EE5609 Assignment 2

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Abstract—**This assignment involves finding the angle betweem a given pair of lines**

The python solution code for this problem can be downloaded from

https://github.com/vimalkb007/EE5609/blob/master/Assignment_2/codes/assignment2_solution.py

The python verification code for this problem can be downloaded from

https://github.com/vimalkb007/EE5609/blob/master/Assignment_2/codes/assignment2_solution_verify.py

1 Problem Statement

Find the angle between the following pair of lines

1)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{1.0.1}$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{1.0.2}$$

2)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{1.0.3}$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{1.0.4}$$

2 Theory

Given two symmetric line equations we can represent them in the vector format.

Using the dot product of the vectors we can find the angle between the two lines. If we have two vectors $\mathbf{u}$, $\mathbf{v}$, the dot product is given by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \tag{2.0.1}$$

From the above 2.0.1 we get the angle between two vectors as

$$\theta = \cos^{-1} \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \tag{2.0.2}$$

3 Solution

From theory, we understand that using dot product we can find the angle between the lines

1)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{3.0.1}$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{3.0.2}$$

The above symmetric equations 3.0.1, 3.0.2 can be represented in the vector form as

$$\mathbf{r_1} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \tag{3.0.3}$$

$$\mathbf{r_2} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix} \tag{3.0.4}$$

As we have to find the angle between the vectors, we will only be taking the direction vectors into consideration. The direction vectors are $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}$. We can find the corresponding magnitude values

$$\|\mathbf{u}\| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38} \tag{3.0.5}$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 8^2 + 4^2} = \sqrt{81} \tag{3.0.6}$$

Using 2.0.2, 3.0.5, 3.0.6 we get

$$\theta = \cos^{-1} \frac{\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}^T \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}}{(\sqrt{38})(\sqrt{81})} \tag{3.0.7}$$

$$\theta = \cos^{-1} \frac{26}{55.4797} \tag{3.0.8}$$

$$\theta = \cos^{-1}(0.4686) \tag{3.0.9}$$

$$\theta = 62.053° \tag{3.0.10}$$

Therefore, the angle between the two lines is 62.053°.See Fig. 1

If we look at the direction vectors we can see that they are not scalar multiples of each other. That means the vectors are not parallel in nature. To check if the lines intersect we need to find if there is a pair value of λ_1 and λ_2, so that $\mathbf{r_1}(\lambda_1) = \mathbf{r_2}(\lambda_2)$.

From 3.0.3 an 3.0.4, we get

$$2 + 2\lambda_1 = -2 - \lambda_2 \implies 2\lambda_1 + \lambda_2 = -4 \tag{3.0.11}$$

$$1 + 5\lambda_1 = 4 + 8\lambda_2 \implies 5\lambda_1 - 8\lambda_2 = 3 \tag{3.0.12}$$

$$-3 - 3\lambda_1 = 5 + 4\lambda_2 \implies -3\lambda_1 - 4\lambda_2 = 8 \tag{3.0.13}$$

We take the two equations 3.0.11, 3.0.12and use Guass-Jordan Elimination Method in order to find the values of λ_1 and λ_2.

$$\left(\begin{array}{cc|c} 2 & 1 & -4 \\ 5 & -8 & 3 \end{array}\right) \tag{3.0.14}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - 5R_1} \left(\begin{array}{cc|c} 2 & 1 & -4 \\ 0 & -21 & 26 \end{array}\right) \tag{3.0.15}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-1}{21}R_2} \left(\begin{array}{cc|c} 2 & 1 & -4 \\ 0 & 1 & \frac{-26}{21} \end{array}\right) \tag{3.0.16}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} 2 & 0 & \frac{58}{21} \\ 0 & 1 & \frac{-26}{21} \end{array}\right) \tag{3.0.17}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{29}{21} \\ 0 & 1 & \frac{-26}{21} \end{array}\right) \tag{3.0.18}$$

We get the values of $\lambda_1 = \frac{29}{21}$ and $\lambda_2 = \frac{-26}{21}$. Checking if these values satisfy the equation 3.0.13.

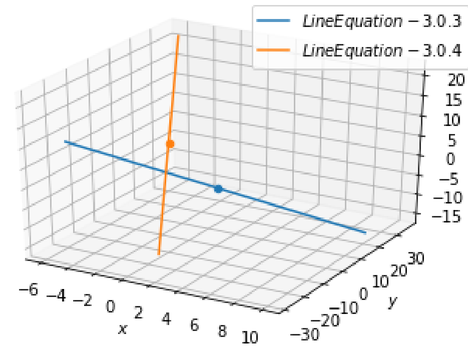

Fig. 1: Graph for equations 3.0.3 3.0.4

$$-3\left(\frac{29}{21}\right) - 4\left(\frac{-26}{21}\right) = \frac{17}{21} \tag{3.0.19}$$

$$\implies \frac{17}{21} \neq 8 \tag{3.0.20}$$

Therefore, the given pair of lines do not intersect.

2)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{3.0.21}$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{3.0.22}$$

The above symmetric equations 3.0.21, 3.0.22 can be represented in the vector form as

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \tag{3.0.23}$$

$$\mathbf{r_2} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix} \tag{3.0.24}$$

As we have to find the angle between the vectors, we will only be taking the direction vectors into consideration. The direction vectors are $\mathbf{u} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$. We can find the corresponding magnitude values

$$\|\mathbf{u}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} \tag{3.0.25}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} \tag{3.0.26}$$

Using 2.0.2, 3.0.25, 3.0.26 we get

$$\theta = \cos^{-1} \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}}{(\sqrt{9})(\sqrt{81})} \tag{3.0.27}$$

$$\theta = \cos^{-1} \frac{18}{27.00} \tag{3.0.28}$$

$$\theta = \cos^{-1}(0.667) \tag{3.0.29}$$

$$\theta = 48.189° \tag{3.0.30}$$

Therefore, the angle between the two lines is 48.189°. See Fig. 2

If we look at the direction vectors we can see that they are not scalar multiples of each other. That means the vectors are not parallel in nature. To check if the lines intersect we need to find if there is a pair value of λ_1 and λ_2, so that $\mathbf{r_1}(\lambda_1) = \mathbf{r_2}(\lambda_2)$.

From 3.0.23 an 3.0.24, we get

$$2\lambda_1 = 5 + 4\lambda_2 \implies 2\lambda_1 - 4\lambda_2 = 5 \tag{3.0.31}$$

$$2\lambda_1 = 2 + \lambda_2 \implies 2\lambda_1 - \lambda_2 = 2 \tag{3.0.32}$$

$$\lambda_1 = 3 + 8\lambda_2 \implies \lambda_1 - 8\lambda_2 = 3 \tag{3.0.33}$$

We take the two equations 3.0.31, 3.0.32and use Guass-Jordan Elimination Method in order to find the values of λ_1 and λ_2.

$$\left(\begin{array}{cc|c} 2 & -4 & 5 \\ 2 & -1 & 2 \end{array}\right) \tag{3.0.34}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 2 & -4 & 5 \\ 0 & 3 & -3 \end{array}\right) \tag{3.0.35}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{3} R_2} \left(\begin{array}{cc|c} 2 & -4 & 5 \\ 0 & 1 & -1 \end{array}\right) \tag{3.0.36}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 4R_2} \left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right) \tag{3.0.37}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \end{array}\right) \tag{3.0.38}$$

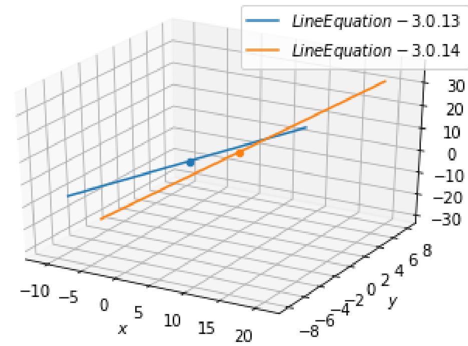

Fig. 2: Graph for equations 3.0.23 3.0.24

We get the values of $\lambda_1 = \frac{1}{2}$ and λ_2 = -1. Checking if these values satisfy the equation 3.0.33.

$$\left(\frac{1}{2}\right) - 8\,(1) = \frac{-15}{2} \tag{3.0.39}$$

$$\implies \frac{-15}{2} \neq 3 \tag{3.0.40}$$

Therefore, the given pair of lines do not intersect.