S : set, RS : free R-module generated by S.

$$RS = \{\sum_{i=1}^{n} r_i s_i,\ n\in\mathbb{N},\ r_i\in R,\ s_i\in S\}$$

Universal Repelling Object:

$\forall$ function $f: S \to N$ $\exists$ unique hom-sm $\varphi: RS \to N$

s.t. $\varphi(s) = f(s)$ $\forall s\in S$.

defined by $\varphi(\sum a_i s_i) = \sum a_i f(s_i)$.

Objects are (N : R module, $f: S \longrightarrow N$).

Morphisms are $\varphi: N_1 \longrightarrow N_2$ s.t.

$$\begin{tikzcd} & N_1 \arrow[d, "\varphi"] \\ S \arrow[ur, "f_1"] \arrow[r, "f_2"'] & N_2 \end{tikzcd}$$ commutes.

let N be any R-module. Let S

be a set of generators of N.

Consider the free module $F_R(S)$ (formal linear combos)

generated by S. Then $\exists$ a homom

$\varphi: F_R(S) \twoheadrightarrow N$ s.t. $\varphi(s) = s$ $\forall s\in S$.

— . .

So $N \cong \mathcal{F}_R(S) / \mathrm{Ker}(\varphi)$.

Thus all R-modules are quotients of free modules.

If N is generated by S, the free module has rank $|S|$.

$M_1 \otimes M_2$ is generated by simple tensors $u_1 \otimes u_2$.

if $M_1 = RS_1$, $M_2 = RS_2$, then $M_1 \otimes M_2 = R\{u_1 \otimes u_2 : u_1 \in S_1, u_2 \in S_2\}$.

$R^n \otimes R^m = R^{mn}$ (where $R^n = \bigoplus_n R = R^{\oplus n}$)

basis for is $u_i \otimes v_j$ where $\{u_i\}_{i=1}^n$ is a basis for R^n, $\{v_j\}_{j=1}^m$ is a basis for R^m.

Note: $\mathrm{Hom}(M_1, M_2) \cong \underset{\text{dual module}}{M_1^*} \otimes M_2$

<u>Extension of Scalars</u>: S : R-algebra. M : R-module

$S \otimes_R M$ has an S-module structure

$\alpha(\beta \otimes u) = (\alpha\beta) \otimes u$.

Let $\alpha \in S$. consider
$$S \times M \longrightarrow S \otimes M$$
$$(\beta, u) \longmapsto (\alpha\beta) \otimes u$$

This map is bilinear so we have a homomorphism
$$S \otimes M \xrightarrow{\varphi_\alpha} S \otimes M$$
$$\beta \otimes u \longmapsto (\alpha\beta) \otimes u$$

So define $\alpha \omega = \varphi_\alpha(\omega)$, $\quad \omega \in S \otimes M$

Then
$$\alpha(\omega_1 + \omega_2) = \alpha\omega_1 + \alpha\omega_2$$
$$(\alpha_1 + \alpha_2)\omega = \alpha_1\omega + \alpha_2\omega$$
$$(\alpha_1\alpha_2)\omega = \alpha_1(\alpha_2\omega)$$

| Direct Check

Examples:

① A : Abelian Group $= \mathbb{Z}$-module.

Then $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is a $\mathbb{Q}$-vector space

② V : $\mathbb{R}$-vector space

$\mathbb{C} \otimes_{\mathbb{R}} V$ is a $\mathbb{C}$-vector space

$\mathbb{C} = \mathbb{R}\{1, i\}$. Let $\{v_1, \dots, v_n\}$ be a basis in V.

Then a basis in $\mathbb{C} \otimes_{\mathbb{R}} V$ is $\{1 \otimes v_1, \dots, 1 \otimes v_n, i \otimes v_1, \dots i \otimes v_n\}$.

as an $\mathbb{R}$-vector space.

As a $\mathbb{C}$-vector space, $\mathbb{C} \otimes_{\mathbb{R}} V$ has the basis $\{1 \otimes v_1, \dots, 1 \otimes v_n\}$.