Lec 9/28

Wednesday, September 28, 2016 9:08 AM

Problem 2 take-home

$\emptyset \neq S \subseteq P$

$S + P = \{c\} + P$ for some $c = \inf S \geq 0$

$B = S + P$

$A = \mathbb{R} \setminus B$

(A, B) a Dedekind cut $A =$ set of lower bounds for S.

$c = \inf S$

NIP = nested intervals property.

Given a sequence of closed finite intervals $\{I_n = [a_n, b_n] : n \in \mathbb{N}^+\}$

Satisfying: (1) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

(2) $\lim_{n \to \infty} (\text{length}(I_n)) = 0 \iff \forall \varepsilon > 0, \exists N$ s.t. $b_n - a_n < \varepsilon \ \forall n > N$

Then (a) $\bigcap_{n=1}^{\infty} I_n = \{c\}$

(b) for any $\delta > 0$ we can find N s.t. $I_n \subseteq (c - \delta, c + \delta) \ \forall n > N$

Theorem: CA $\iff$ NIP & $\mathbb{R}$ has no infinitesimals.

Proof: ☆ CA $\Rightarrow$ NIP & $\mathbb{R}$ has no infinitesimals:

$\Updownarrow$

LUBP $\Rightarrow$

(*) $a_1 \leq a_2 \leq \dots$

(**) $b_1 \geq b_2 \geq \dots$

] because of (i)

Also, for any m, n $a_m < b_n$

Let $p = \max(m, n)$ $a_m \leq a_p < b_p \leq b_n$

by (*) by (**)

Let $p = \max(m,n)$ $\quad a_m \underset{\uparrow}{\leq} a_p < b_p \underset{\uparrow}{\leq} b_n$

by $(*)$ $\qquad$ by $(**)$

let $L = \{a_n : n \geq 1\}$ = set of left endpoints.

L is nonempty and bounded above by any b_n

By LUBP, $c = \sup L$ exists.

for any n, $\quad a_n \leq c$ $\quad$ (since c is an upper bound)

$c \leq b_n$ $\quad$ (since c is the least upperbound and b_n is an upper bound)

So $c \in I_n \ \forall n$

So $c \in \bigcap_{n=1}^{\infty} I_n$

if there were any other element, say d, in $\bigcap_{n=1}^{\infty} I_n$,

Then let $\varepsilon = |d-c|$ then using (ii), chose n so that

$b_n - a_n < \varepsilon = |d-c|$

$d, c \in I_n \Rightarrow |d-c| \leq b_n - a_n < \varepsilon$, which is a contradiction.

Already shown $CA \Rightarrow \mathbb{R}$ has no infinitesimals. ■

☆ NIP & $\mathbb{R}$ has no infinitesimals $\Rightarrow$ CA:

Let (A,B) be a dedekind cut of $\mathbb{R}$.

Pick $a_0 \in A$, $b_0 \in B$. Recursively define $a_n \in A$, $b_n \in B$

Having defined a_n, b_n, let $c_n = \frac{a_n + b_n}{2}$

(1) if $c_n \in A$, $a_{n+1} = c_n$ and $b_{n+1} = b_n$

(2) if $c_n \in B$, $a_{n+1} = a_n$ and $b_{n+1} = c_n$

$\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested intervals.

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

$b_n - a_n = \frac{b_0 - a_0}{2^n} < \frac{b_0 - a_0}{n} \longrightarrow 0$ as $n \to \infty$ because $\mathbb{R}$ has no infinitesimals.

$b_n - a_n < \varepsilon \Leftarrow \frac{b_0 - a_0}{n} < \varepsilon$, so choose $\frac{1}{n} < \frac{\varepsilon}{b_0 - a_0}$

(A,B) Dedekind cut, $\{c\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$ where $a_n \in A$ $b_n \in B$

If $c \in A$ and c is not the maximal element of A, then we can find $\delta > 0$ s.t. $c + \delta \in A \Rightarrow$

$$[a_n, b_n] = I_n \subseteq (c - \delta, c + \delta) \subseteq A$$

for $n > N$

but $b_n \in B$ so this is a contradiction.

if $c \in B$ and c is not the minimal element of B, then

$$[a_n, b_n] = I_n \subseteq (c - \delta, c + \delta) \subseteq B,$$

for $n > N$

again, a contradiction.

so if $c \in A$ then c is the maximal element

" $c \in B$ " " minimal "

so (A,B) has a cut point, namely c.

Intermediate Value Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous and $\min(f(a), f(b)) < d < \max(f(a), f(b))$, then there is a $c \in (a,b)$ s.t. $f(c) = d$.

Proof: without loss of generality, we may assume $d = 0$, $f(a) < 0$, $f(b) > 0$ by replacing f by $f - d$ or $d - f$.

Notation: If f is a function, $S \subseteq \mathbb{R}$, we define

$$f(S) = \{f(x) : x \in S \cap \mathrm{dom}(f)\}$$

$f(S)$ is the image of S under f.

Lemma: If f is continuous at a and $f(a) \neq 0$, we can find $\delta > 0$ so that $f((a-\delta, a+\delta)) \subseteq (0, \infty)$ if $f(a) > 0$ and $f((a-\delta, a+\delta)) \subseteq (-\infty, 0)$ if $f(a) < 0$.

Proof: Pick $\delta > 0$ s.t. $|x - a| < \delta$ & $x \in \mathrm{dom}(f) \Rightarrow |f(x) - f(a)| < \frac{|f(a)|}{2}$

Proof: pick $\delta > 0$ s.t. $|x-a| < \delta$ & $x \in \mathrm{dom}(f) \Rightarrow \left|f(x) - f(a)\right| < \frac{|f(a)|}{2}$

$$\Updownarrow$$

$$f(a) - \frac{|f(a)|}{2} < f(x) < f(a) + \frac{|f(a)|}{2}$$

so therefore, $f(x)$ has same sign as $f(a)$ $\blacksquare$

Recursively define a_n, b_n as follows with $f(a_n) < 0$ and $f(b_n) > 0$.

$a_0 = a$, $b_0 = b$. Having defined a_n and b_n, let $c_n = \frac{b_n + a_n}{2}$.

if $f(c_n) = 0$ then we're done. Take $c = c_n$ and abort.

if $f(c_n) < 0$, take $a_{n+1} = c_n$, $b_{n+1} = b_n$

if $f(c_n) > 0$, take $a_{n+1} = a_n$, $b_{n+1} = c_n$

$$b_n - a_n = \frac{b_0 - a_0}{2^n} \longrightarrow 0.$$

and $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested intervals.

by NIP, $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{c\}$.

Claim: $f(c) = 0$. If $f(c) \neq 0$, we can find $\delta > 0$ s.t.

$$f([a_n, b_n]) \subseteq f((c-\delta, c+\delta)) \subseteq (-\infty, 0) \text{ or } (0, \infty)$$

for $n > N$

This is a contradiction: $f(a_n) < 0$ and $f(b_n) > 0$.

So $C = 0$. $\blacksquare$