Assignment 16

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Download the latex-tikz codes from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_16

1 Problem

(UGC-june2015,77) :
Consider non-zero vector spaces $\mathbf{V_1}, \mathbf{V_2}, \mathbf{V_3}, \mathbf{V_4}$ and linear transformations $\phi_1 : \mathbf{V_1} \to \mathbf{V_2}$, $\phi_2 : \mathbf{V_2} \to \mathbf{V_3}$, $\phi_3 : \mathbf{V_3} \to \mathbf{V_4}$ such that $Ker(\phi_1) = \{0\}$, $Range(\phi_1) = Ker(\phi_2)$, $Range(\phi_2) = Ker(\phi_3)$, $Range(\phi_3) = \mathbf{V_4}$. Then

1) $\sum_{i=1}^{4} (-1)^i \ dim \ \mathbf{V_i} = 0$
2) $\sum_{i=2}^{4} (-1)^i \ dim \ \mathbf{V_i} > 0$
3) $\sum_{i=1}^{4} (-1)^i \ dim \ \mathbf{V_i} < 0$
4) $\sum_{i=1}^{4} (-1)^i \ dim \ \mathbf{V_i} \neq 0$

2 Definition and Result used

Kernel and Nullity	Given a linear transformation $L : \mathbf{V} \to \mathbf{W}$ between wo vector spaces $\mathbf{V}$ and $\mathbf{W}$, the kernel of L is the set of all vectors $\mathbf{v}$ of $\mathbf{V}$ for which $L(\mathbf{v}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $\mathbf{W}$. i.e. $$Ker(L) = \{\mathbf{v} \in \mathbf{V} \mid L(\mathbf{v}) = 0\}$$ Nullity of the linear transformation is the dimension of the kernel of the linear transformation i.e. $$nullity(L) = dim(Ker(L))$$
Range and Rank	Given a linear transformation $L : \mathbf{V} \to \mathbf{W}$ between wo vector spaces $\mathbf{V}$ and $\mathbf{W}$, the range of L is the set of all vectors $\mathbf{w}$ in $\mathbf{W}$ given as $$Range(L) = \{\mathbf{w} \in \mathbf{W} \mid \mathbf{w} = L(\mathbf{v}), \mathbf{v} \in \mathbf{V}\}$$ The rank of a linear transformation L is the dimension of it's range, i.e. $$rank(L) = dim(Range(L))$$

Rank-Nullity Theorem	Let $\mathbf{V}$, $\mathbf{W}$ be vector spaces, where $\mathbf{V}$ is finite dimensional. Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then $$rank(L) + nullity(L) = dim(\mathbf{V})$$

3 Solution

Inference from the Given Data	$Ker(\phi_1) = \{0\}$ $\implies nullity(\phi_1) = 0$ $Range(\phi_1) = Ker(\phi_2)$ $\implies rank(\phi_1) = nullity(\phi_2)$ $Range(\phi_2) = Ker(\phi_3)$ $\implies rank(\phi_2) = nullity(\phi_3)$ $Range(\phi_3) = \mathbf{V_4}$ $\implies rank(\phi_3) = dim(\mathbf{V_4})$ Now talking about the linear transformations we can use rank-nullity theorem to determine the corresponding dimensions of the vector space. $\phi_1 : \mathbf{V_1} \to \mathbf{V_2}$ $\implies rank(\phi_1) + nullity(\phi_1) = dim(\mathbf{V_1})$ $\implies rank(\phi_1) = dim(\mathbf{V_1}) \qquad (\because nullity(\phi_1) = 0)$ $\phi_2 : \mathbf{V_2} \to \mathbf{V_3}$ $\implies rank(\phi_2) + nullity(\phi_2) = dim(\mathbf{V_2})$ $\implies rank(\phi_2) + rank(\phi_1) = dim(\mathbf{V_2}) \qquad (\because rank(\phi_1) = nullity(\phi_2))$ $\implies rank(\phi_2) + dim(\mathbf{V_1}) = dim(\mathbf{V_2}) \qquad (\because rank(\phi_1) = dim(\mathbf{V_1}))$ $\phi_3 : \mathbf{V_3} \to \mathbf{V_4}$ $\implies rank(\phi_3) + nullity(\phi_3) = dim(\mathbf{V_3})$

	$\implies rank(\phi_3) + rank(\phi_2) = dim(\mathbf{V_3}) \quad (\because rank(\phi_2) = nullity(\phi_3))$ $\implies rank(\phi_3) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) = dim(\mathbf{V_3}) \; (\because rank(\phi_2) + dim(\mathbf{V_1}) = dim(\mathbf{V_2}))$ $\implies dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) = dim(\mathbf{V_3}) \quad (\because rank(\phi_3) = dim(\mathbf{V_4}))$ From the above equation we can infer that $dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$
Option 1	It is given that $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} = 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) = 0$ This statement we already proved above. $\therefore$ this statement is **True**.
Option 2	It is given that $\sum_{i=2}^{4} (-1)^i \; dim \; \mathbf{V_i} > 0$ $\implies dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) > 0$ Our original derived equation is $dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$ $\implies dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) = dim(\mathbf{V_1})$ It is given in the question that the vector spaces are non-zero in nature. $\implies dim(\mathbf{V_1}) > 0$ $\therefore dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) > 0$ $\therefore$ this statement is **True**.
Option 3	It is given that $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} < 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) < 0$ This is contrary to our original derived equation i.e. $dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$

	$\therefore$ this statement is **False**.
Option 4	It is given that $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} \neq 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) \neq 0$ This is contrary to our original derived equation i.e. $$dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$$ $\therefore$ this statement is **False**.
Conclusion	From our observation we see that Options 1) and 2) are True.

4 Example

Linear Transforms Example	Let $\phi_1 : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ defined as $$\phi_1 \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{pmatrix}$$ $$\implies \phi_1 \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ For the above transformation ϕ_1 the kernel and the range are $$Ker(\phi_1) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \qquad \implies nullity(\phi_1) = 0$$ $$Range(\phi_1) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \implies rank(\phi_1) = 2$$ We can verify the rank-nullity theorem here as $$\begin{aligned} & nullity(\phi_1) + rank(\phi_1) \\ \implies & 0 + 2 \\ \implies & 2 = dim(\mathbf{R}^2) \end{aligned}$$

Let $\phi_2 : \mathbf{R}^3 \to \mathbf{R}^3$ defined as

$$\phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 - 2x_2 + 4x_3 \\ 3x_1 - 3x_2 + 6x_3 \end{pmatrix}$$

$$\implies \phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For the above transformation ϕ_2 the kernel and the range are

$$Ker(\phi_2) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \implies nullity(\phi_2) = 2$$

$$Range(\phi_2) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \qquad \implies rank(\phi_2) = 1$$

We can verify the rank-nullity theorem here as

$$\begin{aligned} & nullity(\phi_2) + rank(\phi_2) \\ \implies & 2 + 1 \\ \implies & 3 = dim(\mathbf{R}^3) \end{aligned}$$

In the above two transformations ϕ_1 and ϕ_2, we can see the following conditions being satisfied

$$Ker(\phi_1) = \{0\}, \; Range(\phi_1) = Ker(\phi_2)$$

Let $\phi_3 : \mathbf{R}^3 \to \mathbf{R}^2$ defined as

$$\phi_3 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} x_1 + x_2 - x_3 \\ 2x_1 + \frac{1}{2}x_2 - x_3 \end{pmatrix}$$

$$\implies \phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For the above transformation ϕ_3 the kernel and the range are

$$Ker(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \qquad \implies nullity(\phi_3) = 1$$

$$Range(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\} \qquad \implies rank(\phi_3) = 2$$

	We can verify the rank-nullity theorem here as $$\begin{aligned} & nullity(\phi_3) + rank(\phi_3) \\ \implies & 1 + 2 \\ \implies & 3 = dim(\mathbf{R}^3) \end{aligned}$$ With the above ϕ_3 transformation we were able to satisfy the other conditions as well i.e. $$Range(\phi_2) = Ker(\phi_3), \ Range(\phi_3) = \mathbf{V_4}$$ Now, when we can check whether the derived equation statisfies or not. That is, $$\begin{aligned} & -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) \\ \implies & -dim(\mathbf{R}^2) + dim(\mathbf{R}^3) - dim(\mathbf{R}^3) + dim(\mathbf{R}^2) \\ \implies & -2 + 3 - 3 + 2 = 0 \end{aligned}$$ $\therefore$ the condition is getting satisfied.