EE5609: Matrix Theory Assignment 10

Vimal K B
AI20MTECH12001

***Abstract*—This document explains the concept of linear operators, and significance of one-to-one and onto functions.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_10

1 Problem

Let $\mathbf{T}$ be a linear operator on the finite-dimensional space $\mathbf{V}$. Support there is a linear operator $\mathbf{U}$ on $\mathbf{V}$ such that $\mathbf{TU} = \mathbf{I}$. Prove that $\mathbf{T}$ is invertible and $\mathbf{U} = \mathbf{T}^{-1}$. Give an example which shows that this is false when $\mathbf{V}$ is not finite-dimensional.

2 Theorem

Theorem 2.1. *Let f be a function from X into Y. We say that f is invertible if there is a function g from Y to X such that*

1) *$g \circ f$ is the identity function on X i.e. $g \circ f = I$. Here, g will be onto and f will be one-one.*
2) *$f \circ g$ is the identity function on Y i.e. $f \circ g = I$. Here, f will be onto and g will be one-one.*

Theorem 2.2. *Let V and W be finite dimensional vector spaces such that dim V = dim W. If T is a linear transformation from V into W, then the following are equivalent:*

1) *T is non-singular*
2) *T is onto*

If any of the above two condition is satisfied then T is invertible.

3 Solution

1) We are given $\mathbf{V}$ which is a finite dimensional vector space, with the following linear operators defined as:-

$$\mathbf{T} : \mathbf{V} \to \mathbf{V} \tag{3.0.1}$$
$$\mathbf{U} : \mathbf{V} \to \mathbf{V} \tag{3.0.2}$$

The linear operators also satifies the condition

$$\mathbf{TU} = \mathbf{I} \tag{3.0.3}$$

Where $\mathbf{I}$ is an Identity transformation. This identity transformation can be written as

$$\mathbf{I} : \mathbf{V} \to \mathbf{V} \tag{3.0.4}$$
$$\implies \mathbf{TU} : \mathbf{V} \to \mathbf{V} \tag{3.0.5}$$
$$\implies \mathbf{T}\left[\mathbf{U}\left(\mathbf{V}\right)\right] = \mathbf{V} \tag{3.0.6}$$

From theorem (2.1) we can say that $\mathbf{U}$ must be one-one and $\mathbf{V}$ must be onto.
From theorem (2.2) we can say that $\mathbf{T}$ is invertible.

Now we know that

$$\mathbf{TT}^{-1} = \mathbf{I} \tag{3.0.7}$$

Comparing (3.0.3) and (3.0.7) we get

$$\mathbf{TT}^{-1} = \mathbf{I} = \mathbf{TU} \tag{3.0.8}$$

Multiply both sides with $\mathbf{T}^{-1}$

$$\mathbf{T}^{-1}\left(\mathbf{TT}^{-1}\right) = \mathbf{T}^{-1}\left(\mathbf{TU}\right) \tag{3.0.9}$$
$$\mathbf{T}^{-1}\mathbf{I} = \left(\mathbf{T}^{-1}\mathbf{T}\right)\mathbf{U} \tag{3.0.10}$$
$$\mathbf{T}^{-1} = \mathbf{IU} \tag{3.0.11}$$
$$\therefore \mathbf{T}^{-1} = \mathbf{U} \tag{3.0.12}$$

2) Let $\mathbf{D}$ be a differential operator $\mathbf{D} : \mathbf{V} \to \mathbf{V}$, where $\mathbf{V}$ is a space of polynomial functions in one variable over $\mathbf{R}$.

$$\mathbf{D}(c_0 + c_1 x + \dots + c_n x^n) = c_1 + c_2 x + \dots + c_n x^{n-1} \tag{3.0.13}$$

And $\mathbf{U} : \mathbf{V} \to \mathbf{V}$ be linear operator such that

$$\mathbf{U}(c_0 + c_1 x + \dots + c_n x^n) = c_0 x + \frac{c_1 x^2}{2} + \dots + \frac{c_n x^{n+1}}{n+1} \tag{3.0.14}$$

Therefore, the linear operator $\mathbf{UD} : \mathbf{V} \to \mathbf{V}$ will be $\mathbf{UD}(c_0 + c_1 x + \dots + c_n x^n)$

$$\begin{aligned} &= \mathbf{U}[\mathbf{D}(c_0 + c_1 x + \dots + c_n x^n)] \\ &= \mathbf{U}\left[c_1 + c_2 x + \dots + c_n x^{n-1}\right] \\ &= c_1 x + \frac{c_2 x^2}{2} + \dots + \frac{c_n x^n}{n} \\ &= c_1 x + c_2 x^2 + \dots + c_n x^n \\ &\neq \mathbf{I} \end{aligned} \tag{3.0.15}$$

Now, the linear operator $\mathbf{DU} : \mathbf{V} \to \mathbf{V}$ will be $\mathbf{DU}(c_0 + c_1 x + \dots + c_n x^n)$

$$\begin{aligned} &= \mathbf{D}[\mathbf{U}(c_0 + c_1 x + \dots + c_n x^n)] \\ &= \mathbf{D}\left[c_0 x + \frac{c_1 x^2}{2} + \dots + \frac{c_n x^{n+1}}{n+1}\right] \\ &= c_0 + \frac{2 c_2 x}{2} + \dots + \frac{(n+1) c_n x^n}{n+1} \\ &= c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \\ &= \mathbf{I} \end{aligned} \tag{3.0.16}$$

From (3.0.15) and (3.0.16) we see that $\mathbf{DU} = \mathbf{I}$, but $\mathbf{UD} \neq \mathbf{I}$.