Lec 11/5

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$Q \subsetneq R$ primary ideal if $ab \in Q$, $a \notin Q \Longrightarrow b^n \in Q$ for some $n \geq 1$.

$I \subsetneq R$ irreducible ideal if $I = \underbrace{I_1 \cap I_2}_{\text{ideals in } R} \Longrightarrow I = I_1$ or $I = I_2$.

$\mathrm{Rad}(I) := \{a \in R : a^n \in I \text{ for some } n \geq 1\}$

(1) Q primary $\Longrightarrow \mathrm{Rad}(Q)$ is prime.

Pf: to show: $\underbrace{ab \in \mathrm{Rad}(Q),\ a \notin \mathrm{Rad}(Q)} \Longrightarrow b \in \mathrm{Rad}(Q)$

meaning: $a^n b^n \in Q$ for some n, but $a^m \notin Q$ for any m.

So $a^n \notin Q \Rightarrow (b^n)^\ell \in Q$ for some ℓ, so $b \in \mathrm{Rad}(Q)$.

Another proof: $R \xrightarrow{\pi} R/Q \supset$ zero divisor = nilpotent

$\mathrm{Rad}(Q) = \pi^{-1}(\text{nilpotents in } R/Q)$, so $R/\mathrm{Rad}(Q) \cong (R/Q)/\text{its nilradical}$ which has no zero divisors

(2) $\mathrm{Rad}(P^n) = P$ for any Prime $P \subsetneq R$, $n \geq 1$.

$\left(\begin{array}{l} I \subset P \Longrightarrow \mathrm{Rad}(I) \subset P \\ \text{since } a \in \mathrm{Rad}(I) \Rightarrow a^\ell \in I \subset P \Rightarrow a \in P \end{array} \right)$

$\rightarrow$ since $P^n \subset P$, $\mathrm{Rad}(P^n) \subset P$.

but $P \subset \mathrm{Rad}(P^n)$ obviously.

(3) Let $Q_1, \ldots, Q_\ell$ be primary ideals in R s.t. $\mathrm{Rad}(Q_1) = \cdots = \mathrm{Rad}(Q_\ell) = P$,

then $Q_1 \cap \cdots \cap Q_\ell = Q$ is primary & $\mathrm{Rad}(Q) = P$.

[eg in $\mathbb{Z}$: $(p^{k_1}) \cap \cdots \cap (p^{k_\ell}) = (p^{\min(k_1, \ldots, k_\ell)})$]

[eg in $K[x,y]$: (x^2, xy, y^n) are all primary & have same radical: $\mathrm{Rad}((x^2,xy,y^n)) = (x,y)$] ($n \geq 2$)

Pf $Q = Q_1 \cap \cdots \cap Q_\ell$ is primary: $ab \in Q,\ a \notin Q \Longrightarrow b^n \in Q$ for some $n \geq 1$ (to prove)

i.e. $ab \in Q_j\ \forall j$, $a \notin Q_k$ for some k. So $b^N \in Q_k$ for some $N \geq 1$.

So $b \in \mathrm{Rad}(Q_k) = \mathrm{Rad}(Q_j)\ \forall j$. So $b^{N_j} \in Q_j\ \forall j$.

So $b^{\max(N_1,\ldots,N_\ell)} \in Q$.

Lemma (2.5) $\mathrm{Rad}(I_1 \cap I_2) = \mathrm{Rad}(I_1) \cap \mathrm{Rad}(I_2)$ so we are done.

Theorem If R is Noetherian and $I \subsetneq R$ is a proper ideal then $\exists$ primary ideals $Q_1, \ldots, Q_\ell \subsetneq R$ s.t.

(1) $I = Q_1 \cap \cdots \cap Q_\ell$

(2) All $\{\mathrm{Rad}(Q_i) = P_i\}_{1 \leq i \leq \ell}$ are distinct

(3) No term is redundant: $Q_i \not\supset \bigcap_{\substack{1 \leq j \leq \ell \\ j \neq i}} Q_j \quad (\forall i \in \{1, \ldots, \ell\})$

(4) **Cor** If $I \subsetneq R$ is a proper ideal & $Q_1, \ldots, Q_\ell$; $P_1, \ldots, P_\ell$ are as in the thm, define $\mathrm{Min}(I) = \{\underset{\text{Prime}}{P \subsetneq R} : I \subset P \text{ and } P \text{ is minimal}\}$ → $I \subset P' \subset P \Longrightarrow P' = P$ (P' Prime)

then $\underbrace{\mathrm{Min}(I)}_{\text{this set depends only on } I} \subset \{P_1, \ldots, P_\ell\}$ → depends on theorem.

Pf: $P \in \mathrm{Min}(I) \Longrightarrow P \supset Q_1 \cap \cdots \cap Q_\ell \Longrightarrow$ one $Q_j \subset P$.

$\underset{(2)}{\Longrightarrow} \mathrm{Rad}(Q_j) \subset P \Longrightarrow P_j = P$.

$\overset{\shortparallel}{P_j}$ also prime

Corollary^2: $|\mathrm{Min}(I)| < \infty$

(5) $\{P_1, \dots, P_\ell\}$ are uniquely determined by I.

Let $\{P_1, \dots, P_k\} = \mathrm{Min}(I)$

Then

(6) $\{Q_1, \dots, Q_k\}$ is uniquely determined by I.

eg $R = \overset{\text{field}}{K}[x,y]$. $I = (x^2, xy) \subsetneq R$ is a non-primary ideal

Since in R/I, y is a zero divisor but <u>not</u> nilpotent.

$I \subset P \implies x^2 \in P \Rightarrow x \in P$

so $(x) \supset (x^2, xy)$, $\mathrm{Min}(I) = \{(x)\}$

and (x) is prime since $K[x,y]/(x) \cong K[y]$ integral domain

take $Q_1 = (x)$.

$I = (x) \cap \underbrace{(x^2, xy, y^n)}$ for any $n \geq 1$

J_n = primary . $\mathrm{Rad}(J_n) = (x, y)$

$\{P_1, \dots, P_\ell\}$ depend only on I.

"Associated Primes" $\mathrm{Assoc}(I)$

$\{P_1, \dots, P_k\}$ are minimal primes containing I, $\mathrm{Min}(I) \subset \mathrm{Assoc}(I)$

$\{Q_1, \dots, Q_k\}$ are also uniquely determined by I

"Primary Components of I"

$P_{k+1}, \dots, P_\ell$ are "Embedded Components of I"