Lec 9/26

Monday, September 26, 2016 9:08 AM

Limits at ∞, infinite limits

Definition: $\lim_{x \to \infty} f(x) = L$ if

$\forall \varepsilon > 0, \exists M \in \mathbb{R}$ so that $x > M \Rightarrow x \in \mathrm{dom}(f)$ and $|f(x) - L| < \varepsilon$.

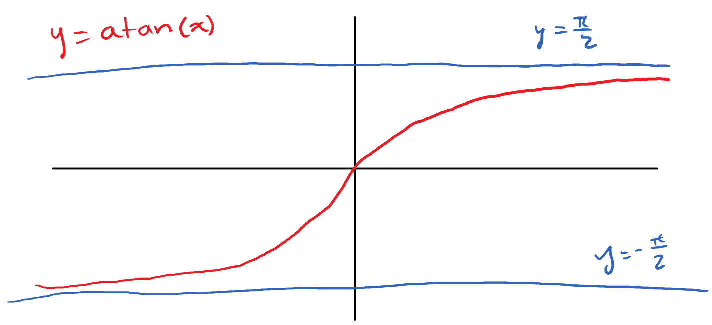
$\lim_{x \to -\infty} f(x) = L$ if

$\forall \varepsilon > 0, \exists m \in \mathbb{R}$ so that $x < m \Rightarrow x \in \mathrm{dom}(f)$ and $|f(x) - L| < \varepsilon$.

Geometric interpretation

$\lim_{x \to \pm\infty} f(x) = L$ means that $y = L$ is a horizontal asymptote to f.
on the right or left ($+$: right, $-$: left).

Ex:



$$\lim_{x \to \infty} \operatorname{atan}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \operatorname{atan}(x) = \frac{-\pi}{2}$$

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

Theorem: $\lim_{x \to \infty} f(x) = L \iff \lim_{u \to 0^+} f\left(\frac{1}{u}\right) = L$

$\lim_{x \to -\infty} f(x) = L \iff \lim_{u \to 0^-} f\left(\frac{1}{u}\right) = L$

Proof: if $\lim_{x \to \infty} f(x) = L$ then $\forall \varepsilon > 0\ \exists M$ so that

$x > M \Rightarrow x \in \mathrm{dom}(f)$ and $|f(x) - L| < \varepsilon$

By replacing M by larger # if necessary, let $M>0$.

If we take $u=\frac{1}{x}$ then $x>M \Rightarrow u=\frac{1}{x}<\frac{1}{M}$

$x \in dom(f) \Leftrightarrow \frac{1}{u} \in dom(f)$

$|f(x)-L|<\varepsilon \Leftrightarrow |f(\frac{1}{u})-L|<\varepsilon$

so $0<|u-0|<\frac{1}{M} \Rightarrow |f(\frac{1}{u})-L|<\varepsilon$

Ex: find: $\lim_{x\to\infty} \frac{2x^2+5x+6}{3x^2-7x+2}$

$$= \lim_{x\to\infty} \frac{x^2(2+\frac{5}{x}+\frac{6}{x^2})}{x^2(3-\frac{7}{x}+\frac{2}{x^2})}$$

$$= \lim_{u\to 0^+} \frac{2+5u+6u^2}{3-7u+2u^2} = \frac{2}{3}$$

Limit theorems for limits at infinity.

Suppose $\lim_{x\to\pm\infty} f(x)=L$, $\lim_{x\to\pm\infty} g(x)=K$

(a) $\lim_{x\to\pm\infty}(f(x)+g(x)) = L+K$

(b) $\lim_{x\to\infty}(f(x)g(x)) = LK$

(c) $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{K}$ if $K\neq 0$

Proof: rewrite these limits as $\lim_{u\to 0^+} f(\frac{1}{u})$, $u=\frac{1}{x}$

then use standard limit theorems.

Theorem: If $\lim_{x\to\infty} f(x)=L$ and $\lim_{v\to L} g(v)=g(L)$ then

$\lim_{x\to\infty} g(f(x)) = g(L)$.

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Definition: we say that $\lim_{x \to a} f(x) = \infty$ if

$\forall M, \exists \delta > 0$ so that

$0 < |x-a| < \delta \Rightarrow x \in \mathrm{dom}(f)$ and $f(x) > M$

(similar thing for $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a} f(x) = -\infty$)

Geometric interpretation: f has a vertical asymptote at a.

Theorem: $\lim_{x \to a} f(x) = \infty \iff f$ is positive on $(c,a) \cup (a,d)$

and $\lim_{x \to a} \frac{1}{f(x)} = 0$.