

Bellman-Ford shortest path with negative edge length

Saturday, February 8, 2020 10:20 PM

Provided that there are no cycles with negative summation of the edge cost

The Bellman-Ford Algorithm

Let A = 2-D array (indexed by i and v)

Base case: $A[0,s] = 0$; $A[0,v] = +\infty$ for all $v \neq s$

For $i = 1, 2, 3, \dots, n-1$:

For each $v \in V$:

$$A[i,v] = \min \left\{ A[i-1,v], \min_{(u,v) \in E} \{A[i-1,u] + c_{uv}\} \right\}$$

As discussed: if G has no negative cycle, then algorithm is correct [with final answers = $A[n-1,v]$'s]

Quiz

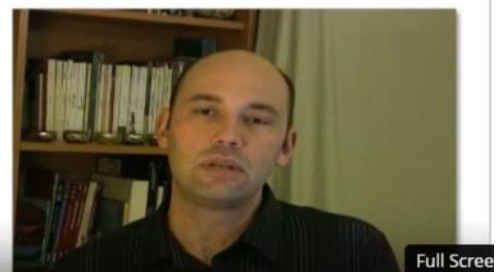
Question: What is the running time of the Bellman-Ford algorithm? [Pick the strongest true statement.] $\{m = \# \text{ of edges}, n = \# \text{ of vertices}\}$

(A) $O(n^2)$ →

(B) $O(mn)$

(C) $O(n^3)$

(D) $O(m^2)$



Stopping Early

Note: Suppose for some $j < n-1$,

$$A[j, v] = A[j-1, v] \text{ for all vertices } v.$$

\Rightarrow for all v , all future $A[i, v]$'s will be the same

\Rightarrow can safely halt (since $A[n-1, v]$'s = correct shortest path distances)

By running one additional iteration \rightarrow till $A[n, v]$, if none of the distances for all the vertices have changed then there is no negative value cycle in the graph

Proof of Claim

(\Rightarrow) already proved in correctness of Bellman-Ford

(\Leftarrow) Assume $A[n-1, v] = A[n, v]$ for all $v \in V$.

(assume also these are finite $(< +\infty)$)

Let $d(v)$ denote the common value of $A[n-1, v]$ and $A[n, v]$.

Recall algorithm: $A[n, v] = \min \left\{ A[n-1, v], \min_{(u,v) \in E} \{ A[n-1, u] + c_{uv} \} \right\}$

$d(v)$ $d(u)$

Thus:
 $d(v) \leq d(u) + c_{uv}$
 for all edges $(u, v) \in E$

Now: consider an arbitrary cycle C .

$$\sum_{(u,v) \in C} c_{uv} \geq \sum_{(u,v) \in C} (d(u) - d(v)) = 0.$$



QED

Equivalently:
 $d(v) - d(u) \leq c_{uv}$

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Reference: <https://www.coursera.org/learn/algorithms-npcomplete/lecture/AB5wH/detecting-negative-cycles>