# Lec 8/21

Sunday, August 12, 2018 20:16

Midterms: Sept 18, Oct 10, Nov 20. Final: Dec 13.

<u>Defn.</u> a group $G$ is a set & a $\underbrace{\text{binary operation}}_{\text{group operation / multiplication}}$ $\begin{array}{l} G\times G \longrightarrow G \\ (a,b) \longmapsto ab \end{array}$ and $\underbrace{e\in G}_{\text{unit / neutral / identity}}$ satisfying

i) associative : $(ab)c = a(bc)$ $\forall a,b,c \in G$.

ii) Unit : $ea = ae = a$ $\forall a \in G$

iii) inverse: $\forall a \in G$, $\exists\, a^{-1}$ s.t. $aa^{-1} = a^{-1}a = e$

<u>Ex</u>: i) $G = \mathbb{Z}$, operation is $+$, $e = 0$, inverse of $a$ is $-a$

ii) $G = (0,\infty)$ operation is $\times$, $e = 1$, inverse of $a$ is $\frac{1}{a}$

<u>Non-ex</u>: $G = \mathbb{R}_{>0}$, $a * b = a^b$. $(a^b)^c \neq a^{(b^c)}$.

$G = \mathbb{Z}_{\geq 0}$, $a * b = a + b$ no inverse!

<u>Defn</u>: $G$ is said to be abelian if $a * b = b * a$ $\forall a,b \in G$.

<u>ex of non-abelian group</u>: $G = GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{R}, \\ ad-bc \neq 0 \end{array} \right\}$. operation is matrix multiplication.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Examples</u>: "Symmetries of a structure" "Remark: Symmetries are always associative"

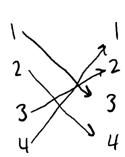
eg: "Structure" = finite set $\{1,\dots,n\}$, "Symmetries" = bijections/permutations $X \xrightarrow{\sigma} X$

associativity is guaranteed since composition of maps is automatic.

identity: $e(i) = i$ $\forall i = 1,\dots,n$. this group is called $S_n$ - Symmetry group on $n$ symbols

Various ways of writing permutations:

$n=4$



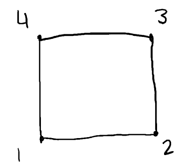
$\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{matrix} \in S_4$

Cyclic notation: $(1324) \in S_4$

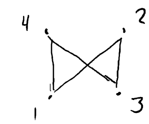
Multiplying: $(123)(34) = (1234)$

eg: $n \geq 3$, define $D_{2n}$ = group of symmetries of an $n$-gon. "Structure" = edges

$n=4$



$D_8 \not\ni (23)$



Definition: $|G|$ = cardinality of $G$ / order of group.

$\sigma \in D_8$ $\quad \sigma(1) \in \underbrace{\{1,2,3,4\}}_{}$ (n options). suppose $\sigma(1) = 3$. then $\sigma(2) \in \{2,4\}$ (2 options)

So $|D_{2n}| = 2n$. → dihedral group.

Examples: "generators & relations"

eg: free group on 2 letters (Paradoxical!). 2 symbols $\alpha, \beta$. (generators) Group is any word in these letters

$F_2 \ni w = \alpha\beta^{-3}\alpha^2$, etc. $\quad \alpha^{m_1}\beta^{m_2}\alpha^{m_3}\cdots$ exponents integers

also $\phi$ = empty word.

(relations)

$*$ = Concatenation. rule: $\alpha^k\alpha^l = \alpha^{k+l}$, $\alpha^0 = \phi$. (same w/ $\beta$).

eg: $w_1 = \alpha^{-1}\beta\alpha^7$ $\quad w_2 = \alpha^{-3}\beta^{-1}$. $\quad w_1w_2 = \alpha^{-1}\beta\alpha^4\beta^{-1}$, $\quad w_2w_1 = \alpha^{-3}\beta^{-1}\alpha^{-1}\beta\alpha^7$

inverse: $(\alpha\beta\alpha^3\beta^{-1})^{-1} = \beta\alpha^{-3}\beta^{-1}\alpha^{-1}$.