Lec 9/25

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Lemma: $|\mathrm{Aut}_{gp}(W)|$ is even because
finite abelian

$f \quad \begin{matrix} z \longmapsto -z \\ \hat{W} \cong \hat{W} \end{matrix}$

for ab. grps only

$f^2 = id$ so $\langle f \rangle \leq \mathrm{Aut}_{gp}(W)$

$\Rightarrow |\mathrm{Aut}_{gp}(W)|$ is even

$\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) = ?$

enough to answer $\mathrm{Aut}_{gp}(\mathbb{Z}/p^r\mathbb{Z}) = ?$

p: odd $\Rightarrow$ cyclic of order $p^{r-1}(p-1) = \phi(p^r)$

$p = 2 \Rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r-2}\mathbb{Z}$.

- 5 has order 2^{r-2}; i.e. $\begin{matrix} \mathbb{Z}/2^r\mathbb{Z} \longrightarrow \mathbb{Z}/2^r\mathbb{Z} \\ 1 \longmapsto 5 \end{matrix}$ applied 2^{r-2} times is id, but nothing less.

$\Rightarrow \mathrm{Aut}(\mathbb{Z}/2^r\mathbb{Z}) \cong \mathbb{Z}/2^{r-1}\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r-2}\mathbb{Z}$

- $2^{r-1} \pm 1$ are elements of order 2.
- $(2^{r-1} \pm 1)^2 \equiv 1 \pmod{2^r}$

So $\begin{matrix} 1 \longmapsto 2^{r-1} \\ 1 \longmapsto 2^{r-1} \end{matrix}$ are 2 different maps of order 2,

So the group cannot be $\mathbb{Z}/2^{r-1}\mathbb{Z}$ (only has 1 elt of order 2).

Classify all groups of order $18 = 2 \cdot 3^2$ (say G is one)

Sylow theorems: $\exists P \leq G$ w/ $|P| = 2$, and $\exists Q \leq G$ w/ $|Q| = 9$.

$Q \trianglelefteq G$ reason 1: Sylow thm 3: $n_3 \equiv 1 \bmod 3$, $n_3 | 2 \Rightarrow n_3 = 1$.

reason 2: if $H' \leq H$ st. $|H/H'| = 2$, then $H' \trianglelefteq H$.

$P \cap Q = \{e\}$ ✓ relatively prime order

$PQ = QP = G$ ✓ $|PQ|$ is div. by 2 and 9.

$G = Q \rtimes_\alpha P$ for some α.

$\alpha : P \longrightarrow \mathrm{Aut}_{gp}(Q)$

recall $|H| = p^r \Rightarrow Z(H) \neq \{e\}$ ✓

$P \cong \mathbb{Z}/2\mathbb{Z}$; $Q \cong \mathbb{Z}/9\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$

<u>Case 1</u>:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Aut}_{gp}(Q) \\ \wr\| & & \wr\| \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/6\mathbb{Z} \end{array}$$

generator $\longmapsto \mathrm{id}_Q$

generator $\longmapsto \{x \bmod 9 \longmapsto -x \bmod 9\}$

generator of $\mathrm{Aut}_{gp}(Q)$

$\{1 \mapsto 5\}$ has order 6, and

$\{1 \mapsto 5\}^3 = \{x \mapsto -x\}$.

$y^9=e$, y_e ; $x^2=e$, x: $Q \rtimes_\alpha P$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}/18\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong D_{18}$
α	trivial	$z \longmapsto -z$ i.e. $xyx^{-1} = y^{-1}$

since it's $\langle x, y \mid x^2 = y^9 = e,\ xyx^{-1} = y^{-1} \rangle$

<u>Case 2</u>:

$Q \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

$\mathrm{Aut}_{gp}(Q) \cong GL_2(\mathbb{F}_3)$

$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ represents an automorphism sending

$(1, 0) \longmapsto (\alpha, \gamma)$
$(0, 1) \longmapsto (\beta, \delta)$

always a homomorphism, isomorphism iff $\alpha\delta - \beta\gamma \neq 0$

So count all matrices which square to identity.

We still counting gp-hom's $\alpha : P \to \mathrm{Aut}_{gp}(Q) \cong GL_2(\mathbb{F}_3)$

generator $\longmapsto X$ s.t. $X^2 = \mathbb{1}$.

$(\det X)^2 = 1$ gives no info : $\det X = \pm 1 = \mathbb{F}_3 \setminus \{0\}$.

determinant.

$X = X^{-1}$: $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \pm \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

So if $\det X = 1$ then $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

if $\det X = -1$ then $\alpha + \delta = 0$, $\alpha\delta - \beta\gamma = -1$

$$\left.\begin{array}{l} \alpha = \delta = 0 \Rightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ \alpha = 1, \delta = -1 \Rightarrow X = \begin{bmatrix} 1 & 0 \\ x & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix} \\ \alpha = -1, \delta = 1 \Rightarrow X = \begin{bmatrix} -1 & 0 \\ x & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & x \\ 0 & 1 \end{bmatrix} \end{array}\right\}$$

all are in one conjugacy class: (use eigenvalues maybe.)

all are conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

<u>Lemma</u>: Let H & N be two groups,

$\alpha, \beta : H \longrightarrow \mathrm{Aut}_{gp}(N)$ gp homs

Assume $\exists\, T \in \mathrm{Aut}_{gp}(N)$ s.t.

$$\alpha(h)(n) = T\left(\beta(h)\left(T^{-1}(n)\right)\right) \quad \forall h \in H,\ n \in N.$$

i.e. $\forall h \in H$, $\alpha(h) = T \circ \beta(h) \circ T^{-1}$

Then $N \rtimes_\alpha H \cong N \rtimes_\beta H$

<u>Pf</u> $(T(n), h) \longleftarrow (n, h)$: f is an isomorphism $\square$

So $P \longrightarrow \mathrm{Aut}_{gp}(Q) \cong GL_2(\mathbb{F}_3)$ has three options.

generator $\longrightarrow X$

these rel^{n}s are just P & Q.

if $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q \rtimes_\alpha P = \left\langle x, y_1, y_2 \;\middle|\; \begin{array}{l} x^2 = e,\ y_1^3 = y_2^3 = e,\ y_1 y_2 = y_2 y_1 \\ \underbrace{x y_1 x^{-1} = y_1,\quad x y_2 x^{-1} = y_2}_{\text{this comes from } \alpha.} \end{array} \right\rangle$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ———————— $x y_1 x^{-1} = y_1^{-1}$, $x y_2 x^{-1} = y_2^{-1}$

call this gp. A

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ——— $xy_1x^{-1} = y_1^{-1}$, $xy_2x^{-1} = y_2^{-1}$

call this gp. A

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ——— $xy_1x^{-1} = y_1$, $xy_2x^{-1} = y_2^{-1}$

call this gp. B

These are the 5 groups of order 18:

$$\mathbb{Z}/18\mathbb{Z},\ D_{18},\ \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},\ A,\ B.$$

Another approach:

Start from a gp hom $j : H \longrightarrow N$

$$\rightsquigarrow \alpha : H \longrightarrow \mathrm{Aut}_{gp}(N)$$
$$h \longmapsto \{n \longmapsto j(h)\,n\,j(h)^{-1}\}$$

$$\Rightarrow \quad N \rtimes_{\alpha} H \cong N \times H.$$