Diffusion MRI, ADC, and some mathematics

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General case:

The simplest method to calculate ADC from DWIs is to fit a line in coordinates (b-value, log(DWI)) and ADC is given as the slope a of the line

$$\log(\mathrm{DWI}_i) = \mathrm{ADC} \cdot b_i + c.$$

The known equation for the slope with n measurement points x_i and measurements y_i is

$$\mathrm{ADC} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \tag{1}$$

where $\bar{x}$ and $\bar{y}$ are the averages of measurement points x_i and measurements y_i.
Consider only the nominator of ADC equation with assumption that one measurement m point equals to average of all measurement points i.e. $x_m = \bar{x}$.
Separate a point $i = m$ from the sum:

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1, i\neq m}^{n}(x_i - x_m)(y_i - \bar{y}) + \cancelto{0}{(x_m - x_m)}(y_m - \bar{y}) \\
&= \sum_{i=1, i\neq m}^{n}(x_i y_i - x_m y_i - x_i \bar{y} + x_m \bar{y}) \\
&= \sum_{i=1, i\neq m}^{n}\left[x_i y_i - x_m y_i + \overbrace{(x_m - x_i)}^{d}\,\bar{y}\right]
\end{aligned}$$

Now, only $\bar{y}$ depends on the measurement y_m so focus on the factor d

$$\begin{aligned}
\sum_{i=1, i\neq m}^{n}(x_m - x_i) &= \sum_{i=1, i\neq m}^{n}(x_m - x_i) \\
&= x_m - x_1 + x_m - x_2 + \ldots + x_m - x_n \\
&= (n-1)\,x_m - \sum_{i=1, i\neq m}^{n} x_i
\end{aligned}$$

Remembering that $x_m = \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

$$\begin{aligned}
(n-1)\,x_m - \sum_{i=1, i\neq m}^{n} x_i &= \left(n\frac{\sum_{i=1}^{n} x_i}{n} - x_m\right) - \sum_{i=1, i\neq m}^{n} x_i \\
&= \left(\sum_{i=1}^{n} x_i - x_m\right) - \sum_{i=1, i\neq m}^{n} x_i
\end{aligned}$$

Again, separating a point $i = m$ from the first sum

$$
\begin{aligned}
\left(\sum_{i=1}^{n} x_i - x_m\right) - \sum_{i=1,i\neq m}^{n} x_i &= \left(\sum_{i=1,i\neq m}^{n} (x_i + x_m) - x_m\right) - \sum_{i=1,i\neq m}^{n} x_i \\
&= \sum_{i=1,i\neq m}^{n} x_i + \cancelto{0}{x_m - x_m} - \sum_{i=1,i\neq m}^{n} x_i \\
&= \sum_{i=1,i\neq m}^{n} x_i - \sum_{i=1,i\neq m}^{n} x_i \\
&= 0
\end{aligned}
$$

Combining this to the ADC equation 1

$$
\mathrm{ADC} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\sum_{i=1,i\neq m}^{n} (x_i y_i - x_m y_i)}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{2}
$$

we see that ADC does not depend on measurement y_m thus adding any b-value measurement that is the average of other b-values cannot enhance the ADC map if it is calculated based on monoexponential model with linear least squares approach.

Practical example (Siemens recommends this):

In the reference Siemens recommends using b-values of $0\,\mathrm{s/mm^2}$, $500\,\mathrm{s/mm^2}$, and $1000\,\mathrm{s/mm^2}$ instead of just $0\,\mathrm{s/mm^2}$ and $1000\,\mathrm{s/mm^2}$ to enchance the SNR of ADC map. Let's see how ADC is affected by such enhancement starting without any assumptions made in the general case i.e. equation 1:

$$
\begin{aligned}
\mathrm{ADC} &= \frac{\sum_{i=1}^{3} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{3} (x_i - \bar{x})^2} \\
&= \frac{(x_1 - \bar{x})(y_1 - \bar{y}) + (x_2 - \bar{x})(y_2 - \bar{y}) + (x_3 - \bar{x})(y_3 - \bar{y})}{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}
\end{aligned}
$$

Now, $x_1 = 0$, $x_2 = 500$, and $x_3 = 1000 = 2x_2$ giving average $\bar{x} = \frac{0+500+1000}{3} = 500 = x_2$.

$$
\begin{aligned}
\mathrm{ADC} &= \frac{(0 - x_2)(y_1 - \bar{y}) + \cancelto{0}{(x_2 - x_2)}(y_2 - \bar{y}) + (2x_2 - x_2)(y_3 - \bar{y})}{(0 - x_2)^2 + \cancelto{0}{(x_2 - x_2)^2} + (2x_2 - x_2)^2} \\
&= \frac{-x_2(y_1 - \bar{y}) + x_2(y_3 - \bar{y})}{2x_2^2} \\
&= \frac{-y_1 + \bar{y} + y_3 - \bar{y}}{2x_2}
\end{aligned}
$$

$$
\mathrm{ADC} = \frac{y_3 - y_1}{2x_2} \tag{3}
$$

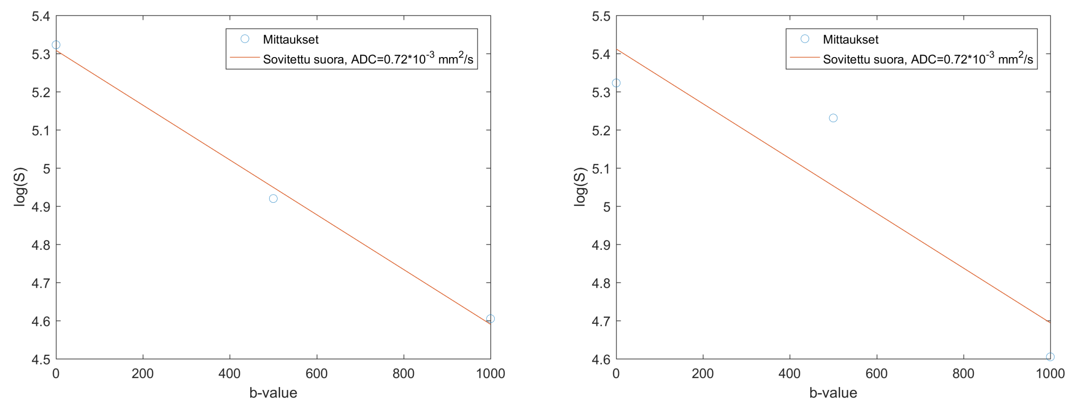
Or by using the generalized equation 2 for the case where any measurement $x_m = \bar{x}$ where $m = 2$:

$$
\begin{aligned}
\mathrm{ADC} &= \frac{\sum_{i=1,i\neq 2}^{3} (x_i y_i - x_m y_i)}{\sum_{i=1}^{3} (x_i - \bar{x})^2} \\
&= \frac{x_1 y_1 - x_2 y_1 + x_3 y_3 - x_2 y_3}{(x_1 - x_2)^2 + \cancelto{0}{(x_2 - x_2)^2} + (x_3 - x_2)^2} \\
&= \frac{x_1 y_1 - x_2 y_1 + x_3 y_3 - x_2 y_3}{(x_1 - x_2)^2 + (x_3 - x_2)^2}
\end{aligned}
$$

we see that ADC does not depend on the second measurement y_2. In the specific case of $x_1 = 0$ and $x_3 = 2x_2$ this simplifies further and equals to 3:

$$
\begin{aligned}
\mathrm{ADC} &= \frac{0y_1 - x_2y_1 + 2x_2y_3 - x_2y_3}{(0-x_2)^2 + (2x_2 - x_2)^2} \\
&= \frac{y_3 - y_1}{2x_2}
\end{aligned}
$$

Figure 1 shows simulation case of this example: only the intersection point on log(DWI) axis changes with y_2.



Kuva 1: Simulation results for ADC measurement with b-values 0, 500, and 1000. On the left the ground truth and on the right the b-value 500 measurement is shifted without any affect on the slope i.e. ADC. (Finnish 101: "Mittaukset" == "Measurements" and "Sovitettu suora" == "fitted line")

Conclusion

When measuring ADC it actually matters which b-values are used due to the used linear equation and the fit method. While this is extremely specific case it is clinically relevant since it is recommended method at least by one MRI manufacturer (Siemens) and such "middle point" feels really intuitive way to improve SNR. This should be noted always when using more than two measurement points (b-values) i.e. when there is a possibility to find infinite number of solutions for ADC or any other value.

Reference

http://clinical-mri.com/wp-content/uploads/software_hardware_updates/Graessner.pdf