Cyclic module $\cong R/I$

Thm If M is finitely generated $M = R\{u_1, \dots, u_n\}$ then M is a quotient module of R^n.

Pf homomorphism $\varphi : R^n \longrightarrow M$ $\varphi(a_1, \dots, a_n) = a_1 u_1 + \dots + a_n u_n$ is surjective so $M \cong R^n / \ker(\varphi)$.

If R is comm., $\mathrm{Hom}_R(M,N)$ is an R-module.

$\mathrm{Hom}_R(M,M) =: \mathrm{End}_R(M)$.

Additional operation: $\cdot$ = composition. $(\varphi\psi)(u) = \varphi(\psi(u))$.
multiplication

Then $\mathrm{End}_R(M)$ is a ring & is an R-algebra if R is commutative.

If R is a field & M is an n-dim V.S. over R,

then $\mathrm{End}_R(M) \cong M_n(R) \leftarrow$ $n \times n$ matrices over R.

M is also an $\mathrm{End}_R(M)$-module. by $\varphi \cdot u = \varphi(u)$.

Def. A module is irreducible, or simple, if it has no submodules (except 0 & itself).

Schur's Lemma: If M, N are simple then any $\varphi \in \mathrm{Hom}_R(M,N)$ is either 0 or is invertible (isomorphism).

Pf. $\mathrm{Ker}(\varphi)$ is a submodule of M.
$\varphi(M)$ is a submodule of N.

Example: Let T be a linear transformation of a vector space V over F.
Then V is an $F[X]$ module by $xu = T(u)$ & $P(x)(u) = P(T)(u)$.

Submodules of V: T-invariant subspaces of V.
(i.e. $T(W) \subseteq W$).

V is a simple $F[x]$-module if $\forall u \in V$, $\mathrm{Span}\{u, T(u), T^2(u), \dots\} = V$.

Let V, W be F-vector spaces, let $T \in \mathrm{End}_F(V)$, $S \in \mathrm{End}_F(W)$.
then V, W are $F[x]$-modules by
$$xu = T(u) \quad u \in V$$
$$xv = S(v) \quad v \in W.$$

What is $\mathrm{Hom}_R(V,W)$?

$$\varphi: V \longrightarrow W \qquad \left.\begin{aligned} \varphi(u_1 + u_2) &= \varphi(u_1) + \varphi(u_2) \quad \forall u_1, u_2 \in V \\ \varphi(au) &= a\varphi(u) \quad \forall a \in R, u \in V \end{aligned}\right\} \varphi \text{ is a linear transformat.}$$

$$\varphi(xu) = x\varphi(u)$$
$$\parallel$$
$$\varphi(T(u)) = S(\varphi(u)) \qquad \forall u \in V.$$

$$\varphi T = S\varphi$$
$$\varphi T \varphi^{-1} = S$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ W \arrow[r, "S"'] & W \end{tikzcd}$$

$$S \circ \varphi = \varphi \circ T$$

(φ is an intertwining transformation $V \to W$)

A diagram of mappings is commutative if

$$\begin{tikzcd} & M_2 \arrow[rr] \arrow[dd] & & M_3 \arrow[dd] \\ M_1 \arrow[ur] \arrow[dr] & & & \\ & M_5 \arrow[uurr] \arrow[dr] & & M_4 \arrow[dl] \\ & & M_6 & \end{tikzcd}$$

$\forall$ two vertices M_i, M_j, the composition along any sequence of arrows from M_1 to M_2 gives the same result.

$$\begin{tikzcd} M_1 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1"'] & M_2 \arrow[d, "\varphi_4"] \arrow[dr, "\varphi_5"] & \\ M_3 \arrow[r, "\varphi_3"'] & M_4 \arrow[r, "\varphi_6"'] & M_5 \end{tikzcd}$$

is commutative if

$$\varphi_4 \circ \varphi_1 = \varphi_3 \circ \varphi_1$$
$$\varphi_5 = \varphi_6 \circ \varphi_4$$
$$\vdots$$

Let

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \longrightarrow \cdots$$

be a sequence of module homomorphisms.

The sequence is exact at M_i if

$$\varphi_{i-1}(M_{i-1}) = \operatorname{Ker}(\varphi_i),$$

(Then $\varphi_i \circ \varphi_{i-1} = 0$).

A sequence is exact if it is exact at every step.

Examples: $0 \longrightarrow N \xrightarrow{\varphi} M$ is exact at term N iff φ is injective.

$M \xrightarrow{\varphi} K \longrightarrow 0$ is exact at K iff φ is surjective.

$0 \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$ is exact iff φ is an isomorphism.

exact sequences of the form $0 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} K \longrightarrow 0$ are called short exact sequences.

it is exact iff φ is injective & ψ is surjective

$N \subseteq M$ $\qquad K \cong M/_{\text{something}}$

$$\varphi(N) = \ker(\psi)$$

$$K \cong M/\varphi(N) \text{ "} \cong M/N \text{"}$$

The short five lemma.

Let the diagram of homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"'] & B' \arrow[r, "\psi'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

be commutative & have exact rows.

Then (a) if α and γ are epimorphisms (surjective), β is too.

(b) if α and γ are monomorphisms (injective), β is too.

(c) if α and γ are isomorphisms, β is too.

(C)

$$\begin{tikzcd} & & B \arrow[dd] \arrow[dr] & & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & B' \arrow[ur] & & \end{tikzcd}$$

commutes & has exactness $\Rightarrow$ $B \cong B'$.

<u>Proof:</u>

(a): Let $b' \in B'$.