Lec 11/8

Tuesday, November 8, 2016 9:15 AM

for $x > 0$

$$e^x \geq \sum_{j=0}^{n} \frac{x^j}{j!}$$

Proposition for any polynomial $p(x) = \sum_{i=0}^{n} a_n x^n$

$$\lim_{x \to \infty} \frac{e^x}{|p(x)|} = \infty$$

$e^x \geq \frac{x^{n+1}}{(n+1)!}$ for $x > 0$. $\quad \frac{e^x}{|p(x)|} > \frac{x^{n+1}}{|p(x)|} \longrightarrow \infty$

Theorem: $e = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!} \overset{\text{def}}{=} \sum_{j=0}^{\infty} \frac{1}{j!}$

Proof: $(1 + \frac{1}{n})^n \leq e$

$$\begin{aligned}
(1+\tfrac{1}{n})^n &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \\
&= \sum_{j=0}^{n} \frac{n!}{(n-j)!\, j!\, n^j} \\
&= \sum_{j=0}^{n} \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} \frac{1}{n^j} \\
&= \sum_{j=0}^{n} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-j+1}{n}\right)\frac{1}{j!} \\
&\leq \sum_{j=0}^{n} \frac{1}{j!}
\end{aligned}$$

by above

so $(1+\frac{1}{n})^n \leq \sum_{j=0}^{n} \frac{1}{j!} \leq e$

$\downarrow \qquad \downarrow \qquad \downarrow$

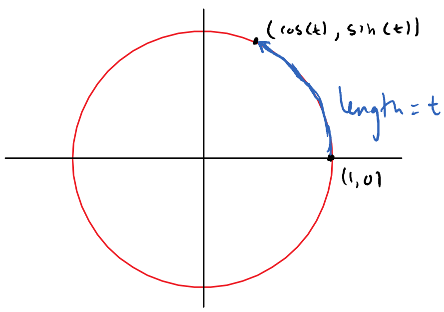
$e \qquad e \qquad e$

by sq. thm.

Ch 15: Trigonometric Functions

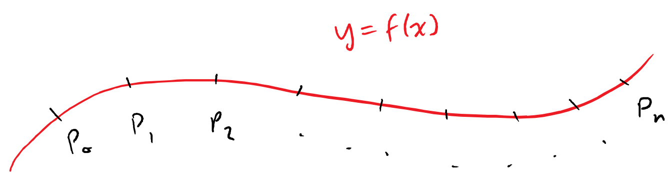
Let a particle move ccw around the unit circle at unit speed (starting at $(1,0)$).

Then it's position at time t is $(\cos(t), \sin(t))$.



Remark: This notion of stuff makes sense for arbitrary smooth curves.

Length of graph $y = f(x)$ $\quad a \le x \le b$.



$P_i = (x_i, f(x_i))$

$\{x_i\}$ a partition of $[a,b]$.

$\Lambda = \text{length} \approx \sum_{i=1}^{n} \overline{P_{i-1}P_i}$ and

$$\overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$
$$= \sqrt{(x_i - x_{i-1})^2}\sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}$$
$$= (x_i - x_{i-1})\sqrt{1 + f'(c_i)^2} \quad \text{for some } \underset{\in}{c_i}\ (x_{i-1}, x_i)$$

$$L(g,P) \le \sum_{i=1}^{n} \overline{P_{i-1}P_i} \le U(g,P$$

where $g(x) = \sqrt{1 + f'(x)^2}$

$$\left| \int_a^b \sqrt{1+f'(x)^2}\, dx - \sum_{i=1}^{n} \overline{P_{i-1}P_i} \right| \le U(g,P) - L(g,P) < \varepsilon$$

Back to circle: $x = \pm\sqrt{1-y^2}$

$$t = \text{length of } \overline{PQ} = \int_0^{\sin(t)} \sqrt{1 + \frac{d}{dy}\left(\pm\sqrt{1-y^2}\right)^2}\, dy$$

$$= \int_0^{\sin(t)} \sqrt{1 + \frac{\pm y}{\sqrt{1-y^2}}^2}\, dy$$

$$= \int^{\sin(t)} \sqrt{\frac{1-y^2+y^2}{1-y^2}}\, dy$$

$$= \int_0^{\sin(t)} \sqrt{\frac{1-y^2+y^2}{1-y^2}}\, dy$$

$$= \int_0^{\sin(t)} \sqrt{\frac{1}{1-y^2}}\, dy$$

$$t = \int_0^{\sin(t)} \frac{1}{\sqrt{1-y^2}}\, dy$$

This is similar to how we defined $\exp(t)$:

$$t = \int_1^{\exp(t)} \frac{1}{y}\, dy$$

Definition Let $\Lambda(w) = \int_0^w \frac{1}{\sqrt{1-y^2}}\, dy$ defined on $(-1, 1)$

not defined at ± 1.

<u>Step 1</u> want to extend the domain of Λ to $[-1, 1]$

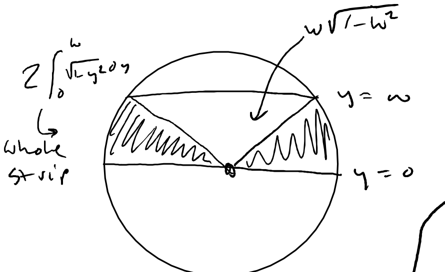
$$\int_0^w \frac{1}{\sqrt{1-y^2}}\, dy = \int_0^w \frac{1-y^2+y^2}{\sqrt{1-y^2}}\, dy \qquad u \leftarrow y \quad \frac{y}{\sqrt{1+y^2}} \rightarrow dv$$

$$= \int_0^w \sqrt{1-y^2}\, dy + \underbrace{\int_0^w \frac{y^2}{\sqrt{1-y^2}}\, dy}_{\text{integrate by parts.}}$$

$u = y \qquad v = -\sqrt{1-y^2}$

$du = dy \qquad dv = \frac{y}{\sqrt{1-y^2}}$

$$= \int_0^w \sqrt{1-y^2}\, dy + \left[y\left(-\sqrt{1-y^2}\right)\right]_0^w - \int_0^w \left(-\sqrt{1-y^2}\right) dy$$



$$= 2\int_0^w \sqrt{1-y^2}\, dy - w\sqrt{1-w^2}$$

$\Lambda(w)$:

which makes sense for $w = \pm 1$.

let $u = -y$.

so $\Lambda(1) = \frac{\pi}{2} \qquad \Lambda(-1) = \frac{-\pi}{2}$ since $\Lambda(-w) = -\Lambda(w)$.

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$\Lambda(w)$: so $\Lambda(1) = \frac{\pi}{2}$ (definition) $\Lambda(-1) = \frac{-\pi}{2}$ since $\Lambda(-w) = -\Lambda(w)$.

for $w \in (-1, 1)$, $\Lambda'(w) = \frac{1}{\sqrt{1-w^2}} > 0$.

Hence $\Lambda : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is increasing and 1-1 and onto.

initial. **Definition** $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] := \Lambda^{-1}$

In other words, if $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $w \in [-1, 1]$

Then $w = \sin(t) \Leftrightarrow t = \Lambda(w) = 2\int_0^w \sqrt{1-t^2} - w\sqrt{1-w^2}$

for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have $\sin'(t) = (\Lambda^{-1})'(t) = \frac{1}{\Lambda'(\Lambda^{-1}(t))} = \frac{1}{\Lambda'(\sin(t))}$

$$= \frac{1}{\frac{1}{\sqrt{1-\sin(t)^2}}} = \sqrt{1-\sin(t)^2} =: \cos(t)$$

Definition: $\cos : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ by $\cos(t) = \sqrt{1-\sin(t)^2}$