Lec 9/17

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#3: $G = GL_2(\mathbb{F}_5)$. $x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$G \circlearrowright G$ by conj. $\mathrm{Stab}_G(x) = \{g \in G : gx = xg\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{matrix} a & 2b \\ c & 2d \end{matrix} = \begin{matrix} a & b \\ 2c & 2d \end{matrix} \implies b = c = 0. \quad a, d \text{ free.}$$

So $|\mathrm{Stab}_G(x)| = \left|\left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : ad \neq 0\right\}\right| = 16$.

$$|\mathrm{Orbit}(x)| = \frac{|G|}{|\mathrm{Stab}_G(x)|}$$

In this context, $\mathrm{Stab}_G(x) = Z_G(x)$.

#7 G: abelian group $\supset H = \{g \in G : 2g = 0\}$

$$\sum_{g \in G} g = \sum_{h \in H} h = x$$

$$\|$$

$$\sum_{g \text{ s.t. } 2g=0} g + \sum_{\substack{g \text{ s.t.} \\ g \neq -g}} (g + (-g)) = \sum_{h \in H} h + 0.$$

<u>Part 2</u> $x = 0$ unless $H = \{0, x\}$.

$$H \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$$

result: $|H| = p^n$ and H is abelian

$$\Rightarrow H \cong \mathbb{Z}/p^{l_1} \times \cdots \times \mathbb{Z}/p^{l_k}$$

<u>Last year's MT1</u>: $G \geq H$, $\underset{|G/H|}{(G : H)} = n$, and H contains no nontrivial subgroups which are normal in G.

(T.S.) G is iso. to a subgroup of S_n.

$G \circlearrowright G/H$ by left multiplication,

and these actions are all permutations.

So $G \xrightarrow{f} S_n$.

now we prove that f is injective $\leftarrow \operatorname{Ker}(f) = \{e\}$

Suppose $g \in \operatorname{Ker}(f)$. then $g g_j H = g_j H$ $\forall j = 1, \ldots, n$.

So $g \in H$. so $\operatorname{Ker}(f) \leq H$, but $\operatorname{Ker}(f) \trianglelefteq G$,

So $\operatorname{Ker}(f) = \{e\}$.

group action: $G \times X \xrightarrow{\text{set map}} X$

$$G \xrightarrow{\text{gp hom}} \mathrm{Aut}_{set}(X).$$

$$\mathrm{Ker}(f) \subseteq \bigcap_{j=1}^{n} g_j H g_j^{-1}$$

Things to Remember:

① Order of $x \in G$ divides $|G|$. $H \leq G$ satisfies $|H| \,\big|\, |G|$ since $|G/H| = \frac{|G|}{|H|}$.

② $\varphi : G \xrightarrow{\text{gp hom}} G_2$
$$\begin{cases} \mathrm{Ker}(\varphi) \trianglelefteq G_1 \\ \mathrm{Ker}(\varphi) = \{e\} \iff \varphi \text{ is } 1\text{-}1 \\ G_1/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi) \leq G_2 \end{cases}$$

③ $G \circlearrowright X$. $|X| = \sum_{\text{Orbits}} |\mathcal{O}|$ in fact, those summands are equal.

$$= \sum_{\text{Orbits}} \frac{|G|}{|\mathrm{Stab}_G(x_0)|}$$

$$\# \text{ of orbits} = \frac{1}{|G|} \sum_{g \in G} |X^g|$$ ← Burnside's Lemma

$|G| = p^r$, $G \circlearrowright X$ (finite set) $\Rightarrow |X| \equiv |X^G| \pmod{p}$.

④ Sylow Theorems:

① $\mathrm{Syl}_p(G) \neq \emptyset$

② $G \circlearrowright \mathrm{Syl}_p(G)$ is transitive
conjugation

③ $\# \mathrm{Syl}_p(G) \equiv 1 \pmod{p}$, and divides $|G|$

for (4.2):
We proved if $|G| = n = p^r m$, $r \geq 1$, $(m,p)=1$,

$H \leq G$, $P \leq G$, $|H| = p^k \leq p^r$, $P \in \mathrm{Syl}_p(G)$

then $\exists g \in G$ s.t. $H \subset gPg^{-1}$

Q: Is $Z(G) \subset N$ for any $N \trianglelefteq G$?

A: No. obviously.

Let us try to come up w/ an example where $Z(G) \cap N = \{e\}$.

$Z(G) \times N$. $\quad \mathbb{Z}/2 \times D_{2\cdot3} =: G$.

So $Z(G) = \mathbb{Z}/2 \times \{e\}$, and $N = D_{2\cdot 3}$ works.