Prerequisite Information

- Properties of Random Variables
  - If X and Y are two continuous random variables described by probability density functions $f_X(x)$ and $f_Y(y)$ respectively, then the joint distribution is defined as follows.

$$f_{XY}(x,y) \stackrel{\text{def}}{=} \mathbb{P}\left(X = x \bigcap Y = y\right) = f_X(x|Y = y)f_Y(y) = f_X(x)f_Y(y|X = x)$$

  - If X and Y are three continuous random variables described by probability density functions $f_X(x)$ and $f_Y(y)$ respectively, then the following property holds

$$f_X(x) = \int_{y\in\Omega_Y} f_{XY}(x,y)dy$$

    where Ω_Y is the set of all possible values of the variable Y. This leads to the next property.

  - If X, Y, and Z are three continuous random variables described by probability density functions $f_X(x)$, $f_Y(y)$, and $f_Z(z)$ respectively, then the following property holds

$$f_{XY}(x,y) = \int_{z\in\Omega_Z} f_{XYZ}(x,y,z)dz$$

- Ratio Distribution density function
  - If X and Y are two continuous random variables described by probability density functions $f_X(x)$ and $f_Y(y)$ respectively, then the ratio distribution defined as $Z = {}^{X}\!/_{Y}$ has a density function that can be calculated through the following formula. (See Proof)

$$f_Z(z) = \int_{t\in\Omega_Y} |t| f_{XY}(zt,t)dt$$

Proof of Density function for the Yield variable

In the current model of the Yield, the random variable is given by the following expression

$$Y = \frac{\varepsilon N + X_Q}{E_r + X_P + \left(\frac{V}{\varepsilon}\right) X_Q}$$

where E_r, V, and ε are constants and N, X_Q, and X_P are independent normally distributed variables distributed as $N \sim \mathrm{N}(\mu_N, \sigma_N^2)$, $X_Q \sim \mathrm{N}\left(0, \sigma_Q^2\right)$, and $X_P \sim \mathrm{N}(0, \sigma_P^2)$. Thus, the known density functions are $f_N(x) = \frac{1}{\sigma_N\sqrt{2\pi}} e^{-\frac{(x-\mu_N)^2}{2\sigma_N^2}}$, $f_{X_Q}(x) = \frac{1}{\sigma_Q\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_Q^2}}$, and $f_{X_P}(x) = \frac{1}{\sigma_P\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_P^2}}$.

A variable switch was introduced where $\mathrm{A} = \varepsilon N + X_Q$ and $\mathrm{B} = E_r + X_P + \left(\frac{V}{\varepsilon}\right) X_Q$. Thus,

$$Y = \frac{\varepsilon N + X_Q}{E_r + X_P + \left(\frac{V}{\varepsilon}\right) X_Q} = \frac{A}{B}$$

Using the Ratio distribution density function formula gives the distribution of the Yield as

$$f_Y(y) = \int_{-\infty}^{\infty} |t| f_{AB}(yt, t) dt$$

Now, the joint distribution $f_{AB}(a, b)$ must be calculated. Using the third listed property of random variables, we have that

$$f_{AB}(a, b) = \int_{-\infty}^{\infty} f_{ABX_Q}(a, b, q) dq$$

In order to calculate the joint distribution $f_{ABX_Q}(a, b, q)$, we use the first listed property of random variables, where

$$f_{ABX_Q}(a, b, q) = f_{AB}\left(a, b | X_Q = q\right) * f_{X_Q}(q)$$

Then, because A and B are independent without X_Q, the conditional joint distribution can be calculated as

$$f_{AB}\left(a, b | X_Q = q\right) = f_A\left(a | X_Q = q\right) * f_B\left(b | X_Q = q\right)$$

The remaining conditional variables are then

$$A \mid (X_Q = q) = \varepsilon N + q \sim \mathrm{N}(\varepsilon\mu_N + q, \varepsilon^2\sigma_N^2)$$

$$B \mid (X_Q = q) = E_r + X_P + \left(\frac{V}{\varepsilon}\right)q \sim \mathrm{N}\left(E_r + \left(\frac{V}{\varepsilon}\right)q, \sigma_P^2\right)$$

Then, the probability density functions to plug in are

$$f_{X_Q}(x) = \frac{1}{\sigma_Q\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_Q^2}}$$

$$f_A(x|X_Q = q) = \frac{1}{\varepsilon\sigma_N\sqrt{2\pi}} e^{-\frac{\left(x-(\varepsilon\mu_N+q)\right)^2}{2\varepsilon^2\sigma_N^2}}$$

$$f_B(x|X_Q = q) = \frac{1}{\sigma_P\sqrt{2\pi}} e^{-\frac{\left(x-\left(E_r+\left(\frac{V}{\varepsilon}\right)q\right)\right)^2}{2\sigma_P^2}}$$

Thus, summarizing previous steps

$$f_Y(y) = \int_{-\infty}^{\infty} |t| f_{AB}(yt, t)dt = \int_{-\infty}^{\infty} |t| \int_{-\infty}^{\infty} f_{ABX_Q}(yt, t, q)dq\, dt$$

$$= \int_{-\infty}^{\infty} |t| \int_{-\infty}^{\infty} f_{AB}(yt, t|X_Q = q) * f_{X_Q}(q)dq\, dt$$

$$= \int_{-\infty}^{\infty} |t| \int_{-\infty}^{\infty} f_A(yt|X_Q = q) * f_B(t|X_Q = q) * f_{X_Q}(q)dq\, dt$$

$$= \frac{1}{\varepsilon\sigma_P\sigma_Q\sigma_N(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} |t| \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{q^2}{\sigma_Q^2} + \frac{\left(yt-(\varepsilon\mu_N+q)\right)^2}{\varepsilon^2\sigma_N^2} + \frac{\left(\varepsilon t-(\varepsilon E_r+Vq)\right)^2}{\varepsilon^2\sigma_P^2}\right)} dq\, dt$$

Evaluating this expression gives the result of

$$f_Y(y) = \frac{e^{-\gamma}}{2 * \alpha(y) * \sqrt{\pi k}} * g\left(\frac{\beta(y)}{2\sqrt{\alpha(y)}}\right)$$

Where

$$k = {\sigma_P}^2{\sigma_Q}^2 + V^2{\sigma_N}^2{\sigma_Q}^2 + \varepsilon^2{\sigma_P}^2{\sigma_N}^2$$

$$\alpha(x) = \frac{1}{2k}\left({\sigma_Q}^2\left(x\left(\frac{V}{\varepsilon}\right) + 1\right)^2 + {\sigma_P}^2x^2 + \varepsilon^2{\sigma_N}^2\right)$$

$$\beta(x) = \frac{1}{k}\left(\left(\frac{V}{\varepsilon}\right){\sigma_Q}^2(E_r x + \varepsilon\mu_N) + x\varepsilon\mu_N\left(\left(\frac{V}{\varepsilon}\right)^2{\sigma_Q}^2 + {\sigma_P}^2\right) + E_r\left({\sigma_Q}^2 + \varepsilon^2{\sigma_N}^2\right)\right)$$

$$\gamma = \frac{1}{2k}\left({\sigma_Q}^2(E_r + V\mu_N)^2 + \varepsilon^2\left({\mu_N}^2{\sigma_P}^2 + {E_r}^2{\sigma_N}^2\right)\right)$$

$$g(x) = \frac{1}{\sqrt{\pi}} + xe^{x^2}\,\mathrm{erf}(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{\Gamma\left(n + \frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{n!\,(2x)^{2n}}{(2n)!}$$