Lec 1/6

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L^p spaces

$\|f\|_p := \left(\int |f|^p\right)^{1/p}$ is a norm on

$L^p(X, M, \mu) := \{f : X \to \mathbb{C}$ mble s.t. $\|f\|_p < \infty\}$

for $p \in [1, \infty)$.

Remember ($p=1$):

let f^- and f^+ be neg & pos parts of f.

when $\int f^-$ and $\int f^+$ are finite, f is $\int$ble.

equivalently, $\underbrace{\int |f|}_{\|f\|_1} < \infty$.

More generally, for $E \in M$, we say f is $\int$ble on E

if $\int_E |f| < \infty$.

Prop^n (2.21) $L^1(X)$ is a vector space. The integral is a linear ftl on it.

Prop^n Let $f, g \in L^1(X)$. TFAE:

(i) $\int_E f = \int_E g \quad \forall\ E \in M$

(ii) $\int |f-g| = 0$

(iii) $f=g$ a.e.

pf: (2) $\Leftrightarrow$ (3) follows from

"if $f \in L^+$ then $\int f = 0$ iff $f=0$ a.e."

(Prop^n 2.16 in Folland)

(2) $\Rightarrow$ (1)

Assume $\int |f-g| = 0$. Then $\left| \int_E f - \int_E g \right| \leq \int \chi_E |f-g| \leq \int |f-g| = 0.$

(1) $\Rightarrow$ (3) (contrapositive)

Assume $f=g$ a.e. is false. Then

$E = \{ \mathrm{Re}(f-g)^+ > 0 \}$ has positive measure

(maybe it is Im or $^-$ but no loss of generality).

Then $\mathrm{Re}\left(\int_E f - \int_E g \right) = \int_E \mathrm{Re}(f-g)^+ > 0$

Since $\mathrm{Re}(f-g)^- = 0$ on E. $\square$

So $L^1(X)$ is actually the vector space of equivalence classes of functions ($f \sim g \Leftrightarrow f=g$ a.e.).

Back to L^p. We define $L^p(X)$ as before, but taking equivalence classes wrt $\sim$.

<u>Exercise</u> show $L^p(X)$ is a vector space.

$\|\cdot\|_p$ defines a norm on $L^p(X)$ if

(0) $\|f\|_p = 0 \Leftrightarrow f=0$ a.e.,

(1) $\|af\|_p = |a|\,\|f\|_p$,

(2) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. $\leftarrow$ Minkowski's Inequality

<u>Lemma (Hölder's Ineq.)</u>: suppose $p \in (1,\infty)$ and f,g are mble fns on X. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \quad \text{where } \frac{1}{p}+\frac{1}{q}=1.$$

<u>Proof</u> wlog suppose $\|f\|_p = \|g\|_q = 1$ (if either is 0, or ∞ the ineq. is trivial).

We use a calculus lemma:

$$\left\{ \begin{array}{l} \text{if } a,b \geq 0 \text{ and } 0 \leq \lambda \leq 1 \text{ then} \\ a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b \end{array} \right\}$$

<u>Proof</u> Exercise (it's in Folland).

We set $a = |f|^p$, $b = |g|^q$. $\lambda = \frac{1}{p}$.

$$(|f|^p)^{1/p} \cdot (|g|^q)^{1/q} \leq \frac{1}{p}|f|^p + \frac{1}{q}|g|^q$$

$$|fg| \leq \frac{1}{p}|f|^p + \frac{1}{q}|g|^q$$

$$\leadsto \quad \|fg\|_1 \leq \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

$$= \|f\|_p \|g\|_q$$

$\square$

<u>Minkowski's Inequality</u>: for $1 \leq p < \infty$ and $f, g \in L^p$,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

<u>Pf</u> Trivial for $p=1$. So assume $p>1$.

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leq (|f|+|g|)\,|f+g|^{p-1}$$

$$= |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$$

Apply Hölder:

$$\int |f+g|^p \leq (\|f\|_p + \|g\|_p)\, \| |f+g|^{p-1} \|_q$$

to be continued...