Mathematics & Statistics

Academic Year: 2012-2013 Marks: 80
Date: March 2013

Question 1: [12]

Question 1: Select and write the correct answer from the given alternatives in each of the following: [6]

Question 1.1.1: [2]

The principal solution of the equation cot x= $-\sqrt{3}$ is

$\frac{\pi}{6}$

$\frac{\pi}{3}$

$\frac{5\pi}{6}$

$\frac{-5\pi}{6}$

Solution:

$\frac{5\pi}{6}$

$\cot x = -\sqrt{3}$

$\cot x = -\cot x$

we know $\cot x$ is negative in 2nd and 4th quadrant so, we convert this 2nd or 4th

now

$\cot x = -\cot x = \cot\left(\pi - \frac{5\pi}{6}\right)$

$x = \frac{5\pi}{6}$

also

$\cot x = \cot\left(2\pi - \frac{11\pi}{6}\right)$

$x = \frac{11\pi}{6}$

only option given is $\frac{5\pi}{6}$

Question 1.1.2: [2]

If the vectors $-3\hat{i}+4\hat{j}-2\hat{k}, \hat{i}+2\hat{k}, \hat{i}-p\hat{j}$ are coplanar, then the value of of p is
(A) -2
(B) 1
(C) -1
(D) 2

Solution: (d) 2

Question 1.1.3: If the line y =x + k touches the hyperbola $9x^2 - 16y^2 = 144$, then k = ________ **[2]**
7
-7
$\pm\sqrt{7}$
$\pm\sqrt{19}$

Solution: C)$\pm\sqrt{7}$

Question 1.2: Attempt any THREE of the following: **[6]**

Question 1.2.1: Write down the following statements in symbolic form : **[2]**
(A) A triangle is equilateral if and only if it is equiangular.
(B) Price increases and demand falls

Solution: (a) p ≡ A triangle is equilateral & q ≡ A triangle is equiangular
∴Symbolic form $p \leftrightarrow q$

(b) Let p ≡ Price increases & q ≡ Demand falls
∴ Symbolic form $p \wedge q$

Question 1.2.2: **[2]**

If $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ then find A^{-1} by adjoint method.

Solution:

Given :$A == \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14 \neq 0$

$\therefore A^{-1}$ exist

$M_{11} = 3 \quad A_{11} = (-1)^2 3 = 3$

$M_{12} = 4 \quad A_{11} = (-1)^3 3 = -4$

$M_{21} = -2 \quad A_{11} = (-1)^3 (-2) = 2$

$M_{22} = 2 \quad A_{11} = (-1)^2 3 = 2$

$Adj.(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{Adj(A)}{|A|}$

$= \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

Question 1.2.3: Find the separate equations of the lines represented by the equation $3x^2 - 10xy - 8y^2 = 0$ [2]

Solution:

Given pairs of lines $3x^2 - 10xy - 8y^2 = 0$

$3x^2 - 12xy + 2xy - 8y^2 = 0$

$3x(x - 4y) + 2y(x - 4y) = 0$

$(x - 4y)(3x + 2y) = 0$

Separated equations

$3x + 2y = 0$ and $x - 4y = 0$

Question 1.2.4: Find the equation of the director circle of a circle $x^2 + y^2 = 100.$ [2]

Solution:

Given circle is

$x^2 + y^2 = 100$

$\therefore a^2 = 100$

$a = 10$

therefore Equation of director circle is

$x^2 + y^2 = 2a^2$

$\therefore x^2 + y^2 = 200$

Question 1.2.5: Find the general solution of the equation 4 $\cos^2 x = 1$ **[2]**

Solution:

Given:

$4\cos^2 x = 1$

$\cos^2 x = \frac{1}{4}$

$\cos^2 x = \cos^2\left(\frac{\pi}{3}\right)$

$\left[\text{Using} \quad \cos^2 x = \cos^2 \alpha \Rightarrow x = n\pi \pm \alpha\right]$

$x = n\pi \pm \frac{\pi}{3} \quad \text{where} \quad n \in Z$

Question 2.1: Attempt any TWO of the following: **[6]**

Question 2.1.1: Without using the truth table show that $P \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ **[3]**

Solution: L.H.S = $p \leftrightarrow q$

$\equiv (p \rightarrow q) \wedge (q \rightarrow p)$ ....(Bi conditional Law)

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ ....(Conditional Law)

$\equiv [\sim p \wedge (\sim q \vee p)] \vee [q \wedge (\sim q \vee p)]$ ....(Distributive Law)

$\equiv [(\sim p \wedge \sim q)] \vee (\sim p \wedge p)] \vee [(q \wedge \sim q) \vee (q \wedge p)]$ ....(Distributive Law)

$\equiv [(\sim p \wedge \sim q) \vee F] \vee [F \vee (q \wedge p)]$ ....(Complement Law)

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$ ....(Identity Law)

$\equiv (\sim p \wedge \sim q) \vee (p \wedge q)$ ....(Commutative Law)

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ ....(Commutative Law)

$\equiv$ R.H.S.

Question 2.1.2: [3]

If θ is the measure of acute angle between the pair of lines given by $ax^2+2hxy+by^2=0$, then prove that

$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|, a+b\neq 0$

Solution:

Let m_1 and m_2 be the slopes of the lines represented by the equation

$ax^2+2hxy+by^2=0$ ....(1)

Then their separate equation are

$y=m_1x$ and $y=m_2x$

therefore their combined equation is

$(y-m_1x)(y-m_2x)=0$

i.e $m_1m_2x^2-(m_1+m_2)xy+y^2=0$ ....(2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we get

$$\frac{m_1m_2}{a}=\frac{1}{b}=\frac{m_1+m_2}{2h}$$

$$\therefore m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}$$

Thus $(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$

$$(m_1 - m_2)^2 = \left(-\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$= (m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2}$$

Let the angle between y = m_1x and y =m_2x be θ .

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

$$= \left|\frac{\sqrt{m_1 - m_2}^2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{4\sqrt{h^2 - ab}}{b^2}}{1 + \frac{a}{b}}\right|,$$

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|, \quad \text{if} \quad a + b \neq 0$$

Hence to proved.

Question 2.1.3: Show that the line x+ 2y + 8 = 0 is tangent to the parabola y^2 = 8x. Hence find the point of contact **[3]**

Solution: Given line is

$2y = -x - 8$

$y = -\frac{1}{2}x - 4$

$\therefore m = -\frac{1}{2}, c = -4$

equation of Parabola

$y^2 = 8x$

$a = 2$

$\therefore \frac{a}{m} = \frac{2}{-\frac{1}{2}} = -4 = c$

Hence x +2y +8 = 0 is tangent to the Parabola y^2 = 8x

Point tocontact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$\frac{a}{m^2} = \frac{2}{\frac{1}{4}} = 8$

$\frac{2a}{m} = \frac{2 \times 2}{-\frac{1}{2}} = -8$

Point of contat (8, –8)

Question 2.2 | Attempt any TWO of the following : **[8]**

Question 2.2.1: The sum of three numbers is 9. If we multiply third number by 3 and add to the second number, we get 16. By adding the first and the third number and then subtracting twice the second number from this sum, we get 6. Use this information and find the system of linear equations. Hence, find the three numbers using matrices. **[4]**

Solution: Let the three numbers be x, y, z
From the first condition
x + y + z = 9
From the second condition
y + 3z = 16
From the third condition
x − 2y + 2 = 6

x + y + z = 9
y + 3z = 16
x − 2y + z = 6

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\16\\6\end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\0 & -3 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\16\\-3\end{bmatrix}$$

$x+y+z=9$

$y= \ 3z= \ 16$

$-3y=-3 \Rightarrow y=1$

$1+3z=16$

$z=5$

$x+1+5=9$

$x=3$

$\therefore x=3, y=1, z=5$

Question 2.2.2: Find the general solution of cos x +sin x =1. [4]

Solution:

cos x + sinx = 1

$$\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos x \frac{\cos \pi}{4} + \sin x \frac{\sin \pi}{4} = \frac{\cos \pi}{4}$$

$$\cos\left(x-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)$$

$\text{by} \quad \cos y = \cos \alpha$

$y = 2n\pi \pm \alpha$

$$x-\frac{\pi}{4}=2n\pi \pm \frac{\pi}{4}$$

$$x-\frac{\pi}{4}=2n\pi + \frac{\pi}{4} \text{ or } x-\frac{\pi}{4}=2n\pi - \frac{\pi}{4}$$

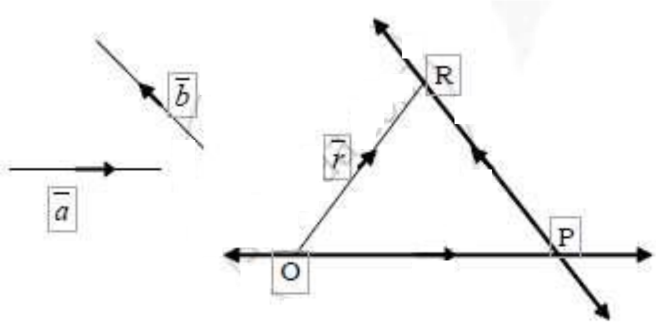
$$x=2n\pi+\frac{\pi}{2} \text{ or } x=2n\pi$$

Question 2.2.3: If $\bar{a}$ and $\bar{b}$ are any two non-zero and non-collinear vectors then prove that any vector $\bar{r}$ coplanar with $\bar{a}$ and $\bar{b}$ can be uniquely expressed as $\bar{r} = t_1\bar{a} + t_2\bar{b}$, where t_1 and t_2 are scalars **[4]**

Solution: Let $\bar{r} = \overline{OR}$ be any vector in a plane.

Let bar and barb be two non-zero non-collinear vectors in the same plane.
We have to prove that bar can be expressed as a linear combination of bar and barb and the linear combination is unique.

Passing through the point O, draw a line parallel to $\bar{a}$, and passing through the point R draw another line || to $\bar{b}$ and let them intersect at the point P.



$\therefore \overline{OP} || \bar{a}$ and $\overline{PR} || \bar{b}$

$\therefore \overline{OP} = t_1\bar{a}$ and $\overline{PR} = t_2\bar{b}$

(where t_1 and t_2 are any two scalars)

In triangle OPR by triangle law

$\overline{OR} = \overline{OP} + \overline{PR}$ i.e $\bar{r} = t_1\bar{a} + t_2\bar{b}$

Thus barr is expressed as a linear combination of $\bar{a}$ and $\bar{b}$

To prove the uniqueness of the linear combination.

If possible let $\alpha\prime$ and $\beta\prime$ be two scalars

Such that: $\bar{r} = \alpha\prime a + \beta\prime b,$

$where t_1 \neq \alpha\prime \text{ and } t_2 \neq \beta\prime$

$\therefore \quad \text{we get} \;\; t_1\bar{a} + t_2\bar{b} = \alpha\prime\bar{a} + \beta\prime\bar{b}$

$(t_1 \alpha\prime)\bar{a} = (t_2 - \beta\prime)\bar{b}$

Dividing throughout by $t_1 - \alpha\prime$ we get :

$$\bar{a} = \left(\frac{\beta\prime - t_2}{t_1 - \alpha\prime}\right)\bar{b}$$

This is of the form $\bar{a} = k\bar{b}$, which shows that $\bar{a}\&\bar{b}$ are collinear which is a contradiction.

Our assumption that $t_1 \neq \alpha\prime$ is wrong.

$t_1 = \alpha\prime$. Similarly, we can prove that $t_2 = \beta\prime$

∴ The linear combination is unique.

Question 3.1: Attempt any TWO of the following : [6]

Question 3.1.1: Using truth table examine whether the following statement pattern is tautology, contradiction or contingency $(p \wedge \text{-}q) \leftrightarrow (p \rightarrow q)$ [3]

Solution:

p	q	$\sim q$	$p \wedge \sim q$	$p \rightarrow q$	$(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

All the entries in the last column of the above truth table are F.

$(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$ is is a contradiction $(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$ is is a contradiction

Question 3.1.2: Find k if the length of the tangent segment from (8,-3) to the circle $x^2 + y^2 - 2x + ky - 23 = 0$ is $\sqrt{10}$ units. [3]

Solution:

Length of tangent from a point (x, y)

$$PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Given that $PT = \sqrt{10}$

$$10 = 8^2 + (-3)^2 - 2(8) + k(-3) - 23$$

$$10 = \ 64 + 9 - 16 - 3k - 23$$

$$24 - 3k = 0$$

$$3k = 24$$

$$k = 8$$

Question 3.1.3: Show that the lines given by $\frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1}$ and $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ intersect. Also find the co-ordinates of the point of intersection. **[3]**

Solution: The given lines are intersecting direction ratio of lines not proportional

$$Let \ \frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1} = \lambda$$

therefore The coordinates of a point on the line are $(-10\lambda - 1, -\lambda - 3, \lambda + 4)$

Similarly, let, $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4} = \mu$

The coordinates of a point on this line are $(-\mu - 10, -3\mu - 1, 4\mu + 1)$

Since two lines intersect for some value of λ and μ.

$$\therefore (-10\lambda - 1, -\lambda - 3, \lambda + 4) = (-\mu - 10, -3\mu - 1, 4\mu + 1)$$

$$-10\lambda - 1 = -\mu - 10, -\lambda - 3 = -3\mu - 1, \lambda + 4 = 4\mu + 1$$

$$-10\lambda + \mu = -9.........(1)$$

$$-\lambda + 3\mu = 2............(2)$$

$$\lambda - 4\mu = -3.............(3)$$

Solving equation (1) and (2), we get $\lambda = \ I, \mu = 1$ and 3rd equation holds for these values.

∴ The lines intersect at the point (–11, –4, 5)

Question 3.2 | Attempt any TWO of the following: **[8]**

Question 3.2.1: Find the equation of the locus of the point of intersection of two tangents drawn to the hyperbola $\frac{x^2}{7} - \frac{y^2}{5} = 1$ such that the sum of the cubes of their slopes is 8. **[4]**

Solution:

Let $p(x_1, y_1)$ be a point on the locus. The equation of the hyperbola is $\frac{x^2}{7} - \frac{y^2}{5} = 1$ for which $a^2 = 7, b^2 = 5$

The equation of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m are

$y = mx \pm \sqrt{a^2m^2 - b^2}$

the equation of the tangents to the given hyperbola are

$y = mx \pm \sqrt{7m^2 - 5}$

If these tangents pass through the point $p(x_1, y_1)$ we get

$y_1 = mx_1 \pm \sqrt{7m^2 - 5}$

y

$y_1 - mx_1 = \pm\sqrt{7m^2 - 5}$

On squaring both sides, we get,

$(y_1 - mx_1)^2 = 7m^2 - 5$

$y_1^2 - 2mx_1y_1 + m^2x_1^2 = 7m^2 - 5$

$(x_1^2 - 7)m^2 - 2mx_1y_1 + (y_1^2 + 5) = 0$

The roots m_1, m_2 of this quadratic equation in m are the slopes of the tangents drawn from the point P to the hyperbola.

From this quadratic equation,

$$\begin{cases} m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - 7} \\ \text{and } m_1m_2 = \frac{y_1^2+5}{x_1^2-7} \end{cases} \text{..........} (1)$$

We are given that,

sum of the cubes of the slopes = 8

$m_1^3 + m_2^3 = 8$

$(m_1 + m_2)^3 - 3m_1m_2(m_1 + m_2) = 8$

$$\left(\frac{2x_1y_1}{x_1^2-7}\right)^3-3\left(\frac{y_1^2+5}{x_1^2-7}\right)\left(\frac{2x_1y_1}{x_1^2-7}\right)=8 \ldots\ldots\ldots\ldots\ldots . by(1)$$

$$8x_1^3y_1^3-6x_1y_1\left(y_1^2+5\right)\left(x_1^2-7\right)=8\left(x_1^2-7\right)^3$$

$$8x_1^3y_1^3-6x_1y_1\left(y_1^2+5\right)\left(x_1^2-7\right)-8\left(x_1^2-7\right)^3=0$$

the equation of the locus of $P(x_1, y_1)$ is

$$8x^3y^3-6xy\left(y_1^2+5\right)\left(x_1^2-7\right)-8\left(x_1^2-7\right)^3=0$$

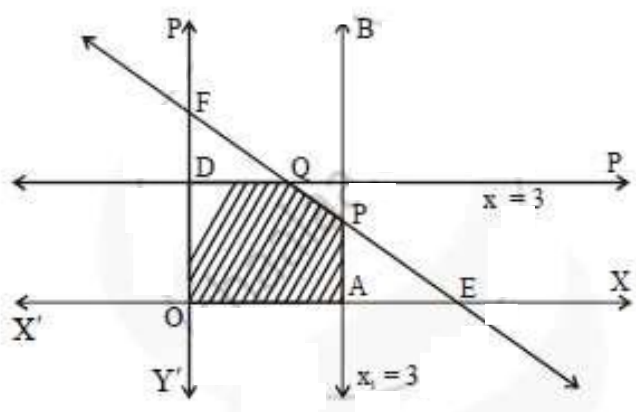
Question 3.2.2: Solve the following L.P.P graphically: [4]

Maximize: Z = 10x + 25y

Subject to: x ≤ 3, y ≤ 3, x + y ≤ 5, x ≥ 0, y ≥ 0

Solution: First we draw the lines AB, CD and EF whose equations are x = 3, y = 3 and x + y = 5 respectively.

Line	Equation	Point on the X-axis	Point on the Y-axis	Sign	Region
AB	x = 3	A(3,0)	-	≤	origin side of line AB
CD	y = 3	-	D(0,3)	≤	origin side of line CD
EF	x + y = 5	E(5,0)	F(0,5)	≤	origin side of line EF



The feasible region is OAPQDO which is shaded in the figure.

The vertices of the feasible region are O (0,0), A (3, 0), P, Q and D (0, 3)

P is the point of intersection of the lines x + y = 5 and x = 3

Substituting x = 3 in x + y = 5, we get, 3+ y=5

y = 2

P≡ (3, 2)

Q is the point of intersection of the lines x + y = 5 and y = 3

Substituting y = 3 in x + y = 5, we get,

x + 3 = 5

x = 2

Q ≡ (2,3)

The values of the objective function z = 10x + 25y at these vertices are

Z(O) =10(0)+ 25(0)= 0

Z(A) =10(3) + 25(0) = 30

Z(P) =10(3) + 25(2) = 30 + 50 = 80

Z(Q) =10(2) + 25(3) = 20 + 75 = 95

Z(D) =10(0) + 25(3) =75

Z has max value 95, when x = 2 and y = 3.

Question 3.2.3: Find the equation of the planes parallel to the plane x + 2y+ 2z + 8 =0 which are at the distance of 2 units from the point (1, 1, 2) **[4]**

Solution:

The equation of the plane parallel to the plane $x + 2y + 2z + 8 = 0$ is $x + 2y + 2z + \lambda = 0.$

Now the distance of this plane from the point (1, 1, 2)

$$= \left|\frac{1(1) + 2(1) + 2(2) + \lambda}{\sqrt{1^2 + 2^2 + 3^2}}\right|$$

$$= \left|\frac{1 + 2 + 4 + \lambda}{\sqrt{9}}\right|$$

$$= \left|\frac{7 + \lambda}{\sqrt{9}}\right|$$

Given that

$$\left|\frac{7+\lambda}{3}\right|=2$$

$$\frac{7+\lambda}{3}=\pm 2$$

$\lambda = \pm 6 - 7$

$$\lambda=\pm 6-7$$

$$\lambda=-1 \text{ or } \lambda=-13$$

Hence eq. of plane x + 2y + 2z -1 = 0 or x + 2y + 2z - 13 = 0

Question 4.1 | Select and write the correct answer from the given alternatives in each of the following: [6]

Question 4.1.1: [2]

Function $f(x)=x^2-3x+4$ has minimum value at

(A) 0

(B)-3/2

(C) 1

(D)3/2

Solution: (d) 3/2

Question 4.1.2: [2]

$$\int \frac{1}{x}\log x dx = \ldots\ldots\ldots\ldots\ldots$$

(A)log(log x)+ c

(B) 1/2 $(\log x)^2$+c

(C) 2log x + c

(D) log x + c

Solution:

(B) $\frac{1}{2}(\log x)^2+c$

Question 4.1.3: [2]

Order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}}=7\frac{d^2y}{dx^2}$ are respectively

(A) 2, 3

(B) 3, 2

(C) 7, 2

(D) 3, 7

Solution:

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}} = 7\frac{d^2y}{dx^2}$$

Cubing on both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^7 = 7\left(\frac{d^2y}{dx^2}\right)^3$$

By definition of degree and order Degree: 3 ; Order: 2

Question 4.2: Attempt any THREE of the following: [6]

Question 4.2.1: If x=at^2, y= 2at, then find dy/dx. **[2]**

Solution:

We have, $y = 2at$

$$\frac{dy}{dt} = 2a\frac{d}{dt}(t) = 2a(1) = 2a$$

also $x = at^2$

$$\frac{dx}{dx} = a\frac{d}{dt}(t^2) = a(2t) = 2at$$

now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

Question 4.2.2: Find the approximate value of $\sqrt{8.95}$ **[2]**

Solution:

$Let f(x) = \sqrt{x}$

$f(a+h) = \sqrt{a+h}$

we choose a=9 and h=-0.05

Then $\sqrt{8.95} = f(a+h)$

But , $f(a+h) \approx f(a) + hf\prime(a)$

$\sqrt{8.95} \approx f(a) + hf\prime(a)..........(1)$

Now $f(a) = \sqrt{a} = \sqrt{9}$ and $h = -0.05$

we have $f(X) = \sqrt{x}$

we have $f(X)=\sqrt{x}$

$$f\prime(x)=\frac{1}{2\sqrt{x}}$$

$$f\prime(a)=f\prime(9)=\frac{1}{2\sqrt{9}}=\frac{1}{2\times 3}=\frac{1}{6}$$

from(1) $\sqrt{8.95}\approx 3+(-0.05)\left(\frac{1}{6}\right)$

=3-0.0083=2.9917

$$\Rightarrow \sqrt{8.95}\approx 2.9917$$

Question 4.2.3: Find the area of the region bounded by the parabola $y^2 = 16x$ and the line x = 3. **[2]**

Solution:

$$y^2=16x$$

$$area=2\int_0^3 ydx$$

$$=2\int_0^{34}\sqrt{x}dx$$

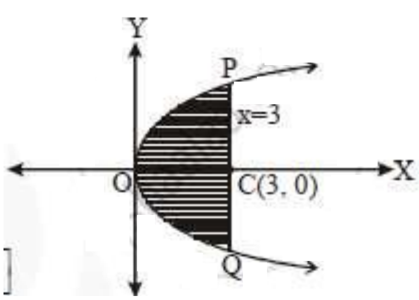
$$=8\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^3$$

$$=\frac{16}{3}\left[3^{\frac{3}{2}}\right]$$

$$=\frac{16}{3}\left[3\times 3^{\frac{1}{2}}\right]$$

$$=16\left[3^{\frac{1}{2}}\right]$$

$$=16\sqrt{3}sq.\,units$$



Question 4.2.4: For the bivariate data r = 0.3, cov(X, Y) = 18, $\sigma_x = 3$, find σ_y . **[2]**

Solution:

r= 0.3, cov(X,Y) =18, $\sigma_x = 3, \sigma_y = ?$

$$r = cov\frac{x, y}{\sigma_x . \sigma_y}$$

$$0.3 = \frac{18}{3 . \sigma_y}$$

$$\sigma_y = \frac{18}{0.3 \times 3}$$

$$\sigma_y = \frac{180}{9}$$

$$\sigma_y = 20$$

Question 4.2.5: Triangle bounded by the lines y = 0, y = x and x = 4 is revolved about the X-axis. Find the volume of the solid of revolution. [2]

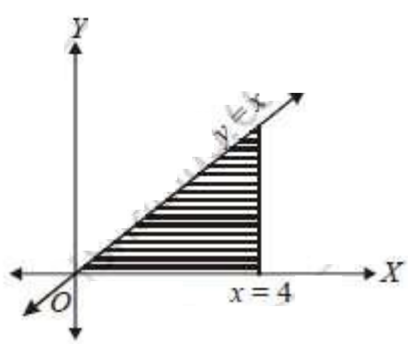
Solution: Given line is y = x

$$V = \pi \int_0^4 y^2 dx$$

$$= \pi \int_0^4 x^2 dx$$

$$= \frac{\pi}{3} [x^3]_0^4$$

$$= \frac{64\pi}{3} cubicunits.$$



Question 5.1 | Attempt any Two of the following: [6]

Question 5.1.1: A function f (x) is defined as [3]
f (x) = x + a, x < 0
= x, 0 ≤x ≤ 1
= b- x, x ≥1
is continuous in its domain.
Find a + b.

Solution: f (x) is continuous in its domain.

f (x) is continuous at x = 0 & x = 1
Since f(x) is continuous at x = 0

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\lim_{x\to 0} (x+a) = \lim_{x\to 0} x = 0$$

$$0 + a = 0$$

$$a = 0$$

Also f (x) is continuous at x = 1

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\lim_{x\to 1} (x+a) = \lim_{x\to 1} (b-x) = b-1$$

$$1 = b - 1$$

$$b = 2$$

$$a + b = 2$$

Question 5.1.2: [3]

If $x = a\left(t - \frac{1}{t}\right), y = a\left(t + \frac{1}{t}\right)$, then show that $\frac{dy}{dx} = \frac{x}{y}$

Solution:

$$x = a\left(t - \frac{1}{t}\right), y = a\left(t + \frac{1}{t}\right)$$

$$\frac{x}{a} = t - \frac{1}{t} \text{ and } \frac{y}{a} = t + \frac{1}{t}$$

we have

$$\left(t + \frac{1}{t}\right)^2 = \left(t - \frac{1}{t}\right)^2 + 4$$

$$\left(\frac{y}{a}\right)^2 = \left(\frac{x}{a}\right)^2 + 4$$

$$\frac{y^2}{a^2} - \frac{x^2}{a^2} = 4$$

$$y^2 - x^2 = 4a^2$$

Differentiating w.r.t. x

$$2y\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = 2\frac{x}{2}y$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Question 5.1.3: [3]

Evaluate : $\int \frac{1}{3+5\cos x}dx$

Solution:

Let $I = \int \frac{1}{3+5\cos x}dx \quad \text{put} \quad \tan\left(\frac{x}{2}\right) = t$

then $dx = \frac{2}{1+t^2}dt \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{2\frac{dt}{1+t^2}}{3+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{\frac{dt}{1+t^2}}{\frac{3(1+t^2)+5(1-t^2)}{1+t^2}}$$

$$= 2\int \frac{dt}{3+3t^2+5-5t^2}$$

$$= 2\int \frac{dt}{8-2t^2}$$

$$= \int \frac{dt}{2^2-t^2}$$

$$= \frac{1}{2(2)}\log\left|\frac{2+t}{2-t}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{2+\tan\left(\frac{x}{2}\right)}{2-\tan\left(\frac{x}{2}\right)}\right| + c$$

Question 5.2: Attempt any TWO of the following: [8]

Question 5.2.1: An insurance agent insures lives of 5 men, all of the same age and in good health. The probability that a man of this age will survive the next 30 years is known to be 2/3. Find the probability that in the next 30 years at most 3 men will survive. [4]

Solution:

$$p=\frac{2}{3}, q=1-\frac{2}{3}=\frac{1}{3}, n=5$$

$$P(x \leq 3)=1-P(x>3)$$

$$=1-[P(4)-+P(5)]$$

$$=1-\left[{}^{5}C_{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{1}+{}^{5}C_{5}\left(\frac{2}{3}\right)^{5}\right]$$

$$=1-\left[5 \times \frac{16}{81} \times \frac{1}{3}+1 \times \frac{32}{343}\right]$$

$$=1-\frac{16}{243} \times 7$$

$$=1-\frac{112}{243}$$

$$=\frac{243-112}{243}$$

$$=\frac{131}{243}$$

$$=0.539$$

Question 5.2.2: The surface area of a spherical balloon is increasing at the rate of $2cm^2$ / sec. At what rate is the volume of the balloon is increasing when the radius of the balloon is 6 cm? **[4]**

Solution: Let r be the radius, A be the surface area and V be the volume of the spherical balloon at time t seconds.

$$\text{Then}, A=4\pi r2$$

$$d\frac{A}{dt}=4\pi\left(2r\left(\frac{dr}{dt}\right)\right)$$

$$Given$$

$$2=8\pi rd\frac{r}{dt}$$

$$rd\frac{r}{dt}=\frac{2}{8}\pi=\frac{1}{4}\pi \ldots\ldots\ldots\ldots (1)$$

$$Now, V=\frac{4}{3}\pi r^{3}$$

$$\frac{dV}{dt}=4\frac{\pi}{3}\times 3r^2\frac{dr}{dt}=(4\pi r)\left(r\frac{dr}{dt}\right)$$

Given : $r=6,(d)\frac{V}{dt}=?$

$$\therefore\frac{dV}{dt}=4\pi(6)\left(\frac{1}{4\pi}\right)=6..........om(1)$$

Therefore, the volume of the balloon is increasing at the rate of 6cm^3/sec.

Question 5.2.3: The slope of the tangent to the curve at any point is equal to y+ 2x. Find the equation of the curve passing through the origin. **[4]**

Solution: Since dy/dx represents the slope of a tangent at any point (x, y) on a given curve, according to given condition

$$\frac{dy}{dx}=y+2x$$

$\frac{dy}{dx}-y=2x$ which is in the form of linear differential equation $\frac{dy}{dx}+Py=Q$

where P=-1 ,Q=2x

$$I.f=e^{\int Pdx}=e^{\int -1dx}=e^{-x}$$

The solution is given by

$$y.e^{-x}=\int Q(IF)dx+C\prime$$

$$y.e^{-x}=2\int x.e^{-x}dx+c$$

Consider $\int x.e^{-x}dx$,Integrating by part

$$=x\frac{e^{-x}}{-1}-\int\left[\frac{d}{dx}x\int e^{-x}dx\right]dx$$

$$=-xe^{-x}-\int\frac{e^{-x}}{-1}dx$$

$$=-xe^{-x}-e^{-x}$$

$$ye^{-x}=2(-xe^{-x}-e^{-x})+c$$

$$y=-2x-2+ce^{x}$$

$2x + \ y + 2 = ce^x$ is the general solution. Since the curve is passing through origin, x = 0, y = 0

$$0 + \ 0 + 2 = c.e^0 \Rightarrow c = 2$$

$$2x + y + \ 2 = 2e^x$$

is the equation of the required curve.

Question 6.1: Attempt any TWO of the following : **[6]**

Question 6.1.1: **[3]**

If u and v are two functions of x then prove that

$$\int uvdx = u\int vdx - \int\left[d\frac{u}{dx}\int vdx\right]dx$$

Hence evaluate, $\int xe^x dx$

Solution:

Let $\int vdx = w.....(1)$

$$then \quad \frac{dw}{dx} = v.....(2)$$

$$Now\frac{d}{dx}(u,w) = u.\frac{d}{dx}(w) + w\frac{d}{dx}(u)$$

$$= u.v + w\frac{du}{dx}.......\text{from}(2)$$

By definition of integration.

$$u.w = \int\left[u.v + w\frac{du}{dx}\right]dx$$

$$= \int u.vdx + \int w.\frac{du}{dx}dx$$

$$\int u.vdx = u.w - \int w\frac{du}{dx}dx$$

$$= u\int vdx - \int\left[\frac{du}{dx}\int v.dx\right]dx$$

[next section only required for question 2]

Hence, $\int xe^x dx = x.\int e^x dx - \int\left[\frac{d}{dx}x.\int e^x dx\right]dx$

$$= xe^x - \int 1 \times e^x dx$$

$$= xe^x - e^x + c$$

Question 6.1.2: The time (in minutes) for a lab assistant to prepare the equipment for a certain experiment is a random variable X taking values between 25 and 35 minutes with p.d.f **[3]**

$$f(x) = \frac{1}{10}, 25 \leq x \leq 35 = 0$$
$$= 0 \text{ otherwise}$$

What is the probability that preparation time exceeds 33 minutes? Also find the c.d.f. of X.

Solution:

$$P(x \geq 33) = \int_{33}^{35} \frac{1}{10} dx$$

$$= \int_{33}^{35} \frac{1}{10} \mathrm{dx}$$

$$= \frac{35}{10} - \frac{33}{10}$$
$$= \frac{2}{10}$$
$$= \frac{1}{5}$$

probability that preparation time exceeds 33 minutes $= \frac{1}{5}$

$$F(x) - \int_{25}^{x} \frac{1}{10} dx$$

$$= \int_{25}^{x} \frac{x}{10} \mathrm{dx}$$

$$= \frac{x - 25}{10}$$

Question 6.1.3: The probability that a certain kind of component will survive a check test is 0.6. Find the probability that exactly 2 of the next 4 tested components survive **[3]**

Solution:

$$p = 0.6, q = 1 - p = 1 - 0.6 = 0.4$$

$$n = 4, r = 2$$

$$p(X = r) = {}^{n}C_{r} p^{r} q^{n-r}$$

$p(X=2) = {}^4C_2(0.6)^2(0.4)^2$

$= \frac{4\times 3}{1\times 2}(0.36)(0.16)$

$= 0.3456$

Question 6.2 | Attempt any TWO of the following: [8]

Question 6.2.1: [4]

If $ax^2 + 2hxy + by^2 = 0$, show that $\frac{d^2y}{dx^2} = 0$

Solution:

$ax^2 + 2hxy + by^2 = 0 ....(1)$

Differentiate w.r.t. x

$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$

$\therefore ax + hx\frac{dy}{dx} + by\frac{dy}{dx} + hy = 0$

$(hx + by)b\frac{y}{dx} = -1(ax + hy)$

$\therefore \frac{dy}{dx} = -\frac{ax + hy}{hx + by} \text{........................} (2)$

From (1),we have

$ax^2 + hxy + hxy + by^2 = 0$

$x(ax + hy) + y(hx + by) = 0$

$x(ax + hy) = -y(hx + by)$

$-\frac{ax + hy}{hx + by} = \frac{y}{x}$

Put in (2),

$\frac{dy}{dx} = \frac{y}{x}$

Differentiate w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y}{x^2}$$

$$= \frac{x\left(\frac{y}{x}\right) - y}{x^2}$$

$$= \frac{0}{x^2}$$

$$= 0$$

Question 6.2.2: Find the area of the region common to the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$ **[4]**

Solution:

$x^2 + y^2 = 9$ and $y^2 = 8x$

$x^2 + 8x = 9$

$x^2 + 8x - 9 = 0$

$\therefore (x+9)(x-1) = 0$

$\therefore x = 1$ or $x = -9$

$\therefore y = \pm 2\sqrt{2}$

$\therefore$ The points of intersections are

$P\left(1, 2\sqrt{2}\right)$ and $Q\left(1, -2\sqrt{2}\right)$

$y^2 = 8x$

$\therefore\ y = \sqrt{8}\sqrt{x} = 2\sqrt{2}x^{\frac{1}{2}} \to f_1(x)$

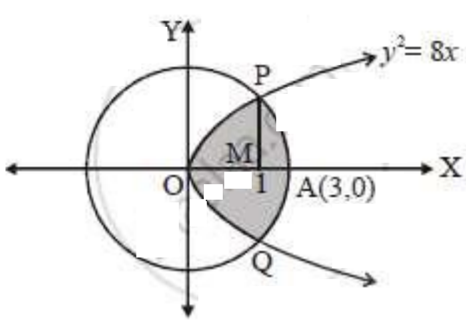
and $x^2 + y^2 = 9 \therefore y^2 = 9 - x^2$

$\therefore y = \sqrt{9 - x^2} \to f_2(x)$

Required area,

= Area OPAQO = 2 Area OPAMO

= 2(Area OPMO + Area APMA)



$$= 2\left[\int_0^1 f_1(x)dx + \int_1^3 f_2(x)dx\right]$$

$$= 2\left[\int_0^{12} \sqrt{2}x^{\frac{1}{2}}dx + \int_1^3 \sqrt{9-x^2}dx\right]$$

$$= 2\left[2\sqrt{2}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^1 + \left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)_1^3\right)\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \left(\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1) - \frac{1}{2}\sqrt{8} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \frac{9}{2}.\frac{x}{2} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= 2\left[\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right) sq.\, units\right.$$

Question 6.2.3: For 10 pairs of observations on two variables X and Y, the following data are available: **[4]**

$$\sum(x-2) = 30, \sum(y-5) = 40, \sum(x-2)^2 = 900, \sum(y-5)^2 = 800, \sum(x-2)(y-5) = 480$$

Find the correlation coefficient between X and Y.

Solution:

$$\sum(x-2) = 30, \sum(y-5) = 40, \sum(x-2)^2 = 900, \sum(y-5)^2 = 800, \sum(x-2)(y-5) = 480$$

Let x – 2 = u and y – 5 = v

$$\therefore \sum u = 30, \sum v = 40, \sum u^2 = 900, \sum v^2 = 800, \sum uv = 480, m = 10$$

The correlation coefficient between X and Y.

$$= r_{xy} = r_{uv} = \frac{m\sum uv - \sum u \sum v}{\sqrt{m\sum u^2 - 1(\sum u)^2}\sqrt{m\sum v^2 - (\sum v)^2}}$$

$$= \frac{10 \times 480 - 1200}{\sqrt{10 \times 900 - 900} \times \sqrt{10 \times 800 - 1600}}$$

$$= 0.5$$