Rec 10/1

Tuesday, October 1, 2019 10:22

- Prove that L^1 is complete

- Prove that L^1 is a Banach Space
  (i.e. a complete normed linear space).

$\|f\|_1 = \int_X |f| \, d\mu$ is a norm on $\mathcal{L}^1$ = eq. classes.

$\{f_n\}$ cauchy if $\forall \varepsilon > 0$, $\|f_n - f_m\| < \varepsilon$ if $n, m > N(\varepsilon)$.

convergence: $\|f_n - f\| \longrightarrow 0$ as $n \longrightarrow \infty$.

<u>1st Step:</u> find f

<u>Easier:</u> look at a series in L^+.

$\{x_n\} \subset L^+$, $\sum_1^\infty x_n = S \in L^+$, $S_n = \sum_1^n x_k \nearrow S$.

<u>Defn</u> A series $\{x_n\}$ converges assolutely if $\sum_1^\infty \|x_n\| = B < \infty$.

<u>Claim:</u> If series $\{x_n\}$ converges assolutely, then it converges to a limit $\sum_1^\infty x_n$ in L_1.

<u>Hint:</u> Look at $\sum |x_n|$

$$\underset{\text{absolute value}}{\left\| \sum |x_n| \right\|} \le \sum \|x_n\| = B < \infty$$

So by MCT, $\sum_1^\infty |x_n| \in L^+ \cap L^1$.

$$\left| \sum_1^N x_n \right| \le \sum_1^N |x_n| \le \sum_1^\infty |x_n| \in L^1,$$

so DCT says $\sum_1^\infty x_n \in L^1$.

Showing that series converg leads to convergence of sequences

$$\{f_j\} \rightsquigarrow \quad x_1 = f_1,\ x_2 = f_2 - f_1,\ \ x_3 = f_3 - f_2,\ \ldots,\ x_k = f_k - f_{k-1}\ \ \ldots$$

$$\sum_1^n x_j = f_n .$$

Problem: $\{f_j\}$ may be Cauchy but the series $\sum x_j$ may not be convergent.

Solution: go to a subsequence of f_{n_k}.

(form a "rapidly converging" sequence).

Let N_k be s.t. $m, n > N_k \Rightarrow \|f_m - f_n\| < \frac{1}{2^k}$

choose $n_1 > N_1$, $n_2 > N_2$, $n_3 > N_3$, etc.

then $\|f_{n_{j+1}} - f_{n_j}\| < \frac{1}{2^j}$

Let $x_1 = f_{n_1}$, $x_2 = f_{n_2} - f_{n_1}$, etc.

Then $\sum_1^\infty x_j$ converges $\overset{\text{in } L'}{\uparrow}$. so $f_{n_j} \longrightarrow \sum_1^\infty x_n$ in L'.

So $\underset{\text{cauchy}}{f_n} \longrightarrow \lim_j f_{n_j}$ as well, in L'.

$\{f_n\}$ Cauchy in L' $\rightsquigarrow$ $\{f_n\}$ cauchy in measure.

$\rightsquigarrow$ $\{f_n\}$ has a limit f in measure, and $f_{n_k} \longrightarrow f$ a.e.

this gives a candidate for $\lim f_n$ in L'.

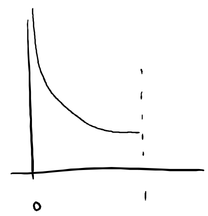
is $f \in L'$?

does $f_n \to f$ in L'?

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(x) = \sum_1^\infty \frac{1}{2^n} f(x - r_n) \qquad \text{where } \{r_n\} = \mathbb{Q}.$$

g is finite a.e. is $g(x) = \infty$ anywhere?



Can choose $\{r_n\}$ to make $g(x) = \infty$ for any x.

Can do 2 values, or countably many.

Could you make it finite everywhere by a different algorithm?