EE5609: Matrix Theory Assignment 14

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Abstract—**This document proves the property of projection.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_14

1 Problem

Prove that if $\mathbf{E}$ is the projection on $\mathbf{R}$ along $\mathbf{N}$, then $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$

2 Theorem

Theorem 2.1. *If $\mathbf{V}$ is a vector space, a projection of $\mathbf{V}$ is a linear operator $\mathbf{E}$ on $\mathbf{V}$ such that $\mathbf{E}^2 = \mathbf{E}$. Let $\mathbf{R}$ be the range and let $\mathbf{N}$ be the nullspace of $\mathbf{E}$. Then the vector space $\mathbf{V}$ can be written as $\mathbf{V} = \mathbf{R} \bigoplus \mathbf{N}$. This operator is called as projection on $\mathbf{R}$ along $\mathbf{N}$.*

3 Solution

It is given that $\mathbf{E}$ is the projection. From thorem 2.1, the linear operator $\mathbf{E}$ will satisfy $\mathbf{E}^2 = \mathbf{E}$. Let's check whether $\mathbf{I}-\mathbf{E}$ is also a projection.

$$\begin{aligned}(\mathbf{I}-\mathbf{E})^2 &= \mathbf{I}^2 + \mathbf{E}^2 - 2\mathbf{I}\mathbf{E} \\ &= \mathbf{I} + \mathbf{E} - 2\mathbf{E} \\ &= (\mathbf{I}-\mathbf{E})\end{aligned} \tag{3.0.1}$$

From (3.0.1), we can say that $(\mathbf{I}-\mathbf{E})$ is also a projector. But $(\mathbf{I}-\mathbf{E})$ is called as the "Complementary Projector", i.e.

$$range(\mathbf{I}-\mathbf{E}) = null(\mathbf{E}) \tag{3.0.2}$$

$$null(\mathbf{I}-\mathbf{E}) = range(\mathbf{E}) \tag{3.0.3}$$

Lets take a vector $\mathbf{v}$ such that $\mathbf{E}\mathbf{v} = 0$, where $\mathbf{v}$ is in the null space of $\mathbf{E}$. Then,

$$\begin{aligned}(\mathbf{I}-\mathbf{E})\mathbf{v} &= \mathbf{v} - \mathbf{v}\mathbf{E} \\ &= \mathbf{v}\end{aligned} \tag{3.0.4}$$

In other words, any $\mathbf{v}$ in the nullspace of $\mathbf{E}$ is also in the range of $(\mathbf{I}-\mathbf{E})$. We know that any $\mathbf{x} \in range(\mathbf{I}-\mathbf{E})$ is characterized by

$$\begin{aligned}\mathbf{x} &= (\mathbf{I}-\mathbf{E})\mathbf{v} \quad \textit{, for some } \mathbf{v} \\ &= \mathbf{v} - \mathbf{E}\mathbf{v} \\ &= -\ (\mathbf{E}\mathbf{v} - \mathbf{v})\end{aligned} \tag{3.0.5}$$

Now we need to check if $\mathbf{x}$ is in the nullspace of $\mathbf{E}$. i.e. $\mathbf{E}\mathbf{x} = 0$

$$\begin{aligned}\mathbf{E}(-\ (\mathbf{E}\mathbf{v} - \mathbf{v})) &= -(\mathbf{E}^2\mathbf{v} - \mathbf{E}\mathbf{v}) \\ &= -(\mathbf{E}\mathbf{v} - \mathbf{E}\mathbf{v}) \quad (\because \mathbf{E} \textit{ is a projection}) \\ &= 0\end{aligned} \tag{3.0.6}$$

Thus, if $\mathbf{x} \in range(\mathbf{I}-\mathbf{E})$, then $\mathbf{x} \in null(\mathbf{E})$.

Therefore, we can say that $null(\mathbf{E}) = range(\mathbf{I}-\mathbf{E})$.

We can use the same argument as above for proving (3.0.3), by taking $\mathbf{E} = \mathbf{I} - (\mathbf{I}-\mathbf{E})$.

$\therefore$ we can say that $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$.

As an example, lets take the below matrix.

$$\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \tag{3.0.7}$$

We can check that the matrix in (3.0.7) satisfies the condition $\mathbf{A}^2 = \mathbf{A}$. Thus, $\mathbf{A}$ is a projection matrix.

$$(\mathbf{I}-\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$

$$(\mathbf{I}-\mathbf{A}) = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{3.0.8}$$

We can check that the matrix in (3.0.8) satisfies the condition $(\mathbf{I}-\mathbf{A})^2 = (\mathbf{I}-\mathbf{A})$. Thus, $(\mathbf{I}-\mathbf{A})$ is a projection matrix.

Null space of $\mathbf{A}$ is given by

$$null(\mathbf{A}) = b\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad (\text{'}b\text{' } \textit{is any real number}) \tag{3.0.9}$$

Range of $(\mathbf{I}-\mathbf{A})$ is given by

$$\begin{aligned} range(\mathbf{I}-\mathbf{A}) &= a\begin{pmatrix} \frac{1}{3} \\ \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix} \quad (\text{'}a\text{' } \textit{is any real number}) \\ &= a\left(\frac{-1}{3}\right)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\ &= a\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \tag{3.0.10}$$

$\therefore$ from (3.0.9) and (3.0.10), we can say that $(\mathbf{I}-\mathbf{A})$ is a projection matrix on $\mathbf{N}$ along $\mathbf{R}$.