Lec 2/19

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Defn Let X, Y be top. sp and let $f: X \to Y$ be cts.

to say f is null-homotopic in Y means f is homotopic in Y to some constant map from X into Y.

If f is null-homotopic in Y, $f(X) \subseteq \underbrace{[[q]]}_{\text{path component of } q \text{ in } Y}$ for some $q \in X$.

and $\forall r \in Y$, f is homotopic in Y to the fn $\begin{array}{c} x \\ \Downarrow \\ X \xrightarrow[g_r]{} r \end{array}$ iff $r \in [[q]]$.

Pf: Suppose $H: f \simeq g_r$ in Y. then $\forall x \in X$, the map $\overset{[0,1]}{\underset{t}{\Downarrow}} \mapsto H(x,t) \in Y$ is a cts path from $f(x)$ to $g_r(x) = r$, etc. □

Notation: $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

eg Let X be a top. sp. and let $f: X \to \mathbb{C}^\times$ be cts.

Then f is null-homotopic in $\mathbb{C}^\times$ iff $f \simeq 1$ in $\mathbb{C}^\times$.

EG let X be a top. sp. and let $f: X \to \mathbb{C}^\times$ be cts.
Suppose there is a branch of $\log f$. Then f is null-homotopic.

Pf let g be the branch of $\log f$. $g: X \to \mathbb{C}$, g is cts, and $e^g = f$. Let $H(x,t) = (1-t)g(x)$. $H: g \simeq 0$ in $\mathbb{C}$

or $H: X \times [0,1] \to \mathbb{C}^\times$ by $H(x,t) = e^{(1-t)g(x)}$. $H: f \simeq 1$ in $\mathbb{C}^\times$.

Conversely, let X be a topological space. let $f: X \xrightarrow{\text{cts}} \mathbb{C}^\times$, and suppose $f \simeq 1$ in $\mathbb{C}^\times$.

Then there is a branch of $\log f$.

(to be proved soon)

Reminder Let X be a top. sp., let $f: X \to \mathbb{C}^\times$ be cts, and suppose g_1, g_2 are branches of $\log f$. Then $g_2 = g_1 + 2\pi i K$ where $K: X \to \mathbb{Z}$ is continuous (and so locally constant). Hence if X is connected, K is constant.

eg let $V = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. There is a branch of $\log z$ in V, it is $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z = g(z)$

$$\operatorname{Arg} z = \operatorname{Tan}^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$$

eg let V, g be as in the preceeding example.

let $a \in \{i, -1, -i\}$. Then there is a branch of $\log z$ in aV.

pf define g_a on aV by $g_a(z) = g\left(\frac{z}{a}\right) + b$ where $b = \begin{cases} i\frac{\pi}{2} & \text{if } a = i \\ i\pi & \text{if } a = -1 \\ -i\frac{\pi}{2} & \text{if } a = -i \end{cases}$

Remark in $V \cap iV$, g and g_i must agree, in $V \cap -iV$, g, g_{-i} agree. Define G on $V \cap iV \cap -iV$ by

square root of $4f$.

$g^2 = 4f$

$2g'g = 4f'$

$$G(z) = \begin{cases} g & V \\ g_i & iV \\ g_{-i} & -iV \end{cases}$$

$\int \left|\frac{g'}{g}\right|$

G is a branch of $\log z$ in $V \cap iV \cap -iV$.

$\left|\frac{f'}{\ }\right|$

Notation: $\forall z \in \mathbb{C}^\times$ $\operatorname{Log} z = \ln|z| + i\theta$ where $\theta \in (-\pi, \pi]$.

on $V \cap iV \cap -iV$, Log agrees w/ G.

the principal logarithm.

$g' = \frac{f'}{f}$

$g = \log f$

Remark Likewise, $\forall\ \theta \in \mathbb{R}$, there is a branch of $\log z$ in $e^{i\theta} V$.

Hence $\forall\ z_0 \in \mathbb{C}^\times$, there is a branch of $\log z$

in $D(z_0, |z_0|) \subseteq e^{i \arg z_0} V$.

$\log f =$

$\sqrt{f} = e^{\frac{1}{2}\log f}$

$\log \sqrt{f} = \frac{1}{2}\log f$

$2 \log \sqrt{f} = \log f$

Theorem Let $f : [0,1] \longrightarrow \mathbb{C}^\times$ be cts.

Then there is a branch of $\log f$.

$\log \sqrt{f}$

Pf Let $E = \{ a \in [0,1] :$ there is a branch of $\log(f|_{[0,a]}) \}$.

$0 \in E$ since we can take $\log f = \log(f(0))$.

Suppose $a \in E$, $a < 1$. Then $f(a)$ is nonzero so $\exists$ branch of $\log z$ in $\underbrace{D(f(a), |f(a)|)}_{D}$, call it h.

let g be a branch of $\log(f|_{[0,a]})$. $g(a)$ and $h(f(a))$ are both logarithms of a, so $\exists\ k \in \mathbb{Z}$ s.t. $h(f(a)) = g(a) + 2\pi i k$. replace h by $h - 2\pi i k$

so $h(f(a)) = g(a)$. Since f is cts, $\exists$ $b \in (a,1]$ s.t. $f([a,b]) \subseteq D$.

So we can take $\tilde{g}(x) = \begin{cases} g(x) & x \leq a \\ h(f(x)) & x > a \end{cases}$ on $[0,b]$. thus $b \in E$.

Suppose $\forall x < b$, $x \in E$ (for some $b \in (0,1]$). f is cts at b so $\exists$ an $a < b$ s.t. $f([a,b]) \subseteq D$. then let $\tilde{g} = \begin{cases} g & x < a \\ h \circ f & x \geq a \end{cases}$ so $b \in E$

thus $E = [0,1]$ so $\exists$ a $\log$ of f in $\mathbb{C}^\times$. $\square$