Lec 11/13

Wednesday, November 13, 2019 10:20

$S \subseteq H \rightsquigarrow S^\perp = \{x \in H \mid \langle x, s\rangle = 0 \ \forall s \in S\}$

Facts about $\perp$:

① $S \subseteq T \Rightarrow T^\perp \supseteq S^\perp$

② $\overline{S} = S^{\perp\perp}$ and $S^\perp = S^{\perp\perp\perp}$

Pf by ①, since $S \subseteq S^{\perp\perp}$, $S^{\perp\perp\perp} \supseteq S^\perp$. but $S^\perp \subseteq (S^\perp)^{\perp\perp}$. ✓

Let $M \subseteq H$ be a subspace.

③ $M \cap M^\perp = 0$

④ $H = \overline{M} \oplus M^\perp$

Pf let $x \in H$. since $\overline{M}$ closed & cnvx, $\exists!\, m \in \overline{M}$ minimizing dist to x. claim: $x \in M^\perp$ so $x = \underset{\overline{M}}{\underset{\cap}{m}} + \underset{M^\perp}{\underset{\in}{(x-m)}} \Rightarrow H = \overline{M} + M^\perp$, and ③ implies $+ = \oplus$.

If $n \in M$, $\alpha \in \mathbb{C}$, $\|x-m\|^2 \le \|x-(m-\alpha n)\|^2 = \|(x-m)+\alpha n\|^2$

$\Rightarrow x - m \perp n$ by something from a while ago.

⑤ $\overline{M} = M^{\perp\perp}$. if $x \in M^{\perp\perp}$ $\exists!\, m \in \overline{M}$ & $y \in M^\perp$ s.t. $x = m + y$.

then $0 = \langle x, y\rangle = \langle m+y, y\rangle = \overset{0}{\cancel{\langle m, y\rangle}} + \langle y, y\rangle \rightsquigarrow y = 0$.

Thm (Riesz Rep): Let H be a Hilbert sp. for $y \in H$, define $\langle \cdot, y \rangle : H \to \mathbb{C}$, $x \mapsto \langle x, y \rangle$.

① $\langle \cdot, y \rangle \in H^*$, $\|\langle \cdot, y \rangle\| = \|y\|$.

② $\forall \varphi \in H^*$, $\exists! y \in H$ s.t. $\varphi = \langle \cdot, y \rangle$

③ $y \longmapsto \langle \cdot, y \rangle$ is a conj linear isometric iso $H \longrightarrow H^*$

pf ① $\langle \cdot, y \rangle$ is clearly linear. by CS, $|\langle x, y \rangle| \leq \|x\| \|y\| \Rightarrow \|\langle \cdot, y \rangle\| \leq \|y\|$.

Taking $x = y$, $|\langle y, y \rangle| = \|y\|^2 \Rightarrow \|\langle \cdot, y \rangle\| = \|y\|$. ✓

② If $\langle \cdot, y \rangle = \langle \cdot, y' \rangle$ then $\langle \cdot, y - y' \rangle = 0 \Rightarrow y = y'$.

Suppose $\varphi \in H^*$. If $\varphi = 0$, $y = 0$ works.

otherwise, $\ker \varphi \subseteq H$ closed proper subspace.

pick $z \in (\ker \varphi)^\perp$ w/ $\varphi(z) = 1$. $\forall x \in H$,

$x - \varphi(x) z \in \ker \varphi$. So

$$
\begin{aligned}
\langle x, z \rangle &= \langle x - \varphi(x) z + \varphi(x) z, z \rangle \\
&= \cancel{\langle x - \varphi(x) z, z \rangle}^{\,0} + \langle \varphi(x) z, z \rangle \\
&= \varphi(x) \|z\| = \varphi(x)
\end{aligned}
$$
✓

③ $y \longmapsto \langle \cdot, y \rangle$ is isometric by ① & onto by ②

It's obviously conjugate-linear. ✓

Exercise H^* w/ the inner product

$$\big\langle \langle \cdot, y \rangle, \langle \cdot, x \rangle \big\rangle_{H^*} := \langle x, y \rangle_H$$

is a Hilbert space.

Exercise: H is reflexive

Def A subset $E \subset H$ is orthonormal if $e, f \in E \Rightarrow \langle e, f \rangle = \delta_{e=f}$.

Observe: $\|e-f\| = \sqrt{2}$ if $e \neq f$ in E.

Thus, if H is separable, any ON set is countable.

Exercise: Suppose $E \subseteq H$ is ON and $\{e_1, \dots, e_n\} \subset E$.

① if $x = \sum^n c_i e_i$, $\langle x, e_j \rangle = c_j$, and $\|x\|^2 = \sum^n |c_i|^2$

② $\{e_1, \dots, e_n\}$ is lin indp $\Rightarrow$ E is lin indp

③ $\forall x \in H$, $\sum \langle x, e_i \rangle e_i$ is the ! elt of span $\{e_1, \dots, e_n\} = M$ of minimal dist to x

④ $\|x\|^2 \geq \sum^n |\langle x, e_i \rangle|^2$

Thm for an ON set $E \subset H$, TFAE:

① E is maximal (E is an O.N. Basis)

② $M := \{\text{finite linear combinations of elts of } E\} = \underset{\text{algebraic span}}{\operatorname{Span}(E)}$ is dense in H

③ $\langle x, e \rangle = 0 \ \forall e \in E \Rightarrow x = 0$

④ $\forall x \in H$, $x = \sum_{e \in E} \langle x, e \rangle e$ where this sum has at most

④ $\forall x \in H$, $x = \sum_{e \in E} \langle x, e \rangle e$ where this sum has at most countably many nonzero terms & converges in $\|\cdot\|$-top regardless of ordering.

⑤ $\forall x \in H$, $\|x\|^2 = \sum_{e \in E} |\langle x, e \rangle|^2$.

Pf ① ⇒ ② if $\overline{\mathrm{Span}(E)} \neq H$, pick $e \in \mathrm{Span}(E)^\perp$ w/ $\|e\| = 1$,

Then $E \cup \{e\}$ is ON.

② ⇒ ③ Observe $\langle e, x \rangle = 0 \ \forall e \in E \Rightarrow \langle \cdot, x \rangle |_{\mathrm{Span}(E)} = 0$.

Since $\mathrm{Span}(E)$ is dense, $\langle \cdot, x \rangle = 0$ everywhere, so $x = 0$.

③ ⇒ ① let $x \in E^\perp$. Then $\langle x, e \rangle = 0 \ \forall e \in E$, so $x = 0$. Thus E is max'l.

③ ⇒ ④ $\forall e_1, \dots, e_n \in E$, $\|x\|^2 \geq \sum_{i=1}^n |\langle x, e_i \rangle|^2$ by Bessel. so $\forall$ ctble $F \subset E$,

$\|x\|^2 \geq \sum_{f \in F} |\langle x, f \rangle|^2$. Hence $\{e \in E \mid \langle x, e \rangle \neq 0\}$ is ctble.

let (e_n) be an emmeration of . Then

$$\left\| \sum_m^n \langle x, e_i \rangle e_i \right\|^2 = \sum_m^n |\langle x, e_i \rangle|^2 \longrightarrow 0 \quad \text{as} \quad m, n \to \infty.$$

So $\sum_{i=1}^\infty \langle x, e_i \rangle e_i$ converges since H is complete.

set $y := x - \sum \langle x, e_i \rangle e_i$. Then $\langle y, e \rangle = 0 \ \forall e \in E \Rightarrow y = 0$. limit $= x$, not dep. on enumeration

④ ⇒ ⑤ $\|x\|^2 - \sum^n |\langle x, e_i \rangle|^2 = \| x - \sum^n \langle x, e_i \rangle e_i \|^2 \longrightarrow 0$.

⑤ ⇒ ③ is immediate.

Facts:

① Every ON set can be extended to ONB.

② H is separable iff $\exists$ ctble ONB.

③ $H \cong K$ iff they have ONB's of the same cardinality.

Def $u: H \longrightarrow K$ is unitary if it is a linear isomorphism

s.t. $\langle ux, uy \rangle_K = \langle x, y \rangle_H \quad \forall x, y \in H.$

$\iff$ u is an iso iso

$\iff$ $u^* = u^{-1}$

Lemma: suppose $X_0 \subset X$ (banach) is dense subspace & $T_0 : X_0 \longrightarrow Y$

s.t. T_0 bdd. Then $\exists!$ ext $T: X \longrightarrow Y$ bdd s.t. $\|T\| = \|T_0\|$ & $T|_{X_0} = T_0$

Know how to prove this for quiz

④ If $E \subset H$ is an ONB, then $H \cong \ell^2(E) = \{ f: E \to \mathbb{C} \mid \sum_{e \in E} |f(e)|^2 < \infty \}$ (counting measure)

$$\overset{H}{\underset{x}{\Downarrow}} \longmapsto \left[\hat{x} : E \to \mathbb{C} \text{ by } \hat{x}(e) = \langle e, x \rangle \right]$$

claim: $x \longmapsto \hat{x}$ is a unitary isomorphism.