Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2022-06-08

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Allowed aid:

- A basic calculator
- BETA mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

<u>NB:</u> **Only one problem per sheet.** Write your name and personal number if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s), \quad n=1,2,3,\dots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \big(1 - L(sI - A + KC + BL)^{-1}B\big)\,\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point, (leaving blank yields 0 points). Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) Consider a control system $G_c(s) = \frac{10}{s+10}$. If the reference signal is

$$r(t) = \begin{cases} r_0, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

then $y(t) = r_0(1 - e^{-10t})$

ii) The following system is observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

iii) When a true system $G^0(s)$ is different from the model $G(s)$ it is impossible to ensure that a controlled designed for $G(s)$ will stabilize the closed-loop system.

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) The main advantage of feedback controllers is that they can supress unmeasured disturbances and mitigate model inaccuracies.

ii) Open-loop controllers can avoid oscillations.

iii) Systems with time-delays are minimum phase.

(3 p)

Proposed solution to problem 1

Part A

i) True.
   Check determinant of observability matrix.

ii) True.

iii) False.

Part B

i) True.

ii) True.

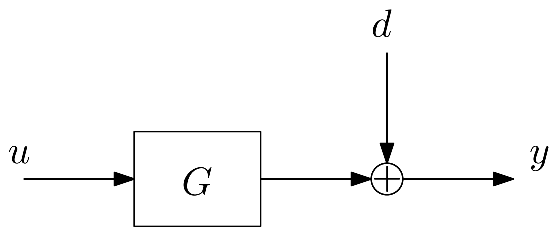
iii) True.

Problem 2 (6/30)

We want to control the rudder angle $y(t)$ of an aircraft subject to turbulence. The system is described in the figure below, where we use the following model:

$$G(s) = \frac{s}{s+3}$$

We want $y(t)$ to follow a reference $r(t)$.



a) First, we consider using a controller which we describe in the Laplace domain as

$$U(s) = \frac{s+3}{s}R(s)$$

Derive the open-loop system from the reference signal $R(s)$ and disturbance $D(s)$ to output $Y(s)$.

(1 p)

b) Show that the open-loop system is stable and that when there is no disturbance, $d(t) \equiv 0$, then the control error is zero.

(1p)

c) Next, we consider using a feedback controller

$$U(s) = K(R(s) - Y(s))$$

and determine K such that the closed-loop system is stable.

(3 p)

d) Mention two advantages of using the feedback controller in c) over the controller in a).

(1 p)

Proposed solution to problem 2

a) To derive the transfer function between $R(s)$ and $D(s)$ and the output $Y(s)$ we note that

$$Y(s) = D(s) + G(s)U(s) = D(s) + G(s)F(s)U(s).$$

b) The open loop stability depends on whether $G(s)F(s)$ contains unstable poles:

$$G(s)F(s) = \frac{\cancel{s}}{\cancel{s+3}} \frac{\cancel{s+3}}{\cancel{s}} = 1$$

Note that this transfer function is input-output stable **but not asymptotically stable**. Indeed, the cancellation of the pole $s = 0$ — which is not strictly stable (it is just marginally stable) — with the corresponding zero produces a non-reachable/non-observable part with unstable dynamics. When there is no disturbance $Y(s) = 1R(s)$, so the steady-state error is null.

c) The proportional controller $F(s) = K$ yields the following closed-loop transfer function

$$G_c(s) = \frac{KG(s)}{1 + KG(s)} = \frac{sK}{(K+1)s + 3}.$$

The closed loop is stable as long as $s = \frac{-3}{K+1} < 0$, i.e., $K > -1$.

d) See the slides or Glad & Ljung.

Problem 3 (6/30)

Consider a continuous-time state-space model of a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{1}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

a) Show that the system is controllable

(1 p)

b) Assume that all the states are measured and the system is controlled by a state-feedback controller $u = -Lx$. Find value of matrix L such that poles of the closed-loop system are located at -2 and -3.

(2 p)

c) Show that the system is observable

(1 p)

d) Assume that a state observer is designed as follows

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + K\left(y - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \right). \tag{3}$$

Find value of matrix K such that poles of the state observer are located at -5 and -7.

(2 p)

Proposed solution to problem 3

Let us reformulate the system model as follows

$$\dot{x} = Ax + Bu, \tag{4}$$

$$y = Cx, \tag{5}$$

where

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

a) Controllability matrix is represented as follows

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}. \tag{6}$$

Since $\text{rank}(\mathcal{C}) = 2$, the same as the dimension of matrix A, the system is controllable.

b) By using a full state-feedback controller $u = -Lx$, where $L = [l_1,\ l_2]$, the closed-loop system becomes

$$\begin{aligned} \dot{x} = (A - BL)x &= \left(\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} \right) x, \\ &= \begin{bmatrix} 1 & -1 \\ 3 - l_1 & 2 - l_2 \end{bmatrix} x \end{aligned} \tag{7}$$

In order to assign poles of the closed-loop system at -2 and -3, we need to hold the following equation $\forall \lambda \in \mathbb{C}$

$$\begin{aligned} \det\left(\lambda I - \begin{bmatrix} 1 & -1 \\ 3 - l_1 & 2 - l_2 \end{bmatrix} \right) &= (\lambda + 2)(\lambda + 3), \\ l_1 = -9, \quad l_2 &= 8. \end{aligned} \tag{8}$$

c) Observability matrix is represented as follows

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}. \tag{9}$$

Since $\text{rank}(\mathcal{O}) = 2$, the same as the dimension of matrix A, the system is observable.

d) The model of state estimation error $\tilde{x} = x - \hat{x}$ is represented as follows

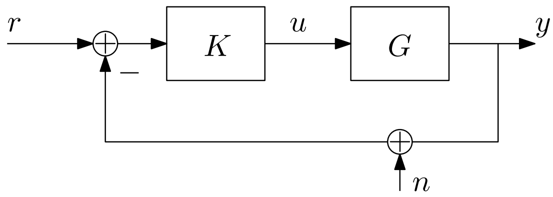
$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = (A - KC)\tilde{x} = \left(\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \tilde{x}. \tag{10}$$

In order to assign poles of the observer at -5 and -7, we need to hold the following equation $\forall \lambda \in \mathbb{C}$

$$
\begin{aligned}
\det\left(\lambda I - \left[\begin{array}{cc} 1-k_1 & -1 \\ 3-k_2 & 2 \end{array}\right]\right) &= (\lambda+5)(\lambda+7), \\
k_1 = 15, \quad k_2 &= -60.
\end{aligned} \tag{11}
$$

Problem 4 (6/30)

We are controlling a power plant $G(s) = \frac{s}{s+5}$ with a P-controller that uses a sensor to observe the output $y(t)$. The sensor introduces a noise $n(t)$ (see the figure below).



a) Design the parameter K so that the closed-loop system from reference to output is stable.

(1 p)

b) Sketch the frequency response of the resulting complementary sensitivity function.

(3 p)

c) Comment on how the system behaves if the noise $n(t)$ is

- low frequency, versus
- high frequency.

(2 p)

Proposed solution to problem 4

a) Open-loop transfer function

$$G_o(s) = K \frac{s}{s+5}$$

The closed-loop transfer function

$$G_c(s) = \frac{Ks}{(K+1)s+5}$$

is stable as long as the pole of $G_c(s)$, which is $s = \frac{-5}{K+1} < 0$, is stable $\Rightarrow K > -1$. We pick, for example, $K = 5$.

b) See the slides.

c) The transfer function between the measurement noise $N(s)$ and the output $Y(s)$ is the complementary sensitivity function. In this case, at low frequencies $|G_o(i\omega)| \approx 0$, so $|T(i\omega)| \approx 0$. This means that low-frequency noise is attenuated, but it also means that the static error is big (because of the derivative action). At high frequencies, $|G_o(i\omega)| \approx |K|$, and hence $|T(i\omega)| \approx \frac{|K|}{1+|K|}$. The noise gets (very slightly) attenuated.

Problem 5 (6/30)

Consider a continuous-time state space model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

a) After discretizing the state-space model (using zero order hold) with sampling time T, we obtain the discrete-time state-space model as follows

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = G \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Fu(k),$$

$$y(k+1) = H \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Find values of matrices G, F, and H.

(2 p)

b) For what values of T is the discretized system observable?

(1 p)

c) For what values of T is the discretized system controllable?

(1 p)

d) Consider the discrete-time state-space model in a) with sampling time $T =$ 1s. Find a discrete-time state-feedback controller $u(k) = -Lx(k)$ such that poles of the continuous-time system are located at -1 and -2.

(2 p)

Proposed solution to problem 5

a) We have

$$H = C \tag{12}$$

For the state transition matrix we have

$$F = \exp(AT) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

First we have

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+2} & \frac{2}{s^2+2} \\ \frac{-1}{s^2+2} & \frac{s}{s^2+2} \end{bmatrix}$$

which then yields after inverse Laplace transforming

$$F = \begin{bmatrix} \cos(\sqrt{2}T) & \sqrt{2}\sin(\sqrt{2}T) \\ \frac{-1}{\sqrt{2}}\sin(\sqrt{2}T) & \cos(\sqrt{2}T) \end{bmatrix} \tag{13}$$

Since A is invertible, one computes

$$G = A^{-1}(F - I)B \tag{14}$$

$$= \begin{bmatrix} 0 & -1 \\ 0.5 & 0 \end{bmatrix} \left(\begin{bmatrix} \cos(\sqrt{2}T) & \sqrt{2}\sin(\sqrt{2}T) \\ \frac{-1}{\sqrt{2}}\sin(\sqrt{2}T) & \cos(\sqrt{2}T) \end{bmatrix} - I \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} 1 - \cos(\sqrt{2}T) \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}T) \end{bmatrix}. \tag{16}$$

b) Observability matrix is represented as follows

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\sqrt{2}T) - 1 & \sqrt{2}\sin(\sqrt{2}T) \end{bmatrix} \tag{17}$$

In order to ensure that the discretized system is observable, we need to guarantee

$$\sqrt{2}\sin(\sqrt{2}T) \neq 0,$$
$$T \neq \frac{k\pi}{\sqrt{2}}, \quad k \in \mathbb{Z}. \tag{18}$$

c) Controllability matrix is represented as follows

$$\mathcal{C} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 1 - \cos(\sqrt{2}T) & \cos(\sqrt{2}T) - \cos^2(\sqrt{2}T) + \sin^2(\sqrt{2}T) \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}T) & \frac{-1}{\sqrt{2}}\sin(\sqrt{2}T) + \sqrt{2}\sin(\sqrt{2}T)\cos(\sqrt{2}T) \end{bmatrix} \tag{19}$$

In order to ensure that the discretized system is controllable, we need to guarantee

$$\sin(\sqrt{2}T) \neq 0, \quad \text{or} \quad \cos(\sqrt{2}T) \neq 0,$$
$$T \neq \frac{k\pi}{\sqrt{2}}, \quad k \in \mathbb{Z}. \tag{20}$$

d) With $T = 1$ and poles of continuous-time system at -1 and -2, we have poles of discretized system are e^{-1} and e^{-2}. By using a full state-feedback controller $u(k) = -Lx(k)$, where $L = [l_1,\ l_2]$, we need to hold the following equation $\forall\lambda$

$$\det\Big(\lambda I - (F - GL)\Big) = (\lambda - e^{-1})(\lambda - e^{-2}),$$
$$l_1 = -0.6762, \quad l_2 = 0.5433. \tag{21}$$