Lec 9/14

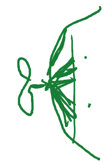
Friday, September 14, 2018 11:27

Practice Problems: 4.5 # 5, 6, 8, 30, 19-23.

Sylow Theorems problems:

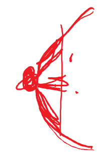
1. Can we prove a group is not simple.

eg: $|G| = 216 = 2^3 \cdot 3^3$.



Sylow thm #3: $n_2 \equiv 1 \mod 2$, $n_2 \mid 27$. $n_2 \in \{1, 3, 9, 27\}$

$n_3 \equiv 1 \mod 3$, $n_3 \mid 8$. $n_3 \in \{1, 4\}$



If $n_3 = 1$, we are done. ($P_3 \trianglelefteq G$) otherwise, $G \overset{\text{conjugation}}{\curvearrowright} Syl_3(G)$ (4 elts), so $G \xrightarrow{\varphi} S_4$

and $|S_4| = 24 < 216$, so $Ker(\varphi) \neq \{e\}$. also, $Ker(\varphi) \neq G$

Since the action is transitive (all sylow 3-subgrps are conjugate by #2)

So $Ker(\varphi) \trianglelefteq G$.

if φ is injective, $\varphi(G)$ is a subgroup so $|G|$ divides $|S_4|$.

2. Examples of Sylow p-subgps. $D_8 \subset S_4$ is a sylow 2-subgp.

$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \subset GL_2(\mathbb{F}_5)$ is a sylow 5-subgp.

$\underline{Ex}$ $G = D_{2m}$, m is odd. $G = \left\{ \begin{matrix} \boxed{e, r, r^2, \ldots, r^{m-1}} \\ \underbrace{s, sr, sr^2, \ldots, sr^{m-1}}_{\text{all order 2 elements.}} \end{matrix} \right\}$ $= \langle r \rangle \cong \mathbb{Z}/_m\mathbb{Z}$ $\trianglelefteq D_{2m}$

So all sylow 2-subgrps are $\{e, sr^i\}$ for some i. so $n_2(D_{2m}) = m$.

If m is not odd, eg $m = 6$, then r^3 is also of order 2 (and is in the center).

Also, Sylow 2-subgps have size 4 now & all contain r^3. $\longrightarrow$ conjugation cannot move the center

eg: $\{e, s, r^3, sr^3\} \cong \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$

$\{e, sr, r^3, sr^4\}$

$\{e, sr^2, r^3, sr^5\}$

Lemma: if $x \in G$ is central of order p, then $x \in P$ $\forall P \in Syl_p(G)$.

Pf There is a sylow p-subgr containing x

Since $|\langle x\rangle| = p$ so by Syl #2, $\langle x\rangle \subset gPg^{-1} = P'$

and every Sylow p-subgp is a conjugate of P',

and conjugation can't move center.

Lemma: $n_2\left(D_{2^a m}\right) = m$ if m is odd and $a \geq 1$.

Hint: Consider the map $D_{2^{a+1}m} \xrightarrow{f} D_{2^{a+1}m} \big/ \langle r^{2^a m}\rangle \cong D_{2^a m}$

check: f sets up a bijection

$$Syl_2(D_{2^{a+1}m}) \longleftrightarrow Syl_2(D_{2^a m})$$

and proceed by induction.

Ex: $G = GL_n(\mathbb{F}_p)$ $n \geq 2$, p prime

$$\begin{array}{cccc} \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \end{bmatrix} \\ c_1 \quad c_2 & & & c_n \end{array}$$

$$|G| = \underbrace{(p^n-1)\,(p^n-p)\cdots(p^n-p^{n-1})}_{\text{options}} = p^{\overbrace{1+2+\cdots+n-1}^{=\frac{n(n-1)}{2}}}\left((p^n-1)(p^{n-1}-1)\cdots(p-1)\right)$$

$\frac{n(n-1)}{2}$ slots

So a Sylow p-subgp is $\left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} : * \text{ is any of the } p^{\frac{n(n-1)}{2}} \text{ triangles} \right\}$

how many? $n_p(GL_2(\mathbb{F}_p)) = p + 1$

$$P_0 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \in \left\{ \begin{pmatrix} 1+\alpha x & -\alpha^2 x \\ x & 1-\alpha x \end{pmatrix} : x \in \mathbb{F}_p \right\}.$$

$$Syl_p(G) = \{ g P_0 g^{-1} \mid g \in G \}$$

$$\text{Every } g \in GL_2 \text{ has the form } \overset{B}{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}} \text{ or } \overset{\text{in fact, } \left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]}{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & -\frac{\Delta}{c} \end{bmatrix} \quad \text{if } c \neq 0$$

$$GL_n(k) = \bigsqcup_{\sigma \in S_n} B X_\sigma B \qquad \text{Bruhat decomposition.}$$

$$g P_0 g^{-1} = P_0 \quad \forall g \in B.$$

$$g = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad g P_0 g^{-1} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}^{-1}$$

Conjecture: $$n_p(GL_n(\mathbb{F}_p)) = \underbrace{\frac{p^n - 1}{p-1}}_{[n]_p} \cdot \underbrace{\frac{p^{n-1}-1}{p-1}}_{[n-1]_p} \cdot \cdots \cdot \underbrace{\frac{p-1}{p-1}}_{[1]_p} \longrightarrow n! \text{ as } p \to 1.$$