Agenda.

1st October 2019

Vimal Kumarasamy

Joint probability.
Pdf to CDF.
Marginal probability.
Conditional probability.

Joint probability

X & Y are 2 continuous random variables

pdf of X, Y

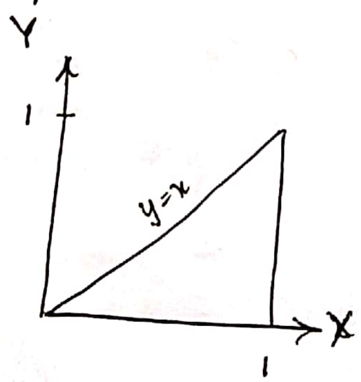
$f(X,Y) = c(X+Y)$

on support X ranges between 0 and 1
$Y = X$

given pdf $f(X,Y) = c(X+Y)$
what could be the value of c?

pdf when integrated gives CDF which should add upto 1 (across support).

Lets start with drawing the support and respective pdf.



the support was given as x ranging from 0 to 1 & y = x.
The area trapped by the support is a triangle, on top of which the probability function c(X+Y) is distributed.

Pdf to CDF

we know that CDF should addupto 1 across support. There are 2 dimensions to the support, so we would need 2 integration.

$$\text{CDF} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,Y)\, dy\, dx = 1.$$

(similar to squishing along both Y axis and x axis resulting to 1).

① while squishing across Y axis, the range of support of Y is from 0 to x.

(note: the support should have been written as Y ranges from 0 to X).

so this range becomes the integral limit.

② while squishing across x axis, the range of support of x is clear 0 to 1.

$$\therefore \text{CDF} = \int_{X=0}^{X=1} \int_{Y=0}^{Y=X} c(X+Y)\, dY\, dx = 1.$$

$$= c \int_{X=0}^{X=1} \left[XY + \frac{Y^2}{2} \right]_{Y=0}^{Y=X} dx$$

$$= c \int_{X=0}^{X=1} \left[X^2 + \frac{X^2}{2} \right] dx = \frac{3c}{2} \int_{X=0}^{X=1} X^2\, dx$$

$$= \frac{3c}{2} \left[\frac{X^3}{3} \right]_0^1$$

$$= \frac{3c}{2} (1/3)$$

$$= \frac{c}{2} = 1$$

$\therefore$ $C = 2$

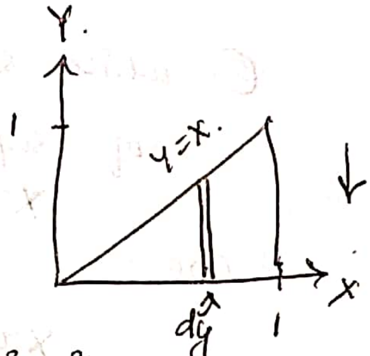
$C=2$ makes the pdf and CDF meaningful.

Marginal Probability

There are 2 dimensions on the support. If we squish one support, then we have only 1 Random Variable, based on which the pdf can be computed.

$$c=2 \Rightarrow f(x,y) = 2x+2y$$

Squishing across Y-axis, leaves us with just x Variable → so probability is distributed across just x (ranging from 0 to 1).

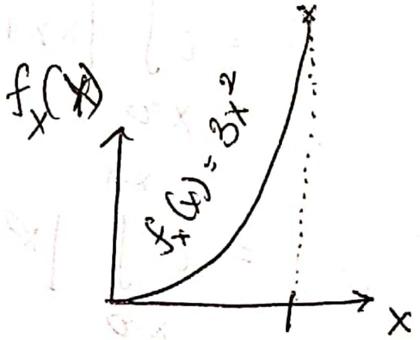


Y ranges between 0 & X.

so. $$f_X(x) = \int_{Y=0}^{Y=x} (2x+2y)\, dY$$

$$= \left[2xY + \frac{2Y^2}{2}\right]_{Y=0}^{Y=x} = 2x^2 + x^2$$

$$f_X(x) = 3x^2.$$



so the pdf of $f_X(x)$ is shown here

the reason ~~of~~ why at $x=1$, $f_X(x=1) = 3$ (highest) is because squishing across Y axis gathers so much of mass at $x=1$, compared at any other points so its the highest at $x=1$.

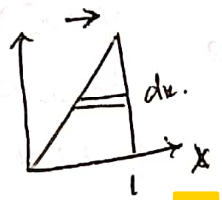
now lets squish the joint probability across X-axis, so x Variable is removed and pdf Varies across Y (ranging between 0 and x).

$$f_Y(Y) = \int_{X=Y}^{X=1} (2X+2Y)\, dx$$

$$= \left[\frac{2x^2}{2} + 2xY\right]_{Y}^{1} = 1+2Y-Y^2-2Y^2$$

$$f_Y(Y) = 1+2Y-3Y^2.$$

<u>Note:</u> the range we have taken here for squishing across x variable is from $X=Y$ to $X=1$. the reason is in this diagram

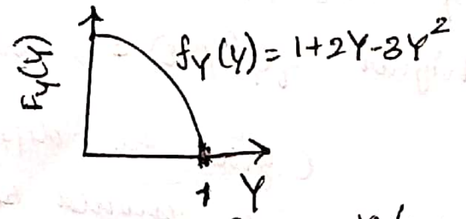


we are computing the area of the small rectangle of thickness dx (extremely small) and the length of the rectangle is from $x=Y$ and $x=1$.

so pdf $F_Y(Y)$ when $Y=1$ $F_Y(Y=1)=0$.

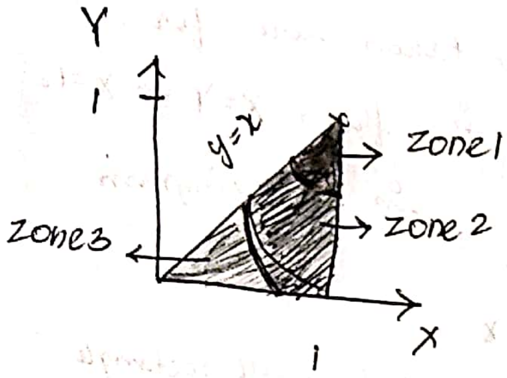
at $Y=1$ the apex is too thin, and gathers no mass while squishing, whereas at $Y=0$, there is a huge mass that gets gathered so $F_Y(Y=0)=1$ (maximum).

so the pdf $F_Y(Y)$ looks like



$$\left.\begin{aligned} f_X(X=1) &= 3 \\ f_Y(Y=0) &= 1 \end{aligned}\right\}$$ these are the maximum values of the respective pdf.

the reason pdf at $X=1$ is way higher than pdf at $Y=0$ is, the pdf near $X=0.8$ to 1 and $Y=0.8$ to 1 are large number (joint prob $f(X,Y) = 2X+2Y$, which gets squished across Y axis and lands near $X=1$ is $f_X(x)$. whereas while squishing across X axis, only smaller such points are gathered ~~at~~ near $Y=1$. even though more such points are gathered near $Y=0$, the quantity of such points (X ranging from 0 to 1, Y ranging from 0 to 0.2) is much smaller resulting in $f_Y(Y=0)=1$, which is much smaller than $f_X(x=1)=3$.



joint probability $f_{X,Y}(x,y) = 2(x+y)$ so across the support (X axis and Y axis) the height is pdf. In zone1 the height is huge, zone 2 height is moderate, zone 3 the height is low. so when squished across Y axis, the entire zone 1, part of zone 2 falls near $x=1$, thus $f_X(x=1)$ is high. while squishing across X axis, just a part of zone 1 gathers near $Y=1$ in $f_Y(Y=1)$, and the top is too thin to gather any mass, so $f_Y(Y=1)=0$. near $Y=0$, it gathers entire of zone 3, (wide) and zone 2, thus it comparatively higher $f_Y(Y=0)=1$, but not as high as $f_X(x=1)=3$.

(given the different pdf points when $x=1$ and $Y=0$, what would be the expected value of x? and what would be the expected value of Y?).

Discussing on the expected value of X at the further right corner ~~and~~ would be obvious. But what about the expected value of Y? we can discuss with the actual number in hand during the next class.

Conditional probability.

pdf of Y for a given value of X

$$f_{Y|X}(Y) = \frac{f(x,y)}{f_X(x)} = \frac{2x+2Y}{3x^2}$$

pdf of X for a given value of Y

$$f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{2X+2Y}{1+2Y-3Y^2}$$

these formulae, when we substitute for the respective conditioning, becomes a function on its own. $f_{Y|X}(Y) = \frac{2X+2Y}{3X^2}$ when $X = 0.1$, would be $\frac{2(0.1)+2Y}{3(0.01)} = \frac{2Y+0.2}{0.03}$ which is a pdf across the line $X = 0.1$.

Note: conditional probability would be more fun if the conditioning is done across a range.