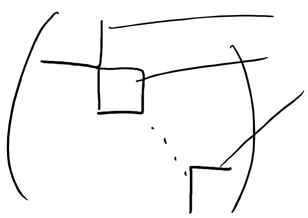
A - matrix over F_1 can be considered as over F_2.

RNF stays the same

EDF differs

JNF stays same (if it exists in both fields).

Rat. Norm. form of A over F_1 is



$p_1 \mid p_2 \mid \cdots \mid p_m \in F_1[x]$

$\cap$

it also ok as $F_2[x]$, and rep^n is unique.