Functoriality of H.

Simplicial map $f : X \to Y$

$\Downarrow$

Chain map $C_k(f) : C_k(X) \longrightarrow C_k(Y)$

$$\sigma \longmapsto \begin{cases} f(\sigma) & \text{if } |f(\sigma)| = k \\ 0 & \text{o.w.} \end{cases}$$

<u>Lemma</u>:

$$\begin{tikzcd} C_k(X) \arrow[r, "\partial_k^X"] \arrow[d, "C_k(f)"'] & C_{k-1}(X) \arrow[d, "C_{k-1}(f)"] \\ C_k(Y) \arrow[r, "\partial_k^Y"'] & C_{k-1}(Y) \end{tikzcd}$$

commutes $\forall k$.

~~Pf~~ 2 cases.

Note that $f(\sigma)$ is a k-simplex iff f is inj on σ

Then it's easy.

if f is not injective on σ then $C_k(f)(\sigma) = 0$.

<u>OTOH</u>: $C_{k-1}(f)\left(\partial_k^X(\sigma)\right) = \sum C_{k-1}(f)\left(\{v_0, \dots, \hat{v}_j, \dots, v_k\}\right)$

$= 0$ (want).

2 cases: 1: f is injective on $\{v_0, \dots, \hat{v}_j, \dots, v_k\}$ for some j.

WLOG, $f(v_0) = f(v_1)$, and $f(v_j) \neq f(v_i)$ for $i \neq j$ otherwise.

Then $\forall j \neq 0, 1$, we get 0.

and the 0 & 1 terms cancel.

2: f isn't. Then it's all 0. $\square$

Theorem: sp. map $f: X \to Y \rightsquigarrow H_k(f): H_k(X) \to H_k(Y)$

~~pf~~ For $v \in \mathrm{Ker}(\partial_k^X)$, define

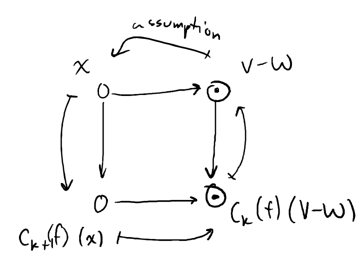
$$H_k(f)(v + \mathrm{Im}(\partial_{k+1}^X)) = C_k(f)(v) + \mathrm{Im}(\partial_{k+1}^Y).$$

Check: $C_k(f)(v) \in \mathrm{Ker}(\partial_k^Y)$.

if $v - w \in \mathrm{Im}(\partial_{k+1}^X)$, then $C_k(f)(v) - C_k(f)(w) \in \mathrm{Im}(\partial_{k+1}^Y)$.

The first thing is true by the lemma.

The second thing is also true by the lemma:



Categories:

eg: $\mathrm{Sim} = (\mathcal{O}, \mathcal{M})$ where $\mathcal{O} = \{$simplicial complexes$\}$
$\mathcal{M} = \{$simplicial maps$\}$

$\mathrm{Vec}_F = (\mathcal{O}, \mathcal{M})$ where $\mathcal{O} = \{$V. spaces over $F\}$
$\mathcal{M} = \{$linear maps$\}$

$\mathrm{Top} = (\mathcal{O}, \mathcal{M})$ where $\mathcal{O} = \{$Topological spaces$\}$
$\mathcal{M} = \{$continuous maps$\}$

A functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ sends objects to objects & morphisms to morphisms:

$$f : X \longrightarrow Y$$
$$\downarrow$$
$$F(f) : F(X) \longrightarrow F(Y)$$

So: $H_k : \mathrm{Sim} \longrightarrow \mathrm{Vec}_{F_2}$ is a functor $\forall k$.

BB Theorem: If $\mathcal{X}, \mathcal{Y}$ are geom sp. cpx with corresponding abstract sp. cpx X, Y,
Then $\mathcal{X} \underset{\text{h.e.}}{\simeq} \mathcal{Y} \Rightarrow H_k(X) \cong H_k(Y) \quad \forall k$.