Lec 2/6

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MLE problem for $N(\mu, \sigma^2)$

↖↑ 2 parameters to estimate.

Want to maximize $L(\mu, \sigma^2)$

In general, to maximize $\overset{\text{smooth}}{\downarrow}$ $f(x,y)$, look for places where $\nabla f = 0$. these are crit. pts.

then if at C.P. $|H| > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ then local max.

↳ Hessian aka Jacobian of ∇f.

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§11.4 estimations of proportions.

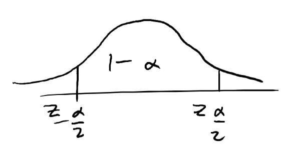
Recall binomial dist: $X \sim Bin(n, \theta)$. X = # successes.

by CLT, $\dfrac{X - n\theta}{\sqrt{n\theta(1-\theta)}} \sim N(0,1)$ as $n \to \infty$

$$X \sim N(n\theta, n\theta(1-\theta)) \quad \text{as } n \to \infty$$

We often care about θ. can construct CI for θ using approx.

$$P\left(z_{\frac{\alpha}{2}} < \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}} < z_{\frac{\alpha}{2}}\right) \simeq 1 - \alpha$$



So a CI for θ is $\dfrac{X}{n} \pm z_{\frac{\alpha}{2}} \sqrt{\dfrac{\theta(1-\theta)}{n}}$

but boundary points contain θ.

so replace θ with $\hat{\theta} = \frac{X}{n}$ (MLE)

11.6 So a $(1-\alpha)\times 100\%$ CI for θ where $X \sim Bin(n,\theta)$, n is known & large, is

$$\frac{X}{n} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{X(n-X)}{n^3}}$$

$$\frac{X}{n} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} \qquad \text{where} \qquad \hat{\theta} = \frac{X}{n}.$$

find a 95% CI with 100 boxes & 10 w/ wicking.

$$\hat{\theta} = 0.1, \qquad Z_{\frac{\alpha}{2}} = 1.96$$

$$0.1 \pm 1.96 \sqrt{\frac{0.1(0.9)}{100}} \qquad \frac{0.3}{10}$$

$$= 0.1 \pm 1.96 \cdot 0.03$$

$$= 0.1 \pm 0.0588$$

$$= (0.0412,\ 0.1588)$$

Exam question: Suppose we want to estimate θ so that we can be 95% confident that we are "off" by no more than 0.04. Determine # of boxes needed.

$$X \pm 1.96 \sqrt{\frac{x(1-x)}{n^3}}$$

Note:
$\theta = 0.5$ is worst case.

$$\frac{1.96\sqrt{x(1-x)}}{n^3} \leq \frac{1.96 \cdot 0.5}{n^3} = \frac{0.98}{n^3}.$$

want $\frac{0.98}{n^3} \leq 0.04 \Rightarrow \left(\frac{0.98}{0.04}\right)^{1/3} \leq n$

$= 600.25$, take $n = 601$

↳ can also use $\hat{\theta} = 0.1$ as in previous example, to get a less conservative n.
↳ use this method unless specified.

§11.5 Estimation of difference between proportions.

$$\hat{\theta}_1 = \frac{X_1}{n_1} \overset{\text{approx}}{\sim} N\left(\theta_1, \frac{\theta_1(1-\theta_1)}{n_1}\right)$$

$$\hat{\theta}_2 = \frac{X_2}{n_2} \overset{\text{approx}}{\sim} N\left(\theta_2, \frac{\theta_2(1-\theta_2)}{n_2}\right)$$

$$\hat{\theta}_1 - \hat{\theta}_2 \overset{\text{approx}}{\sim} N\left(\underbrace{\theta_1 + \theta_2}_{\mu}, \underbrace{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}}_{\sigma^2}\right)$$

Pivot idea; $\frac{\hat{\theta}_1 - \hat{\theta}_2 - \mu}{\sigma} \overset{\text{approx}}{\sim} N(0,1)$.

<u>Thm 11.8</u> $X_1 \sim Bin(n_1, \theta_1)$, $X_2 \sim Bin(n_2, \theta_2)$ and n_1, n_2 large, then

$$\hat{\theta}_1 + \hat{\theta}_2 \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}}$$ is a $(1-\alpha)\times 100\%$ CI for $\theta_1 - \theta_2$.

Use $\hat{\theta}_1$ and $\hat{\theta}_2$ to approximate θ_1 and θ_2 in the error.