Lec 10/16

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Def: $S \in L(V,V)$ is symmetric if $\langle S(v), w\rangle = \langle v, S(w)\rangle \Rightarrow \alpha_{ij} = \alpha_{ji}$

Lemma if $W \subseteq V$ is irreducible wrt S, $\dim(W) = 1$.

Thm! : $\exists$ OB of V of eigenvectors for S.

Proof: let W be irreducible, then $V = W \oplus W^\perp$, induction.

$x \in \mathbb{R}^n$. $x = x_1 e_1 + \cdots + x_n e_n$

$$Q(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \cdots + \alpha_n x_n^2 + 2\alpha_{12} x_1 x_2 + \cdots + 2\alpha_{n(n-1)} \alpha_n \alpha_{n-1}$$

$$= \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j} 2\alpha_{ij} x_i x_j = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \quad \text{where } \alpha_{ii} = \alpha_i, \ \alpha_{ij} = \alpha_{ji}$$

$(V, \langle,\rangle)$ Quadratic form on V is $Q: V \to \mathbb{R}$ s.t. $\underbrace{Q(v) = B(v,v)}_{\text{invariant terms}}$ (Polarization)

with $B: V \times V \to \mathbb{R}$ a symmetric bilinear form.

(without referring to a basis)

how to polarize:

$$Q(v+w) = B(v+w, v+w) = B(v,v) + 2B(v,w) + B(w,w) = Q(v) + Q(w) + 2B(v,w)$$

$$\text{So } B(v,w) = \frac{Q(v+w) - Q(v) - Q(w)}{2}$$

example: $Q(v) = \|v\|^2 = \langle v, v\rangle$ is a quadratic form.

Q could not be positive definite so not all Q are norms.

Thm 2 Let Q be a quadratic form on a V.S. V. Then $\exists OB(V) = \{u_1, \dots, u_n\}$ wrt which $Q(x) = \sigma_1 x_1^2 + \sigma_2 x_2^2 + \dots + \sigma_n x_n^2$ where $x = x_1 u_1 + \dots + x_n u_n$

Proof. Let $OB(V) = \{v_1, \dots, v_n\}$. $\beta_{ij} := B(v_i, v_j) = \beta_{ji}$. Then $\exists !\ S: V \to V$ s.t. $S \rightsquigarrow (\beta_{ij})$ under this basis. S is symmetric so $\exists$ basis $\{u_1, \dots, u_n\}$ s.t. $S \rightsquigarrow \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & \sigma_n \end{pmatrix}$, $\langle S(u_i), u_j \rangle = \sigma_i \delta_{ij} \Rightarrow Q(u_i) = \sigma_i$