Assignment 15

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Download the latex-tikz codes from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_15

1 Problem

(UGC-dec2018,106) :
Consider a Markov chain with transition probability matrix **P** given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

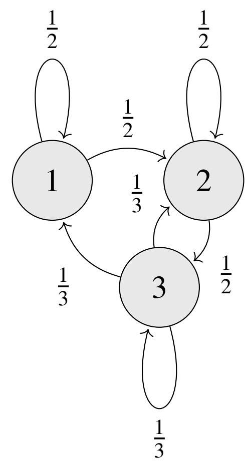
For any two states i and j, let $\mathbf{P}_{ij}^n$ denote the n-step transition probability of going from i to j. Identify the correct statements

1) $\lim_{n\to\infty} \mathbf{P}_{11}^n = \frac{2}{9}$
2) $\lim_{n\to\infty} \mathbf{P}_{21}^n = 0$
3) $\lim_{n\to\infty} \mathbf{P}_{32}^n = \frac{1}{3}$
4) $\lim_{n\to\infty} \mathbf{P}_{13}^n = \frac{1}{3}$

2 Definition and Result used

Irreducible Markov Chain	A Markov chain is **irreducible** if all the states communicate with each other, i.e., if there is only one communication class.
Aperiodic Markov Chain	If there is a self-transition in the chain ($p^{ii} > 0$ for some i), then the chain is called as **aperiodic**
Stationary Distribution	A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix **P**, it satisfies $$\pi = \pi\mathbf{P}$$

3 Solution

Drawing Transition diagram	
Checking whether the chain is Irreducible and Aperiodic	Here, All the states are accessible to one another. $\implies$ They are in the same communication class. So, it is Irreducible. There exists the non- zero self-transition, which means that the chain is Aperiodic. We know that if the Markov Chain is irreducible and aperiodic then $\pi_j = \lim_{n\to\infty} P\{X_n = j\},\ j = 1, ..., N$ These are the stationary probabilities.
Finding the Stationary Probability Distributions	Stationary Probability can be represented as $\pi = \pi\mathbf{P}$ $\implies \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\mathbf{P}$ Equating the above equation we get $\frac{1}{2}v_1 - \frac{1}{3}v_3 = 0$ $\frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 = 0$ $\frac{1}{2}v_2 - \frac{2}{3}v_3 = 0$ We see that summation of second and the third equation gives us the first equation only. And we know that the probability distribution will sum up to 1. $v_1 + v_2 + v_3 = 1$

	Therefore, we get the equation form as $$\begin{pmatrix}1 & 1 & 1\\ \frac{1}{2} & 0 & \frac{-1}{3}\\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$
Solving the linear equtions	The above linear equation can be solved using Gauss-Jordan method as $$\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ \frac{1}{2} & 0 & \frac{-1}{3} & 0\\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0\end{array}\right)$$ $$\xleftrightarrow{R_2\leftarrow R_2-\frac{1}{2}R_1}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2}\\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0\end{array}\right)$$ $$\xleftrightarrow{R_3\leftarrow R_3-\frac{1}{2}R_1}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2}\\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2}\end{array}\right)$$ $$\xleftrightarrow{R_2\leftarrow \frac{-1}{2}R_2}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & 1 & \frac{5}{3} & 1\\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2}\end{array}\right)$$ $$\xleftrightarrow{R_3\leftarrow R_3+R_2}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & 1 & \frac{5}{3} & 1\\ 0 & 0 & \frac{3}{2} & \frac{1}{2}\end{array}\right)$$ $$\xleftrightarrow{R_3\leftarrow \frac{3}{2}R_3}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & 1 & \frac{5}{3} & 1\\ 0 & 0 & 1 & \frac{1}{3}\end{array}\right)$$ $$\xleftrightarrow{R_2\leftarrow R_2-\frac{5}{3}R_3}\left(\begin{array}{ccc\|c}1 & 1 & 1 & 1\\ 0 & 1 & 0 & \frac{4}{9}\\ 0 & 0 & 1 & \frac{1}{3}\end{array}\right)$$ $$\xleftrightarrow{R_1\leftarrow R_1-R_3}\left(\begin{array}{ccc\|c}1 & 1 & 0 & \frac{2}{3}\\ 0 & 1 & 0 & \frac{4}{9}\\ 0 & 0 & 1 & \frac{1}{3}\end{array}\right)$$ $$\xleftrightarrow{R_1\leftarrow R_1-R_2}\left(\begin{array}{ccc\|c}1 & 0 & 0 & \frac{2}{9}\\ 0 & 1 & 0 & \frac{4}{9}\\ 0 & 0 & 1 & \frac{1}{3}\end{array}\right)$$ $\therefore$, stationary probability distribution π is given by $$\pi=\begin{pmatrix}\frac{2}{9} & \frac{4}{9} & \frac{1}{3}\end{pmatrix}$$

Observations	Since the given transition probability matrix $\mathbf{P}$ is irreducible and aperiodic, then $\lim_{n\to\infty} \mathbf{P}^n$ converges to a matrix with all rows identical and equal to π. We were able to find π as $\begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ $\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ From the above matrix, we get $\lim_{n\to\infty} \mathbf{P}^n_{11} = \frac{2}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{21} = \frac{2}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{32} = \frac{4}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{13} = \frac{1}{3}$
Conclusion	From our observation we see that Options 1) and 4) are True.