$S \otimes R[x] \cong S[x]$ as S-algebras.

S - commutative & unital, same for R.

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ as $\mathbb{C}$-algebras.

Flat Modules: K is flat if the functor $\mathrm{Tor}_0(\cdot, K)$ which maps $M \longmapsto M \otimes K$ and $\varphi \longmapsto \varphi \otimes 1_K$ is exact: That is, $\forall$ short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

The sequence

$$0 \longrightarrow A \otimes K \longrightarrow B \otimes K \longrightarrow C \otimes K \longrightarrow 0$$

is exact. This is so iff $\forall$ exact

$$0 \longrightarrow N \longrightarrow M,$$

the sequence

$$0 \longrightarrow N \otimes K \longrightarrow M \otimes K$$

is exact.

We proved: if R is an integral domain, Then any flat module is torsion-free.

Criterion if $R = \mathrm{PID}$.

Eg: K - torsion free but not flat.

Let $R = F[x, y]$, $I = (x, y)$.

field.

$0 \longrightarrow I \longrightarrow R$ is exact, but

$$\begin{tikzcd} 0 \arrow[r] & I \otimes I \arrow[r] \arrow[dr] & R \otimes I \cong I \\ & & I^2 \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

is not exact

because $\underbrace{x \otimes y - y \otimes x}_{w} \longmapsto 0$

Let $V = I/I^2 = F^2$ with basis $\{x, y\}$.

V is an F-vector space.

$V \otimes V$ is a factor of $I \otimes I$:

we have surj maps

$I \otimes I \twoheadrightarrow I \otimes V \twoheadrightarrow V \otimes V$.

Basis in $V \otimes V$ is $\{x \otimes x, y \otimes x, x \otimes y, y \otimes y\}$.

image of w in $V \otimes V$ is nonzero, so $w \neq 0$.

<u>Criterion of flatness</u>: K is flat iff $\forall$ (finitely generated) ideal

$I \subseteq R$, the homomorphism

$$\begin{aligned} I \otimes K &\longrightarrow R \otimes K \cong K \\ a \otimes u &\longmapsto au \end{aligned}$$ is injective.

<u>Proof</u>: Assume K is s.t. the map above is always injective.

① It suffices to prove that $\forall$ finitely generated N, M with injection $\varphi : N \longrightarrow M$, the homomorphism $\varphi \otimes 1_K : N \otimes K \to M \otimes K$ is injective.

pf (a) Let N, M be general modules. Let $\varphi : N \to M$ be injection. Let $w \in N \otimes K$, $w = \sum_{i=1}^{n} u_i \otimes v_i \neq 0$.

we need to check that $\sum_{i=1}^{n} \varphi(u_i) \otimes v_i \neq 0$.

Let $N' = R \cdot \{u_1, \dots, u_n\} \subseteq N$. Then if we know

$$\begin{tikzcd} N' \otimes K \arrow[r] & N \otimes K \arrow[r] & M \otimes K \\ w \arrow[r, mapsto] \arrow[rr, bend right, "(1)"'] & w \arrow[r, mapsto, "(2)"] & \varphi \otimes 1_K (w) \end{tikzcd}$$

(1) is injective, then (2) is injective, so we can deal with N' instead

(b) to prove that $\varphi \otimes 1_K (w) \neq 0$ in $M \otimes K$, we can deal with finitely generated submodule of M.

if $\varphi \otimes 1_K (w) = 0$ in $M \otimes K$ then $\varphi \otimes 1_K (w) = 0$ in $M' \otimes K$ where M' is finitely generated.

Lets say "K is flat for M" if $\forall$ injection $N \xrightarrow{\varphi} M$, $N \otimes K \xrightarrow{\varphi \otimes 1_K} M \otimes K$ is injective.

We are given that K is flat for R.

We need to show that K is flat $\forall$ finitely gen'd module.

② if K is flat for M_1 & M_2 then K is flat for $M = M_1 \oplus M_2$.

pf: Let $N \to M$ be injective. consider the diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

exact

exact

identify N with a submodule of M.

so $N_1 = N \cap M_1$

$N_2 = N \bmod M_1$

$= (N+M_1)/M_1 \subseteq M_2$.

diagram is commutative.

multiply by K:

mutiply by K:

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & \\
& N_1 \otimes K \arrow[r] \arrow[d, "\alpha"] & N \otimes K \arrow[r] \arrow[d, "\beta"] & N_2 \otimes K \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M_1 \otimes K \arrow[r, "\varphi"'] & M \otimes K \arrow[r] & M_2 \otimes K \arrow[r] & 0
\end{tikzcd}$$

$M \otimes K = (M_1 \otimes K) \oplus (M_2 \otimes K)$

injectivity of α, γ, φ is sufficient for β to be injective.

So if K is flat for R, K is flat for R^n $\forall n$.

③ if K is flat for M and $M_2 = M/M_1$ then K is flat for M_2.

(Note: any finitely generated module is a quotient of R^n so K is flat).

Pf

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & 0 \\
\text{let} & 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r, "\pi"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

be exact

where $N = \pi^{-1}(N_2) \leq M$.

Then

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_1 \otimes K \arrow[r] \arrow[d, "\alpha"] & N \otimes K \arrow[r] \arrow[d, "\beta"] & N_2 \otimes K \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M_1 \otimes K \arrow[r] \arrow[d] & M \otimes K \arrow[r, "\psi"] & M_2 \otimes K \arrow[r] & 0 \\
& 0 & & &
\end{tikzcd}$$

if α, ψ surj, β - inj $\Longrightarrow$ γ injective. $\square$

<u>Theorem</u>: if R is integral domain, F is it's field of fractions, then F is a flat R-module.

<u>Proof</u>: Let $0 \longrightarrow N \xrightarrow{\varphi} M$ be exact. Any element of $N \otimes F$ has form $u \otimes \frac{1}{d}$ for some $d \in R$.

if $\varphi \otimes 1_F (u \otimes \frac{1}{d}) = \varphi(u) \otimes \frac{1}{d} = 0$, then $\varphi(u) \otimes 1 = 0$,

so $\varphi(u)$ is a torsion element of M.

So $\exists\, c \neq 0$ s.t. $c\varphi(u) = \varphi(cu) = 0$, so $cu = 0$ since φ is inj, so $u \otimes \frac{1}{d} = cu \otimes \frac{1}{cd} = 0$.