Probability

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1 Introduction

Probability is the area of mathematics which deals with chance. This means it supplies us with a way to treat or analyse the outcomes and behaviours of systems which are not deterministic, that is, we do not have concrete equations which predict exactly what will happen. For example, the current i in an RC circuit is modelled as

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{RC} = \frac{\omega V}{R}\cos(\omega t).$$

This model has been produced from the underlying physics of an electrical circuit and we would expect, saving measurement error, that it will tell us exactly what the current i is at a time t. This is a deterministic situation or a deterministic equation. However, there are situations where things are not amenable to this type of analysis. This happens when events happen randomly or with chance, so the result of a process is not definitely going to be one specific outcome.

There are several ideas central to the topic, such as events and the probability of their occurrence, experiments and outcomes. These ideas will be defined first; everything we come to study in this area, including Hypothesis

Testing, can only be properly understood if these fundamental definitions are kept in mind.

1.1 Basic Definitions

We will start by defining events, experiments and probability.

1.1.1 Definition - An Event

An event is a well-defined occurrence that may or may not happen. Well defined means that we know exactly whether or not it has occurred.

A good example comes from the arena of gambling and specifically throwing dice. If a fair die is thrown, there are six possible outcomes, in other words, one of six possible numbers will be left facing up (any other result is highly unlikely and can be ignored). Any possible events in this situation can be defined in terms of the numbers that come up. The simplest possible events to look at are the numbers that actually comes up. There are six well-defined possible events; that of each one of the six numbers coming face up. Alternatively, an event could be the occurrence of an even number, or a number less than 4, or some other classification of the number. In each of these cases, when a number comes up on the dice, it is immediately clear whether or not the criteria defining the event have been satisfied. For an example of a more ambiguous event, consider a "White Christmas." Although our image of a White Christmas may involve fields of white and other clichés, meteorologists have an exact definition; it is a certain amount of snow at certain weather stations throughout Ireland. Now that we know what an event is, the next step is to define exactly what its probability is. We will define this idea in two steps, the first giving a general idea of probability, the second giving a quantifiable version.

1.1.2 First Definition of Probability

The probability of an event is a measure of how likely it is to occur. It is a number between 0 and 1. A probability of 0 for an event means it will not occur, while a probability of 1 means it definitely will occur.

1.1.3 Notation

Mathematically, an event can be denoted by any symbol. Thus the event of a particular number showing up on a dice could be represented as E. With this in mind, the probability of event E is written mathematically as

$$P[E].$$

1.1.4 Example of Notation

The meteorological service has stated that 'there is a 35 % chance of rain in Dublin today'. To write this as a proper mathematical statement, we set R to be the event that there is rain in Dublin today, presumably defined as a minimal amount of rain recorded at weather stations in the Dublin area. Then to say the probability of this event happening is 35%, that is, 0.35, we write:

$$P[R] = 0.35.$$

Take note of how the '35% chance', written in lay-man's language, becomes a probability of 0.35.

The probability of an event is best defined if it may occur many times. For example, in the case of the dice, we are concerned with the result of a particular action; the throwing of the dice. To allow us to follow this line of thought for the definition, we need another definition, that of an experiment.

1.1.5 Definition of an Experiment

An action, with a well-defined set of outcomes, that can be repeated a large number of times, is called an experiment.

The importance of this idea of an experiment is that we can now think of an event as the result of an experiment, defined in terms of one or more of the outcomes of the experiment. In the case of the dice, throwing the dice is the experiment, with the numbers which come up as the outcomes. With this idea of an event happening as the result of an experiment, we can go back to the definition of the probability of an event and quantify it as the proportion of times it occurs if the experiment is repeated many times. We will now frame a definition of probability and, with this in mind, find the probability of a given number coming up. Keep in mind that the probability of event E is written mathematically as $P[E]$.

1.1.6 Second Definition of Probability

Let E denote an event, which is the possible outcome of an experiment. The probability $P[E]$ is the proportion of times event E occurs, as the experiment is repeated a very large number of times. Mathematically, if the experiment is repeated N times, and event E turns up N_E times, then the probability of E occurring, is the fraction

$$P[E] = \frac{N_E}{N}$$

evaluated as the number N gets bigger. We can write this formally as a limit:

$$P[E] = \lim_{N \to \infty} \frac{N_E}{N}.$$

1.2 The Example of Dice

Rolling a dice is a very good way of illustrating this idea of the probability of an event being defined by the idea of an experiment. In the case of the throw of one dice, there are six faces on the dice, one of which is left facing up when it is rolled. We therefore take the experiment as the throwing of the dice, so the outcomes for the experiment are:

$$\{1, 2, 3, 4, 5, 6\}.$$

The various possible events involving one dice can be defined in terms of these outcomes.

Let us take the event as being the number left facing up. The set of possible events is therefore

$$\{1, 2, 3, 4, 5, 6\}.$$

This is the same as the list of outcomes.

To determine the probabilities of these events, the dice could be thrown many times and the outcomes counted. The probabilities would then come from using the second, quantitative definition of probability quoted above.

However, it would be expected that after a large number of throws, each side of a fair dice would come up a roughly equal number of times. Indeed, this is the definition of a fair dice. Thus the probabilities of various events may be derived from this knowledge.

Let the dice be thrown N times, where N is a large number. Let n be the number of times a particular side ends up facing up. Then it follows that if each face comes up approximately the same number of times, we expect that

$$6n \approx N.$$

From the second, quantitative definition of probability, the proportion of times each number comes up is approximately

$$\frac{n}{N} \approx \frac{1}{6}.$$

As the number of throws N increases without limit, the ratio becomes closer to the value $\frac{1}{6}$. This means we can define the probability as follows: if E is the event of a given number coming up when the dice is thrown, then if the dice is fair,

$$P[E] = \frac{N_E}{N} = \frac{1}{6}.$$

1.2.1 Empirical Data and Probabilities

The probability in the previous example came up as a result of applying the logic of the second definition to the situation and drawing upon our knowledge of that situation. Now let is look at some empirical examples, in other words, cases where we rely on data and can make the same kind of logical steps. A

Empirical Data	Probabilities
Clinical data suggests that 8 % of the population born in Ireland has O Negative blood-type.	The probability that a person chosen at random born in Ireland has O neg blood-type is 0.08.
A recent census established that 6.2 % of the population of Ireland have no religion.	The probability that a person in Ireland, chosen at random, has no religion is 0.062.
A company manufactures a certain type of component, by the thousands. A random sample of 200 components are taken and 5 are found to be faulty.	The company management concludes that the probability of an individual component being faulty is 0.025.

very important word used in this table is *random* - each of the proportions estimated is based on a randomly selected group of subjects, so there is no reason to think they are any different from the overall population.

1.2.2 Example - Two dice

We will now look at this definition of a probability with the example of the throw of two dice. For this case, we will calculate the probability of the result adding up to 5. Doing this will introduce some important concepts that allow us to analyse any particular scenario rigorously.

Firstly, let E be the event that we roll two dice and we get a sum of five. The experiment is rolling two dice and seeing what the numbers are. The outcomes are the pairs of numbers that come up when we do the experiment. With each die having 6 possible outcomes, there are 36 possible outcomes in all;

$$(1,1),\ (1,2),\ \ldots,\ (1,6),\ (2,1),\ (2,2),\ldots,\ (2,6),$$

etc...

$$(6,1),\ (6,2),\ldots,\ (6,6).$$

To find the probability, we would normally need the number N, the number of times the experiment is repeated, and N_E, the number of times the event E turns up. Then the probability of E occurring, is the fraction

$$P[E] = \frac{N_E}{N},$$

evaluated as the number N increases.

However, the logic of the situation will give us this ratio. If every face on the dice is equally likely to come up, then each one of these 36 pair of numbers is equally likely to come up. Thus the ratio $\frac{N_E}{N}$ will head towards the proportion of the outcomes that satisfy the condition of the event.

Of the 36 pairs, 4 of them satisfy this condition that the two numbers add to 5:

$$(1,\ 4),\ (2,\ 3),\ (3,\ 2) \text{ and } (4,\ 1).$$

Then

$$P[E] = \frac{N_E}{N} = \frac{4}{36} = \frac{1}{9}.$$

This example shows that the probability of a particular event can often be found by considering all the possible outcomes of the experiment being run, and seeing how they compare. This is particularly useful if there is a basic list of outcomes, all of which are equally likely.

1.2.3 The Product of Two Numbers

In the same scenario, rolling two dice, calculate the probability that the two numbers, when multiplied, give 6.

This problem will be approached in the same way as the previous example. We have 36 possible outcomes, the list of all possible pairs of numbers. They are all equally like to come up. The possible ways to get a product of 6 are:

(1,6), (2,3), (3,2) and (6, 1).

There are four possible outcomes that give 6, so the probability is

$$P[E] = \frac{N_E}{N} = \frac{4}{36} = \frac{1}{9}.$$

1.3 The Sample Space

Consider the following problem. Two fair dice are rolled; calculate the probability that the sum of the two numbers is less than 6.

Once again we need to find the ratio $\frac{N_E}{N}$ for very large numbers of repetitions of the experiment. We will again look at the list of possible outcomes and see what proportion of them satisfy the condition for the event.

The most methodical way of counting the number of outcomes that satisfy the condition is to calculate the sum for each possible outcome in a table. Every possible outcome, the numbers coming up and their relevant sum, will be listed. Here it is: Such a table is an example of a *sample space*, since every

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

possible outcome is here with the corresponding sum. Every possible event is a subset of this table. All possible events then come from the list of possible values of the sum:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

This is called the *event space*.

To find the probability of this event, count those outcomes where the sum is below 6. There are 10 such results where the sum is below 6, so the probability is $10/36 = 0.277778$.

If the probability required was that of a product or difference, a similar sample and event space could be drawn up, but with the product or difference calculated. This table is the sample and event space for 'the sum'.

This example shows how drawing up a sample space and event space can be a simple and direct way to calculate a probability. The following is a formal definition of these ideas.

1.3.1 Definition of Sample Space

The sample space is the list or set of all possible outcomes of an experiment. An event is composed of subsets of the sample space.

Any event which arises as a result of that experiment is a subset of the event space, in other words, it can be built up from the list of outcomes.

1. For the case of rolling one dice, the sample space and event space are identical, the numbers 1 to 6.

2. The sample space for the experiment of 'rolling two dice and taking the sum' is the list

   $$(1{,}1),\ (1{,}2),\ ...,\ (1{,}6),\ (2{,}1),\ (2{,}2),\ldots,\ (6{,}1),\ (6{,}2),\ldots,\ (6{,}6).$$

   The event space for this experiment is the list of numbers:

   $$2,\ 3,\ 4,\ 5,\ ...\ 11,\ 12.$$

3. The sample space for the experiment of 'rolling two dice and taking the product' is the list

   $$(1{,}1),\ (1{,}2),\ ...,\ (1{,}6),\ (2{,}1),\ (2{,}2),\ldots,\ (6{,}1),\ (6{,}2),\ldots,\ (6{,}6).$$

   The event space for this experiment is the list of numbers:

   $$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ ...\ 30,\ 36.$$

1.3.2 Difference Event Space

In the same scenario, rolling two dice, calculate the probability that the difference of the two numbers is 2.

This calculation is done by drawing up the sample space and the event space for the two dice and for 'the difference'. Then find the probability by counting, as before, the outcomes that satisfy the criterion for this event. Here is the sample space for 'the difference'. The event space for this experiment is the list of numbers:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

From the table, the sample space, there are 8 results where the difference is 2, so the probability is:

$$\frac{8}{36} = \frac{2}{9} = 0.222\ldots$$

1.4 Complementary Events

We will now look at a scenario related to the previous example. Calculate the probability that the sum is equal to or greater than 6.

The event we are investigating here comes from the same sample space we have been using. It is the exact opposite of the previous event, the sum less than 6. It is known that there are 10 outcomes with the sum less than 6, therefore the number of outcomes equal to or greater than 6 is

$$36 - 10 = 26.$$

The probability of the sum being equal to or greater than 6 is then 26/36. We were able to do this subtraction because, with the aid of the event space, we knew all possible outcomes for the two dice. The two events were the exact opposite of each other, which allows us to count the correct outcomes quickly. This is an example of a particularly important rule.

1.4.1 Two Complementary Events

In an experiment with a number of possible outcomes, let A be an event and let B be the exactly opposite event. Therefore B consists of every outcome that is not included in A. The event B is called the complement of A. Then

$$P[A] + P[B] = 1.$$

In other words, when two distinct events between them cover all possibilities eventualities, the sum of their probabilities is 1. Now we take this to its logical conclusion with several events.

1.4.2 Example - Rain in Dublin

We saw at the start the statement from the meteorological service that 'there is a 35 % chance of rain in Dublin today'. We set R to be the event that there is rain in Dublin today. If the probability of this event happening is 35%, that is:

$$P[R] = 0.35.$$

We can now identify the complementary event, which is that it does not rain in Dublin today, based on the same criteria. Call this event D. Then

$$P[D] = 1 - 0.35 = 0.65.$$

This statement is in a way quite obvious, but it illustrates an important idea.

1.4.3 The General Case

Calculate the probability of the following events; the sum is less than 7, the sum is equal to 7 and then the sum is greater than 7.

The solution to this question uses the same sample space for the sum to count the outcomes which give the correct respective sum. The counts are:

15, 6 and 15 outcomes respectively,

so that the probabilities are:

15/36, 6/36 and 15/36 respectively.

It is the case that these three probabilities add up to 1, as they should from the counting process. A generalisation of the rule for complementary events can now be based on this example of the dice.

Say an experiment has n distinct (in other words, not overlapping) possible events, which we will call E_1, E_2, to E_n. Then

$$P[E_1] + P[E_2] + \ldots P[E_n] = 1.$$

When all distinct possibilities have been taken care of, the sum of the probabilities is 1.

1.5 The Birthday Enigma

As the examples of the dice showed, if an event is the result of a simple experiment, then finding its probability is a matter of seeing how many possible outcomes the experiment can produce and then how many it can produce which satisfy the criteria of the event. If all the outcomes are equally likely, the probability is then the ratio of these two numbers. Here is an example which gives what seems like an unusual result. This example also uses the idea of complementary events we have just discussed.

In a group of 25 people, calculate the probability that two or more share the same birth date.

This event is more likely than not. In fact, the probability passes 0.5 for 22 or 23 people. To see why, consider a slightly more abstract, but simpler, version of the same question.

1.5.1 A Simpler Question

A random number generator produces a number from 1 to 100. It is run 20 times. Calculate the probability that the same number came up twice or more.

To work on this question, it is simpler to look at the converse question - find the probability that no number came up twice. With this in mind, let S be the event that two or more of the numbers are the same. Let T be the opposite event, that none of the numbers generated are the same.

We start by looking at the structure of the scenario. The sample space is the list or set of all possible lists of 20 numbers. Any list of 20 numbers, produced by the number generator being run 20 times will be in this list. The event is the subset of this list made up of all those lists of 20 numbers where none of the numbers are the same.

The probability $P[T]$ is then the ratio of the number of these two sets, because each list of 20 numbers is equally likely to come up.

In other words, we find how many ways in which the 20 numbers can come up, with no restrictions, this being the total number of outcomes, the quantity N. Then find the number of outcomes which satisfy the criteria for the event, in other words, the number of ways they can pick the numbers with no two the same; N_E. Because we are implicitly assuming that each outcome is equally likely to come up, we can then say that the probability is the ratio

$$P[T] = \frac{N_T}{N}.$$

To calculate the two numbers required, look at the two lists:

1. N: This is the set of lists of 20 numbers, with no restriction on them. Therefore there is a choice of 100 for the first number, 100 for the second and so on to the 20th. The number in this set is 100^{20}.

2. N_T: This is the set of lists of 20 numbers, with no two the same. Therefore there is a choice of 100 for the first number, 99 for the second, 98 for the third and so on to the 20th. The number in this set is $100 \times 99 \times 98 \ldots \times 81$.

The first number is

$$N = 100^{20} = 10^{40}.$$

The second number N_T is denoted by the symbol

$$^{100}P_{20} = 100 \times 99 \times \ldots \times 81.$$

The 100 in the symbol shows that the list of numbers being multiplied starts at 100 and the 20 shows there are 20 numbers being multiplied, decreasing by 1 each time.

So our ratio is

$$P[T] = \frac{N_T}{N} = \frac{100 \times 99 \times \ldots \times 81}{100 \times 100 \times \ldots \times 100} = \frac{^{100}P_{20}}{100^{20}} = \frac{1.3 \times 10^{39}}{10^{40}} = 0.13.$$

The calculation of $^{100}P_{20} = 1.3 \times 10^{39}$ can be done quickly on most scientific calculators.

So we have found that $P[T] = 0.13$ so $P[S] = 0.87$. There is a probability of 0.87 that two or more of the numbers produced by running the Random Number Generator 20 times will be the same.

1.5.2 Birthdays

Now return to the original question of birthdays. In a group of 25, calculate the probability that two or more people share a birthday. Let S be the event that two or more of the people share the same birthday and let T be the opposite event, that none are the same. Then, from the same logic as in the previous example, the probability of nobody having the same birthday among this group is

$$P[T] = \frac{N_T}{N} = \frac{365 \times 364 \times \ldots \times 341}{365 \times 365 \times \ldots \times 365} = \frac{{}^{365}P_{25}}{365^{25}} = \frac{4.92 \times 10^{63}}{1.14 \times 10^{64}} = 0.43.$$

This then means that $P[S] = 0.57$. Therefore in a group of 25 people, the probability that two or more people share the same birthday is 0.57.

2 The Laws of Probability

Having established the basic ideas underpinning probability, in this section we will look at the fundamental laws that probability calculations obey. In particular, this will show us the ways in which probabilities involving two or more events can be calculated, depending on how the events are related to one another. We have already seen one example of this, the rules for complementary events.

1. In an experiment with a number of possible outcomes, let A be an event and let B be the exactly opposite event, the complement of A. Then

$$P[A] + P[B] = 1.$$

2. If an experiment has n distinct possible events E_1, E_2, to E_n, then

$$P[E_1] + P[E_2] + \ldots P[E_n] = 1.$$

We will now see a few more examples of these types of laws.

2.1 Mutually Exclusive Events

Before we look at this, we have to define the type of events we are looking at.

2.1.1 Definition – Mutually Exclusive Events

Two events A and B, the possible results of the same experiment, are said to be *mutually exclusive* if it is impossible for them to happen together. If two events A and B are not mutually exclusive, that is, A and B could occur together, then they are said to be *mutually non-exclusive* events.

With this definition, we can now write down two laws governing the probability of the event A or B.

2.1.2 The Addition Law

If two events A and B are mutually exclusive, then the following law holds:

$$P[A \text{ or } B] = P[A] + P[B].$$

If the two events are not mutually exclusive, the law above is broadened so that the probability of A or B occurring is given by:

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

This means the probability of one of event A or event B occurring is given by the sum of their two probabilities. If the events are not mutually exclusive, the probability of one of event A or event B occurring is given by the sum of their two probabilities, less the probability of them both occurring.

2.1.3 An Example of the Addition Law

If a single dice is thrown, determine the probability of getting a multiple of 2, a multiple of 3 and then one or the other.

To start, define the basic events and calculate their probabilities.

- Let A be the event of getting a multiple of 2 and B be the event of getting a multiple of 3.

- The probability of scoring a multiple of 2 is that of getting a 2, 4 or 6. Then $P[A] = 3/6$.
- The probability of scoring a multiple of 3 is that of getting a 3 or a 6. Then $P[B] = 2/6$.

Now both of these events could happen at the same time; a number can be a multiple of 2 and 3 and one of these is on the die: 6. It then follows that A and B are non-exclusive events. The event 'A and B' means that a number is a multiple of 2 and 3, so this is equivalent to getting a 6. Thus we must use the second version of the addition rule

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B].$$

The result is:

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B] = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

In this case, finding the probability $P[A \text{ or } B]$ from scratch would not require too much calculation. It would simply be a matter of counting the number of ways a number on the dice could be a multiple of 3 or a multiple of 2. The ways of getting this are 3 and 6 for the first, and 2, 4 and 6 for the second. This is a list of 4 distinct numbers, so the answer is

$$P[A \text{ or } B] = \frac{4}{6} = \frac{2}{3}.$$

This confirms the answer already found and is also 'computationally' the same. Looking at this alternative calculation shows why the law works and where it comes from.

2.2 Independent and Dependent Events

The next concept in probability concerns identifying which events do or don't affect the occurrence of others. An example would be the rolling of a dice on two occasions. The outcome of the first throw will not affect the probabilities for the second throw; the dice is picked up and thrown again with no link to the last throw. There is no mechanism by which the second can be influenced by the first.

Now consider the following scenario. A bag contains three apples and three oranges, all of roughly the same size. A person reaches in and takes out and retains one of the fruit randomly. Clearly the probability of this person picking an apple is 0.5.

Now consider the same action - the person reaches in again and picks a fruit randomly. To calculate the probability that the second fruit was an apple, we need to know whether the first was an apple or an orange, since it was not returned to the bag.

Let us recast this in terms of events.

Let A_1 be the event that, when the first person chooses a fruit and retains it, they take away an apple. Let A_2 be the event that, when the second person chooses a fruit and retains it, they take away an apple.

Clearly the probability $P[A_2]$ depends on whether or not event A_1 happened. We say that event A_2 is dependent on event A_1.

The dependence arises because the person picking a fruit first retains their apple or orange. If the fruit was returned to the bag, the probability of the second event, picking an apple, would be 0.5 and would be independent of whether the first had happened or not.

2.2.1 Definition - Independent Events

Two events are independent if the occurrence of one event does not affect the probability of the occurrence of the second event. If the outcome of one event does affect the probability of the second event, they are said to be dependent.

Let us calculate the probabilities of the two events described above.

- The bag initially holds 3 oranges and 3 apples.
- For A_1, the experiment is picking a fruit at random. The event is that the fruit picked is an apple. We can say that $P[A_1] = 0.5$.
- We now calculate $P[A_2]$. The experiment is selecting a fruit at random from the remaining 5. If A_1 did happen, then there are 2 apples and 3 oranges left, so that $P[A_2] = 2/5 = 0.4$. If A_1 did not happen, then there are 3 apples and 2 oranges left, so that $P[A_2] = 3/5 = 0.6$.

We now identify how to calculate probabilities for these scenarios.

2.2.2 The Multiplication Law of Probability (1)

For two independent events A and B, the probability of the occurrence of both events A and B, is given by

$$P[A \text{ and } B] = P[A].P[B].$$

The probability of the occurrence of both events is the product of the two individual probabilities.

2.2.3 An Example of Independent Events

A single fair dice is thrown 4 times. Calculate the probability of getting the number 5 facing up four times in a row.

The event of rolling the die each time is an independent event. Each time, the probability of getting a 5 is $\frac{1}{6}$. Thus using the multiplicative law, the probability of getting this four times in a row is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6^4}.$$

2.2.4 Dependant Events - Conditional Probability

To deal with dependent events, some notation will be needed to indicate when the probability of one event depends on whether another event has happened. This idea comes up a lot in calculations of failure probabilities in complex systems of components, where the failure of the system depends on different structures or 'pathways' within the system.

Consider the situation of two events A and B, where the occurrence or not of event A does effect the probability of event B. In other words, B is dependent on A. We will set up a notation for this.

2.2.5 Notation - Conditional Probability

Let A and B be two events, where event B is dependent on event A. The probability of event B, given that event A has already occurred is denoted by the notation:

$$P[B|A].$$

This is 'the probability of B, given A.'

For two independent events A and B, by definition, the fact that A has already occurred does not affect the probability of event B. It then follows that

$$P[B|A] = P[B].$$

If the events are dependent, these probabilities are not the same.

2.2.6 The Multiplication Law of Probability (2)

For two events A and B, where event B is dependent on A, the probability of the occurrence of both events A and B, is given by

$$P[A \text{ and } B] = P[A].P[B|A].$$

The probability of the occurrence of both events is the probability of A times the probability of B, given that A has occurred.

2.2.7 An Example of Dependent Events

A box contains five 10 kilo-Ohm resistors and twelve 20 kilo-Ohm resistors. Determine the following probabilities:

- The probability of randomly picking a 10 kilo-Ohm resistor from the box.
- The probability of randomly picking a 10 kilo-Ohm resistor from the box and then a 20 kilo-Ohm resistor.

Before starting the calculations, we label the two events; let event A denote the event of picking a 10 kilo-Ohm resistor and let B denote the event of picking a 20 kilo-Ohm resistor.

The first probability is just $P[A]$ and since the total number of resistors is 17, it is

$$P[A] = \frac{5}{17}.$$

To find the probability of both, that is, $P[A \text{ and } B]$, observe that B depends on A, since A is the event that is happening first. The probability law for dependent events must be used:

$$P[A \text{ and } B] = P[A].P[B|A].$$

To find $P[B|A]$, we need the probability that a second resistor picked from the box will be a 20 kilo-Ohm resistor, providing that the first one was a 10 kilo-Ohm resistor. For this case,

$$P[B|A] = \frac{12}{16} = \frac{3}{4}.$$

So then the probability of both events, picking a 20 kilo-Ohm resistor after getting a 10 kilo-Ohm resistor is:

$$P[A \text{ and } B] = P[A].P[B|A] = \frac{5}{17} \times \frac{3}{4} = \frac{15}{68}.$$

2.3 Reliability of Complex Systems

The laws of probability we have just encountered are essential for analysing probabilities of failure for systems made up of several components, each of which has a known probability of failure. The applications of this work are clear - many engineering systems are composed of numbers of individual components, any one of which could fail.

Here are a few examples. In each of the following systems, all components are independent. There are two types, A and B. The probability that each individual component of type A will fail is 0.03 and the probability for type B is 0.05. For each system, the aim is to calculate the probability that the system overall will work.

2.3.1 Example 1

Here is a system made of two components in series in figure 1.

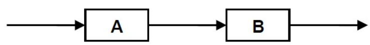

Figure 1: System 1

For this example, clearly both components have to work so that the system overall works. This means that the event of the system working is the event 'A and B'. We will use the symbol S to mean the event 'The System works'. We will use the symbol A to represent the event that component A works and the same for B. So

$$P[S] = P[A \text{ and } B].$$

The probability A works is $1 - 0.03 = 0.97$ and 0.95 for B:

$$P[A] = 0.97, \ \ P[B] = 0.95.$$

The two events A and B are independent, so the probability they both work is:

$$P[A \text{ and } B] = P[A] \times P[B] = 0.97 \times 0.95 = 0.9215.$$

This is the probability the system works.

2.3.2 Example 2

Here is a slightly more complex system in figure 2

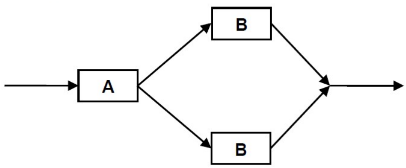

Figure 2: System 2

In this case, going from left to right, the first stage, that is, the first component A, has to work, followed by the second stage, which is either one B or the other. For the system overall, both stages must work. We will use S_1 to refer to stage 1 and the same with S_2. They are both independent therefore:

$$P[S] = P[S_1 \text{ and } S_2] = P[S_1]P[S_2].$$

Firstly, stage 1 is equivalent to component A:

$$P[S_1] = P[A].$$

For stage 2:

$$P[S_2] = P[B \text{ or } B].$$

In the second stage, the two components of type B working are not mutually exclusive. So to find the probability this part of the system works, we have to use the second version of the addition law, this is just

$$P[B \text{ or } B] = P[B] + P[B] - P[B]P[B].$$

Putting in the known probabilities:

$$P[B \text{ or } B] = 0.95 + 0.95 - 0.95 \times 0.95 = 0.9975.$$

The laws then mean that the probability the whole system works is

$$P[S] = P[A](P[B] + P[B] - P[B]P[B]) = 0.97 \times 0.9975 = 0.9676.$$

2.3.3 Example 3

Here is another more complex system in figure 3

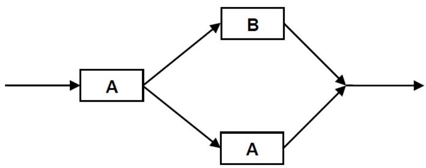

Figure 3: System 3

In this case, going from left to right, the first component A has to work, followed by the second stage, which is either the B or the second A. For the system overall, both stages must work.

$$P[S] = P[S_1] \times P[S_2].$$

Firstly,

$$P[S_1] = P[A].$$

For the second stage to work, A or B must work. Components A working and B working are not mutually exclusive so

$$P[S_2] = P[A \text{ or } B]$$

We have to use the second version of the addition law, this is just

$$P[A \text{ or } B] = P[A] + P[B] - P[A]P[B] = 0.97 + 0.95 - 0.97 \times 0.95 = 0.9985.$$

The laws then mean that the probability the whole system works is

$$P[S] = P[S_1] \times P[S_2] = P[A] \times 0.9985 = 0.97 \times 0.9985 = 0.968,545.$$

2.3.4 Example 4

Another complex system:

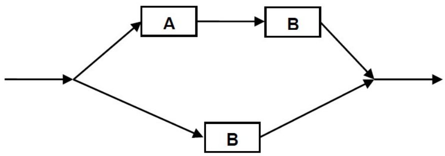

Figure 4: System 4

In this example, for the system to work; The lower B has to work, or the upper A and B together, or both scenarios. Set $S_1 = A$ and B so the system probability can be written as

$$P[S] = P[S_1 \text{ or } B]$$

Looking at S_1 :

$$P[S_1] = P[A \text{ and } B] = P[A]P[B] = 0.97 \times 0.95 = 0.9215.$$

For the system overall,

$$P[S] = P[S_1 \text{ or } B]$$

Using the addition law, this is

$$P[S_1 \text{ or } B] = P[S_1] + P[B] - P[S_1]P[B].$$

Using the values we have:

$$P[S_1 \text{ or } B] = 0.9215 + 0.95 - 0.9215 \times 0.95 = 0.996075.$$

3 Permutations and combinations

We will now introduce some very important ideas for our practice of counting outcomes for experiments involving selections.

3.1 Factorial Notation

We will firstly recall an important notation for arrangements, the factorial symbol.

3.1.1 Definition: n Factorial

For an integer number n, the number $n!$ is defined as the product of n with every integer below it down to 1:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

This number is called n-factorial.

3.1.2 Examples of Factorial Calculations

- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$
- $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040.$

- These values increase rapidly: $10! = 10 \times 9 \ldots 2 \times 1 = 3,628,800$.

It can be seen that when calculating the value of 6!, it was not necessary to find the value of $5 \times 4 \times 3 \times 2 \times 1$; this was already done for 5!. In other words,

$$6! = 6 \times 5!$$

The general case:

$$n! = n \times (n-1)!$$

3.2 Permutations

A tourist is visiting the south-west of Ireland, and has time to visit four towns out of Cork, Killarney, Ennis, Limerick, Tralee, Bantry. Calculate how many possible trips are there if the order in which the towns are visited is taken into account.

There are 6 towns and 4 choices are made, so the number of ways in which the tourist could choose the towns is

- 6 choices for the first town, then
- 5 for the second,
- 4 for the third and
- 3 for the fourth.

The result is 6 by 5 by 4 by 3, which is 360. Each particular choice of towns, in a particular order, is a *permutation* of four of the six names in the list.

3.2.1 Definition: Permutation

A permutation of r items from a list of n is a choice of r items in a particular order. The number of permutations of r items from a list of n is

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1).$$

There are r numbers being multiplied here. This number is defined as ${}^nP_r$:

$$ {}^nP_r = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1). $$

The number can be written in terms of the factorial notation as follows:

$$ {}^nP_r = \frac{n!}{(n-r)!}, $$

but when n is high it is the previous version which gives us the quickest way to calculate the number.

3.2.2 Example - Permutation of Towns

For the example of the tourist, the number of ways of choosing 4 towns out of 6, in a particular order, was

$$ {}^6P_4 = 6 \times 5 \times 4 \times 3 = 360. $$

In terms of the factorial notation:

$$ {}^6P_4 = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}. $$

The 2 by 1 term cancels from above and below the line giving the same result. This is useful as a way to define the number ${}^6P_4$, it is generally not a practical way to carry out the calculation.

3.3 Combinations

In the example of the list of towns, we now look at the case where the order in which the towns were visited does not matter. This selection is called a combination. We will now calculate how many ways the tourist can visit the towns independent of the order they were visited in.

3.3.1 Definition: Combination

A Combination of r items from a list of n is a choice of r items in no particular order. The number of permutations of r items from a list of n is

$$\frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)}{r \times (r-1) \times (r-2) \times \ldots \times 2 \times 1}.$$

There are r numbers being multiplied above the line and below the line in this calculation. This number is defined as ${}^nC_r$:

$${}^nC_r = \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)}{r \times (r-1) \times (r-2) \times \ldots \times 2 \times 1} = \frac{n!}{(n-r)!r!}.$$

It can be proved that this ratio will always give an integer value.

This value for the number of combinations can be explained as follows for the example of the towns. We need to find how many ways the towns can be visited when the order does not matter. The number ${}^6P_4$ was the number of ways to visit the towns in a specific order. This is now divided by the number of ways the 4 towns can be ordered. The 4 towns can be reordered $4 \times 3 \times 2 \times 1$ times, in other words 4!. Therefore the number of combinations is

$$\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = 15.$$

It can be written as

$$\frac{6!}{2!4!}.$$

This is the number ${}^6C_4$.

3.3.2 An Example from Cards

From a pack of 52 cards, 5 cards are to be dealt in a given hand. A hand is a set of 5 cards, irrespective of the order in which they were obtained, then this is equivalent to a combinations. Therefore we must calculate how many combinations of 5 cards can be dealt.

The answer, in the notation we have adopted in terms of the factorial notation, is:

$$^{52}C_5 = \frac{52!}{5! \times 47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}.$$

Again, the first term here is a definition, rather than a useful way to calculate this number. To carry out the calculation, there will always be cancellations that will reduce the ratio to a single integer:

$$^{52}C_5 = 52 \times 51 \times 10 \times 49 \times 2 = 2,598,960.$$

3.3.3 Example - Soccer Teams

A five-a-side soccer team is to be selected from a panel of 8 players. Calculate the number of possible teams.

Firstly, treat each possible team as a list of names in no particular order, in other words, a combination. The number of ways of selecting 5 players out of 8 is given by:

$$^{8}C_5 = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1}.$$

When all the numbers are cancelled down, this is 56.

It is notable that when we look at the calculation for $^{8}C_5$, we could cancel down the 4 and 5 to get the following expression:

$$^{8}C_5 = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}.$$

This is exactly the set-up for $^{8}C_3$. It reflects the fact that picking 5 players from 8 is equivalent to leaving out 3 players from the 8 and so the number of ways of picking 5 players from 8 is the same as the number of ways of leaving out 3 players:

$$^{8}C_5 =^{8} C_3.$$

This in turn is a reflection of the definition of the quantity $^{n}C_r$

$$^{n}C_r = \frac{n!}{(n-r)!r!}.$$

This definition shows that we can interchange the numbers r and $n-r$ and the value of the term will not change:

$$^nC_r = \frac{n!}{(n-r)!r!} \Rightarrow^n C_{n-r}$$

This in turn is a reflection of the definition of the quantity ^nC_r:

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This definition shows that we can interchange the numbers r and $n-r$ and the value of the term will not change:

$$^nC_r = \frac{n!}{(n-r)!r!} \Rightarrow^n C_{n-r}$$

So in this case:

$$^8C_5 = \frac{8!}{5! \times 3!} = \frac{8!}{3! \times 5!} =^8 C_3.$$

So far it has been assumed that the 5-a-side soccer team is just a list of five names. If the same five players, but in different positions, counts as a different team then each team is a permutation. The number of possible teams is then

$$^8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6,720.$$

3.3.4 Another Example of Combinations

A class consists of 10 left-handers and 15 right-handers. Calculate the probability that

1. If four names are chosen at random, all are right-handed.

2. If six names are chosen at random, half are right-handed.

Solution:

1. For this case, the sample space is the list of all combinations of 4 names from the 25. The number of outcomes is then $^{25}C_4$. The number of ways

of having all 4 people be right-handed is ${}^{15}C_4$. Therefore the probability of this event is:

$$\frac{{}^{15}C_4}{{}^{25}C_4} = \frac{1,365}{12,650} = 0.108.$$

By comparison, the probability that all 4 names were left-handers would be:

$$\frac{{}^{10}C_4}{{}^{25}C_4} = 0.0166.$$

This is considerably lower, as we would expect.

2. If six names are chosen at random, half are right-handed. Now the sample space is the set of all combinations of 6 names from the 25. The number of outcomes is ${}^{25}C_6$.

   The event here is that 3 right-handed and 3 left-handed people are picked. The probability of this event is

$$\frac{{}^{15}C_3 \times {}^{10}C_3}{{}^{25}C_6}.$$

   The result of this calculation is 0.31.

3.4 Lotteries

The definitions of permutations and combinations are used to calculate the odds for lotteries. In a lottery, the organiser has a mechanism to generate a selection of r numbers from a much larger set of n numbers. A participant chooses their own selection of r numbers and wins a prize depending on how many of the r numbers they match the selection generated by the organiser. Typically a Jackpot is awarded for matching all of the r numbers.

The probability of winning the jackpot will be the number of combinations which satisfy the criteria for the event, which is 1, divided by the number of possible combinations of the r numbers, which is ${}^{n}C_r$.

3.4.1 An Example of a Lottery

A small club is running a lottery. They give one prize for picking 4 correct numbers out of 20. Calculate the probability of winning.

To reiterate, the experiment here is selecting the 4 numbers at random. The sample space is the full list of selections of 4 numbers at a time from the 20. The event is the selection of 4 specific numbers. Let symbol W denote the event of winning this prize. The number of ways of selecting 4 numbers out of 20 is given by

$$^{20}C_4,$$

so the probability of winning this prize is

$$P[W] = \frac{1}{^{20}C_4}.$$

To evaluate this number:

$$^{20}C_4 = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845.$$

Therefore the probability of winning such a lottery is $1/4845$.

This result could also be produced by looking at it as a problem in conditional probabilities. The event of winning the lottery prize is the following sequence of dependent events together:

The first number is correct *and* The second is correct given the first was *and* The third is correct given the first and second were *and* The fourth is correct given the first, second and third were.

Calculate these conditional probabilities and multiply them according to the rule for dependent events:

$$P[W] = \frac{4}{20}\frac{3}{19}\frac{2}{18}\frac{1}{17} = \frac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}.$$

To see that this is giving the same answer as before, note that

$$\frac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17} = \frac{1}{\frac{20\times19\times18\times17}{4\times3\times2\times1}} = \frac{1}{^{20}C_4}.$$

The two answers are the same.

Returning to the Club lottery, the probability for picking 4 numbers from 24 would be

$$P[W] = \frac{1}{{}^{24}C_4}.$$

To evaluate this number:

$${}^{24}C_4 = \frac{24 \times 23 \times 22 \times 21}{4 \times 3 \times 2 \times 1} = 10,626.$$

For 28 numbers, the number of combinations is

$${}^{28}C_4 = \frac{28 \times 27 \times 26 \times 25}{4 \times 3 \times 2 \times 1} = 20,475.$$

In the case of large amount of numbers to select from, a second prize is often introduced to encourage participation. Consider the event of matching 3 numbers of the 4 in the last case of drawing from 28 numbers.

The sample space is still the full list of combinations of 4 numbers from the 28, so the number of outcomes is still the same; ${}^{24}C_4$. The number of ways in which the event can happen is the number of ways of choosing 3 of the 4 correct numbers and one of the incorrect 24 numbers.

$${}^{4}C_3 \times^{24} C_1$$

. Therefore the probability is

$$\frac{{}^{4}C_3 \times^{24} C_1}{{}^{28}C_4}.$$

This probability will be significantly higher than that of the jackpot of matching all four numbers.