Group Presentations are a thing.

The Free Group on a set S exists. It's called $F(S)$.

Let G be a gp, $\varphi : S \xrightarrow{\text{any set map}} G$.

Then $\exists !$ homomorphism $\Phi : F(S) \longrightarrow G$
s.t. the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, hook, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(S) \arrow[d, "\Phi"] \\ & G \end{tikzcd}$$

Take S to be a set of generators for G.
i.e. $S \subseteq G$, $G = \langle S \rangle$.

Def: A presentation of G is a pair $(\underset{\text{generates } G}{S}, \overset{\subseteq F(S)}{R})$
s.t. the normal closure of $\langle R \rangle \leq F(S)$
is the kernel of Φ (which extends $S \hookrightarrow G$).

Def the normal closure of H in G is smallest normal subgp of G containing H.

So $G \underset{\Phi}{\cong} F(S)/K$ where K = normal closure of $\langle R \rangle$.

Def G is finitely presented if $|S|, |R| < \infty$.

ex $D_n = \langle r, s \mid r^n, s^2, rsrs \rangle$

Group Actions:

$G \circlearrowright S \longleftrightarrow$ homomorphism $G \xrightarrow{T} \mathrm{Sym}(S)$.

The action is said to be effective (or faithful) if the homomorphism T is injective.

The kernel of the action is the kernel of T.

Ex the action of G by left mult is faithful.

Ex the action of G by right mult:

$g \cdot s = sg^{-1}$ $\quad (g_1 g_2) \cdot s = s(g_1 g_2)^{-1} = s g_2^{-1} g_1^{-1} = g_1 \cdot (g_2 \cdot s)$.

(can't use $g\cdot s = sg$!).

Ex $G \circlearrowright G$ by conjugation:

$$g \cdot s = gsg^{-1} = \underbrace{{}^{g}s}_{\text{jacobson's notation}}$$

The kernel of the action is $Z(G)$.

The action is effective if $Z(G) = 1$.

Ex The action of G on a space of left cosets of G:

let $H \leq G$, $G/H = \{xH \mid x \in G\}$

$$g \cdot (xH) = gxH.$$

In general, this is not effective.

The kernel of this action is

$$\{g \in G \mid gxH = xH \quad \forall x \in G\}$$

$$= \{g \in G \mid x^{-1}gx \in H \quad \text{for all } x \in G\}$$

$= \{ g \in G \mid g \in xHx^{-1} \text{ for all } x \in G \}$

$$= \underbrace{\bigcap_{x \in G} xHx^{-1}}.$$

this is
a normal subgp of G.

It's also a subgp of H.

<u>Claim</u> this is the largest normal subgp of G contained in H.

~~Pf~~ if $N \leq H$ is normal in ...