Matrix Theory EE5609 Assignment 11

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***Abstract*—This document solves a problem of linear combinations.**

All the codes for the figure in this document can be found at

https://github.com/vimalkb007/EE5609/tree/master/Assignment_11

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with rank r. If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

1) $m = r$
2) the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$
3) the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$
4) $m \geq n$ implies $m = n$

2 Theorem

Theorem 2.1. *Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:*

1) ***Unique Solution****: If rank[A,b] = rank(A) = $n \leq m$, then and only then the system has a unique solution. In this case, indeed as many as $m - n$ equations are redundant. And the solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$. This is called as* ***Exactly Determined****.*
2) ***No Solution****: If rank[A,b] > rank(A) which necessarily implies $\mathbf{b} \neq 0$ and $m >$ rank(A), then and only then the system has no solution. This is called as* ***Overdetermined****.*

3 Solution

If the columns of an $m \times n$ matrix $\mathbf{A}$ span $\mathbf{R}^m$ then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of $\mathbf{A}$ is defined to be

$$Null(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} = 0\} \tag{3.0.1}$$

Let $\mathbf{A}$ be given as

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \tag{3.0.2}$$

Reduced Row Echelon form is

$$RREF\left(\mathbf{A}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.3}$$

$\therefore$ the only possible nullspace of the matrix $\mathbf{A}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\mathbf{B}$ be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \tag{3.0.4}$$

Reduced Row Echelon form is

$$RREF\left(\mathbf{B}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.5}$$

$\therefore$ the rank of matrix $\mathbf{B} = 3$.

Options	Observations
$m = r$	The rank of any matrix **A** is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of $\mathbf{R}^m$. $\therefore$ This statement is **True**.
the column space of **A** is a proper subspace of $\mathbf{R}^m$	Any subspace of a vector space **V** other than **V** itself is considered a proper subspace of **V**. Which means that linear combination of **A** will span less than m. That will make the resultant **b** span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting. $\therefore$ This statement is **False**.
the null space of **A** is a non-trivial subsapce of $\mathbf{R}^n$ whenever $m = n$	From (3.0.2) we see that even when $m = n$ then also we are getting a trivial nullspace. $\therefore$ This statement is **False**.
$m \geq n$ implies $m = n$	It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (2.1) we see that the corresponding system will be **Exactly Determined** system. As an example, it will look like (3.0.4). $\therefore$ This statement is **True**.