Lec 9/27

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Useful Reformulations of Completeness axiom

(A, B) Dedekind cut

(1) $A \neq \emptyset \neq B$

(2) $\mathbb{R} = A \cup B$

(3) $a \in A,\ b \in B \Rightarrow a < b$

Completeness axiom: if (A, B) is a Dedekind cut of $\mathbb{R}$, either
A has a greatest element or $(-\infty, c] \cup (c, \infty)$
B has a least element. $(-\infty, c) \cup [c, \infty)$

Least upper bound property

If a nonempty set is bounded above, then it has a least upper bound.

- x is an upper bound for S if, $\forall s \in S$, $s \leq x$.
- S is bounded above if it has at least one upper bound.
- x is the least upper bound if
  - (a) x is an upper bound for S
  - (b) $\forall y < x$, y is not an upper bound for S
    $\Leftrightarrow \exists s \in S \ni s > y$

Remark: If S has an upper bound x and $x \in S$, then x is the maximal element of S.

However, often the least upper bound of S is not an element of S.

Ex: Let $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} = \{\frac{n-1}{n} : n \in \mathbb{N}\}$

$\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\} = \{\frac{n-1}{n} : n \in \mathbb{N}\}$

1 is the least upper bound, but 1 is not in S.

$\frac{n-1}{n} = 1 - \frac{1}{n} < 1$ so 1 is an upperbound.

1 is the least upper bound for S; suppose that $y < 1$

Pick n so that $\frac{1}{n} < \underbrace{1-y}_{\text{positive}}$. Then $y < 1 - \frac{1}{n} = \frac{n-1}{n}$

so y is not an upper bound.

Supremum := least upper bound.

Notation: least upper bound of S is $\sup S$ or $\text{lub } S$

Completeness axiom $\iff$ Least upper bound property

$\Rightarrow$ Suppose S is nonempty and bounded above.

Let B = set of all upper bounds for S. $= \{x : x \geq s \ \forall s \in S\}$

$A = \mathbb{R} \setminus B = \{y : y$ is not an upper bound for $S\} = \{y : y < s$ for some $s \in S\}$

(A, B) is a dedekind cut

(1) B nonempty since S bounded above

A nonempty since $s \in S \Rightarrow s - 1 \in A$

(2) $\mathbb{R} = A \cup B$ since $A = \mathbb{R} \setminus B$

(3) if $a \in A$ and $b \in B$, then a is not an upper bound for S and b is an upper bound for S, so, for some $s \in S$

$a < s \leq b$ so $a < b$.

by the completeness axiom, either A has a maximal element or B has a minimal element (which is the least upper bound)

A has no maximal element $c \in A$. Since $c \in A$, we can find $s \in S$ s.t. $c < s$. Then, $c < \frac{s+c}{2} < s$ so $\frac{s+c}{2} \in A$ and c is not the maximal element.

So B has a least element - the least upper bound.

$B = [c, \infty)$, $\sup S = c$.

$\Leftarrow$ Let (A,B) be a dedekind cut of $\mathbb{R}$.

We'll show that $c = \sup A$ is a cut point of (A,B).

If $c \in B$, then it has to be the least element of B (because every $b \in B$ is an upper bound to A).

If $c \in A$, then it has to be the greatest element of A (because if $c = \sup A$ and $c \in A$ then c is the max element of A).

So either $A = (-\infty, c]$, $B = (c, \infty)$

or $A = (-\infty, c)$, $B = [c, \infty)$

Greatest lower bound Property:

If $S \neq \emptyset$ is bounded below, then S has a $\underbrace{\text{greatest lower bound}}_{\text{Infimum}}$.

(called $\inf S$ (or $\text{glb}\, S$)).

If $\inf S \in S$ then $\inf S$ is the minimal element of S.

CA $\Leftrightarrow$ GLBP $\Leftrightarrow$ LUBP

Proof: GLBP $\Leftrightarrow$ LUBP

$\inf S = -\sup(-S)$ where $-S = \{-x : x \in S\}$

NIP: nested intervals property.

Suppose $I_n = [a_n, b_n]$ and

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

(ii) $\lim [\text{length}(I_n)] = 0 \overset{\text{def}}{\Leftrightarrow} \forall \varepsilon > 0, \exists N$ so that $b_n - a_n < \varepsilon$ for all $n > N$

$n \to \infty$

Then $\bigcap_{n=1}^{\infty} I_n = \{c\}$ for some $c \in \mathbb{R}$

moreover, $\forall \delta > 0$, $\exists N$ so that $I_n = [a_n, b_n] \subseteq (c-\delta, c+\delta)$ $\forall n > N$

$CA \Leftrightarrow \mathbb{R}$ satisfies NIP and has no infinitessimals.