Outline:

1. Gauss's law explanation
    a. plate geometry
2. modeling g measurements from rainfall
    a. graphs
3. gauss's law spherical applications (dark matter/galaxies)
4. matching luminosity graph to velocity v radius for stars orbital velocity
    a. "galactic rotation curve"

1. <u>Gauss's Law Overview</u>

Gauss's Law is a technique of representing force fields beyond imaginary surfaces. Often used with electric fields, it can be used with gravitational fields as well. One requirement for this 'imaginary' surface is that the field lines must be perpendicular to the surface. Thus, the surfaces which Gauss's Law can be most easily applied to (called Gaussian Surfaces) are perfect spheres and flat planes (or plates).[1] These restrictions are in fact propitious for this project, as planets and layers of rainfall can be modeled as spheres and plates, respectively.

The general form of Gauss's Law, modified for gravity, is: →

$$\oint gdA = 4G\pi m_{enclosed}$$

where g (gravity) and A are vectors, and m is the mass enclosed within the Gaussian surface. G is the gravitational constant. Integration is simple:

$$gA = 4\pi Gm_{enclosed}$$

Now there is a trick to plate geometry where it differs from spherical geometry: In a Gaussian surface, field lines point at *all* points along surface. For a sphere, this would simples be inward (or outward, but in the case of gravity, always inward) at all points on a surface. But on a flat plate, field lines would not only go against top of surface but also underneath it. Thus, the force of g vector (gravity) must be doubled:

$$2gA = 4G\pi m_{enclosed}$$

Mass can also be represented by the simple formula m = density*volume. Here, we'll use the symbol σ for density, called 'areal density'; basically a 2D version of density used in this situation for a 2D plate. 2D volume is also just area, so m can be represented by the product of σ and A:

$$2gA = 4G\pi\sigma A$$

Simplifying to:

$$g = 2\pi G\sigma$$

Which, for the sake of visual simplicity, can be written as:

$$g = 4.19345_E{}^{-10}\sigma \;[$$

Where σ is density in kg/cm^2

Now theoretical gravity can be calculated for a plate of rainfall.

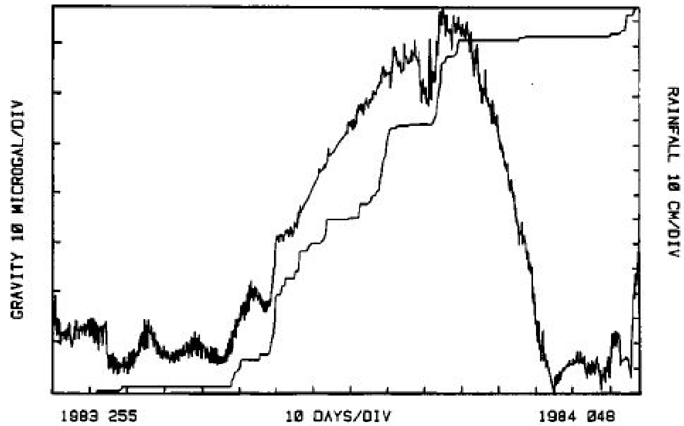
2. **Modelling g measurements from rainfall**

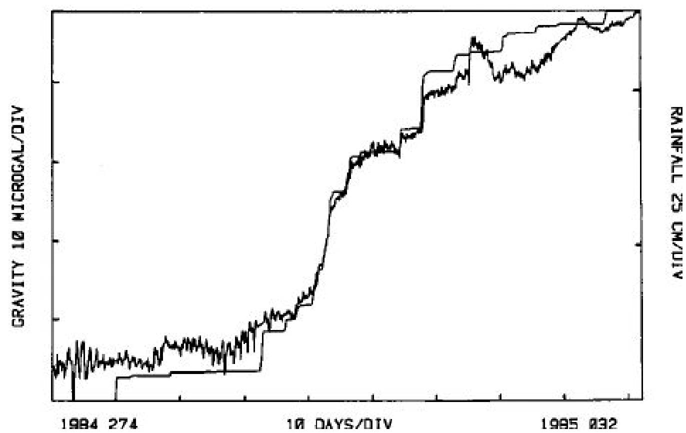
According to the documentary *The Amazing World of Gravity*, "heavy rainfall [...] could cause gravity to increase slightly"[2] (0:11:27). Gravity of the Earth changes not only with radial distance from the center of our planet but with rainfall.

The density (σ) of water is about 997 kg/m^2. Plugging in to $g = 4.19345_E{}^{-10}\sigma$ gives

$$g = 4.181_E{}^{-7} m/s^2 = .4\mu gal$$

"An infinite sheet of water 1 cm thick produces a gravitational field of 0.4μgal. Thus 1 cm of rainfall will increase gravity by 0.4μgal at locations where it falls more rapidly than it can drain away laterally. Clear correlations with rainfall at this ratio are observed at locations where lateral drainage is slow"[3]. Measurements confirm the calculations. The graph below shows the relationship between rainfall during a rainy season in a geothermal field in northern California and gravity as a function of time.[3]





3. **<u>Gauss's law spherical applications (dark matter/galaxies)</u>**

In order to best analyze gravity for for other objects (including galaxies and other planets), the g equation should be written in terms of r, to account for varying masses.

Volume of sphere: $\frac{4}{3}\pi r^3$

$$M = \rho V$$

From Newton's Law of Gravity:

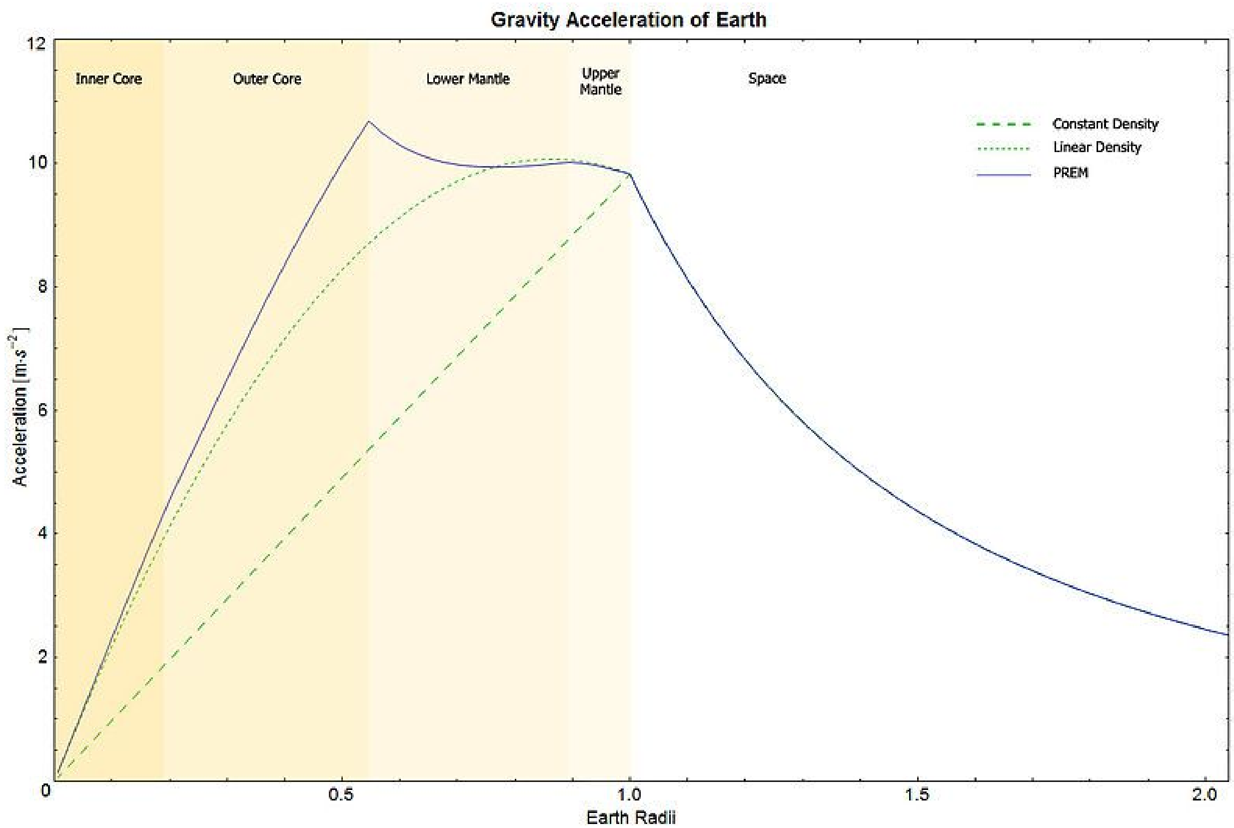
$$g = \frac{GM}{r^2}$$

$$g = \frac{G\rho V}{r^2}$$

$$g = \frac{G\rho\frac{4}{3}\pi r^3}{r^2}$$

$$g = \frac{4}{3}\pi G\rho r$$

With a sphere of uniform density, ρ is just a constant. For nonuniform densities, ρ can be a complex function (or even several functions), as demonstrated here on a graph of gravity vs radius for the Earth[6]:



Just like the earth, densities of galaxies can be very complicated, with various contributing phenomena including 'spiral density', which describes the slowed orbital velocities of stars through areas of spiral arms were there is a buildup of gas and dust.[7] Another potential issue is with spiral galaxies having disk shapes, not spheres. Because of this, elliptical galaxies were initially proposed to be the points of application. However, ellipticals having even more complex characteristics regarding galactic density, irregular/random star orbits, and technical issues of observation (ellipticals are in general further away from the Milky Way).[8] On the other hand, the stellar orbits within spiral galaxies can be approximated to be circular, and orbits are fairly uniform. Because of this, spirals are the easiest type of galaxies to study in this project.

Dark matter density as a function of radius:

Continuing the thought process, the theoretical mass density of dark matter (assuming it is spherically distributed) as a function of radius can be calculated:

$$\rho = \frac{M}{V}$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$M = \frac{4}{3}\pi\rho R^3$$

Using Newton's Laws: $$\frac{v^2}{R} = \frac{G(\frac{4}{3}\pi\rho R^3)}{R^2}$$

$$\rho = \frac{3v^2}{4\pi G R^2}$$

Assuming v(R) is constant, the density of dark matter is $\rho(R) \propto \frac{1}{R^2}$

4. **Matching luminosity graph to velocity v radius for stars orbital velocity.**

A galaxy's luminosity can be calculated by measuring the apparent brightness of individuals stars, especially 'standard candles' (such as Cepheids), certain stars whose luminosity or *absolute magnitude* is known regardless of their distances. In general terms, the process is as thus:

brightness → luminosity → mass (M/L ratio) → density m(r) → v(r) (rotation curve)

with brightness being the initial data directly collected and velocity is the end derivation. Density can be used to attain a velocity vs. radius function in a similar as previously shown in this chapter, with rewriting mass as a function of density and radius.

More specifically, this process plays out with the following equations:

Apparent Brightness: $B = \frac{L}{4\pi d^2}$, where d is the distance to the star, L is the luminosity of the star. [5]

The mass-to-light ratio of main sequence stars are used to calculate the mass of stars.[11] Additionally, by using Newton's Law of Gravitation, the rotational velocity of stars can be predicted in relationship with the radius from the center.

Mass-to-Light Ratio: $L = L_{sun}(\frac{M}{M_{sun}})^a$, $3 \square a \square 4$

Newton's Law of Gravitation:

$$F_g = G\frac{Mm}{r^2}$$

$$F = m\frac{v^2}{r}$$

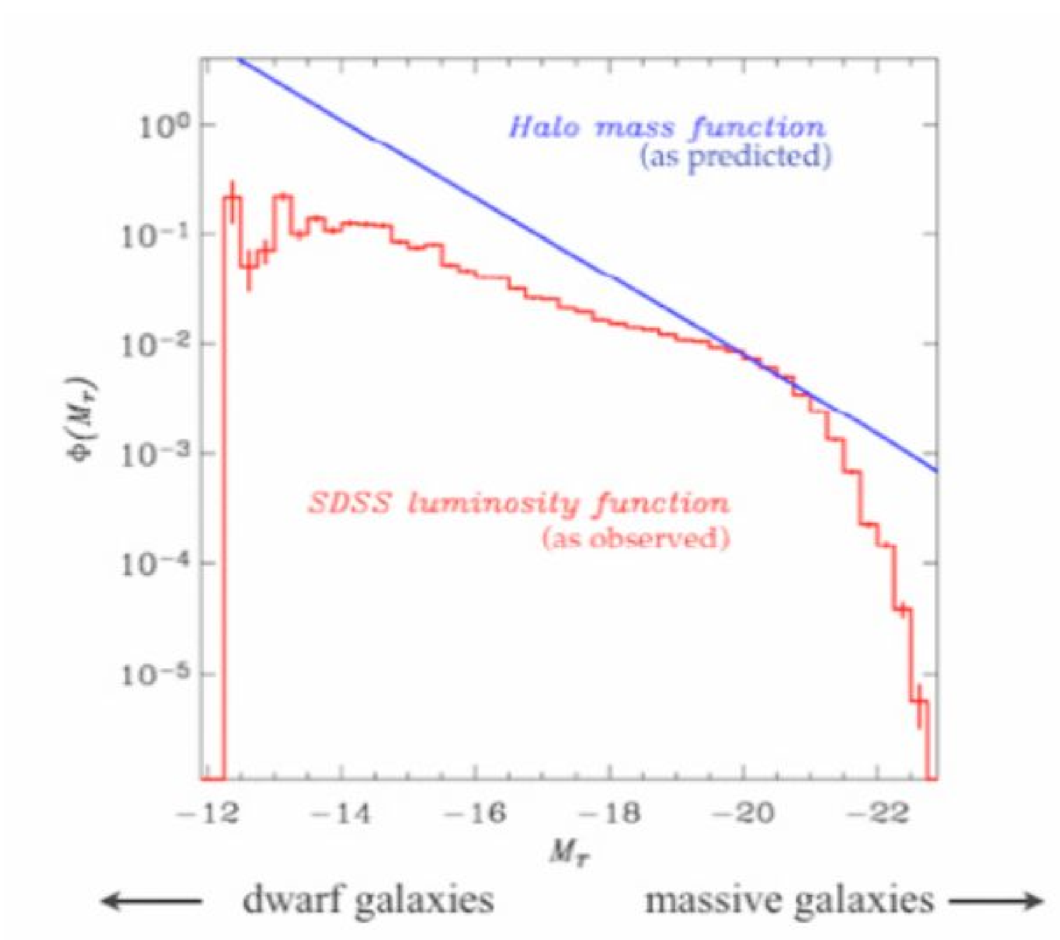
$$m\frac{v^2}{r} = G\frac{Mm}{r^2}$$

$$v = (G\frac{M}{r})^{1/2}$$

Thus, for Keplerian Rotation Curves [8]: $v \propto r^{-\frac{1}{2}}$

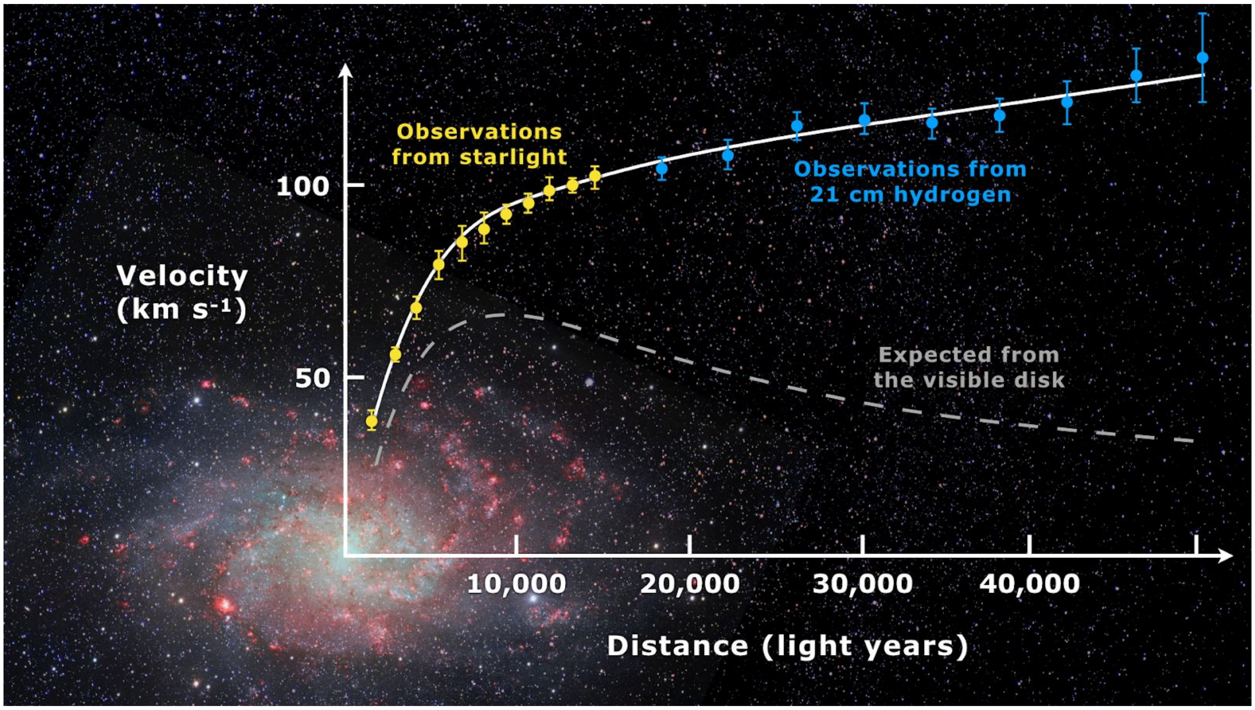
So if distance r is known (via measuring a star's brightness from earth) and its orbital velocity v is known (via spectroscopy and measuring Doppler shift) then the calculated mass within the orbit, M, can be calculated. This number is usually much higher than what would be expected from estimates based on a galaxy's total luminosity. In other words, the mass of the galaxy, as indicated by the velocities of the stars, is much higher than what should be accounted for by visible matter alone.

Luminosity vs. Mass Curve:

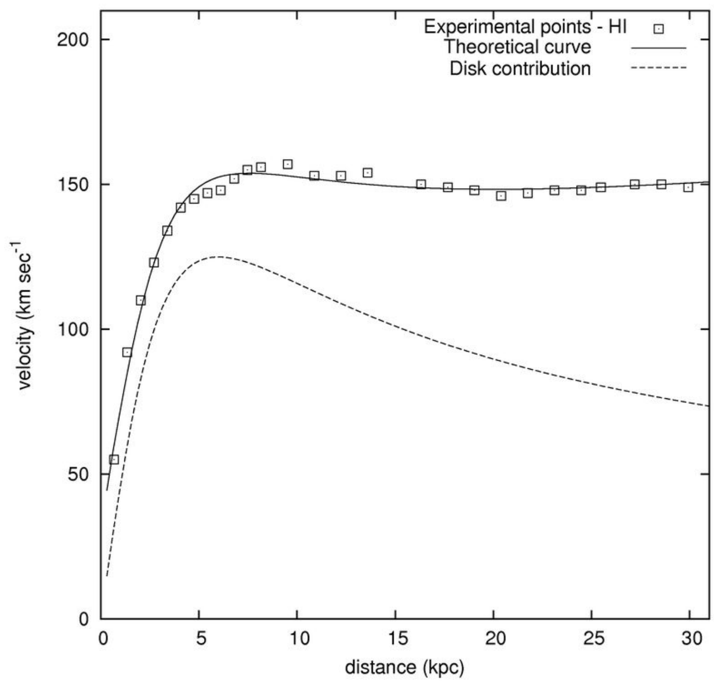


source: https://www.astro.princeton.edu/~jgreene/AST542/Alex2013.pdf

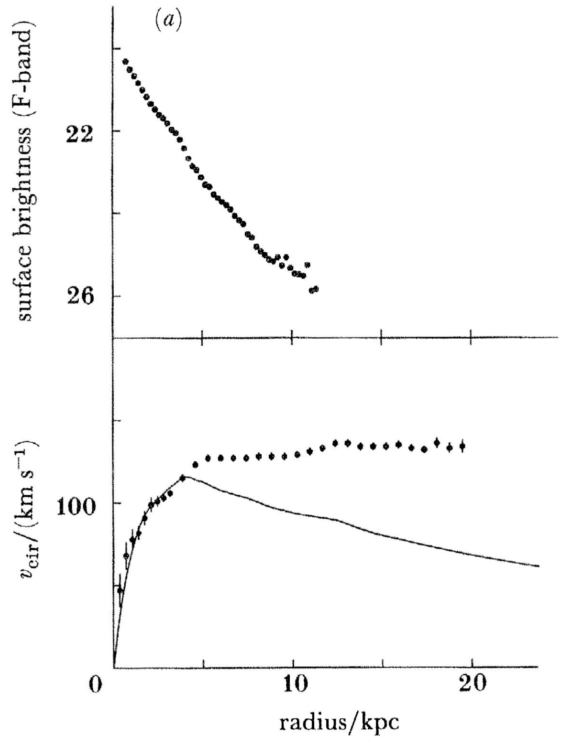
Rotation curve of spiral galaxy Messier 33:



Rotation curve for the galaxy NGC 3198[9]:



Light profile for NGC 3198[12]:



If a galaxy's luminosity decreases exponentially with radius:

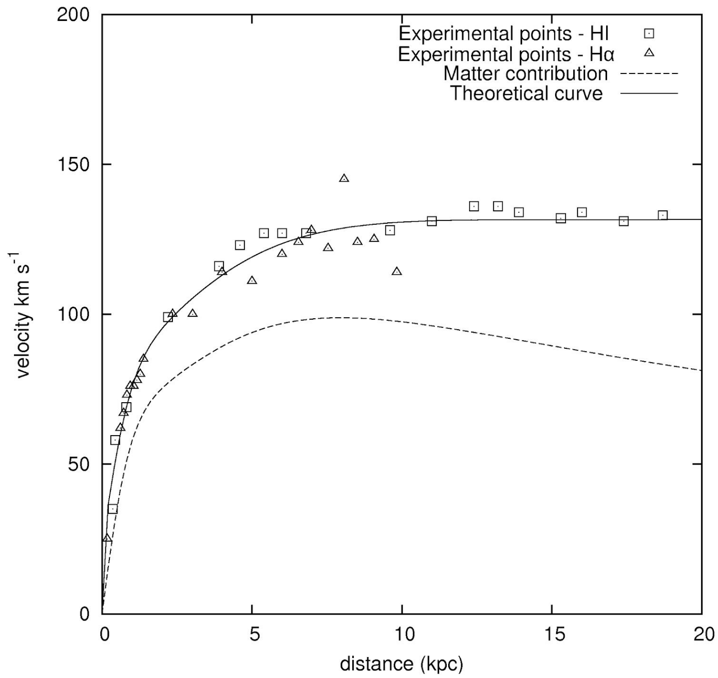
$L = ae^{-r}$, where a is an arbitrary constant, a>0.

"The maximum mass-to-light ratio for the disk is $M/L_B = 3.6$".[13]

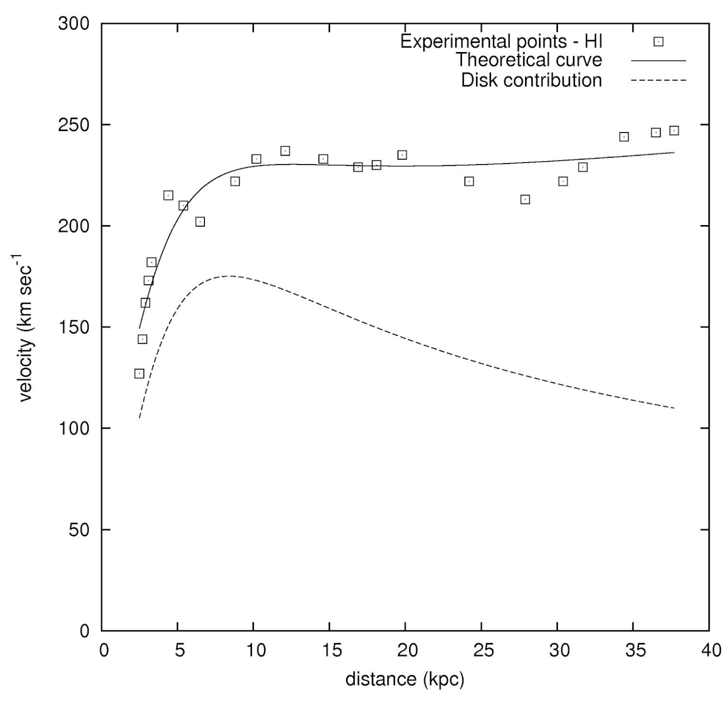
Expected velocity at the bulge: $v \sim r^{\frac{1}{2}}$

Expected velocity at the outer parts: $v \sim r^{-\frac{1}{2}}$

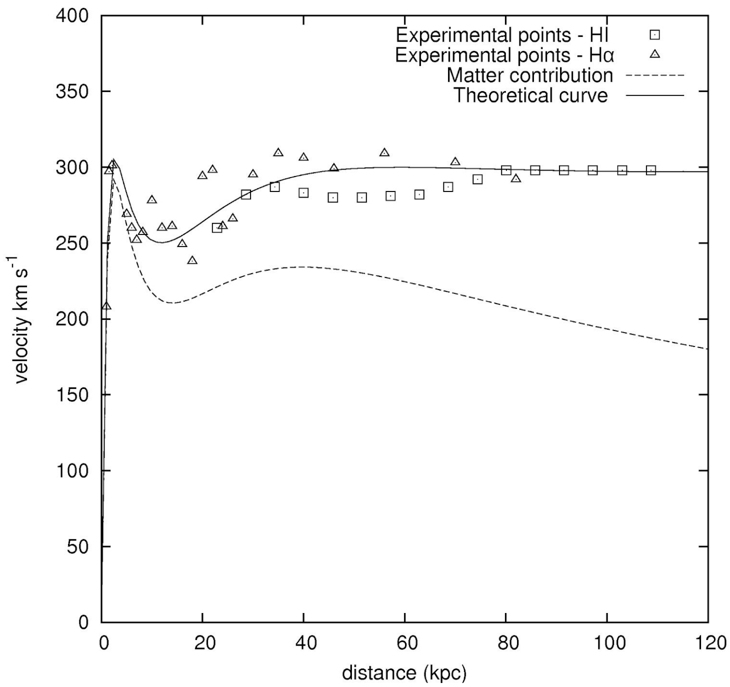
Rotation curve for NGC 2403:



Rotation curve for NGC 4725:

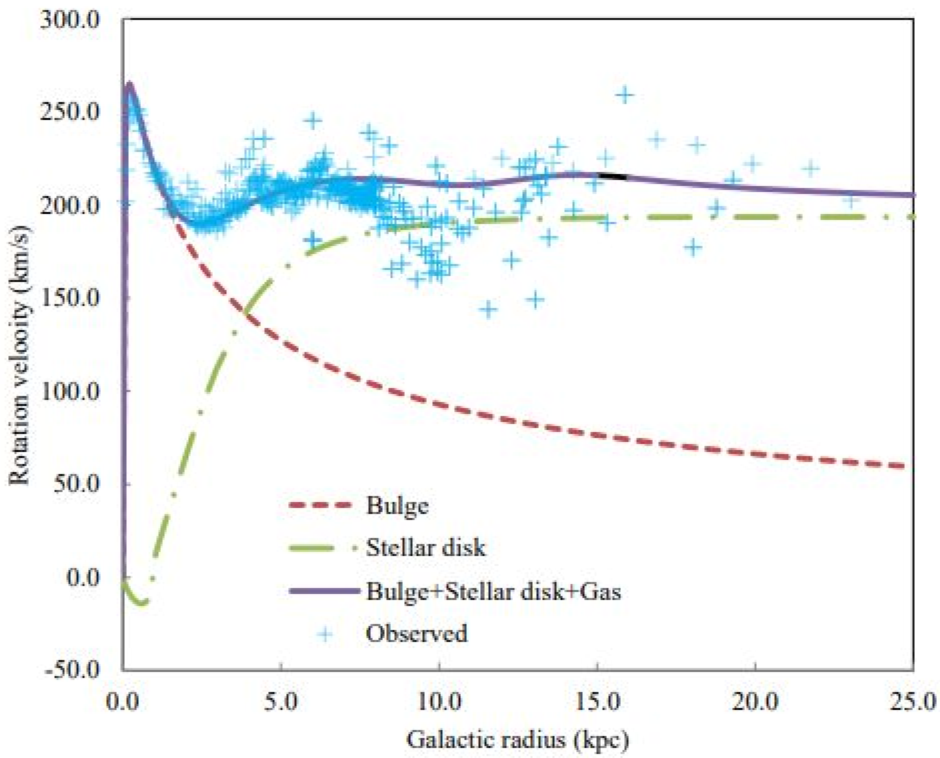


Rotation curve for UGC 2885:



Observations does not confirm the Keplerian Rotation Curve. The measured velocity is almost constant as the radius from the center increases, suggesting that the mass is proportional to the radius.

But do these curves alone truly indicate the existence of dark matter. An analytical study conducted by Enbang Li at the University of Wollongong in New South Wales, Australia, produced similarly-shaped rotation curves *without* taking dark matter (extra mass) into account. The study considered our own Milky Way over a range of radius = 25 kpc (81.5 ly). The densities were segregated into three groups: bulge (with black hole), disk, and 'gas' (presumably to account for halo matter). But instead of using the M-L ratio (previously discussed) to model galactic mass distribution, this study uses star density distribution.[14]



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