Rec 8/27

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Arithmetic on $\overline{\mathbb{R}}$.

$x = \infty$.

$x \pm \infty = \pm\infty$ if $x \neq \pm\infty$.

$\infty + \infty = \infty$ ✓ $\infty - \infty$ X (undefined)

$0 \times \infty = 0$

$\infty \times \infty = \infty$

$\infty \times x = \pm\infty$ (depending on sign of x).

$\frac{x}{\infty}$? undefined if $x = \pm\infty$.

0 if x is finite ?

but... $\frac{a}{b} \cdot b = a$ normally,

and $\frac{x}{\infty} \cdot \infty = 0$ not x.

So, leave $\frac{x}{\infty}$ undefined.

Note: $(a-a) \times \infty = \infty - \infty$.

$\parallel$

$0 \times \infty = 0$.

So things are bad.

$a = \lim a_n$ in $\overline{\mathbb{R}}$.

$\forall\, \varepsilon > 0,\ \exists N$ s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$ in $\mathbb{R}$

or $\forall M > 0,\ \exists N$ s.t. $n > N \Rightarrow a_n > M$ $(a_n \to \infty)$.

$\overline{\lim}$, $\underline{\lim}$ "$\overline{\mathbb{R}}$ is complete".

$$\overline{\lim}\, a_n = \inf_{k} \sup_{n \geq k} a_n$$

$$\underline{\lim}\, a_n = \sup_{k} \inf_{n \geq k} a_n$$

$$\lim a_n = \underbrace{\overline{\lim}\, a_n = \underline{\lim}\, a_n}$$

if these two are equal.
otherwise, it doesn't exist.

$\lim (s_n + t_n) = \lim s_n + \lim t_n$ (in $\mathbb{R}$)

fails in $\overline{\mathbb{R}}$.

restrict to $\overline{\mathbb{R}}_+ = [0, \infty]$, then it works.

Proof (1) $s_n, t_n \to$ finite

(2) one $\to \infty$, other $\to$ finite

(could also do something with bounding limsups & infs).

π - system: subsets of X, closed under finite intersections.

Λ - System: (Dynkin System, d-system)

(1) $X \in \Lambda$.

(2) $E \in \Lambda \Rightarrow E^c \in \Lambda$.

(3) $\{E_i\}$ disjoint $\Rightarrow \bigcup_i^{\infty} E_i \in \Lambda$.

eg $\mathcal{F}_\sigma \cup \mathcal{G}_\delta$ looks like it...

?

$$\left\{ \begin{matrix} (0,2), & (0,2)^c, & X \\ (1,3), & (1,3)^c, & \emptyset \end{matrix} \right\}$$

$$\varepsilon = \sum_1^{\infty} \varepsilon_i .$$

$$\varepsilon_i = \frac{\varepsilon}{2^i},$$

$$\text{or} \quad \varepsilon_i = \frac{3\varepsilon}{(\pi i)^2}, \qquad \sum \varepsilon_i = \frac{\varepsilon}{2}, \qquad \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

Define an o.m. on $\mathbb{R}$ as follows:

$$\rho : \mathcal{E} \to [0,\infty], \quad \mathcal{E} = \{(a,b) : a<b\}$$

$\rho((a,b)) = F(b) - F(a)$ where F is

cts increasing fn on $\mathbb{R}$. (eg $F(x) = x$)

Extend to μ^*.

<u>Claim</u>: $\mu^*(A) = 0$ if A is countable

$A = \{x_1, x_2, \dots\}$

Choose $\varepsilon>0$, define ε_i.

Construct elementary sets $E_i = (x_i - \delta_i, x_i + \delta_i)$

where $|x - x_i| < \delta_i \Rightarrow |F(x) - F(x_i)| < \varepsilon_i$.

Now $\mu^*(A) \leq \sum_i \mu^*(E_i) = \sum_i \rho(E_i)$

$$= \sum_{i=1}^{\infty} \left(F(x_i+\delta) - F(x_i) + F(x_i) - F(x_i - \delta_i)\right)$$

$$< 2 \sum \varepsilon_i = \varepsilon.$$

So $\mu^*(A) = 0$.

Suppose $A = \mathbb{Q} \cap [0,1]$.