Lec 4/18

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on (a,b) only finitely many discontinuities, all at most jump

Theorem Suppose f is periodic, piecewise smooth,

Suppose that $f'(\theta)$ is defined and bounded on $(a-\delta, a) \cup (a, a+\delta)$

for some $\delta > 0$. then the Fourier series for f converges to $\frac{1}{2}[f(a+) + f(a-)]$

Proof: as before, $S_N^f(a) - \frac{1}{2}[f(a+) + f(a-)] = C'_{-N-1} - C'_N$

Where C'_n are fourier coefficients of a function

$$g_a(\varphi) = \begin{cases} \dfrac{f(\varphi + a) - f(a-)}{e^{i\varphi} - 1} & \text{if } \varphi \in [-\pi, 0) \\[2ex] \dfrac{f(\varphi + a) - f(a+)}{e^{i\varphi} - 1} & \text{if } \varphi \in (0, \pi] \end{cases}$$

by hypothesis $g_a(\varphi)$ has finite discontinuities, and it is integrable

Provided it is bounded near 0.

we may suppose $\delta < \frac{\pi}{4}$

by MVT for real functions, for φ in $(-\delta, 0)$:

$$|g_a(\varphi)| = \left| \frac{f(\varphi+a) + f(a-)}{e^{i\varphi} - 1} \right| = \left| \frac{(f_1'(\alpha_1 + a) + i f_2'(\alpha_2 + a))\varphi}{(-\sin(\beta_1) + i\cos(\beta_2))\varphi} \right|$$

$$\leq \frac{|f_1'(\alpha_1 + a)| + |f_2'(\alpha_2 + a)|}{|\cos(\beta_2)|} \leq \frac{2M}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}M$$

f' is bounded

Similarly for $\varphi \in (0, \delta) \Rightarrow g_a(\varphi)$ is bounded around 0.

$\Rightarrow g_a(\varphi)$ integrable on $[-\pi, \pi] \Rightarrow |g_a(\varphi)|^2$ integrable $\Rightarrow \sum_{i=1}^{n} |C_n'|^2 < \infty$

$$\Rightarrow \lim_{N \to \infty} (C'_{-N-1} - C'_N) = 0.$$

$$\Rightarrow \lim_{N \to \infty} S_N^f(a) = \frac{1}{2}[f(a+) + f(a-)].$$

$$\sum_{n=-\infty}^{\infty} c_n e^{ina}$$

Term-by-Term integration / differentiation of fourier series.

$$\frac{\theta}{2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta) \quad \text{for } \theta \in (-\pi, \pi)$$

Term by term differentiation gives

$$\frac{1}{2} \overset{?}{=} \underbrace{\sum_{n=-\infty}^{\infty} (-1)^{n+1} \cos(n\theta)}$$

never converges, terms don't go to 0.

(if $\lim_{n\to\infty} \cos(n\theta) = 0$ then $\lim_{n\to\infty} \cos(2n\theta) = 0$

$$\| $$

$$\lim_{n\to\infty} (2\cos^2(n\theta) - 1) = -1$$

contradiction)

Fundamental Theorem of Calculus

If $f: [a,b] \longrightarrow \mathbb{C}$ continuous, f' defined on (a,b) & integrable,

then $\int_a^b f'(t)\,dt = f(b) - f(a)$

<u>Improvement</u>: Assume only that f' is defined & integrable on $(a,b) \setminus \{x_1, \dots x_k\}$.

applying FTC to $[a, x_1], \dots, [x_i, x_{i+1}], \dots [x_{k-1}, x_k], [x_k, b]$, we get the same result:

$$\int_a^b f'(t)\,dt = f(b) - f(x_k) + f(x_k) - f(x_{k-1}) + \dots - f(a).$$

$= f(b) - f(a)$.

<u>Integration by parts</u>: If $f, g : [a,b] \to \mathbb{C}$ continuous, f', g' def/int on (a,b) minus $\{x_1, \dots, x_n\}$

then $\int_a^b (f(t)g(t))' dt = \int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt$

$\| $

$[f(t)g(t)]_a^b$

$$\Rightarrow \int_a^b f'(t)\,g(t)\,dt = [f(t)\,g(t)]_a^b - \int_a^b f(t)\,g'(t)\,dt$$

<u>Theorem</u> If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic & continuous,

and f' is defined and integrable on $(-\pi, \pi) \setminus$ finite set

then if c_n, c_n' are fourier coefficients of f and f' respectvely,

then $c_n' = in c_n$ for all n.

<u>Proof</u>: Integration by parts:

$$c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(n) e^{-in\theta}\, d\theta = \frac{1}{2\pi}\left(\left[f(\theta)\, e^{-in\theta}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(\theta)(-in\, e^{-in\theta})\, d\theta\right)$$

$$= \frac{1}{2\pi} f(\pi)(-1)^n - \frac{1}{2\pi} f(-\pi)(-1)^n + in\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta\right)$$

$$= 0 + in\, c_n$$

no cancel.

So the reason $\frac{\theta}{2}$ doesn't work is it's discontinuous at odd multiples of π.

If f' is piecewise smooth then

$$f'(\theta) = \left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \right)' = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} in\, c_n e^{in\theta+} + \sum_{n=-\infty}^{\infty} in c_n e^{in\theta-} \right]$$