EE5609: Matrix Theory Assignment 12

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Abstract—**This document explains the relation between linear operators, and diagonalizability.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_12

1 Problem

Let $\mathbf{T}$ be the linear operator on $\mathbf{R}^4$ which is represented in the standard basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$$

Under what conditions on a, b and c in $\mathbf{T}$ is diagonalizable?

2 Theorem

Theorem 2.1. *A linear operator $\mathbf{T}$ on a $n-$ dimensional space $\mathbf{V}$ is diagonalizable, if and only if $\mathbf{T}$ has an n distinct characteristic vectors (or) null spaces corresponding to the characteristic values.*

Theorem 2.2. *Let $\mathbf{T}$ be a linear operator on a finite-dimensional space $\mathbf{V}$. Let $c_1, c_2, ..., c_k$ be the distinct characteristic values of $\mathbf{T}$ and let $\mathbf{W_i}$ be the null space of $(\mathbf{T} - c_i\mathbf{I})$. The following are equivalent:*

1) *$\mathbf{T}$ is diagonizable*
2) *$dim\ \mathbf{W_1} + ... + dim\ \mathbf{W_k} = dim\ \mathbf{V}$*

3 Solution

Let the given matrix be,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \tag{3.0.1}$$

As per theorem 2.1, we need to find the characteristic polynomial for the matrix $\mathbf{A}$. Characteristic equation is given by $det\,(x\mathbf{I} - \mathbf{A})$.

$$det\,(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x-0 & 0 & 0 & 0 \\ -a & x-0 & 0 & 0 \\ 0 & -b & x-0 & 0 \\ 0 & 0 & -c & x-0 \end{vmatrix} \tag{3.0.2}$$

$$det\,(x\mathbf{I} - \mathbf{A}) = x^4 \tag{3.0.3}$$

The characteristic equation will be,

$$det\,(x\mathbf{I} - \mathbf{A}) = 0 \tag{3.0.4}$$

$$x^4 = 0 \tag{3.0.5}$$

From (3.0.5) we get the characteristic value as $c_1 = 0$ with a multiplicity of 4.

The basis for the characteristic value $c_1 = 0$ can be obtained by solving the equation

$$(\mathbf{A} - c_1\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.0.6}$$

i.e.

$$(\mathbf{A} - (0)\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.0.7}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{0} \tag{3.0.8}$$

Solving the above equation we get

$$ax = 0, \;\; by = 0, \;\; cz = 0 \tag{3.0.9}$$

$$\tag{3.0.10}$$

From theorem 2.2, for $\mathbf{T}$ to be diagonalizable, the null space $\mathbf{W_1}$ of $\mathbf{A}$ has $dim\ \mathbf{W_1} = 4$, which is only possible if in equation (3.0.9) we get $a = 0$, $b = 0$ and $c = 0$.

$\therefore$ **A** is diagonalizable only if

$$a = b = c = 0 \tag{3.0.11}$$

i.e. **A** is a zero matrix.