$X = E_q \qquad Y = E_p - kE_q \qquad k = \frac{v}{\varepsilon}$

$E_q \sim N(\mu_x, \sigma_x^2) \qquad E_p \sim N(\mu_p, \sigma_p^2)$

$\downarrow$

$X \sim N(\mu_x, \sigma_x^2) \qquad Y \sim N(\mu_p - k\mu_x, \sigma_p^2 + k\sigma_x^2)$

$Y|X \sim N(\mu_p - kx, \sigma_p^2)$

$Z = \frac{X}{Y} \longrightarrow Z \sim ?$

$$X_{PDF}(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \qquad Y_{PDF}(y) = \frac{1}{\sqrt{(\sigma_p^2 + k\sigma_x^2)2\pi}} e^{-\frac{(y-(\mu_p - k\mu_x))^2}{2(\sigma_p^2 + k\sigma_x^2)}}$$

$$Y_{PDF}(y|x) = \frac{1}{\sigma_p\sqrt{2\pi}} e^{-\frac{(y-(\mu_p-kx))^2}{2\sigma_p^2}}$$

Joint

$$XY_{PDF}(x,y) = \frac{1}{2\pi\sigma_p\sigma_x} e^{-\left(\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-(\mu_p-kx))^2}{2\sigma_p^2}\right)}$$

$$Z_{PDF}(z) = \int_{-\infty}^{\infty} |t|\, XY_{PDF}(zt, t)\, dt$$

$$XY_{PDF}(x,y) = \frac{1}{2\pi\sigma_p\sigma_x} e^{-C(x,y)} \longrightarrow Z_{PDF}(z) = \frac{1}{2\pi\sigma_x\sigma_p}\int_{-\infty}^{\infty} |t|\, e^{-C(zt,t)}\, dt$$

$$C(zt,t) = \frac{(zt-\mu_x)^2}{2\sigma_x^2} + \frac{(t(1+kz)-\mu_p)^2}{2\sigma_p^2} = \frac{\sigma_p^2(z^2t^2 - 2\mu_x zt + \mu_x^2) + \sigma_x^2(t^2(1+kz)^2 - 2\mu_p t(1+kz) + \mu_p^2)}{2\sigma_x^2\sigma_p^2}$$

$$= t^2\left(\frac{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right) - 2t\left(\frac{z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right) + \left(\frac{\mu_x^2\sigma_p^2 + \mu_p^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)$$

$$= \left(\frac{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)\left(t - \left(\frac{z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}\right)\right)^2 + \left(\frac{\mu_x^2\sigma_p^2 + \mu_p^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right) - \frac{(z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2)^2}{2\sigma_x^2\sigma_p^2(z^2\sigma_p^2 + (1+kz)^2\sigma_x^2)}$$

$$= \left(\frac{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)\left(t - \left(\frac{z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}\right)\right)^2 + \left(\frac{(z\mu_p - (1+kz)\mu_x)^2}{2(z^2\sigma_p^2 + (1+kz)^2\sigma_x^2)}\right)$$

$$Z_{PDF}(z) = \frac{1}{2\pi\sigma_x\sigma_p}\int_{-\infty}^{\infty} |t|\, e^{-\left(\left(\frac{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)\left(t - \left(\frac{z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}\right)\right)^2 + \left(\frac{z^2\mu_p^2 + (1+kz)^2\mu_x^2}{2(z^2\sigma_p^2 + (1+kz)^2\sigma_x^2)}\right)\right)} dt$$

$$= \frac{e^{-\left(\frac{(z\mu_p - (1+kz)\mu_x)^2}{2(z^2\sigma_p^2 + (1+kz)^2\sigma_x^2)}\right)}}{2\pi\sigma_x\sigma_p}\int_{-\infty}^{\infty} |t|\, e^{-\left(\frac{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)\left(t - \left(\frac{z\mu_x\sigma_p^2 + (1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2 + (1+kz)^2\sigma_x^2}\right)\right)^2} dt$$

Remember: $\mathrm{Erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\,dt$

$$\int_{-\infty}^{\infty} |t|\, e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_x^2\sigma_p^2}\right)\left(t-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)\right)^2} dt \qquad u = t-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right) \quad du = dt$$

$$= \int_{-\infty}^{\infty} \left|u+\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)\right| e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)u^2} du$$

$$= \int_{-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)}^{\infty} \left(u+\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)\right) e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)u^2} du$$

$$+ \int_{-\infty}^{-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)} \left(-u-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)\right) e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)u^2} du$$

$$= \int_{-\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)\sigma_x^2}\right)}^{\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)} \left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right) e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)u^2} du \qquad v_1 = u\sqrt{\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}} \quad dv_1 = du\sqrt{\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}}$$

$$+ \int_{\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)}^{\infty} 2u\, e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)u^2} du \qquad v_2 = u^2 \quad dv_2 = 2u\,du$$

$$= \frac{(z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2)\sigma_x\sigma_p\sqrt{2}}{(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)^{3/2}} \int_{-\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{\sigma_p\sigma_x\sqrt{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\sqrt{2}}}^{\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{\sigma_p\sigma_x\sqrt{2(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)}}} e^{-v_1^2}\,dv_1$$

$$+ \left[-\frac{2\sigma_p^2\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2} e^{-\left(\frac{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}{2\sigma_p^2\sigma_x^2}\right)v_2}\right]_{v_2=\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2}\right)^2}^{\infty}$$

$$= \frac{(z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2)\sigma_x\sigma_p\sqrt{2\pi}}{(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)^{3/2}} \mathrm{Erf}\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{\sigma_p\sigma_x\sqrt{2(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)}}\right)$$

$$+ \frac{2\sigma_p^2\sigma_x^2}{z^2\sigma_p^2+(1+kz)^2\sigma_x^2} e^{-\frac{(z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2)^2}{2\sigma_x^2\sigma_p^2(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)}}$$

$$P_z(z) = \frac{\sigma_p\sigma_x}{\pi(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)} e^{-\frac{1}{2}\left(\left(\frac{\mu_x}{\sigma_x}\right)^2+\left(\frac{\mu_p}{\sigma_p}\right)^2\right)}$$

$$+ \frac{(z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2)}{\sqrt{2\pi}(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)^{3/2}} e^{-\frac{(z\mu_p-(1+kz)\mu_x)^2}{2(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)}} \mathrm{Erf}\left(\frac{z\mu_x\sigma_p^2+(1+kz)\mu_p\sigma_x^2}{\sigma_p\sigma_x\sqrt{2(z^2\sigma_p^2+(1+kz)^2\sigma_x^2)}}\right)$$

$$P_z\left(z, \frac{\mu_p}{\sigma_p}, \frac{\mu_x}{\sigma_x}, \frac{\sigma_p}{\sigma_x}\right) = \frac{1}{\pi\left(z^2\left(\frac{\sigma_p}{\sigma_x}\right)+(1+kz)^2\left(\frac{\sigma_x}{\sigma_p}\right)\right)} e^{-\frac{1}{2}\left(\left(\frac{\mu_x}{\sigma_x}\right)^2+\left(\frac{\mu_p}{\sigma_p}\right)^2\right)}$$

$$+ \frac{\left(\frac{\sigma_p}{\sigma_x}\right)\left(z\left(\frac{\mu_x}{\sigma_x}\right)\left(\frac{\sigma_p}{\sigma_x}\right)+(1+kz)\left(\frac{\mu_p}{\sigma_p}\right)\right)}{\sqrt{2\pi}\left(z^2\left(\frac{\sigma_p}{\sigma_x}\right)^2+(1+kz)^2\right)^{3/2}} e^{-\frac{\left(z\left(\frac{\mu_p}{\sigma_p}\right)\left(\frac{\sigma_p}{\sigma_x}\right)-(1+kz)\left(\frac{\mu_x}{\sigma_x}\right)\right)^2}{2\left(z^2\left(\frac{\sigma_p}{\sigma_x}\right)^2+(1+kz)^2\right)}}$$

$$\mathrm{Erf}\left(\frac{z\frac{\mu_x}{\sigma_x}+(1+kz)\frac{\mu_p}{\sigma_p}\frac{\sigma_x}{\sigma_p}}{\sqrt{2\left(z^2+(1+kz)^2\left(\frac{\sigma_x}{\sigma_p}\right)^2\right)}}\right)$$