Lec 9/19

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Exam problems:

#4 : $|G| = pq$ ($p<q$ odd primes, $q \not\equiv 1 \bmod p$) $\Longrightarrow$ $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$.

→ use Sylow theorems (part 3).

$$\underset{\langle x \rangle}{\overset{}{P}} \trianglelefteq G , \quad \underset{\langle y \rangle}{Q} \trianglelefteq G, \quad P \cap Q = \{e\}, \quad P \cdot Q = G .$$

$P = \langle x \rangle$, $Q = \langle y \rangle$, $|xy| = \operatorname{lcm}(|x|, |y|) = pq$.

3rd Isomorphism Theorem

$H \leq G$, $N \trianglelefteq G$: group

Then $$H/H \cap N \cong H \cdot N / N$$

Hidden in the statement.

(i) $H \cdot N = N \cdot H$ subgp in G.

Pf. $h \cdot n = (hnh^{-1}) \cdot h$

(ii) $H \cap N \trianglelefteq H$ and $N \trianglelefteq H \cdot N$.

Pf. $h \in H$, $x \in H \cap N$ $\Rightarrow$ $hxh^{-1} \in H$, $\in N$.

$$\begin{tikzcd} H \arrow[r, phantom, "\subset"] & H \cdot N \arrow[r, "\text{natural projection}"] & H \cdot N / N \\ h \arrow[r, mapsto] \arrow[urr, bend right, "\text{gp hom } f(h) = hN"'] & h \cdot 1 & \end{tikzcd}$$

f is surjective because every element in HN/N is of the form $h \cdot n N = h \cdot N$.

$\operatorname{Ker}(f) = H \cap N$. So 1^{st} iso says

$$H/H \cap N \cong H \cdot N/N \qquad \left(\text{Source}/\text{Kernel} \cong \text{Image}\right)$$

"Semidirect Product"

Definition We say G is a ~~semi~~direct product of H and N if $H \trianglelefteq G \trianglerighteq N$ and $H \cdot N \,(= N \cdot H) = G$ and $H \cap N = \{e\}$.

Direct product $\subset$ Semidirect product

Eg: $G = D_{2n} \geq H = \{e, s\}$ $\qquad D_{2n}$ is a semidirect product.

$\trianglerighteq$
$N = \{e, r, \ldots, r^{n-1}\}$.

Eg: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : ac \neq 0 \right\} \leq GL_2$.

$G \geq \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} : ac \neq 0 \right\}$

$\trianglerighteq \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : \text{any } x \right\}$.

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$

Eg: $|G| = 21 = 3 \cdot 7$.

Sylow Thm (part 3)

$n_3 \equiv 1 \mod 3$	$n_7 \equiv 1 \mod 7$
$n_3 \mid 7$	$n_7 \mid 3$
$\Downarrow$	$\Downarrow$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$n_3 = 1, 7$ $\qquad\qquad\qquad\qquad$ $n_7 = 1$

If $n_3 = 1$, $n_7 = 1$, $G \cong \mathbb{Z}/_{21}\mathbb{Z}$.

$n_3 = 7$, $n_7 = 1$. Let $N \trianglelefteq G$ be <u>the</u> Sylow 7-subgr.

Let $H \leq G$ be <u>a</u> Sylow 3-subgr.

$\downarrow$

G is a semidirect product of N and H.

$H = \{e, x, x^2\}$, $N = \{e, y, \ldots, y^6\}$

$xyx^{-1} = y^t$ (could be anything $0 \leq t \leq 6$

but, needs to be invertible so $t \neq 0$.

$N \xrightarrow{\sim} xNx^{-1}$

$t \neq 1$ since that's the other case $(n_3 = 1)$

$x^3 = e$ implies $t^3 \equiv 1 \bmod 7$.

$t = 2$ works

$t = 3$ doesn't

$t = 4$ works

$t = 5$ no

$t = 6$ no

G is defined as follows:

$$\left\{ y^i x^j \;\middle|\; \begin{array}{c} 0 \leq i \leq 6 \\ 0 \leq j \leq 2 \end{array} \right\}$$

$$x y^k x^\ell = y^{2k} x^{\ell+1}$$

$$G = \left\langle x, y \;\middle|\; \begin{array}{c} x^3 = y^7 = e \\ xyx^{-1} = y^2 \end{array} \right\rangle$$

would get another example if 4 not 2.

'would get another example if 4 not 2.

Summary: To build semidirect product, we need

$\begin{cases} N \\ H \end{cases}$ and $H \curvearrowright N$

2 groups

by group isomorphisms

$H \xrightarrow{\alpha} \{\text{all gp iso's } N \xrightarrow{\sim} N\}$

given $H, N, \alpha : H \xrightarrow{\text{gp hom}} \mathrm{Aut}_{\text{group}}(N)$,

$$G = N \times H = \{(n,h) \mid n \in N, h \in H\}$$

$$(n_1, h_1) \cdot (n_2, h_2) \overset{\text{defn}}{=} \left(n_1 \cdot \alpha(h_1)(n_2), h_1 h_2\right)$$

(if $\alpha(h_1) = id_N$ for all h_1, then this is a direct product)

In our example: $H = \mathbb{Z}/3$, $N = \mathbb{Z}/7$

$H \longrightarrow \{N \xrightarrow{\sim} N\}$

$x \longmapsto (y \mapsto y^2)$

$$n_1 h_1 n_2 h_2 = n_1 \underbrace{\boxed{h_1 n_2 h_1^{-1}}}_{\text{given by } \alpha(h_1)(n_2)} h_1 h_2$$

Theorem: G defined above is a group and G is a semidirect product of N & H.

$\{(e,h) \mid h \in H\} \leq G$, $\{(n,h) \mid n \in N\} \trianglelefteq G$

Every semidirect product arises this way.

Notation: $G = H \ltimes_{\alpha} N$

Pf: $\left((n_1, h_1) \cdot (n_2, h_2)\right) \cdot (n_3, h_3)$

$$\left(n_1 \cdot \alpha(h_1)(n_2),\ h_1 h_2\right) \cdot (n_3, h_3)$$

$$\left(n_1 \cdot \alpha(h_1)(n_2) \cdot \alpha(h_1 h_2)(n_3),\ h_1 h_2 h_3\right)$$

since $\alpha(h_1)$ is a gp iso $\longrightarrow$ $\left(n_1 \cdot \alpha(h_1)\left(n_2 \cdot \alpha(h_2)(n_3)\right),\ h_1 h_2 h_3\right)$

$$(n_1, h_1) \cdot \left((n_2, h_2) \cdot (n_3, h_3)\right).$$