Lec 10/23

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<u>Quiz Prep</u>:

<u>Ex</u>: $R = \mathbb{Z}[x]$, $I = \{ f(x) : 6 \mid f(0) \}$

<u>T/F</u> I is an ideal: yes $I = \mathrm{Ker}(\varphi)$ where $\varphi : R \longrightarrow R/I$

$I = (6, x)$

Since all $f(x) = 6b_0 + x(a_1 + a_2 x + \cdots + a_n x^{n-1})$

so $I \subset 6 \cdot \mathbb{Z} + x \cdot \mathbb{Z}[x] = (6, x)$

and $(6,x) \subset I$ since $6, x \in I$.

is I prime? $R/I \cong \mathbb{Z}/6\mathbb{Z} \leftarrow \underbrace{\text{not integral domain}}$ $\Rightarrow$ not prime.

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}/6\mathbb{Z}$$
$$f(x) \longmapsto f(0) \bmod 6$$

$J = \{ f(x) : f(0) \text{ is NOT divisible by } 3 \}$

not a subgroup. $1 \in J$ so if it were an ideal it would be $\mathbb{Z}[x]$.

$f \cdot g \in J$ if $f, g \in J$.

$I \cap R^\times \neq \emptyset \implies I = R$ if $I \subset R$ is an ideal.

Prime & Maximal Ideals

$I \subsetneq R$ (proper ideal) if R/I is integral domain (field)

I is prime (maximal)

Maximal: equiv $I \subset J \subset R \Rightarrow J = I$ or R.

Prime: equiv $ab \in I \Rightarrow a \in I$ or $b \in I$.

Prop: Maximal ideals exist.

Let R be a commutative ring ($1_R \in R$, $1_R \neq 0_R$).

Let $J \subsetneq R$ be a proper ideal. then $\exists M \subsetneq R$ a maximal ideal s.t. $J \subset M$.

Zorn's Lemma:

set w/ relation $\leq$ s.t. $i \leq i$

$(\forall i, j, k \in \mathbb{I})$ $i \leq j$ & $j \leq k \Rightarrow i \leq k$

$i \leq j$ & $j \leq i \Rightarrow i = j$

Set up: We have a partially ordered set $\emptyset \neq \mathbb{I}$

Every chain has an upper bound $\longrightarrow$ **Hypotheses**: $\forall\, i_0 \leq i_1 \leq i_2 \leq \cdots$ in $\mathbb{I}$ there is $j \in \mathbb{I}$ s.t. $j \geq i_\ell$ $\forall \ell$.

the set has a maximal element $\longrightarrow$ **Conclusion**: $\exists\, j \in \mathbb{I}$ s.t. $j \not\leq k$ for any $k \in \mathbb{I} \setminus \{j\}$.

Proof of prop. $\mathbb{I}$ = set of all proper ideals containing J ($J \in \mathbb{I}$ so $\mathbb{I} \neq \emptyset$).

Partial order = inclusion ($I_1 \leq I_2 \Leftrightarrow I_1 \subset I_2$).

ZL Hypothesis: Given $I_0 \subset I_1 \subset I_2 \subset \cdots$ in $\mathbb{I}$.

Take $I = \bigcup_{n=0}^{\infty} I_n \supset J$. if I is not proper, $1_R \in I$ but then 1_R came from some I_n, a contradiction.

I is an ideal since $x, y \in I \Rightarrow x, y \in I_m$ for some large enough m and so $x \pm y$, $rx \in I_m \subset I$ $\forall r \in R$.

ZL Conclusion: $\exists\, M \in \mathbb{I}$ s.t. $M \supset I$ $\forall$ $I \in \mathbb{I}$

Later we'll prove the same result for a class of rings (Noetherian rings) without using Zorn's Lemma.

Lemma: (1) any two maximal ideals are coprime

(2) if $f: R_1 \to R_2$ is a ring hom (on commutative rings) and $P_2 \subsetneq R_2$ is a prime ideal,

then $P_1 = f^{-1}(P_2) \subsetneq R$ is also a prime ideal.

proper since $1_{R_1} \notin P_1$

<u>Pf</u>: (1) $M_1 + M_2$ is an ideal which contains both of them so $M_1+M_2 = R$.

(2) $R_1 \xrightarrow{f} R_2 \xrightarrow{\pi} R_2/P_2 \Rightarrow f \circ \pi = \bar{f}$ is a ring hom.

$P_1 = \ker(\bar{f})$ and so by 1st iso thm we get an injective map $R_1/P_1 \overset{i}{\hookrightarrow} R_2/P_2$. means injective

R_2/P_2 is an integral domain so R_1/P_1 is an integral domain too.

Subring of integral domain is integral domain. $\longrightarrow$ ($a,b \in R_1/P_1$ s.t. $ab = 0 \Rightarrow i(ab) = 0 = i(a)i(b) \Rightarrow$ one of $i(a)$ or $i(b) = 0$.

but i is injective so only 0 goes to 0).

(2) alternative pf: T.S. $ab \in P_1 \Longrightarrow a \in P_1$ or $b \in P_1$

$ab \in P_1 \Rightarrow f(ab) = f(a)f(b) \in P_2 \Rightarrow f(a) \in P_2$ or $f(b) \in P_2 \Rightarrow a \in P_1$ or $b \in P_1$.

<u>Optional</u>: Geometrically:

"Comm rings = (cts) functions on (top) spaces X"

"Ideals = functions vanishing on a subset $Y \subseteq X$"

$R = \mathbb{R}[x,y]$ = (poly) fns on (v.space) $\mathbb{R}^2$

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$f(x,y) \longleftrightarrow (a,b) \longmapsto f(a,b)$

$(y - x^2) \subsetneq R \longleftrightarrow X_1 = \{(a,b) \in \mathbb{R}^2 : b = a^2\}$

$(y) \subsetneq R \longleftrightarrow X_2 = \{(a,b) \in \mathbb{R}^2 : b = 0\}$

$X_1 \cap X_2 = \{(0,0)\}$

$(y, x^2) \longleftarrow X_1 \cup X_2$