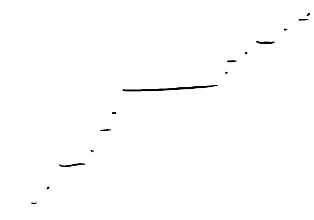
Rec 9/19

Thursday, September 19, 2019 10:21

$\overline{\mathbb{R}}$ looks like $[\longrightarrow]$.

Non-Borel Set?

φ "devil's staircase"



cts, increasing (not strictly)

$\psi(x) = \varphi(x) + x$ cts, strictly increasing.

$$\mathcal{C} \sqcup \mathcal{O} = [0,1] \longrightarrow [0,2].$$

$$\psi(\mathcal{C}) \sqcup \psi(\mathcal{O}) = [0,2]$$

$$m(\psi(\mathcal{O})) = 1$$

$\Rightarrow m(\psi(\mathcal{C})) = 1$. Let $W = \psi(\mathcal{C})$

Fact: If $m(W) > 0$, then $W \supset$ non mble set.

Why? W is uncountable.

Consider eq classes under $\sim$ "rational relation"

collection of eq classes, choose $N = \{1 \text{ member of each in } W\}$

Then leave W and do same trick as before.

Alternative: $W - W \supset I \supset N$

W mble $\Rightarrow$ $W - W$ mble?

$N \subset W = \psi(\mathcal{C})$

$\psi^{-1}(N) \subset \mathcal{C}$, and $m(\mathcal{C}) = 0$, $\mathcal{L}$ is complete,

So $\psi^{-1}(N)$ is mble.

$$\psi : \underbrace{\psi^{-1}(N)}_{\text{mble}} \longmapsto \underbrace{N}_{\text{non-mble}}$$

So $\Psi^{-1}(N)$ is not Borel.

(Ψ is cts injection so it takes open intervals to open intervals Borel sets to Borel sets).

(1) f is $\overline{\mathcal{M}}$-mble, $g=f$ a.e.

Then g is $\overline{\mathcal{M}}$-mble.

pf $f-g=0$ on E, $\mu(E^c)=0$.

$$g^{-1}((a,\infty)) = (\underbrace{g^{-1}((a,\infty))\cap E}_{=f^{-1}((a,\infty))\cap E}) \cup (\underbrace{g^{-1}((a,\infty))\cap E^c}_{\text{measure } 0})$$

it's not true if $\overline{\mathcal{M}}$ is replaced by $\mathcal{M}$.

(2) Show that if f is $\overline{\mathcal{M}}$-mble, then $\exists$ an $\mathcal{M}$-mble g with $f=g$ a.e.

(prop 2.12 in folland)

(a) assume $f=\chi_E$ where $E\in\overline{\mathcal{M}}$.

$E = F\cup N$ where $F\in\mathcal{M}$, $\mu(N)=0$.

Let $g = \chi_F$.

(b) f simple fn. ✓

(c) f arbitrary.

$\{\phi_n\}$ simple fns in $\overline{\mathcal{M}}$ which increase to f.

$\{\psi_n\}$ " " $\mathcal{M}$ " " g.

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