

# Assignment 15

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Download the latex-tikz codes from

[https://github.com/vimalkb007/EE5609/tree/master/Assignment\\_16](https://github.com/vimalkb007/EE5609/tree/master/Assignment_16)

## 1 PROBLEM

(UGC-june2015,77) :

Consider non-zero vector spaces  $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$  and linear transformations  $\phi_1 : \mathbf{V}_1 \rightarrow \mathbf{V}_2$ ,  $\phi_2 : \mathbf{V}_2 \rightarrow \mathbf{V}_3$ ,  $\phi_3 : \mathbf{V}_3 \rightarrow \mathbf{V}_4$  such that  $\text{Ker}(\phi) = \{0\}$ ,  $\text{Range}(\phi_1) = \text{Ker}(\phi_2)$ ,  $\text{Range}(\phi_2) = \text{Ker}(\phi_3)$ ,  $\text{Range}(\phi_3) = \mathbf{V}_4$ . Then

- 1)  $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i = 0$
- 2)  $\sum_{i=2}^4 (-1)^i \dim \mathbf{V}_i > 0$
- 3)  $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i < 0$
- 4)  $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i \neq 0$

## 2 DEFINITION AND RESULT USED

Kernel and Nullity	<p>Given a linear transformation <math>L : \mathbf{V} \rightarrow \mathbf{W}</math> between two vector spaces <math>\mathbf{V}</math> and <math>\mathbf{W}</math>, the kernel of <math>L</math> is the set of all vectors <math>\mathbf{v}</math> of <math>\mathbf{V}</math> for which <math>L(\mathbf{v}) = \mathbf{0}</math>, where <math>\mathbf{0}</math> denotes the zero vector in <math>\mathbf{W}</math>. i.e.</p> $\text{Ker}(L) = \{\mathbf{v} \in \mathbf{V} \mid L(\mathbf{v}) = \mathbf{0}\}$ <p>Nullity of the linear transformation is the dimension of the kernel of the linear transformation i.e.</p> $\text{nullity}(L) = \dim(\text{Ker}(L))$
Range and Rank	<p>Given a linear transformation <math>L : \mathbf{V} \rightarrow \mathbf{W}</math> between two vector spaces <math>\mathbf{V}</math> and <math>\mathbf{W}</math>, the range of <math>L</math> is the set of all vectors <math>\mathbf{w}</math> in <math>\mathbf{W}</math> given as</p> $\text{Range}(L) = \{\mathbf{w} \in \mathbf{W} \mid \mathbf{w} = L(\mathbf{v}), \mathbf{v} \in \mathbf{V}\}$ <p>The rank of a linear transformation <math>L</math> is the dimension of its range, i.e.</p> $\text{rank}(L) = \dim(\text{Range}(L))$

Rank-Nullity Theorem	<p>Let <math>\mathbf{V}</math>, <math>\mathbf{W}</math> be vector spaces, where <math>\mathbf{V}</math> is finite dimensional. Let <math>L : \mathbf{V} \rightarrow \mathbf{W}</math> be a linear transformation. Then</p> $\text{rank}(L) + \text{nullity}(L) = \dim(\mathbf{V})$
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### 3 SOLUTION

Inference from the Given Data	<p><math>\text{Ker}(\phi_1) = \{0\}</math></p> <p><math>\implies \text{nullity}(\phi_1) = 0</math></p> <p><math>\text{Range}(\phi_1) = \text{Ker}(\phi_2)</math></p> <p><math>\implies \text{rank}(\phi_1) = \text{nullity}(\phi_2)</math></p> <p><math>\text{Range}(\phi_2) = \text{Ker}(\phi_3)</math></p> <p><math>\implies \text{rank}(\phi_2) = \text{nullity}(\phi_3)</math></p> <p><math>\text{Range}(\phi_3) = \mathbf{V}_4</math></p> <p><math>\implies \text{rank}(\phi_3) = \dim(\mathbf{V}_4)</math></p> <p>Now talking about the linear transformations we can use rank-nullity theorem to determine the corresponding dimensions of the vector space.</p> <p><math>\phi_1 : \mathbf{V}_1 \rightarrow \mathbf{V}_2</math></p> <p><math>\implies \text{rank}(\phi_1) + \text{nullity}(\phi_1) = \dim(\mathbf{V}_1)</math></p> <p><math>\implies \text{rank}(\phi_1) = \dim(\mathbf{V}_1) \quad (\because \text{nullity}(\phi_1) = 0)</math></p> <p><math>\phi_2 : \mathbf{V}_2 \rightarrow \mathbf{V}_3</math></p> <p><math>\implies \text{rank}(\phi_2) + \text{nullity}(\phi_2) = \dim(\mathbf{V}_2)</math></p> <p><math>\implies \text{rank}(\phi_2) + \text{rank}(\phi_1) = \dim(\mathbf{V}_2) \quad (\because \text{rank}(\phi_1) = \text{nullity}(\phi_2))</math></p> <p><math>\implies \text{rank}(\phi_2) + \dim(\mathbf{V}_1) = \dim(\mathbf{V}_2) \quad (\because \text{rank}(\phi_1) = \dim(\mathbf{V}_1))</math></p> <p><math>\phi_3 : \mathbf{V}_3 \rightarrow \mathbf{V}_4</math></p> <p><math>\implies \text{rank}(\phi_3) + \text{nullity}(\phi_3) = \dim(\mathbf{V}_3)</math></p>
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	$\begin{aligned} \Rightarrow \text{rank}(\phi_3) + \text{rank}(\phi_2) &= \dim(\mathbf{V}_3) & (\because \text{rank}(\phi_2) = \text{nullity}(\phi_3)) \\ \Rightarrow \text{rank}(\phi_3) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) &= \dim(\mathbf{V}_3) & (\because \text{rank}(\phi_2) + \dim(\mathbf{V}_1) = \dim(\mathbf{V}_2)) \\ \Rightarrow \dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) &= \dim(\mathbf{V}_3) & (\because \text{rank}(\phi_3) = \dim(\mathbf{V}_4)) \end{aligned}$ <p>From the above equation we can infer that</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$
Option 1	<p>It is given that</p> $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i = 0$ $\Rightarrow -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) = 0$ <p>This statement we already proved above.</p> <p><math>\therefore</math> this statement is <b>True</b>.</p>
Option 2	<p>It is given that</p> $\sum_{i=2}^4 (-1)^i \dim \mathbf{V}_i > 0$ $\Rightarrow \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) > 0$ <p>Our original derived equation is</p> $\begin{aligned} \dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) &= 0 \\ \Rightarrow \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) &= \dim(\mathbf{V}_1) \end{aligned}$ <p>It is given in the question that the vector spaces are non-zero in nature.</p> $\Rightarrow \dim(\mathbf{V}_1) > 0$ $\therefore \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) > 0$ <p><math>\therefore</math> this statement is <b>True</b>.</p>
Option 3	<p>It is given that</p> $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i < 0$ $\Rightarrow -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) < 0$ <p>This is contrary to our original derived equation i.e.</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$

	<p><math>\therefore</math> this statement is <b>False</b>.</p>
Option 4	<p>It is given that</p> $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i \neq 0$ $\implies -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) \neq 0$ <p>This is contrary to our original derived equation i.e.</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$ <p><math>\therefore</math> this statement is <b>False</b>.</p>
Conclusion	<p>From our observation we see that</p> <p>Options 1) and 2) are True.</p>