EE5609: Matrix Theory Assignment 14

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Abstract—**This document proves the property of projection.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_14

1 Problem

Prove that if $\mathbf{E}$ is the projection on $\mathbf{R}$ along $\mathbf{N}$, then $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$

2 Theorem

Theorem 2.1. *If $\mathbf{V}$ is a vector space, a projection of $\mathbf{V}$ is a linear operator $\mathbf{E}$ on $\mathbf{V}$ such that $\mathbf{E}^2 = \mathbf{E}$. Let $\mathbf{R}$ be the range and let $\mathbf{N}$ be the nullspace of $\mathbf{E}$. Then the vector space $\mathbf{V}$ can be written as $\mathbf{V} = \mathbf{R} \bigoplus \mathbf{N}$. This operator is called as projection on $\mathbf{R}$ along $\mathbf{N}$.*

3 Solution

It is given that $\mathbf{E}$ is the projection. From thorem 2.1, the linear operator $\mathbf{E}$ will satisfy $\mathbf{E}^2 = \mathbf{E}$. Let's check whether $\mathbf{I}-\mathbf{E}$ is also a projection.

$$\begin{aligned}(\mathbf{I}-\mathbf{E})^2 &= \mathbf{I}^2 + \mathbf{E}^2 - 2\mathbf{I}\mathbf{E}\\ &= \mathbf{I} + \mathbf{E} - 2\mathbf{E}\\ &= (\mathbf{I}-\mathbf{E})\end{aligned} \tag{3.0.1}$$

From (3.0.1), we can say that $(\mathbf{I}-\mathbf{E})$ is also a projector. But $(\mathbf{I}-\mathbf{E})$ is called as the "Complementary Projector", i.e.

$$\mathit{range}(\mathbf{I}-\mathbf{E}) = \mathit{null}(\mathbf{E}) \tag{3.0.2}$$

$$\mathit{null}(\mathbf{I}-\mathbf{E}) = \mathit{range}(\mathbf{E}) \tag{3.0.3}$$

Lets take a vector $\mathbf{v}$ such that $\mathbf{E}\mathbf{v} = 0$, where $\mathbf{v}$ is in the null space of $\mathbf{E}$. Then,

$$\begin{aligned}(\mathbf{I}-\mathbf{E})\mathbf{v} &= \mathbf{v} - \mathbf{v}\mathbf{E}\\ &= \mathbf{v}\end{aligned} \tag{3.0.4}$$

In other words, any $\mathbf{v}$ in the nullspace of $\mathbf{E}$ is also in the range of $(\mathbf{I}-\mathbf{E})$. We know that any $\mathbf{x} \in \mathit{range}(\mathbf{I}-\mathbf{E})$ is characterized by

$$\begin{aligned}\mathbf{x} &= (\mathbf{I}-\mathbf{E})\mathbf{v} \quad \textit{, for some } \mathbf{v}\\ &= \mathbf{v} - \mathbf{E}\mathbf{v}\\ &= -\ (\mathbf{E}\mathbf{v} - \mathbf{v})\end{aligned} \tag{3.0.5}$$

Now we need to check if $\mathbf{x}$ is in the nullspace of $\mathbf{E}$. i.e. $\mathbf{E}\mathbf{x} = 0$

$$\begin{aligned}\mathbf{E}(-\ (\mathbf{E}\mathbf{v} - \mathbf{v})) &= -(\mathbf{E}^2\mathbf{v} - \mathbf{E}\mathbf{v})\\ &= -(\mathbf{E}\mathbf{v} - \mathbf{E}\mathbf{v}) \quad (\because \mathbf{E} \textit{ is a projection})\\ &= 0\end{aligned} \tag{3.0.6}$$

Thus, if $\mathbf{x} \in \mathit{range}(\mathbf{I}-\mathbf{E})$, then $\mathbf{x} \in \mathit{null}(\mathbf{E})$.

Therefore, we can say that $\mathit{null}(\mathbf{E}) = \mathit{range}(\mathbf{I}-\mathbf{E})$.

We can use the same argument as above for proving (3.0.3), by taking $\mathbf{E} = \mathbf{I} - (\mathbf{I}-\mathbf{E})$.

$\therefore$ we can say that $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$.