EE5609: Matrix Theory
Assignment 12

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Abstract—**This document explains the relation between linear operators, and diagonalizability.**

Download all solutions from

https://github.com/vimalkb007/EE5609/tree/master/Assignment_12

1 Problem

Let $\mathbf{T}$ be the linear operator on $\mathbf{R}^4$ which is represented in the standard basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$$

Under what conditions on a, b and c in $\mathbf{T}$ is diagonalizable?

2 Theorem

Theorem 2.1. *A linear operator* $\mathbf{T}$ *on a* $n-$ *dimensional space* $\mathbf{V}$ *is diagonalizable, if and only if* $\mathbf{T}$ *has an n distinct characteristic vectors (or) null spaces corresponding to the characteristic values.*

Theorem 2.2. *Let* $\mathbf{T}$ *be a linear operator on a finite-dimensional space* $\mathbf{V}$. *Let* $c_1, c_2, ..., c_k$ *be the distinct characteristic values of* $\mathbf{T}$ *and let* $\mathbf{W_i}$ *be the null space of* $(\mathbf{T} - c_i\mathbf{I})$. *The following are equivalent:*

1) $\mathbf{T}$ *is diagonizable*
2) $dim\ \mathbf{W_1} + ... + dim\ \mathbf{W_k} = dim\ \mathbf{V}$

3 Solution

Let the given matrix be,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \tag{3.0.1}$$

As per theorem 2.1, we need to find the characteristic polynomial for the matrix $\mathbf{A}$. Characteristic equation is given by $det\,(x\mathbf{I} - \mathbf{A})$.

$$det\,(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x-0 & 0 & 0 & 0 \\ -a & x-0 & 0 & 0 \\ 0 & -b & x-0 & 0 \\ 0 & 0 & -c & x-0 \end{vmatrix} \tag{3.0.2}$$

$$det\,(x\mathbf{I} - \mathbf{A}) = x^4 \tag{3.0.3}$$

The characteristic equation will be,

$$det\,(x\mathbf{I} - \mathbf{A}) = 0 \tag{3.0.4}$$

$$x^4 = 0 \tag{3.0.5}$$

From (3.0.5) we get the characteristic value as $c_1 = 0$ with a multiplicity of 4.

The basis for the characteristic value $c_1 = 0$ can be obtained by solving the equation

$$(\mathbf{A} - c_1\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.0.6}$$

i.e.

$$(\mathbf{A} - (0)\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.0.7}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{0} \tag{3.0.8}$$

Solving the above equation we get

$$ax = 0,\ \ by = 0,\ \ cz = 0 \tag{3.0.9}$$

We know that the null space of $(\mathbf{A} - (0)\mathbf{I})$, is spanned by the vector $\mathbf{x}$, where the basis for the space $\mathbf{W_1}$ need to satisfy the condition of (3.0.9). If we assume that

$$a \neq 0 \ , \ b \neq 0 \ , \ c \neq 0 \tag{3.0.10}$$

This will correspond that the elements in the basis of the vector **x** will be

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix} \tag{3.0.11}$$

Which implies that the *dim* $\mathbf{W_1} = 1$. From theorem 2.2, for **T** to be diagonalizable, the null space $\mathbf{W_1}$ of **A** must have the *dim* $\mathbf{W_1} = 4$, since *dim* $\mathbf{R^4} = 4$. So, there is a contradiction with (3.0.10).

$\therefore$ **A** is diagonalizable only if

$$a = b = c = 0 \tag{3.0.12}$$

i.e. **A** is a zero matrix.