Oblig 1 FYS 4150

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Abstract

1 Introduction

2 Theory

$$-\frac{v_{i+1}+v_{i-1}-2v_i}{h^2} = f_i \qquad \text{for } i = 1,\dots,n, \tag{1}$$

where $f_i = f(x_i)$.

Rewriting the Equation 2 as a linear set of equations of the form

$$\mathbf{A}\mathbf{v} = \tilde{\mathbf{b}},$$

where $\mathbf{A}$ is an $n \times n$ tridiagonal matrix which we rewrite as

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix},$$

and $\tilde{b}_i = h^2 f_i$.

Writing out the equations:

$$\begin{aligned} i = 1 &: -v_0 + 2v_1 - v_2 = f_1 h^2 \\ i = 2 &: -v_1 + 2v_2 - v_3 = f_2 h^2 \\ &\vdots \\ i = n &: -v_{n-1} + 2v_n - v_{n+1} = f_n h^2 \end{aligned}$$

where $v_0 = 0$ and $v_{n+1} = 0$.

Adding some zeros to the equations for illustrative purposes:

$$\begin{aligned} i = 1 &: & 2v_1 - v_2 + 0 + 0 + 0 + 0 + \cdots + 0 &= f_1 h^2 \\ i = 2 &: & -v_1 + 2v_2 - v_3 + 0 + 0 + \cdots + 0 &= f_2 h^2 \\ i = 3 &: & 0 - v_2 + 2v_3 - v_4 + 0 + \cdots + 0 &= f_3 h^2 \\ &\vdots & &\vdots \\ i = n-1 &: & 0 + \cdots + 0 - v_{n-2} + 2v_{n-1} - v_n &= f_{n-1} h^2 \\ i = n &: & 0 + \cdots + 0 + 0 + 0 - v_{n-1} + 2v_n &= f_n h^2 \end{aligned}$$

We can now recognize the equations as a product between a matrix an a vector that equals another vector.

Defining vectors that fit:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \tilde{\mathbf{b}} = \begin{bmatrix} f_1 h^2 \\ f_2 h^2 \\ \vdots \\ f_n h^2 \end{bmatrix}$$

The matrix is the matrix $\mathbf{A}$. Then we have:

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} f_1 h^2 \\ f_2 h^2 \\ f_3 h^2 \\ \vdots \\ f_{n-1} h^2 \\ f_n h^2 \end{bmatrix} \Longrightarrow \mathbf{A}\mathbf{v} = \tilde{\mathbf{b}}$$

Putting the solution into the Poisson equation:

The Poisson equation:

$$-u''(x) = f(x)$$

In this case $f(x) = 100e^{-10x}$.

The solution and derivatives:

$$\begin{aligned} u(x) &= 1 - \left(1 - e^{-10}\right) x - e^{-10x} \\ u'(x) &= \left(1 - e^{-10}\right) + 10e^{-10x} \\ u''(x) &= -100e^{-10x} \end{aligned}$$

The solution is a solution of the Poisson equation:

$$-u''(x) = -(-100)e^{-10x} = 100e^{-10x} = f(x)$$

3 Method

4 Result

5 Discussion

6 Conclusion

7 Refrences