Lec 10/5

Wednesday, October 5, 2016 9:14 AM

Theorem 3: CA $\Leftrightarrow$ CCP & $\mathbb{R}$ has no infinitesimals.

proof: $\Rightarrow$: proved yesterday. (using BWP $\Leftrightarrow$ CA)

$\Leftarrow$: since CA $\Leftrightarrow$ NIP & $\mathbb{R}$ has no infinitesimals.

it suffices to show that CCP $\Rightarrow$ NIP.

Let $I_n = [a_n, b_n]$, $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ and $\lim_{n\to\infty} |a_n - b_n| = 0$.

Then $\{a_n\}$ forms a cauchy sequence

if $m > n$ then $a_m \in [a_n, b_n]$, so $a_m - a_n \leq b_n - a_n < \varepsilon$ if $n > N$.

so $\lim_{n\to\infty} a_n = L$ exists.

$a_n \leq L \leq b_n$ for all n for all n (proof by contradiction).

So $L \in \bigcap_{i=1}^{\infty} I_n$ and because $\lim_{n\to\infty} |a_n - b_n| = 0$,

There cannot be two distinct elements in $\bigcap_{i=1}^{\infty} I_n$.

So $\bigcap_{i=1}^{\infty} I_n = \{L\}$ ∎

Summary

CA $\Leftrightarrow$ LUBP $\Leftrightarrow$ GLBP $\Leftrightarrow$ NIP & no infinitesimals $\Leftrightarrow$ MCP $\Leftrightarrow$ BWP $\Leftrightarrow$ CCP & no infinitesimals.

($\Leftrightarrow$ IVT $\Leftrightarrow$ EVT) ← problem 4 on HW 6.

Remark an ordered field with no infinitesimals is called archimedian.

otherwise it is called non-archimedian.

archimedian fields $\subseteq \mathbb{R}$

There are ultra-non-archimedian fields:

For any countable set of positive elements $\{\varepsilon_1, \varepsilon_2, \dots\}$

There is a positive element $\omega > 0$ s.t. $\omega < \varepsilon_i \ \forall i$.

Non-archimedian is this property for one such $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

In an ultra-non-archimedian field, any Cauchy sequence $\{a_n\}$ is eventually constant: $a_N = a_{N+1} = a_{N+2} = \cdots$

hence it converges trivially. So the field satisfies CCP but has lots of infinitesimals.

If $\{a_n\}$ is a sequence that is not eventually constant, then we can find a subsequence $\{a_{n_j}\}$ s.t. $a_{n_j} \neq a_{n_k}$ if $j \neq k$.

If $\{a_n\}$ is Cauchy, so is any subsequence, but $\{|a_{n_j} - a_{n_{j+1}}| : j = 1, 2, \ldots\}$ is a countable set of positive elements, so there is a ω st. $|a_{n_j} - a_{n_{j-1}}| < \omega$

So $\{a_{n_j}\}$ is not Cauchy.

Differential Calculus

Definition: $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of f at a (if it exists and is finite).

Note: this requires that f is defined on an open interval containing a.

Also: $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

obtained by substituting $x = a+h$ in the previous definition.

to show these definitions are the same, technical point:

Define $$G(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a,\ x \in \operatorname{dom}(f) \\ f'(a) & \text{if } x = a. \end{cases}$$

Then $$\lim_{x \to a} G(x) = f'(a) = G(a)$$

by substitution/composition thm for limits:

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$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} G(a+h) = G\left(\lim_{h\to 0}(a+h)\right) = G(a) = f'(a).$$

Ex: $f(x) = x^{3/2} = (\sqrt{x})^3$ defined on $[0,\infty)$. Since $f'(a)$ requires f to be defined on an open interval around a, we must assume $a>0$.

$$f'(a) = \lim_{x\to a} \frac{x^{3/2} - a^{3/2}}{x-a} = \lim_{x\to a} \frac{x^3 - a^3}{(x-a)(x^{3/2}+a^{3/2})}$$

$$= \lim_{x\to a} \frac{(x-a)(x^2+xa+a^2)}{(x-a)(x^{3/2}+a^{3/2})} = \lim_{x\to a} \frac{x^2+ax+a^2}{x^{3/2}+a^{3/2}} \quad \text{(by localization principle)}$$

$$= \frac{3a^2}{2a^{3/2}} \quad \text{(need to check that } f \text{ is continuous at } a\text{)}.$$

$$= \frac{3}{2} a^{1/2}$$

Want to show $f(x) = x^{3/2}$ is continuous $\forall a>0$. $\left(\lim_{x\to a} x^{3/2} = a^{3/2} \quad \forall a>0\right)$

Since $x^{3/2} = (\sqrt{x})^3$, it suffices to show that $\sqrt{x}$ is continuous $\forall a>0$.

given $\varepsilon>0$, want to find $\delta>0$ s.t.

$$|x-a|<\delta \Rightarrow x\in \mathrm{dom}(\sqrt{x}) \ \& \ |\sqrt{x}-\sqrt{a}|<\varepsilon$$

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{a}} < \varepsilon \iff |x-a| < \sqrt{a}\,\varepsilon$$

choose $\delta = \min(a, \varepsilon\sqrt{a})$

Proposition: $f'(a)$ exists $\Rightarrow$ f is continuous at a.

Proof: if $f'(a)$ exists, then

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}(x-a)\frac{f(x)-f(a)}{x-a}$$

$$= 0\, f'(a) = 0$$

$$\iff \lim_{x\to a} f(x) = f(a)$$