Lec 9/21

Wednesday, September 21, 2016 9:13 AM

Theorem: If $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{K}{L}$.

proof: Use results we already proved about limits:

let $h(u) = \frac{1}{u}$ $u \in (-\infty, 0) \cup (0, \infty)$

$\lim_{u \to L} h(u) = \frac{1}{L}$. h is cts at L and is defined on an open interval around L. (either $(-\infty, 0)$ or $(0, \infty)$).

by composition thm for limits, $\lim_{x \to a} h(g(x)) = \frac{1}{L}$

$= \lim_{x \to a} \frac{1}{g(x)}$. now by product thm for limits,

$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{K}{L}$ ■

One-Sided limits:

Definition: we say that $\lim_{x \to a^-} f(x) = L$ (limit from below / left-hand limit

if, $\forall \varepsilon > 0$, we can find $\delta > 0$ so that

$$0 < a - x < \delta \Rightarrow x \in \text{dom}(f) \text{ and } |f(x) - L| < \varepsilon$$

$$\Updownarrow$$

$$x \in (a - \delta, a)$$

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we say that $\lim_{x \to a^+} f(x) = L$ (limit from above / right-hand limit

if $\forall \varepsilon > 0$ we can find $\delta > 0$ so that

$$0 < x - a < \delta \Rightarrow x \in \text{dom}(f) \text{ and } |f(x) - L| < \varepsilon$$

$$\Updownarrow$$

$$x \in (a, a + \delta)$$

Theorem: $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Proof: $\Rightarrow$ given $\varepsilon > 0$ we can find a $\delta > 0$ so that

$0<|x-a|<\delta \Rightarrow x \in dom(f)$ and $|f(x)-L|<\varepsilon$

$\Updownarrow$

$x \in (a-\delta, a) \cup (a, a+\delta)$

$\Leftarrow$ given $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ then given $\varepsilon > 0$

find 2 deltas and pick the minimum. This bounds x so that on both sides of a $f(x)$ is close enough to L. ∎

Theorem: (One-sided localization principle)

(1) if $f(x) = g(x)$ $\forall x \in (b, a)$ and $\lim_{x \to a^-} g(x) = L$ then $\lim_{x \to a^-} f(x) = L$

(2) if $f(x) = g(x)$ $\forall x \in (a, c)$ and $\lim_{x \to a^+} g(x) = L$ then $\lim_{x \to a^+} f(x) = L$

Proof of (1): Let $\varepsilon > 0$. Since $\lim_{x \to a^-} g(x) = L$, we can find $\delta_1 > 0$ s.t.

$x \in (a-\delta_1, a) \Rightarrow x \in dom(g)$ and $|g(x) - L| < \varepsilon$

let $\delta = \min(\delta_1, a-c)$. Then

$x \in (a-\delta, a) \Rightarrow x \in (a-\delta_1, a) \cap (c, a) \subseteq dom(f)$ and $dom(g)$

$\Downarrow$ $\Downarrow$

$|g(x) - L| < \varepsilon \Rightarrow |f(x) - L| < \varepsilon$

Problem 4 on review sheet:

$$f(x) = \frac{x^2 + 2x - 3|1-x^2| + 1}{2|x+1| - |x^2+x|}$$ (where denominator $\neq 0$)

Find: $\lim_{x \to -1} f(x)$ and $\lim_{x \to 2} f(x)$ if they exist.

first find splice points (points where expressions inside $||$ change signs, $=0$).

$|1-x^2| = 0$ when $x=-1$ and $x=1$

$|x+1| = 0$ when $x=-1$

$|x^2+x| = 0$ when $x=0$ and $x=-1$

3 splice points : $x=-1$, $x=0$, $x=1$ so simplify abs. values. over intervals $(-\infty,-1)$, $(-1,0)$, $(1,\infty)$

(skip $(0,1)$ because we don't need it)

	$(-\infty,-1)$	$(-1,0)$	$(1,\infty)$
$\lvert 1-x^2\rvert$	x^2-1	$1-x^2$	x^2-1
$\lvert x+1\rvert$	$-1-x$	$x+1$	$x+1$
$\lvert x^2+x\rvert$	x^2+x	$-x^2-x$	x^2+x

for $x \in (-\infty,-1) \cap dom(f)$,

$$f(x) = \frac{x^2+2x+3(1-x^2)+1}{-2(x+1)-(x^2+x)}$$

$$= \frac{-2x^2+2x+4}{-x^2-3x-2}$$

$$= \frac{2(x^2-x-2)}{x^2+3x+2}$$

$$= \frac{2\cancel{(x+1)}(x-2)}{\cancel{(x+1)}(x+2)}$$

$$= \frac{2(x-2)}{(x+2)}$$

for $x \in (-1,0)$

$$f(x) = \frac{x^2+2x-3(1-x^2)+1}{2(x+1)+(x^2+x)}$$

$$= \frac{4x^2+2x-2}{x^2+3x+2}$$

$$= \frac{2(2x-1)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}}$$

$$= \frac{2(2x-1)}{(x+2)}$$

for $x \in (1,\infty)$

$$f(x) = \frac{x^2+2x+3(1-x^2)+1}{}$$

$$2(x+1) - (x^2 + x)$$

$$= \frac{-2(x-2)(x+1)}{-(x^2 - x - 2)}$$

$$= \frac{2(x-2)\cancel{(x+1)}}{(x-2)\cancel{(x+1)}}$$

$$= \frac{2(x-2)}{(x-2)}$$

$$f(x) = \frac{2(x-2)}{(x+2)} \quad \text{for } x \in (-2, -1)$$

so by left side localization principle,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{2(x-2)}{(x+2)} = \frac{2(-1-2)}{(-1+2)} = -6$$

$$f(x) = \frac{2(2x-1)}{(x+2)} \quad \forall x \in (-1, 0)$$

so by right side localization principle,

$$\lim_{x \to -1^+} f(x) = \lim_{x-1^+} \frac{2(2x-1)}{(x+2)} = \frac{2(-2-1)}{(-1+2)} = -6$$

so since $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x) = 6$, $\lim\limits_{x \to -1} f(x) = -6$.

$$f(x) = \frac{2(x-2)}{(x-2)} \quad \forall x \in (1,2) \cup (2, \infty)$$

$$= 2 \quad \forall x \in (1,2) \cup (2, \infty)$$

so $\lim\limits_{x \to 2} f(x) = 2$ by 2-sided localization principle.