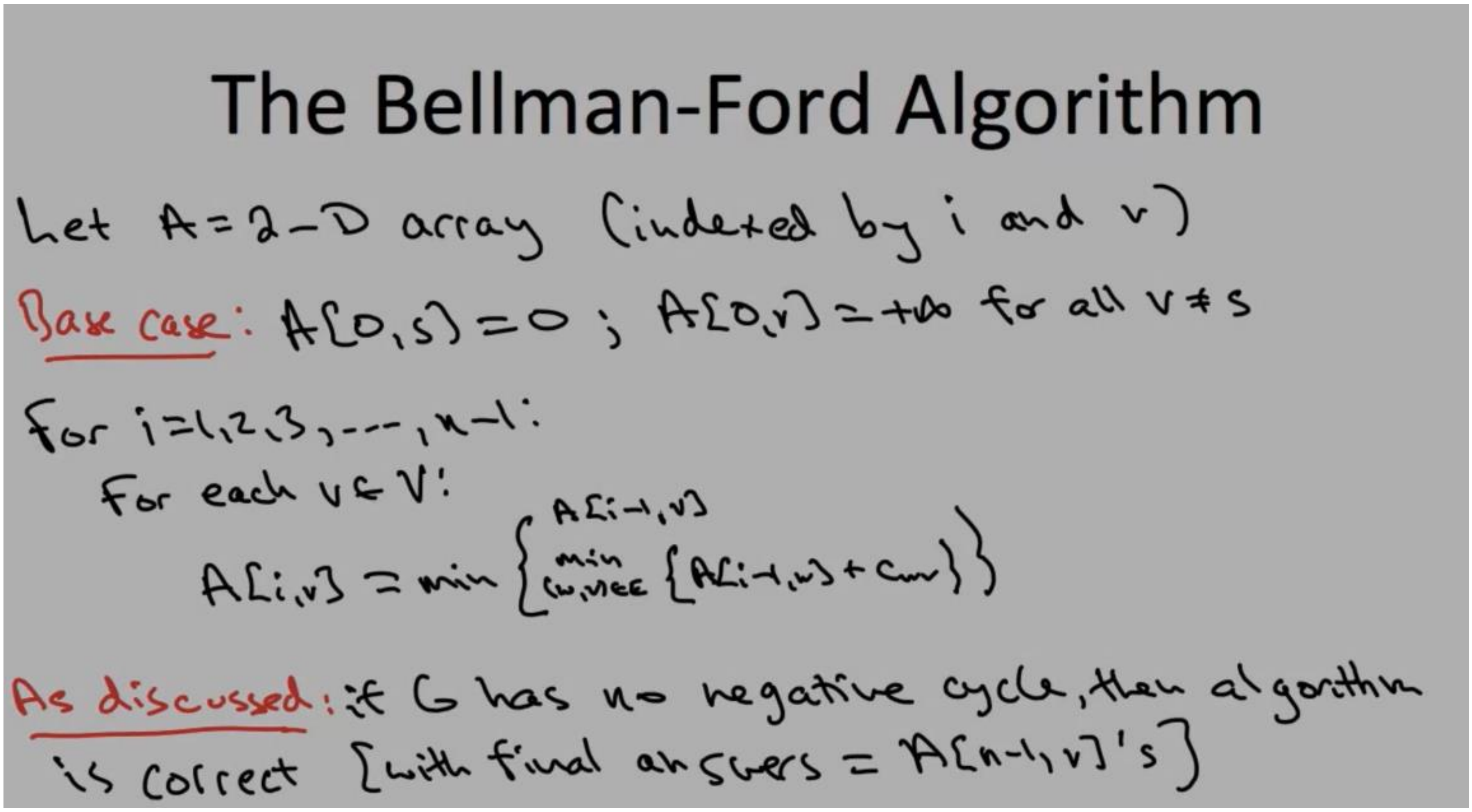
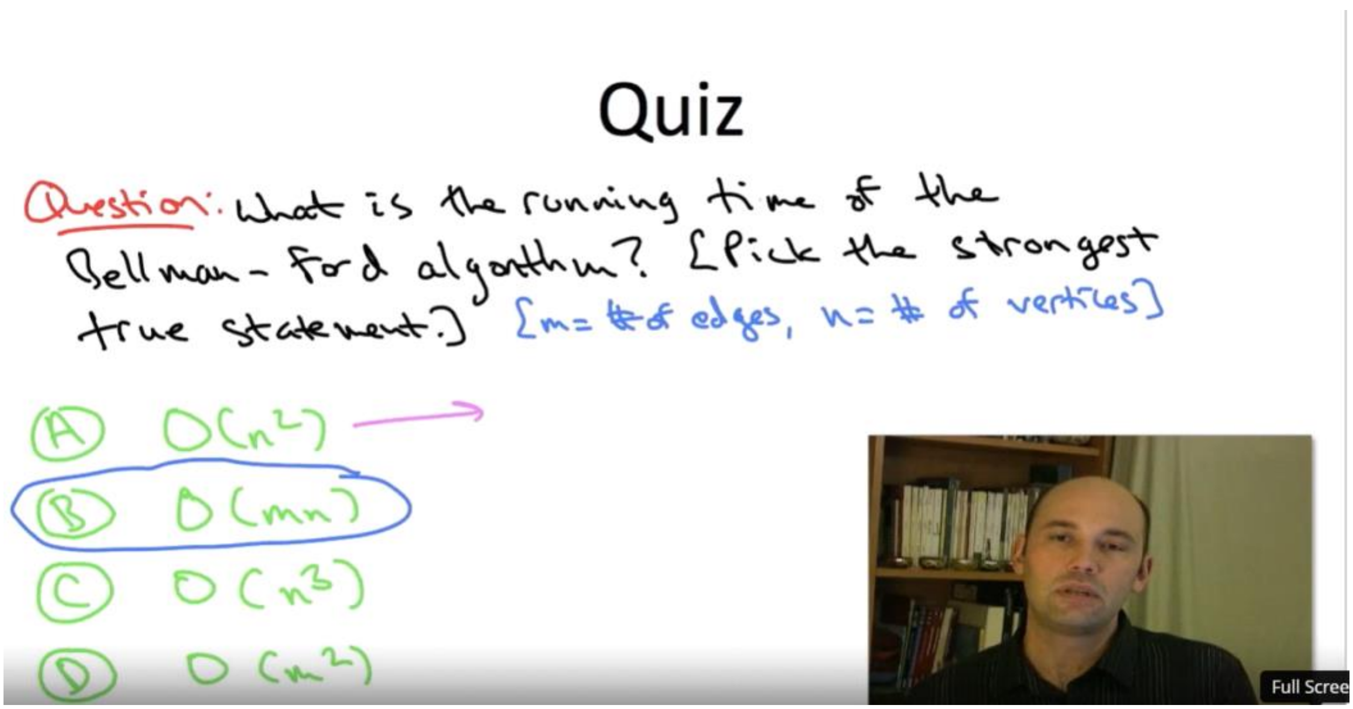
Bellman-Ford shortest path with negative edge length

Saturday, February 8, 2020 10:20 PM

Provided that there are no cycles with negative summation of the edge cost





Stopping Early

Note: Suppose for some $j < n-1$,

$A[j,v] = A[j-1,v]$ for all vertices v.

$\Rightarrow$ for all v, all future $A[i,v]$'s will be the same

$\Rightarrow$ can safely halt (since $A[n-1,v]$'s = correct shortest-path distances)

By running one additional iteration -> till A[n,v], if none of the distances for all the vertices have changed then there is no negative value cycle in the graph

Proof of Claim

($\Rightarrow$) already proved in correctness of Bellman-Ford

($\Leftarrow$) Assume $A[n-1,v] = A[n,v]$ for all $v \in V$. (assume also these are finite ($< +\infty$))

Let $d(v)$ denote the common value of $A[n-1,v]$ and $A[n,v]$.

Recall algorithm:

$$\underbrace{A[n,v]}_{d(v)} = \min \begin{cases} A[n-1,v] \\ \min_{(w,v)\in E} \{ \underbrace{A[n-1,w]}_{d(w)} + c_{wv} \} \end{cases}$$

Thus: $d(v) \le d(w) + c_{wv}$ for all edges $(w,v) \in E$

Equivalently: $d(v) - d(w) \le c_{wv}$

Now: consider an arbitrary cycle C.

$$\sum_{(w,v)\in C} c_{wv} \ge \sum_{(w,v)\in C} (d(v) - d(w)) = 0.$$

QED!

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Reference: https://www.coursera.org/learn/algorithms-npcomplete/lecture/AB5wH/detecting-negative-cycles