Lec 9/9

Friday, September 9, 2016 9:06 AM

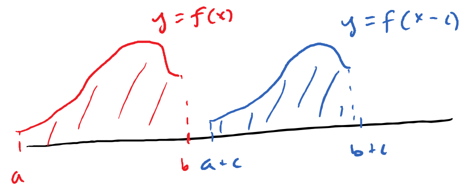
Diagnostic quiz over chapters 1 & 2

Reindexing a sum

$$\sum_{i=m}^{n} a_i = \sum_{k=m+p}^{n+p} a_{k-p}$$

analogous to a shift substitution in a definite integral.

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(u-c)\,du$$



2.3 (d) $(*)$ $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$ (proof by induction)

Induction $n \to n+1$

assume $(*)$ holds

Want to show that

$$(a+b)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i$$

$$(a+b)^{n+1} = (a+b)^n (a+b) = \left(\sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \right)(a+b)$$

$$= \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1}$$

$\downarrow$ trivial reindexing $k = j$ $\qquad$ $\downarrow$ nontrivial reindexing $k = j+1$, $j = k-1$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^k$$

$$= \binom{n}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + \binom{n}{n} b^{n+1}$$

$$= \binom{n}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + \binom{n}{n} b^{n+1}$$

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) a^{n+1-k} b^k$$

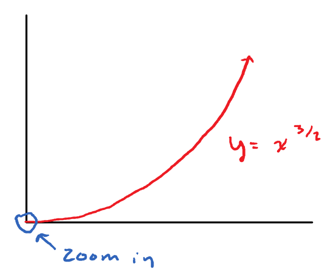
$$= \binom{n+1}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} b^{n+1}$$

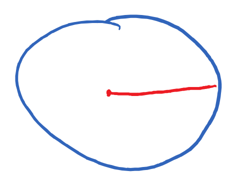
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k \qquad \blacksquare$$

Questions from last time

Let $f : [0, \infty) \to \mathbb{R}$ and $f(x) = x^{3/2} = (\sqrt{x})^3$

		guess:	answer:
Q1:	is f continuous at 0?	no	yes
Q1':	is it true that $\lim_{x \to 0} f(x) = 0$?	no	no (but)
Q2:	is f differentiable at 0?	no	no
Q2':	is it true that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$?	no	no





ε–δ formulations of continuity & limits

Definition: f is continuous at a if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Definition: We say that $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

ambiguity: should $x \in \mathrm{dom}(f)$ be part of hypothesis or conclusion?

if $f(x) = x^{3/2}$ is cts at 0 then it should go in hypothesis.

Right

Definition: f is continuous at a if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|x - a| < \delta$ & $x \in \mathrm{dom}(f)$ $\Rightarrow |f(x) - f(a)| < \varepsilon$

Right

such that $|x-a|<\delta$ & $x \in \mathrm{dom}(f)$ $\Rightarrow$ $|f(x)-f(a)|<\varepsilon$

Right Definition: we say that $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0$ $\exists \delta > 0$

such that $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$ & $x \in \mathrm{dom}(f)$

it is wrong or not useful to have $x \in \mathrm{dom}(f)$ in the wrong place.

Note: this definition implies that $f = \emptyset$ is continuous everywhere

but IVT stipulates the function must be <u>defined</u> as well as continuous.

Zig's approach to limits & continuity:

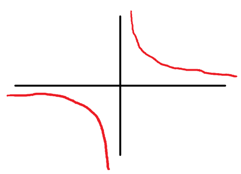
1) understand discontinuity

2) understand continuity

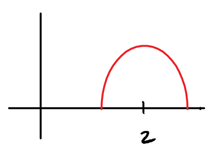
3) understand limits

<u>Ex</u> functions "discontinuous at 0"

1) $f(x) = \frac{1}{x}$



2) $g(x) = \sqrt{1-(x-2)^2}$

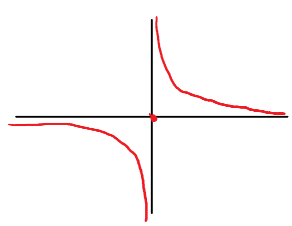


These are non-examples

but $g(x)$ is not discontinuous at 0 so neither is f.

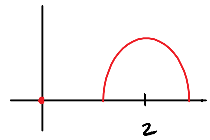
it doesn't make sense to discuss continuity at points where f is undefined

3) $\tilde{f}(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



discontinuous

4) $\hat{g}(x) = \begin{cases} g(x) & 1 \le x < 3 \\ 0 & x = 0 \end{cases}$



Continuous

$$|x-0| < 1 \text{ and } \underbrace{x \in \mathrm{dom}(\hat{g})}_{\substack{\text{not true} \\ \text{so implication} \\ \text{holds.}}} \Rightarrow |\hat{g}(x) - \hat{g}(0)| < \varepsilon$$