

Matrix Theory EE5609

Assignment 11

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Abstract—This document solves a problem of linear combinations.

All the codes for the figure in this document can be found at

https://github.com/vimalkb007/EE5609/tree/master/Assignment_11

1 PROBLEM

Let \mathbf{A} be an $m \times n$ matrix with rank r . If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

- 1) $m = r$
- 2) the column space of \mathbf{A} is a proper subspace of \mathbf{R}^m
- 3) the null space of \mathbf{A} is a non-trivial subspace of \mathbf{R}^n whenever $m = n$
- 4) $m \geq n$ implies $m = n$

2 SOLUTION

If the columns of an $m \times n$ matrix \mathbf{A} span \mathbf{R}^m then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each \mathbf{b} in \mathbf{R}^m .

The **null space** of \mathbf{A} is defined to be

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} = \mathbf{0}\} \quad (2.0.1)$$

Let \mathbf{A} be given as

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \quad (2.0.2)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.3)$$

\therefore the only possible nullspace of the matrix \mathbf{A} is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let \mathbf{B} be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \quad (2.0.4)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.5)$$

\therefore the rank of matrix $\mathbf{B} = 3$.

Options	Observations
$m = r$	<p>The rank of any matrix \mathbf{A} is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of \mathbf{R}^m.</p> <p>\therefore This statement is True.</p>
the column space of \mathbf{A} is a proper subspace of \mathbf{R}^m	<p>Any subspace of a vector space \mathbf{V} other than \mathbf{V} itself is considered a proper subspace of \mathbf{V}. Which means that linear combination of \mathbf{A} will span less than m. That will make the resultant \mathbf{b} span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting.</p> <p>\therefore This statement is False.</p>
the null space of \mathbf{A} is a non-trivial subspace of \mathbf{R}^n whenever $m = n$	<p>From (2.0.2) we see that even when $m = n$ then also we are getting a trivial nullspace.</p> <p>\therefore This statement is False.</p>
$m \geq n$ implies $m = n$	<p>When $m \geq n$, then number of rows will become greater than columns. And it is given that there exists a solution. From (2.0.4) we see that rank will be equal to n. And the \mathbf{b} will span in \mathbf{R}^n.</p> <p>\therefore This statement is True.</p>