Lec 8/31

Wednesday, August 31, 2016 9:05 AM

Review:

$P(n)$ = "if in a set of n dogs, one is a chihuahua then all are chihuahuas"

$P(1)$ is true: $\{D_1\}$ D_1 is a chihuahua

induction does not hold however: $P(n) \Rightarrow P(n+1)$ $\forall n \geq 2$ but not for $n=1$.

Given $P(n)$: $\{D_1, D_2, \dots, D_n\}$ are all chihuahuas if D_1 is a chihuahua

then $\{D_1, D_2, \dots, D_n, D_{n+1}\} \setminus \{D_n\}$ are all chihuahuas

$\{D_1, D_2, \dots, D_n, D_{n+1}\} \setminus \{D_{n+1}\}$ are all chihuahuas

so $\{D_1, D_2, \dots, D_n, D_{n+1}\}$ are all chihuahuas.

$2^n > n$ $\forall n \in \mathbb{N} = \{0, 1, 2 \dots\}$

the induction step:

$2 \cdot 2^n > 2 \cdot n \geq n+1$ only works for $n \geq 1$

a different induction step:

assume $2^n > n$

Then $2^n \geq n+1$

$2^{n+1} = (1+1)2^n = 2^n + 2^n > 2n+1 \geq n+1$

so $2^{n+1} > n+1$

Simple induction:

if $P(0)$ holds

and $P(n) \Rightarrow P(n+1)$ $\forall n$

then $P(n)$ holds $\forall n$

Complete induction:

if $P(0)$ holds

and $P(0) \wedge P(1) \wedge \dots \wedge P(n) \Rightarrow P(n+1)$ $\forall n$

then $P(n)$ holds $\forall n$

* Clear that complete induction $\Leftarrow$ simple induction

Conversely, simple induction for the statement $Q(n) = P(0) \wedge P(1) \wedge \dots \wedge P(n)$ is the same as complete induction for $P(n)$

$Q(n)$ holds $\forall n \Leftrightarrow P(n)$ holds $\forall n$

Ex: Proof by complete induction

Generalized associative law for addition

$P(n) = a_1 + a_2 + \dots + a_n$ is independent of parenthesization

base case: Vacuously true for $n = 1, 2$

true for $n = 3$ by P1

induction: So assume $n \geq 3$ and we know $P(n)$ holds

$S = a_1 + a_2 + \dots + a_n + a_{n+1}$

$T = a_1 + a_2 + \dots + a_n + a_{n+1}$

with different choices of parenthesization

want to show $S = T$

Without loss of generality, we may assume that

$S = ((\cdots((a_1 + a_2) + a_3) + \dots) + a_n) + a_{n+1}$

left to right parenthesization

$T = T_1 + T_2$ (last addition performed in computation of T w/ arbitrary parenthesization)

Case 1: T_2 consists of a single term: $T_2 = a_{n+1}$

then T_1 is some parenthesization of $a_1 + a_2 + \dots + a_n$

by induction, T_1 is independent of parenthesization

$T_1 =$

$$T = T_1 + T_2 = \underbrace{((\cdots((a_1 + a_2) + a_3) + \dots) + a_n)}_{T_1} + \underbrace{a_{n+1}}_{T_2} = S$$

Case 2: T_2 contains more than the single term a.

Case 2: T_2 contains more than the single term a_{n+1}

T_2 has $\leq n$ terms

by Complete induction hypothesis,

$T_2 = T_2'$ where terms are parenthesized left to right

$= T_2'' + a_{n+1}$

$$\text{now} \quad T = T_1 + T_2 = T_1 + (T_2'' + a_{n+1}) = \underbrace{(\overbrace{T_1 + T_2''}^{T_1'}) + \overbrace{a_{n+1}}^{T_2'}}_{\text{This is now in case 1 which we have already covered.}} \quad \text{by P1}$$

$T_1' + T_2' = T$, T_2' is a single term

This completes the induction ∎

Illustration of induction step $5 \to 6$ in case 2

$$T = \underbrace{(a_1 + (a_2 + a_3))}_{T_1} + \underbrace{(a_4 + (a_5 + a_6))}_{T_2}$$

$$T_2 = \underbrace{(a_4 + a_5)}_{T_2''} + a_6 \quad \text{by case } n=3 \text{ or P1}$$

$$T = (a_1 + (a_2 + a_3)) + ((a_4 + a_5) + a_6)$$

$$((a_1 + (a_2 + a_3)) + (a_4 + a_5)) + a_6 \quad \text{by P1}$$

$$= ((((a_1 + a_2) + a_3) + a_4) + a_5) + a_6 \quad \text{by Case 1}$$

One further variation of induction:

well-ordering principle

If $S \subseteq \mathbb{N}$ and $S \neq \emptyset$ then S has a least element

Prove by induction:

by contradiction: Assume S has no least element. let $T = \{x \in \mathbb{N} : \{0,1,\dots,n\} \subseteq \mathbb{N} \setminus S\}$

then $0 \notin S$ because otherwise 0 would be the least element of S.

base case: so $0 \in T$

induction: if $n \in T$ then

$\{0\} \subseteq \mathbb{N} \setminus S$

$\{0,1\} \subseteq \mathbb{N} \setminus S$

$\vdots$

$\{0,1,\dots,n\} \subseteq \mathbb{N} \setminus S$

so $\{0,1,\dots,n\} \subseteq T$

if $n+1 \notin T$ then $n+1 \in S$

but then $n+1$ would be the least element of S

because $0,1,\dots,n \notin S$

so $n+1 \in T$

so $S = \emptyset$ because $T = \mathbb{N} \setminus S = \mathbb{N}$

Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$ i.e. any open interval in $\mathbb{R}$ contains a rat'l.

Proof sketch: let $(a,b) \subseteq \mathbb{R}$

may assume $0 < a < b$ (Why?)

choose q so $\frac{1}{q} < b-a$

(since $\mathbb{R}$ has no infinitesimals)

Let $S = \{n \in \mathbb{N} \mid \frac{n}{q} \geq b\}$

S is non empty (why?)

let p be the least elem. of S

then $\frac{p-1}{q} \in (a,b)$