Lec 4/2

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eg define $x : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $x(u^1, u^2) = (u^1, u^2, u^1 u^2)$.

x is C^∞. $x_1 = (1, 0, u^2)$, $x_2 = (0, 1, u^1)$.

$x_1 \times x_2 = (-u^2, -u^1, 1) \neq 0$ so x is an immersion.

$$(g_{ij}) = (\langle x_i | x_j \rangle) = \begin{pmatrix} 1 + (u^2)^2 & u^1 u^2 \\ u^1 u^2 & 1 + (u^1)^2 \end{pmatrix}.$$

$$g = \det(g_{ij}) = (u^1)^2 + (u^2)^2 + 1 = |x_1 \times x_2|^2$$

$$n = \frac{x_1 \times x_2}{\sqrt{g}} = \frac{(-u^1, -u^2, 1)}{\sqrt{(u^2)^2 + (u^1)^2 + 1}}$$

$$x_{11} = x_{22} = (0,0,0), \quad x_{12} = x_{21} = (0,0,1).$$

$$(L_{ij}) = (\langle x_{ij} | n \rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big/ \sqrt{(u^2)^2 + (u^1)^2 + 1}$$

Let $M = \{ x(u^1, u^2) : (u^1, u^2) \in \mathbb{R}^2 \}$. M is a C^∞ surface in $\mathbb{R}^3$ and x is a C^∞ coord. patch on M.

Consider a unit-speed curve on M:

$$s \longmapsto \gamma(s) = x(\gamma^1(s), \gamma^2(s))$$

with $\gamma(0) = (0,0,0)$. Let $(a^1, a^2) = \left(\frac{d\gamma^1}{ds}, \frac{d\gamma^2}{ds} \right) \Big|_{s=0}$

$$\frac{d\gamma}{ds}\Big|_{s=0} = (a^1, a^2, 0) = T(0) \quad \text{hence } (a^1)^2 + (a^2)^2 = 1.$$

$K_n(0) = 2a^1 a^2$. $K_1(0,0) = 1$, $K_2(0,0) = -1$, so

$K(0,0) = -1$.

max & min principal curvatures.

gaussian curvature, product of principal curvatures.

The mean curvature is

$$H(0,0) = \frac{1}{2}\big(\underbrace{1 + (-1)}_{\text{sum of principal curvatures.}}\big) = 0$$

Now let's find Γ_{ij}^{k}. Reminder $\Gamma_{ij}^{k} = \sum_{\ell} \langle x_{ij} \mid x_\ell \rangle g^{\ell k}$

notation change

$$\begin{pmatrix} u = u^1 \\ v = u^2 \end{pmatrix}$$

$$\text{Now } (g^{\ell k}) = (g_{ij})^{-1} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} = \frac{1}{\sqrt{v^2+u^2+1}}\begin{pmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{pmatrix}$$

$$\text{Thus } \Gamma_{ij}^{k} = \langle x_{ij} \mid x_1 \rangle g^{1k} + \langle x_{ij} \mid x_2 \rangle g^{2k}.$$

Now $\Gamma_{11}^{k} = \Gamma_{22}^{k} = 0$ since $x_{11} = x_{22} = (0,0,0)$.

Also $\Gamma_{12}^{k} = \Gamma_{21}^{k}$ since $x_{12} = x_{21}$, always

it remains to find Γ_{12}^{1} and Γ_{12}^{2}.

$$\begin{aligned}
\Gamma_{12}^{1} &= \langle x_{12} \mid x_1 \rangle g^{11} + \langle x_{12} \mid x_2 \rangle g^{21} \\
&= \langle (0,0,1), (1,0,v) \rangle g^{11} + \langle (0,0,1), (0,1,u) \rangle g^{21} \\
&= v\, g^{11} + u\, g^{21} \\
&= \frac{v + vu^2}{\sqrt{v^2+u^2+1}} - \frac{u^2 v}{\sqrt{v^2+u^2+1}} = \frac{v}{\sqrt{v^2+u^2+1}}
\end{aligned}$$

$$\Gamma_{12}^{2} = v\, g^{12} + u\, g^{22} = \frac{u}{\sqrt{v^2+u^2+1}}$$

<u>Propn 43</u> $\Gamma_{ij}^{k} = \sum_{\ell} g^{k\ell} [ij, \ell]$

where $\underbrace{[ij, \ell]}_{} = \frac{1}{2}\left(\frac{\partial}{\partial u^i} g_{j\ell} + \frac{\partial}{\partial u^j} g_{\ell i} - \frac{\partial}{\partial u^\ell} g_{ij}\right)$

Christoffel symbol of the first kind. Also denoted $\Gamma_{\ell i j}$

hence Γ_{ij}^{k} are intrinsic.

Proof (Gauss)

$$2[ij,\ell] = \frac{\partial}{\partial u^i}\langle x_j \mid x_\ell\rangle + \frac{\partial}{\partial u^j}\langle x_\ell \mid x_i\rangle - \frac{\partial}{\partial u^\ell}\langle x_i , x_j\rangle$$

$$= \Big(\langle x_{ji} \mid x_\ell\rangle + \langle x_j \mid x_{\ell i}\rangle + \langle x_{\ell j} \mid x_i\rangle + \langle x_\ell \mid x_{ij}\rangle$$

$$- \langle x_{i\ell} \mid x_j\rangle - \langle x_i \mid x_{j\ell}\rangle\Big)$$

$$= \langle x_{ij} \mid x_\ell\rangle + \langle x_{\ell i} \mid x_j\rangle + \langle x_{j\ell} \mid x_i\rangle + \langle x_{ij} \mid x_\ell\rangle$$

$$- \langle x_{\ell i} \mid x_j\rangle - \langle x_{j\ell} \mid x_i\rangle$$

$$= 2\langle x_{ij} \mid x_\ell\rangle$$

So $[ij,\ell] = \langle x_{ij} \mid x_\ell\rangle$.

Thus $\Gamma_{ij}^{k} = \sum_\ell \langle x_{ij} \mid x_\ell\rangle g^{\ell k} = \sum_\ell g^{k\ell}[ij \mid \ell]$ $\square$

Note for calculation, it's easier to use extrinsic defn.

Propn 4.4 The geodesic curvature of a surface curve is intrinsic.

Pf Let $\varepsilon_{ij} = \langle x_i \times x_j \mid n\rangle$. Then $\varepsilon_{11} = 0 = \varepsilon_{22}$, and $\varepsilon_{12} = \sqrt{g} = -\varepsilon_{21}$.

thus ε_{ij} is intrinsic.

Now $K_g = \langle S \mid K_g S\rangle = \langle n \times T \mid K_g S\rangle = \langle T \times K_g S \mid n\rangle$

$$= \left\langle \left(\sum_\ell x_\ell \frac{d\gamma^\ell}{ds}\right) \times \left[\sum_k \left(\frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^{k} \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}\right) x_k\right] \,\middle|\, n \right\rangle$$

$$= \sum_{l,k} \left[\frac{d\gamma^l}{ds} \left(\frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} \right) \right] \varepsilon_{lk} \quad ,$$

which is intrinsic. $\square$

A convenient extrinsic formula for κ_g

$\kappa_g = \kappa \cos\alpha$ where α is the angle between n and B

unit normal to M $\downarrow$ n; Binormal to γ $\downarrow$ B

Pf $\kappa_g = \langle S \mid \kappa N \rangle = \langle S \mid T' \rangle = \langle n \times T \mid T' \rangle = \langle n \mid T \times T' \rangle$

$= \kappa \langle n \mid T \times N \rangle = \kappa \langle n \mid B \rangle = \kappa \cos\alpha$.

4-5 Geodesics, 4-6 Parallelism

Propn Let γ be a C^2 unit speed curve on M. Then TFAE:

(a) γ is a geodesic

(b) $\kappa_g \equiv 0$

(c) $\kappa = |\kappa_n|$

(d) T' is perpendicular to M

(e) The tangential component of T' is 0

(f) $\langle n \mid T \times T' \rangle = 0$ (remember $n \times T = S$)

(g) $\displaystyle \frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} = 0$ for each k.

Eg geodesics on a sphere are exactly the great circles.