Assignment 17

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Download the latex-tikz codes from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_17

1 Problem

(UGC-june2015,70) :
An $n \times n$ complex matrix $\mathbf{A}$ satisfies $\mathbf{A}^k = \mathbf{I}_n$. the $n \times n$ identity matrix, where k is a positive integer > 1. Suppose 1 is not an eigenvalue of $\mathbf{A}$. Then which of the following statements are necessarily true?

1) $\mathbf{A}$ is diagonalizable.
2) $\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$, the $n \times n$ zero matrix.
3) $tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$
4) $\mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$

2 Definition and Result used

Minimal Polynomial	The minimal polynomial $\mu_{\mathbf{A}}$ of an $n \times n$ matrix $\mathbf{A}$ over a field $\mathbf{F}$ is the monic polynomial P over the field $\mathbf{F}$ of least degree such that $P(\mathbf{A}) = 0$. Any other polynomial Q with $Q(\mathbf{A}) = 0$ is polynomial multiple of $\mu_{\mathbf{A}}$.
Eigen Value and Minimal Polynomial	If λ is an eigen value of matrix $\mathbf{A}$ then λ will also be the root of the minimal polynomial $\mu_{\mathbf{A}}$.
Diagonalizability and Eigen Values	If $\mathbf{A}$ is an $n \times n$ matrix with n distinct eigenvalues, then $\mathbf{A}$ is diagonalizable
Polynomial and it's Zeros	If a polynomial is of form $x^k - 1$, it can be written as $$x^k - 1 = (x-1)(1 + x + x^2 + ... + x^{k-1})$$ The zeros to the given polynomial will be of the format $$e^{\frac{n2\pi i}{k}} \qquad \text{for } 0 \leq n < k.$$ From this we can see that all the roots of the equation $x^k - 1$ will be distinct.

3 Solution

Inference from the Given Data	We are given that $$\mathbf{A}^k = \mathbf{I}_n$$ This can be written as $$\mathbf{A}^k - \mathbf{I}_n = 0$$ This resembles the polynomial equation of the form $x^k - 1$, So we further write the above equation as $$\implies \mathbf{A}^k - \mathbf{I}_n = 0$$ $$\implies (\mathbf{A} - \mathbf{I}_n)(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}) = 0$$ Let $\mu_{\mathbf{A}}$ be the minimal polynomial of $\mathbf{A}$. It is given that 1 is not an eigenvalue of $\mathbf{A}$. That means $\mu_{\mathbf{A}}$ cannot divide $(\mathbf{A} - \mathbf{I}_n)$. But $\mu_{\mathbf{A}}$ will be able to divide $(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1})$ as it is a polynomial multiple of $\mathbf{A}$ i.e. $(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1})$ is polynomial multiple of $\mu_{\mathbf{A}}$ $$\implies \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$ Since we know that $1 + x + x^2 + ... + x^{k-1}$ will have distinct roots which are not equal to 1.
Option 1	We were able to find that $\implies \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}$ is a polynomial multiple of $\mu_{\mathbf{A}}$ with $k - 1$ distinct roots. Which implies that $\mu_{\mathbf{A}}$ will also have distinct roots. Since, there are distinct roots to the minimal polynomial, it implies that $\mathbf{A}$ will be diagonalizable. $\therefore$ this statement is **True**.
Option 2	We know that $$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$ $$\implies \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = -\mathbf{I}_n$$ $\therefore$ this statement is **False**.
Option 3	We know that

	$$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$ $$\implies \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = -\mathbf{I}_n$$ Taking *trace*() on both sides, we get $$\implies tr(\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}) = tr(-\mathbf{I}_n)$$ $$\implies tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = tr(-\mathbf{I}_n) \qquad (\because \text{ trace() is a linear function})$$ $$\implies tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$$ $\therefore$ this statement is **True**.
Option 4	We know that $$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-2} + \mathbf{A}^{k-1} = 0$$ Multiply the whole equation with $\mathbf{A}^{-(k-1)}$. We get $$\mathbf{A}^{-(k-1)} + \mathbf{A}^{1-(k-1)} + ... + \mathbf{A}^{k-2-(k-1)} + \mathbf{A}^{k-1-(k-1)} = 0$$ $$\implies \mathbf{A}^{-(k-1)} + \mathbf{A}^{1-(k-1)} + ... + \mathbf{A}^{-1} + \mathbf{I}_n = 0$$ $$\implies \mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$$ $\therefore$ this statement is **True**.
Conclusion	From our observation we see that Options 1), 3) and 4) are True.

4 Example

Complex Matrix Example	Let the complex matrix $\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ When $k = 4$, we get $$\mathbf{A}^4 = \mathbf{I}_2$$ The eigen values of the matrix $\mathbf{A}$ are $-i$ and $+i$. Since, there are two distinct eigen values for the matrix $\mathbf{A}$, $\mathbf{A}$ is diagonalizable.

Now checking the equation for $\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}$

$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 \qquad (\because \text{ here } k = 4)$$

$$\Longrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}_2$$

Now checking the equation for $tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$

$$tr(\mathbf{A}) + tr(\mathbf{A}^2) + tr(\mathbf{A}^3) \qquad (\because \text{ here } k = 4)$$

$$\Longrightarrow tr\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + tr\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + tr\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\Longrightarrow 0 + (-2) + 0 = -2$$

Now checking the equation for $\mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$

$$\mathbf{A}^{-1} + \mathbf{A}^{-2} + \mathbf{A}^{-3} \qquad (\because \text{ here } k = 4)$$

$$\Longrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}_2$$