Mathematics & Statistics

Academic Year: 2012-2013 Marks: 80
Date: October 2012

Question 1: [12]

Question 1: Select and write the correct answer from the given alternatives in each of the following [6]

Question 1.1.1: If A = {2, 3, 4, 5, 6}, then which of the following is not true? [2]
(A) $\exists\ x \in A$ such that $x + 3 = 8$
(B) $\exists\ x \in A$ such that $x + 2 < 5$
(C) $\exists\ x \in A$ such that $x + 2 < 9$
(D) $\forall\ x \in A$ such that $x + 6 \geq 9$

Solution: Since, $x = 2 \in A$ does not satisfy $x + 6 \geq 9$.

∴ Option (D) is not true

Question 1.1.2: If $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, then the value of k is [2]

$\frac{1}{2}$
$\frac{11}{2}$
$\frac{5}{2}$
$\frac{-11}{2}$

Solution: Auxiliary equation of the given equation is $2m^2 + km + 3 = 0$.

Slope of the line $2x + y = 0$ is $m = -2$.

∴ $m = -2$ is a root of the auxiliary equation $2m^2 + km + 3 = 0$

∴ $2(-2)^2 - 2k + 3 = 0$

∴ $8 - 2k + 3 = 0$

∴ $K = 11/2$

Question 1.1.3: If a line is inclined at 60° and 30° with the X and Y-axes respectively, then the angle which it makes with Z-axis is [2]

(A) 0
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$
(D) $\frac{\pi}{6}$

Solution: Let α, β, γ be the angles made by a line with X, Y, Z axes respectively

∴ α = 60°, β = 30°

Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

∴ $\cos^2 60° + \cos^2 30° + \cos^2 \gamma = 1$

$\therefore \frac{1}{4} + \frac{3}{4} + \cos^2\gamma = 1$

$\therefore \gamma = \frac{\pi}{2}$

Question 1.2 | Attempt any THREE of the following [6]

Question 1.2.1: [2]

If A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and AX = I then find X by using elementary transformations

Solution: We have, AX = I

$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Applying $R_2 \to \left(-\frac{1}{2}\right) R_2$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Applying $R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$\therefore X = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Question 1.2.2: With usual notations, in ΔABC, prove that a(b cos C − c cos B) = $b^2 - c^2$ **[2]**

Solution: L.H.S. = a(b cos C − c cos B)

$$= a\left\{b\left(\frac{a^2+b^2-c^2}{2ab}\right)-c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right\} \quad \text{....[By cosine rule]}$$

$$= \frac{\left(a^2+b^2-c^2\right)-\left(a^2+c^2-b^2\right)}{2}$$

$$= \frac{1}{2}(2b^2-2c^2)$$

$$= b^2-c^2$$

= R.H.S.

Question 1.2.3: Show that the equation of a tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x_1,y_1)$ on it is $xx_1 + yy_1 = a^2$ **[2]**

Solution: Given equation of circle is $x^2 + y^2 = a^2$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Slope of the tangent at $P(x_1, y_1)$

$$= \left(\frac{dx}{dy}\right)_{x_1\ y_1} = -\frac{x_1}{y_1}$$

Equation of a tangent to the circle at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$\therefore \quad (y - y_1)y_1 = -x_1(x - x_1)$

$\therefore \quad yy_1 - y_1^2 = -xx_1 + x_1^2$

$\therefore \quad xx_1 + yy_1 = x_1^2 + y_1^2$

As $P(x_1, y_1)$ lies on the circle,

$x_1^2 + y_1^2 = a^2$

$\therefore$ equation of the tangent at $P(x_1, y_1)$ is $xx_1 + yy_1 = a^2$.

Question 1.2.4: Find k, if the line 2x − 3y + k = 0 touches the ellipse $5x^2 + 9y^2 = 45$. **[2]**

Solution: Equation of the ellipse is $5x^2 + 9y^2 = 45$.

i.e $\frac{x^2}{9} + \frac{y^2}{5} = 1$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9$, $b^2 = 5$

The equation of tangent is $2x - 3y + k = 0$.

$\therefore \quad y = \frac{2}{3}x + \frac{k}{3}$

Comparing this equation with $y = mx + c$, we get

$m = \frac{2}{3}$, $c = \frac{k}{3}$

Using condition of tangency,

$c^2 = a^2m^2 + b^2$

$\therefore \quad \left(\frac{k}{3}\right)^2 = 9\left(\frac{2}{3}\right)^2 + 5$

$\therefore \quad \frac{k^2}{9} = 4 + 5$

$\therefore k^2 = 81$

$\therefore k = \pm 9$

Question 2.1 | Attempt any TWO of the following: [6]

Question 2.1.1: Using truth table, prove that $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$ [3]

Solution:

1	2	3	4	5	6
p	q	~p	~p ∧ q	p ∨ q	(p∨q) ∧~p
T	T	F	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	F	F	F

The entries in columns 4 and 6 are identical

$\therefore \sim p \wedge q \equiv (p \vee q) \wedge \sim p$

Question 2.1.2: Find the values of p and q, if the following equation represents a pair of perpendicular lines: $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$. [3]

Solution: Given equation is $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$

Comparing with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = p, h = -4, b = 3, g = 7, f = 1, c = q$.

The given equation represents a pair of lines perpendicular to each other

$\therefore a + b = 0$

$\therefore p + 3 = 0$

$\therefore p = -3$

Also, the given equation represents a pair of lines

$$\therefore \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & q \end{vmatrix} = 0$$

$\therefore -3(3q - 1) + 4(-4q - 7) + 7(-4 - 21) = 0$

$\therefore -9q + 3 - 16q - 28 - 175 = 0$

$\therefore -25q - 200 = 0$

$\therefore -25q = 200$

$\therefore q = -8$ [1]x

$\therefore p = -3$ and $q = -8$

Question 2.1.3: Find the equations of tangents to the parabola $y^2 = 12x$ from the point (2, 5). **[3]**

Solution: Equation of the parabola is $y^2 = 12x$

$\therefore 4a = 12$

$\therefore a = 3$

The equation of the tangent to the parabola with slope m is

$$y = mx + \frac{a}{m}$$

$$\therefore y = mx + \frac{3}{m}$$

If this tangent passes through the point (2, 5), then

$$5 = 2m + \frac{3}{m}$$

$\therefore 5m = 2m^2 + 3$

$\therefore 2m^2 - 5m + 3 = 0$

$\therefore 2m^2 - 2m - 3m + 3 = 0$

$\therefore 2m(m - 1) - 3(m - 1) = 0$

$\therefore (m - 1)(2m - 3) = 0$

$\therefore m = 1$ or $m = 3/2$

$\therefore m_1 = 1$ and $m_2 = 3/2$ are the slopes of the required tangents

$\therefore$ the equations of the tangents are

$y - 5 = 1(x - 2)$ and $y - 5 = 3/2\ (x - 2)$

$\therefore y - 5 = x - 2$ and $2y - 10 = 3x - 6$

$\therefore x - y + 3 = 0$ and $3x - 2y + 4 = 0$

Question 2.2 | Attempt any TWO of the following: **[8]**

Question 2.2.1: The cost of 2 books, 6 notebooks and 3 pens is Rs 40. The cost of 3 books, 4 notebooks and 2 pens is Rs 35, while the cost of 5 books, 7 notebooks and 4 pens is Rs 61. Using this information and matrix method, find the cost of 1 book, 1 notebook and 1 pen separately. **[4]**

Solution: Let the cost of 1 book, 1 notebook and 1 pen be Rs x, Rs y and Rs z respectively.

According to the given conditions,

$2x + 6y + 3z = 40$

$3x + 4y + 2z = 35$

$5x + 7y + 4z = 61$

These equations can be written in the matrix form as

$$\begin{bmatrix} 2 & 6 & 3 \\ 3 & 4 & 2 \\ 5 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 35 \\ 61 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$,

$$\begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 2 \\ 5 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 35 \\ 61 \end{bmatrix}$$

Applying $R_1 \to (-1)R_1$,

$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & 4 & 2 \\ 5 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 35 \\ 61 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 5R_1$,

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 10 & 5 \\ 0 & 17 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 50 \\ 86 \end{bmatrix}$$

Applying $R_2 \to \left(\frac{1}{10}\right)R_2$,

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 17 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 86 \end{bmatrix}$$

Applying $R_3 \to R_3 - 17R_2$,

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 1 \end{bmatrix}$$

By equality of matrices,

$x - 2y - z = -5$ ….(i)

$y + z/2 = 5$ ….(ii)

$z/2 = 1$ ….(iii)

From (iii), $z = 2$

Putting z = 2 in (ii), we get

y + 1 = 5

∴ y = 4

Putting y = 4, z = 2 in (i), we get

x − 8 − 2 = − 5

∴ x = 5

Thus, the cost of 1 book, 1 notebook and 1 pen are 5, 4 and 2 respectively

Question 2.2.2: [4]

Prove that $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let $\sin^{-1}\left(-\frac{1}{2}\right)$ = x

$$\therefore \quad \sin x=-\frac{1}{2}=-\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right)$$

The principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $-\frac{\pi}{2}\leq-\frac{\pi}{6}\leq\frac{\pi}{2}$.

$$\therefore \quad x=-\frac{\pi}{6}$$

$$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6} \qquad ....(i)$$

Let $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=y$

$$\therefore \quad \cos y=-\frac{\sqrt{3}}{2}=-\cos\frac{\pi}{6}$$

$$=\cos\left(\pi-\frac{\pi}{6}\right)$$

$$=\cos\frac{5\pi}{6}$$

The principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $0\leq\frac{5\pi}{6}\leq\pi$.

$\therefore \quad y = \frac{5\pi}{6}$

$\therefore \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ ....(ii)

Let $\cos^{-1}\left(-\frac{1}{2}\right) = z$

$\therefore \quad \cos z = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

The principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $0 \le \frac{2\pi}{3} \le \pi$.

$\therefore \quad z = \frac{2\pi}{3}$

$\therefore \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ ....(iii)

From (i) and (ii), we get

$$\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} + \frac{5\pi}{6} = \frac{2\pi}{3}$$

$$= \cos^{-1}\left(-\frac{1}{2}\right) \quad \text{....[From (iii)]}$$

Question 2.2.3: Show that the product of lengths of perpendicular segments drawn from the foci to any tangent to the hyperbola $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is equal to 16. **[4]**

Solution:

Equation of the hyperbola is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Here, $a^2 = 25$, $b^2 = 16$

$\therefore$ a = 5, b = 4

$\therefore$ e = $\frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{41}}{5}$

$\therefore$ ae = $5\left(\frac{\sqrt{41}}{5}\right) = \sqrt{41}$

$\therefore$ Foci are S(ae, 0) ≡ S($\sqrt{41}$, 0) and

and S'(−ae, 0) ≡ S'(− $\sqrt{41}$, 0)

Equation of tangent to the hyperbola with slope m is

y = mx + $\sqrt{a^2m^2 - b^2}$

$\therefore$ y = mx + $\sqrt{25m^2 - 16} = 0$ ....(i)

p_1 = length of perpendicular segment from the focus S($\sqrt{41}$, 0) to the tangent (i)

$$= \left|\frac{m(\sqrt{41}) + (-1)(0) + \sqrt{25m^2 - 16}}{\sqrt{m^2 + 1}}\right|$$

$$\therefore \quad p_1 = \left|\frac{\sqrt{25m^2 - 16} + \sqrt{41}m}{\sqrt{m^2 + 1}}\right|$$

p_2 = length of perpendicular segment from the focus S' ($-\sqrt{41}$, 0) to the tangent (i)

$$= \left|\frac{m(-\sqrt{41}) + (-1)(0) + \sqrt{25m^2 - 16}}{\sqrt{m^2 + 1}}\right|$$

$$\therefore \quad p_2 = \left|\frac{\sqrt{25m^2 - 16} - \sqrt{41}m}{\sqrt{m^2 + 1}}\right|$$

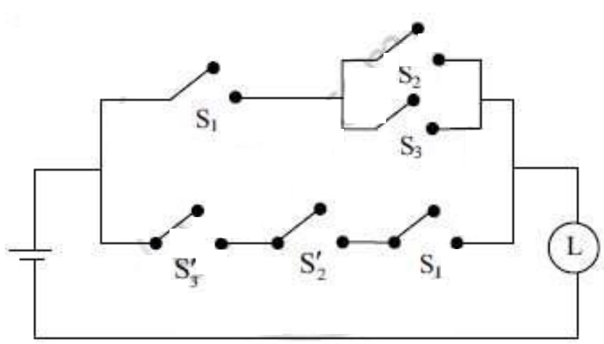
$$\therefore \quad p_1p_2 = \left|\frac{\sqrt{25m^2 - 16} + \sqrt{41}m}{\sqrt{m^2 + 1}}\right| \times \left|\frac{\sqrt{25m^2 - 16} - \sqrt{41}m}{\sqrt{m^2 + 1}}\right|$$

$$= \left|\frac{25m^2 - 16 - 41m^2}{m^2 + 1}\right| = \left|\frac{-16m^2 - 16}{m^2 + 1}\right|$$

$$= \left|\frac{-16(m^2 + 1)}{m^2 + 1}\right| = |-16| = 16$$

Question 3.1 | Attempt any TWO of the following: [6]

Question 3.1.1: Construct the new switching circuit for the following circuit with only one switch by simplifying the given circuit: [3]



Solution: Let p : The switch S_1 is closed.

q : The switch S_2 is closed.

r : The switch S_3 is closed.

~p: The switch 1 S' is closed or the switch S_1 is open.

~q: The switch 2 S' is closed or the switch S_2 is open.

~r : The switch 3 S' is closed or the switch S_3 is open

The logical expression corresponding to the given circuit is

$[p \wedge (q \vee r)] \vee [\sim r \wedge \sim q \wedge p]$

$\equiv [p \wedge (q \vee r)] \vee [\sim(r \vee q) \wedge p]$ ….(De-Morgan's law)

$\equiv [p \wedge (q \vee r)] \vee [\sim(q \vee r) \wedge p]$ ….(Commutative law)

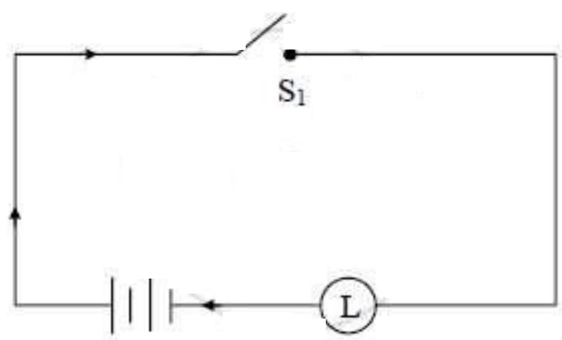
$\equiv [p \wedge (q \vee r)] \vee [p \wedge \sim(q \vee r)]$ ….(Commutative law)

$\equiv p \wedge [(q \vee r) \vee \sim(q \vee r)]$ ….(Distributive law)

$\equiv p \wedge T$ ….(Complement law)

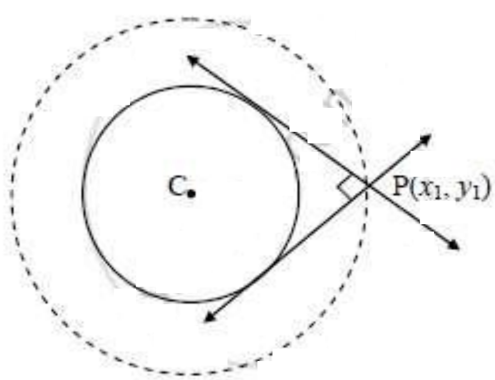
$\equiv p$ ….(Identity law)

The simplified circuit is as



Question 3.1.2: Find the locus of a point, the tangents from which to the circle $x^2 + y^2 = a^2$ are mutually perpendicular **[3]**

Solution:



Let P(x1, y1) be any point on the locus.

Equation of a tangent with slope 'm' to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1 + m^2}$

This tangent passes through $P(x_1, y_1)$

$\therefore y_1 = mx_1 \pm a\sqrt{1 + m^2}$

$\therefore (y_1 - mx_1)^2 = a^2(1 + m^2)$

$\therefore \left(x_1^2 - a^2\right)m^2 - 2x_1y_1m + \left(y_1^2 - a^2\right) = 0$ ....(i)

This is a quadratic equation in 'm'.

Let m1 and m2 be slopes of two tangents drawn from P(x1, y1) to the circle.

Thus, it has two roots say m1 and m2, which are the slopes of tangents drawn from P.

$$\therefore m_1 . m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$$

$$\therefore y_1^2 - a^2 = -x_1^2 + a^2$$

$$\therefore x_1^2 + y_1^2 = 2a^2$$

$\therefore$ the equation of the locus of $P(x_1, y_1)$ is $x^2 + y^2 = 2a^2$.

Question 3.1.3: Find the shortest distance between the lines `[3]`

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Solution: Shortest distance between the lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ is}$$

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

Equations of the given lines are

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Here,

$x_1 = -1, y_1 = -1, z_1 = -1, x_2 = 3, y_2 = 5, z_2 = 7,$

$a_1 = 7, b_1 = -6, c_1 = 1, a_2 = 1, b_2 = -2, c_2 = 1$

Now, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

= 4(−6 + 2) − 6(7 − 1) + 8(−14 + 6)

= − 16 − 36 − 64

= − 116

and $(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$

= (−6 + 2)2 + (1 − 7)2 + (−14 + 6)2

= 16 + 36 + 64

= 116

∴ shortest distance between the given lines

$\left|\frac{-116}{\sqrt{116}}\right|$

$\sqrt{116}$

= $2\sqrt{29}$ units

Question 3.2 | Attempt any TWO of the following: [8]

Question 3.2.1: Find the angle between the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z+2}{4}$ and the plane 2x + y − 3z + 4 = 0. [4]

Solution: The angle θ between the line

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane ax + by + cz + d = 0 is given by

sinθ = $\frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2 + b^2 + c^2}. \sqrt{a_1^2 + b_1^2 + c_1^2}}$

Here , $a_1 = 3, b_1 = 2, c_1 = 4$ and $a = 2, b = 1, c = -3$

$\therefore aa_1 + bb_1 + cc_1 = 2(3) + 1(2) + (-3)(4)$

= 6 + 2 - 12 = -4

$\sqrt{a^2+b^2+c^2} = \sqrt{2^2+1^2+(-3)^2} = \sqrt{4+1+9} = \sqrt{14}$

and $\sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{3^2+2^2+4^2} = \sqrt{9+4+16} = \sqrt{29}$

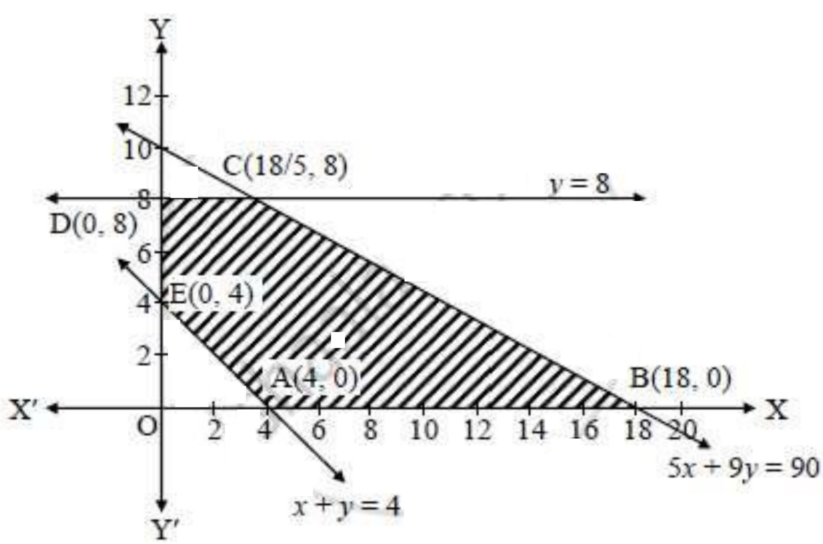
$\therefore \sin\theta = \dfrac{-4}{\sqrt{14}.\sqrt{29}} = \dfrac{-4}{\sqrt{406}}$

$\therefore \theta = \sin^{-1}\left(\dfrac{-4}{\sqrt{406}}\right)$

Question 3.2.2: Solve the following L. P. P. graphically: Linear Programming
Minimize Z = 6x + 2y **[4]**
Subject to
5x + 9y ≤ 90
x + y ≥ 4
y ≤ 8
x ≥ 0, y ≥ 0

Solution: To draw the feasible region, construct table as follows:

Inequality	5x + 9y ≤ 90	x + y ≥ 4	y ≤ 8
Corresponding equation (of line)	5x + 9y = 90	x + y = 4	y = 8
Intersection of line with X-axis	(18, 0)	(4, 0)	–
Intersection of line with Y-axis	(0, 10)	(0, 4)	(0, 8)
Region	Origin side	Non-origin side	Origin side



Shaded portion ABCDE is the feasible region, whose vertices are A(4, 0), B(18, 0), C, D(0, 8) and E(0, 4).

C is the point of intersection of the lines y = 8 and 5x + 9y = 90.

Putting y = 8 in 5x + 9y = 90, we get

5x + 72 = 90

∴ x = 18/5

∴ C = $\left(\frac{18}{5}, 8\right)$

Here, the objective function is Z = 6x + 2y,

Z at A(4, 0) = 6(4) + 2(0) = 24

Z at B(18, 0) = 6(18) + 2(0) = 108

Z at $C\left(\frac{18}{5}, 8\right) = 6\left(\frac{18}{5}\right)$ + 2(8)

= 188/5 = 37.6

Z at D(0, 8) = 6(0) + 2(8) = 16

Z at E(0, 4) = 6(0) + 2(4) = 8

∴ Z has minimum value 8 at E(0, 4).

∴ Z is minimum, when x = 0 and y = 4.

Question 3.2.3: Find the volume of a tetrahedron whose vertices are A(−1, 2, 3), B(3, −2, 1), C(2, 1, 3) and D(−1, −2, 4). **[4]**

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of points A, B, C, D respectively of a tetrahedron.

$$\therefore \quad \bar{a} = -\hat{i} + 2\hat{j} + 3\hat{k}, \ \bar{b} = 3\hat{i} - 2\hat{j} + \hat{k},$$
$$\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}, \ \bar{d} = -\hat{i} - 2\hat{j} + 4\hat{k}$$
$$\text{Now, } \overline{AB} = \bar{b} - \bar{a} = (3\hat{i} - 2\hat{j} + \hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 4\hat{i} - 4\hat{j} - 2\hat{k}$$

$$\overline{AC} = \bar{c} - \bar{a} = (2\hat{i}+\hat{j}+3\hat{k}) - (-\hat{i}+2\hat{j}+3\hat{k})$$
$$= 3\hat{i}-\hat{j}$$
$$\overline{AD} = \bar{d} - \bar{a} = (-\hat{i}-2\hat{j}+4\hat{k}) - (-\hat{i}+2\hat{j}+3\hat{k})$$
$$= -4\hat{j}+\hat{k}$$

Volume of a tetrahedron whose coterminus edges are $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ is $\frac{1}{6}\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$

$\therefore$ Volume of the tetrahedron $= \frac{1}{6}\begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$

$= \frac{1}{6}[4(-1-0)+4(3-0)-2(-12-0)]$

= 1/6 (-4+12 + 24)

= 1/6 (32)

$= \frac{16}{3}$

$\therefore$ Volume of the tetrahedron is $\frac{16}{3}$ cubic units

Question 4.1 | Select and write the correct answer from the given alternatives in each of the following [6]

Question 4.1.1: [2]

If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ = ______

A) $\frac{1+x}{1+\log x}$

B) $\frac{\log x}{(1+\log x)^2}$

C) $\frac{1-\log x}{1+\log x}$

D) $\frac{1-x}{1+\log x}$

Solution: $x^y = e^{x-y}$

Taking logarithm on both sides, we get

$y\log x = (x-y)\log e = x - y$

$\therefore y = \frac{x}{1+\log x}$

$$\frac{dy}{dx}=\frac{(1+\log x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$=\frac{1+\log x-x\times\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$=\frac{1+\log x-1}{(1+\log x)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

Question 4.1.2: [2]

$\int \frac{1}{1+\cos x}$ dx = ____

A) $\tan\left(\frac{x}{2}\right)+c$

B) $2\tan\left(\frac{x}{2}\right)+c$

C) $-\cot\left(\frac{x}{2}\right)+c$

D) $-2\cot\left(\frac{x}{2}\right)+c$

Solution:

$$\int\frac{1}{1+\cos x}dx=\int\frac{1}{2\cos^2\left(\frac{x}{2}\right)}dx$$

$$=\frac{1}{2}\int\sec^2\left(\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[\frac{\tan\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]+c$$

$$=\tan\left(\frac{x}{2}\right)+c$$

Question 4.1.3: If X ~ B (n, p) and E(X) = 12, Var(X) = 4, then the value of n is ______ [2]

(A) 3

(B) 48

(C) 18

(D) 36

Solution: E(X) = np and Var (X) = npq

$\therefore \quad \frac{\text{Var}(X)}{E(X)} = \frac{npq}{np}$

$\therefore \quad \frac{4}{12} = q$

$\therefore \quad q = \frac{1}{3}$

$\therefore \quad p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

Now, $np = 12$

$\therefore \quad n\left(\frac{2}{3}\right) = 12$

$\therefore \quad n = 18$

Question 4.2: Attempt any THREE of the following [6]

Question 4.2.1: Find the equation of tangent to the curve $y = 3x^2 - x + 1$ at P(1, 3). [2]

Solution: Equation of the curve is $y = 3x^2 - x + 1$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x - 1$$

∴ Slope of tangent at P(1, 3) is

$$\left(\frac{dy}{dx}\right)_{(1,3)} = 6(1) - 1 = 5$$

∴ the equation of tangent at P(1, 3) is

$y - 3 = 5(x - 1)$

∴ $y - 3 = 5x - 5$

∴ $5x - y - 2 = 0$

Question 4.2.2: [2]

Evaluate: $\int \frac{1}{x(x-1)} dx$

Solution:

Let $I = \int \frac{1}{x(x-1)} dx = \int \frac{x - x + 1}{x(x-1)} dx$

$$= \int \left(\frac{1}{x-1} - \frac{1}{x} \right) dx$$

$$= \log |x-1| - \log |x| + c$$

$$= \log \left| \frac{x-1}{x} \right| + c$$

Question 4.2.3: Solve the differential equation y − x = dy/dx = 0 **[2]**

Solution:

$$y - x\frac{dy}{dx} = 0$$

$$\therefore \quad y = x\,\frac{dy}{dx}$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{y}$$

Integrating on both sides, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\therefore \quad \log |x| = \log |y| + \log |c|$$

$$\therefore \quad \log |x| = \log |cy|$$

$$\therefore \quad x = cy$$

Question 4.2.4: **[2]**

In a bivariate data, n = 10, $\bar{x}$ = 25, $\bar{y}$ = 30 and $\sum xy$ = 7900. Find cov(X,Y)

Solution:

n = 10, $\bar{x}$ = 25, $\bar{y}$ = 30 and $\sum xy$ = 7900

$$\therefore \quad \text{Cov}(X, Y) = \frac{1}{n}\sum_{i=1}^{10} x_i y_i - \bar{x} \cdot \bar{y}$$

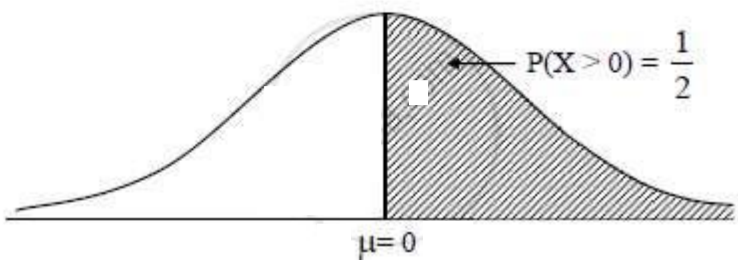
$$= \frac{7900}{10} - (25)(30) = 790 - 750 = 40$$

Question 4.2.5: A random variable X ~ N (0, 1). Find P(X > 0) and P(X < 0). **[2]**

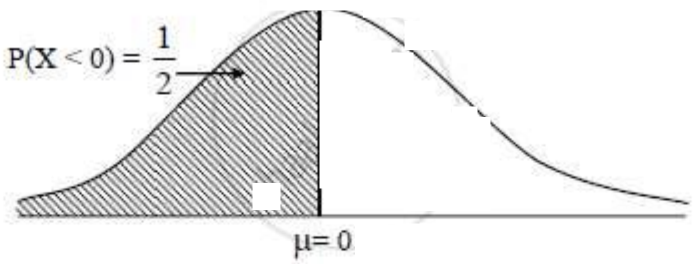
Solution: Given X ~ N(0, 1)

∴ μ = 0

∴ P(X > μ) = P(X > 0) = 1/2 as the distribution is symmetric about μ = 0.



$P(X < \mu) = P(X < 0) = 1/2$ as the distribution is symmetric about $\mu = 0$.



Question 5.1 | Attempt any TWO of the following: [6]

Question 5.1.1: Examine the function for maximum and minimum $f(x) = x^3 - 9x^2 + 24x$. [3]

Solution: $f(x) = x^3 - 9x^2 + 24x$

$\therefore f'(x) = 3x^2 - 18x + 24$

$\therefore f''(x) = 6x - 18$

Now, $f'(x) = 0$

$\therefore 3x^2 - 18x + 24 = 0$

$\therefore x^2 - 6x + 8 = 0$

$\therefore (x - 4)(x - 2) = 0$

$\therefore x = 2$ or $x = 4$

For $x = 2$,

$f''(2) = 6(2) - 18 = 12 - 18 = -6 < 0$

$\therefore$ f is maximum at $x = 2$

$\therefore$ maximum value $= f(2) = (2)^3 - 9(2)^2 + 24(2) = 8 - 36 + 48 = 20$

For $x = 4$,

$f''(4) = 6(4) - 18 = 24 - 18 = 6 > 0$

$\therefore$ f is minimum at $x = 4$

$\therefore$ minimum value $= f(4) = (4)^3 - 9(4)^2 + 24(4) = 64 - 144 + 96 = 16$

Question 5.1.2: If y = f(x) is a differentiable function of x such that inverse function x = f^{-1} (y) exists, then prove that x is a differentiable function of y and **[3]**

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)} \text{ where } \frac{dy}{dx}\neq 0$$

Solution 1: Let δy be the increment in y corresponding to an increment δx in x.

as $\delta x\rightarrow 0, \delta y\rightarrow 0$

Now y is a differentiable function of x.

$$\therefore \lim_{\delta x\rightarrow 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}$$

Now $\frac{\delta y}{\delta x}\times\frac{\delta x}{\delta y}=1$

$$\therefore \frac{\delta x}{\delta y}=\frac{1}{\frac{\delta y}{\delta x}}$$

Taking limits on both sides as $\delta x\rightarrow 0, we\geq t$

$$\lim_{\delta x\rightarrow 0}\frac{\delta x}{\delta y}=\lim_{\delta x\rightarrow 0}\left[\frac{1}{\frac{\delta y}{\delta x}}\right]=\frac{1}{\lim_{dx\rightarrow 0}\frac{\delta y}{\delta x}}$$

$$\lim_{\delta x\rightarrow 0}\frac{\delta x}{\delta y}=\frac{1}{\lim_{dx\rightarrow 0}\frac{\delta y}{\delta x}} \quad \text{....[as } \delta x\rightarrow 0, \delta y\rightarrow 0]$$

Since limit in R.H.S. exists

limit in L.H.S. also exists and we have,

$$\lim_{\delta y\rightarrow 0}\frac{\delta x}{\delta y}=\frac{dx}{dy}$$

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}, \text{ where } \frac{dy}{dx}\neq 0$$

Let $y=\tan^{-1}x$

$$x=\tan y\Rightarrow \cos y=\frac{1}{\sqrt{1+\tan^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

$$\therefore \sec^y.\frac{dy}{dx}=1\Rightarrow\frac{dx}{dy}=\sec^2 y$$

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\sec^2 y}=\cos^2 y\Rightarrow\frac{dy}{dx}=\cos^y$$

$$\frac{d(\tan^{-1}x)}{dx}=\cos^2 y=(\cos y)^2=\left(\frac{1}{\sqrt{1+x^2}}\right)^2$$

$$\therefore \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

Solution 2: 'y' is a differentiable function of 'x'.

Let there be a small change δx in the value of 'x'.

Correspondingly, there should be a small change δy in the value of 'y'.

As δx → 0, δy → 0

Consider, $\frac{\delta x}{\delta y}\times\frac{\delta y}{\delta x}=1$

$$\therefore \quad \frac{\delta x}{\delta y}=\frac{1}{\frac{\delta y}{\delta x}},\ \frac{\delta y}{\delta x}\neq 0$$

Taking $\lim\limits_{\delta x\to 0}$ on both sides, we get

$$\lim_{\delta x\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{1}{\lim\limits_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)}$$

Since 'y' is a differentiable function of 'x'

$$\therefore \quad \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\frac{dy}{dx}$$

As $\delta x\to 0,\ \delta y\to 0$

$$\therefore \quad \lim_{\delta y\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{1}{\lim\limits_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)} \qquad \ldots.(i)$$

∴ limits on R.H.S. of (i) exist and are finite.

Hence, limits on L.H.S. of (i) also should exist and be finite.

$$\therefore \quad \lim_{\delta y\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{dx}{dy} \text{ exists and is finite.}$$

$$\therefore \quad \frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)},\ \frac{dy}{dx}\neq 0$$

Question 5.1.3: The probability distribution of X, the number of defects per 10 metres of a fabric is given by **[3]**

x	0	1	2	3	4
P(X = x)	0.45	0.35	0.15	0.03	0.02

Find the variance of X

Solution:

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	0.45	0	0
1	0.35	0.35	0.35
2	0.15	0.30	0.60
3	0.03	0.09	0.27
4	0.02	0.08	0.32
Total		0.82	1.54

From the table, $\sum p_i x_i$ = 0.82 and $\sum p_i x_i^2$ = 1.54

∴ Var(X) = $\sum p_i x_i^2 - \left(\sum p_i x_i\right)^2$

= 1.54 − $(0.82)^2$

= 1.54 − 0.6724

∴ Var (X) = 0.8676

Question 5.2 | Attempt any TWO of the following: [8]

Question 5.2.1: [4]

If $\sqrt{1-x^2} + \sqrt{1-y^2}$ = a(x − y), show that dy/dx = $\sqrt{\frac{1-y^2}{1-x^2}}$

Solution:

$\sqrt{1-x^2} + \sqrt{1-y^2}$ = a(x − y)

Put x = sin θ, y = sin Φ

∴ θ = $\sin^{-1}$ x, Φ = $\sin^{-1}$ y

$$\therefore \quad \sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$$

$$\therefore \quad \cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\therefore \quad 2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi}{2}\right) = 2a\cos\left(\frac{\theta+\phi}{2}\right)\cdot\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\therefore \quad \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)} = a$$

$$\therefore \quad \cot\left(\frac{\theta-\phi}{2}\right) = a$$

$$\therefore \quad \frac{\theta-\phi}{2} = \cot^{-1} a$$

$$\therefore \quad \theta - \phi = 2\cot^{-1} a$$

$$\therefore \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

Differentiating w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Question 5.2.2: Solve the differential equation $\cos^2 x \frac{dy}{dx} + y = \tan x$ **[4]**

Solution:

$\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\therefore \quad \frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\therefore \quad \frac{dy}{dx} + \sec^2 x . y = \tan x . \sec^2 x$$

The given equation is of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \sec^2 x$ and $Q = \tan x . \sec^2 x$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}$$

∴ Solution of the given equation is

$$y(\text{I.F.}) = \int Q.(\text{I.F.})dx + c$$

$$\therefore \quad ye^{\tan x} = \int \tan x.\sec^2 x.e^{\tan x}dx + c$$

Put $\tan x = t$

$$\therefore \quad \sec^2 x\, dx = dt$$

$$\therefore \quad ye^{\tan x} = \int t\, e^t dt + c$$

$$= t\int e^t dt - \int\left[\frac{d}{dt}(t)\int e^t dt\right]dt + c$$

$$= te^t - \int e^t dt + c$$

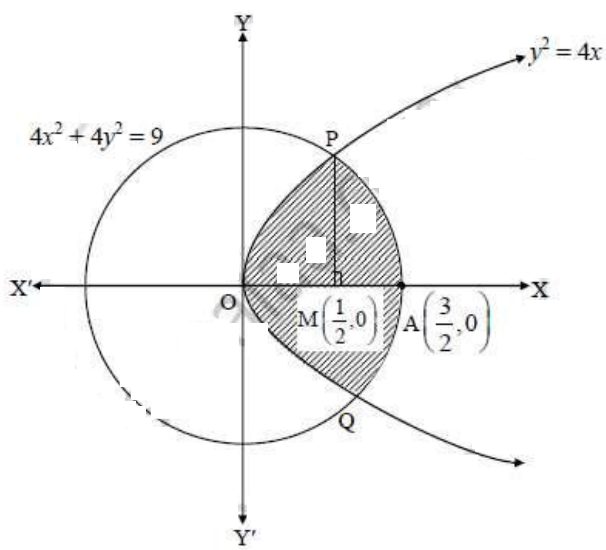
$$= te^t - e^t + c$$

$$\therefore \quad ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

$$\therefore \quad y = \tan x - 1 + c.e^{-\tan x}$$

Question 5.2.3: Find the area of the region bounded by the curves $y^2 = 4x$ and $4x^2 + 4y^2 = 9$ with $x >= 0$. **[4]**

Solution:



Required area is nothing but area bounded by the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 9/4$

To find the points of intersection.

Solving the given equations, we get

$x^2 + 4x - \frac{9}{4} = 0$

$\therefore$ $4x^2 + 16x - 9 = 0$

$\therefore$ $4x^2 + 18x - 2x - 9 = 0$

$\therefore$ $(2x - 1)(2x + 9) = 0$

$\therefore$ $x = 1/2$ or $x = -\frac{9}{2}$ (not possible)

When $x = 1/2$, $y = \pm\sqrt{2}$

$\therefore$ The curves intersect at $P\left(\frac{1}{2}, \sqrt{2}\right)$ and $Q\left(\frac{1}{2}, -\sqrt{2}\right)$

Consider, $y^2 = 4x$

$\therefore y = 2x^{\frac{1}{2}} = y_1$ ....(say)

Also, $x^2 + y^2 = \frac{9}{2}$

$\therefore y^2 = \frac{9}{4} - x^2$

$\therefore$ $y = \sqrt{\frac{9}{4} - x^2} = y_2$ ...(say)

$\therefore$ Required area = A(OPAQO) =2.A(OPAMO)

= 2[A(OPMO) + A(PAMP)]

$$= 2\left[\int_0^{\frac{1}{2}} y_1 dx + \int_{\frac{1}{2}}^{\frac{3}{2}} y_2 dx\right]$$

$$= 2\left[\int_0^{\frac{1}{2}} 2x^{\frac{1}{2}} dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\, dx\right]$$

$$= 2\left\{2 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}\right\}$$

$$= 2\left\{\frac{4}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} + \frac{3}{4}(0) + \frac{9}{8}\sin^{-1}(1) - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right\}$$

$$= 2\left[\frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8} \times \frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= 2\left[\frac{2}{3\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= 2\left[\frac{1}{6\sqrt{2}} + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \left[\frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right] \text{ sq. units.}$$

Question 6.1 | Attempt any TWO of the following **[6]**

Question 6.1.1: Find the approximate value of $\tan^{-1}(1.001)$. **[3]**

Solution: Let $f(y) = \tan^{-1}y$

Differentiating f(y) w.r.t.y, we have

$\Rightarrow f'(y) = \dfrac{1}{1+y^2}$

$y = 1.001 = x + \Delta x$

Here,

$x = 1$

$\Delta x = 0.001$

Therefore, $f(x) = f(1) = \tan^{-1}(1) = \dfrac{\pi}{4}$

Similarly, $f'(x) = f'(1) = \dfrac{1}{1+1^2} = \dfrac{1}{2}$

Now,

$f(y) = f(x + \Delta x) = f(x) + \Delta x. f'(x)$ ...[$\because \Delta x <<< x$]

$\tan^{-1}y = \tan^{-1}(x + \Delta x) = \tan^{-1}x + \Delta x.\left(\dfrac{1}{1+x^2}\right)$

$\therefore$ $\tan^{-1}1.001 = \tan^{-1}(1 + 1.001) = \tan^{-1}1 + (0.001).\tan^{-1}1$

$\Rightarrow \tan^{-1}1.001 = \frac{\pi}{4} + 0.001\left(\frac{1}{2}\right)$

$\Rightarrow \tan^{-1}1.001 = \frac{\pi}{4} + 0.0005 \approx 0.7855$

Hence the approxiate value of $\tan^{-1}0.001$ will be 0.7855.

Question 6.1.2: **[3]**

Examine continuity of the function f(x) at x = 0, where

$$f(x) = \frac{10^x + 7^x - 14^x - 5^x}{1 - \cos 4x}, \quad \text{for} \quad x \neq 0$$
$$= \frac{10}{7}, \quad \text{for } x = 0$$

Solution:

$f(0) = \frac{10}{7}$ ....(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{10^x + 7^x - 14^x - 5^x}{1 - \cos 4x} = \lim_{x\to 0} \frac{5^x(2^x - 1) - 7^x(2^x - 1)}{2\sin^2 2x}$$

$$= \lim_{x\to 0} \frac{(2^x - 1)(5^x - 7^x)}{2\sin^2 2x}$$

$$= \lim_{x\to 0} \frac{\frac{(2^x - 1)}{x}\left(\frac{5^x - 1}{x} - \frac{7^x - 1}{x}\right)}{2\times\frac{4}{4}\frac{\sin^2 2x}{x^2}}$$

$$= \frac{\left(\lim_{x\to 0}\frac{2^x - 1}{x}\right)\left(\lim_{x\to 0}\frac{5^x - 1}{x} - \lim_{x\to 0}\frac{7^x - 1}{x}\right)}{8\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2}$$

$$= \frac{\log 2(\log 5 - \log 7)}{8(1)^2}$$

$$= \frac{\log 2\left(\log\frac{5}{7}\right)}{8} \neq f(0)$$

Since $\lim_{x\to 0} f(x) \neq f(0)$, f is discontinuous at x = 0.

Question 6.1.3: The probability that a person who undergoes kidney operation will recover is 0.5. Find the probability that of the six patients who undergo similar operations, **[3]**
(a) None will recover
(b) Half of them will recover.

Solution 1:

Probability of recovery=P(R)= 0.5
Probability of non-recovery = $P(\overline{R}) = 1 - 0.5 = 0.5$

(a) If there are six patients, the probability that none recovers

$$= {}^6C_0 \times [P(R)]^0 \times \left[P(\overline{R})\right]^6 = (0.5)^6 = \frac{1}{64}$$

(b) Of the six patients, the probability that half will recover

$$= {}^6C_3 \times [P(R)]^3 \times \left[P(\overline{R})\right]^3 = \frac{6!}{3!3!} \times 0.5^3 \times 0.5^3 = 20 \times \frac{1}{64} = \frac{5}{16}$$

Solution 2: Let X be the number of patients who recovered out of 6.

P(patient recovers) = p = 0.5

∴ q = 1 − p = 1 − 0.5 = 0.5

Given, n = 6

∴ X ~ B(6, 0.5)

The p.m.f. of X is given by

P(X = x) = p(x) = ${}^6C_x(0.5)^x(0.5)^{6-x}$, x = 0, 1, 2, ..., 6

a)P(none will recover) = P (X = 0)

= ${}^6C_0(0.5)^0(0.5)^6$

= (1) (1) $(0.5)^6$

= 0.015625

(b) P(half of the patients will recover) = P (X = 3)

= ${}^6C_3(0.5)^3(0.5)^3$

= 20 (0.5)6

= 20 × 0.015625

= 0.3125

Question 6.2 | Attempt any TWO of the following: **[8]**

Question 6.2.1: Prove that: **[4]**

$$\int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$$

Solution:

Let I = $\int \sqrt{x^2+a^2}dx$

$$= \int \sqrt{x^2+a^2}\cdot 1dx$$

$$= \sqrt{x^2+a^2}\int 1dx - \int\left[\frac{d}{dx}\left(\sqrt{x^2+a^2}\right)\cdot\int 1dx\right]dx$$

$$= \sqrt{x^2+a^2}\cdot x - \int\frac{2x}{2\sqrt{x^2+a^2}}\cdot x\,dx$$

$$= x\cdot\sqrt{x^2+a^2} - \int\frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\cdot\sqrt{x^2+a^2} - \int\left(\frac{x^2+a^2}{\sqrt{x^2+a^2}}-\frac{a^2}{\sqrt{x^2+a^2}}\right)dx$$

$$= x\cdot\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

$$\therefore \quad I = x\cdot\sqrt{x^2+a^2} - I + a^2\log\left|x+\sqrt{x^2+a^2}\right|+c_1$$

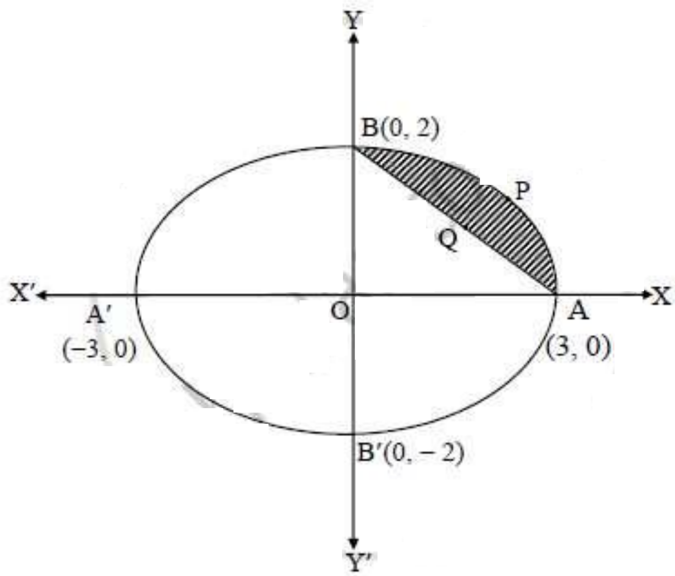
$$\therefore \quad 2I = x\cdot\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right|+c_1$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+\frac{c_1}{2}$$

$$\therefore \quad \int\sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c\text{, where } c = \frac{c_1}{2}$$

Question 6.2.2: Find the volume of the solid generated, when the area between ellipse $4x^2 + 9y^2 = 36$ and the chord AB, with A (3, 0), B (0, 2), is revolved about X-axis. **[4]**

Solution:



Given equation of ellipse is $4x^2 + 9y^2 = 36$

$\therefore y^2 = \frac{4}{9}(9 - x^2)$.....(i)

and equation of the chord is $\frac{x}{3} + \frac{y}{2} = 1$

$\therefore 2x + 3y = 6$

$\therefore 3y = 6 - 2x$

$\therefore y = 2 - \frac{2}{3}x$

$\therefore y^2 = \left(2 - \frac{2}{3}x\right)^2 = 4 - \frac{8}{3}x + \frac{4}{9}x^2$ ....(ii)

Required solid is obtained by revolving the shaded region about the X-axis between x = 0 and x = 3.

Let V_1 = volume of solid obtained by revolving the region OAPBO under the ellipse

V_2
= volume of solid obtained by revolving the region OAQBO under the chord AB.

$\therefore V = V_1 - V_2$

$$= \pi\left[\int_0^3 \frac{4}{9}(9 - x^2)dx - \int_0^3\left(4 - \frac{8}{3}x + \frac{4}{9}x^2\right)dx\right] \quad \text{....[From (i) and (ii)]}$$

$$= \pi\left[\frac{4}{9}\left\{9(x)_0^3 - \left(\frac{x^3}{3}\right)_0^3\right\}\right] - \left[4(x)_0^3 - \frac{8}{3}\left(\frac{x^2}{2}\right)_0^3 + \frac{4}{9}\left(\frac{x^3}{3}\right)_0^3\right]$$

$$= \pi\left\{\frac{4}{9}\left[9(3) - \left(\frac{27}{3}\right)\right] - \left[4(3) - \frac{4}{3}(9) + \frac{4}{27}(27)\right]\right\}$$

$$= \pi\left[\frac{4}{9}(27 - 9) - (12 - 12 + 4)\right]$$

$$= \pi\left[\frac{4}{9}(18) - 4\right]$$

$$= \pi(8 - 4) = 4\pi \text{ cubic units.}$$

Question 6.2.3: Find Karl Pearson's coefficient of correlation between the variables X and Y for the following data **[4]**

X	11	7	9	5	8	6	10
Y	10	8	6	5	9	7	11

Solution: Let X = x_i , Y = y_i

								Total
x_i	11	7	9	5	8	6	10	56
y_i	10	8	6	5	9	7	11	56
x_i^2	121	49	81	25	64	36	100	476
y_i^2	100	64	36	25	81	49	121	476
x_iy_i	110	56	54	25	72	42	110	469

From the table, we have

$$n = 7,\ \sum_{i=1}^{7} x_i = 56,\ \sum_{i=1}^{7} y_i = 56,\ \sum_{i=1}^{7} x_i^2 = 476,\ \sum_{i=1}^{7} y_i^2 = 476,\ \sum_{i=1}^{n} x_i . y_i = 469$$

$$\therefore \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{7} x_i = \frac{56}{7} = 8,\ \ \bar{y} = \frac{1}{n}\sum_{i=1}^{7} y_i = \frac{56}{7} = 8$$

$$\therefore \quad \operatorname{Corr}(X, Y) = \frac{\frac{1}{n}\sum_{i=1}^{7} x_i . y_i - \bar{x}.\bar{y}}{\sqrt{\frac{1}{n}\sum_{i=1}^{7} x_i^2 - \left(\bar{x}\right)^2} . \sqrt{\frac{1}{n}\sum_{i=1}^{7} y_i^2 - \left(\bar{y}\right)^2}}$$

$$= \frac{\frac{469}{7} - (8)(8)}{\sqrt{\frac{476}{7} - (8)^2} . \sqrt{\frac{476}{7} - (8)^2}}$$

$$= \frac{67 - 64}{\sqrt{68 - 64} . \sqrt{68 - 64}}$$

$$= \frac{3}{2 \times 2} = 0.75$$