Lec 9/15

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Fejer's theorem

If $f(x) \nearrow \infty$, $f'(x) \searrow 0$, $xf'(x) \nearrow \infty$ $\Big\}$ then $f(n)$ is u.d. mod 1
$f(n)$ mod 1 is u.d.

a criterion for (x_n) to be base-r normal in terms of u.d.:

x is base-r normal iff $r^n x$ mod 1 is u.d. (exercise)

exercise: $d^*(S) = 0$ where S is the set of square-free numbers.

$$d^*(S) = \liminf_{N-M\to\infty} \frac{|S \cap \{M+1, \dots, N\}|}{N-M} = 0$$

"most misses you can get in a row"

a set $A \subseteq \mathbb{N}$ (or in $\mathbb{Z}$) is thick if it contains arbitrarily long intervals.

$\Updownarrow$ $d^*(A) = 1$ (Exercise: prove this equivalence)

If $d(A) > \frac{1}{2}$ then $x+y=z$ is solvable in A. (Exercise)

Theorem: if $f'(x)$ exists on (a,b), then f' has the Darboux property

(Darboux property: $f(x)$ attains all values in $(f(a), f(b))$ for $x \in (a,b)$).

$\forall c \in [0,1]$ there is a set A_c with density c.

$A = \{ \lfloor \frac{n}{\alpha} \rfloor : n \in \mathbb{N} \}$. then $d(A) = \alpha$. (exercise: prove this)

plus for $\lceil \, \rceil$ and closest integer

So d has "darboux property."

Exercise: $d(C_j) = 0$ where $C_j = \{ p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} : p_i \in P, \alpha_i \in \mathbb{N} \}$.

Fejer's thm: $f(x) = x^c, \log^{1+\varepsilon} x$
$0 < c < 1$

Exercise: $n^c, \log^{1+\varepsilon} n$ are dense mod 1

Harder exercise: n^c is dense mod 1 for all $c \notin \mathbb{N}$. in particular, $n^{3/2}$

freestyle question:

① generalize $d, \bar{d}, d^*$, etc to $\mathbb{Z}_2$

② Prove darboux property for these generalizations.

Midterm structure:

I: Defns / Formulations. (Named theorems / notions)

II: 2-3 proofs.

III: T/F, why?

$d(\{a_n\}) \approx \frac{n}{a_n}$

$\Gamma = \{2^n 3^m, \; n, m \in \mathbb{N}\}$ semigroup

Furstenberg's Dichotomy Thm:

Let $S = \{n_1 < n_2 < \cdots < n_k < \cdots\}$ be a multiplicative semigroup.

Then either S is <u>Lacunary</u> (if $S = \{n_1 < n_2 < \cdots\}$ then $\frac{n_{i+1}}{n_i} > \lambda > 1 \; \forall i$)

Or $\frac{n_{i+1}}{n_i} \to 1$

Exercise : if $\{n_1 < n_2 < \cdots < n_k < \cdots\}$ is the ordering of Γ then $\frac{n_{i+1}}{n_i} \to 1$

<u>Theorem</u> (Furstenberg): $\forall \alpha \notin \mathbb{Q}$, $\Gamma\alpha = \{2^n 3^m \alpha : n, m \in \mathbb{N}\}$ is dense in $[0,1]$.

<u>Def.</u> $A \subset \mathbb{R}$ has measure 0 if there is a system of intervals I_τ, for $\tau \in T$, with total length

$$\sum_{\tau \in T} |I_\tau| < \varepsilon \quad \text{s.t. } A \subseteq \bigcup_{\tau \in T} I_\tau$$

1: enumerable sets have measure 0.

$$x_n \in \left(x_n - \frac{\varepsilon}{2^{n+1}}, \; x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

2: Cantor sets (uncountable, measure 0).