Lec 2/5

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Defns Let X be a set. To say that ρ is a pseudometric on X means that $\rho : X \times X \longrightarrow [0,\infty)$ satisfies the following $\forall x,y,z \in X$:

$$\rho(x,z) \le \rho(x,y) + \rho(y,z) \quad , \quad \rho(x,y) = \rho(y,x) \quad , \quad \text{and} \quad \rho(x,x) = 0.$$

ρ is a metric on X if $\rho(x,y) = 0 \iff x = y$.

Def pseudometric space (X,ρ)

metric space (X,ρ).

eg Let $X \subseteq \mathbb{R}^d$. define $\rho : X \times X \longrightarrow [0,\infty)$ by $\rho(x,y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2}$.

ρ is a metric on X.

eg Let X be the set of Riemann-integrable functions on $[0,1]$. Define $\rho_2 : X \times X \to [0,\infty)$ by $\rho_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}$. ρ_2 is a pseudometric but not a metric: $\rho_2(\_\_\_ , \_\_\_\_) = 0$. (if we restrict this to continuous functions we get a metric).

define $\rho_1 : X \times X \longrightarrow [0,\infty)$ by $\rho_1(f,g) = \int_0^1 |f(x) - g(x)| \, dx$. This is another pseudometric on X.

Let $f_n = n^{2/3} \mathbb{1}_{[0,\frac{1}{n}]}$, $\rho_1(f_n, 0) = n^{2/3}/n = n^{-1/3} \longrightarrow 0$ as $n \longrightarrow \infty$.

but $\rho_2(f_n, 0) = \left(\frac{n^{4/3}}{n}\right)^{1/2} = n^{1/6} \longrightarrow \infty$ as $n \longrightarrow \infty$.

So ρ_1, ρ_2 <u>do not</u> induce the same topology.

Defns Let (X,ρ) be a pseudometric space.

(a) if $x_0 \in X$ and $r > 0$ then $B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$. "open ball".

(b) if $x_0 \in X$ and $r \ge 0$ then $B[x_0, r] = \{x \in X : \rho(x_0, x) \le r\}$ "closed ball".

(c) if $G \subseteq X$, then to say G is <u>open</u> is to say that $\forall x \in G$, $\exists r > 0$ s.t. $B(x,r) \subseteq G$.

(d) if $F \subseteq X$ then to say F is <u>closed</u> is to say that $X \setminus F$ is open.

Propn: Let (X, ρ) be a pseudometric space.

(a) Let $\mathcal{G} = \{G \subseteq X : G \text{ is open}\}$. Then $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$.
$\mathcal{G}$ is closed under finite intersections, $\mathcal{G}$ is closed under arbitrary unions.

(b) Let $\mathcal{F} = \{F \subseteq X : F \text{ is closed}\}$. Then $\emptyset \in \mathcal{F}$, $X \in \mathcal{F}$.
$\mathcal{F}$ is closed under finite unions, $\mathcal{F}$ is closed under arbitrary intersections. deMorgan's laws.

Propn: $B(x_0, r)$ is open, $B[x_0, r]$ is closed.

Defn A topological space is a pair $(X, \mathcal{G})$ such that X is a set, $\mathcal{G} \subseteq P(X)$, $\emptyset \in \mathcal{G}$, $X \in \mathcal{G}$, $\mathcal{G}$ is closed under arbitrary unions & finite intersections. $\mathcal{G}$ is a "Topology."
the elements of $\mathcal{G}$ are called open sets.

Defns Let $(X, \mathcal{G})$ be a topological space.

(a) if $F \subseteq X$, then to say F is closed means $X \setminus F \in \mathcal{G}$ (is open).

(b) if $A \subseteq X$, Then $\text{int}(A) = \bigcup\{G \in \mathcal{G} : G \subseteq A\}$. sometimes $\text{int}(A) = A^\circ$.
this is the largest open set contained in A.

(c) Let $\mathcal{F}$ be the set of closed subsets of X. If $A \subseteq X$ then $cl(A) = \overline{A} = \{F \in \mathcal{F} : F \supseteq A\}$
this is the smallest closed set containing A. $cl(A)$ is the closure of A.

(d) if $x_0 \in X$ and $A \subseteq X$ then to say A is a neighborhood (nbhd) of x_0 means that $x_0 \in A^\circ$.

(e) if $A \subseteq X$ then $Fr(A) = cl(A) \setminus \text{int}(A) = \{x_0 \in X : \text{each nbd of } x_0 \text{ meets both } A \text{ and } X \setminus A\}$.

Remarks let $(X, \mathcal{G})$ be a topological space. let $A \subseteq X$. Then

$\text{int}(A) = \{x_0 \in A : \exists V \text{ nbd of } x_0 \text{ s.t. } V \subseteq A\}$

$cl(A) = \{x_0 \in A : \forall V \text{ nbd of } x_0,\ V \cap A \neq \emptyset\}$

Warning if (X, ρ) is a pseudometric space, $x_0 \in X$, and $r>0$, then $cl(B(x_0, r))$ is not necessarily $B[x_0, r]$. (it may be a proper subset $cl(B(x_0,r)) \subset B[x_0, r]$. eg let $X = \mathbb{Z}$, let $\rho(x,y) = |x-y|$ $\forall x, y \in X$, let $x_0 = 0$, let $r=1$. then $cl(B(x_0, r)) = \{0\}$, but $B[x_0, r] = \{-1, 0, 1\}$

eg let X be any set. Define ρ on $X \times X$ by $\rho(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$. ρ is a metric on X. then $\mathcal{G} = P(X)$ and $\mathcal{F} = P(X)$. $\forall x_0 \in X$, $B(x_0, 1) = \{x_0\}$ and $B[x_0, 1] = X$.
Also, $B(x_0, 1.0001) = X$.

eg let X be any set. let $\mathcal{G} = \{\emptyset, X\}$. then $\mathcal{G}$ is the trivial topology on X and arises from the zero pseudometric on X.

Prop: let (X, ρ) and (Y, σ) be pseudometric spaces. let $f: X \to Y$ and let $x_0 \in X$. then TFAE:

(a) f is continuous at x_0, i.e. $\forall \varepsilon > 0$ $\exists \delta > 0 \ni \forall x \in X$ $\rho(x_0, x) < \delta \Rightarrow \sigma(f(x_0), f(x)) < \varepsilon$.

(b) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f[B(x_0, \delta)] \subseteq B(f(x_0), \varepsilon)$

(c) $\forall$ nhd V of $f(x_0)$ in Y, $\exists$ nhd U of x_0 in X s.t. $f(U) \subseteq V$.

(d) $\forall$ nhd V of $f(x_0)$ in Y, $f^{-1}(V)$ is a nhd of x_0 in X

pf (a) $\Rightarrow$ (b): just rewriting.

(b) $\Rightarrow$ (c): Let V be a nhd of $f(x_0)$ in Y. then V contains a ball $B(f(x_0), \varepsilon)$ for some $\varepsilon > 0$. thus $\exists$ a corresponding $\delta > 0$ s.t. $f[B(x_0, \delta)] \subseteq B(f(x_0), \varepsilon) \subseteq V$. Let $B(x_0, \delta) = U$ and it is shown.

(c) $\Rightarrow$ (d) let V be a nhd of $f(x_0)$ in Y. by (c), $\exists$ U nhd of x_0 in X, $f[U] \subseteq V$. then $f^{-1}[V] = \{x \in X : f(x) \in V\} \supseteq U$ so $f^{-1}(V) \supseteq$ a nhd of x_0 & so is a nhd of x_0, since $x_0 \in int(U) \subseteq U \subseteq f^{-1}(V)$, so $x_0 \in int(f^{-1}(V))$ since $int(f^{-1}(V))$ is union of open subsets, and $int(U)$ is an open subset.

(d) $\Rightarrow$ (a) Let $\varepsilon > 0$. let $V = B(f(x_0), \varepsilon)$. this is a nhd of $f(x_0)$ in Y, so $f^{-1}(B(f(x_0), \varepsilon))$ is a nhd of x_0 in X. thus $\exists \delta > 0$ s.t. $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$
So $\rho(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$, $\forall x \in X$.

Defn Let X and Y be topological spaces. Let $f: X \to Y$.
... V ... of $f(x_0)$ in Y,

Defns Let X and Y be topological spaces. Let $f: X \to Y$.

(a) Let $x_0 \in X$. To say <u>f is cts at x_0</u> means $\forall$ V nhd of $f(x_0)$ in Y, $f^{-1}[V]$ is a nhd of x_0 in X.

(b) Let $A \subseteq X$. To say <u>f is cts on A</u> means $\forall x_0 \in A$, f is cts at x_0.

(c) To say <u>f is cts</u> means f is cts on X.

Remark Let X and Y be top. sp. and let $f: X \to Y$. Then f is cts iff $\forall G \subseteq Y$ open $f^{-1}[G]$ is open in X.

Defn Let X be a top. sp. To say X is <u>connected</u> means $\forall A \subseteq X$, if A is clopen (closed & open) in X, then $A = \emptyset$ or $A = X$.

Theorem Let X be a top. sp. Then TFAE:

(a) X is connected

(b) $\forall$ continuous $f: X \to \mathbb{R}$, $f[X]$ is an interval.

pf (a) $\Rightarrow$ (b). Suppose X is connected. Let $f: X \to \mathbb{R}$ be continuous.

Let $I = f[X]$. Let $a, b \in I$ w/ $a < b$, and let $c \in (a,b)$.

WTS $c \in I$. Suppose not. Then $I \subseteq (-\infty, c) \cup (c, \infty)$, so $X = U \cup V$ where

$U = f^{-1}((-\infty, c))$ and $V = f^{-1}((c, \infty)) = X \setminus U$. Thus U is open, closed, not empty (since $f^{-1}(a) \subseteq U$), not X since $\emptyset \neq V \supseteq f^{-1}(b)$