

Assignment 15

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Download the latex-tikz codes from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_15

1 PROBLEM

(UGC-dec2018,106) :

Consider a Markov chain with transition probability matrix \mathbf{P} given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

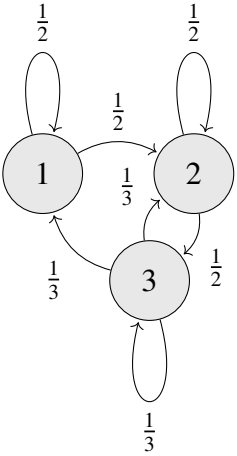
For any two states i and j , let p_{ij}^n denote the n -stp transition probability of going from i to j . Identify the correct statements

- 1) $\lim_{n \rightarrow \infty} p_{11}^n = \frac{2}{9}$
- 2) $\lim_{n \rightarrow \infty} p_{21}^n = 0$
- 3) $\lim_{n \rightarrow \infty} p_{32}^n = \frac{1}{3}$
- 4) $\lim_{n \rightarrow \infty} p_{13}^n = \frac{1}{3}$

2 DEFINITION AND RESULT USED

Irreducible Markov Chain	A Markov chain is irreducible if all the states communicate with each other, i.e., if there is only one communication class.
Aperiodic Markov Chain	If there is a self-transition in the chain ($p^{ii} > 0$ for some i), then the chain is called as aperiodic
Stationary Distribution	A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix \mathbf{P} , it satisfies $\pi = \pi \mathbf{P}$

3 SOLUTION

Drawing Transition diagram	 <pre> graph TD 1((1)) -- 1/2 --> 1 1 -- 1/2 --> 2((2)) 2 -- 1/2 --> 2 2 -- 1/3 --> 3((3)) 3 -- 1/3 --> 1 3 -- 1/2 --> 2 3 -- 1/3 --> 3 </pre>
Checking whether the chain is Irreducible and Aperiodic	<p>Here, All the states are accessible to one another. \Rightarrow They are in the same communication class. So, it is Irreducible.</p> <p>There exists the non- zero self-transition, which means that the chain is Aperiodic.</p> <p>We know that if the Markov Chain is irreducible and aperiodic then $\pi_j = \lim_{n \rightarrow \infty} P\{X_n = j\}, j = 1, \dots, N$ These are the stationary probabilities.</p>
Finding the Stationary Probability Distributions	<p>Stationary Probability can be represented as $\pi = \pi \mathbf{P}$</p> <p>$\Rightarrow (v_1 \quad v_2 \quad v_3) = (v_1 \quad v_2 \quad v_3) \mathbf{P}$</p> <p>Equating the above equation we get</p> $\frac{1}{2}v_1 - \frac{1}{3}v_3 = 0$ $\frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 = 0$ $\frac{1}{2}v_2 - \frac{2}{3}v_3 = 0$ <p>We see that summation of second and the third equation gives us the first equation only.</p> <p>And we know that the probability distribution will sum up to 1.</p> $v_1 + v_2 + v_3 = 1$

Therefore, we get the equation form as

$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{-1}{3} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving the linear equations

The above linear equation can be solved using Gauss-Jordan method as

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{-1}{3} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2} \\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & \frac{2}{3} & \frac{1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow \frac{3}{2}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{5}{3}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$$

\therefore , stationary probability distribution π is given by

$$\pi = \left(\frac{2}{9} \quad \frac{4}{9} \quad \frac{1}{3} \right)$$

Observations	<p>We will write the Stationary Probability distribution as</p> $\lim_{n \rightarrow \infty} p^n = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ $\lim_{n \rightarrow \infty} p_{11}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} p_{21}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} p_{32}^n = \frac{4}{9}$ $\lim_{n \rightarrow \infty} p_{13}^n = \frac{1}{3}$
Conclusion	<p>From our observation we see that</p> <p>Options 1) and 4) are True.</p>