Lec 9/16

Friday, September 16, 2016 9:12 AM

Recall:

Definitions: Let $a \in \text{dom}(f)$. Let $\delta > 0$. Then $\underline{J_{f,a,\delta}}$ = smallest closed interval containing $\{f(x) : x \in [a-\delta, a+\delta] \cap \text{dom}(f)\}$

$\underline{J_{f,a}}$ (jump/oscillation interval) $= \bigcap_{\delta>0} J_{f,a,\delta}$ = "$\lim_{\delta \to 0} J_{f,a,\delta}$"

f is <u>continuous</u> at a if $J_{f,a} = \{f(a)\}$ and <u>discontinuous</u> if $J_{f,a}$ is a proper closed interval.

ε-δ Definition: we say f is continuous at $a \in \text{dom}(f)$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ so that (*) $|x-a| < \delta$ and $x \in \text{dom}(f) \Rightarrow |f(x) - f(a)| < \varepsilon$

These two definitions are equivalent:

$$(*) \iff \{f(x) : x \in (a-\delta, a+\delta) \cap \text{dom}(f)\} \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$$

$$(a-\delta, a+\delta) = \bigcup_{0<\delta'<\delta} [a-\delta', a+\delta']$$

$$(*) \iff \{f(x) : x \in [a-\delta', a+\delta'] \cap \text{dom}(f)\} \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$$

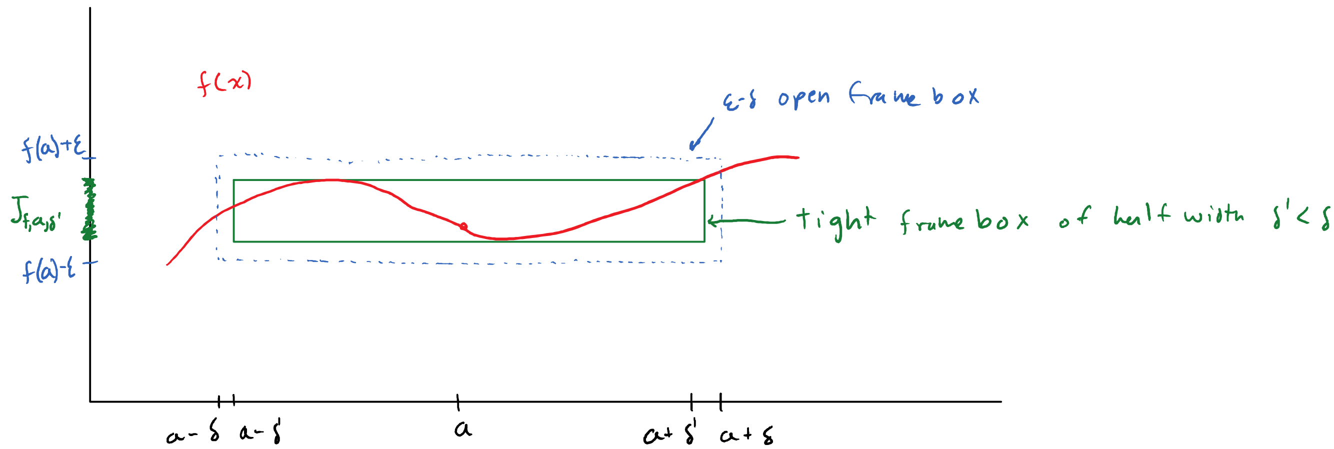
$$\iff J_{f,a,\delta'} \subseteq (f(a) - \varepsilon, f(a) + \varepsilon) \qquad \text{for } 0 < \delta' < \delta$$

$$\iff \underset{\{f(a)\}}{\underset{\cup\!\shortmid}{J_{f,a}}} = \bigcap_{\delta>0} J_{f,a,\delta'} \subseteq \bigcap_{\varepsilon>0} (f(a) - \varepsilon, f(a) + \varepsilon) = \{f(a)\}$$

↑ since ε is arbitrary

$$\iff J_{f,a} = \{f(a)\}$$

Graphical interpretation:



So

$$\{(a,f(a))\} \subseteq \{a\} \times J_{f,a} = \underbrace{\bigcap_{\delta'>0} [a-\delta', a+\delta'] \times J_{f,a,\delta'}}_{\text{tight frame box}} \subseteq \bigcap_{\varepsilon>0} (a-\delta, a+\delta) \times (f(a)-\varepsilon, f(a)+\varepsilon) = \{(a, f(a))\}$$

where δ corresponds to ε

Compare the definitions:

Jump interval definition: 1) more conceptual
2) highlights difference between continuity and discontinuity

ε-δ definition: 1) easier to work with

Remark: suppose f is discontinuous at $a \in dom(f)$. Then ε-δ argument will fail. Let D = distance from $f(a)$ to the furthest endpoint of the jump interval $J_{f,a}$. ($D = \infty$ if $J_{f,a}$ is infinite)

If $\varepsilon > D$ then can find $\delta > 0$ to satisfy implication in ε-δ def.
$\varepsilon \le D$ then you can't.

Example: ε-δ argument to show $f(x) = x^3 - 2x$ is continuous at any a.

Let $\varepsilon > 0$ be given. Want to show $|f(x) - f(a)| < \varepsilon$.

$$|f(x) - f(a)| < \varepsilon$$

$$|(x^3 - 2x) - (a^3 - 2a)| < \varepsilon$$

$$|x^3 - a^3 - 2(x-a)| < \varepsilon$$

$$|(x-a)(x^2+xa+a^2)-2(x-a)|<\varepsilon$$
$$|x-a||x^2+xa+a^2-2|<\varepsilon$$

first approximation to δ: suppose $|x-a|<1$

$$\Downarrow$$
$$|x|-|a|<1$$

so $|x|-|a|\leq|x-a|$

so $|x|\leq|a|+|x-a|\leq|a|+1$

now $|x^2+ax+a^2-2|\leq|x^2|+|a||x|+|a^2|+|-2|\leq(|a|+1)^2+|a|(|a|+1)+|a|^2+2$

$=3|a|^2+3|a|+3$

so $|x-a|<1\Rightarrow|f(x)-f(a)|=|x-a||x^2+ax+a^2-2|\leq|x-a|(3|a|^2+3|a|+3)<\varepsilon$

$|x-a|<\dfrac{\varepsilon}{3|a|^2+3|a|+3}$

so in order to conclude that $|f(x)-f(a)|<\varepsilon$, we can let $\delta=\min\left(1,\dfrac{\varepsilon}{3|a|^2+3|a|+3}\right)$ and make $|x-a|<\delta$.

Note: expression for δ should not involve x.