Lec 10/30

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Let $A \subseteq \mathbb{N}$, $\delta(A) = 1$. Then $\delta(A \cap A-17) = 1$

if $A, B \subset X$, $\mu(X) = 1$, $\mu(A) = \mu(B) = 1$, then $\mu(A \cap B) = 1$.

$\mu(A^c \cup B^c) = 0$ (sub additivity)

Easy: it may happen that $\delta^*(A) = 1 = \delta^*(B)$ but $A \cap B = \emptyset$
(same w/ $\bar{\delta}$).

exercise: show that if $\bar{\delta}(A) = 1$ then $\bar{\delta}(A \cap A-17) = 1$

exercise: show that if $\delta^*(A) = 1$ then $\delta^*(A \cap A-17) = 1$

$A \subset \mathbb{N}$, $\delta^*(A) > 0$, $x(n) = \mathbb{1}_A(n) \in \{0,1\}^{\mathbb{N}}$ shift operator $\sigma x(n) = x(n+1)$

on $\{\sigma^n x,\ n \in \mathbb{Z}\} = X$ there exists a decent measure.

Let $f: \mathbb{Z} \to \mathbb{C}$ be bounded. Then $\exists\, N_i \nearrow \infty$ s.t.

(exercise) $\lim_{i \to \infty} \frac{1}{N_i} \sum_{n=1}^{N_i} f(k)\overline{f(k+n)}$ exists. Then $\varphi(n) = \lim_{i \to \infty} \frac{1}{N_i} \sum_{k=1}^{N_i} f(k)\overline{f(k+n)}$ is p.d.

$\varphi(n)$ is p.d. iff $\forall N$, $\forall\, \xi_1, \dots, \xi_N \in \mathbb{C}$, $\overbrace{\sum_{1 \le n,m \le N} \varphi(n-m)\, \xi_n \bar{\xi}_m}^{\text{this must be real}} \geq 0$

example: $\mu(A \cap T^{-n}A) =: \varphi(n)$ $\mu(A) = \int 1_A \, d\mu$

$$\mu(A \cap T^{-n}A) = \int f(x)\, f(T^n x)\, d\mu \qquad \text{where } f = 1_A.$$

$$= \int 1_{A \cap T^{-n}A}\, d\mu \qquad 1_{A\cap B} = 1_A 1_B$$

$$= \int 1_A \cdot 1_{T^{-n}A}\, d\mu \qquad \text{and} \quad 1_A(T^n x) = 1_{T^{-n}A}(x)$$

since $1_{T^{-1}A}(x) = 1_A(Tx)$

$$= \int f(x)\, f(T^n x)\, d\mu \quad \checkmark$$

since $1_A(Tx) = 1$ iff $Tx \in A$ iff $x \in T^{-1}(A)$. ✓.

Now
$$\sum_{n,m} \varphi(n-m)\, \xi_n \overline{\xi_m} = \sum_{n,m} \xi_n \overline{\xi_m} \int f(x) f(T^{n-m}x)\, d\mu$$

$$= \sum_{n,m} \xi_n \overline{\xi_m} \int f(T^m x) f(T^n x)\, d\mu$$

$$= \sum_{n,m} \int \xi_n f(T^m x)\ \overline{\overline{\xi_m}\, \overline{f(T^n x)}}\, d\mu$$

$$= \int \underbrace{\left(\sum_n \xi_n f(T^n x)\right)}_{F(x)} \underbrace{\left(\sum_m \overline{\xi_m}\, \overline{f(T^m x)}\right)}_{\overline{F}(x)} d\mu$$

$$= \int F \overline{F}\, d\mu \geq 0$$

$\int f(T^n x)\, d\mu = \int f(x)\, d\mu$ since T m.p. (true $\forall f$, esp 1_A, use linear combos of characteristic functions

$1_{T^n A} \cdot 1_{T^m A} = 1_{T^{-m}A \cap T^{-n}A} = 1_{A \cap T^{-(n-m)}A}$

(exercise): $\varphi(n) = \langle U^n f, f \rangle$ is positive definite (should get $\| \sum \xi_n U^n f \|^2$, where $\|a\|^2 = \langle a, a \rangle$).

(remember for midterm)

Furstenberg's Correspondence Principle

(wolog: T invertible & ergodic)

Let $E \subset \mathbb{N}$ (or $\mathbb{Z}$) satisfy $d^*(E) > 0$

Then $\exists$ p.m.p.s. $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ w/ $\mu(A) = d^*(E)$

s.t. $\forall\, n_1, \dots, n_k \in \mathbb{Z}$, $d^*(E \cap (E - n_1) \cap \cdots \cap (E - n_k)) \geq \mu(A \cap T^{-n_1} A \cap \dots \cap T^{-n_k} A)$.

Suppose $\exists\, n$ s.t. $\mu(A \cap T^{-n^2} A) > 0$. Then $d^*(E \cap (E - n^2)) > 0$

$$\mu(A \cap T^{-n} A) = \int_{\mathbb{T}} e^{2\pi i n x} \, d\mu \qquad \text{(Herglotz)}$$

$$\frac{1}{N} \sum \mu(A \cap T^{-n^2} A) = \int_{\mathbb{T}} \frac{1}{N} \sum e^{2\pi i n^2 x} \, d\mu \qquad \text{(Weyl)}$$

Pointwise Ergodic Theorem: Let $(X, \mathcal{B}, \mu, T)$ be a pmps.

Then $\forall$ "nice" $f : X \to \mathbb{R}$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x)$ exists a.e.

("nice" ~ measurable: $f^{-1}(B)$ is borel m-able if B is borel m-able).

$(X, \mathcal{B}, \mu, T)$ (or T itself) is ergodic if T "moves" any nontrivial set. More formally, T is ergodic if

$\underbrace{T^{-1} A = A \text{ a.e.}}_{\mu(A \,\Delta\, T^{-1} A) = 0}$ $(1_A = 1_{T^{-1}A}$ a.e.$)$ $\Rightarrow$ $\mu(A) = \begin{cases} 0 \\ 1 \end{cases}$

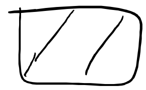
$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$

$\rho(A, B)$ must be 0 even if $A \neq B$

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

$\rho(A,B)$ may be 0 even if $A \neq B$.

exercise: $\rho(A,B) = \mu(A \triangle B)$ is a pseudometric (satisfies triangle inequality)

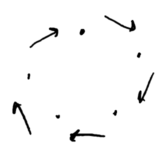
Examples of ergodic T:  $x \mapsto 2x \mod 1$

exercise: this one is

(to check this, use backwards, it's enough to just check intervals or use Criterion ✪ below)

exercise: this one is  $x \to x + \alpha$ iff $\alpha \notin \mathbb{Q}$ (circumference of circle is 1)

and if T is ergodic, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f \, d\mu$ a.e. for any "nice" f. (this is iff T ergodic)

More ergodic T:  (cyclic rotation on finitely many points).

To check if T ergodic it's enough to check:

$$\frac{1}{N} \sum \mu(A \cap T^{-n} B) \longrightarrow \mu(A)\mu(B)$$

✪ or $\frac{1}{N} \sum \mu(A \cap T^{-n} A) \to \mu^2(A)$ for intervals A.