CS1200: Intro. to Algorithms and their Limitations	Anshu & Vadhan

Sender–Receiver Exercise 6: Reading for Receivers

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The goals of this exercise are:

- to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them,
- to reinforce the definition of NP and practice NP-completeness proofs

To prepare for this exercise as a receiver, you should try to understand the theorem statement and definition in Section 1 below, and review the relevant material from lectures. Your partner sender will communicate the proof of Theorem 1.2.

1 The Result

So far, we have seen examples of NP-complete problems in logic (e.g. SAT) and graph theory (e.g. Independent Set). Here you will see an example of a numerical NP-complete problem.

Input	**:** Natural numbers $v_0, v_1, \ldots, v_{n-1}, t$
Output	**:** A subset $S \subseteq [n]$ such that $\sum_{i \in S} v_i = t$, if such a subset S exists

Computational Problem SubsetSum

Example SubsetSum instance: Given the input $(v_0, v_1, \ldots, v_5, t) = (1, 5, 3, 8, 6, 2, 7)$, a solution would be the subset $S = \{0, 4\}$ since $v_0 + v_4 = 1 + 6 = 7 = t$.

Theorem 1.1. *SubsetSum is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

Actually, we will focus on the *vector* version of the problem, where we replace the v_i's with vectors having $\{0, 1\}$ entries and t with a vector of natural numbers.

Input	**:** Vectors $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1} \in \{0,1\}^d$, $\vec{t} \in \mathbb{N}^d$
Output	**:** A subset $S \subseteq [n]$ such that $\sum_{i \in S} \vec{v}_i = \vec{t}$, if such a subset S exists

Computational Problem VectorSubsetSum

We will use the notation $\vec{v}[j]$ to denote the j'th entry of vector $\vec{v}$, so the condition $\sum_{i \in S} \vec{v}_i = \vec{t}$ means that for every $j = 0, 1, \ldots, d-1$, we have $\sum_{i \in S} \vec{v}_i[j] = \vec{t}[j]$.

Example VectorSubsetSum instance: Consider the 3 vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$ and target $\vec{t}$ written in the table below:

	0	1	2	3
$\vec{v}_0$	1	0	0	1
$\vec{v}_1$	1	0	0	1
$\vec{v}_2$	0	1	0	0
$\vec{v}_3$	0	1	1	0
$\vec{t}$	2	1	1	2

A solution is $S = \{0, 1, 3\}$ since $\vec{v}_0 + \vec{v}_1 + \vec{v}_3 = \vec{t}$.

Theorem 1.2. *VectorSubsetSum is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

It will be shown in section how to derive Theorem 1.1 from Theorem 1.2. In today's SRE, we will prove Theorem 1.2.

2 The Proof

VectorSubsetSum is in $\mathsf{NP}_{\mathsf{search}}$:

Reduction to show VectorSubsetSum is $\mathsf{NP}_{\mathsf{search}}$-hard, and example:

Analysis of the reduction: