Lec 1/10

Friday, January 10, 2020 10:46

Dual of L^p (p, q conjugate: $\frac{1}{p} + \frac{1}{q} = 1$)

Given $g \in L^q$ define ϕ_g on L^p by:

$$\phi_g(f) := \int fg.$$

ϕ_g is a bounded operator:

$$\|\phi_g\| := \sup\left\{ \left|\int fg\right| : \|f\|_p = 1 \right\} < \infty.$$

pf $\|\phi_g\| \le \int |fg| \le \|f\|_p \|g\|_q = \|g\|_q \quad \forall\ f \in L^p$ w/ $\|f\|_p = 1$.

Prop^n In fact, $\|\phi_g\| = \|g\|_q$.

pf Define $f = \dfrac{|g|^{q-1}}{\|g\|_q^{q-1}} \overline{\text{sign}(g)}$

[remember: $p(q-1) = q$.

Then $\|f\|_p^p = \dfrac{1}{\|g\|_q^{p(q-1)}} \int |g|^{p(q-1)} = \dfrac{\|g\|_q^q}{\|g\|_q^q} = 1$

So $\|\phi_g\| \ge \left|\int fg\right| = \int fg = \dfrac{1}{\|g\|_q^{q-1}} \int |g|^q = \dfrac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q$

Exercise: prove the prop^n when $q = 1$.

Th^m p, q conjugate exponents, $fg \in L^1$ $\forall f \in \Sigma$ (simple functions with finite-measure support.), and

$$M_q(g) := \sup\left\{ \left|\int fg\right| : f \in \Sigma \text{ and } \|f\|_p = 1 \right\} < \infty.$$

Also, either $S_g := \{g \neq 0\}$ is σ-finite or μ is semifinite.

Then $g \in L^q$ and $M_q(g) = \|g\|_q$.

Pf Next time.