$\alpha(s)$ Lec 1/29

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Pf of Lancret's Theorem (continued).

($\Rightarrow$) done last time

($\Leftarrow$) Suppose $\exists\, c \in \mathbb{R}$ s.t. $\tau \equiv c\kappa$. Then $\exists!\ \theta \in (0, \pi)$, $\cot\theta = c$.

Let $u = T\cos\theta + B\sin\theta$. Then
$$u' = T'\cos\theta + B'\sin\theta$$
$$= \kappa N\cos\theta - \tau N\sin\theta$$
$$= \kappa N\cos\theta - \cot\theta\,\kappa\sin\theta$$
$$= \kappa N\cos\theta - \kappa N\cos\theta = 0$$

So u is constant.

$|u| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$

Finally, $\langle T, u\rangle = \cos\theta$, constant. Thus α is a helix, with axis u and pitch θ.

Reminder (The canonical rep. of a curve).

Let α be a C^3 unit-speed curve whose domain includes 0.

$$\alpha(s) = \alpha(0) + \left[s - \kappa(0)^2\frac{s^3}{6}\right]T(0) + \left[\kappa(0)\frac{s^2}{2} + \kappa'(0)\frac{s^3}{6}\right]N(0) + \kappa(0)\tau(0)\frac{s^3}{6}B(0)$$
$$+\ o(s^3) \quad \text{as } s \to 0$$

Locally Comparing 2 curves (with κ's never 0):

Let α and β be C^3 unit-speed curves in $\mathbb{R}^3$, defined on the same interval I. Let $s_0 \in I$ and let's compare $\alpha(s)$, $\beta(s)$ for s near s_0. To simplify notation, suppose $s_0 = 0$.

From the canonical representations of α and β, we see that

$$\alpha(s) - \beta(s) = [\alpha(0) - \beta(0)] + s\,[T_\alpha(0) - T_\beta(0)] + \frac{s^2}{2}\left[\kappa_\alpha(0)N_\alpha(0) - \kappa_\beta(0)N_\beta(0)\right]$$
$$+ \frac{s^3}{6}\Big\{\left[\kappa_\beta(0)^2T_\beta(0) - \kappa_\alpha(0)^2T_\alpha(0)\right] + \left[\kappa_\alpha'(0)N_\alpha(0) - \kappa_\beta'(0)N_\beta(0)\right]$$

$$+ \frac{s^3}{6}\Big\{\Big[\kappa_\beta(0)^2 T_\beta(0) - \kappa_\alpha(0)^2 T_\alpha(0)\Big] + \Big[\kappa_\alpha'(0) N_\alpha(0) - \kappa_\beta'(0) N_\beta(0)\Big]$$

$$+ \Big[\kappa_\alpha(0)\, \tau_\alpha(0)\, B_\alpha(0) - \kappa_\beta(0)\, \tau_\beta(0)\, B_\beta(0)\Big]\Big\}$$

$$+ o(s^3) \qquad \text{as } s \longrightarrow 0.$$

Consider the following conditions:

(0) $\alpha(0) = \beta(0)$

(1) $T_\alpha(0)\ T_\beta(0) =$

(2) $\kappa_\alpha(0) = \kappa_\beta(0)$ and $N_\alpha(0) = N_\beta(0)$

(3) $\kappa_\alpha'(0) = \kappa_\beta'(0)$, $\tau_\alpha(0) = \tau_\beta(0)$, and $B_\alpha(0) = B_\beta(0)$.

Then:

(a): if (0) holds then $\alpha(s) - \beta(s) = o(1)$ as $s \longrightarrow 0$.

(b): if (0) and (1) hold then $\alpha(s) - \beta(s) = o(s)$ as $s \longrightarrow 0$.

(c): if (0), (1), and (2) hold then $\alpha(s) - \beta(s) = o(s^2)$ as $s \longrightarrow 0$

(d): if all 4 conditions hold then $\alpha(s) - \beta(s) = o(s^3)$ as $s \longrightarrow 0$

Conversely:

(a) Suppose $\alpha(s) - \beta(s) = o(1)$ as $s \to 0$. Then $\underbrace{\lim_{s\to 0}\,[\alpha(s) - \beta(s)]}_{\alpha(0) - \beta(0)} = 0$

then (1) holds.

(b) $\beta(s))$ = Suppose $\alpha(s) - \beta(s)$ o as $s \to 0$ then $_{s \to 0}$ ———

Then (2) ... $\alpha(0) = \beta(0)$

(b') $\Rightarrow$ Suppose $\alpha(s) - \beta(s) = o(s)$ as $s \to 0$ then $\lim_{s \to 0} \left(\frac{\alpha(s)-\beta(s)}{s} \right) = 0$

so $\alpha(s) - \beta(s)$ must $\to 0$. so (0) holds. And then the limit

is of $s \frac{T_\alpha(0) - T_\beta(0)}{s} + o(1)$ so $T_\alpha(s) - T_\beta(s) = 0$ so (1) holds.

(c') Suppose $\alpha(s) - \beta(s) = o(s^2)$ as $s \to 0$. then $\lim_{s \to 0} \left(\frac{\alpha(s) - \beta(s)}{s^2} \right) = 0$

then $\alpha(s) - \beta(s) \to 0$ and $\frac{\alpha(s) - \beta(0)}{s} \to 0$ so (0) and (1) hold.

so the limit is of $s^2 \frac{K_\alpha(0) N_\alpha(0) + K_\beta(0) N_\beta(0)}{2 s^2} + o(1)$ so $K_\alpha(0) N_\alpha(0) = K_\beta N_\beta(0)$.

since N_α, N_β are unit vectors we must have $N_\alpha(0) = \pm N_\beta(0)$ and so

$K_\alpha(0) = \pm K_\beta(0)$ (same $\pm$ in both). but K is positive so both are equal so (2) holds.

(d') Suppose $\alpha(s) - \beta(s) = o(s^3)$ as $s \to 0$, $\lim_{s \to 0} \left(\frac{\alpha(s) - \beta(s)}{s^3} \right) = 0$.

by (0), (1), (2) hold since $\frac{\alpha(s) - \beta(s)}{s^2} \to 0$ as well.

Hence $B_\alpha(0) = T_\alpha(0) \times N_\alpha(0) = T_\beta(0) \times N_\beta(0) = B_\beta(0)$.

And the limit is of $\frac{1}{6} \{ [K_\alpha'(0) - K_\beta'(0)] N_\alpha(0) + K_\alpha(0) [\tau_\alpha(0) - \tau_\beta(0)] B_\alpha(0) \}$

so (3) holds.

thus all conditions are iff.

The Osculating Circle:

let $\alpha: I \to \mathbb{R}^3$ be a C^3 unit-speed curve, with K_α never 0.

Let $s_0 \in I$. We wish to compare the behavior of α near s_0 with

that of a unit-speed circle in $\mathbb{R}^3$. Assume $s_0 = 0$ to simplify notation.

Let β be a unit-speed parametric circle in $\mathbb{R}^3$.

Then $\beta(s) = c + \rho u \cos\frac{s}{\rho} + \rho v \sin\frac{s}{\rho}$ $\forall s$, where c is the center of the circle, ρ is the radius, and u and v are suitable orthonormal vectors. We have $\beta'(s) = -u\sin\frac{s}{\rho} + v\cos\frac{s}{\rho} = T_\beta(s)$.

$$T_\beta'(s) = -\frac{u}{\rho}\cos\frac{s}{\rho} - \frac{v}{\rho}\sin\frac{s}{\rho} \quad \text{so } \kappa_\beta(s) = \frac{1}{\rho} \quad \text{and } N_\beta(s) = -u\cos\frac{s}{\rho} - v\sin\frac{s}{\rho}.$$

Consider the following conditions:

(0) $c + \rho u = \alpha(0)$

(1) $v = T_\alpha(0)$

(2) $\frac{1}{\rho} = \kappa_\alpha(0)$ and $u = -N_\alpha(0)$

Now $\beta(0) = c + \rho u$ and hence

(a) $\alpha(s) - \beta(s) = o(1)$ as $s \longrightarrow$ iff (0) holds.

Next, $T_\beta(0) = v$ and hence

(b) $\alpha(s) - \beta(s) = o(s)$ iff (0) and (1) hold.

Finally, $\frac{1}{\rho} = \kappa_\beta(0)$ and $N_\beta(0) = -u$ and hence

(c) $\alpha(s) - \beta(s) = o(s^2)$ iff (0)–(2) hold

So the parameterized osculating circle to α at a point s_0 is

$$\beta(s) = \alpha(s_0) + \frac{N_\alpha(s_0)}{\kappa_\alpha(s_0)} - \frac{N_\alpha(s_0)}{\kappa_\alpha(s_0)}\cos(\kappa_\alpha(s_0)s) + \frac{T_\alpha(s_0)}{\kappa_\alpha(s_0)}\sin(\kappa_\alpha(s_0)s)$$

Note that the osculating circle lies in the osculating plane.

<u>Propn</u> Let α be a C^3 unit-speed curve in $\mathbb{R}^3$ which lies on a sphere with center $m \in \mathbb{R}^3$ and radius $r \in (0,\infty)$. Then:

(a) κ is never 0, $m - \alpha = \rho N + c B$ and $r^2 = \rho^2 + c^2$ where

$\rho = \frac{1}{K}$ and $c = \langle m - \alpha, B \rangle$.

(b) If τ is never 0, then $m = \alpha + \rho N + \rho' \sigma B$ and $r^2 = \rho^2 + (\rho'\sigma)^2$ where $\sigma = \frac{1}{\tau}$.

(c) If K is constant, α is a circle (a plane $\cap$ the sphere). $\tau \equiv 0$.

Pf (a) $m - \alpha = aT + bN + cB$ for suitable functions a, b, c.

Now $\langle m-\alpha, m-\alpha \rangle = r^2$, so $\frac{d}{ds} \langle m-\alpha, m-\alpha \rangle = 0$

but $\frac{d}{ds} \langle m-\alpha, m-\alpha \rangle = -2 \langle m-\alpha, T \rangle = -2 \langle aT, T \rangle = -2a = 0$ so $a = 0$.

So $m - \alpha = bN + cB$ so $-T = b'N + bN' + c'B + cB' = b'N - b\kappa T + b\tau B + c'B - c\tau N$

so $-T = -b\kappa T + (b' - c\tau)N + (c' + b\tau)B$ (*)

So $b = \frac{1}{\kappa} = \rho$, $\underbrace{0 = b' - c\tau, \quad 0 = c' + b\tau}_{\text{Not used for (a)}}$. so κ is never 0.

so $m - \alpha = \rho N + cB$ and $r^2 = |m-\alpha|^2 = \rho^2 + c^2$. $\square$

(b) from (*), $b' - c\tau = 0$ so if τ is never 0 then $c = \frac{b'}{\tau} = \rho'\sigma$ etc. $\square$

(c) Suppose κ constant. then ρ constant so $\tau = \frac{b'}{c} = \frac{\rho'}{c} = 0$ $\square$