Lec 10/5

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$\mathrm{Aut}_{gp}(D_{2n}) = ?$

$\varphi \in \mathrm{Aut}_{gp}(D_{2n}) \longrightarrow$ $\varphi(s)$ has order 2

$\varphi(r)$ has order n

$\Rightarrow$ $\varphi(r) = r^j$ where $(j, n) = 1$.

$\varphi(s) = sr^i$ for some i.

we show that $\forall j \in (\mathbb{Z}/n\mathbb{Z})^\times$, and $\forall i = 0, \dots, n-1$,

we get a gp iso

$$\begin{aligned} D_{2n} &\longrightarrow D_{2n} \\ s &\longmapsto sr^i \\ r &\longmapsto r^j \end{aligned}$$

easy pf: $(sr^i)(r^j)(sr^i) = r^{-i-j}r^i = r^{-j}$ ✓

$$\left|\mathrm{Aut}_{gp}(D_{2n})\right| = n \cdot \phi(n)$$

$\phi(n) \to \#\{(n, j) = 1\}$

$$|\mathbb{Z}/n\mathbb{Z}| \cdot |\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})|$$

<u>Lemma</u>: $\mathrm{Aut}_{gp}(D_{2n}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z})$

($H \circlearrowleft \mathrm{Aut}_{gp}(H) \rightsquigarrow H \rtimes \mathrm{Aut}_{gp}(H)$ makes sense)

<u>Pf</u> We have to prove $\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) \overset{\text{not } \trianglelefteq}{\leq} \mathrm{Aut}_{gp}(D_{2n}) \trianglerighteq \mathbb{Z}/n\mathbb{Z}$

(i)

these two should not intersect.

(ii) $gxg^{-1} = g(x)$ in $\mathrm{Aut}_{gp}(D_{2n})$

(iii) our gp hom is bijective.

$$
\begin{cases}
\mathrm{Aut}_{gp}(\mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathrm{Aut}_{gp}(D_{2n}) \\
j : \{1 \mapsto j\} \longmapsto \left\{ \begin{matrix} s \mapsto s \\ r \mapsto r^j \end{matrix} \right\} \\
\\
\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{Aut}_{gp}(D_{2n}) \\
1 \longmapsto \left\{ \begin{matrix} s \mapsto sr \\ r \mapsto r \end{matrix} \right\}
\end{cases}
$$

Ex: Check these things (i, ii, & iii).

$$\phi(n) = 2 \iff n = 3, 4, 6$$

$$\cos\left(\frac{2\pi}{m}\right) \in \mathbb{Q} \iff m = 1, 2, 3, 4, 6$$

(root systems of type A_2, B_2, G_2).

$$
\begin{array}{ccc}
G & \xrightarrow{f} & \mathrm{Aut}_{gp}(G) \\
\Psi & & \Psi \\
g & \longmapsto & \{x \longmapsto gxg^{-1}\}
\end{array}
$$

$\mathrm{Ker}(f) = Z(G)$

$\mathrm{Im}(f) =$ inner automorphisms.

Problem: Given a group homomorphism $\psi : G \longrightarrow H$ and a composition series $\Sigma : H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$.

Define $G_j = \psi^{-1}(H_j)$. Prove that $\Sigma' : G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_m = \mathrm{Ker}(\psi) \trianglerighteq \{e\}$ is a composition series of G and $G_i/G_{i+1} \longrightarrow H_i/H_{i+1}$ is an injective gp hom and if ψ is surjective then this is an isomorphism $\forall\ 0 < i < m-1$

and if ψ is surjective then $\overline{\text{this is an}}$ isomorphism $\forall\, 0 \le i \le m-1$.

<u>Proof</u> (a) Each $G_j \le G$ since $\forall f: \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \ge A$, $f^{-1}(A) \le \mathcal{G}_1$.

[to check: $\forall a,b \in f^{-1}(A)$, $a^{-1}b \in f^{-1}(A)$. i.e. $f(a^{-1}b) \in A$. i.e. $f(a)^{-1}f(b) \in A$. $\square$]

(b) Each $G_{j+1} \trianglelefteq G_j$ because $\forall f: \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \trianglerighteq A$, $f^{-1}(A) \trianglelefteq \mathcal{G}_1$.

[to show: $\begin{matrix} g \in \mathcal{G}_1 \\ x \in f^{-1}(A) \end{matrix} \Rightarrow gxg^{-1} \in f^{-1}(A)$, i.e. $f(gxg^{-1}) \in A$, i.e. $f(g)f(x)f(g)^{-1} \in A$ $\square$]

(c)
$$\begin{tikzcd} G_j = \psi^{-1}(H_j) \arrow[r, "\psi \text{ restricted to } G_j"] \arrow[dr, bend right, "\psi_j"'] & H_j \arrow[d, "\pi"] \\ & H_j/H_{j+1} \end{tikzcd}$$
(not necessarily surjective)

$$G_j/\mathrm{Ker}(\psi_j) \xrightarrow{\text{inj. gp hom}} H_j/H_{j+1}$$

to show: $\mathrm{Ker}(\psi_j) = G_{j+1}$

$$\begin{matrix} \psi \\ x \mapsto \psi(x) \mapsto \pi(\underbrace{\psi(x)) = e}) \end{matrix}$$
$$\Updownarrow$$
$$x \in G_{j+1} \iff \psi(x) \in H_{j+1} = \mathrm{Ker}(\pi) \qquad \square$$

<u>Ex</u>: let G be a group, $N \trianglelefteq G$. (not necessarily finite.)

G has a J–H series iff N & G/N have J–H series.

$\ell(G) = \ell(N) + \ell(G/N)$.

<u>Hint</u>: $G \xrightarrow{\pi} G/N$