Lec 8/30

Friday, August 30, 2019 11:32

Let $\alpha \in S_n$, $\alpha = (i_1 \cdots i_r)(j_1 \cdots j_s) \cdots (l_1 \cdots l_u)$

be the (essentially) unique cycle decomposition of α.

define

$$N(\alpha) = (r-1) + (s-1) + \cdots + (l-1)$$

So α can be written as a product of $N(\alpha)$ transpositions.

Thm: The # of transpositions used to write α is either always even or always odd.

proof: By direct verification of the image of each $i \in \{1, \ldots, n\}$,

$$(*) \quad (a\ b)(a\ c_1 \cdots c_h\ b\ d_1 \cdots d_k)$$

$$= (b\ d_1 \cdots d_k)(a\ c_1 \cdots c_k)$$

where a, b, c_r, d_t are all distinct.

So if a, b occur in the same cycle in the unique decomp of α, then

$$\underbrace{N((a\,b)\alpha)}_{h+k} = \underbrace{N(\alpha)}_{h+k+1} - 1$$

multiply $(*)$ by $(a\ b)$ on both sides:

$$(a\ b)(a\ c_1 \cdots c_h)(b\ d_1 \cdots d_k) = (a\ c_1 \cdots c_h\ b\ d_1 \cdots d_k).$$

If a, b occur in disjoint cycles,

$$N((ab)\alpha) = N(\alpha) + 1.$$

Therefore, $\forall$ transp (ab), $N((ab)\alpha) = N(\alpha) \pm 1$. $\square$

$E = \bigsqcup E_n$, $\quad \nu(E) \geq \sum \nu(E_n)$

ε trick $\quad$ if $F_n \subset E_n$, $\;$ ($\frac{\varepsilon}{2^n}$ trick) $\;\bigsqcup F_n \subset E$ $\quad$ so $\nu(E) \geq \mu(\bigsqcup F_n) = \sum \mu(F_n)$

$$\geq \sum \left(\nu(E_n) + \frac{\varepsilon}{2^n}\right)$$

$$= \left(\sum \nu(E_n)\right) + \varepsilon$$

if $F \subset E$, $\quad F \cap E_n \subset E_n$,

$$\sum \nu(E_n) \geq \sum \mu(F \cap E_n) = \mu(F) \geq \nu(E) + \varepsilon$$

if one E_n is ν-infinite, so is E, if E is ν-infinite, so is $\sum \nu(E_n)$

Prop: Let $A_n \subset S_n$ be the set of all even perms of S_n. Then $A_n \leq S_n$ is a subgp of S_n called the alternating group. $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$.

Pf $\quad S_n = A_n \sqcup (ab) A_n$.

Orbits & cosets

orbits of G in S partition S.

If there's one orbit, G acts <u>transitively</u>.

Let $H \le G$.

H acts on G by left multiplication.

an orbit of H is called a coset.

The number of cosets of H in G is $|G:H|$.

<u>Lagrange Thm</u>:

Suppose $K \le H \le G$. Then

$$|G:H| \cdot |H:K| = |G:K|.$$

There is a bijection $Hx \longleftrightarrow x^{-1}H$
by inverting everything.

However $Hx \longleftrightarrow xH$ is <u>not</u> well-defined.

<u>counterexample</u>:

$G = S_3$, $H = \{1, (1\,2)\}$, $x_1 = 1$, $x_2 = (2\,3)$, $x_3 = (1\,3\,2)$.

<u>Def</u> A subgp K of G is normal if $xK = Kx$ $\forall x \in G$.