Lec 11/22

Tuesday, November 22, 2016 9:04 AM

Definition An infinite series $\sum_{j=1}^{\infty} a_j$ converges absolutely if $\sum_{j=1}^{\infty} |a_j|$ converges. $(AC \Rightarrow C)$.

Conditional convergence: convergence but not absolutely.

Proof of $AC \Rightarrow C$: $b_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$ $\quad c_n = \begin{cases} -a_n & \text{if } a_n \leq 0 \\ 0 & \text{if } a_n \geq 0 \end{cases}$

$$\sum_{j=1}^{\infty} |a_j| = \sum_{j=1}^{\infty} b_j + \sum_{j=1}^{\infty} c_j, \quad \sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} b_j - \sum_{j=1}^{\infty} c_j.$$

Corollary: if a series converges conditionally, then the sum of nonnegative terms diverges.
" " " " nonpositive "

Proof: sum or diff of convergent series is convergent. ∎

Alternating Series Test (very simple): Suppose $\sum_{j=1}^{\infty} (-1)^j a_j = a_1 - a_2 + a_3 - a_4 + \cdots$

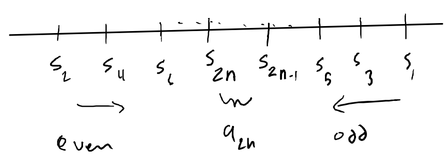
and (1) $a_i > 0$

(2) $a_1 > a_2 > a_3 > \cdots$

(3) $\lim_{j \to \infty} a_j = 0$

Then $\sum_{j=1}^{\infty} (-1)^j a_j$ converges.

Proof



$$S_2 = (a_1 - a_2)$$

$$S_4 = (a_1 - a_2) + (a_3 - a_4)$$

$$S_{2n} = (a_1 - a_2) + (a_1 - a_2) + \cdots + (a_{2n-1} - a_{2n})$$

$\Big\}$ S_{2n} increases

$$S_1 = a_1$$

$$S_3 = a_1 - (a_2 - a_3)$$

$$S_{2n-1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1})$$

$\Big\}$ S_{2n-1} decreases.

$$s_3 = a_1 - (a_2 - a_3)$$

$$\left.\begin{array}{l} s_{2n-1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) \end{array}\right\} s_{2n-1} \text{ decreases.}$$

s_{2n}, s_{2n-1} bounded & monotone so they both converge.

$\lim_{n\to\infty} s_{2n} = L_1 \qquad \lim_{n\to\infty} s_{2n-1} = L_2$

$L_1 - L_2 = \lim_{n\to\infty} s_{2n} - s_{2n-1} = 0$ since $a_n \to 0$. ∎

Example: $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j}$ converges by AST, but not absolutely (harmonic series).
↳ conditionally.

Theorem If a series converges conditionally to S, then the series can be rearranged to converge to any other value T. (or to diverge in any way).

Example: $\sum_{j=1}^{\infty} \frac{(-1)^j}{j} = \log(2)$, can be rearranged to sum to π.

Note: $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ diverges, so does $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$ (LCT with $\sum \frac{1}{n}$ or factor out $-\frac{1}{2}$).

Algorithm: (1): add positive terms until sum is $> \pi$.
(2): add negative terms until sum is $< \pi$.
(3): repeat.

Note: difference from π is less than the last term added and this $\to 0$, so the thing sums to π.

Theorem: The sum of an absolutely convergent series cannot be changed by rearrangement.

Proof: By contradiction: suppose $\sum_{j=1}^{\infty} a_j = S_1$ and $\sum_{j=1}^{\infty} b_j$ is a rearrangement with $\sum_{j=1}^{\infty} b_j = S_2 \neq S_1$.

Pick N large enough so that $\sum_{j=N+1}^{\infty} |a_j| < \frac{|S_2 - S_1|}{3}$

then pick $M \geq N$ large enough so that $\sum_{j=M+1}^{\infty} |b_j| < \frac{|S_1 - S_2|}{3}$

and that any term a_j with $j \leq N$ occurs as b_k with $k \leq M$.

$$\left| S_1 - \sum_{j=1}^{N} a_j \right| = \left| \sum_{j=N+1}^{\infty} a_j \right| \leq \sum_{j=N+1}^{\infty} |a_j| < \frac{|S_1 - S_2|}{3}$$

$$\left|s_2 - \sum_{j=1}^{m} b_j\right| = \left|\sum_{j=N+1}^{\infty} b_j\right| \le \sum_{j=N+1}^{\infty} |b_j| < \frac{|s_1 - s_2|}{3}$$

$$\left|\sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j\right| = \left|\sum{}' b_k\right| \to b_k \text{ where } k \le M \text{ which do not correspond to any } a_j,\ j \le N.$$

They correspond with a_j $j \geq N+1$

$$= \left|\sum{}' a_k\right|$$

$$\le \left|\sum_{j=N+1}^{\infty} a_j\right|$$

$$\le \sum_{j=N+1}^{\infty} |a_j|$$

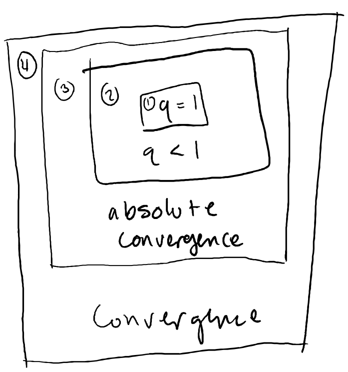
$$< \frac{|s_1 - s_2|}{3}$$

So $|s_1 - s_2| \le \left|s_1 - \sum_{j=1}^{N} a_j\right| + \left|s_2 - \sum_{j=1}^{M} b_j\right| + \left|\sum_{j=1}^{M} b_j + \sum_{j=1}^{N} a_j\right|$

$< \frac{|s_1 - s_2|}{3} \cdot 3$

$< |s_1 - s_2|$ a contradiction.

Convergence of infinite series — Big picture



①: $q = 0$ - Very fast convergence.

② \ ①: $0 < q < 1$ - Reasonably fast convergence.

③ \ ②: $q = 1$, abs. cvg. - Slow convergence but not too slow for low precision.

④ \ ③: $q = 1$, cond. cvg. - Very slow, very bad & unstable.