

Defn group homomorphism:  $f(ab) = f(a)f(b)$ .

Defn <sup>homomorphism</sup>  $f$  is an isomorphism if  $\exists g$  s.t.  $f \circ g = \text{id}$ ,  $g \circ f = \text{id}$ .

in our context, isomorphism = bijective homomorphism. (Ex. show this agrees w/ a book) ↗ is also hom.

$G_1 \cong G_2$  iff  $\exists$  an isomorphism  $f: G_1 \rightarrow G_2$

Lemma if  $f: G_1 \rightarrow G_2$  is a group homomorphism then  $f(e_1) = e_2$ ,  $f(x^{-1}) = f(x)^{-1}$ .

Pf  $f(e_1) = f(e_1) f(e_1) \Rightarrow f(e_1) = e_2$ .

$e_2 = f(e_1) = f(x) f(x^{-1}) \Rightarrow f(x^{-1}) = f(x)^{-1}$ .

Ex. of one-sided inverse:  $\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} \\ \begin{bmatrix} a \\ b \end{bmatrix} & \mapsto & a \end{array}$   $g(x) = \begin{bmatrix} x \\ 0 \end{bmatrix}$  so  $f(g(x)) = x$   
but  $g(f(x)) \neq x$  in general.

Ex.  $S_3 \cong D_6$ .

$\uparrow$   
 $S_3 \leftarrow \langle s, r \mid s^2 = r^3 = e, sr = rs^{-1} \rangle$

$(12) \longleftarrow s$

$(123) \longleftarrow r$

ex. this is an iso.

Ex.  $\det(AB) = \det(A)\det(B)$  so  $\det: GL_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$  is a group homomorphism,  
but not an isomorphism (it's onto but not 1-1).  $\det \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = 1 \forall x \in \mathbb{R}^\times$ .

Ex.  $G_1 = \text{Free}(2) \xrightarrow{p \in \text{Hom}(G_1, G_2)} G_2 = \mathbb{Z}^2$ .  $P(w) = (\# \text{ of } x\text{'s}, \# \text{ of } y\text{'s})$ . (where path ends).

$P$  cannot be injective:  $P(xy x^{-1}) = P(y)$ . also, if it were, then  $\text{Free}(2) \cong \mathbb{Z}^2$   
↑ Not Abelian

## First Isomorphism Theorem

$f: G_1 \rightarrow G_2$  group hom

Defn: Kernel of  $f$ ,  $\text{Ker}(f) = \{x \in G_1 \mid f(x) = e_2\} \subseteq G_1$

Image of  $f$ ,  $\text{Im}(f) = \{f(x) \mid x \in G_1\} \subseteq G_2$

Ex.  $\text{Ker}(\det) = \text{SL}_n(\mathbb{R})$ .

$\text{Ker}(p) = H = \langle \{\alpha \beta \alpha^{-1} \beta^{-1} : \alpha, \beta \in \text{Free}(2)\} \rangle$

Lemma:  $\text{Ker}(f) \trianglelefteq G_1$ ,  $\text{Im}(f) \leq G_2$ .

Pf.  $e_1 \in \text{Ker}(f) \Rightarrow \text{Ker}(f) \neq \emptyset$ .

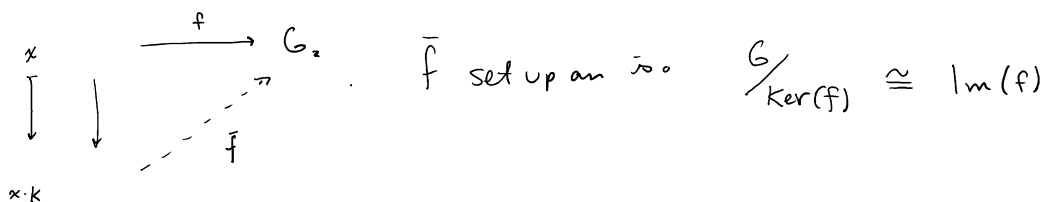
to show  $x, y \in \text{Ker}(f) \Rightarrow x^{-1}y \in \text{Ker}(f)$ :  $f(x^{-1}y) = f(x)^{-1}f(y) = e_2$

So  $\text{Ker}(f) \leq G_1$ . To show  $\text{Ker}(f)$  is normal in  $G_1$ , we have to check  $\forall x \in G_1, \forall k \in \text{Ker}(f)$ ,

$x k x^{-1} \in \text{Ker}(f)$ .  $f(x k x^{-1}) = f(x)f(k)f(x)^{-1} = f(x)f(x)^{-1} = e_2$ .  $\square$

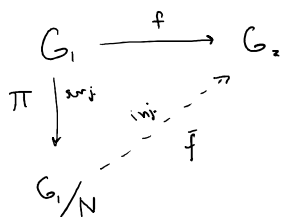
Let  $f$  be a gp homomorphism.

First Isomorphism Theorem:  $\forall f$  defines a unique group hom  $G_1/\text{Ker}(f) \xrightarrow{\bar{f}} G_2$   
 $x \cdot \text{Ker}(f) \mapsto f(x)$



Ex of group hom:  $G \xrightarrow{\pi} G/N$  ( $N \trianglelefteq G$ ).  $\pi$  is gp hom,  $\text{Ker}(\pi) = N$ .  
 $g \mapsto gN$

So  $\forall N \trianglelefteq G$ ,  $\exists$  hom  $f$  s.t.  $N = \text{Ker}(f)$ .  $\pi$  is called "natural projection onto quotient gr."



Pf of Thm. Claim:  $\forall$  gr hom  $g: H_1 \rightarrow H_2$

$$(1) \quad g \text{ is 1-1} \iff \text{Ker}(g) = \{e_1\}$$

$$(2) \quad g \text{ is onto} \iff \text{Im}(g) = H_2$$

$$(\Rightarrow) \quad \text{Suppose } x \in H_1 \text{ s.t. } g(x) = e_2 = g(e_1). \quad g \text{ is 1-1} \Rightarrow x = e_1.$$

$$(\Leftarrow) \quad \text{If } g(x) = g(y) \text{ then } g(x^{-1}y) = e_1 \Rightarrow x^{-1}y = e_1 \Rightarrow x = y.$$

$\bar{f}$  is well-defined - It does not depend on choice of  $x$ . Suppose  $xK = yK$ .

$$\text{then } x^{-1}y \in K \Rightarrow f(x^{-1}y) = e_2 \Rightarrow f(x) = f(y).$$

well-defined bc  $K$  is normal

$$\bar{f}(x_1K * x_2K) = \bar{f}((x_1 * x_2)K) = f(x_1x_2)$$

$\parallel$

$$\bar{f}(x_1K) \bar{f}(x_2K) = f(x_1)f(x_2) \quad // \checkmark$$

$$\text{Ker}(\bar{f}) = \{eK\} \quad \text{so } \bar{f} \text{ is 1-1.}$$

$$\text{So } G_1 / \text{Ker}(f) \cong \text{Im}(f) \quad (\text{obviously}) \quad \square$$

$$\underline{\text{Ex.}} \quad GL_2(\mathbb{R}) \xrightarrow[\det]{\text{hom}} \mathbb{R}^* \quad \text{so by thm, } GL_2(\mathbb{R}) / SL_2(\mathbb{R}) \cong \mathbb{R}^*.$$

$$\underline{\text{Ex.}} \quad \text{Free}(2) \xrightarrow{p} \mathbb{Z}^2.$$

$$H = \langle \{\alpha\beta\alpha^{-1}\beta^{-1} \mid \alpha, \beta \in \text{Free}(2)\} \rangle. \quad H \subseteq \text{Ker}(p).$$

$$* \quad \underbrace{\text{Free}(2) / \langle \alpha\beta\alpha^{-1}\beta^{-1} = e \rangle}_H \cong \mathbb{Z}^2 \quad \text{so } H \text{ must equal } \text{Ker}(p)$$

$$\text{since } \text{Free}(2) / \text{Ker}(p) = \mathbb{Z}^2 = G$$

2 generators, they commute