Lec 10/21

Monday, October 21, 2019 10:22

<u>Stone - Weierstrass</u>: X cpt Hausdorff, so $C(X)$ a Banach alg.

Let $A \subset C(X)$ be a sub alg s.t. A

- separates pts: $\forall x, y \in X$, $\exists a \in A$ s.t. $a(x) \neq a(y)$,
- is closed under conjugation.

① If A contains a non-vanishing fn, $\overline{A}^{\|\cdot\|_\infty} = C(X)$.

② If every $a \in A$ vanishes at $x_0 \in X$, $\bar{A} = \{ f \in C(X) \mid f(x_0) = 0 \}$.

<u>Step 1</u>: the fn $x \longmapsto |x|$ on $\mathbb{R}$ can be uniformly approximated by a polynomial which vanishes at 0 on any cpt $K \subset \mathbb{R}$.

~~pf~~: We'll show for $R > 0$, $\exists$ seq (p_n) of poly's s.t. $p_n \longrightarrow |\cdot|$ unif on $[-R, R]$ & $p_n(0) = 0$ $\forall n$. Wlog, $R = 1$.

Define $q(t) = 1 - |t|$ on $[-1, 1]$. It suffices to find $\overset{\text{polys}}{q_n} \longrightarrow q$ unif s.t. $q_n(0) = 1$. Observe:

$(\ast)$ q takes values in $[0,1]$ and $(1 - q(t))^2 = t^2$ $\forall t \in [-1, 1]$.

For given $t \in [-1, 1]$, consider $(1 - s)^2 = t^2$.

It has two solutions: $s = 1 \pm |t|$. exactly <u>one</u> solution, $1 - |t|$, lies in $[0, 1]$. Hence $q(t)$ is the <u>unique</u> fn on $[-1, 1]$ satisfying $(\ast)$. Rewrite $(\ast)$ as

$(\ast\ast)$ q takes values in $[0,1]$ and $q(t) = \frac{1}{2}(1 - t^2 + q(t)^2)$.

Define q_n inductively by

- $q_0(t) = 1 \quad \forall\ |t| \leq 1$

- $q_{n+1}(t) = \frac{1}{2}\left(1 - t^2 + q_n(t)^2\right)$

$$\begin{cases} q_1(t) = 1 - \frac{1}{2}t^2 \\ q_0 - q_1 = \frac{1}{2}t^2 \geqslant 0 \end{cases}$$

By induction, $\forall n \geqslant 0$,

- q_n takes values in $[0,1]$

- $q_n(0) = 1$

- $q_n - q_{n+1} = \frac{1}{2}\left(q_{n-1}^2 - q_n^2\right) = \frac{1}{2}\left(q_{n-1} - q_n\right)\left(q_{n-1} + q_n\right) > 0$

So (q_n) is monotone decreasing by construction

Let $\tilde{q}$ be the ptwise limit, which exists.

Observe $\tilde{q}$ takes values in $[0,1]$ and by construction $\tilde{q}$ satisfies $(**)$. So $\tilde{q} = q$ by uniqueness.

As $q_n \searrow q$ on $[-1,1]$, $q_n \longrightarrow q$ uniformly by Dini. $\square$

<u>Step 2</u>: If $A \subset C(X,\mathbb{R})$ is a closed (in ∞-norm) subalgebra, then A is a lattice: i.e. $\forall a,b \in A$, $\max\{a,b\}$ and $\min\{a,b\} \in A$.

Pf: Suppose $a \in A$ and $a \not\equiv 0$. Then $\frac{a}{\|a\|_\infty} : X \longrightarrow [-1,1]$. Let $\varepsilon > 0$. by Step 1, $\exists$ polynomial p on $[-1,1]$ s.t. $p(0) = 0$ and $\big||t| - p(t)\big| < \varepsilon \quad \forall t \in [-1,1]$. Hence

$$\left| \frac{|a(x)|}{\|a\|_\infty} - p\left(\frac{a(x)}{\|a\|_\infty}\right) \right| < \varepsilon \quad \forall x \in X.$$

$$\Rightarrow \quad \left\| \frac{|a|}{\|a\|_\infty} - p\left(\frac{a}{\|a\|_\infty}\right) \right\|_\infty < \varepsilon.$$

Since $p(0) = 0$, $p\left(\frac{a}{\|a\|_\infty}\right) \in \text{span}\{a^n \mid n \in \mathbb{N}\} \subset A$.

Since A is closed & $\varepsilon > 0$ was arbitrary,

$\frac{|a|}{\|a\|_\infty} \in A \rightsquigarrow |a| \in A$.

Now if $a, b \in A$, $\max\{a,b\} = \frac{1}{2}\left[a + b + |a-b|\right]$

$\min\{a,b\} = \frac{1}{2}\left[a + b - |a-b|\right]$ $\square$

<u>Step 3</u>: Suppose $A \subset C(X, \mathbb{R})$ is a real vector space & a lattice. if $f \in C(X,\mathbb{R})$ can be approximated at every $x \neq y \in X$ by some $a_{xy} \in A$, then $f \in \overline{A}^{\|\cdot\|_\infty}$.

~~pf~~: Let $\varepsilon > 0$. For $x \neq y \in A$, pick $a_{xy} \in A$ s.t.

$$|f(x) - a_{xy}(x)| < \varepsilon, \quad |f(y) - a_{xy}(y)| < \varepsilon.$$

Then x, y are both in

$$\mathcal{U}_{xy} := \{z \in X \mid f(z) < a_{xy}(z) + \varepsilon\}$$

$$V_{xy} := \{ z \in X \mid a_{xy}(z) < f(z) + \varepsilon \}$$

Fix $x \in X$. The sets $(\mathcal{U}_{xy})_{y \in X}$ cover X.

Since X cpt, $X \subset \bigcup_{i=1}^{m} \mathcal{U}_{xy_i}$. Set $a_x := \bigvee_{i=1}^{m} a_{xy_i} \in A$ ← sup

By construction, $f(z) < a_x(z) + \varepsilon$ for all $z \in X$.

Also, $a_x(z) < f(z) + \varepsilon$ $\forall z \in \bigcap_{i=1}^{m} V_{xy_i}$ ← call this W_x.

Varying $x \in X$, $(W_x)_{x \in X}$ is an open cover. So $X \subset \bigcup_{j=1}^{n} W_{x_j}$.

Now $a := \bigwedge_{j=1}^{n} a_{x_j}$ satisfies $\|f - a\|_\infty < \varepsilon$. □

<u>Step 4</u>: Suppose $A \subset C(X, \mathbb{R})$ is a subalg which separates points.

① if A contains a nonvanishing fn, $\overline{A} = C(X, \mathbb{R})$

② if every $a \in A$ has a zero, $\exists x_0 \in X$ s.t. $\overline{A} = \{ f \mid f(x_0) = 0 \}$.

Pf Suppose $x \neq y$ in X. Since pt eval is an $\mathbb{R}$-alg hom $A \to \mathbb{R}$,

$A_{xy} = \{ (a(x), a(y)) \mid a \in A \} \subset \mathbb{R}^2$ is a subalgebra.

The only subalgebras of $\mathbb{R}^2$ are:

$\mathbb{R}^2$, $(0,0)$, $\mathbb{R} \times \{0\}$, $\{0\} \times \mathbb{R}$, $\Delta = \{ (x,x) \mid x \in \mathbb{R} \}$.

Since A separates pts, $A_{xy} \neq (0,0)$ or Δ.

<u>Claim</u>: $A = \mathbb{R}^2$ $\forall x \neq y$ except for at most one possible $x_0 \in X$.

<u>Case 1</u>: $A_{xy} = \mathbb{R}^2$ $\forall x \neq y$. [This is the case for ①].

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$\forall f \in C(X,\mathbb{R})$, $\exists a_{xy} \in A$ s.t. $f(x) = a_{xy}(x)$ and $f(y) = a_{xy}(y)$.

by step 2, $\bar{A}$ is a lattice, and by step 3, f

can be unif approx by $\bar{A}$, so $f \in \bar{A}$.

Case 2: $\exists x_0 \in X$ s.t. $A_{x_0 y} = \{0\} \times \mathbb{R}$. Then the argument in case 1 applies $\forall f \in C(X,\mathbb{R})$ s.t. $f(x_0) = 0$.

Step 5: $A \subset C(X,\mathbb{C})$ sep pts & closed under complex conj.

Pf Apply Step 4 to $A_{sa} = \{a \in A \mid a = \bar{a}\}$.

since A closed under complex conj, $\mathrm{Re}\, a, \mathrm{Im}\, a \in A$. So

$A = A_{sa} \oplus i A_{sa}$ and $C(X,\mathbb{C}) = C(X,\mathbb{R}) \oplus i\, C(X,\mathbb{R})$. $\square$