Lec 9/5

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$$\begin{aligned} \alpha_{11} x_1 + \cdots + \alpha_{1n} x_n &= 0 \\ \vdots \qquad\qquad & \vdots \\ \alpha_{m1} x_1 + \cdots + \alpha_{mn} x_n &= 0 \end{aligned} \qquad (\star)$$

m eqns, n unknowns.

$$A = (\alpha_{ij}) = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \qquad c_i \in \mathbb{F}^m$$

$$\text{rank } A = \dim S(c_1, \ldots, c_n)$$

$(\star\star)$ $\quad X = (x_1, \ldots, x_n)$ s.t.

$$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = 0$$

select a basis $\overbrace{c_1, \ldots, c_r}^{\text{maybe rearrange}}$ of $S(c_1, \ldots, c_n)$

each of the $n-r$ eqns give a soln to $x_1 c_1 + \cdots + x_n c_n = 0$

$$\left\{ \begin{aligned} \lambda_1^{(r+1)} c_1 + \cdots + \lambda_r^{(r+1)} c_r - c_{r+1} &= 0 \\ \vdots \qquad\qquad & \\ \lambda_1^{(n)} c_1 + \cdots + \lambda_r^{(n)} c_r - c_n &= 0 \end{aligned} \right.$$

all of type vectors

They are lin. indp. bc of the -1s.

$$\left\{ \begin{aligned} V_{r+1} &= (\lambda_1^{(r+1)}, \lambda_2^{(r+1)}, \ldots, \lambda_r^{(r+1)}, -1, 0, \ldots, 0) \\ V_{r+2} &= (\lambda_1^{(r+2)}, \lambda_2^{(r+2)}, \ldots, \lambda_r^{(r+2)}, 0, -1, 0, \ldots, 0) \\ &\vdots \\ V_n &= (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_r^{(n)}, 0, \ldots, 0, -1) \end{aligned} \right.$$

(the Null space)

the solutions of ($\circledast$) form a subspace $\mathcal{N}$ of $\mathbb{F}^n$

Let $X = (\xi_1, \ldots, \xi_n)$ be an arbitrary solution.

$$\xi_{r+1} v_{r+1} + \xi_{r+2} v_{r+2} + \cdots + \xi_n v_n - X = (\mu_1, \mu_2, \ldots, \mu_r, 0, \ldots, 0) \in \mathcal{N}$$

So $\mu_1 c_1 + \cdots + \mu_r c_r = 0 \Rightarrow \mu_i = 0 \quad \forall i$

So $X = -\xi_{r+1} v_{r+1} - \xi_{r+2} v_{r+2} - \cdots - \xi_n v_n$

showing $\{v_{r+1}, \ldots, v_n\}$ generates $\mathcal{N}$ and so

$\dim \mathcal{N} = n - \text{rank } A$

(column) $\text{rank } A = \dim$ (space of columns)

(row) $\text{rank } A = \dim$ (space of rows)

Theorem: for any $m \times n$ matrix A over a field F, the row rank of A = rank A

Proof: $\text{rank } A = n - \dim \mathcal{N}$. Taking A to REF to get

obtaining sys. of. eqns

$$A' = \begin{pmatrix} 1 & * & * & \cdots \\ 0 & \cdots 1 & * & \cdots \\ 0 & 0 & \cdots 1 & \cdots \\ 0 & 0 & 0 \cdots 1 & \cdots \\ \hline 0 & 0 & 0 & \cdots 0 \\ \vdots & & & \\ 0 & 0 & 0 & \cdots 0 \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} p = \text{row rank } A'$$

$$x_1 + \alpha'_{12} x_2 + \cdots + \alpha'_{1n} x_n = 0$$
$$x_2 + \cdots + \alpha'_{2n} x_n = 0$$
$$\ddots$$

$\text{rank } A' = n - \dim \mathcal{N}' = n - \dim \mathcal{N} = \text{rank } A$

rowrank A = rowrank A' (by theorem of last week)

Now column vectors of A' are in $\mathbb{F}^{\text{row rank } A'}$, so

$\ldots$

$$\text{rank}\,(A') \le \text{row rank}\,(A')$$

so $$\text{rank}\,(A) \le \text{row rank}\,(A)$$

Transposing A to get $A^T = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ $(n \times m)$

and
$$\begin{array}{ccc} \text{rank}\,(A^T) & \le & \text{row rank}(A^T) \\ \parallel & & \parallel \\ \text{row rank}(A) & \le & \text{rank}(A) \end{array}$$
by above argument.

$$\Rightarrow \text{rank } A = \text{row rank } A \quad ∎$$