Mathematics & Statistics

Academic Year: 2015-2016 Marks: 80
Date & Time: 26th February 2016, 11:00 am
Duration: 3h

Question 1: [12]

Question 1: Select and write the most appropriate answer from the given alternatives in each of the following sub-questions [6]

Question 1.1.1: The negation of p ∧ (q → r) is ____________. [2]
p ∨ (~q ∨ r)
~p ∧ (q → r)
~p ∧ (~q → ~r)
~p ∨ (q ∧ ~r)

Solution: ~ [P ∧ (q → r)

=~[(P)] ∨ [~ (q → r)] ...(By De Morgan's law)
=~[(P)] ∨ [~ (~q ∨ r)] ...(By Conditional Law)
=~[(P)] ∨ [(q ∧ ~r)] ...(By De Morgan's law)

~ [P ∧ (q → r)] = ~ P ∨ (q ∧ ~r)

Question 1.1.2: [2]

If $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$ then x is

1. -1/2
2. 1
3. 0
4. 1/2

Solution: (c)

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$$

$$(1-x) = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$$

$$(1-x) = \cos\left(2\sin^{-1}x\right)$$

$$(1-x) = \cos\left(\cos^{-1}\left(1-2x^2\right)\right)$$

$$(1-x) = 1-2x^2$$

$$2x^2 - x = 0$$

$$x(2x-1)=0$$

$$x=0 \text{ or } 2x-1=0$$

$$x=0 \text{ or } x=\frac{1}{2}$$

$$\text{for } x=\frac{1}{2}$$

$$\sin^{-1}(1-x)-2\sin^{-1}x=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)=-\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

So x=1/2 is not solution of the given equation

for x=0

$$\sin^{-1}(1-x)-2\sin^{-1}x=\sin^{-1}(1)-2\sin^{-1}(0)=\frac{\pi}{2}-0=\frac{\pi}{2}$$

So x = 0 is a valid solution of the given equation.

Question 1.1.3: The joint equation of the pair of lines passing through (2,3) and parallel to the coordinate axes is **[2]**

1. xy -3x - 2y + 6 = 0
2. xy +3x + 2y + 6 = 0
3. xy = 0
4. xy - 3x - 2y - 6 = 0

Solution: (a)
Equation of the coordinate axes are x = 0 and y = 0.
The equations of the lines passing through (2, 3) and parallel to coordinate axes are,
x = 2 and y = 3.
i.e. x - 2 = 0 and y - 3 = 0
The joint equation is given as

(x-2)(y-3)=0

xy-3x-2y+6=0

Question 1.2: Attempt any 3 of the following **[6]**

Question 1.2.1: **[2]**

Find $(AB)^{-1}$ if

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & -2 & -3\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ 1 & 2\\ 1 & -2\end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}1&2&3\\1&-2&-3\end{bmatrix}, B=\begin{bmatrix}1&-1\\1&2\\1&-2\end{bmatrix}$$

$$AB=\begin{bmatrix}1&2&3\\1&-2&-3\end{bmatrix}\begin{bmatrix}1&-1\\1&2\\1&-2\end{bmatrix}$$

$$=\begin{bmatrix}6&-3\\-4&1\end{bmatrix}$$

$$(AB)^{-1}(AB)=I$$

$$(AB)^{-1}\begin{bmatrix}6&-3\\-4&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\text{Using } R_1\to\frac{1}{6}R_1$$

$$(AB)^{-1}\begin{bmatrix}1&-\frac{1}{2}\\-4&1\end{bmatrix}=\begin{bmatrix}\frac{1}{6}&0\\0&1\end{bmatrix}$$

$$\text{Using } R_2\to R_2+4R_1$$

$$(AB)^{-1}\begin{bmatrix}1&-\frac{1}{2}\\0&-1\end{bmatrix}=\begin{bmatrix}\frac{1}{6}&0\\\frac{2}{3}&1\end{bmatrix}$$

$$\text{using } R_2\to(-1)R_2$$

$$(AB)^{-1}\begin{bmatrix}1&-\frac{1}{2}\\0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{6}&0\\-\frac{2}{3}&-1\end{bmatrix}$$

$$\text{Using } R_1\to R_1+\left(\frac{1}{2}\right)R_2$$

$$(AB)^{-1}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-\frac{1}{6}&-\frac{1}{2}\\-\frac{2}{3}&-1\end{bmatrix}$$

$$(AB)^{-1}I=\begin{bmatrix}-\frac{1}{6}&-\frac{1}{2}\\-\frac{2}{3}&-1\end{bmatrix}$$

$$(AB)^{-1}=\begin{bmatrix}-\frac{1}{6}&-\frac{1}{2}\\-\frac{2}{3}&-1\end{bmatrix}$$

Question 1.2.2: Find the vector equation of the plane passing through a point having position vector $3\hat{i}-2\hat{j}+\hat{k}$, and perpendicular to the vector $4\hat{i}+3\hat{j}+2\hat{k}$ **[2]**

Solution: We know that the vector equation of a plane passing through a point $A(\bar{a})$ and normal to $\bar{r}.\bar{n}=\bar{a}.\bar{n}$

Here $\bar{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{n} = 4\hat{i} + 3\hat{j} + 2\hat{k}$

The vector equation of the required plane is

$$\bar{r}.\bar{n} = \bar{a}.\bar{n}$$

$$\bar{r}.\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = \left(3\hat{i} - 2\hat{j} + \hat{k}\right).\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right)$$

$$\bar{r}.\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 12 - 6 + 2$$

$$\bar{r}.\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 8$$

The vector equation of the required plane is $\bar{r}.\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 8$

Question 1.2.3: If $\bar{p} = \hat{i} - 2\hat{j} + \hat{k}$ and $\bar{q} = \hat{i} + 4\hat{j} - 2\hat{k}$ are position vector (P.V.) of points P and Q, find the position vector of the point R which divides segment PQ internally in the ratio 2:1 **[2]**

Solution 1: R is the point which divides the line segment joining the points PQ internally in the ratio 2:1.

$$\bar{r} = \frac{2(\bar{q}) + 1(\bar{p})}{2+1}$$

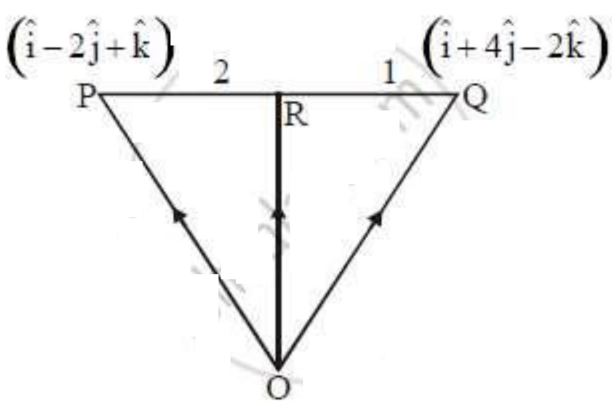
$$= \frac{2\left(\hat{i} + 4\hat{j} - 2\hat{k}\right) + 1\left(\hat{i} - 2\hat{j} + \hat{k}\right)}{3}$$

$$= \frac{3\hat{i} + 6\hat{j} - 3\hat{k}}{3}$$

$$\bar{r} = \hat{i} + 2\hat{j} - \hat{k}$$

The position vector of point R is $\hat{i} + 2\hat{j} - \hat{k}$

Solution 2:



Position vector of point R in

$$\overrightarrow{OR} = \frac{\overrightarrow{OQ} \times 2 + 1 \times \overrightarrow{OP}}{2+1}$$

$$\overrightarrow{OR} = \frac{2\left(\hat{i} + 4\hat{j} - 2\hat{k}\right) + 1\left(\hat{i} - 2\hat{j} + \hat{k}\right)}{3}$$

$$\overrightarrow{OR} = \frac{2\hat{i} + 8\hat{j} - 4\hat{k} + \hat{i} - 2\hat{j} + \hat{k}}{3}$$

$$\overrightarrow{OR} = \frac{3\hat{i} + 6\hat{j} - 3\hat{k}}{3}$$

$$\overrightarrow{OR} = \hat{i} + 2\hat{j} - \hat{k}$$

Question 1.2.4: Find k, if one of the lines given by $6x^2 + kxy + y^2 = 0$ is $2x + y = 0$ **[2]**

Solution: Let m_1 be the slope of $2x + y = 0$.

$\therefore m_1 = -2$

$$6x^2 + kxy + y^{=}0$$

$$\therefore a = 6, h = \frac{k}{2}, b = 1$$

$$m_1 + m_2 = -\frac{2h}{b} = -k$$

$$-2 + m_2 = -k$$

$$m_2 = -k + 2$$

$$Now, m_1 m_2 = \frac{a}{b}$$

$$(-2)(-k+2) = 6$$

$$2k - 4 = 6$$

$$2k = 10$$

$$k = \frac{10}{2}$$

$$k = 5$$

The value of k is 5.

Question 1.2.5: **[2]**

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \text{ and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$$

are at right angle then find the value of k

Solution: Given equations of the line are:

Let $\bar{a}$ and $\bar{b}$ be vectors in the direction of lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ respectively

$\therefore \bar{a} = -3\hat{i} + 2k\hat{j} + 2\hat{k}$ and $\bar{b} = 3k\hat{i} + \hat{j} - 5\hat{k}$

$\bar{a}.\bar{b} = -9k + 2k - 10 = -7k - 10$

Given lines are at right angle

$\theta = 90^\circ$

$\cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}|.|\bar{b}|}$

$\bar{a}.\bar{b} = 0$

$-7k - 10 = 0$

$k = -\frac{10}{7}$

Question 2: [14]

Question 2.1 | Attempt any TWO of the following [6]

Question 2.1.1: Examine whether the following logical statement pattern is a tautology, contradiction, or contingency. [5]
$[(p \rightarrow q) \wedge q] \rightarrow p$

Solution: Consider the statement pattern: $[(p \rightarrow q) \wedge q] \rightarrow p$
No. of rows = 2n = 2 × 2 = 4
No. of column = m + n = 3 + 2 = 5

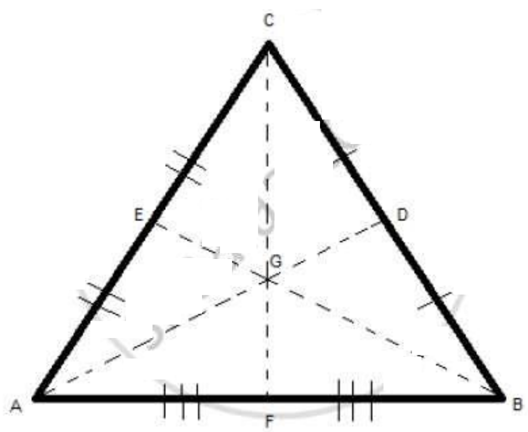
Thus the truth table of the given logical statement :
$[(p \rightarrow q) \wedge q] \rightarrow p$

p	**q**	**$p \rightarrow q$**	**$(p \rightarrow q) \wedge q$**	**$[(p \rightarrow q) \wedge q] \rightarrow p$**
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

From the above truth table we can say that given logical statement: $[(p \rightarrow q) \wedge q] \rightarrow p$ is contingency.

Question 2.1.2: By vector method prove that the medians of a triangle are concurrent.
[3]

Solution:



Let A, B and C be the vertices of a triangle.
Let D, E and F be the midpoints of the sides BC, AC and AB respectively.

Let $\overline{GA} = a, \overline{GB} = b, \overline{GC} = c, \overline{GD} = d, \overline{GE} = e$ and $\overline{GF} = f$ be position vectors of points A, B, C, D, E and F respectively.

Therefore, by midpoint formula,

$$\bar{d} = \frac{\bar{b} + \bar{c}}{2}, \bar{e} = \frac{\bar{a} + \bar{c}}{2} \text{ and } \bar{f} = \frac{\bar{a} + \bar{b}}{2}$$

$$2\bar{d} = \bar{b} + \bar{c}, 2\bar{e} = \bar{a} + \bar{c} \text{ and } 2\bar{f} = \bar{a} + \bar{b}$$

$$2\bar{d} + \bar{a} = \bar{a} + \bar{b} + \bar{c}, 2\bar{e} + \bar{b} = \bar{a} + \bar{b} + \bar{c} \text{ and } 2\bar{f} + \bar{c} = \bar{a} + \bar{b} + \bar{c}$$

$$\frac{2\bar{d} + \bar{a}}{3} = \frac{2\bar{e} + \bar{b}}{3} = \frac{2\bar{f} + \bar{c}}{3} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

$$Let \quad \bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

$$\therefore \quad \text{We have } \bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3} = \frac{(2)\bar{d} + (1)\bar{a}}{3} = \frac{(2)\bar{e} + (1)\bar{b}}{3} = \frac{(2)\bar{f} + (1)\bar{c}}{3}$$

If G is the point whose position vector is $\bar{g}$, then from the above equation it is clear that the point G lies on the medians AD, BE, CF
and it divides each of the medians AD, BE, CF internally in the ratio 2:1. Therefore, three medians are concurrent.

Question 2.1.3: Find the shortest distance between the lines [3]

$r = \left(4\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$

and

$r = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$

where λ and μ are parameters

Solution: Equation of lines are,

$r = (4\bar{i} - \bar{j}) + \lambda(\bar{i} + 2\bar{j} - 3\bar{k})$ &

$\bar{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$

∴ above lines passes through

$\bar{a}_1 = (4\bar{i} - \bar{j})\text{and}\bar{a}_2 = (\bar{i} - \bar{j} + 2\bar{k})$

and parallel to

$\bar{b}_1 = \bar{i} + 2\bar{j} - 3\bar{k}$ & $\bar{b}_2 = \bar{i} - 4\bar{j} - 5\bar{k}$

Shortest distance = $\left|\frac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|(\bar{b}_1 \times \bar{b}_2)|}\right|$

$\Rightarrow \bar{a}_2 - \bar{a}_1 = -3\bar{j} + 2\bar{k}$

$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix} = 2\bar{i} + 2\bar{j} + 2\bar{k}$

$\therefore |\bar{b}_1 \times \bar{b}_2| = 2\sqrt{3}$

Shortest distance = $\left|\frac{(-3\bar{i} + 2\bar{k}).(2\bar{i} + 2\bar{j} + 2\bar{k})}{2\sqrt{3}}\right|$

$= \left|\frac{-6 + 4}{2\sqrt{3}}\right|$

$= \left|-\frac{2}{2\sqrt{3}}\right|$

$\text{d} = \frac{1}{\sqrt{3}}\text{units}$

Question 2.2 | Attempt any TWO of the following : [8]

Question 2.2.1: [4]

In Δ ABC with the usual notations prove that $(a - b)^2 \cos^2\left(\frac{C}{2}\right) + (a + b)^2 \sin^2\left(\frac{C}{2}\right) = c^2$

Solution:

LHS= $(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)$

$$=a^2\left[\cos^2\left(\frac{C}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right]+b^2\left[\cos^2\left(\frac{C}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right]-2ab\left[\cos^2\left(\frac{C}{2}\right)-\sin^2\left(\frac{C}{2}\right)\right]$$

$$=a^2+b^2-a^2-b^2+c^2$$

$$=c^2$$

=RHS

Hence proved

Question 2.2.2: [4]

Minimize $z=4x+5y$ subject to $2x+y\geq 7, 2x+3y\leq 15, x\leq 3, x\geq 0, y\geq 0$ solve using graphical method.

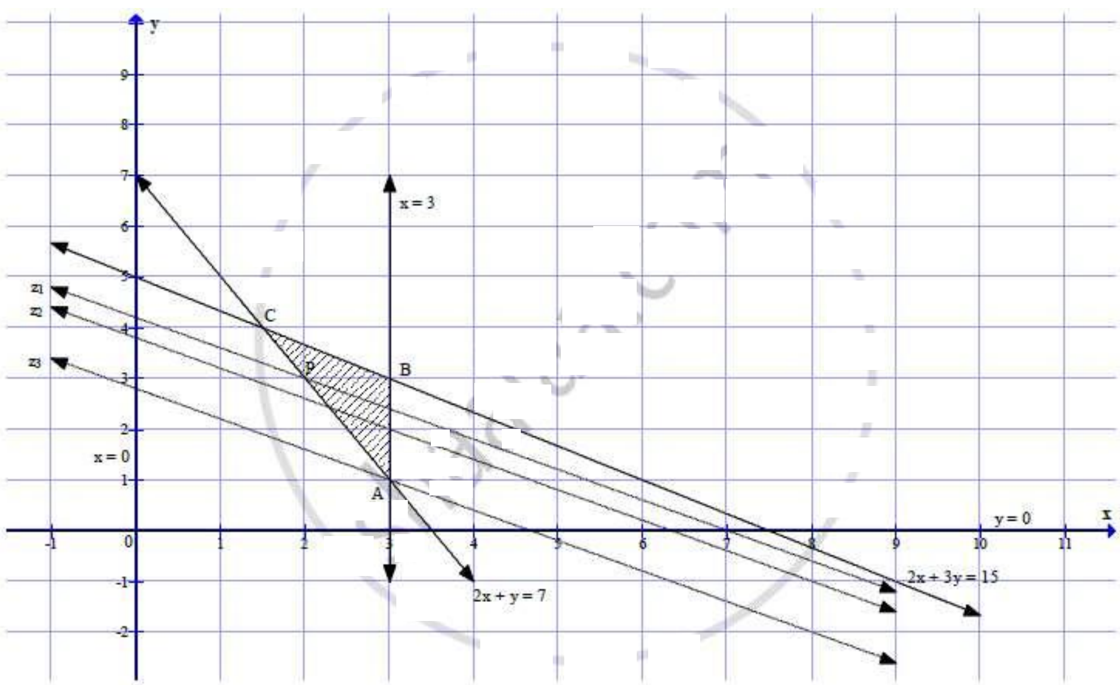
Solution: Consider equations obtained by converting all in equations representing the constraints.

$2x+y=7$ i.e $\frac{x}{3.5}+\frac{y}{7}=1$

$2x+3y=15$ i.e $\frac{x}{7.5}+\frac{y}{5}=1$

$x=3, x=0, y=0$

Plotting these lines on graph we get the feasible region.



From the graph we can see that ABC is the feasible region.

Take any one point on the feasible region say P (2, 3) .
Draw initial is cost line z passing through the point (2, 3) .

∴z1=4(2)+5(3)=8+15=23

$\therefore z_1 = 4(2) + 5(3) = 8 + 15 = 23$

$\therefore$ intial isocost line is $4x + 5y = 23.$

Since the objective function is of minimization type, from the graph we can see that the line z_3 contains only one point A(3, 1) of the feasible region ABC.

Minimum value of z = 4(3)+ 5(1)= 12 + 5 =17
z is minimum when x = 3 and y = 1.

Question 2.2.3: The cost of 4 dozen pencils, 3 dozen pens and 2 dozen erasers is Rs. 60. The cost of 2 dozen pencils, 4 dozen pens and 6 dozen erasers is Rs. 90 whereas the cost of 6 dozen pencils, 2 dozen pens and 3 dozen erasers is Rs. 70. Find the cost of each item per dozen by using matrices. **[4]**

Solution: Let Rs.'x', Rs.'y' and Rs.'z' be the cost of one dozen pencils, one dozen pens and one dozen erasers.
Thus, the system of equations are:

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

Let us write the above equations in the matrix form as:

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} \text{ i.e } AX = B$$

Using $R_z \to R_2 - \frac{1}{2}R_1$ and $R_3 \to R_3 - \frac{3}{2}R_1$

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & \frac{5}{2} & 5 \\ 0 & -\frac{5}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \\ -20 \end{bmatrix}$$

Using $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & \frac{5}{2} & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \\ 40 \end{bmatrix}$$

As matrix A is reduced to its upper triangular form we can write

4x + 3y + 2z = 60..........(i)

$\frac{5}{2}$y + 5z = 60..........(ii)

0x + 0y + 5z = 40

z = 8.....(iii)

Substituting (iii) in (ii) we get,

$\frac{5}{2}$y + 5(8) = 60

y = $\frac{60-40}{5} \times 2 = 8$

y = 8 .....(iv)

Substituting (iii) and (iv) in (i) we get,

4x + 3 (8) + 2 (8) = 60

4x = 60 - 24 - 16

x = $\frac{20}{4} = 5$

∴ x = 5

∴ Cost of one dozen pencils, one dozen pens and one dozen erasers is Rs. 5, Rs. 8 and Rs. 8 respectively.

Question 3: [14]

Question 3.1 | Attempt any TWO of the following: [6]

Question 3.1.1: Find the volume of tetrahedron whose coterminous edges are $7\hat{i} + \hat{k}; 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } 4\hat{i} + 3\hat{j} + \hat{k}$ [3]

Solution:

Volume of tetrahedron whose conterminus edges are $\bar{a}, \bar{b}$ and $\bar{c}$ is $\frac{1}{6}[\bar{a}\bar{b}\bar{c}]$.

Here $\bar{a} = 7\hat{i} + \hat{k}; \bar{b} = 2\hat{i} + 5\hat{j} - 3\hat{k}; \bar{c} = 4\hat{i} + 3\hat{j} + \hat{k}$.

Volume of tetrahedron $= \frac{1}{6}[\bar{a}, \bar{b}, \bar{c}]$

$$= \frac{1}{6}\begin{vmatrix} 7 & 0 & 1 \\ 2 & 5 & -3 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{6}[7(5+9) - 0(2+12) + 1(6-20)]$$

$$= \frac{1}{6}[98 - 0 - 14]$$

$$= \frac{1}{6}[84]$$

= 14

Hence volume of tetrahedron is 14 cubic units.

Question 3.1.2: Without using truth tabic show that ~(p v q)v(~p ∧ q) = ~p **[3]**

Solution: ~(p v q)v(~p ∧ q)

≡~(p v q)v~(p ∨ ~q) by De Morgan's Law

≡~[(p ∨ q) ∧ (p ∨ ~q)] by De Morgan's Law

≡~{[(p ∨ q) ∧ p] ∨ [(p ∨ q)∧ ~q)]} by Distributive Law

≡ ~{[p] ∨ [(p ∨ q) ∧ ~q]} by Absorption Law

≡ ~{[p] ∨ [(p∧ ~q) ∨ (q ∧ ~q)]} by Distributive Law

≡~{[p] ∨ [(p ∧ ~q) ∨ F]} by Complement Law

≡~{[p] ∨ [(p ∧ ~q)]} by Identity Law

≡~p ∧ (~p ∨ q) by De Morgan's Law

≡ ~p by Absorption Law

Question 3.1.3: Show that every homogeneous equation of degree two in x and y, i.e., $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through origin if $h^2 - ab \geq 0$. **[3]**

Solution: Consider a homogeneous equation of the second degree in x and y,

$ax^2 + 2hxy + by^2 = 0$...................... (1)

Case I: If b = 0 (i.e., a ≠ 0, h ≠ 0), then the equation (1) reduce to $ax^2 + 2hxy = 0$
i.e., $x(ax + 2hy) = 0$

Case II: If a = 0 and b = 0 (i.e. h ≠ 0), then the equation (1) reduces to 2hxy = 0, i.e., xy = 0 which represents the coordinate axes and they pass through the origin.

Case III: If b ≠ 0, then the equation (1), on dividing it by b, becomes $\frac{a}{b}x^2 + \frac{2hxy}{b} + y^2 = 0$

$$\therefore y^2 + \frac{2h}{b}xy = -\frac{a}{b}x^2$$

On completing the square and adjusting, we get $y^2 + \frac{2h}{b}xy + \frac{h^2x^2}{b^2} = \frac{h^2x^2}{b^2} - \frac{a}{b}x^2$

$$\left(y + \frac{h}{b}x\right)^2 = \left(\frac{h^2 - ab}{b^2}\right)x^2$$

$$\therefore y + \frac{h}{b}x = \pm\frac{\sqrt{h^2 - ab}}{b}x$$

$$\therefore y = -\frac{h}{b}x \pm \frac{\sqrt{h^2 - ab}}{b}x$$

$$\therefore y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

$\therefore$ equation represents the two lines $y = \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right)x$ and $y = \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right)x$

The above equation are in the form of y = mx
These lines passing through the origin.
Thus the homogeneous equation (1) represents a pair of lines through the origin, if $h^2 - ab \geq 0$.

Question 3.2 | Attempt any TWO of the following [8]

Question 3.2.1: If a line drawn from the point A(1, 2, 1) is perpendicular to the line joining P(1, 4, 6) and Q(5, 4, 4) then find the co-ordinates of the foot of the perpendicular. [4]

Solution: Let M be the foot of the perpendicular drawn from the point A (1, 2, 1) to the line joining P (1, 4, 6) and Q (5, 4, 4) .

Equation of a line passing through the points (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z1}$$

Equation of the required line passing through P (1, 4, 6) and Q(5, 4, 4) is

$$\frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}=\lambda$$

$x=4\lambda+1; y=4; z=-2\lambda+6$

$\therefore$ Coordinates of M are$(4\lambda+1,4,-2\lambda+6)$ ....(1)

The direction ratios of AM are

$4\lambda+1-1,4-2,-2\lambda+6-1$

i.e $4\lambda,2,-2\lambda+5$

The direction ratios of given line are 4,0,-2.
Since AM is perpendicular to the given line

$\therefore 4(4\lambda)+0(2)+(-2)(-2\lambda+5)=0$

$\therefore \lambda=\frac{1}{2}$

Putting $\lambda=\frac{1}{2}$ in (i) , the coordinates of M are (3,4,5).

Length of perpendicular from A on the given line

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z1}$$

Equation of the required line passing through P (1, 4, 6) and Q(5, 4, 4) is

$$\frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}=\lambda$$

$x=4\lambda+1; y=4; z=-2\lambda+6$

$\therefore$ Coordinates of M are$(4\lambda+1,4,-2\lambda+6)$ ....(1)

The direction ratios of AM are

$4\lambda+1-1,4-2,-2\lambda+6-1$

i.e $4\lambda,2,-2\lambda+5$

The direction ratios of given line are 4,0,-2.
Since AM is perpendicular to the given line

$\therefore 4(4\lambda)+0(2)+(-2)(-2\lambda+5)=0$

$\therefore \lambda=\frac{1}{2}$

Putting $\lambda=\frac{1}{2}$ in (i) , the coordinates of M are (3,4,5).

Length of perpendicular from A on the given line

$AM=\sqrt{(3-1)^2+(4-2)^2+(5-1)^2}=\sqrt{24} units.$

Question 3.2.2: Find the vector equation of the plane passing through the points $\hat{i}+\hat{j}-2\hat{k}, \hat{i}+2\hat{j}+\hat{k}, 2\hat{i}-\hat{j}+\hat{k}.$ Hence find the cartesian equation of the plane. **[4]**

Solution:

Let

$\overline{AB}=\bar{b}-\bar{a}=\left(\hat{i}+2\hat{j}+\hat{k}\right)-\left(\hat{i}+\hat{j}-2\hat{k}\right)=\hat{j}+3\hat{k}$

$\overline{AC}=\bar{c}-\bar{a}=\left(2\hat{i}-\hat{j}+\hat{k}\right)-\left(\hat{i}+\hat{j}-2\hat{k}\right)=\hat{i}-2\hat{j}+3\hat{k}$

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 3\\ 1 & -2 & 3\end{vmatrix}$$

$=\hat{i}(3+6)-\hat{j}(0-3)+\hat{k}(0-1)$

$=9\hat{i}+3\hat{j}-\hat{k}$

Then the equation of required plane is,

$\bar{r}.\bar{n}=\bar{a}.\bar{n}$

$\bar{r}.\left(9\hat{i}+3\hat{j}-\hat{k}\right)=\left(\hat{i}+\hat{j}-2\hat{k}\right).\left(9\hat{i}+3\hat{j}-\hat{k}\right)$

$\bar{r}.\left(9\hat{i}+3\hat{j}-\hat{k}\right)=9+3+2$

$\bar{r}.\left(9\hat{i}+3\hat{j}-\hat{k}\right)=14$

The cartesian equation of the plane is given by,

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(9\hat{i}+3\hat{j}-\hat{k}\right)=14,$

9x+3y-z=14

The cartesian equation of the plane is 9x + 3y - z = 14.

Question 3.2.3: Find the general solution of $\sin x + \sin 3x + \sin 5x = 0$ **[4]**

Solution:

$\sin x + \sin 3x + \sin 5x = 0$

$\therefore (\sin x + \sin 5x) + \sin 3x = 0$

$\therefore 2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{5x-x}{2}\right) + \sin 3x = 0$

$\therefore 2\sin 3x \cos 2x + \sin 3x = 0$

$\therefore (2\cos 2x + 1)\sin 3x = 0$

$\therefore$ sin 3x = 0 or 2 cos 2x + 1 = 0

$\therefore$ sin 3x = 0 ...(i) or cos 2x = $-\frac{1}{2}$ ....(ii)

For (ii) cos 2x = - cos $\frac{\pi}{3}$

$\therefore \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right)$ .....(by allied angles) cos 2x = cos $\frac{2\pi}{3}$

∴ from (i) and (ii) we get

∴ sin 3x = 0 or cos 2x = cos $\frac{2\pi}{3}$

∴ 3x = n π , n ∈ Z or 2x = 2mx $\pm\frac{2\pi}{3}$, $\mathbf{where\ \ m \in Z.}$

Here the required solution is

$\therefore \mathbf{x} = \frac{\mathbf{n}\pi}{3}$ or $\mathbf{x} = \mathbf{m}\pi \pm \frac{\pi}{3}, \mathbf{where n, m} \in \mathbf{Z}.$

Question 4: [12]

Question 4.1: Select and write the most appropriate answer from the given alternatives in each of the following sub-questions : [6]

Question 4.1.1: [2]

if the function

$$f(x) = k + x, f \text{ or } x < 1$$
$$= 4x + 3, f \text{ or } x \geq 1$$

id continuous at x=1 then k=

(a) 7

(b) 8

(c) 6

(d) -6

Solution: (c)

$f(1) = 4(1) + 3 = 7$

$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} h(1 - h) = \lim_{h \to 0} k + 1 - h = k + 1$

For the function to be continuous at x 1,

$f(1) = \lim_{x \to 1^-} f(x)$

7 = k + 1

k = 6

Question 4.1.2: [2]

The equation of tangent to the curve y $= y = x^2 + 4x + 1$ at (-1,-2) is..............

(a) 2x -y = 0 (b) 2x+y-5 = 0

(c) 2x-y-1=0 (d) x+y-1=0

Solution: (a)

$y = x^2 + 4x + 1$

Differentiating w.r.t 'x', we get

$\frac{dy}{dx} = 2x + 4$

$\frac{dy}{dx}\ |_{x \to -1} = 2(-1) + 4 = 2$

Hence, slope of tangent at (-1, -2) is 2.

So equation of tangent line is

y -(-2)= 2(x-(- 1))

2x - y = 0

Question 4.1.3: Given that X ~ B(n= 10, p). If E(X) = 8 then the value of p is .... [2]

(a) 0.6

(b) 0.7

(c) 0.8

(d) 0.4

Solution: (c)

Since X ~ B(n = 10, p),

E(x) = np

10p = 8

p = 0.8

Question 4.2 | Attempt any THREE of the following: [6]

Question 4.2.1: [2]

if $y = x^x$ find $\frac{dy}{dx}$

Solution:

$y = x^x$

Taking logarithms of both sides

$\log y = \log x^x$

$\log y = x \log x$

Differentiating both sides with respect to 'x', we get

$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x(1)$

$\frac{dy}{dx} = y(1 + \log x)$

$\frac{dy}{dx} = x^x(1 + \log x)$

Question 4.2.2: The displacement 's' of a moving particle at time 't' is given by s = 5 + 20t — 2t^2. Find its acceleration when the velocity is zero. [2]

Solution:

$s = 5 + 20t - 2t^2$

$v = \frac{ds}{dt} = 20 - 4t$

$v = 0$

$20 - 4t = 0$

$4t = 20$

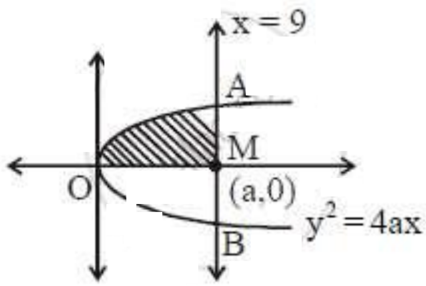
$t = 5$

$a = \frac{dv}{dt} = -4, \textbf{which is contatnt}$

Hence, acceleration is -4 when velocity is zero.

Question 4.2.3: Find the area bounded by the curve y^2 = 4ax, x-axis and the lines x = 0 and x = a. [2]

Solution:



Area bounded = $\int_0^a y dx$

$= \int_0^a \sqrt{4ax} dx$

$= 2\sqrt{a} \int_0^a x^{\frac{1}{2}} dx$

$= 2\sqrt{a} \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_0^a$

$= 2\sqrt{a} \times \frac{2}{3} \times a^{\frac{3}{2}}$

$= \frac{4}{3} a^{\frac{1}{2}+\frac{3}{2}}$

$= \frac{4}{3} a^2$

Question 4.2.4: The probability distribution of a discrete random variable X is: **[2]**

X=x	1	2	3	4	5
P(X=x)	k	2k	3k	4k	5k

Find P(X≤4)

Solution: Given X is discreate r . v.

∴ P(x) is p.m.f.

p.m.f = ∑ pi = 1

i.e. k + 2k + 3k + 4k + 5k = 1

i.e. 15 k = 1

i.e. k = $\frac{1}{15}$

P(X≤ 4) = P(X =1) + P(X = 2) + P(X = 3) + P(X = 4)

= k + 2k + 3k + 4k

= 10k

$=10 \times \frac{1}{15}$

= $\frac{2}{3}$

Question 4.2.5: [2]

Evaluate : $\int \frac{\sin x}{\sqrt{36-\cos^2 x}} dx$

Solution:

$$\int \frac{\sin x}{\sqrt{36-\cos^2 x}} dx$$

Substitute, cosx = t

∴ - sin x dx = dt

∴ sin x dx = - dt

The integral becomes

$$\int \frac{-dt}{\sqrt{36-t^2}}$$

$= -\int \frac{dt}{\sqrt{6^2-t^2}}$

$= -\sin^{-1}\left(\frac{t}{6}\right) + C$

$= -\sin^{-1}.\left(\frac{\cos x}{6}\right) + c$

Question 5: [12]

Question 5.1: Attempt any TWO of the following [6]

Question 5.1.1: If y=f(u) is a differentiable function of u and u = g(x) is a differentiable function of x then prove that y = f (g(x)) is a differentiable function of x and [3]

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Solution: Let δx be a small increment in x.
Let δy and δu be the corresponding increments in y and u respectively

As $\delta x \to 0, \delta y \to 0, \delta u \to 0$.
As u is differentiable function, it is continuous.

Consider the incrementary ratio $\frac{\delta y}{\delta x}$

We have , $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

Taking limit as $\delta x \to 0$, on both sides,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\partial ty}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} ...(1)$$

Since y is a differentiable function of u , $\lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ exists

and $\lim_{\delta u \to 0} \frac{\delta y}{\delta x}$ exists as u is a differentiable function of x.

Hence, R.H.S. of (1) exists

now $\lim_{\delta u \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du}$ and $\lim_{\delta u \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx}$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{du} \times \frac{du}{dx}$$

Since R.H.S. exists, L.H.S. of (1) also exists and

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Question 5.1.2: The probability that a person who undergoes kidney operation will recover is 0.5. Find the probability that of the six patients who undergo similar operations, **[3]**
(a) None will recover
(b) Half of them will recover.

Solution: Probability of recovery=P(R) = 0.5

Probability of non-recovery = $P(\bar{R}) = 1 - 0.5 = 0.5$

(a) If there are six patients, the probability that none recovers

$$= {}^6C_0 \times [P(R)]^0 \times \left[P(\bar{R})\right]^6 = (0.5)^6 = \frac{1}{64}$$

(b) Of the six patients, the probability that half will recover

$$= {}^6C_3 \times [P(R)]^3 \times \left[P(\bar{R})\right]^3 = \frac{6!}{3!3!} \times 0.5^3 \times 0.5^3 = 20 \times \frac{1}{64} = \frac{5}{16}$$

Question 5.1.3: [3]

Evaluate : $\int_0^\pi \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

Solution:

$$I = \int_0^\pi \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \ldots\ldots\ldots\ldots (i)$$

$$I = \int_0^\pi \frac{\pi - x}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$$

$$I = \int_0^\pi \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \ldots\ldots\ldots . (ii)$$

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx$$

Adding (i) and (ii), we get

$$2\mathrm{I} = \int_0^\pi \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$2\mathrm{I} = \int_0^\pi \frac{\pi}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$2I = \int_0^{\pi} \frac{\pi \sec^2 x}{a^2 + b^2 \tan^2 x} \quad \text{........} \quad \frac{1}{b^2} \int_0^{\pi} \frac{\pi \sec^2 x dx}{\left(\frac{a}{b}\right)^2 + \tan^2 x}$$

$$2I = \frac{\pi}{b^2} \int \frac{dt}{\left(\frac{a}{b}\right)^2} + t^2 \quad \text{.........} \quad [\tan x = t \ \rightarrow \sec^2 x dx \ = dt]$$

$$2I = \frac{\pi}{b^2} \left[\left(\frac{b}{a} \right) \tan^{-1} \left(b \frac{t}{a} \right) \right]_0^{\pi}$$

$$2I = \frac{\pi}{ab} \left[\tan^{-1} \left(\frac{b}{a} \tan x \right) \right]_0^{\pi}$$

$$2I = \frac{\pi}{ab} (0 - 0) = 0$$

2 I = 0

I = 0

Question 5.2: Attempt any TWO of the following [8]

Question 5.2.1: Discuss the continuity of the following functions. If the function have a removable discontinuity, redefine the function so as to remove the discontinuity [4]

$$f(x) = \frac{4^x - e^x}{6^x - 1} \text{ for } x \neq 0$$
$$= \log\left(\frac{2}{3}\right) \text{ for } x=0$$

Solution:

$$f(0) = \log\left(\frac{2}{3}\right) \quad \text{.....Given .....(1)}$$

$$\lim_{x \to 0^- f(x)} = \lim_{x \to 0^-} \frac{4^x - e^x}{6^x - 1}$$

$$= \lim_{x \to 0} \frac{(4^x - 1) - (e^x - 1)}{6^x - 1}$$

$$\lim_{x \to 0} \frac{\frac{4^x - 1}{x} - \frac{e^x - 1}{x}}{\frac{6^x - 1}{x}} \quad \text{.....}[x \rightarrow 0 \text{ , } x \neq 0]$$

$$\lim_{x\to 0}\frac{\lim_{x\to 0}\frac{4^x-1}{x}-\lim_{x\to 0}\frac{e^x-1}{x}}{\lim_{x\to 0}\frac{6^x-1}{x}}$$

$$=\frac{(\log 4)-1}{\log 6} \quad \text{.......}\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log e\right]$$

$$\therefore \lim_{x\to 0} f(x)=\frac{(\log 4)-\log e}{\log 6}$$

$$\therefore \lim_{x\to 0}=\frac{\log 4}{\log e.\log 6}$$

$$\therefore \lim_{x\to 0}=\frac{\log 4}{1.\log 6}$$

$$\therefore \lim_{x\to 0}=\log\left(\frac{2}{3}\right)$$

From (1) and (2) , lim_(x->0) f(x) ≠ f(0)

∴ f is discontinuous at x = 0

Here lim_(x->0) f(x) exists but not equal to f(0). Hence, the discontinuity at x = 0 is removable and can be removed by redefining the function as follows :

$$f(x)=\frac{4^x-e^x}{6^x-1} \text{ for } x\neq 0$$

$$=\log\left(\frac{2}{3}\right) \text{ for } x=0$$

Question 5.2.2: **[4]**

Prove that : $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

Solution:

$$\int\sqrt{a^2-x^2}dx$$

Substitute x = asinθ ...(i)

dx = acosθ dθ

The integral becomes

$$\int\sqrt{a^2-a^2\sin^2\theta}a\cos\theta d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\, a\cos\theta d\theta$$

$$= a^2 \int \cos^2\theta d\theta$$

$$= a^2 \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= a^2 \left[\int \frac{1}{2} d\theta + \int \frac{\cos 2\theta}{2} d\theta \right]$$

$$= \frac{a^2\theta}{2} + \frac{a^2}{4}\sin 2\theta + C$$

$$\text{From } \ (i), \theta = \sin^{-1}\left(\frac{x}{a}\right), \sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{a}\right)\sqrt{1-\left(\frac{x^2}{a^2}\right)} = \frac{2x}{a^2}\sqrt{a^2-x^2}$$

Substituting these values, we get

$$\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{4} \times \frac{2x}{a^2}\sqrt{a^2-x^2} + C$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C \quad \text{(Proved)}$$

Question 5.2.3: A body is heated at 110°C and placed in air at 10°C. After 1 hour its temperature is 60°C. How much additional time is required for it to cool to 35°C? **[4]**

Solution:

Let $\theta^{\circ}\mathrm{C}$ he the temperature of the body at any time t.

Temperature of air is 10° C i.e $\theta_{\circ} = 10$

According to Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto \theta - \theta_{\circ}$$

$$\therefore \frac{d\theta}{dt} = -k(\theta - \theta_{\circ}), k > 0$$

$$\therefore \frac{\mathrm{d}\theta}{\mathrm{dt}} = -\mathrm{K}(\theta - 10)$$

$$\therefore \frac{1}{\theta - 10}\mathrm{d}\theta = -kdt$$

Integrating both sides, we get

$$\int \frac{1}{\theta - 10} d\theta = -k \int dt$$

$$\therefore \log(\theta - 10) = -Kt + C$$

When t = 0 , θ = 110° C

∴ log (110 - 10) = - K × 0 + C ∴ C = log 100

∴ log (θ - 10) = - Kt + log 100

$$\log\left(\frac{\theta - 10}{100}\right) = -Kt$$

Also θ = 60° C ; when t = 1

$$\therefore \log\left(\frac{60 - 10}{100}\right) = -K \times 1$$

$$\therefore K = -\log\left(\frac{1}{2}\right)$$

$$\therefore \log\left(\frac{\theta - 10}{100}\right) = t \log\left(\frac{1}{2}\right)$$

when θ = 35° C , then

$$\log\left(\frac{35 - 10}{100}\right) = t \log\left(\frac{1}{2}\right)$$

$$\log\left(\frac{25}{100}\right) = t \log\left(\frac{1}{2}\right)$$

$$\log\left(\frac{1}{4}\right) = t \log\left(\frac{1}{2}\right)$$

- log 4 = - t log 2

t = $\frac{\log 4}{\log 2} = 2$

The additional time required for body to cool to 35° C = (2 - 1) = 1 hour

Question 6: [14]

Question 6.1 | Attempt any TWO of the following : [6]

Question 6.1.1: [3]

Prove that: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Solution:

$$LHS = \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx ....(1)$$

Substitute x = a + t in the second integral

dx=dt

When x = a, t = 0.

When x = 2a, t = a.

$$\therefore \int_a^{2a} f(x)dx = \int_0^a f(a+t)dt$$

$$= \int_0^a f(a+(a-t))dt \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^a f(2a-t)dt$$

$$\int_a^{2a} f(x)dx = \int_0^a f(2a-x)dx \left(\because \int_0^a f(t)dt = \int_0^a f(x)dx\right)$$

Using the above in (1), we get

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

$$= \int_0^a f(x)dx + \int_0^a f(2a - x)dx = RHS(\text{Proved})$$

Question 6.1.2: **[3]**

Evaluate: $\int \frac{1 + \log x}{x(2 + \log x)(3 + \log x)} dx$

Solution:

$$\int \frac{1 + \log x}{x(2 + \log x)(3 + \log x)} dx$$

Substitute logx = t..................(1)

$$\therefore \frac{1}{x} dx = dt$$

Hence, the integral becomes

$$\int \frac{1 + t}{(2 + t)(3 + t)} dt$$

$$= \int \frac{2 + t - 1}{(2 + t)(3 + t)} dt$$

$$= \int \frac{2 + t}{(2 + t)(3 + t)} dt - \int \frac{1}{(2 + t)(3 + t)} dt$$

$$= \int \frac{1}{3 + t} dt - \int \frac{(t + 3) - (t + 2)}{(2 + t)(3 + t)} dt$$

$$= \int \frac{1}{3 + t} dt - \left[\int \frac{t + 3}{(2 + t)(3 + t)} dt - \int \frac{t + 2}{(2 + t)(3 + t)} dt\right]$$

$$= \int \frac{1}{3 + t} dt - \int \frac{1}{2 + t} dt + \int \frac{1}{3 + t} dt$$

$$= 2\int \frac{1}{3 + t} dt - \int \frac{1}{2 + t} dt$$

$$= 2\int \frac{1}{3 + t} dt - \int \frac{1}{2 + t} dt$$

Substituting the value of 't' from (1), we get

$$\int \frac{1 + \log x}{x(2 + \log x)(3 + \log x)} dx$$

$$2\ln(3 + \log x) - \ln(2 + \log x) + C$$

$$= \log\left|\frac{(3 + \log x)^2}{2 + \log x}\right| + C$$

Question 6.1.3: [3]

If $y=\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, find dy/dx

Solution:

$y=\cos^{-1}\left(2x\sqrt{1-x^2}\right)$

put $x=\sin\theta$

$\theta=\sin^{-1}x$

$=\cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$=\cos^{-1}(\sin 2\theta)$

$=\cos^{-1}\left(\cos\left(\frac{\pi}{2}-2\theta\right)\right)$

$y=\frac{\pi}{2}-2\theta=\frac{\pi}{2}-2\sin^{-1}x$

Differentiating with respect to 'x', we get

$$\frac{dy}{dx}=0-\frac{2}{\sqrt{1-x^2}}=\frac{-2}{\sqrt{1-x^2}}$$

Question 6: [8]

Question 6.2: Attempt any TWO of the following : [4]

Question 6.2.1: Solve the differential equation cos(x +y) dy = dx hence find the particular solution for x = 0 and y = 0.

Solution: Cos (x + y)dy = dx

$\therefore \frac{dy}{dx} = \frac{1}{\cos(x+y)}$

Let x + y = t

$\therefore 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dt}{dx} - 1 = \frac{1}{\cos t}$

$\frac{dt}{dx} = \frac{1}{\cos t} + 1$

$\frac{dt}{dx} = \frac{1+\cos t}{\cos t}$

$\therefore \frac{\cos t}{1+\cos t} dt = dx$

Integrating both side.

$\therefore \int \frac{\cos t}{1+\cos t} dt = \int dx$

$\therefore \int \frac{\cos t(1-\cos t)}{\sin^2 t} dt = x + c$

$\therefore \int (\cos ect.\cot t - \cot^2 t) dt = x + c$

$\therefore \int (\cos ect.\cot t - \cos ec^2 t + 1) dt = x + c$

$\therefore$ - cosect + cot t + t = x + c

$\therefore \frac{\cos t}{\sin t} - \frac{1}{\sin t} + t = x + c$

- tan$\left[\frac{x+y}{2}\right]$ + x + y = x + c

$\therefore$ -tan$\left[\frac{x+y}{2}\right]$ + y = c

Putting x = 0, y = 0

$\therefore$ -tan$\left[\frac{0+0}{2}\right]$ + 0 = c

$\therefore$ c = 0

$\therefore$ y = tan$\left[\frac{x+y}{2}\right]$

Question 6.2.2: A wire of length l is cut into two parts. One part is bent into a circle and other into a square. Show that the sum of areas of the circle and square is the least, if the radius of circle is half the side of the square. **[4]**

Solution: Length of the wire is 'l'.
Let the part bent to make circle is of length 'x',
and the part bent to make square is of length 'l - x'.
Circumference of the circle = 2πr = x

$r = \frac{x}{2\pi}$

Area of the circle $= \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$

Perimeter of the square = $4a = l - x \Rightarrow a = \frac{l-x}{4}$

Area of the square = $\left(\frac{1-x}{4}\right)^2 = \frac{(1-x)^2}{16}$

Sum of the areas A(x) = $\frac{x^2}{4\pi} + \frac{(l-x)^2}{16}$

For extrema, $\frac{dAx}{dx} = 0$

$\frac{2x}{4\pi} + \frac{2(l-x)(-1)}{16} = 0$

$\frac{4(2x) + 2\pi(x-l)}{16} = 0$

$4x + \pi x - \pi l = 0$

$x = \frac{\pi l}{4+\pi}$

Since there is one point of extremum, it has to be the minimum in this case.

$r = \frac{x}{2\pi} = \frac{l}{2(4+\pi)}$.........(1)

Side of the square $a = \frac{l-x}{4} = \frac{l - \frac{\pi l}{4+\pi}}{4} = \frac{l}{4+\pi}$..................(2)

From (1) and (2), we get that the radius of the circle is half the side of the square, for least sum of areas. (Proved)

Question 6.2.3: The following is the p.d.f. (Probability Density Function) of a continuous random variable X : **[4]**

$f(x) = \frac{x}{32}, 0 < x < 8$

= 0 otherwise

(a) Find the expression for c.d.f. (Cumulative Distribution Function) of X.

(b) Also find its value at x = 0.5 and 9.