Assignment-6

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***Abstract*—This document explains how to factorize a matrix using QR decomposition.**

Download the latex-tikz codes from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_QR

1 Problem

Perform QR decomposition of $\begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$

2 Explanation

Let $\mathbf{a}$ and $\mathbf{b}$ be columns of a $\mathbf{V}$. Then, the matrix $\mathbf{V}$ can be decomposed in the form as:

$$\mathbf{V} = \mathbf{QR} \tag{2.0.1}$$

such that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.4}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.5}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{2.0.7}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.0.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.0.9}$$

Then, the matrix can be represented as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.10}$$

3 Solution

$\mathbf{V} = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$ where

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{3.0.2}$$

Now, from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{2} \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.0.4}$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}}{\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|^2} = \sqrt{2} \tag{3.0.5}$$

$$\tag{3.0.6}$$

Now, by (2.0.8)

$$\mathbf{u_2} = \frac{\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \sqrt{2}\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \sqrt{2}\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.0.7}$$

From (2.0.9),

$$k_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{8}{\sqrt{2}} \tag{3.0.8}$$

Verification for $\mathbf{Q}^T\mathbf{Q} = I$
Now,

$$\mathbf{Q} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.0.9}$$

Now, we observe that $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.10)$$

Now, by (2.0.1) we can write matrix $\mathbf{V}$ as

$$\begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \frac{8}{\sqrt{2}} \end{pmatrix} \quad (3.0.11)$$

which is the required $\mathbf{QR}$ decomposition of $\mathbf{V}$.