Lec 12/5

Monday, December 5, 2016 9:16 AM

nth order Taylor Polynomial at a:

$$P_{n,a,f}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j$$

Advantages:

1) applies to not-necessarily analytic functions

2) Generalizes easily to functions of several vars:

$$P_{n,a,f}(\vec{x}) = \sum_{j=0}^{n}\left(\frac{1}{j!}\sum_{i_1,i_2,\dots,i_j=1}^{k} \frac{\partial^j f}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}(\vec{a})(x_{i_1}-a)(x_{i_2}-a)\cdots(x_{i_n}-a)\right)$$

taylor series $\lim_{n\to\infty} P_{n,a,f}(\vec{x})$ difficult.

Theorem If $f^{(n)}(a)$ exists, then $\lim_{x\to a} \frac{f(x)-P_{n,a,f}(x)}{(x-a)^n} = 0$

Proof: Induction on n:

base case $n=1$: $\lim_{x\to a} \frac{f(x)-(f(a)+f'(a)(x-a))}{(x-a)}$

$$= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} - f'(a)\right]$$

$$= f'(a) - f'(a) \quad \text{by def. of derivative}$$

induction: $n \to n+1$. assume $\lim_{x\to a} \frac{g(x)-P_{n,a,g}(x)}{(x-a)^n} = 0 \;\forall g$ which have $P_{n,a,g}(x)$ defined

Assume $f^{(n+1)}(a)$ is defined.

$$\lim_{x\to a} \frac{f(x)-P_{n+1,a,f}(x)}{(x-a)^{n+1}} \qquad \frac{0}{0} \text{ limit}$$

$$= \lim_{x \to a} \frac{f'(x) - P'_{n+1,a,f}(x)}{(n+1)(x-a)^n}$$

observe that $P'_{n+1,a,f} = P_{n,a,f'}$

$$= \frac{1}{n+1} \lim_{x \to a} \frac{f'(x) - P_{n,a,f'}(x)}{(x-a)^n}$$

$$= \frac{1}{n+1} \cdot 0 = 0$$

So Theorem$(n) \Rightarrow$ Theorem$(n+1)$ so Theorem holds $\forall n$ $\boxtimes$

btw: $$P'_{n,a,f}(x) = \sum_{j=1}^{n} \frac{f^{(j)}(a)}{j!} j (x-a)^{j-1}$$

re index, cancel,

$$= \sum_{j=0}^{n-1} \frac{(f')^{(j)}(a)}{j} (x-a)^j$$

Recall 2nd derivative test for local maxima/minima:

If c is a critical pt of f and $f''(c)$ is defined, then

1) $f''(c) > 0 \Rightarrow f$ has local min

2) $f''(c) < 0 \Rightarrow f$ has local max

3) $f''(c) = 0 \Rightarrow$ inconclusive

Higher order derivative test for local maxima/minima.

If c is a crit. pt. of f and $0 = f'(c) = f''(c) = \cdots = f^{(n-1)}(c)$

but $f^{(n)}(c) \neq 0$.

Then

1) if n is odd, there is neither a max nor a min at c.

2) if n is even, then

a) $f^{(n)}(c) > 0 \Rightarrow$ local min

b) $f^{(n)}(c) < 0 \Rightarrow$ local max

<u>Proof:</u> $P_{n,c,f}(x) = f(c) + \underset{0}{\underset{\shortparallel}{f'(c)}}(x-c) + \underset{0}{\underset{\shortparallel}{\cdots}} + \frac{f^{(n)}(c)}{n!}(x-a)^n$

$= f(c) + \frac{f^{(n)}(c)}{n!}(x-c)^n$ → odd power ⇒ change signs

So conclusion holds for taylor polynomial.

by thm above, $\lim_{x\to c} \frac{f(x) - \left(f(c) + \frac{f^{(n)}(c)}{n!}(x-c)^n\right)}{(x-c)^n} = 0$

$= \lim_{x\to c} \left[\frac{f(x)-f(c)}{(x-c)^n}\right] - \frac{f^{(n)}(c)}{n!} = 0$

⇒ in some open interval (a,b) containing c,

$\frac{f(x)-f(c)}{(x-c)^n}$ has the same sign as $f^{(n)}(c)$

So looking through all cases, theorem holds:

1) n odd ⇒ $(x-c)^n$ sign change ⇒ $f(x)-f(c)$ sign change ⇒ no min/max.

2) n even ⇒ $(x-c)^n$ positive ⇒ $f(x) - f(c)$ has same sign as $f^{(n)}(c)$ in open interval ⇒ Local min/max depending on sign. ◻

If $f^{(n)}(a)$ exists, denote $R_{n,a,f}(x) = f(x) - P_{n,a,f}(x)$

$R_{n,a,f}(x)$ is nth order remainder for taylor approximation.

Theorem if $f^{(n+1)}(x)$ is defined on an open interval $(b,c) \in a$, then $\forall x \in (b,c)$, $\exists x_0$ between x and a s.t. $R_{n,a,f}(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-c)^{n+1}$
(generalized MVT)

Proof: induction on n:

B.C. $n=0$: $R_{0,a,f}(x) = f(x) - \overset{P_{0,a,f}(x)}{\overset{\|}{f(a)}}$

$$= f'(x_0)(x-a) \quad \text{for some } x_0 \text{ btwn } x \text{ and } a.$$

$$= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-a)^{n+1}$$

induction! Assume for n:

$$\frac{R_{n+1,a,f}(x)}{\frac{(x-a)^{n+1}}{(n+1)!}} = \frac{R_{n+1,a,f}(x) - \overset{0}{R_{n+1,a,f}(a)}}{\frac{(x-a)^{n+1}}{(n+1)!} - \underbrace{\frac{(a-a)^{n+1}}{(n+1)!}}_{0}}$$

by Cauchy MVT
$$= \frac{R_{n+1,a,f}{}'(x_1)}{\frac{(x_1-a)^{n+1}}{n!}} \quad \text{for some } x_1 \text{ btwn. } x \text{ and } a.$$

$$= \frac{R_{n,a,f'}(x_1)}{\frac{(x_1-a)^n}{n!}}$$

apply ind. hyp. for $g = f'$
$$= (f')^{(n)}(x_0) = f^{(n+1)}(x_0) \quad \text{for some } x_0 \text{ btwn } x_1 \text{ and } a. \Downarrow x \text{ and } a$$

so $R_{n+1,a,f}(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-a)^{n+1}$ for some x_0 btwn x and a.

this completes the proof. ∎

Theorem if $f^{(n+1)}(x)$ is defined & integrable over some open interval $(b,c) \ni a$, then $R_{n,a,f}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$

$(b,c) \ni a$, then $R_{n,a,f}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$

Proof: by induction:

B.C. $n=0$ FTC

Ind. IBP