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$\mathbb{Q}(\sqrt[n]{a}) \cap \mathbb{Q}(\omega) \neq \mathbb{Q}$ (for HW problem, it's false).

Group is polycyclic if it has a subnormal series with cyclic factors.

A finite group is polycyclic iff it is solvable

$\left(\begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \\ \text{in general} \end{array} \right)$

prove ($\Leftarrow$) for finite group:

Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$, G_{i+1}/G_i is abelian.

Then $G_{i+1}/G_i = C_1^{i+1} \times \cdots \times C_{k_{i+1}}^{i+1}$ where C_j^{i+1} is cyclic

so $1 \trianglelefteq \underbrace{C_1^{i+1}}_{= H_1^{i+1}} \trianglelefteq \underbrace{(C_1^{i+1} \times C_2^{i+1})}_{= H_2^{i+1}} \trianglelefteq \cdots \trianglelefteq \underbrace{(C_1^{i+1} \times \cdots \times C_{k_{i+1}}^{i+1})}_{= H_{k_{i+1}}^{i+1}} = G_{i+1}/G_i$

Let $\tilde{H}_j^{i+1}$ = preimage of $H_j^{i+1} \subseteq G_{i+1}/G_i$ in G_{i+1}

Then $\forall j$, $\tilde{H}_{j+1}^{i+1}/\tilde{H}_j^{i+1} \cong H_{j+1}^{i+1}/H_j^{i+1} \cong C_{j+1}^{i+1}$

cyclic factors

So $1 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq \tilde{H}_1^{i+1} \trianglelefteq \cdots \trianglelefteq \tilde{H}_{k_{i+1}}^{i+1} = G_{i+1} \trianglelefteq \cdots$

so $1 \trianglelefteq \cdots \trianglelefteq G_i \trianglelefteq H_1^{i+1} \trianglelefteq \cdots \trianglelefteq \tilde{H}^{i+1}_{k_{i+1}} = G_{i+1} \trianglelefteq \cdots$

An extension is polycyclic if it is contained in a Galois extension w/ polycyclic group.

<u>Theorem</u> K/F is polycyclic iff it is a tower of cyclic extensions

~~Pf~~ ($\Rightarrow$) if $K \subseteq E$ s.t. $\overset{=G}{\mathrm{Gal}(E/F)}$ is polycyclic,

Let $H = \mathrm{Gal}(E/K) \leq G$.

Then $\exists$ subnormal series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k = G$$

with cyclic factors.

} to be proved next time

$\forall i$, let $L_i = \mathrm{Fix}(H_i)$.

Then $\underbrace{F \subseteq L_{k-1} \subseteq L_{k-2} \subseteq \cdots \subseteq L_0 = K}_{\text{tower of cyclic extensions of } F}$.

($\Leftarrow$) let K/F be a tower of cyclic extensions, then let E be the Galois Closure of K/F. then E is also a tower of Galois extensions whose groups are subgroups of cyclic groups (so are cyclic).

Theorem if K/L, L/F are polycyclic, then K/F is polycyclic.

If L_1/F, L_2/F are polycyclic, then $L_1 \cap L_2/F$ & $L_1 L_2/F$ are polycyclic.

Extension is polyradical if it is a tower of simple radical extensions $F(\sqrt[n]{a})/F$.

Theorem If K/F is polyradical, it is polycyclic.

If K/F is polycyclic & contains all roots of unity of degrees dividing $[K:F]$, then K/F is polyradical (assume $\operatorname{char} F = 0$ or $> [K:F]$).

<u>proof</u> Let K/F be a tower of radical extensions of degrees $n_1, \dots, n_k$. Let $\tilde{F} = F(\omega_1, \dots, \omega_k)$ where ω_i is a primitive root of unity of degree n_i.

Let $\tilde{K} = K\hat{F}$. Then $\check{K}/\tilde{F}$ is a tower of radical extensions & so is a tower of cyclic extensions, and $\hat{F}/F$ is a composite of cyclotomic extensions, and so it is a tower of cyclic extensions.

So $\tilde{K}/F$ is contained in a polycyclic Galois extension, and K is also contained there.

If K/F is a tower of cyclic extensions & all roots of 1 are in F, then it is a tower of radical extensions. $\square$

<u>Def</u> $f \in F[x]$ is solvable in radicals if each of its roots is contained in a polyradical

extension (then they all are in one, the composition).

α is expressible by radicals if it is contained in a polyradical extension.

<u>Theorem</u> (assuming char F is "good"), $f \in F[x]$ is solvable in radicals iff $Gal(f)$ is solvable (= polycyclic).

If f is irreducible & one of its roots is expressible by radicals, then f is solvable in radicals.

<u>proof.</u> If f is solvable by radicals, all roots of f are contained in polyradical K/F.

(Separable!) Then K/F is contained in a polycyclic Galois extension E/F.

Let L be the splitting field of f.

Then $L \subseteq K \subseteq E$. Then $Gal(L/F)$ is a quotient group of $Gal(E/F)$. solvable

So $Gal(L/F) = Gal(f)$ is solvable.

Conversely, if $Gal(f)$ is solvable, let L be the splitting field of f, let

$\tilde{L} = L(\underbrace{\omega_1, \dots, \omega_k}_{\text{suitable roots of } 1})$. Then $Gal(\tilde{L}/F)$

is solvable, $(= Gal(F(\omega_1, \dots, \omega_k)/F) \ltimes Gal(L/F))$.

So $\tilde{L}/F$ is polycyclic, roots of f are in a polyradical extension $\tilde{L}$. $\square$