Lec 9/11

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Exam: everything up to (and including) Sylow theorems.

Sylow Theorems: $|G| = n = p^r m$, $p \nmid m$.

(1) $\exists P \leq G$, $|P| = p^r$. P is a Sylow p-subgroup.

(2) P_1, P_2 Sylow p-subgroups $\Longrightarrow \exists g \in G$ s.t. $P_2 = g P_1 g^{-1}$

(3) $\underbrace{\#\text{Sylow p-subgroups}}_{n_p} \equiv 1 \pmod{p}$.

Perhaps Memorize These pfs.

Ideas of proofs

	group	set
(1)	G	p^r-element subsets of G
(2)	P_1	G/P_2

haven't done

(3) not: $G \circlearrowright_{\text{Conjugation}} Syl_p(G)$: set of sylow p-subgroups

(transitive by (2))

Since $n_p \mid |G| \Rightarrow n_p \mid m$

Instead: $P \circlearrowright_{\text{Conjugation}} Syl_p(G)$, $P \in Syl_p(G)$

Recall: If $H \circlearrowright X$, $|X| \equiv |X^H| \pmod{p}$

H p-group

$|X^H|$ # fixed pts.

claim: P-fixed points in $Syl_p(G) = \{P\}$

Suppose $\forall \sigma \in P$, $Q \in Syl_p(G)$ s.t. $Q = \sigma Q \sigma^{-1}$.

(i.e. let Q be a fixed point of this action).

normalizer of Q in G $\longrightarrow$ $N_G(Q) := \{ g \in G : gQg^{-1} = Q \}$

$$\begin{matrix} N_G(Q) & \\ \text{IV} & \text{IV} \\ P & Q \end{matrix}$$

P Q are both sylow p-subgroups in $N_G(Q)$.

By (2) applied to $\left\{ \begin{array}{l} G \longrightarrow N_G(Q) \\ P_1 \longrightarrow P \\ P_2 \longrightarrow \end{array} \right\}$, $\exists\ g \in N_G(Q)$ s.t. $g\overset{\overset{Q}{\|}}{Q}g^{-1} = P$ □

Let G be a group s.t. $|G| = 45$. What is G?

$45 = 3^2 . 5$

(1) there is a subgroup P of size 9, and Q of size 5.

(3) # Sylow 3-subgps $=: n_3 \Longrightarrow n_3 \equiv 1 \mod 3$, $n_3 \mid 5$

$n_5 \equiv 1 \mod 5$, $n_5 \mid 9$.

So $n_3 = 1 = n_5$. So P & Q are unique.

If there is only one Sylow p-subgroup, then that sylow p-subgroup is normal

Since gPg^{-1} is another Sylow p-subgroup.

$|G| = 45 \Rightarrow G$ has two normal subgroups $\underset{\text{size } 9}{P}$ & $\underset{\text{size } 5}{Q}$.

(1) $\langle P, Q \rangle = G$ since $9 \cdot 5 \mid \langle P, Q \rangle \leq 45$.

(2) $P \cap Q = \{e\}$ since $\forall x \in P \cap Q$, $|x| \mid 9$ and $|x| \mid 5$, and $|x| \leq 5$, $|x| \leq 9$.

<u>Lemma</u>: If $\begin{matrix} N_1 \trianglelefteq \\ N_2 \trianglelefteq \end{matrix} H$ s.t. $N_1 \cap N_2 = \{e\}$ then $ab = ba$ $\forall a \in N_1, b \in N_2$

<u>pf</u>: $\underbrace{\overbrace{a\,b\,a^{-1}}^{\in N_2}}_{}\, \overset{\in N_2}{b^{-1}}$, $\underset{N_1}{a}$, $\underbrace{b\,a^{-1}\,b^{-1}}_{\in N_1}$ $\Rightarrow$ $aba^{-1}b^{-1} \in N_1 \cap N_2 = \{e\}$

So (3) $\forall a \in P, b \in Q$ $ab = ba$.

So $G = \{ab \mid a \in P, b \in Q\}$: $(ab)(a'b') = (aa')(b'b)$.

$|Q| = 5 \Rightarrow Q \cong \mathbb{Z}/5\mathbb{Z}$

$|P| = 9 \Rightarrow$???

<u>Prop</u>: $|H| = 9 \Rightarrow H \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$ or $\mathbb{Z}/9\mathbb{Z}$

So $|G| = 45 \Rightarrow G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

or $G \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

<u>Pf</u> $|H| = 9 = 3^2 \Longrightarrow Z(H)$ is nontrivial so $|Z(H)| = 3$ or 3^2.

<u>Case 1</u>: $|Z(H)| = 3 \Rightarrow |H/Z(H)| = 3$, so $H/Z(H) = \{\bar{1}, \bar{g}, \bar{g}^2\}$

Say $\bar{g} = \sigma Z$ so $\sigma^3 \in Z$ which has 3 elts. $\nearrow$ σ generates $Z(\sigma)$

Say $\bar{y} = \sigma Z$ so $\sigma^3 \in Z$ which has 3 elts. σ generates $Z(G)$

either $\sigma^3 = e \in Z$ or not, in which case $(\sigma^3)^3 = e \Rightarrow |Z \quad| = 9$ $\times$.

Case 2: $|Z(H)| = 9$, pick $x \in Z \setminus \{e\}$, ord $(x) = 3$ or 9 in which case $H = \mathbb{Z}/9\mathbb{Z}$

We are in situation where $|H| = 9$, $H_1 \trianglelefteq H$ w/ $|H_1| = 3$, $H \leq Z(H)$

pick $\sigma_2 \in H$, $\sigma_2^3 \in H_1$ ... $\Rightarrow H = \{\sigma_1^i \sigma_2^j \mid 0 \leq i, j \leq 2\} \Rightarrow H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

$\sigma_2 H_1 = \bar{y} \neq \bar{1}$.

$\square$

Prop. $|H| = p^2$ p prime $\Longrightarrow H = \mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

but not true for p^3. $|D_8| = 2^3$

For Later: $|G| = p^r$ and G is abelian

$\Longrightarrow G \cong \mathbb{Z}/p^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{r_k}\mathbb{Z}$ where $r_1 + \cdots + r_k = r$.