Matrix Theory EE5609
Assignment 11

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***Abstract*—This document solves a problem of linear combinations.**

All the codes for the figure in this document can be found at

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https://github.com/vimalkb007/EE5609/tree/master/Assignment_11
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1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with rank r. If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

1) $m = r$
2) the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$
3) the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$
4) $m \geq n$ implies $m = n$

2 Solution

If the columns of an $m \times n$ matrix $\mathbf{A}$ span $\mathbf{R}^m$ then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of $\mathbf{A}$ is defined to be

$$Null(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \,|\, \mathbf{Ax} = 0\} \tag{2.0.1}$$

Let $\mathbf{A}$ be given as

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \tag{2.0.2}$$

Reduced Row Echelon form is

$$RREF\,(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$\therefore$ the only possible nullspace of the matrix $\mathbf{A}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\mathbf{B}$ be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \tag{2.0.4}$$

Reduced Row Echelon form is

$$RREF\,(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.5}$$

$\therefore$ the rank of matrix $\mathbf{B} = 3$.

Options	Observations
$m = r$	The rank of any matrix **A** is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of $\mathbf{R}^m$. $\therefore$ This statement is **True**.
the column space of **A** is a proper subspace of $\mathbf{R}^m$	Any subspace of a vector space **V** other than **V** itself is considered a proper subspace of **V**. Which means that linear combination of **A** will span less than m. That will make the resultant **b** span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting. $\therefore$ This statement is **False**.
the null space of **A** is a non-trivial subsapce of $\mathbf{R}^n$ whenever $m = n$	From (2.0.2) we see that even when $m = n$ then also we are getting a trivial nullspace. $\therefore$ This statement is **False**.
$m \geq n$ implies $m = n$	When $m \geq n$, then number of rows will become greater than columns. And it is given that there exists a solution. Which implies that the rows will be dependent. From (2.0.4) we see that rank will be equal to n. And the **b** will span in $\mathbf{R}^n$. $\therefore$ This statement is **True**.