Lec 10/18

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(X, ρ) metric space

<u>Thm</u> TFAE:

① X is compact

② X is sequentially compact

③ X is complete & <u>totally bounded</u>.

$\forall \varepsilon, \exists x_1, \dots, x_n \in X$ s.t. $X \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$

<u>Corollary</u>. Suppose (X, ρ) is complete & $A \subset X$.

$\bar{A}$ is compact iff A is totally bdd.

A totally bdd $\Rightarrow$ $\bar{A}$ totally bdd quiz question.

<u>Def</u> (X, τ) <u>locally cpt</u> if $\forall x \in X$, $\exists$ open $\mathcal{U} \ni x$ s.t. $\bar{\mathcal{U}}$ is cpt.

<u>LCH</u>: locally cpt Hausdorff.

<u>Exercises</u>: X is LCH

① $\forall$ open $\mathcal{U} \subset X$, $x \in \mathcal{U}$, $\exists$ open $V \subset X$ s.t. $x \in V \subset \bar{V} \subset \mathcal{U}$ and $\bar{V}$ cpt.

② $\underset{\text{cpt}}{K} \subset \underset{\text{open}}{\mathcal{U}} \subset X \Rightarrow \exists \underset{\text{open}}{V} \supset K$ s.t. $K \subset V \subset \underset{\text{open}}{\bar{V}} \subset \mathcal{U}$.

③ (Urysohn) if $K \subset U \subset X$ as above, $\exists$ cts $f : X \to [0,1]$

s.t. $f|_K = 1$ and $f|_{U^c} = 0$.

- actually, $f = 0$ outside a cpt set containing K.

($f \in C_c(X)$ s.t. $f|_K = 1$).

④ (Tietze) if $K \subset X$ is cpt & $f \in C(K)$, $\exists F \in C_c(X)$ s.t. $F|_K = f$.

Def Suppose X is LCH. A fn $f \in C(X)$ vanishes at ∞

if $\forall \varepsilon > 0$, $\{|f| \geq \varepsilon\}$ is cpt.

$C_0(X) = \{$cts $f : X \longrightarrow \mathbb{C}$ which vanish at $\infty\}$

$C_b(X) = \{$cts bdd $f : X \longrightarrow \mathbb{C}\}$

$C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$.

The uniform/∞ norm on $C_b(X)$ is

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$$

- check it's a norm.

<u>Prop</u>: Suppose X is LCH.

① $C_b(X)$ is complete wrt $\|\cdot\|_\infty$.

② $C_0(X) \subset C_b(X)$ is a closed linear subspace.

③ $\overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$.

<u>Pf</u>

① if (f_n) is uniformly cauchy,

- $(f_n(x))$ is cauchy in $\mathbb{C}$, let $f(x) := \lim f_n(x)$ cts.
  $\downarrow$
  since $f_n \to f$ unif
- $(\|f_n\|) \subset [0,\infty)$ bdd by Δ-ineq.
  $|f_m - f_n| < \varepsilon$
  $\downarrow$
  $|f - f_n| \leq \varepsilon$
- $\sup|f(x)| < \sup\|f_n\| < \infty$. ✓

② Suppose $(f_n) \subset C_0(X)$ s.t. $f_n \to f$ in $C_b(X)$.

Let $\varepsilon > 0$. Pick N s.t. $n \geq N \Rightarrow \|f - f_n\| < \frac{\varepsilon}{2}$.

$K := \{|f_N| \geq \frac{\varepsilon}{2}\}$ is cpt.

Then $\underset{\substack{\text{closed} \\ \Downarrow \\ \text{cpt}}}{\{|f| \geq \varepsilon\}} \subset \{|f_N| \geq \frac{\varepsilon}{2}\}$ ✓

③ $f \in C_0(X)$. Then $K = \{|f| \geq \varepsilon\}$ is cpt.

by LCH Urysohn's Lemma, $\exists$ cts $g : X \longrightarrow [0,1]$

s.t. $g|_K = 1$ & $g \in C_c(X)$.

then $gf \in C_c(X)$ & $\|f-gf\|_\infty < \varepsilon$.