

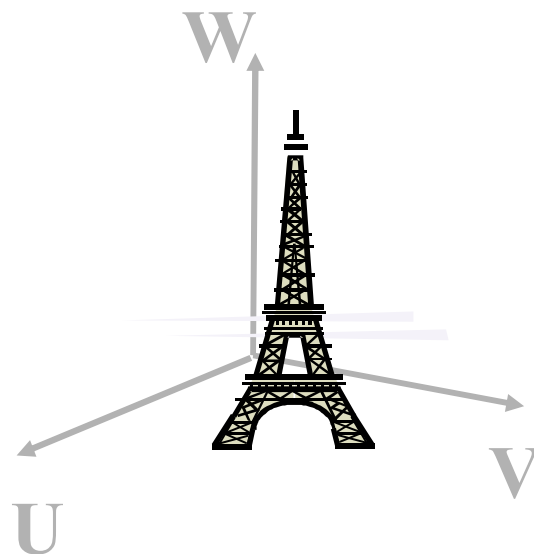
# **Lecture 12:**

# **Camera Projection**

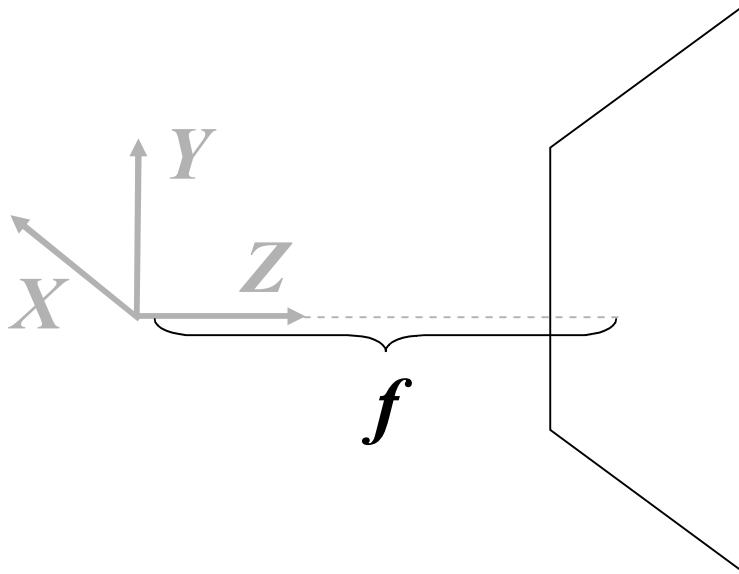
Reading: T&V Section 2.4

# Imaging Geometry

**Object of Interest  
in World Coordinate  
System (U,V,W)**



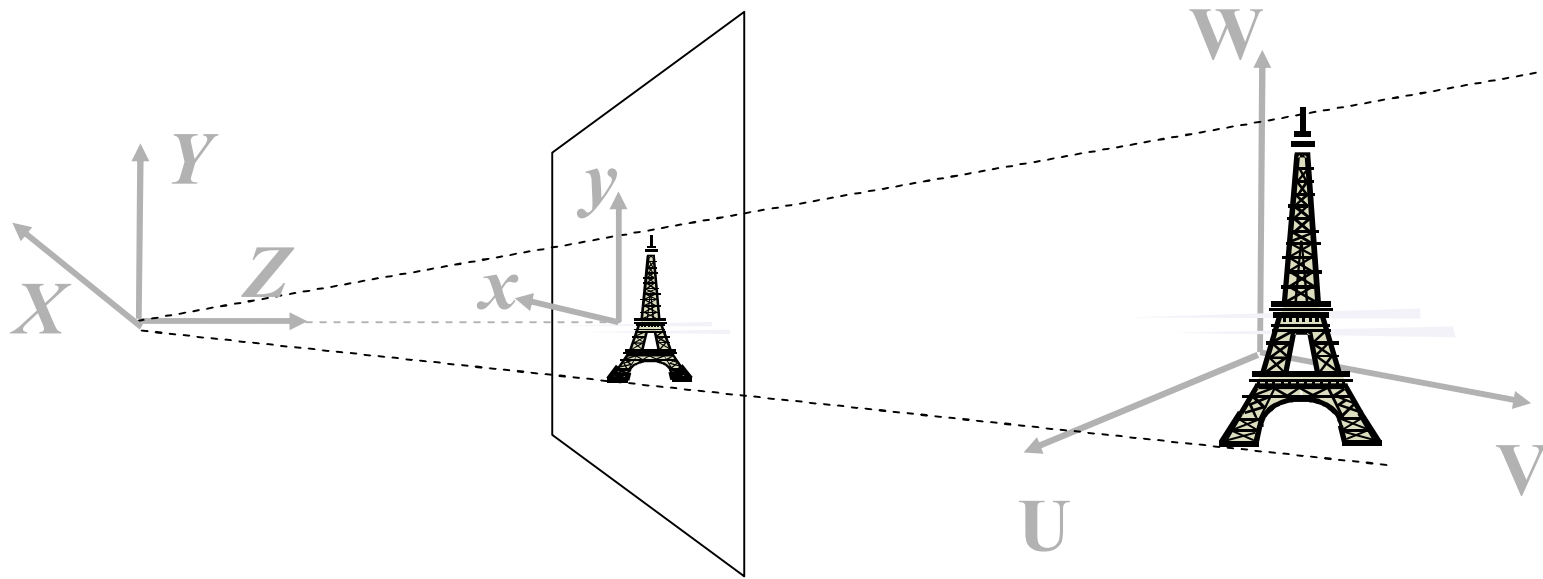
# Imaging Geometry



## Camera Coordinate System ( $X, Y, Z$ ).

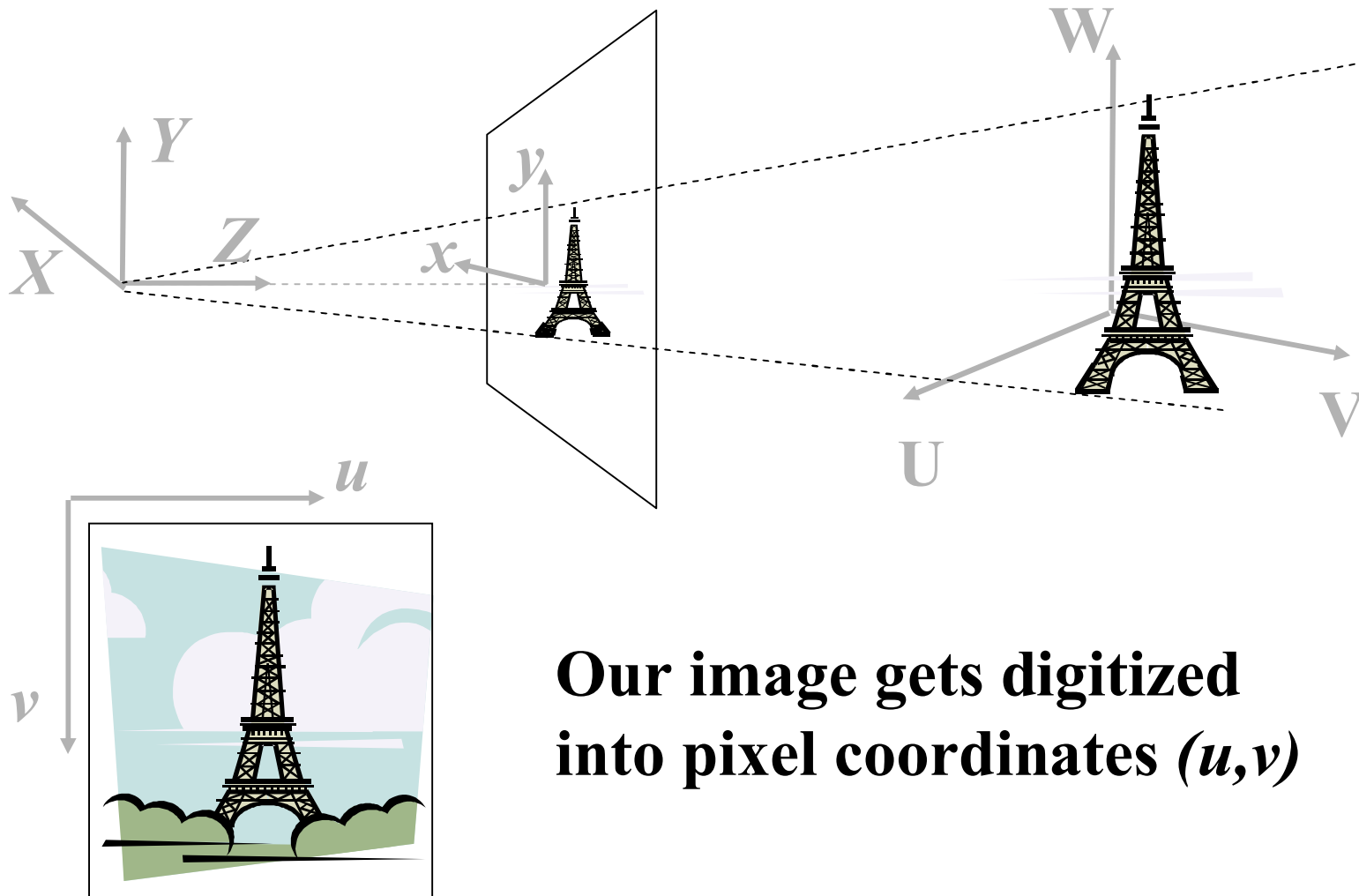
- $Z$  is optic axis
- Image plane located  $f$  units out along optic axis
- $f$  is called focal length

# Imaging Geometry



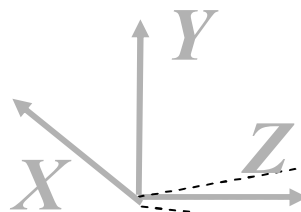
**Forward Projection onto image plane.  
3D  $(X, Y, Z)$  projected to 2D  $(x, y)$**

# Imaging Geometry

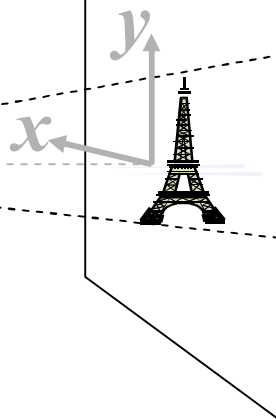


# Imaging Geometry

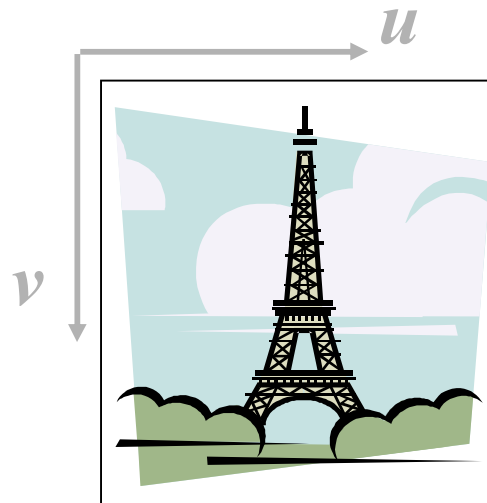
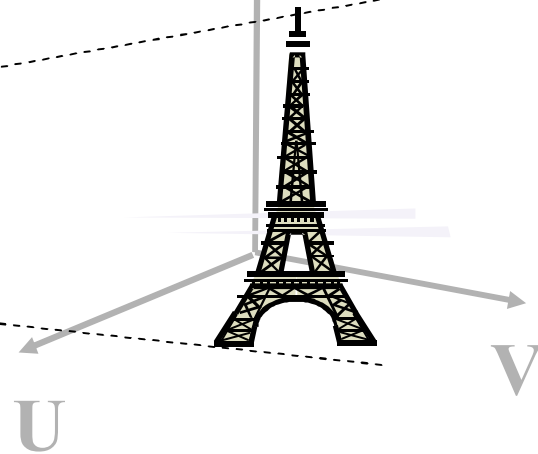
**Camera  
Coordinates**



**Image (film)  
Coordinates**

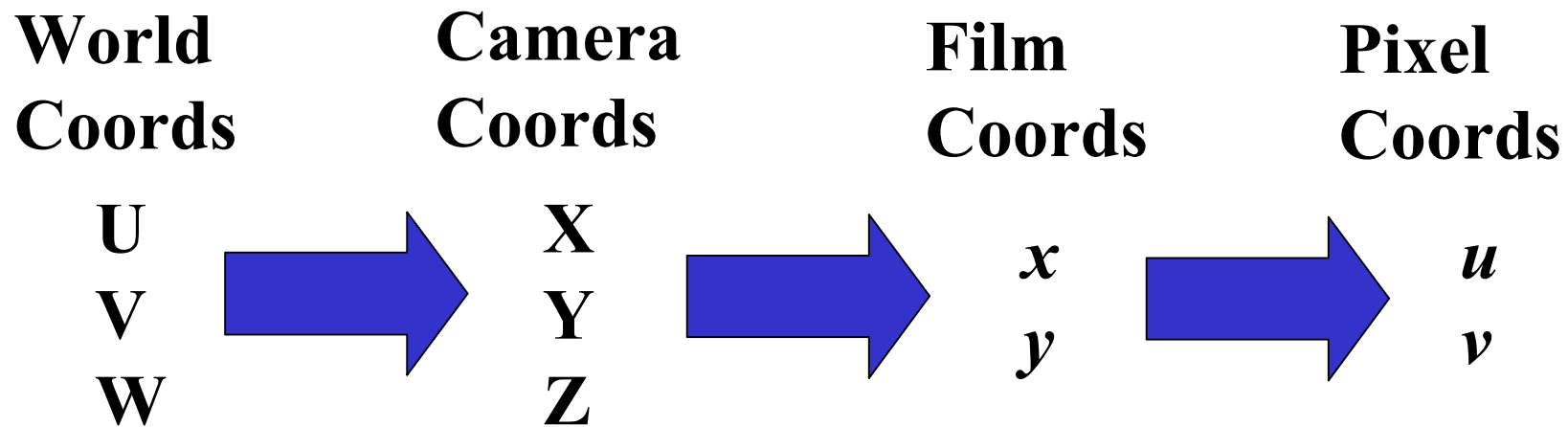


**World  
Coordinates**



**Pixel  
Coordinates**

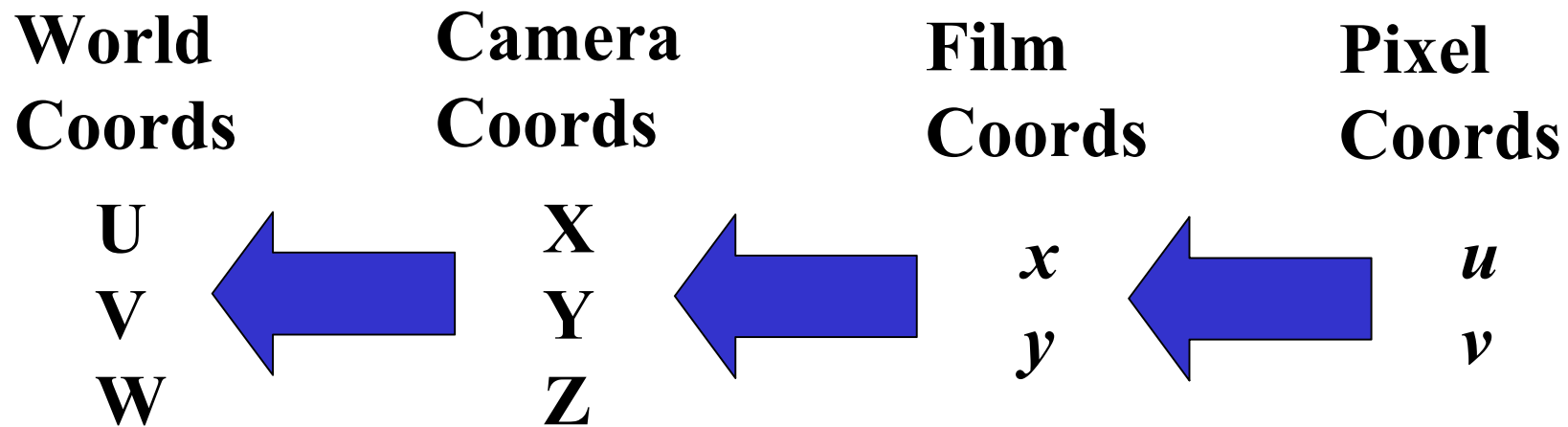
# Forward Projection



**We want a mathematical model to describe how 3D World points get projected into 2D Pixel coordinates.**

**Our goal: describe this sequence of transformations by a big matrix equation!**

# Backward Projection

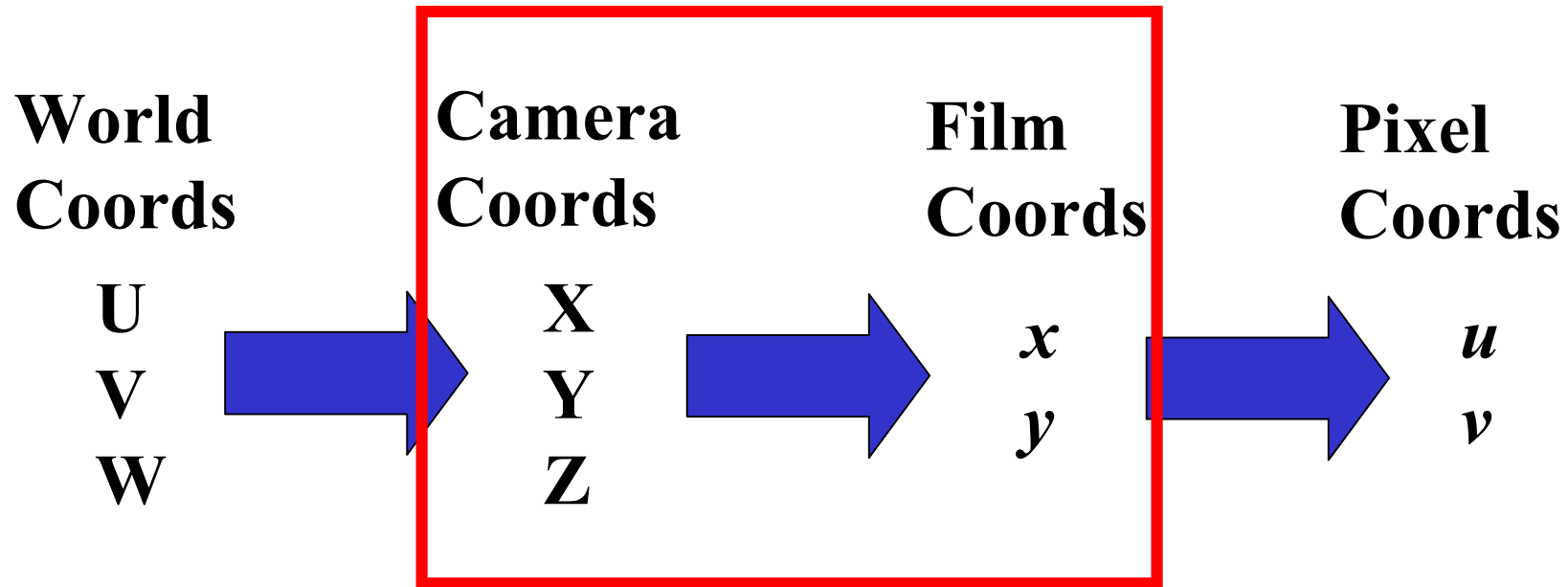


**Note, much of vision concerns trying to derive backward projection equations to recover 3D scene structure from images (via stereo or motion)**

**But first, we have to understand forward projection...**



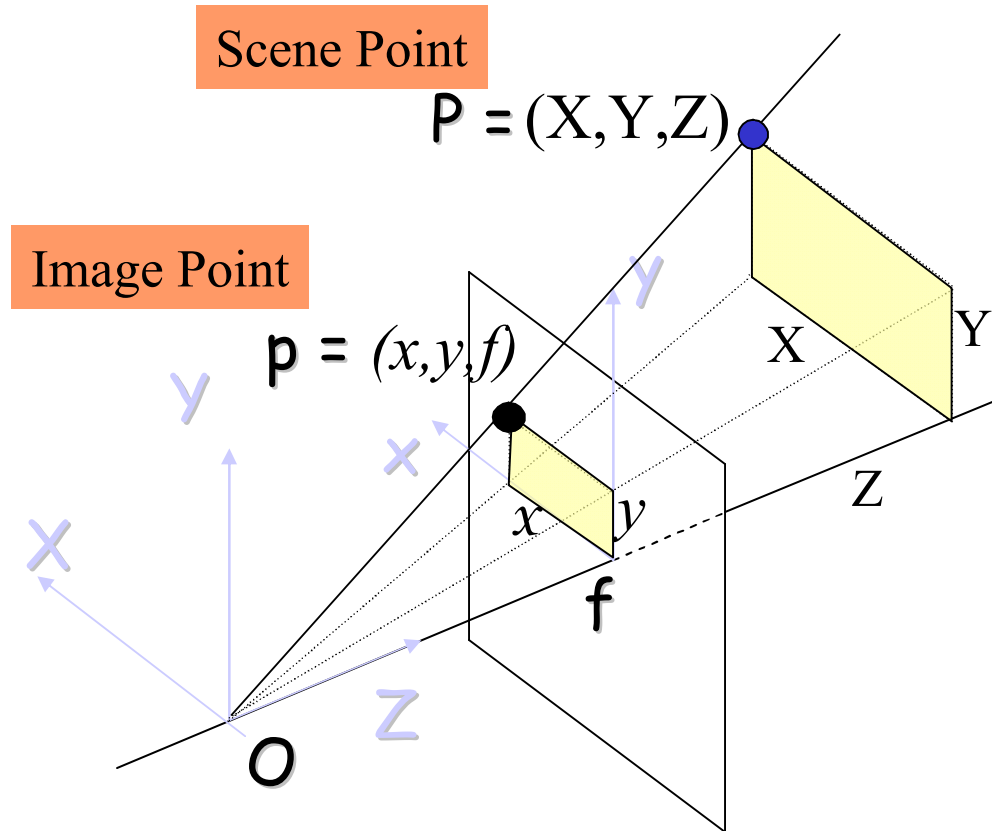
# Forward Projection



**3D-to-2D Projection**  
• perspective projection

We will start here in the middle, since we've already talked about this when discussing stereo.

# Basic Perspective Projection

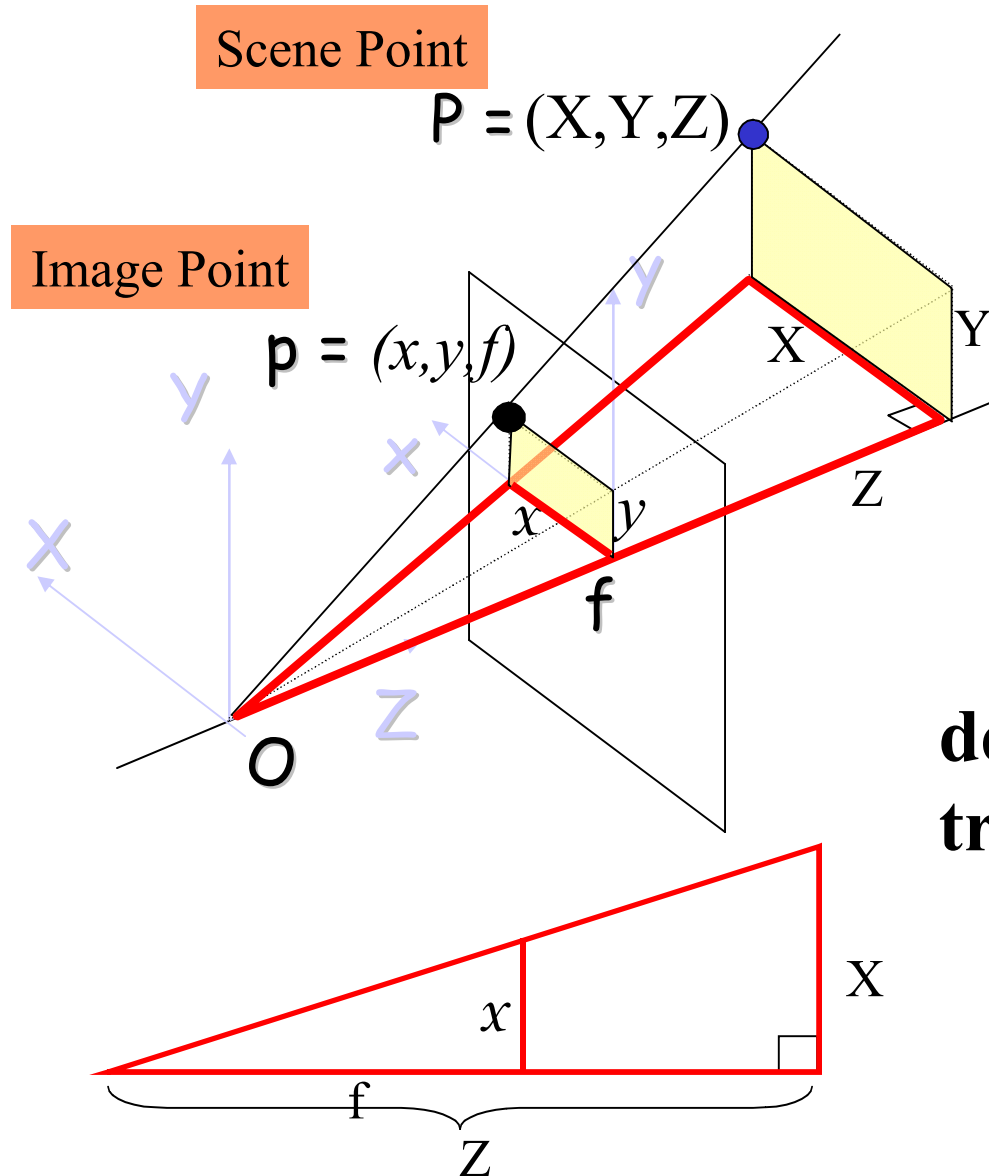


Perspective Projection Eqns

$$x = f \frac{X}{Z}$$

$$y = f \frac{Y}{Z}$$

# Basic Perspective Projection



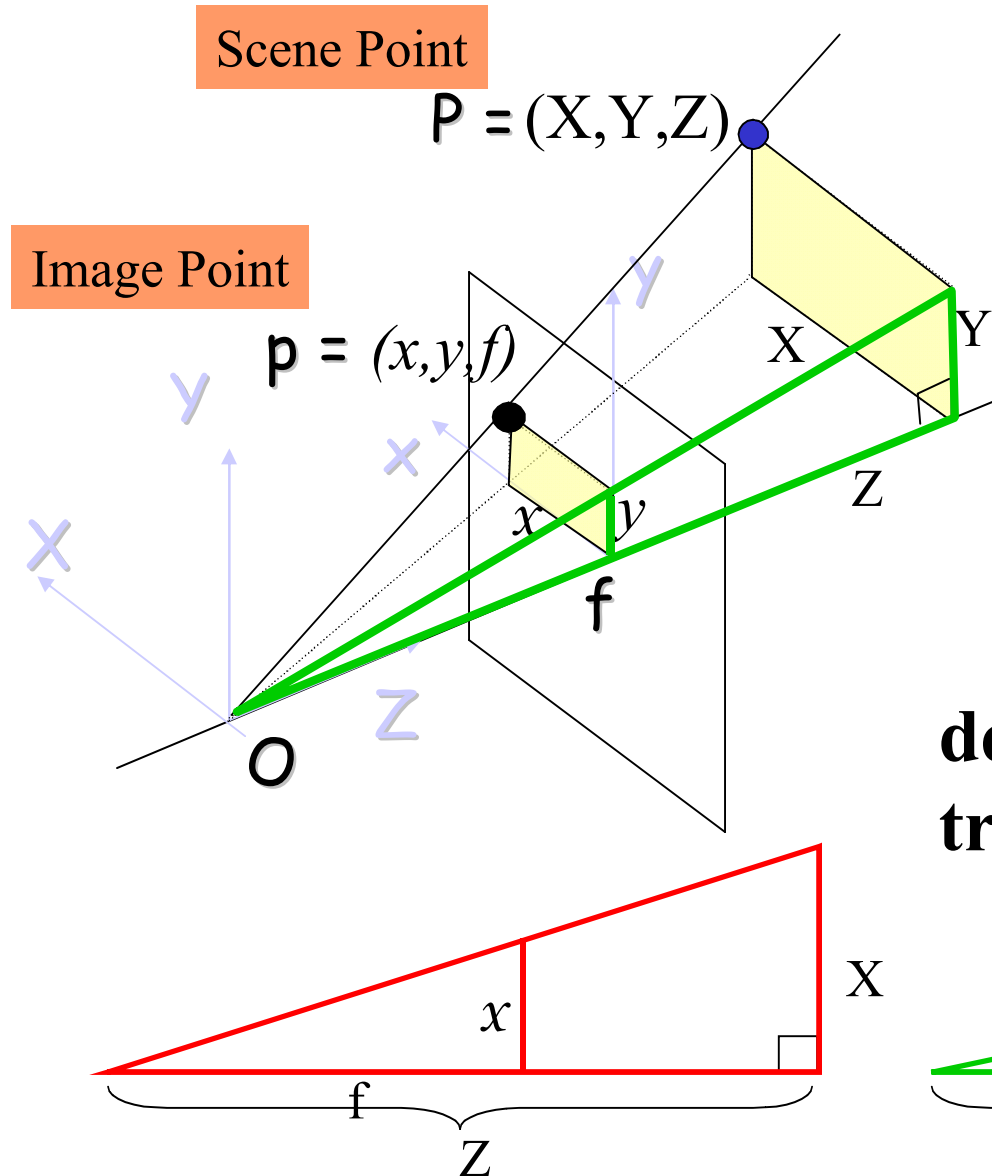
Perspective Projection Eqns

$$x = f \frac{X}{Z}$$

$$y = f \frac{Y}{Z}$$

derived via similar  
triangles rule

# Basic Perspective Projection

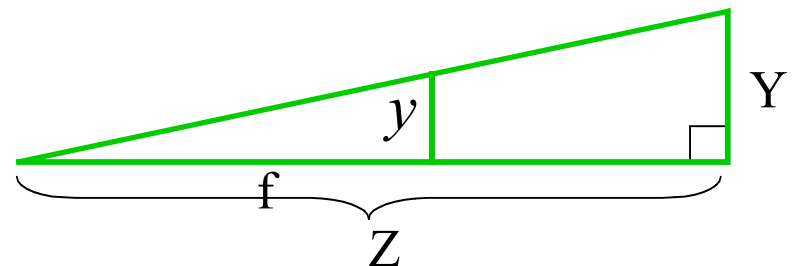
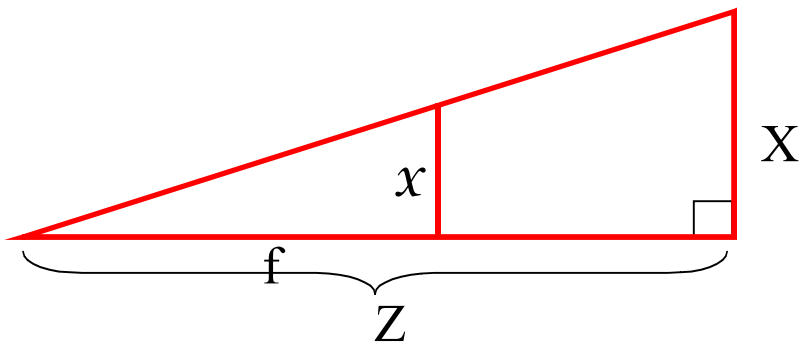


Perspective Projection Eqns

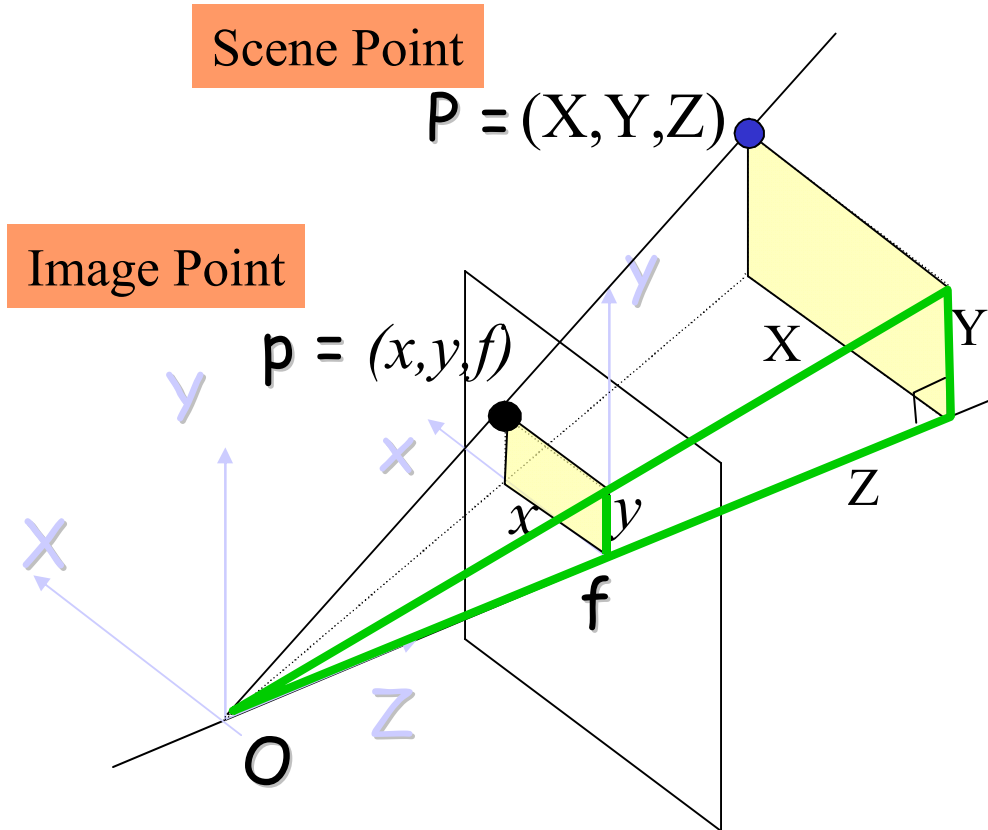
$$x = f \frac{X}{Z}$$

$$y = f \frac{Y}{Z}$$

derived via similar  
triangles rule



# Basic Perspective Projection



## Perspective Projection Eqns

$$x = f \frac{X}{Z}$$

$$y = f \frac{Y}{Z}$$

**So how do we represent this as a matrix equation?  
We need to introduce homogeneous coordinates.**

# Homogeneous Coordinates

Represent a 2D point  $(x,y)$  by a 3D point  $(x',y',z')$  by adding a “fictitious” third coordinate.

By convention, we specify that given  $(x',y',z')$  we can recover the 2D point  $(x,y)$  as

$$x = \frac{x'}{z'} \quad y = \frac{y'}{z'}$$

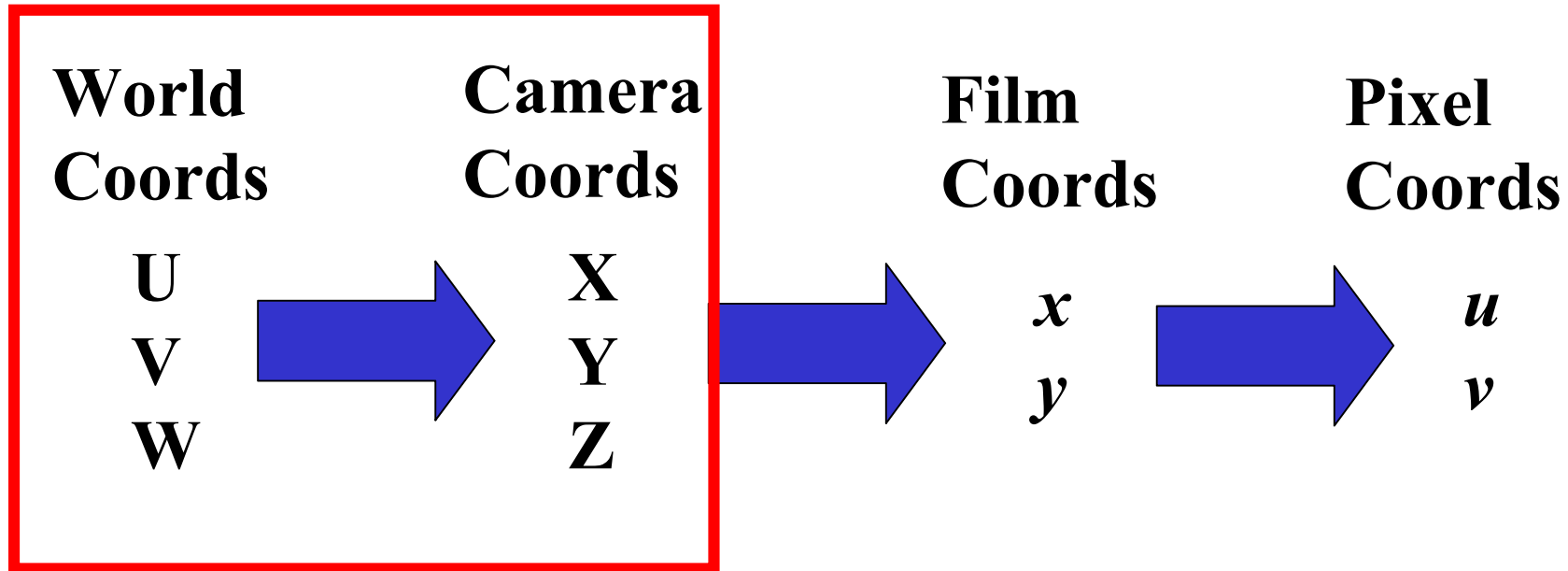
Note:  $(x,y) = (x,y,1) = (2x, 2y, 2) = (k x, k y, k)$   
for any nonzero  $k$  (can be negative as well as positive)

# Perspective Matrix Equation

(in Camera Coordinates)

$$\begin{aligned} x &= f \frac{X}{Z} \\ y &= f \frac{Y}{Z} \end{aligned} \iff \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

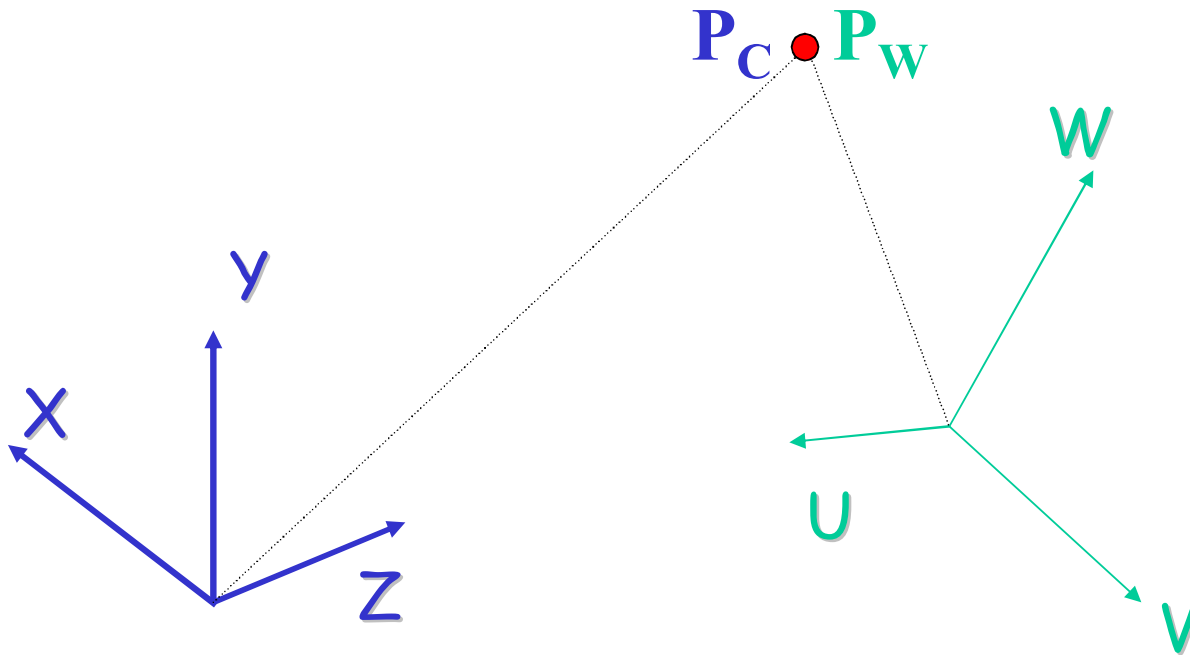
# Forward Projection



**Rigid Transformation (rotation+translation)**  
between world and camera coordinate systems

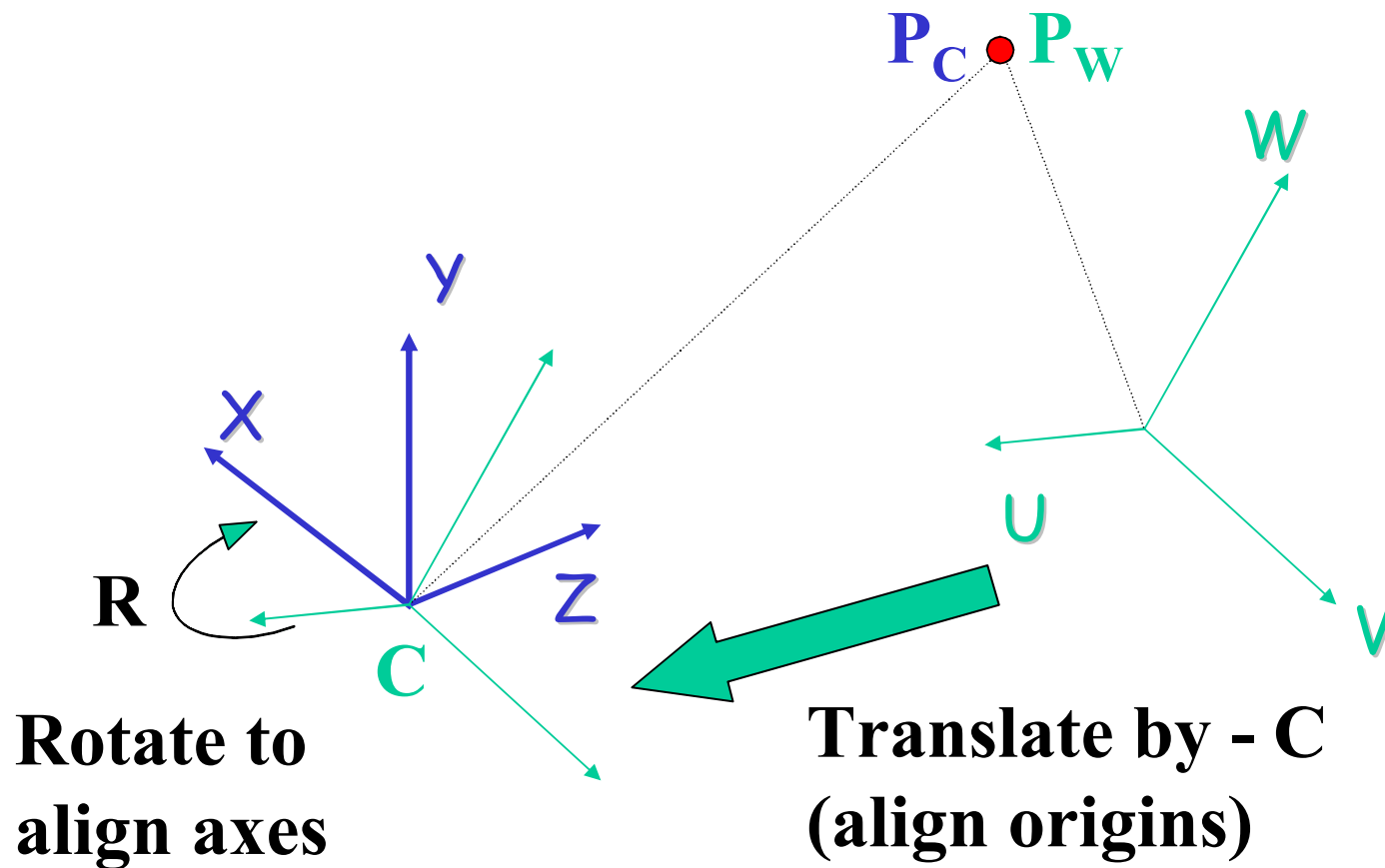


# World to Camera Transformation



Avoid confusion:  $P_W$  and  $P_C$  are not two different points. They are the same physical point, described in two different coordinate systems.

# World to Camera Transformation



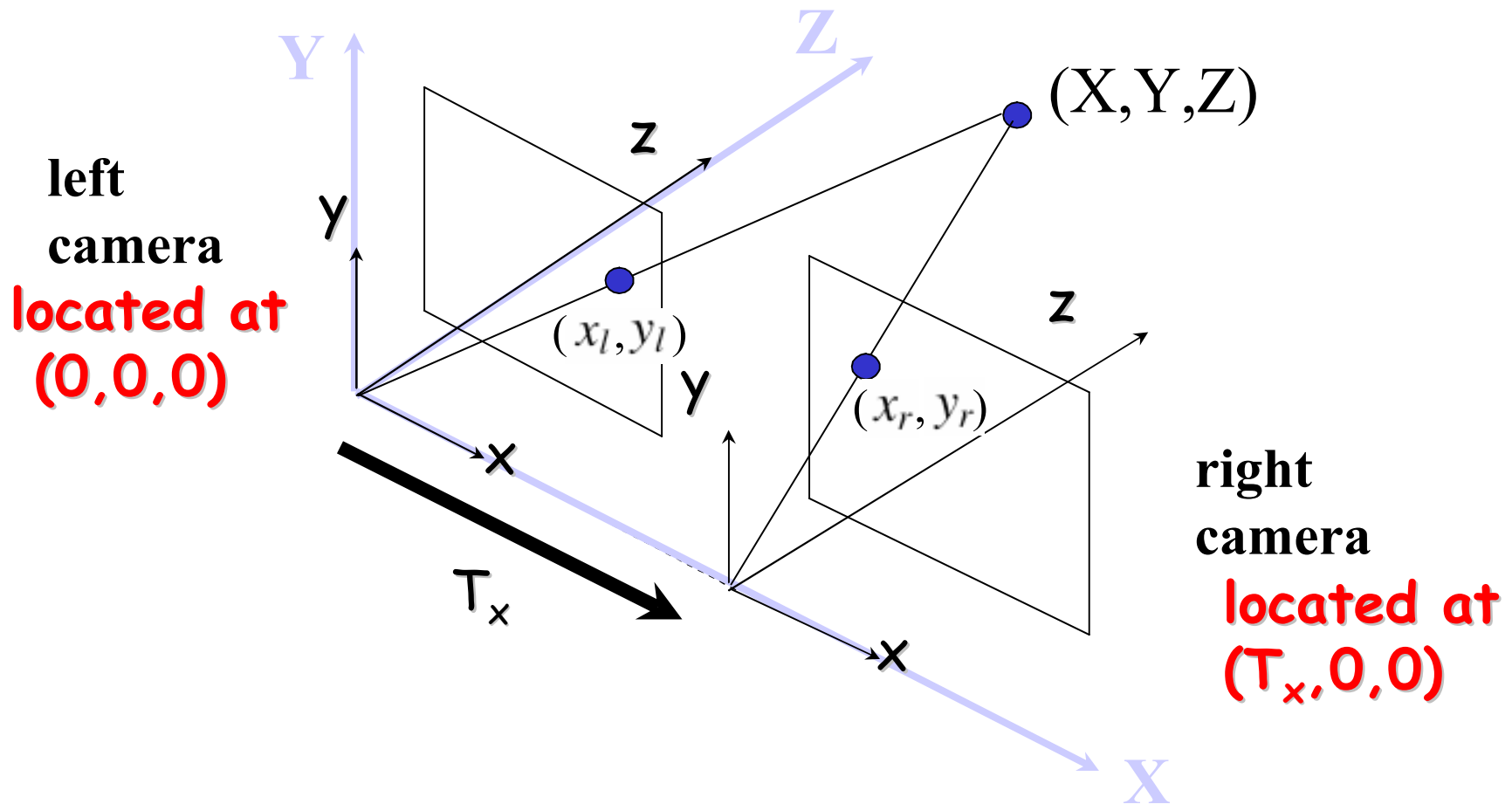
$$P_C = R ( P_W - C )$$

# Matrix Form, Homogeneous Coords

$$\mathbf{P}_C = \mathbf{R} ( \mathbf{P}_W - \mathbf{C} )$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

## Example: Simple Stereo System



Left camera located at world origin  $(0,0,0)$   
and camera axes aligned with world coord axes.

# Simple Stereo, Left Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned  
with world axes

located at world  
position (0,0,0)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

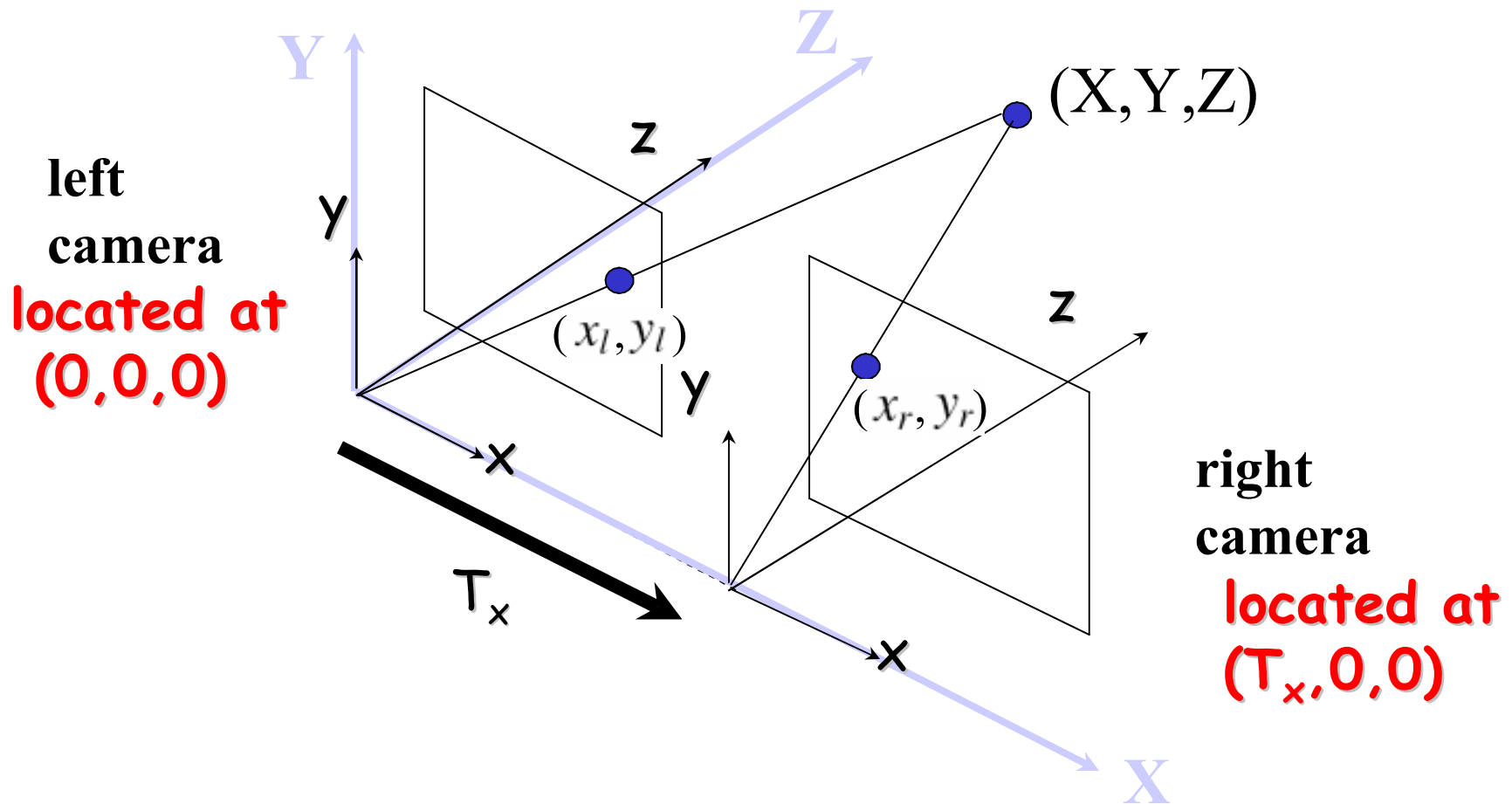
# Simple Stereo Projection Equations

**Left camera**

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_l = f \frac{X}{Z} \quad y_l = f \frac{Y}{Z}$$

## Example: Simple Stereo System



Right camera located at world location  $(T_x, 0, 0)$   
and camera axes aligned with world coord axes.

# Simple Stereo, Right Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned  
with world axes

located at world  
position  $(T_x, 0, 0)$

$$= \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Simple Stereo Projection Equations

## Left camera

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_l = f \frac{X}{Z} \quad y_l = f \frac{Y}{Z}$$

## Right camera

$$\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_r = f \frac{X - T_x}{Z} \quad y_r = f \frac{Y}{Z}$$

# Bob's sure-fire way(s) to figure out the rotation

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cancel{1} & 0 & 0 & \cancel{-c_x} \\ \text{forget about this} \\ \text{while thinking} \\ \text{about rotations} \\ \cancel{0} & 0 & 0 & \cancel{1} \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

$$\mathbf{P}_C = \mathbf{R} \mathbf{P}_W$$

This equation says how vectors in the world coordinate system (including the coordinate axes) get transformed into the camera coordinate system.

# Figuring out Rotations

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \quad \mathbf{P}_C = \mathbf{R} \mathbf{P}_W$$

what if world x axis (1,0,0) corresponds to camera axis (a,b,c)?

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & r_{12} & r_{13} & 0 \\ \mathbf{b} & r_{22} & r_{23} & 0 \\ \mathbf{c} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix}$$

we can immediately write down the first column of R!

# Figuring out Rotations

and likewise with world Y axis and world Z axis...

same axis in camera coords

axis is world coords

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

world X axis (1,0,0)  
in camera coords

world Y axis (0,1,0)  
in camera coords

world Z axis (0,0,1)  
in camera coords

# Figuring out Rotations

Alternative approach: sometimes it is easier to specify what camera X,Y,or Z axis is in world coordinates. Then do rearrange the equation as follows.

$$\mathbf{P}_C = \mathbf{R} \mathbf{P}_W \Rightarrow \mathbf{R}^{-1} \mathbf{P}_C = \mathbf{P}_W \Rightarrow \mathbf{R}^T \mathbf{P}_C = \mathbf{P}_W$$

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

# Figuring out Rotations

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \quad \mathbf{R}^T \mathbf{P}_C = \mathbf{P}_W$$

what if camera X axis (1,0,0) corresponds to world axis (a,b,c)?

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{a} & r_{21} & r_{31} & 0 \\ \mathbf{b} & r_{22} & r_{32} & 0 \\ \mathbf{c} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix}$$

we can immediately write down the first column of  $\mathbf{R}^T$ ,  
(which is the first row of  $\mathbf{R}$ ).

# Figuring out Rotations

and likewise with camera Y axis and camera Z axis...

same axis in camera coords

axis is world coords

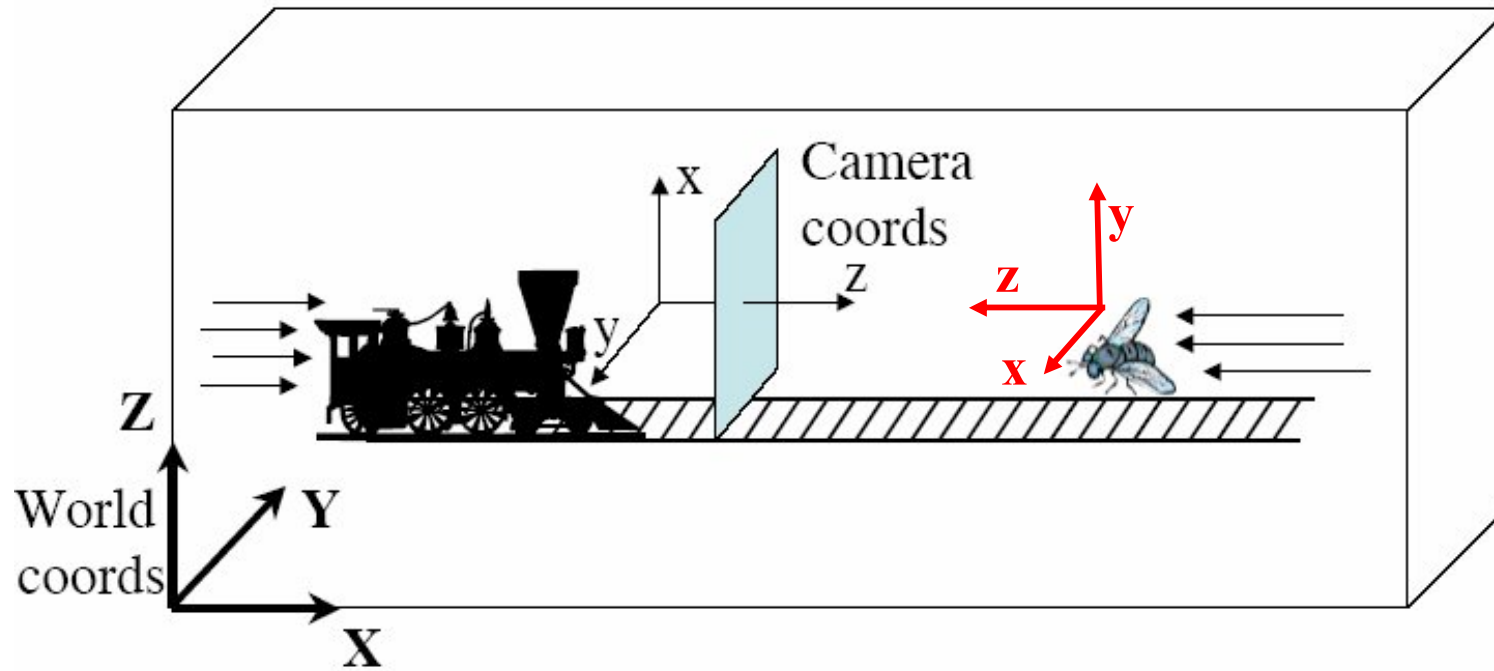
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera X axis (1,0,0)  
in world coords

camera Y axis (0,1,0)  
in world coords

camera Z axis (0,0,1)  
in world coords

# Example



$$R_{\text{train}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{\text{fly}} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

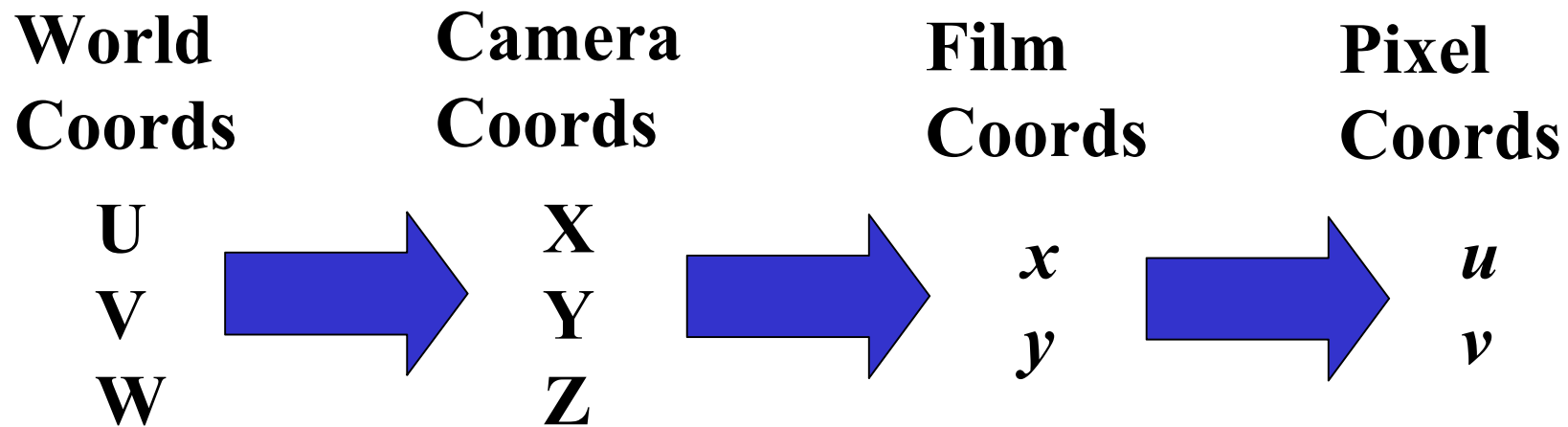


## Note: External Parameters also often written as R,T

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

$$\begin{aligned} & \mathbf{R} (\mathbf{P}_W - \mathbf{C}) \\ &= \mathbf{R} \mathbf{P}_W - \mathbf{R} \mathbf{C} \\ &= \mathbf{R} \mathbf{P}_W + \mathbf{T} \end{aligned} \quad \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Summary



**We now know how to transform 3D world coordinate points into camera coords, and then do perspective project to get 2D points in the film plane.**

**Next time: pixel coordinates**