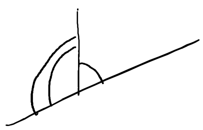
Lec 1/10

Wednesday, January 10, 2018 14:21

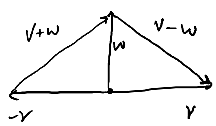
Let $\langle \cdot \mid \cdot\cdot \rangle$ be an inner product on a v.s. $V/\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

$v, w \in V$ are orthogonal if $\langle v \mid w \rangle = 0$.



$\perp$ if $) =))$.

Suppose first that $\mathbb{K}=\mathbb{R}$



$v \perp w$ iff $\|v+w\| = \|v-w\|$.

↑ in the "real" sense.

this happens iff $\|v+w\|^2 = \|v-w\|^2$

$$\|v\|^2 + \langle v \mid w \rangle + \langle w \mid v \rangle + \|w\|^2 = \|v\|^2 - \langle v, w \rangle - \langle w, v \rangle + \|w\|^2$$

$$\langle v, w \rangle = 0$$

Now suppose $\mathbb{K}=\mathbb{C}$. iff $\forall \alpha, \beta \in \mathbb{C}$, $\langle \alpha v \mid \beta w \rangle = 0$,

iff $\forall \alpha, \beta \in \mathbb{C}$, $\alpha v \perp \beta w$ in the "real" sense (planes are orthogonal).

$v \perp w$ in the "real" sense iff $\operatorname{Re} \langle v \mid w \rangle = 0$

the geometrical meaning is stronger in $\mathbb{C}$ sense.

<u>Pythagorean Theorem</u>:

Let $v, w \in V$ with $\langle v, w \rangle = 0$.

then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Let $v_1, \dots, v_n \in V$ with $\langle v_j | v_k \rangle = 0 \quad \forall j \neq k$ then

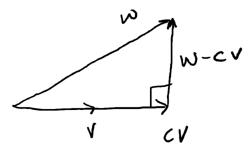
$$\| v_1 + \dots + v_n \|^2 = \|v_1\|^2 + \dots + \|v_n\|^2$$

Note: $\langle v | w \rangle = v^* w$,
antilinear in the first argument
linear in the second.

Schwarz's Inequality:

Let $v, w \in V$. Then $|\langle v | w \rangle| \le \|v\| \|w\|$.

Proof: choose $c \in K$ s.t. $\langle v | w - cv \rangle = 0$. if $v = 0$, any c will do.
(if $v \neq 0$ then $\langle v | v \rangle \neq 0$ and $c = \dfrac{\langle v | w \rangle}{\langle v | v \rangle}$ will do)



Then $\|w - cv\|^2 + \|cv\|^2 = \|w\|^2$ by pyth.

so $|\langle v | w \rangle| = |\langle v | (w - cv) + cv \rangle| = |\langle v | cv \rangle| = |c| \|v\|^2 = \|v\| \|cv\|$

$$\le \|v\| \left(\|w\|^2 + \|w - cv\| \right)^{1/2} = \|v\| \|w\|. \qquad \square$$

Corollary: even if $\langle \cdot | \cdot \rangle$ is only positive semidefinite ($\langle v | v \rangle \ge 0 \;\; \forall v \in V$.) we still have Schwarz's inequality.

[Example of $\langle \cdot | \cdot \rangle$ which is only +-s.d.: let $f, g : [0,1] \longrightarrow \mathbb{C}$,

$$\langle f | g \rangle = \int_0^1 \bar{f}(x) g(x) \, dx$$

Pf of cor: Let $v, w \in V$. wts $|\langle v | w \rangle| \le \|v\| \|w\|$. if $\langle v | v \rangle > 0$, earlier pf goes through. suppose $\langle v | v \rangle = 0$. Then $\forall c \in \mathbb{K}$,

We have $0 \le \|w - cv\|^2 = \|w\|^2 - 2 \mathrm{Re} \langle cv | w \rangle + |c|^2 \|v\|^2$

$$= \|w\|^2 - 2 \mathrm{Re} (\bar{c} \langle v | w \rangle).$$

Considering $c = t\langle v|w\rangle$, we see that $\forall t \in \mathbb{R}$, we have

$2t\,|\langle v|w\rangle|^2 \leq \|w\|^2$, hence $|\langle v|w\rangle| = 0$. $\square$

Adjoints:

Suppose V is a finite-dim inner product space over $\mathbb{C}$.

Let T be a linear operator on V. Then there is a unique

map T^* s.t. $\langle v|Tw\rangle = \langle T^*v|w\rangle \quad \forall v, w \in V$.

T^* is a linear map, the adjoint of T.

$$(cT)^* = \bar{c}\,T^*, \quad (T_1 + T_2)^* = T_1^* + T_2^*$$

$$(T_1T_2)^* = T_2^*T_1^* \qquad (T^*)^* = T$$

Suppose $\mathbb{K} = \mathbb{C}$.

<u>Spectral Theorem</u>: There is an orthonormal basis for V consisting of eigenvectors for T iff $T^*T = TT^*$ (T is <u>normal</u>.)

$\forall T$, (not necessarily normal) $v \in Ker(T^*)$ iff $T^*v = 0$ iff $\langle v|Tw\rangle = 0 \;\forall w \in \mathbb{C}$.

iff $v \in Rng(T)^\perp$.

Thus $Ker(T^*) = Rng(T)^\perp$

$Ker(T) = Rng(T^*)^\perp$

If T is normal then

$$\|Tv\|^2 = \langle Tv \mid Tv \rangle = \langle T^*Tv \mid v \rangle = \langle TT^*v \mid v \rangle = \langle T^*v \mid T^*v \rangle = \|T^*v\|$$

Thus $\operatorname{Ker}(T) = \operatorname{Ker}(T^*)$

and $\operatorname{Ker}(T) = \operatorname{Rng}(T)^\perp$.

Prop Suppose T normal. Let $\lambda \in \mathbb{C}$. Let $M = \overbrace{\{x \in V : Tx = \lambda x\}}^{\text{eigenspace for } \lambda \text{ wrt } T.}$.

T maps M into itself. M is also equal to $\{x \in V : T^*x = \bar{\lambda} x\}$

and T also maps $M^\perp$ into itself.

pf $M = \ker(T - \lambda I)$, $(T - \lambda I)^* = T^* - \bar{\lambda} I$, and $T - \lambda I$ is normal too

$= \operatorname{Ker}(T^* - \bar{\lambda} I)$. This proves first part. Then T^* also maps M to itself. Let $v \in M^\perp$. Then $\forall\, w \in M$, $\langle Tv \mid w \rangle = \langle v \mid \underbrace{T^*w}_{\in M} \rangle = 0$

hence $Tv \in M^\perp$.

Prop Suppose T is normal and $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$.

Let $M_k = \{x \in V : Tx = \lambda_k x\}$ for $k = 1, 2$. Then $M_1 \perp M_2$.