

# Project 2

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## Abstract

This is the abstract.

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## 1 Introduction

## 2 Theory and method

In this project we investigate a fermionic system of  $N=2, 6$  and  $12$  electrons. It is a so-called closed shell-system. The Hamiltonian used to model this system is

$$\hat{H} = \sum_{i=1}^N \left( -\frac{1}{2} \nabla_i^2 + \frac{1}{2} \omega^2 r_i^2 \right) + \sum_{i<j} \frac{1}{r_{ij}}, \quad (1)$$

where

$$\hat{H}_0 = \sum_{i=1}^N \left( -\frac{1}{2} \nabla_i^2 + \frac{1}{2} \omega^2 r_i^2 \right)$$

is the single particle part and

$$\hat{H}_1 = \sum_{i<j} \frac{1}{r_{ij}}, \quad (2)$$

represent the interaction potential between particles. The Hamiltonian is written in atomic units, which implies that  $\hbar = 1, m = 1$ , the unit of length is  $a_0 = \dots$  and the unit of energy is  $\dots$ . We also have  $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$  and  $\omega$  is the oscillator frequency. Later we will study the dependence of the system on the oscillator frequency.

### 2.1 Two particle system

The single particle wave function in two dimensions is

$$\phi_{n_x, n_y}(x, y) = A H_{n_x}(\sqrt{\omega}x) H_{n_y}(\sqrt{\omega}y) \exp(-\omega(x^2 + y^2)/2). \quad (3)$$

where the functions  $H_{n_x}(\sqrt{\omega}x)$  are Hermite polynomials, while  $A$  is a normalization constant. The relevant Hermite polynomials in this project are listed in Appendix D.  $\omega$  is the trap frequency.

For the lowest-lying state,  $E_{00}$  (see Fig. 1), we have  $n_x = n_y = 0$  and an energy  $\epsilon_{n_x, n_y} = \omega(n_x + n_y + 1) = \omega$ , the total energy of the lowest-lying state is hence  $2\omega$  because there is room for two electrons with opposite spins.

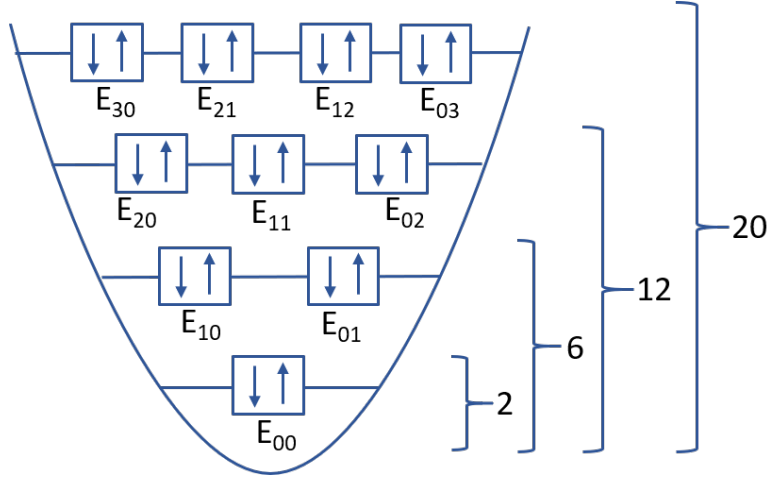


Figure 1: •

The expectation value can be found by solving the equation

$$\langle E \rangle = \frac{\int d\mathbf{r}_1 d\mathbf{r}_2 \psi_T^*(\mathbf{r}_1, \mathbf{r}_2) \hat{H}(\mathbf{r}_1, \mathbf{r}_2) \psi_T(\mathbf{r}_1, \mathbf{r}_2)}{\int d\mathbf{r}_1 d\mathbf{r}_2 \psi_T^*(\mathbf{r}_1, \mathbf{r}_2) \psi_T(\mathbf{r}_1, \mathbf{r}_2)}. \quad (4)$$

We will use Variational Monte Carlo (VMC) methods to evaluate the Eq. 4 [1]. The exact wave function for two not interacting electrons in the ground state is given by

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = C \exp(-\omega(r_1^2 + r_2^2)/2),$$

where  $r_i = \sqrt{x_i^2 + y_i^2}$  and  $C$  is a normalization constant. The trial wavefunction we use for the not interacting case is

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = C \exp(-\alpha\omega(r_1^2 + r_2^2)/2). \quad (5)$$

with the parameter  $\alpha$ . From the exact wave function we know that  $\alpha = 1$  for the situation without interaction. On the other hand, for the interacting case, the trial wave function for the two-electron system is

$$\psi_T(\mathbf{r}_1, \mathbf{r}_2) = C \exp(-\alpha\omega(r_1^2 + r_2^2)/2) \exp\left(\frac{ar_{12}}{(1 + \beta r_{12})}\right), \quad (6)$$

where we introduce another parameter,  $\beta$ , and a spin factor,  $a$ .  $a$  is 1 when the two electrons have anti-parallel spins and 1/3 when they have the parallel spins (this is not relevant before we introduce more particles to the system, as can be seen from Fig. 1).

## 2.2 More particles

Since we are looking at closed shell systems, the next amount of particles are six. We can see this from Fig. 1, there are room for two electrons with opposite spin in two different states, in addition to the two in the lowest lying state. The trial wave function is now given by

$$\psi_T(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_6) = \text{Det}(\phi_1(\mathbf{r}_1), \phi_2(\mathbf{r}_2), \dots, \phi_6(\mathbf{r}_6)) \prod_{i < j}^6 \exp\left(\frac{ar_{ij}}{(1 + \beta r_{ij})}\right), \quad (7)$$

where

$$\text{Det}(\phi_1(\mathbf{r}_1), \phi_2(\mathbf{r}_2), \dots, \phi_6(\mathbf{r}_6)) = \begin{vmatrix} \phi_1(\mathbf{r}_1) & \phi_2(\mathbf{r}_1) & \cdots & \phi_6(\mathbf{r}_1) \\ \phi_1(\mathbf{r}_2) & \phi_2(\mathbf{r}_2) & \cdots & \phi_6(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{r}_6) & \phi_2(\mathbf{r}_6) & \cdots & \phi_6(\mathbf{r}_6) \end{vmatrix}$$

is the Slater determinant. This determinant occurs because electron are indistinguishable particles and they are antisymmetric ... . The functions,  $\phi_i(\mathbf{r}_j)$ , are given by Eq. 3 and the notation is explained in Tab. 1.

Table 1: The relation between the notation used in the determinant (left) compared to Eq. 3 (right).

$\phi_1$	$\phi_{n_x=0, n_y=0}$	$\phi_7$	$\phi_{n_x=2, n_y=0}$
$\phi_2$	$\phi_{n_x=0, n_y=0}$	$\phi_8$	$\phi_{n_x=2, n_y=0}$
$\phi_3$	$\phi_{n_x=1, n_y=0}$	$\phi_9$	$\phi_{n_x=1, n_y=1}$
$\phi_4$	$\phi_{n_x=1, n_y=0}$	$\phi_{10}$	$\phi_{n_x=1, n_y=1}$
$\phi_5$	$\phi_{n_x=0, n_y=1}$	$\phi_{11}$	$\phi_{n_x=0, n_y=2}$
$\phi_6$	$\phi_{n_x=0, n_y=1}$	$\phi_{12}$	$\phi_{n_x=0, n_y=2}$

Similarly if we include another "shell" in our system we get 12 particles and the trial wavefunction is

$$\psi_T(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{12}) = \text{Det}(\phi_1(\mathbf{r}_1), \phi_2(\mathbf{r}_2), \dots, \phi_{12}(\mathbf{r}_{12})) \prod_{i < j}^{12} \exp\left(\frac{ar_{ij}}{(1 + \beta r_{ij})}\right). \quad (8)$$

The determinant have the same structure as for six particles and the relation to the single-particle wave functions are shown in Tab. 1.

### 2.3 One-body density

The radial one-body density is a measure of the spacial distribution of the electrons with respect to the distance from the middle of the harmonic oscillator trap. To calculate the radial one-body density, we want to sample the position of the electrons. The distance from the origin to a set cut-off is seperated into bins with a length  $\Delta r$ . For every Monte Carlo step, the distance between the electron's position and the origin is calculated, and the bin that corresponds to the current distance get a count. In the end, you have an array corresponding to the different bins with counts corresponding to how many times an electron was found to have that particular distance to the origin. This array is normalized by dividing by the number of Monte Carlo steps. However, to get the density, we have to divide the number in the bins with the area or volume the bin represents. Because we have two-dimensional

problem in this project and we calculate the radial one-body density, we divide bin  $i$  with the area  $A = \pi(r_i + \Delta r)^2 - \pi r_i^2$  where  $r_i$  is distance from the origin to bin  $i$ . *normalized to the number of particles. Mention. Should happen automatically... Does not happen automatically for me, but I have notice that Even has the same scale on the y-axis.*[2]

## 2.4 Virial theorem

The virial theorem gives a relation between the time-average total kinetic energy,  $\langle T \rangle$ , and the time-average external potential energy,  $\langle V_{ext} \rangle$ , that is

$$2 \langle T \rangle = n \langle V_{ext} \rangle \quad (9)$$

where  $V(r) = Br^n$ . In our case when we do not include interaction  $n = 2$  from the external potential term in Eq. 1, and therefore the average kinetic energy should be equal to the average potential energy. This can be used as a test to see if the simulations executed are correct.

## 2.5 Trap frequency

The trap frequency changes the external potential felt by the electrons. Figure 2 shows how a larger trap frequency,  $\omega$ , results in a narrower external potential. In this narrow harmonic oscillator trap, the electrons are forced to be closer to each other. Later, we will investigate how  $\omega$  affects the energy when the electrons are interacting with each other through the potential given in Eq. 2

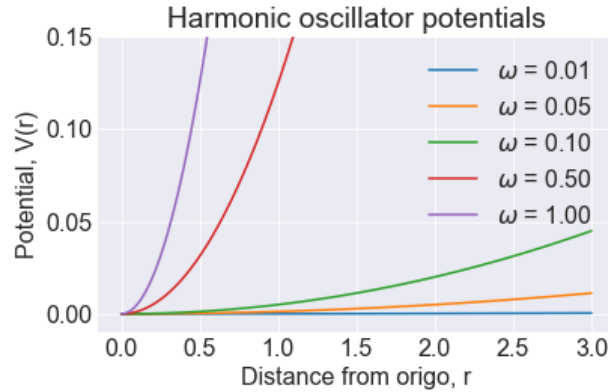


Figure 2: Illustration of how the potential changes when the trap frequency changes.

## 2.6 Evaluating the error

Standard error of the mean (SEM):

$$\text{SEM} = \frac{\sigma}{\sqrt{N}} \quad (10)$$

where  $N$  is the number of observations, in our case the number of Monte Carlo cycles.

### 3 Results and discussion

#### 3.1 Two electrons in two dimensions

We start with the simple case of two electrons in a harmonic oscillator trap. These electrons do not interact with each other and the trial wavefunction is given by Eq. 5.

##### 3.1.1 Brute force sampling

First, brute force sampling was used to calculate the new position and evaluate the metropolis ratio. The double derivative of the wavefunction, used to calculate the kinetic energy part of the expectation energy, was evaluated both analytically and numerically. Table 2 shows the energy for different values for the parameter  $\alpha$ . The numbers show that the standard error of the mean (SEM) is underestimating the deviations. From Tab. 2 one can observe that including the correlations, e.g. correlations in the random number generator, increases the deviation, giving us  $\sigma_b$ . This value is also an estimate of the error, but probably a more true estimate of the error.

From Tab. 2 one can also observe that  $\alpha = 1.0$  gives zero standard deviation and is therefore the optimal parameter. By comparing the results for the analytical and the numerical cases one can observe that the SEM and  $\sigma_b$  is similar for both cases, especially around the ground state ( $\alpha = 1.0$ ). If the expectation energies from the analytical case and the numerical cases are compared, they differ with values at the scale of  $10^{-3}$ , which is reasonable with a  $\sigma_b$  around  $10^{-3}$  to  $10^{-2}$ . At last one can observe, both from the individual CPU time measurements and the mean CPU time of these 10 measurements (though with different  $\alpha$ s), that the analytical solution of the double derivative is much faster and more efficient than the numerical case.

Table 2: Comparing the results for analytical/numerical evaluation of the double derivative. Energy are in atomic units (a.u.) and CPU time is in units of seconds. Number of MC cycles are  $2^{21}$ .

Analytical:					Numerical:				
$\alpha$ :	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	CPU time:	$\alpha$ :	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	CPU time:
0.50	2.49402	0.00073	0.01022	5.57812	0.50	2.49991	0.00073	0.01093	18.20310
0.60	2.26441	0.00052	0.00690	5.76562	0.60	2.26412	0.00053	0.00727	18.21880
0.70	2.13118	0.00035	0.00448	5.92188	0.70	2.13039	0.00036	0.00436	18.62500
0.80	2.05016	0.00022	0.00263	5.67188	0.80	2.04993	0.00022	0.00269	18.46880
0.90	2.01015	0.00010	0.00116	5.96875	0.90	2.01160	0.00010	0.00118	18.46880
1.00	2.00000	0.00000	0.00000	5.62500	1.00	2.00000	0.00000	0.00000	18.31250
1.10	2.00871	0.00009	0.00102	6.20312	1.10	2.00825	0.00009	0.00097	18.35940
1.20	2.03402	0.00018	0.00175	6.34375	1.20	2.03308	0.00018	0.00170	18.31250
1.30	2.07259	0.00026	0.00244	5.95312	1.30	2.06460	0.00026	0.00243	20.23440
1.40	2.11041	0.00034	0.00311	6.15625	1.40	2.11803	0.00033	0.00308	19.00000
Mean CPU time: 5.91875					Mean CPU time: 18.62033				

### 3.1.2 Including importance sampling

Figure 3 compare the expectation value of the energy and the acceptance rate of brute force sampling and importance sampling. It can be observed from the right part of the figure that the acceptance rate of both methods increase with decreasing step size, but one can also observe that the acceptance is lower for importance sampling than brute force sampling at large step sizes. These observations could indicate that a small step size would be ideal for both methods.

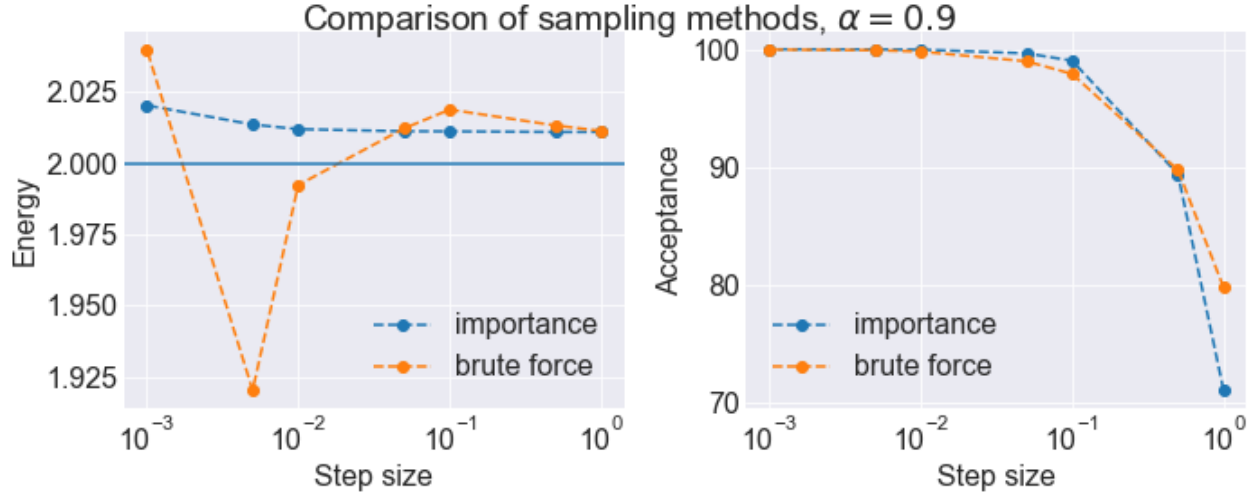


Figure 3: Left: Expectation energies after  $2^{21}$  MC cycles for different step sizes. Right: Percentage of accepted steps for different step sizes. Here importance sampling and brute force sampling is compared.

From the left part of the figure it can be observed that one of the expectation values for the energies are lower than the ground state energy ( $dl = 0.005$  with brute force sampling) when these calculations were done with  $\alpha = 0.9$ . However, in Tab. 3 which compare the result of brute force sampling and importance sampling for different step sizes one can observe that the standard deviation from the blocking method is larger for the case of brute force sampling with a step size of 0.005. However, the SEM does not indicate anything to be special about this result.

Table 3: Comparing the results for importance/brute force sampling. Energy are in atomic units (a.u.) and CPU time is in units of seconds. Here the parameter  $\alpha$  is set to 0.9 and number of MC cycles are  $2^{21}$ .

	Brute force:					Importance:				
$dl$ :	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	Acc.:	$t_{CPU}$ :	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	Acc.:	$t_{CPU}$ :
1.000	2.010	0.00010	0.00066	79.832	6.625	2.011	0.00010	0.00022	71.078	8.531
0.500	2.011	0.00010	0.00112	89.794	7.266	2.011	0.00010	0.00023	89.343	8.672
0.100	2.015	0.00011	0.00549	97.919	7.078	2.011	0.00010	0.00047	99.049	8.859
0.050	2.007	0.00011	0.00946	99.001	7.016	2.011	0.00010	0.00068	99.662	8.188
0.010	2.002	0.00007	0.01510	99.788	6.953	2.007	0.00010	0.00136	99.968	8.281
0.005	1.942	0.00006	0.02140	99.916	6.578	2.010	0.00010	0.00199	99.989	8.547
0.001	2.118	0.00002	0.00371	99.974	6.531	2.007	0.00010	0.00410	99.999	8.016
Mean CPU time: 6.86384						Mean CPU time: 8.44197				

I took a closer look at the actual local energies for the brute force sampling method. Figure 4a shows how the energy is not stable for steps sizes smaller than 0.01, so even though the step sizes 0.001 and 0.01 seems to give reasonable expectation values for the energy (see Tab. 3 and Fig. 3), Fig. 4a seems to show that that is sort of a lucky shot. I also saw this by running the calculation with brute force sampling and the step size, 0.005, with different seeds for the random number generator. The expectation energy for five different runs where  $\langle E_L \rangle = 1.91487, 2.03452, 1.90805, 1.88356$  and  $1.9284$ . From Fig. 4b one can observe that even  $dl = 0.1$  seems to be too small since it also results in the local energy varying slowly and taking longer "trips" to higher energies and using many steps to get back down again, but for this step size the "trips" to higher energies are more frequent than for the smaller step sizes. I concluded that a step size of 0.5 is the best choice for the brute force sampling because it gives resonable changes of the local energy and an accptance rate of  $\sim 90 \%$  (see Fig. 3).

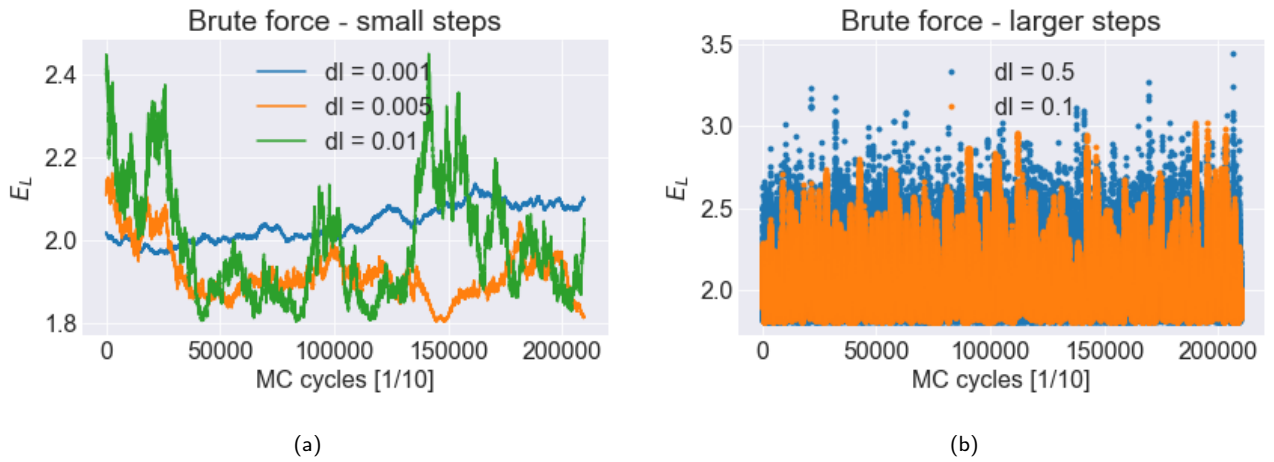


Figure 4: The local energy for every tenth MC cycle for brute force sampling at different step sizes,  $dl$ . a) shows the smaller step sizes and b) some that are a bit larger.

Proceeding to evaluate the energy, Table 4 shows how the energy changes with different trap frequencies,  $\omega$ . In this table one can observe that there is no large difference between the results from brute force sampling and importance sampling. However, to be able to compare the sampling methods more thoroughly it is better to look at the case where the system is not in the ground state.

*JEG KOM HIT :)*

Table 4: Ground state energy of two electrons in harmonic oscillator trap. Number of MC cycles are  $2^{23}$ .

$\omega$	Brute force:					Importance:				
	$\alpha$	$\langle E_L \rangle$	$\bar{r}_{12}$	$\langle T \rangle$	$\langle V_{ext} \rangle$	$\alpha$	$\langle E_L \rangle$	$\bar{r}_{12}$	$\langle T \rangle$	$\langle V_{ext} \rangle$
1.00	1	2.00	1.250	1.0008	0.9992	1	2.00	1.254	0.9982	1.0018
0.50	1	1.00	1.775	0.4971	0.5029	1	1.00	1.781	0.4973	0.5027
0.10	1	0.20	3.967	0.0996	0.1004	1	0.20	4.046	0.0987	0.1013
0.05	1	0.10	5.638	0.0497	0.0503	1	0.10	5.534	0.0512	0.0488
0.01	1	0.02	12.631	0.0099	0.0101	1	0.02	12.488	0.0101	0.0099



Table 5: Comparing the results for brute force sampling/importance sampling. Energy are in atomic units (a.u.) and CPU time is in units of seconds. Number of MC cycles are  $2^{21}$ .

	Brute force:				Importance:			
$\alpha$ :	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	CPU time:	$\langle E_L \rangle$ :	SEM:	$\sigma_B$ :	CPU time:
0.50	2.49074	0.00072	0.01030	6.26562	2.52950	0.00075	0.01952	8.53125
0.60	2.27645	0.00053	0.00721	6.53125	2.26531	0.00052	0.01281	8.62500
0.70	2.12481	0.00036	0.00451	6.59375	2.12449	0.00036	0.00813	8.76562
0.80	2.04873	0.00022	0.00265	6.68750	2.04797	0.00022	0.00454	9.37500
0.90	2.01135	0.00010	0.00115	6.82812	2.01171	0.00010	0.00203	8.81250
1.00	2.00000	0.00000	0.00000	6.75000	2.00000	0.00000	0.00000	8.76562
1.10	2.00863	0.00009	0.00097	6.59375	2.00954	0.00009	0.00171	8.40625
1.20	2.03343	0.00018	0.00174	7.29688	2.03276	0.00018	0.00305	8.50000
1.30	2.07103	0.00026	0.00246	6.75000	2.07363	0.00026	0.00440	8.67188
1.40	2.11725	0.00033	0.00317	6.70312	2.10697	0.00034	0.00536	8.67188
	Mean CPU time: 6.7000				Mean CPU time: 8.7125			

### 3.1.3 Including optimization

Minimization rate of 0.5 seemed to be ideal. It resulted in the fewest steps until the parameter value stabilized both for guesses close to the optimal value and for guesses far away from the optimal value, but for the smallest trap frequencies I had to use  $\gamma = 0.1$  or  $0.2$ . The parameters were optimized by trying out different first guesses for  $\alpha$  and  $\beta$  and tuning  $\gamma$  so that the parameters stabilized during the first 200 iterations. The optimal parameters were extracted from the mean of the last 50 iterations. An example run is shown in Fig. ?? for  $\omega = 0.5$ .

### 3.1.4 Including interaction

Table 6: Ground state energy of two interacting electrons in harmonic oscillator trap found with brute force sampling. Number of MC cycles are  $2^{23}$

$\omega$	$\alpha$	$\beta$	$\langle E_L \rangle$	SEM	$\sigma_B$	$\bar{r}_{12}$	$\langle T \rangle$	$\langle V_{ext} \rangle$	$\langle V_{int} \rangle$
1.00	0.98847	0.39965	3.0068	0.00001	0.00009	1.636	0.8944	1.2990	0.8135
0.50	0.98061	0.31091	1.6674	0.00001	0.00010	2.481	0.4488	0.7051	0.5135
0.10	0.94693	0.17764	0.4486	0.00001	0.00011	6.695	0.1003	0.1767	0.1716
0.05	0.92747	0.13815	0.2609	0.00000	0.00011	10.389	0.0533	0.0997	0.1076
0.01	0.88398	0.07287	0.0777	0.00000	0.00006	29.177	0.0129	0.0284	0.0364

Table 7: Ground state energy of two interacting electrons in harmonic oscillator trap found with importance sampling. Number of MC cycles are  $2^{23}$

$\omega$	$\alpha$	$\beta$	$\langle E_L \rangle$	SEM	$\sigma_B$	$\bar{r}_{12}$	$\langle T \rangle$	$\langle V_{ext} \rangle$	$\langle V_{int} \rangle$
1.00	0.98846	0.39954	3.0069	0.00001	0.00018	1.643	0.8931	1.3052	0.8086
0.50	0.98082	0.31068	1.6674	0.00001	0.00019	2.481	0.4547	0.6997	0.5130
0.10	0.94734	0.17810	0.4486	0.00001	0.00023	6.724	0.0989	0.1787	0.1710
0.05	0.92262	0.14090	0.2610	0.00000	0.00021	10.333	0.0495	0.1024	0.1091

### 3.2 One-body density

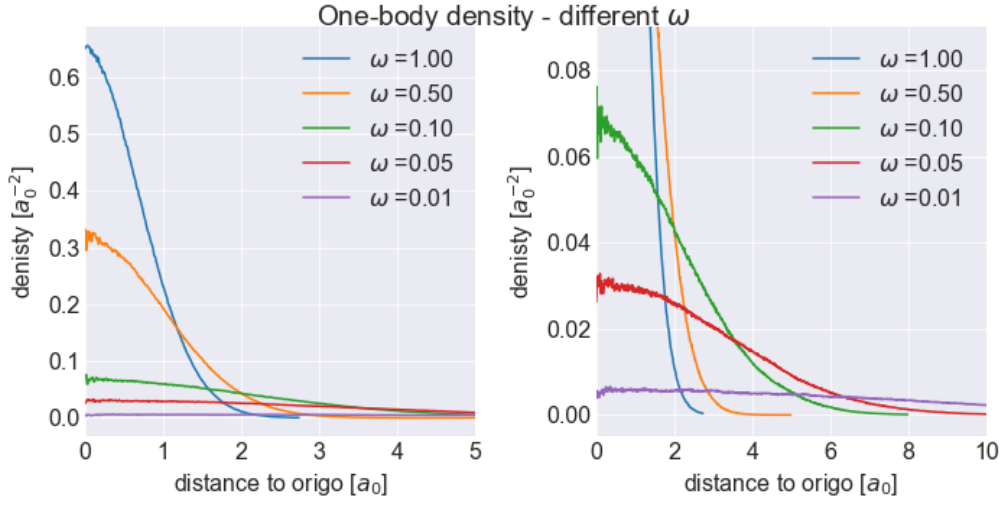


Figure 5: •

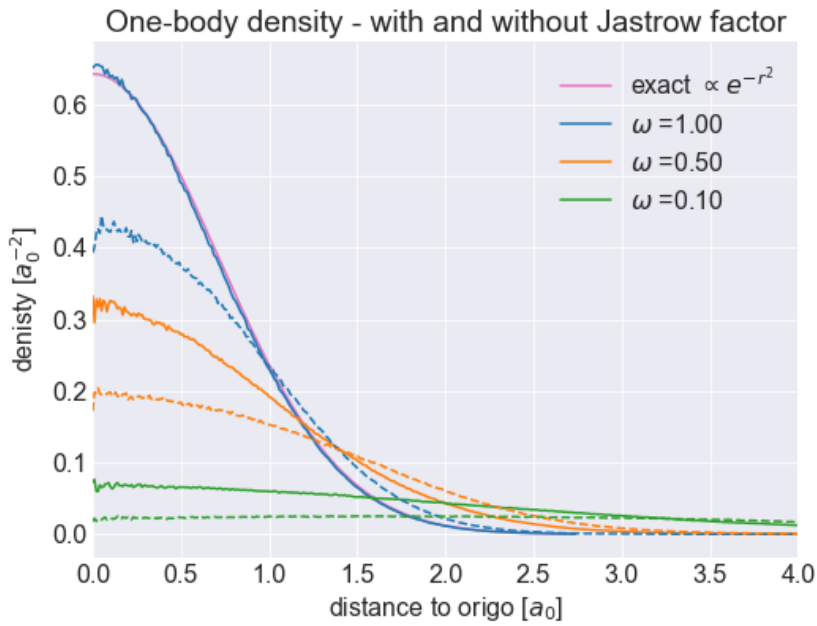


Figure 6: Dashed lines are with the Jastrow factor. Normal lines is without Jastrow factor.

### 3.3 Extending to more particles

#### 3.3.1 Six particles

Table 8: Number of MC cycles are  $2^{23}$ .

$\omega$	$\langle E_L \rangle$	$\langle T \rangle$	$\langle V_{ext} \rangle$
1.00	10.00	4.9865	5.0135
0.50	5.00	2.4973	2.5027
0.10	1.00	0.4861	0.5139
0.05	0.50	0.2483	0.2517
0.01	0.10	0.0454	0.0546

Table 9: Ground state energy of two interacting electrons in harmonic oscillator trap found with importance sampling. Number of MC cycles are  $2^{23}$

$\omega$	$\alpha$	$\beta$	$\langle E_L \rangle$	SEM	$\sigma_B$	$\langle T \rangle$	$\langle V_{ext} \rangle$	$\langle V_{int} \rangle$
1.00	0.71567	0.49372	20.4492	0.00022	0.00813	2.3429	10.7076	7.3988
0.50	0.75823	0.34260	11.9868	0.00011	0.00522	1.3226	5.8094	4.8548
0.10	0.78852	0.15041	3.6542	0.00003	0.00416	0.2951	1.7035	1.6556
0.05	0.76518	0.10733	2.2223	0.00003	0.00436	0.1178	1.0882	1.0162

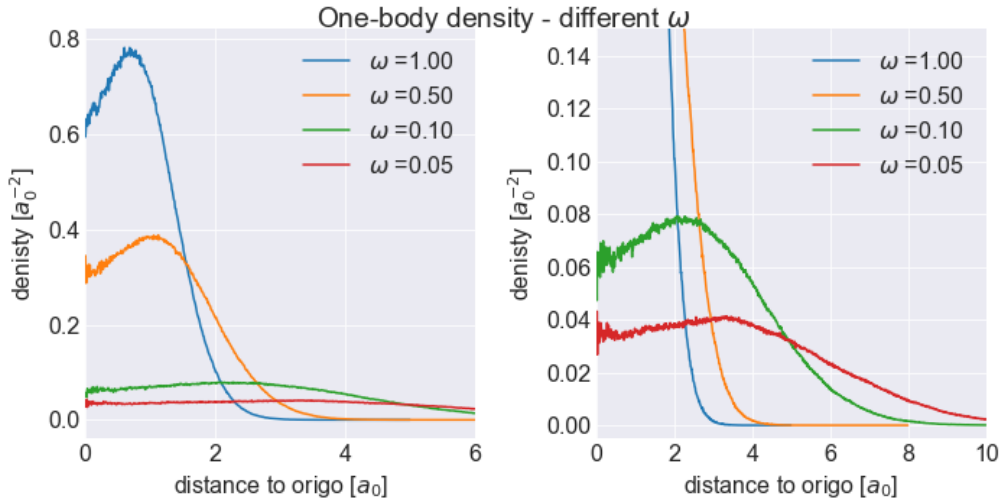


Figure 7: •

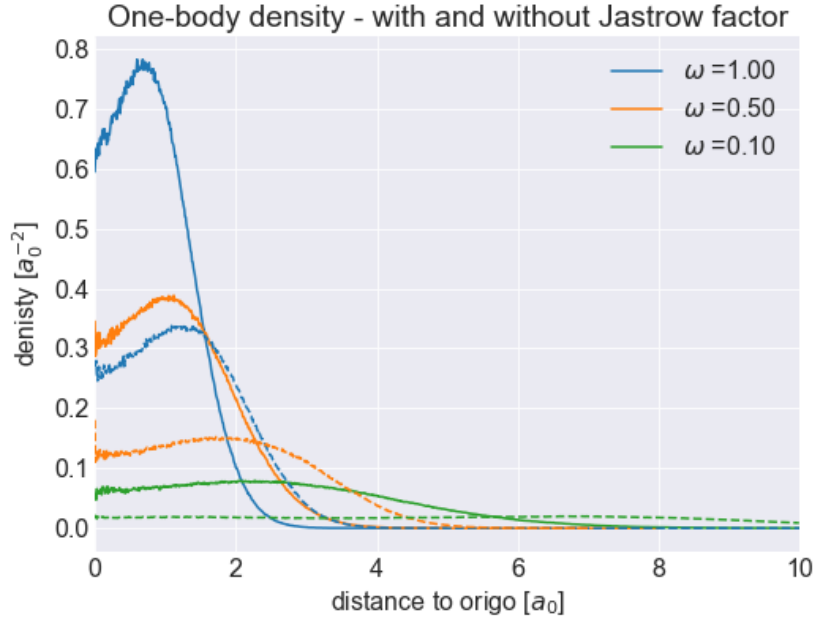


Figure 8: Dashed lines are with the Jastrow factor. Normal lines is without Jastrow factor.

### 3.3.2 Twelve particles

Table 10: Number of MC cycles are  $2^{23}$ .

$\omega$	$\langle E_L \rangle$	$\langle T \rangle$	$\langle V_{ext} \rangle$
1.00	28.00	14.0117	13.9883
0.50	14.00	7.0463	6.9537
0.10	2.80	1.4084	1.3916
0.05	1.40	0.6901	0.7099
0.01	0.28	0.1419	0.1381

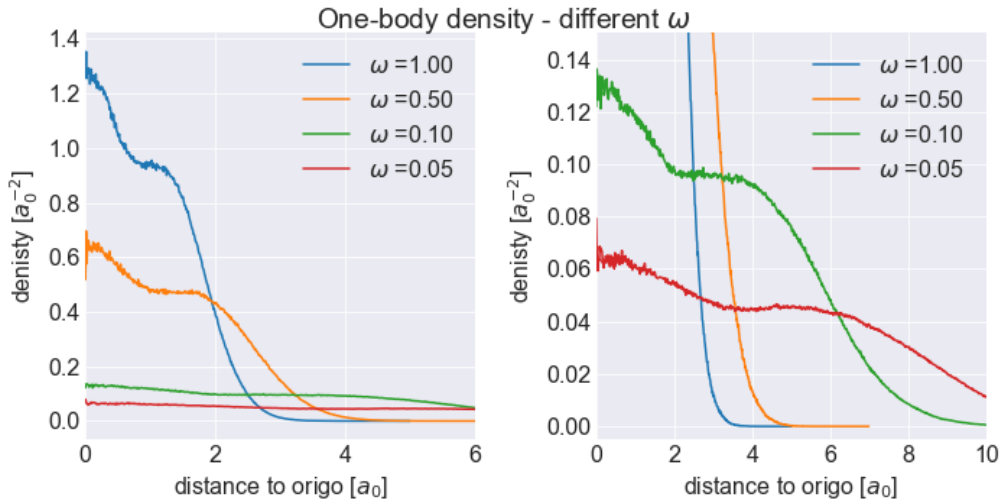


Figure 9: Had to use MC 2 24 to get smooth graphs.

# Appendices

## A Iteration - derivatives with regards to parameters

$$\frac{\partial \psi_T}{\partial \alpha} = -\frac{\omega}{2} \sum_i^N r_i^2 \quad (11)$$

$$\frac{\partial \psi_T}{\partial \beta} = -\sum_{i<j}^N \frac{a_{ij} r_{ij}^2}{(1 + \beta r_{ij})^2} \quad (12)$$

## B Dealing with the Slater determinant efficiently

Determining a determinant numerically is a costly operation, so we want to do some alteration to increase the efficiency of the code.

### B.1 Slater determinant

The Slater determinant contains the single-particle wave function of the number of particles included in the system evaluated at the position for all particles included because electrons are indistinguishable. The determinant is written as

$$D = \text{Det}(\phi_1(\mathbf{r}_1), \phi_2(\mathbf{r}_2), \dots, \phi_N(\mathbf{r}_N)) = \begin{vmatrix} \phi_1(\mathbf{r}_1) & \phi_2(\mathbf{r}_1) & \cdots & \phi_N(\mathbf{r}_1) \\ \phi_1(\mathbf{r}_2) & \phi_2(\mathbf{r}_2) & \cdots & \phi_N(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{r}_N) & \phi_2(\mathbf{r}_N) & \cdots & \phi_N(\mathbf{r}_N) \end{vmatrix}$$

Hence the rows represent different positions,  $r_i$ , and the columns represent different states. To simplify the calculations we want to have all states with spin up in one determinant and all states with spin down in another determinant. For six particles we then get

$$D = D_{\uparrow} D_{\downarrow} = \begin{vmatrix} \phi_1(\mathbf{r}_1) & \phi_3(\mathbf{r}_1) & \phi_5(\mathbf{r}_1) \\ \phi_1(\mathbf{r}_2) & \phi_3(\mathbf{r}_2) & \phi_5(\mathbf{r}_2) \\ \phi_1(\mathbf{r}_3) & \phi_3(\mathbf{r}_3) & \phi_5(\mathbf{r}_3) \end{vmatrix} \begin{vmatrix} \phi_2(\mathbf{r}_4) & \phi_4(\mathbf{r}_4) & \phi_6(\mathbf{r}_4) \\ \phi_2(\mathbf{r}_5) & \phi_4(\mathbf{r}_5) & \phi_6(\mathbf{r}_5) \\ \phi_2(\mathbf{r}_6) & \phi_4(\mathbf{r}_6) & \phi_6(\mathbf{r}_6) \end{vmatrix}.$$

We see this from Tab. 1 and Eq. 3. *Mister anti-symmetrien, men expectation value er lik.*

The trial wavefunction can therefore be rewritten to

$$\psi_T = D_{\uparrow} D_{\downarrow} \psi_C$$

where  $\psi_C$  is the correlation part of the trial wavefunction. Now we only have to update one of these matrices when we move a particle, depending on which spin the particle has.

## B.2 The Metropolis ratio

In the metropolis test we calculate the ratio between the wavefunction before and after a proposed move, but now the wavefunction includes a determinant which is costly to calculate. We therefore want to utilize some relations from linear algebra to simplify the ratio and make the algorithm more efficient. The ratio between the Slater determinant part of the wavefunction,  $\psi_{SD}$ , is

$$R = \frac{\psi_{SD}(\mathbf{r}^{new})}{\psi_{SD}(\mathbf{r}^{old})} = \frac{\sum_i^N d_{ij}(\mathbf{r}^{new}) C_{ij}(\mathbf{r}^{new})}{\sum_i^N d_{ij}(\mathbf{r}^{old}) C_{ij}(\mathbf{r}^{old})}. \quad (13)$$

where  $d_{ij} = \psi_i(j)$

Here we have used the fact that when you calculate a determinant, you break it down into a sum of smaller determinants times a factor:

$$D = \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{NN} \end{vmatrix} = \sum_i^N d_{ij} C_{ij}.$$

So if  $d_{ij} = d_{11}$  then

$$C_{11} = \begin{vmatrix} d_{22} & d_{23} & \cdots & d_{2N} \\ d_{32} & d_{33} & \cdots & d_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N2} & d_{N3} & \cdots & d_{NN} \end{vmatrix}.$$

We observe in Eq. 13 that if we move particle  $j$  from  $r_j^{old}$  to  $r_j^{new}$  the matrix  $C_{ij}$  is unchanged, we have only changed the  $d_{ij}$  in the original determinant  $D$  that is not included in  $C_{ij}$ . Equation 13 is then

$$R = \frac{\sum_i^N d_{ij}(\mathbf{r}^{new})}{\sum_i^N d_{ij}(\mathbf{r}^{old})} \quad (14)$$

We can simplify this even further with the relation

$$\sum_{k=1}^N d_{ik} d_{kj}^{-1} = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (15)$$

The ratio can be rewritten as

$$R = \frac{\sum_i^N d_{ij}(\mathbf{r}^{new}) d_{ij}(\mathbf{r}^{old})^{-1}}{\sum_i^N d_{ij}(\mathbf{r}^{old}) d_{ij}(\mathbf{r}^{old})^{-1}} = \sum_i^N d_{ij}(\mathbf{r}^{new}) d_{ij}(\mathbf{r}^{old})^{-1}. \quad (16)$$

The consequence of these calculations are that we now only have to calculate the invers values of the determinant once to know the values for  $d_{ij}(\mathbf{r}^{old})^{-1}$  and then update only the row of the position that was changed in the Slater determinant and calculate the invers of the determinant again only if the move is accepted.

### B.3 Updating the inverse of the Slater determinant

After a move is accepted in the Metropolis test, the row in the Slater determinant representing that particle is updated, but the inverse of the Slater determinant also needs to be updated because the Slater determinant has changed. This could be done by simply calculating the inverse of the determinant, but this is costly and there is a more efficient way. The elements of the determinant  $d_{kj}^{-1}$  (hva betyr den  $^{-1}$ ? At den skal opphøyes i minus 1 eller er det notasjon på at det er et element i den inverse matrisen?) can be found through

$$d_{kj}^{-1}(\mathbf{r}^{new}) = \begin{cases} d_{kj}^{-1}(\mathbf{r}^{old}) - \frac{d_{kj}^{-1}(\mathbf{r}^{old})}{R} \sum_{l=1}^N d_{il}^{-1}(\mathbf{r}^{new}) d_{lj}^{-1}(\mathbf{r}^{old}) & \text{if } i \neq j \\ \frac{d_{kj}^{-1}(\mathbf{r}^{old})}{R} \sum_{l=1}^N d_{il}^{-1}(\mathbf{r}^{old}) d_{lj}^{-1}(\mathbf{r}^{old}) & \text{if } i = j \end{cases},$$

where  $i$  is the number of the row representing the particle that was moved.

## C Energies

$$E_{n_x n_y} = \hbar\omega(n_x + n_y + \frac{d}{2}) \quad (17)$$

where  $d$  is the number of dimensions. In this project  $d = 2$ .

Table 11: The exact energies for the non-interacting case with different number of particles in a closed shell system.

Energies	
$E_{00}$	$\hbar\omega$
$E_{10} = E_{01}$	$2\hbar\omega$
$E_{20} = E_{02} = E_{11}$	$3\hbar\omega$
$E_{30} = E_{03} = E_{21} = E_{12}$	$4\hbar\omega$
$E_{N=2} = 2E_{00}$	$2\hbar\omega$
$E_{N=6} = E_{N=2} + 2E_{10} + 2E_{01}$	$10\hbar\omega$
$E_{N=12} = E_{N=6} + 2E_{20} + 2E_{02} + 2E_{11}$	$28\hbar\omega$
$E_{N=20} = E_{N=12} + 2E_{30} + 2E_{03} + 2E_{21} + 2E_{12}$	$60\hbar\omega$

## D Hermite polynomials and the wavefunction derivatives

The relevant Hermite polynomials

$H_0(\sqrt{\omega}x)$	1
$H_1(\sqrt{\omega}x)$	$2\sqrt{\omega}x$
$H_2(\sqrt{\omega}x)$	$4\omega x^2 - 2$
$H_3(\sqrt{\omega}x)$	$8\omega\sqrt{\omega}x^3 - 12\sqrt{\omega}x$

$$\phi_{n_x, n_y}(x, y) = AH_{n_x}(\sqrt{\omega}x)H_{n_y}(\sqrt{\omega}y) \exp(-\omega(x^2 + y^2)/2).$$

Table 12:  $\psi_{n_x n_y}$

Trial wavefunctions for the different states

$\psi_{00}$	$A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{01}$	$2\sqrt{\omega}x A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{10}$	$2\sqrt{\omega}y A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{20}$	$(4\omega x^2 - 2) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{02}$	$(4\omega y^2 - 2) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{11}$	$4\omega xy A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{30}$	$(8\omega\sqrt{\omega}x^3 - 12\sqrt{\omega}x) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{03}$	$(8\omega\sqrt{\omega}y^3 - 12\sqrt{\omega}y) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{21}$	$(8\omega\sqrt{\omega}x^2y - 4\sqrt{\omega}y) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\psi_{12}$	$(8\omega\sqrt{\omega}xy^2 - 4\sqrt{\omega}x) A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$



Table 13:  $\psi_{n_x n_y}$ 

The derivative of the trial wavefunctions for the different states

$\nabla\psi_{00}$	$(-\alpha\omega x, -\alpha\omega y)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{01}$	$-(\sqrt{\omega}(a\omega x^2 - 1), \alpha\omega^{3/2}xy)2A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{10}$	$-(\alpha\omega^{3/2}xy, \sqrt{\omega}(a\omega y^2 - 1))2A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{20}$	$-(2\alpha\omega^2x^3 - \alpha\omega x - 4\omega x, 2\alpha\omega^2x^2y - \alpha\omega y)2A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{02}$	$-(2\alpha\omega^2xy^2 - \alpha\omega x, 2\alpha\omega^2y^3 - \alpha\omega y - 4\omega y)2A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{11}$	$(-4\omega y(\alpha\omega x^2 - 1), -4\omega x(\alpha\omega y^2 - 1))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{30}$	$(-4\sqrt{\omega}(2\alpha\omega^2x^4 - 3(\alpha + 2)\omega x^2 + 3), -4\alpha\omega^{3/2}xy(2\omega x^2 - 3))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{03}$	$(-4\sqrt{\omega}(-4\alpha\omega^{3/2}xy(2\omega y^2 - 3), 2\alpha\omega^2y^4 - 3(\alpha + 2)\omega y^2 + 3))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{21}$	$(-4\sqrt{\omega}(\alpha\omega x^2(2\omega xy - 1) - 4\omega xy + 1), -4\omega^{3/2}x(2x(\alpha\omega y^2 - 1) - \alpha y))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla\psi_{12}$	$(-4\omega^{3/2}y(2y(\alpha\omega x^2 - 1) - \alpha x), -4\sqrt{\omega}(\alpha\omega y^2(2\omega xy - 1) - 4\omega xy + 1))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$

Table 14:  $\psi_{n_x n_y}$ 

The double derivative of the trial wavefunctions for the different states

$\nabla^2\psi_{00}$	$(\alpha^2\omega^2r^2 - \alpha\omega)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{01}$	$2\alpha\omega^{3/2}x(\alpha\omega r^2 - 4)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{10}$	$2\alpha\omega^{3/2}y(\alpha\omega r^2 - 4)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{20}$	$2\omega(\alpha^2\omega(2\omega x^2 - 1)r^2 + \alpha(2 - 12\omega x^2) + 4))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{02}$	$2\omega(\alpha^2\omega(2\omega y^2 - 1)r^2 + \alpha(2 - 12\omega y^2) + 4))A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{11}$	$4\alpha\omega^2xy(\alpha\omega r^2 - 6)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{30}$	$4\omega^{3/2}x(\alpha^2\omega(2\omega x^2 - 3)r^2 - 4\alpha(4\omega x^2 - 3) + 12)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{03}$	$4\omega^{3/2}y(\alpha^2\omega(2\omega y^2 - 3)r^2 - 4\alpha(4\omega y^2 - 3) + 12)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{21}$	$4\omega^{3/2}(\alpha^2\omega xr^2(2\omega xy - 1) + 4\alpha x(1 - 4\omega xy) + 4y)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$
$\nabla^2\psi_{12}$	$4\omega^{3/2}(\alpha^2\omega yr^2(2\omega xy - 1) + 4\alpha y(1 - 4\omega xy) + 4x)A \exp\left(\frac{-\alpha\omega r^2}{2}\right)$

## References

- [1] Vilde M Reinertsen. Project 1, 2020.
- [2] Even Marius Nordhagen. Studies of quantum dots using machine learning. Master's thesis, University of Oslo, 2019.