Lec 11/19

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Rings

Ideals

Ring Homs

PID — a D where every I is P.

Local Rings: Ring which has exactly one max'l ideal. (proper)

Coprime Ideals: $I_1 + I_2 = R \iff I_1$ & I_2 are coprime.

$f: R_1 \longrightarrow R_2$ $\quad f(0)=0, \quad f(1)=1, \quad f(a+b)=f(a)+f(b), \quad f(ab)=f(a)f(b).$

$R, \underset{\text{ideal}}{I \subsetneq R}, \rightsquigarrow \pi: \begin{array}{ccc} R & \longrightarrow & R/I \\ r & \longmapsto & r \bmod I \end{array}$

$R, \underset{\substack{\text{mult.} \\ \text{closed}}}{S \subsetneq R} \rightsquigarrow j: \begin{array}{ccc} R & \longrightarrow & S^{-1}R \\ r & \longmapsto & \frac{r}{1} \end{array}$

$R \hookrightarrow R[x]$

"constant poly"

Chinese remainder thm: I, J coprime

$\Longrightarrow \quad IJ = I \cap J \quad$ and $\quad R/IJ \xrightarrow{\cong} R/I \times R/J$.

#4) A: PID, $\overset{\substack{(a) \\ \shortparallel}}{P} \subsetneq A$ prime, nonzero. P is max'l.

if $(a) \subsetneq (b)$, $a = bx$, so $x \in (a)$ so $x = az$

meaning $a = baz \Rightarrow bz = 1 \Rightarrow (b) = A$.

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ & & \cup \\ & & (0) \text{ max'l} \end{array}$$

$f^*((0)) = (0) \subsetneq \mathbb{Z}$ not max'l.

$a \in M$: $M + (a) = M$.

$a \notin M$: $M + (a) = R$.

#6) What are prime ideals in $\mathbb{Q}[x]/(x^3)$?

$0 \in I \Rightarrow x^3 \in I \overset{I \text{ is prime}}{\Rightarrow} x \in I$, so $(x) \subset I$ but $(x) \subset \mathbb{Q}[x]$ is max'l

So $(x) = I$.

$$\mathbb{Q}[x] \Big/ \underbrace{(x^2+1)(x^3-2)}_{\text{max'l \& so coprime ideals.}} \underset{\text{Chinese Remainder Theorem}}{\cong} \mathbb{Q}[x]/(x^2+1) \times \mathbb{Q}[x]/(x^3-2)$$

#17) $n = \mathrm{Char}(R)$. $a \longmapsto a^n$ is a ring hom.

$\uparrow$

prime $\quad (a+b)^n = a^n + b^n$ if n is prime.

$$(ab)^n = a^n b^n$$

$$0^n = 0$$

$$1^n = 1$$