Lec 11/20

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Last Time: X LCH. every Radon integral φ on $C_c(X)$ is $\int \cdot \, d\mu$ for a $!$ Radon meas. μ

Lemma: X LCH and μ Radon on $(X, \mathcal{B}_X)$.

Define $\varphi(f) := \int f \, d\mu$ on $C_c(X)$. TFAE:

① φ extends ctsly to $C_0(X)$

② φ bdd wrt $\|\cdot\|_\infty$

③ $\mu(X)$ is finite

Pf ②$\Leftrightarrow$③ follows from

$$\mu(X) = \sup\left\{ \int f \, d\mu \mid f \in C_c(X), \ 0 \le f \le 1 \right\}.$$

Corollary: positive linear ftls $\varphi \in C_0(X)^*$ are of the form $\varphi = \int \cdot \, d\mu$ where μ is a finite Radon meas $\Leftrightarrow$ finite reg Borel.

Prop suppose $\varphi \in C_0(X, \mathbb{R})^*$. Then $\exists$ positive $\varphi_\pm \in C_0(X, \mathbb{R})^*$ s.t. $\varphi = \varphi_+ - \varphi_-$

Pf for $f \in C_0(X, [0, \infty))$, define $\varphi_+(f) = \sup\{ \varphi(g) \mid 0 \le g \le f \}$.

Since $|\varphi(g)| \le \|\varphi\| \|g\|_\infty \le \|\varphi\| \|f\|_\infty$ $\forall$ $0 \le g \le f$ and since $\varphi(0) = 0$,

$0 \le \varphi_+(f) \le \|\varphi\| \cdot \|f\|_\infty$. Observe

① $\varphi_+(cf) = c\,\varphi_+(f) \quad \forall\, c \geq 0.$

② $\forall f_1, f_2 \in C_0(X, [0,\infty)),\ \varphi_+(f_1+f_2) = \varphi_+(f_1) + \varphi_+(f_2).$

<u>Pf</u> whenever $0 \leq g_i \leq f_i$, $0 \leq g_1+g_2 \leq f_1+f_2$.

So $\varphi_+(f_1+f_2) \geq \varphi_+(f_1) + \varphi_+(f_2)$.

but if $0 \leq g \leq f_1+f_2$, set $g_1 = g \wedge f_1$ & $g_2 = g - g_1$.

Then $0 \leq g_i \leq f_i$ and so $\varphi_+(f_1+f_2) \leq \varphi_+(f_1) + \varphi_+(f_2)$.

for $f \in C_0(X,\mathbb{R})$, define $\varphi_+(f) = \varphi_+(f_+) - \varphi_+(f_-)$.

If $f = \underset{= f_+ - f_-}{g-h}$ where $g, h \geq 0$ then $g + f_- = f_+ + h$,

so $\varphi_+(g) + \varphi_+(f_-) = \varphi_+(h) + \varphi_+(f_+) \Rightarrow \varphi_+(f) = \varphi_+(g) - \varphi_+(h)$.

This implies φ_+ is linear on $C_0(X,\mathbb{R})$.

Finally, $|\varphi_+(f)| \leq \max\{\varphi_+(f_+), \varphi_+(f_-)\} \leq \|\varphi\| \cdot \|f\|_\infty$.

$\Rightarrow \|\varphi_+\| \leq \|\varphi\|$. Set $\varphi_- = \varphi - \varphi_+$.

It's also positive: $\underset{f \geq 0}{\varphi_-(f)} = \varphi(f) - \sup\{\varphi(g) \mid 0 \leq g \leq f\}$
$= \inf\{\varphi(f-g) \mid 0 \leq g \leq f\} \geq 0.$ $\square$

<u>Corollary</u>: If $\varphi \in C_0(X,\mathbb{R})^*$, $\exists$ finite Radon measures μ_1, μ_2 on X s.t. $\varphi(f) = \int f\,d\mu_1 - \int f\,d\mu_2$

$$= \int f \, d(\mu_1 - \mu_2)$$

?

<u>HW cor</u>: for $\varphi \in C_0(X)^*$, $\exists$ finite Radon measures $\mu_0, \mu_1, \mu_2, \mu_3$, s.t. $\forall f \in C_0(X)$,

?

$$\varphi(f) = \sum_{k=1}^{3} i^k \int f d\mu_k = \int f \, d\Big(\sum_{k=1}^{3} i^k \mu_k\Big)$$

<u>Signed Measures</u>: $(X, \mathcal{M})$ mble space.

A function $\nu : \mathcal{M} \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ is called a <u>signed measure</u> if

- ν takes at most one value in $\{\pm\infty\}$.
  [if ν only takes finite values, call ν finite].
- $\nu(\emptyset) = 0$
- $\forall$ disjoint seq $(E_n) \subset \mathcal{M}$, $\nu(\amalg E_n) = \sum \nu(E_n)$, where
  - * the sum <u>converges absolutely</u> if $|\nu(\amalg E_n)| < \infty$.
  - * otherwise, $\sum \nu(E_n)$ diverges.

<u>Examples</u>:

1. If μ_1, μ_2 are measures on $(X, \mathcal{M})$ w/ at least one finite, $\nu = \mu_1 - \mu_2$ is a signed meas.

2. Suppose μ is a measure on $(X, \mathcal{M})$ and $f : X \to \overline{\mathbb{R}}$ is mble

s.t. at least one of $\int f_\pm \, d\mu < \infty$. $\leftarrow$ "extended μ $\int$ble"

Then $\nu(E) := \int_E f \, d\mu$ is a s.m.

Suppose ν is a signed measure on $(X, \mathcal{M})$.

call $E \in \mathcal{M}$ either

positive
negative if $\forall$ mble $F \subset E$ $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$.
null

Observe $N \in \mathcal{M}$ is null $\Leftrightarrow$ N both pos & neg.

Facts: ① E pos $\Rightarrow$ $\nu(E) \geq 0$.

② E pos, $F \subset E$ $\Rightarrow$ F pos.

④ E_n pos $\Rightarrow$ $\bigcup E_n$ pos.

Pf: disjointify $\bigcup E_n = \coprod \overset{\text{pos}}{F_n}$

if $G \subset \bigcup E_n = \coprod F_n$,

$\nu(G) = \nu(\coprod G \cap F_n) = \sum \overset{\text{pos}}{\nu(G \cap F_n)} \geq 0$.

③ observe: σ-additivity holds for (E_n) disjoint union of positive sets (or negative).

⑤ If $0 < \nu(E) < \infty$, $\exists$ positive $F \subseteq E$ s.t. $\nu(F) > 0$.

Pf If E positive, done! Else let $n_1 \in \mathbb{N}$ be minimal s.t. $\exists E_1 \subset E$ and $\nu(E_1) < \frac{-1}{n_1}$.

If $E \setminus E_1$ positive, done. Otherwise, let $n_2 \in \mathbb{N}$ be minimal st. $\exists E_2 \subset E \setminus E_1$ and $\nu(E_2) < \frac{-1}{n_2}$.

Iterate to get either $E \setminus \coprod_{i=1}^{k} E_i$ pos for some k, or (E_i) disjoint w/ $\nu(E_i) < -\frac{1}{n_i}$ $\forall i$.

Since $|\nu(E)| < \infty$, $\sum |\nu(E_i)| < \infty$.

So $\sum \frac{1}{n_i}$ converges, so $n_i \longrightarrow \infty$ as $i \to \infty$.

Since $\nu(E) > 0$ & $\nu(E_i) < 0$ $\forall i$, $\nu(F := E \setminus \amalg E_i) > 0$.

Claim: F is positive.

Let $G \subset F$ be mble. Then $\nu(G) \geq \frac{-1}{n_i}$ $\forall i$.

So $\nu(G) \geq 0$.

$\square$

Thm (Hahn Decomposition). Let ν be a signed measure on $(X, \mathcal{M})$. $\exists$ positive set P s.t. P^c is negative.

If $Q \in \mathcal{M}$ is another pos set w/ Q^c negative, then $P \Delta Q$ and $P^c \Delta Q^c$ are ν-null.

$X = P \amalg P^c$ "Hahn Decomposition" unique up to ν-null sets.

Pf of uniqueness: Suppose $P, Q \in \mathcal{M}$ pos s.t P^c, Q^c neg.

Then $P \Delta Q = [P \setminus Q] \amalg [Q \setminus P]$

$$= \underbrace{\underbrace{[\underset{+}{P} \cap \underset{-}{Q^c}]}_{\text{null}} \uplus \underbrace{[\underset{+}{Q} \cap \underset{-}{P^c}]}_{\text{null}}}_{\text{null}}$$