EE5609: Matrix Theory Assignment 9

Vimal K B
AI20MTECH12001

***Abstract*—This document explains the concept of vector space over complex numbers.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_9

1 Problem

Let $\mathbf{V}$ be the vector space over the complex numbers of all functions from $\mathbf{R}$ into $\mathbf{C}$, i.e. the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly dependent.

(b) Let $g_1(x) = 1$, $g_2(x) = \cos(x)$, $g_3(x) = \sin(x)$. Find an invertible 3×3 matrix P such that $g_i = \sum_{i=1}^{3} P_{ii} f_i$

2 Solution

(a) To check for independence, lets represent the function in a polynomial format as

$$\alpha f_1 + \beta f_2 + \gamma f_3 = 0 \tag{2.0.1}$$
$$\alpha + \beta e^{ix} + \gamma e^{-ix} = 0 \tag{2.0.2}$$

Multiply the whole equation with e^{ix} to get $\beta(e^{ix})^2 + \alpha e^{ix} + \gamma = 0$.

Let $y = e^{ix}$, which makes the equation as $\beta y^2 + \alpha y + \gamma = 0$. The above quadratic polynomial in y can be zero for atmost two values of y. But $y = e^{ix}$, and e^{ix} takes infinitely many different values as x varies in $\mathbf{R}$. So (2.0.2) cannot be zero for all $y = e^{ix}$. Which implies there is a contradiction.

Then the only case of $\alpha = \beta = \gamma = 0$, can satisfy (2.0.2). Therefore, f_1, f_2, f_3 are linearly independent.

(b) We need to find the coordinates of vectors g_i where $i = 1, 2, 3$ in ordered basis

$$B = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \tag{2.0.3}$$

It is given that $g_1 = 1$, which can be written as

$$g_1 = f_1 \tag{2.0.4}$$
$$\implies g_1 = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.5}$$

We can use the following identities:-

$$\cos(x) = \frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix} \tag{2.0.6}$$
$$\sin(x) = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix} \tag{2.0.7}$$

Comparing equations (2.0.6) and (2.0.7) with f_2, f_3, we can write g_2 and g_3 as

$$g_2 = \frac{1}{2} f_2 + \frac{1}{2} f_3 \tag{2.0.8}$$
$$\implies g_2 = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{2.0.9}$$
$$g_3 = \frac{1}{2i} f_2 - \frac{1}{2i} f_3 \tag{2.0.10}$$
$$\implies g_3 = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2i} \\ \frac{-1}{2i} \end{pmatrix} \tag{2.0.11}$$

Therefore, the required matrix P is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & \frac{-1}{2i} \end{pmatrix} \tag{2.0.12}$$