Lec 1/20

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<u>Recall</u>: (Thm 10.3): If $\hat{\theta}$ is an unbiased estimator of θ and $Var(\hat{\theta}) \to 0$ as $n \to \infty$, then $\hat{\theta}$ is a consistent estimator for θ.

Considering the following example.

Ex: $X_1, \ldots, X_n$ iid sample from $f(x;\theta) = \frac{1}{2}(1+\theta x)$, $\underset{\text{w.r.t.}}{(x,\theta)} \in (-1,1)^2$

$E(\bar{X}) = \frac{\theta}{3}$, so $\hat{\theta} = 3\bar{X}$ is unbiased.

$$Var(\hat{\theta}) = 9\,Var(\bar{X}) = \frac{9}{n} Var(X_i) = \frac{9}{n} E(X_1^2) - \underbrace{E(X_1)^2}_{\frac{\theta}{3}^2} = \frac{9}{n}\left(\frac{1}{3} - \left(\frac{\theta}{3}\right)^2\right) \to 0 \text{ as } n \to \infty$$

$$\int_{-1}^{1} x^2 \frac{1}{2}(1-\theta x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,dx - \cancel{\frac{\theta}{2}\int_{-1}^{1} x^3\,dx} \to 0$$

$$= \frac{1}{2}\left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

So $\hat{\theta}$ is a consistent estimator for θ.

<u>Remark</u>: Thm 10.3 is a sufficient condition for consistency, but not a necessary one.

Consistent estimators need not be unbiased or even asymptotically unbiased.

see Exercise 10.41

§10.5 Sufficiency:

$\hat{\theta}$ is sufficient for θ if it uses all information in a sample relevant to estimation of θ.

If all knowledge about sample $(X_1, \dots, X_n)$ can be known by just knowing $\hat{\theta}$.

Ex: want to est. μ, σ^2 in $N(\mu, \sigma^2)$. consider $\bar{X} = \dfrac{X_1 + \cdots + X_n}{n}$.

intuitively, $\bar{X}$ is not sufficient (gives no info of variance, only gives center).

for estimating σ^2.

Def: the statistic $\hat{\Theta}$ (T) is a (T) sufficient estimator (statistic) of param θ iff for each value of $\hat{\theta}$ (i.e $\hat{\theta} = \hat{\theta}(x_1, \dots, x_n)$), the conditional pmf/pdf of RS $X_1, \dots, X_n$ given $\hat{\Theta}$ is "independent" of θ.

Note that conditional pmf/pdf $f(x_1, \dots, x_n \mid \hat{\theta}) = \dfrac{f(x_1, \dots, x_n, \hat{\theta})}{g(\hat{\theta})} = \dfrac{f(x_1, \dots, x_n)}{g(\hat{\theta})}$ b.c $\hat{\theta}$ is a func of $x_1, \dots, x_n$.

$g(\hat{\theta})$: pmf/pdf of $\hat{\theta}$

Example: Let $X_1, \dots, X_n$ be iid Poisson(λ). Show that $\sum_{i=1}^{n} X_i$ ($\hat{\lambda}$) is sufficient for λ.

sol: by def. want to show that cond. dist of $X_1, \dots, X_n$ given $\hat{\lambda}$ is indp. of λ.

note: $\hat{\lambda} \sim$ Poisson $(n\lambda)$. now $f(x_1, \dots, x_n \mid \hat{\lambda}) = \dfrac{f(x_1, \dots, x_n)}{g(\hat{\lambda})}$

$$= \frac{f(x_1) \cdot \cdots \cdot f(x_n)}{g(\hat{\lambda})} = \frac{\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}}{\frac{(n\lambda)^{\Sigma x_i} e^{-n\lambda}}{(\Sigma x_i)!}} = \frac{(\Sigma x_i)!}{\prod x_i!} \cdot \frac{1}{n^{\Sigma x_i}}$$

Doesn't dep. on λ. ✓

So $\hat{\lambda}$ is sufficient for λ.