Lec 8/29

Wednesday, August 29, 2018 11:29

<u>Defn</u> group homomorphism : $f(ab) = f(a)f(b)$.

<u>Defn</u> homomorphism f is an isomorphism if $\exists g$ s.t. $f \circ g = id$, $g \circ f = id$.

g is also hom.

in our context, isomorphism = bijective homomorphism. [Ex. show this agrees w/ above]

$G_1 \cong G_2$ iff $\exists$ an isomorphism $f: G_1 \to G_2$

<u>Lemma</u> if $f: G_1 \to G_2$ is a group homomorphism then $f(e_1) = e_2$, $f(x^{-1}) = f(x)^{-1}$.

<u>Pf</u> $f(e_1) = f(e_1) f(e_1) \Rightarrow f(e_1) = e_2$.

$e_2 = f(e_1) = f(x) f(x^{-1}) \Rightarrow f(x^{-1}) = f(x)^{-1}$.

<u>Ex.</u> of one-sided inverse : $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$, $\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto a$ $\quad g(a) = \begin{bmatrix} a \\ 0 \end{bmatrix}$ so $f(g(x)) = x$

but $g(f(x)) \neq x$ in general.

<u>Ex.</u> $S_3 \cong D_6$.

$$S_3 \longleftarrow \langle s, r \mid s^2 = r^3 = e,\ srs = r^{-1} \rangle$$

$(12) \longleftarrow s$

$(123) \longleftarrow r$

<u>ex.</u> this is an <u>iso</u>.

<u>Ex.</u> $\det(AB) = \det(A)\det(B)$ so $\det : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^\times$ is a group homomorphism,

but not an isomorphism (it's onto but not 1-1). $\det\begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} = 1 \ \forall x \in \mathbb{R}^\times$.

<u>Ex.</u> $G_1 = \text{Free}(2) \xrightarrow{P \in \text{Hom}(G_1, G_2)} G_2 = \mathbb{Z}^2$. $\quad P(w) = (\#\text{ of } x\text{'s}, \#\text{ of } y\text{'s})$. (where path ends).

P cannot be injective : $P(xyx^{-1}) = P(y)$. also, if it were, then $\text{Free}(2) \cong \mathbb{Z}^2$

Not Abelian

<u>First Isomorphism Theorem</u>

$f : G_1 \longrightarrow G_2$ group hom

Defn. Kernel of f, $\mathrm{Ker}(f) = \{x \in G_1 \mid f(x) = e_2\} \subseteq G_1$

Image of f, $\mathrm{Im}(f) = \{f(x) \mid x \in G_1\} \subseteq G_2$

ex. $\mathrm{Ker}(\det) = SL_n(\mathbb{R})$.

$\mathrm{Ker}(p) = H = \langle\{\alpha\beta\alpha^{-1}\beta^{-1} : \alpha, \beta \in \mathrm{Free}(2)\}\rangle$

Lemma: $\mathrm{Ker}(f) \trianglelefteq G_1$, $\mathrm{Im}(f) \leq G_2$.

Pf. $e_1 \in \mathrm{Ker}(f) \Rightarrow \mathrm{Ker}(f) \neq \phi$.

to show $x, y \in \mathrm{Ker}(f) \Rightarrow x^{-1}y \in \mathrm{Ker}(f)$: $f(x^{-1}y) = f(x)^{-1}f(y) = e_2$

So $\mathrm{Ker}(f) \leq G$. To show $\mathrm{Ker}(f)$ is normal in G_1, we have to check $\forall x \in G$, $\forall k \in \mathrm{Ker}(f)$, $xkx^{-1} \in \mathrm{Ker}(f)$. $f(xkx^{-1}) = f(x)f(k)f(x)^{-1} = f(x)f(x)^{-1} = e_2$. $\square$

First Isomorphism Theorem: Let f be a gp homomorphism. f defines a unique group hom $G_1/_{\mathrm{Ker}(f)} \xrightarrow{\bar{f}} G_2$

$x \cdot \mathrm{Ker}(f) \longmapsto f(x)$

$$\begin{tikzcd} x \arrow[d] & G_1 \arrow[r, "f"] \arrow[d] & G_2 \\ x \cdot K & G_1/K \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$\bar{f}$ set up an iso $G/_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$

Ex of group hom: $\begin{matrix} G \xrightarrow{\pi} G/_N \\ g \longmapsto gN \end{matrix}$ $(N \trianglelefteq G)$. π is gp hom, $\mathrm{Ker}(\pi) = N$.

So $\forall N \trianglelefteq G$, $\exists$ hom f s.t. $N = \mathrm{Ker}(f)$. π is called "natural projection onto quotient gp."

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "\pi"', "\text{surj}"] & G_2 \\ G_1/N \arrow[ur, dashed, "\bar{f}"', "\text{inj}"] & \end{tikzcd}$$

Pf of Thm. Claim: $\forall$ gr hom $g: H_1 \to H_2$

(1) g is 1-1 $\iff$ $\mathrm{Ker}(g) = \{e_1\}$

(2) g is onto $\iff$ $\mathrm{im}(g) = H_2$

($\Rightarrow$) Suppose $x \in H_1$ s.t. $g(x) = e_2 = g(e_1)$. g is 1-1 $\Rightarrow x = e_1$.

($\Leftarrow$) If $g(x) = g(y)$ then $g(x^{-1}y) = e_1 \Rightarrow x^{-1}y = e_1 \Rightarrow x = y$.

$\bar{f}$ is well-defined - It does not depend on choice of x. Suppose $xK = yK$.

then $x^{-1}y \in K \Rightarrow f(x^{-1}y) = e_2 \Rightarrow f(x) = f(y)$.

well-defined bc K is normal.

$$\bar{f}(x_1K * x_2K) = \bar{f}((x_1 * x_2)K) = f(x_1x_2)$$

$$\parallel$$

$$\bar{f}(x_1K)\,\bar{f}(x_2K) = f(x_1)f(x_2) \quad /\!/\checkmark$$

$\mathrm{Ker}(\bar{f}) = \{eK\}$ so $\bar{f}$ is 1-1.

So $G_1/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ (obviously) $\square$

Ex. $GL_2(\mathbb{R}) \xrightarrow[\text{det}]{\text{hom:}} \mathbb{R}^\times$. So by thm, $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \cong \mathbb{R}^\times$.

Ex. $\mathrm{Free}(2) \xrightarrow{p} \mathbb{Z}^2$.

$\mathrm{Free}(2) \geq H = \langle\{\alpha\beta\alpha^{-1}\beta^{-1} \mid \alpha, \beta \in \mathrm{Free}(2)\}\rangle$. $H \subseteq \mathrm{Ker}(p)$.

$$* \left[\underbrace{\mathrm{Free}(2) / \overset{H}{\underset{\psi}{xyx^{-1}y^{-1} = e}}}_{\text{2 generators, they commute}} \cong \mathbb{Z}^2 \right.$$

so H must equal $\mathrm{Ker}(p)$ since $\mathrm{Free}(2)/\mathrm{Ker}(p) = \mathbb{Z}^2 = (*)$