Lec 3/7

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Two comments about the last proof

1: we don't need to make $h=1$ on $\mathbb{R}^2 \setminus (-a,a)^2$, we can simply extend h to $[-a,a]^2 \setminus E$ and then make h constant on $\{tz : t \geq 1\}$ for each $z \in \partial([a,a]^2)$.

2: union of 3 or fewer edges is contractible.

Lemma:

Let $a \in \mathbb{C}$, let $\varepsilon > 0$. Let $f : D(a,3\varepsilon) \setminus \{a\} \longrightarrow \mathbb{C}^\times$ be cts.

Then $\exists\, \ell \in \mathbb{Z}$ and $g : D(a,3\varepsilon) \setminus \{a\} \xrightarrow{\text{cts}} \mathbb{C}^\times$ s.t. $g = f$ on $D(a,3\varepsilon) \setminus D(a,2\varepsilon)$ and $g(z) = (z-a)^\ell \ \forall z \in D(a,\varepsilon) \setminus \{a\}$.

Pf Define $\gamma : S^1 \longrightarrow \mathbb{C}^\times$ by $\gamma(\omega) = f(a + 2\varepsilon\omega)$.

Then γ is a loop in $\mathbb{C}^\times$. Let $\ell = \operatorname{ind}(\gamma)$. Define $\beta : S^1 \longrightarrow \mathbb{C}^\times$ by $\beta(\omega) = (\varepsilon\omega)^\ell$. $\operatorname{ind}(\beta) = \ell$ as well so $\beta \simeq \gamma$ in $\mathbb{C}^\times$. Let $H : \beta \simeq \gamma$ in $\mathbb{C}^\times$.

Define $g : D(a,3\varepsilon) \setminus \{a\} \xrightarrow{\text{cts}} \mathbb{C}^\times$ by

$$g(z) = \begin{cases} f(z) & \text{if } z \in D(a,3\varepsilon) \setminus D(a,2\varepsilon) \\ H\left(\frac{z-a}{|z-a|}, \frac{|z-a| - \varepsilon}{\varepsilon}\right) & \text{if } z \in D(a,2\varepsilon) \setminus D(a,\varepsilon) \\ (z-a)^\ell & \text{if } z \in D(a,\varepsilon) \setminus \{a\} \end{cases}$$

$\square$

Notation Let $K \subseteq \mathbb{C}^\times$ be cpt. $R(K,\mathbb{C}^\times)$ is the subgroup of $C(K,\mathbb{C}^\times)$ generated by $\{\mathrm{id}_K - a : a \in \mathbb{C} \setminus K\}$. This is rational functions on K (poles & zeroes in complement of K).

Corollary: Let $K \subseteq \mathbb{C}$ be cpt. Let $f \in C(K, \mathbb{C}^\times)$.

Then $\exists\, h \in R(K, \mathbb{C}^\times)$ s.t. $\frac{f}{h}$ can be extended to a continuous map from $\mathbb{C}$ into $\mathbb{C}^\times$.

Pf We know that $\exists$ a finite set $E \subseteq \mathbb{C} \setminus K$ s.t. f can be extended to a continuous map $g : \mathbb{C} \setminus E \longrightarrow \mathbb{C}^\times$.

For each $a \in E$ there is $\varepsilon_a > 0$ s.t. $D(a, 3\varepsilon_a) \subseteq \mathbb{C} \setminus (K \cup E)$.

Now $\forall a \in E$, g is continuous on $D(a, 3\varepsilon_a) \setminus \{a\}$. Hence there is a family $(l_a)_{a \in E}$ and a cts fn $\tilde{g} : \mathbb{C} \setminus E \longrightarrow \mathbb{C}^\times$ s.t. $\tilde{g} = \overset{f}{\overset{\|}{g}}$ on K and $\forall a \in E, \forall z \in D(a, \varepsilon_a) \setminus \{a\}$, $\tilde{g}(z) = (z-a)^{l_a}$.

Define $\tilde{h}$ on $\mathbb{C} \setminus E$ by $\tilde{h}(z) = \prod_{a \in E} (z-a)^{l_a}$.

Then $\tilde{h}$ is a cts map from $\mathbb{C} \setminus E$ into $\mathbb{C}^\times$ and $\forall a \in E$, $\forall z \in D(a, \varepsilon_a)$,

$$\frac{\tilde{g}(z)}{\tilde{h}(z)} = \frac{(z-a)^{l_a}}{\prod_{b \in E} (z-b)^{l_b}} = \prod_{b \in E \setminus \{a\}} (z-b)^{-l_b}$$

So $\tilde{g}/\tilde{h}$ can be extended to a cts map from $\mathbb{C}$ into $\mathbb{C}^\times$, say u.

now let $h = \tilde{h}|_K$. Then $h \in R(K, \mathbb{C}^\times)$ and also $\forall z \in K$,

$$\frac{f(z)}{h(z)} = \frac{\tilde{g}(z)}{\tilde{h}(z)} = u(z),$$ and so u is an extension of $\frac{f}{h}$ to a cts map from $\mathbb{C}$ into $\mathbb{C}^\times$. $\square$

Corollary Let $K \subseteq \mathbb{C}$ be compact. Let $f : K \longrightarrow \mathbb{C}^\times$ be cts. Then $\exists\, h \in R(K, \mathbb{C}^\times)$ s.t. $f \simeq h$ in $\mathbb{C}^\times$

Pf by prev. corollary, $\exists\, h \in R(K, \mathbb{C}^\times)$, $\frac{f}{h}$ can be extended to a cts map from $\mathbb{C}$ into $\mathbb{C}^\times$ and so since $\mathbb{C}$ is contractible, $\frac{f}{h} \simeq 1$ so $f \simeq h$ in $\mathbb{C}^\times$.

Notation Let $K \subseteq \mathbb{C}$ be cpt. Then $\rho(K, \mathbb{C}^\times) \overset{\text{def}}{=} \{[h] : h \in R(K, \mathbb{C}^\times)\}$

where $[h]$ is the homotopy class of h in $C(K, \mathbb{C}^\times)$.

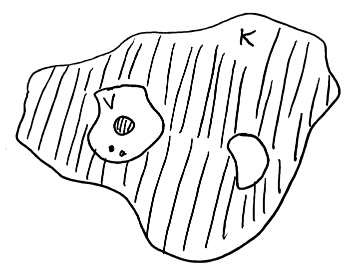
Corollary Let $K \subseteq \mathbb{C}$ be cpt. Then $\pi(K, \mathbb{C}^\times) = \rho(K, \mathbb{C}^\times)$.

Pf Let $f \in C(K, \mathbb{C}^\times)$. By previous corollary, $\exists\, h \in R(K, \mathbb{C}^\times)$ s.t. $f \simeq h$.

thus $[f] \simeq [h]$ $\square$

Thus $\pi(K, \mathbb{C}^\times)$ is generated by $\{[\mathrm{id}_K - a] : a \in \mathbb{C} \setminus K\}$.

Propn Let $K \subseteq \mathbb{C}$ be cpt. Let V be a bounded component of $\mathbb{C} \setminus K$. Let $a \in V$, let $\ell \in \mathbb{Z}$, and let $f = (\mathrm{id}_K - a)^\ell$. Suppose f can be extended to a cts fn g from $K \cup V$ into $\mathbb{C}^\times$. Then $\ell = 0$.



She fell in love with the drummer

Pf First let us show that in fact f can be extended to a cts map h from $\mathbb{C}$ to $\mathbb{C}^\times$.

Note that $K \cup V$ is closed since its complement is the union of the other components of $\mathbb{C} \setminus K$.