Lec 11/30

Wednesday, November 30, 2016 9:08 AM

Theorem: If $\{f_n\}$ are integrable over $[a,b]$ and $\{M_n\}$ s.t.

1) $|f_n(x)| \le M_n$ for $x \in [a,b]$
2) $\lim_{n\to\infty} M_n = 0$

Then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = 0 = \int_a^b (\lim_{n\to\infty} f_n(x))\,dx$

Remark: Also true if $b < a$.

Theorem1 If $\sum_{n=1}^{\infty} c_n(x-a)^n$ has nonzero radius of convergence R and $|x-a| < R$ then

$$\int_a^x \sum_{n=0}^{\infty} c_n(t-a)^n\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(t-a)^{n+1}$$

First need another theorem:

Theorem2 the function defined by a power series is continuous everywhere on its interval of convergence.

(Proof postponed)

Proof of first thm: $\sum_{j=0}^{\infty} c_j(t-a)^j = \sum_{j=0}^{n} c_j(t-a)^j + R_n(t)$, where $R_n(t) = \sum_{j=n+1}^{\infty} c_j(t-a)^n$

$$\int_a^x \sum_{j=0}^{\infty} c_j(t-a)^j\,dt = \sum_{j=0}^{n} \int_a^x c_j(t-a)^j\,dt + \int_a^x R_n(t)\,dt$$

$$= \sum_{j=0}^{n} \frac{c_j}{j+1}(t-a)^{j+1} + \int_a^x R_n(t)\,dt$$

$\sum_{j=1}^{\infty} c_j(t-a)^j = \lim_{n\to\infty}($ " " " $)$, So this thm amounts to showing

that $\lim_{n\to\infty} \int_a^x R_n(t)\,dt = 0.$

Applying Thm 1 w/ $\{f_n\} = \{R_n\}$, we have

$|R_n(t)| \le \sum_{j=n+1}^{\infty} |c_j||t-a|^j$ (triangle)

$$\le \sum_{j=n+1}^{\infty} |c_j||x-a|^j = M_n$$

$\{M_n\}$ satisfies (i) in thm 1 by def.

And $\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{j=1}^{\infty} |c_j||x-a|^j = 0$ since $\sum_{j=0}^{\infty} c_j(x-a)^j$

converges absolutely since $|x-a| < R \Rightarrow q_x < 1 \Rightarrow$ A.C.

So by thm 1, $\lim_{n\to\infty} \int_a^x R_n(t)\,dt = 0.$ ∎

theorem 3 If $\sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$ has radius of convergence $R>0$, and $|x-a|<R$ then

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Proof: Let $g(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$. We have shown that $g(x)$ has same R.C. as $f(x)$.

So by Thm 2, $\int_a^x \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}\,dx = \sum_{n=1}^{\infty} \frac{nc_n}{n-1+1}(x-a)^{n-1+1} = \sum_{n=1}^{\infty} c_n(x-a)^n = f(x) - c_0$

by FTC, $f'(x) = g(x)$ ∎

Examples:

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \quad \text{for } |t|<1. \qquad \text{(geometric series)}.$$

$$\sum_{n=0}^{\infty} \underbrace{(-t)^n}_{=(-1)^n(t)^n} = \frac{1}{1+t} \quad \text{for } |t|<1.$$

$|x|<1$, $\int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\,dt = \int_0^x \frac{1}{1+t}\,dt$

$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(1+x)$. reindexing: $\sum_{j=1}^{\infty} (-1)^{j-1}\frac{x^j}{j} = \ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Interval of convergence: $(-1, 1]$.

thm 2 not good enough for $x=1$.

Theorem (Abel) A power series is continuous over its interval of convergence.

So $\ln(2) = \lim_{x\to 1^-} \ln(1+x) = \lim_{x\to 1^-} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{j} \overset{\text{abel.}}{=} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$\ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x) \qquad x \in (-1,1).$

and $\frac{1-x}{1+x}$ can be any positive real as x varies.

$$\frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x.$$

$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ for $|t|<1$. Replace t by $-t^2$

$$\sum_{n=0}^{\infty} (-t)^{2n} = \frac{1}{1+t^2} \qquad |t|<1$$

$$\int_0^x \sum_{n=0}^{\infty} (-t^2)^n \, dt = \arctan(x)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } x \in (-1,1).$$

int. of convergence : $(-1, 1]$. (slow convergence)

So by abel, $\arctan(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

better: $\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$ ratio test: interval of convergence $= \mathbb{R}$.

$= 0$

$f'(x) = f(x)$. $f(0) = 1$, so $f(x) = e^x$

$g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. $g''(x) + g(x) = 0$, $g(0) = 0$, $g'(0) = 1 \Rightarrow g(x) = \sin(x)$.

$\cos(x) = g'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. intervals of convergence: $\mathbb{R}$.