M : R-module

Theorem: The intersection of any collection of submodules of M is a submodule.

Proof: If $\{N_\alpha : \alpha \in \Lambda\}$ is a collection of submodules of M and $u, v \in \bigcap_{\alpha \in \Lambda} N_\alpha$, $a \in R$, we have $u, v \in N_\alpha \Rightarrow u - v \in N_\alpha \Rightarrow u - v \in \bigcap_{\alpha \in \Lambda} N_\alpha$ and $au \in N_\alpha \Rightarrow au \in \bigcap_{\alpha \in \Lambda} N_\alpha$.

Definition: The sum of a family $\{N_\alpha\}$ of submodules is the set of finite sums $u_{\alpha_1} + \dots + u_{\alpha_k}$ for some $\alpha_1, \dots, \alpha_k \in \Lambda$ and $u_{\alpha_i} \in N_{\alpha_i}$.

Notation: $\sum_{\alpha \in \Lambda} N_\alpha$.

Theorem: $\sum_{\alpha \in \Lambda} N_\alpha$ is a submodule

Definition: Let $S \subset M$. The minimal submodule of M containing S is called the submodule generated by S.

This submodule is RS = finite linear combinations of S w/ coefficients from R.

If $M = RS$ for some S, we say S is a generating set of M (or a set of generators).
If S is finite, M is finitely generated.

If $M = Ru$ for some $u \in M$, then M is cyclic.

If $\{N_\alpha\}$ is a system of submodules then $R(\bigcup_\alpha N_\alpha) = \sum_\alpha N_\alpha$.

Factorization: Let N be a submodule of M. The factor module (or quotient module) M/N is the set $\{u + N : u \in M\}$ of cosets of N in M with

$a(u+N) = au + N$.

Example: if I is a left ideal of ring R, R/I is a left R-module.

Subexample: Let $R = \mathrm{Mat}_{n,n}(F)$ (field). Let $I = \left(\begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix}\middle| *\right)$. I is a left ideal.

So $R/I = \left(\begin{smallmatrix}\cdot \\ \vdots\end{smallmatrix}\right)$ (first columns of matrices) is an R-module.

$\cong$

F^n (an R-module by left multiplication of matrices).

Torsion Elements: Let M be an R-module. $u \in M$ is a torsion element if $\exists a \in R$, $a \neq 0$, s.t. $au = 0$.

Example: G additive-abelian group $\Rightarrow$ G is a $\mathbb{Z}$-module. $g \longmapsto ng$.

torsion elements are elements of finite order.

If u, v are torsion elements, $au = 0$, $bv = 0$ for $a, b \neq 0$,

then $ab(u+b) \overset{?}{=} 0$. If R is commutative, yes. If R is not, maybe not.

also, if $ab = 0$ then we don't have torsion.

Theorem If R is an integral domain, then torsion elements of M form the torsion submodule.

Notation: $\mathrm{Tor}(M)$.

Counterexamples: ① R has zero divisors. ② R is non-commutative

① $R = M = \mathbb{Z}_6$. Torsion elements are $\{0, 2, 3, 4\}$ – not a subgroup bc $3 - 2 = 1$.

② Next time.

If $M = \mathrm{Tor}(M)$, M is called a torsion module.

If $\mathrm{Tor}(M) = 0$, M is said to be torsion-free.

Theorem: If R is an integral domain, $M/\mathrm{Tor}(M)$ is torsion-free.

Proof: Let $a\bar{u} = \bar{0}$ for $u \in M$, $\bar{u} = u + \mathrm{Tor}(M)$, $a \neq 0$. then $\overline{au} = au + \mathrm{Tor}(M) = \bar{0}$

So $au \in \mathrm{Tor}(M)$. So $\exists\, b \neq 0$ s.t. $bau = 0$, and $ba \neq 0$ so $u \in \mathrm{Tor}(M)$ So $\bar{u} = \bar{0}$.

Annihilators: Let M be an R-module, $S \subseteq M$. The Annihilator of S in R is

$\{a \in R : aS = 0 \quad (as = 0 \ \forall s \in S)\} = \operatorname{Ann}(S)$.

Claim: $\operatorname{Ann}(S)$ is a left ideal of R.

Pf: trivial.

Claim: if N is a submodule of M, $\operatorname{Ann}(N)$ is a 2-sided ideal.

Pf Exercise.

In particular, $\operatorname{Ann}(M)$ is a 2-sided ideal in R.

then $R/\operatorname{Ann}(M)$ is a ring, and M is a module over this ring.

Def: Let $S \subseteq R$, the annihilator of S in M is $\operatorname{Ann}(S) = \{u \in M : Su = 0\}$.

Claim: if S is a right ideal in R, $\operatorname{Ann}(S)$ is a submodule of M.