# FYS3110 Hjemmeeksamen

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## 1 Problem 1

### 1.1

Have that

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

So I can show that

$$\begin{aligned}(\hat{H} + cI)|\psi\rangle &= \hat{H}|\psi\rangle + cI|\psi\rangle \\ &= E|\psi\rangle + c|\psi\rangle \\ &= (E + C)|\psi\rangle\end{aligned}$$

### 1.2

For $\hat{H}$ to be a hermitian operator we require $\hat{H} = \hat{H}^\dagger$.

I find that

$$\begin{aligned}\hat{H}^\dagger =& \alpha_e^*|\nu_e\rangle\langle\nu_e| + g_{e\mu}^*|\nu_\mu\rangle\langle\nu_e| + g_{e\tau}^*|\nu_\tau\rangle\langle\nu_e| \\ & g_{\mu e}^*|\nu_e\rangle\langle\nu_\mu| + \alpha_\mu^*|\nu_\mu\rangle\langle\nu_\mu| + g_{\mu\tau}|\nu_\tau\rangle\langle\nu_\mu| \\ & g_{\tau e}^*|\nu_e\rangle\langle\nu_\tau| + g_{\tau\mu}^*|\nu_\tau\rangle\langle\nu_\mu| + \alpha_\tau^*|\nu_\tau\rangle\langle\nu_\tau|\end{aligned}$$

Given the following relation for the outer products:

$$((c_1|\phi_1\rangle\langle\psi_1|) + (c_2|\phi_2\rangle\langle\psi_2|))^\dagger = (c_1^\star|\psi_1\rangle\langle\phi_1|) + (c_2^\star|\psi_2\rangle\langle\phi_2|)$$

Comparing the terms in $\hat{H}$ and $\hat{H}^\dagger$ we see that we need the coefficients of corresponding terms to be equal in order to meet the requirement. This gives the equations $\alpha_e = \alpha_e^*$, $\alpha_\mu = \alpha_\mu^*$ and $\alpha_\tau = \alpha_\tau^*$. We also need $g_{e\mu} = g_{\mu e}^*$, $g_{e\tau} = g_{\tau e}^*$ and $g_{\mu\tau} = g_{\tau\mu}^*$.

This leads us to the following conditions on the $\alpha$'s and $g$'s:

$$\begin{aligned}
&\alpha \in \mathbb{R} \\
&g_{ij} = g_{ji}^*, \quad \text{for} \quad i, j = e, \mu, \tau
\end{aligned}$$

### 1.3

The observable we are measuring here is the neutrino type, of which the spectrum of possible measurements is discrete: electron, muon or tau neutrino. The probability of measuring the electron type associated with the orthonormal state $|\nu_e\rangle$ is then

$$\begin{aligned}
|c_e|^2 &= |\langle \nu_e | \nu \rangle|^2 \\
&= |a\delta_{ee} + b\delta_{e\mu} + c\delta_{e\tau}|^2 \\
&= |a|^2
\end{aligned}$$

### 1.4

Neglectiing the tau neutrino and using the representation $|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we now have the Hamiltonian

$$\begin{aligned}
\hat{H} &= \alpha_e |\nu_e\rangle\langle \nu_e| + g_{e\mu} |\nu_e\rangle\langle \nu_\mu| \\
&+ g_{\mu e} |\nu_\mu\rangle\langle \nu_e| + \alpha_\mu |\nu_\mu\rangle\langle \nu_\mu| \\
&= \alpha_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + g_{e\mu} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&+ g_{\mu e} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_\mu \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \alpha_e [\frac{1}{2}(I + \sigma_z)] + g_{e\mu} [\frac{1}{2}(\sigma_x + i\sigma_y)] \\
&+ g_{\mu e} [\frac{1}{2}(\sigma_x - i\sigma_y)] + \alpha_\mu [\frac{1}{2}(I - \sigma_z)] \\
&= \frac{1}{2}(\alpha_e + \alpha_\mu) I + \frac{1}{2}(g_{e\mu} + g_{\mu e})\sigma_x \\
&+ \frac{i}{2}(g_{e\mu} - g_{\mu e})\sigma_y + \frac{1}{2}(\alpha_e - \alpha_\mu)\sigma_z \\
&= \alpha I + c\sigma_x + d\sigma_y + \beta\sigma_z
\end{aligned}$$

We now have the Hamiltonian expressed as a linear combination of the identity matrix $I$ and the Pauli matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$. In the last line I have used the condition $g_{e\mu} = g_{\mu e}^* = c - id$ from 1.2 and introduced $\alpha = \frac{\alpha_e + \alpha_\mu}{2}$ and $\beta = \frac{\alpha_e - \alpha_\mu}{2}$ to ease notation. In short we have

$$\hat{H} = \begin{pmatrix} \alpha_e & g_{e\mu} \\ g_{\mu e} & \alpha_\mu \end{pmatrix} = \begin{pmatrix} \alpha_e & c - id \\ c + id & \alpha_\mu \end{pmatrix} = \alpha I + c\sigma_x + d\sigma_y + \beta\sigma_z = \alpha I + |\vec{r}|\hat{r} \cdot \vec{\sigma}$$

where the purpose of the last expression is a more compact notation where $\vec{r} = (c, d, \beta)$, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ is a unit vector and $\vec{\sigma}$ is the Pauli vector.

### 1.5

In order to find this probability we want to express the system's development in time, given by (setting $t_0 = 0$)

$$|\nu(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\nu(0)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\nu_e\rangle$$

I choose to solve this by finding the energy eigenstates of the Hamiltonian, express $|\nu_e\rangle$ in these, and thereby substitute the operator $\hat{H}$ in the exponent with its eigenvalue $E_n$, which must correspond to whichever eigenstate $|E_n\rangle$ the factor $e$ is in front of.

Start by finding the eigenvalues $E_n$, which are the roots of the polynomial $det(H - EI)$

$$0 = det(H - EI) \tag{1}$$
$$0 = (\alpha_e - E)(\alpha_\mu - E) - c^2 - d^2 \tag{2}$$
$$E_\pm = \frac{\alpha_e + \alpha_\mu \pm \sqrt{(\alpha_e - \alpha_\mu)^2 + 4g_{e\mu}g_{\mu e}}}{2} \tag{3}$$
$$E_\pm = \alpha \pm \sqrt{c^2 + d^2 + \beta^2} \tag{4}$$
$$E_\pm = \alpha \pm |\vec{r}| \tag{5}$$

I set $E_1 = E_+$ and $E_2 = E_-$. We now need to find eigenvectors $\vec{E}_i$ corresponding to this. Solving $H\vec{E}_i = E_i\vec{E}_i$ I find suitable eigenvectors $|E_1\rangle = \begin{pmatrix} 1 \\ \frac{E_1 - \alpha_e}{q_{\mu e}} \end{pmatrix}$ and $|E_2\rangle = \begin{pmatrix} 1 \\ \frac{E_2 - \alpha_e}{g_{\mu e}} \end{pmatrix}$. This gives us

$$|\nu_e\rangle = \frac{E_2 - \alpha_e}{E_2 - E_1}|E_1\rangle + \frac{\alpha_e - E_1}{E_2 - E_1}|E_2\rangle$$

I can now express

$$|\nu(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\nu_e\rangle \tag{6}$$
$$= \frac{E_2 - \alpha_e}{E_2 - E_1}e^{-\frac{i}{\hbar}E_1 t}|E_1\rangle + \frac{\alpha_e - E_1}{E_2 - E_1}e^{-\frac{i}{\hbar}E_2 t}|E_2\rangle \tag{7}$$

Finally I find the coefficient $c_\mu$

$$\begin{aligned} c_\mu &= \langle \nu_\mu | \nu(t) \rangle \\ &= \frac{(E_2 - \alpha_e)(E_1 - \alpha_e)}{g_{\mu e}(E_2 - E_1)} e^{-\frac{i}{\hbar}\alpha t} (e^{-\frac{i}{\hbar}|\vec{r}|t} - e^{\frac{i}{\hbar}|\vec{r}|t}) \end{aligned}$$

Where we use Euler's formuale on the last paranthesis to get $-2sin(\frac{|\vec{r}|}{\hbar}t)$. Finally this gives the probability

$$\begin{aligned} P_{e\to\mu}(t) &= |c_\mu|^2 \\ &= 4\frac{c^2 + d^2}{4(c^2 + d^2 + \beta^2)} sin^2(\frac{|\vec{r}|}{\hbar}t) \\ &= \frac{|g|^2}{|\vec{r}|^2} sin^2(\frac{|\vec{r}|}{\hbar}t) \end{aligned}$$

where we notice the interesting fact that the angular frequency $\omega = 2\frac{E_1 - E_2}{\hbar}$, which resembles the quantum relation $E = h\nu$. As I understand it we would expect the frequency for this propability amplitude to be proportional to the difference in the two possible energy values over $\hbar$, when looking at this kind of two state neutrino system, so it's an encouraging observation.

In order to avoid finding the eigenvectors I could have used the fact that there is an identity which expresses e raised to the pauli vector as a sine and cosine expression. The factor of e raised to $\alpha$ could have been removed since it would correspond to adding a constant to the Hamiltonian which, as we showed in a), does not alter the eigenvectors and only changes the eigenvalues by the same constant. In my case as well we see that this factor becomes irrelevant to the probability at the moment we take the absolute value.

### 1.6

At an uspecified distance from a reactor producing only electron neutrinos the probability of a detector detecting muon neutrinos is given as $10^{-4}$ after some unknown time t. Thus in this case we have $P_{e\to\mu}(t) = 10^{-4}$. We want to use this information to find a limit on the paramter ratio

$$\frac{|g_{e\mu}|}{|\alpha_\mu - \alpha_e|} = \frac{|g|}{|\alpha_e - \alpha_\mu|} = \frac{\sqrt{c^2 + d^2}}{2|\beta|}$$

We get

$$P_{e\to\mu}(t) = \frac{|g|^2}{|\vec{r}|^2} sin^2(\frac{|\vec{r}|}{\hbar}t) = \frac{1}{10000}$$

$$(\frac{\alpha_\mu - \alpha_e}{|g|})^2 = 4[10^4 sin^2(\frac{|\vec{r}|}{\hbar}t) - 1]$$

We see that this is useful to describe a limit of the parameter ratio because no matter the values in the argument of sine, the sine value has 1 as maximum value and 0 as minimum value, and what we have outside of sine here is just constants. We see that looking at sine equal to 0 does not make sense here because the that would make the right hand side negative while the left hand side is necessarily positive. Thus we look at the sine equal to 1 case, which gives us the following upper lower bound of the ratio

$$\frac{|g|}{|\alpha_e - \alpha_\mu|} \geq \frac{1}{2\sqrt{10^4 - 1}} \approx 0.005$$

### 1.7

I solved this in the same manner as 1.5, but as a 3x3 eigenvalue problem, in Matlab. See the attatched code. See figure 1 for the plot of the computed probability. The relevant combinations would be those affecting the amplitude and angular frequency of the probability amplitude. It would be like the dimensionless ratio we look at in 1.6 - we see that it reflects the ratio between the terms which constituets the frequency and amplitude of the probability expression.

## 2 Problem 2

In this problem we have the Hamiltonian

$$\hat{H} = -b_x\sigma_x - b_z\sigma_z = \begin{pmatrix} -b_z & -b_x \\ -b_x & b_z \end{pmatrix} \tag{8}$$

### 2.1

We recognize the similarities between this spin system and the neutrino system in problem 1, both being two state systems. In this problem we have a state

$$|\chi\rangle = a|\chi_-\rangle + b|\chi_+\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

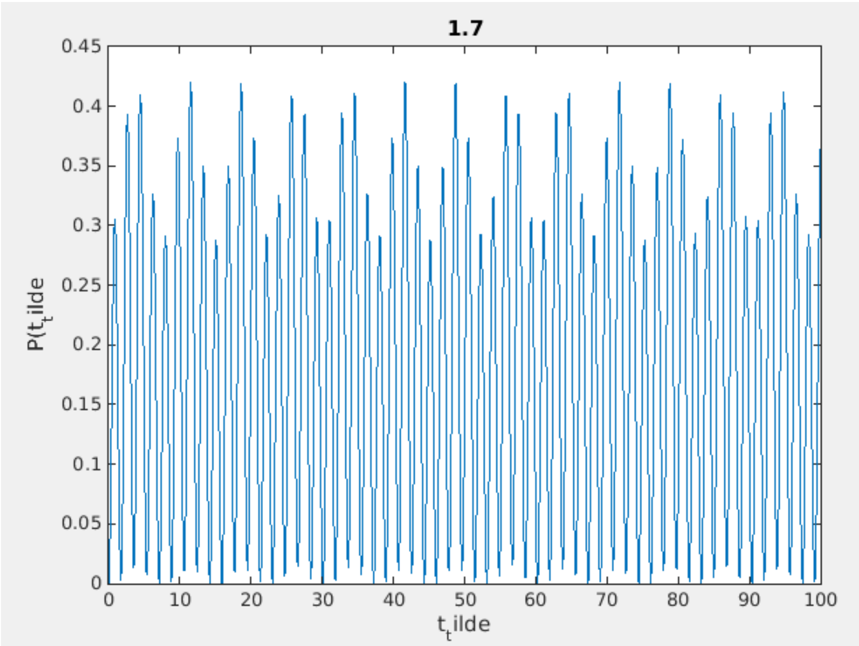


Figure 1: Exercise 1.7

Which at the time t=0 is given to be in the state

$$|\chi(t=0)\rangle = |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

And we are looking for the probability $P_{\chi_+ \to \chi_-}(t)$ of finding the system has changed to the state $|\chi_-\rangle$ after a time t. We want this state because it is the eigenvector of $\hat{S}_z$ corresponding to the eigenvalue $-\frac{\hbar}{2}$, the measurement of $S_z$ we want the probability for.

We see that the Hamiltonian is equivalent to the one in 1.5 if we set $\alpha_\mu = -\alpha_e = b_z$ and $g_{e\mu} = g_{\mu e} = -bx$, giving $c = -b_x$ and $d = 0$. Due to this, putting these values into the answer found in 1.5 will give us the probability. I find

$$P_{\chi_+ \to \chi_-}(t) = \frac{b_x^2}{b_x^2 + b_z^2} sin^2(\frac{\sqrt{b_x^2 + b_z^2}}{\hbar} t)$$

**2.2**

In general we know that the expectation value of an observable Q should be the sum over all possible outcomes of the eigenvalue $q_n$ times the probability $P_n = |c_n|^2$of getting that eigenvalue: In this case then I get

$$\begin{aligned}
\langle S_z \rangle &= \frac{\hbar}{2}P(S_z = \frac{\hbar}{2}) + \frac{-\hbar}{2}P(S_z = \frac{-\hbar}{2}) \\
&= \frac{\hbar}{2}[2P(S_z = \frac{\hbar}{2}) - 1] \\
&= \frac{hbar}{2}[2\frac{b_x^2}{b_x^2 + b_z^2}sin^2(\frac{\sqrt{b_x^2 + b_z^2}}{\hbar}t) - 1]
\end{aligned}$$

Where I have used that we must have $P(S_z = \frac{-\hbar}{2}) = 1 - P(S_z = \frac{\hbar}{2})$.

```
g = 4E-10;
alp_e = 1E-10;
alp_mu = 5E-10;
alp_tau = 6E-10;
hbar = 6.582119514E-16 % eVs

t_marked = linspace(0, 100, 1000);
t = t_marked*hbar/(alp_mu-alp_e);
H = [alp_e, g, g; g, alp_mu, g; g, g, alp_tau;];

[E_vecs, vals] = eig(H);
E_vals = [vals(1,1), vals(2,2), vals(3,3)];

v_e = transpose([1, 0, 0]);
c = E_vecs\v_e; % coeff for v_e expanded in E_vecs

E1 = E_vecs(:,1);
E2 = E_vecs(:,2);
E3 = E_vecs(:,3);

p = -sqrt(-1).*t./hbar;
v_t_mu = c(1).*exp(p.*E_vals(1)).*E1(2) + c(2).*exp(p.*E_vals(2)).*E2(2) + c(3).*exp(

P_t = (abs(v_t_mu)).^2;

plot(t_marked, P_t)
xlabel('t_tilde')
ylabel('P(t_tilde')
title('1.7')
```