A C^k surface in $\mathbb{R}^3$ is a set $M \subseteq \mathbb{R}^3$ s.t. for each $p \in M$ there is a relatively open neighborhood V of p in M and a C^k coordinate patch $x : U \overset{open}{\subseteq} \mathbb{R}^2 \xrightarrow{onto} V \overset{open}{\subseteq} M$.

Klein Bottle can't be embedded in $\mathbb{R}^3$

$\underline{\text{Generalization}}$ C^k $\underbrace{\text{m-dimensional manifold}}_{\text{m-fold}}$ in $\mathbb{R}^n$

Let M be a C^k m-fold in $\mathbb{R}^n$, let $p \in M$, let $x : U \overset{open}{\subseteq} \mathbb{R}^m \longrightarrow \overset{p}{\overset{\cap}{V}} \overset{open}{\subseteq} M$ be a C^k coordinate patch. Let $u_0 \in U$ with $x(u_0) = p$

Then $T_pM := \{v \in \mathbb{R}^n : \substack{v \text{ tangent to} \\ M \text{ at } p}\} = \text{Span}\{x'(u_0)e_i\}_{i=1}^m =: \text{Range}(x'(u_0)) = \{\alpha'(0) : \alpha \text{ is a } C^1 \text{ curve in } M \text{ and } \alpha(0) = p\}$.

(to see the last eq., consider $\beta = x^{-1} \circ \alpha$ which is a C^1 curve in U, and apply chain rule to $\alpha = x \circ \beta$.

The tangent bundle for M is $TM = \{(p, v) : p \in M, v \in T_pM\} \subseteq \mathbb{R}^{2n}$.

TM is a C^{k-1} 2m-fold in $\mathbb{R}^{2n}$

The first fundamental form, or metric tensor, on M is the fn $p \longmapsto I_p$ where for each p, I_p is the fn on $T_pM \times T_pM$ defined by $I_p(X, Y) = \langle X, Y \rangle$.

I_p is an inner product on T_pM

If $x : U \overset{open}{\subseteq} \mathbb{R}^m \longrightarrow V \overset{open}{\subseteq} M$ is a C^1 patch in M, then we write x_j for $\frac{\partial x}{\partial u^j}$ and $g_{ij} = \langle x_i, x_j \rangle$. for each $u \in U$, the vectors $x_1(u), \dots, x_m(u)$ form a basis for $T_{x(u)}M$

If $X, Y \in T_{x(u_0)}M$ then $X = \sum_{i=1}^m X^i x_i(u_0)$ and $Y = \sum_{i=1}^m Y^i x_i(u_0)$

So $\langle X, Y \rangle = \sum_i \sum_j X^i Y^j \langle x_i(u_0), x_j(u_0) \rangle = \sum_{i,j} g_{i,j}(u_0) X^i Y^j$

Thus $(g_{i,j}(u_0))$ is the matrix of the inner product $I_{x(u_0)}$ on T_pM with respect to the basis $\{x_1(u_0), \ldots, x_m(u_0)\}$.

Hence $(g_{ij}(u_0))$ is symmetric & strictly positive definite.

In particular, it has an inverse.

We write g for $\det(g_{ij}) > 0$ and g^{kl} for the kl-th entry of the inverse of (g_{ij}).

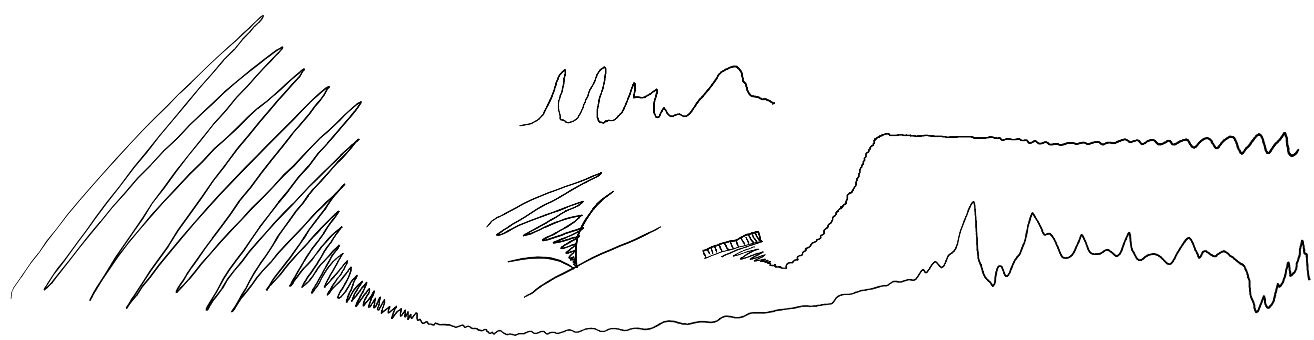
Now let $m=2$ and $n=3$

Lemma 3.4 (a) $g = |x_1 \times x_2|^2$.

(b) $(g^{kl}) = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$

(c) $\sum_k g_{ik} g^{jk} = \delta_i^j$

Pf



(a) $|x_1 \times x_2|^2 = (|x_1||x_2|\sin\theta)^2 = |x_1|^2|x_2|^2(1-\cos^2\theta) = |x_1|^2|x_2|^2 - (|x_1||x_2|\cos\theta)^2$

$= |x_1|^2|x_2|^2 - \langle x_1, x_2\rangle\langle x_2, x_1\rangle = g_{11}g_{22} - g_{12}g_{21} = g.$

(b) and (c) are obvious.