4. Pair of Straight Lines

Pair of straight Lines

Equation of a pair of straight lines

The general form of equation of second degree is:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Quadratic form of equation of a pair of straight lines

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can also be written as:

$$by^2 + 2(hx + f)y + ax^2 + 2gx + c = 0$$

By quadratic formula, we get:

$$y = \frac{-2(hx+f) \pm 2\sqrt{(hx+f)^2 - b(ax^2 + 2gx + c)}}{2b}$$
$$\Rightarrow y = \frac{-(hx+f) \pm \sqrt{(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc}}{b}$$

The nature of lines can be determined w.r.t. the value of $h^2 - ab$ as follows:

Case	**Nature of lines**
$h^2 - ab > 0$	Pair of non-parallel lines
$h^2 - ab = 0$ (i) $f^2 - bc > 0$ (ii) $f^2 - bc = 0$ (iii) $f^2 - bc < 0$	Parallel non-coincident lines Coincident lines Imaginary lines
$h^2 - ab < 0$	Pair of imaginary lines

General form of a pair of straight lines passing through the origin

The homogeneous equation of a pair of straight lines passing through the origin is

$ax^2 + 2hxy + by^2 = 0$.

Angle between the lines

Let $y - m_1x = 0$ and $y - m_2x = 0$ be the lines represented by the general equation $ax^2 + 2hxy + by^2 = 0$.

Then the angle θ between these lines is given as follows:

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

Points to be remembered

Equation of the pair of straight lines passing through the origin and the perpendicular to pair of lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Area of the triangle formed by the pair of lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|$.

Point of intersection of the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

Distance between $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$ and the origin is $\sqrt{\frac{g^2 + f^2 - c(a + b)}{h^2 - ab}}$.

Equation of the pair of lines passing through (x_1, y_1) and parallel to the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$.

Equation of the pair of lines passing through (x_1, y_1) and perpendicular to the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$.

Equation of the pair of lines passing through the origin and at a distance of d units from (x_1, y_1) is $d^2(x^2 + y^2) = (y_1x - x_1y)^2$.

If d_1 and d_2 are the perpendicular distances from (x_1, y_1) to the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $d_1d_2 = \frac{|ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c|}{\sqrt{(a - b)^2 + 4h^2}}$

If d_1 and d_2 are the perpendicular distances from (x_1, y_1) to the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$, then $d_1d_2 = \frac{|ax_1^2 + 2hx_1y_1 + by_1^2|}{\sqrt{(a - b)^2 + 4h^2}}$.

If the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are equidistant from the origin, then $f^4 - g^4 = c(bf^2 - ag^2)$.

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ bisect the angle between the co-ordinate axes, then $(a + b)^2 = 4h^2$.