Rec 9/26

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- every finite domain is a division ring

↳ nonzero elements can be cancelled, so the nonzero elts form a gp.

↳ $|R|=n$, the set $\{x, x^2, x^3, \dots, x^n\}$ contains a duplicate

$x^i = x^j$, $i<j$. Then $x^i(1-x^{j-i})=0$, $x^i \neq 0 \Rightarrow x^{j-i}=1$,

So x^{j-i-1} is inverse for x.

idempotent: $e^2=e$

nilpotent: $\exists\, n$ s.t. $x^n=0$.

Jordan-Hölder Theorem:

If $1=H_k \lhd H_{k-1} \lhd \cdots \lhd H_1 = G$ and $1=G_\ell \lhd G_{\ell-1} \lhd \cdots \lhd G_1 = G$ are two comp. series for G, then $\ell = k$ and $\exists\, \sigma \in S_{k-1}$ s.t. $G_i/G_{i+1} \cong H_{\sigma(i)}/H_{\sigma(i)+1}$.

We can use this to prove FTA: arithmetic.

Idea $n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_\ell$.

$$\mathbb{Z}_n \rhd \mathbb{Z}_{\frac{n}{p_1}} \rhd \mathbb{Z}_{\frac{n}{p_1 p_2}} \rhd \cdots \rhd \mathbb{Z}_{p_k} \rhd 1$$

$$\mathbb{Z}_n \rhd \mathbb{Z}_{\frac{n}{q_1}} \rhd \mathbb{Z}_{\frac{n}{q_1 q_2}} \rhd \cdots \rhd \mathbb{Z}_{q_\ell} \rhd 1$$

Prime Ideal

R/p is an integral domain

$IJ \subset P \Rightarrow I \subset P$ or $J \subset P$.

- $(I:J) = \{ r \in R \mid rJ \subset I \}$

- $\operatorname{ann}(x) = \{ r \mid rx = 0 \} = \{ r \mid r(x) \subset (0) \} = (0 : x)$

- $Z(R) = \bigcup_{x \in R} \operatorname{ann}(x)$

$a \in R$ is irreducible if (TFAE)

① $a = bc \Rightarrow b$ or c is a unit

② $a = bc \Rightarrow (a) = (b)$ or $(a) = (c)$

③ (a) is maximal in the set of proper principal ideals.

$\mathbb{Z}_4$, $0, 2$ 0-divisors

$\mathbb{Z}_6$, $3^2 = 3$, $3^3 = 3$, et cetera

$\mathbb{Z}_6[X]$ $(2x+3)(3x+2) = x$

both irreducible

Central Series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

(1) $G_{i+1} \trianglelefteq G_i$

(2) $G_i / G_{i+1} \leq Z(G/G_{i+1})$

G is nilpotent iff it has a central series.