Lec 12/1

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Definition a function $f(x)$ is analytic at $x=a$ if there is a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ with radius of convergence $R>0$ s.t. $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a|<R$.

Theorem If $f(x)$ is analytic at $x=a$, then there is a unique power series centered at a representing f. Specifically, $C_n = \frac{f^{(n)}(a)}{n!}$.

Proof: Since the derivative of a power series is another power series w/ same radius of convergence, for any $m \geq 0$ we have $f^{(m)}(x) = \sum_{n=m}^{\infty} c_n \frac{m!}{(n-m)!}(x-a)^{n-m}$. plugging in $x=a$, we get $f^{(m)}(a) = C_m m! + 0 + \cdots$ so $C_m = \frac{f^{(m)}(a)}{m!}$ ∎

So if f is analytic at a, then all derivatives of f at a must exist. It's power series expansion is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{valid for} \quad |x-a|<R$$

Called taylor series. when $a=0$, we get:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Called Maclaurin series.

$f^{(n)}(a)$ must be defined for all n (necessary but not sufficient)

Non-Example: $f(x) = x^{\alpha}$ where α positive real, not integer.

not analytic at 0.

$f^{(m)}(x) = \alpha(\alpha-1)\cdots(\alpha-m)x^{\alpha-m}$ where $m > \alpha$

does not exist at 0.

Need additional condition(s) for f to be analytic at a.

There is a formal taylor series at a if $f^{(m)}(a)$ exists $\forall m$.

but

1) R could $= 0$

$$f(x) = \begin{cases} \dfrac{\int_x^0 \frac{e^{1/t}}{t}\,dt}{x e^{1/x}} & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ \dfrac{\int_x^1 \frac{e^{1/t}}{t}\,dt}{x e^{1/x}} & \text{for } x > 0 \end{cases}$$

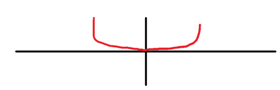
mth derivative exists at 0, $= (m!)^2$

Maclaurin series is $\sum_{m=0}^{\infty} m!\, x^m$ which has $R = 0$.

$$q_x = \lim_{m\to\infty} \left| \frac{(m+1)!\, x^{m+1}}{m!\, x^m} \right| = x \lim_{m\to\infty} m+1 = \infty \text{ if } x \neq 0.$$

2) $R > 0$, but $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ may only hold for $x = a$.

example: $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



$$f^{(n)}(x) = \frac{P_n(x)}{x^{3n}} e^{-1/x^2}, \quad x \neq 0, \quad P_n(x) \text{ poly. } P_n(0) \neq 0.$$

(prove this by induction on n (product rule, chain rule, etc)).

Prove by induction that $f^{(n)}(0) = 0$ for all n.

Assume $f^{(n-1)}(0) = 0$. Then $f^{(n)}(0) = \lim_{h\to 0} \dfrac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{P_{n-1}(h)}{h^{3n-3}} e^{-1/h^2}}{h}$$

$$= \lim_{h\to 0} \left(h^{3n} P_{n-1}(h)\right)\left(\frac{e^{-1/h^2}}{h^{2m}}\right) \quad \text{where } m = 3n-2$$

$\downarrow$ 0 ✓ $\qquad \downarrow$ 0 ?

$$\lim_{u\to\infty} e^{-u} u^m \quad \text{where } u = \tfrac{1}{h^2}$$

$$= \lim_{u\to\infty} \frac{u^m}{e^u} \to 0 \text{ by repeating L'H.}$$

hence Maclaurin series for this is just 0 series.
but clearly the function is nonzero.

Example: Let $\alpha \in \mathbb{R}$, show that $f(x) = (1+x)^\alpha$ is analytic at 0.

Find Maclaurin series: $f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} =$$

So maclaurin series is: $\displaystyle\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{n!\,\Gamma(\alpha-n)} x^n$

(if $\alpha \in \mathbb{N}$, this reduces to $\displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ by binom. thm.)

Can't conclude yet that this works:

① Check radius of convergence $R \neq 0$:

$$q = \lim_{n\to\infty} \left| \frac{\frac{\alpha(\alpha-1)\cdots(\alpha-n)\,x^{n+1}}{(n+1)!}}{\frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))\,x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x(\alpha-n)}{(n+1)} \right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{\alpha-n}{n+1}\right| = |x| \quad \text{so } R = 1.$$

② Show $f(x) =$ Maclaurin series for $|x|<1$. Show that both $f(x) = (1+x)^\alpha$ and maclaurin series both satisfy same differential equation w/ same initial condition.

$$\frac{dy}{dx} = \frac{\alpha y}{1+x} \qquad y = 1 \text{ when } x = 0.$$

Let $\displaystyle y = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(n-1)!} x^{n-1}$$

$$(1+x)\frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(n-1)!} x^n$$

$m = n-1$ $\qquad$ $m = n$

$$= \sum_{m=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-m)}{m!} x^m + \sum_{m=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{(m-1)!} x^m$$

$$= \underset{\substack{\uparrow \\ m=0 \\ \text{term}}}{\alpha} + \sum_{m=1}^{\infty} \left[\frac{\alpha(\alpha-1)\cdots(\alpha-m)}{m!} + \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{(m-1)!} \right] x^m$$

$$\sum^{\infty} \alpha(\alpha-1)\cdots(\alpha-(m-1))$$

$m=0$ term

$$= \alpha + \sum_{m=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{(m-1)!} \left[\frac{\alpha-m}{m} + \frac{m}{m} \right] x^m$$

$$= \alpha + \sum_{m=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{(m-1)!} \cdot \frac{\alpha}{m} \cdot x^m$$

$$= \alpha \left(\sum_{m=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{m!} x^m \right) = \alpha y. \quad \checkmark$$

Let $g(x) = (1+x)^{-\alpha} y$

$$g'(x) = -\alpha(1+x)^{-\alpha-1} y + (1+x)^{-\alpha} \frac{\partial y}{\partial x}$$

$$= -\alpha(1+x)^{-\alpha-1} y + (1+x)^{-\alpha} \left(\frac{\alpha y}{1+x} \right)$$

$$= -\alpha(1+x)^{-\alpha-1} y - (1+x)^{-\alpha-1} \alpha y = 0$$

so $g'(x) = 0$ for $|x| < 1$, $\Rightarrow g(x) = C$, $g(0) = 1$, so $y = (1+x)^{\alpha}$