Rec 10/15

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Directed Sets

(1) $\mathbb{N}$

(2) Partitions of $[a,b]$ ordered by refinement $= \mathcal{P}$

(3) $\mathcal{N} =$ nhds of $x \in X$, $\mathcal{U} \leq V \Leftrightarrow \mathcal{U} \geq V$.

(4) $I \times J$ $(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha'$ & $\beta \leq \beta'$.

Nets

- Mappings $\underset{\text{directed set}}{I} \longrightarrow \underset{\text{set}}{X}$, denoted $(x_\alpha)_{\alpha \in I}$ or $\langle x_\alpha \rangle_{\alpha \in I}$

eg

(1) $I = \mathbb{N} \rightsquigarrow$ sequences

(2) $I = \mathcal{P}$, $f : [a,b] \to \mathbb{R}$, $S_p = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$. $(S_p)_{p \in \mathcal{P}}$.

If S is a subset of X and $\langle x_\alpha \rangle_{\alpha \in I}$ is a net

then $\langle x_\alpha \rangle$ is <u>eventually</u> in S if $\exists\, \alpha_0 \in I$ s.t.

$x_\beta \in S$ whenever $\beta \geq \alpha$.

Also, $\langle x_\alpha \rangle$ is <u>frequently</u> in S if $\forall \alpha \in I$, $\exists \beta \geq \alpha$ s.t. $x_\beta \in S$.

X a topological space

$\rightsquigarrow$ $\langle x_\alpha \rangle$ converges to $x \in X$ if $\forall$ nhd S of x, $\langle x_\alpha \rangle$ is eventually in S.

$\rightsquigarrow$ $x \in X$ is a cluster point of $\langle x_\alpha \rangle$ if $\forall$ nhd S of x, $\langle x_\alpha \rangle$ is frequently in S.

(X, τ) top space.

<u>Prop 1</u> for $A \subset X$, TFAE

① x is an accumulation point of A $\rightarrow$ $\forall$ open $U \ni x$, $A \cap U \setminus \{x\} \neq \emptyset$.

② $\exists$ a net in $A \setminus \{x\}$ that converges to x

<u>pf</u> <u>① $\Rightarrow$ ②</u>

Choose $I =$ nbhd base at x, $\overbrace{U \geq V \text{ if } U \subseteq V}^{\text{reverse inclusion}}$.

For the mapping $U \mapsto x_U$ take x_U to be any point in $A \cap U \setminus \{x\}$.

For any nhd $S \ni x$, $\exists\, T \in I$ with $x \in T \subset S$.

<u>claim</u>: $\langle x_U \rangle$ is eventually in S.

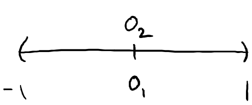
pf if $\mathcal{U} \geq T$, so $\mathcal{U} \subset T$, then $\mathcal{U} \subset S$ and $x_{\mathcal{U}} \in \mathcal{U} \subset S$.

② ⟹ ①

Given a net $\langle x_\alpha \rangle$ w/ $x_\alpha \in A \setminus \{x\}$ that converges to x,
This means $\forall$ nhd S of x, $x_\alpha \in S$ for $\alpha \geq \alpha_0$ (for some α_0).
So $x_\alpha \in S \cap A \setminus \{x\}$. So x is an accumulation pt.

Cor "If a net $\langle x_\alpha \rangle \subset A$ converges, then the limit $x \in A$" means the same thing as "A is closed".

Prop 2 X is Hausdorff iff any convergent net has a unique limit.

eg (# 37) 

not Hausdorff

Pf "$\Longrightarrow$" is clear: if $\langle x_\alpha \rangle \longrightarrow x$ then $\forall y \neq x$, choose disjoint open sets $\mathcal{U} \ni x$, $V \ni y$. x_α is eventually in $\mathcal{U}$, and if $\langle x_\alpha \rangle \longrightarrow y$ then x_α eventually in V too.
so $\exists$ $\underbrace{x_\beta}_{\text{Directed Set stuff}} \in \mathcal{U}$ and in V, but this is a contradiction.

"$\Longleftarrow$" Contrapositive. Suppose X is not Hausdorff. so $\exists$ $x \neq y$ s.t. if $\mathcal{U} \ni x$, $V \ni y$ are nhds then $\mathcal{U} \cap V \neq \emptyset$.

...

Let I = nbhd base for x, J = nbhd base for y.

Define a net on $I \times J$ s.t. $x_{(U,V)} \in U \cap V$.

Then $\langle x_{(U,V)} \rangle \longrightarrow x$, $\langle x_{(U,V)} \rangle \longrightarrow y$. $\square$

<u>Convergence of Real-valued functions</u>

Uniform convergence is in metric sp. w/ $\|f\| = \sup_x |f(x)|$.

Pointwise convergence ??

<u>Prop 3</u> $f: X \longrightarrow Y$ cts $((X,\tau), (Y,\theta)$ top spaces$)$.

$\Leftrightarrow$ $\forall$ convergent net $\langle x_\alpha \rangle \longrightarrow x$ in X,

$\langle f(x_\alpha) \rangle \longrightarrow \langle f(x) \rangle$ in Y.