Black-Litterman Model

⟨I⟩ $U = W^T R - \frac{1}{2} A W^T S W$

U - Utility

$W^T R$ - Expected Returns

A - Price of Risk / Risk aversion Parameter

S - variance-covariance Matrix

$W^T S W$ - variance of the portfolio

we want to maximize this ↑ subject to constraint where sum of all weights is equal to 1.

$$\frac{dU}{dW} = R - \frac{1}{2} \cdot 2ASW = R - A \cdot SW = 0$$

$$R = \underbrace{A}_{\downarrow} SW$$

$$\frac{\text{Excess Return of the market}}{\text{variance of the market}} = \frac{E(r_m) - r_f}{\sigma_m^2}$$

Excess Implied returns as vector - π

$\pi \rightarrow$ Implied Equilibrium excess returns

(Just changing the name R to π to be consistent with B-L)

⟨II⟩ Let us assume three asset portfolio comprised of three assets - A, B, C.

$r_A > r_B$ by 1% ; $r_C > r_A$ by 0.5% → 2 views expressed as vector

r_A ↓ return on asset A

views Vector → Q → views vector = $\begin{bmatrix} .01 \\ .005 \end{bmatrix}_{2\times1}$

Matrix establishing the link between our views and what the views are about. → P → link Matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} \text{view 1} \\ \text{view 2} \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}_{2\times3} \end{array}$$

for each +ve view = 1

for each -ve view = -1

(sum of all -ve view = 0)

$S^{-1} =$

$$\underset{\text{views}}{Q} + \underset{\text{error component}}{\varepsilon} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.005 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

$$\underset{\text{normally distributed}}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}} \sim N\left[\underset{\text{zero mean}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \underbrace{\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}}_{\text{Uncertainity}}\right] \rightarrow \Omega \text{ (Uncertainity about our views)}$$

(III) $\Omega = \underset{\substack{\text{scalar} \\ (\tau) \\ 0.25}}{\tau} P S P^T$ confidence of views $\rightarrow \Omega^{-1}$

we use the value of $\tau = 1$

τ = scalar (1)

π = 3×1 vector

S = 3×3

Q = 2×1

P = 2×3

Ω = 2×2

Computing the weighted average about these different vectors.

1. π
2. Q

1. $(\tau S)^{-1}$ ← confidence about π
2. $P^T \Omega^{-1}$ → weight on Q.

$$\boxed{\underbrace{(\tau S^{-1})\pi}_{\text{weighted } \pi} + \underbrace{P^T \Omega^{-1} Q}_{\text{weighted } Q}}$$

weighted average on Q ←

→ 2nd term in Black-Litterman model.

1st (term) formula comes from → Sum of weights = 1

$$\hookrightarrow \left[(\tau S)^{-1} + P^T \Omega^{-1} P\right]^{-1}$$

IV) Complete Black-Litterman formula :-

$$\underbrace{E(r) - r_f}_{\text{Excess Returns}} = \underbrace{\left[(\tau S)^{-1} + P^T \Omega^{-1} P\right]^{-1}}_{\substack{\text{Sum of weights} = 1 \\ \text{(1st term)}}} \underbrace{\left[(\tau S^{-1})\pi + P^T \Omega^{-1} Q\right]}_{\substack{\text{weighted average} \\ \text{(2nd term)}}}$$

(1st term):

$(3\times3) + \underbrace{(3\times2)\times(2\times2)}\times(2\times3)$

$= (3\times3) + (3\times2)\,(2\times3)$

$= (3\times3) + (3\times3)$

$= (3\times3)$ matrix

(2nd term):

$\underbrace{(3\times3)\times(3\times1)} + (3\times2)\times(2\times2)\times(2\times1)$

$= (3\times1) + (3\times2)\times(2\times1)$

$= (3\times1) + (3\times1)$

$= (3\times1)$ matrix

$= (3\times3) + (3\times1) = (3\times1)$ vector.

V) A simple Demonstration :-

Single Asset → Asset 'A'

1. Implied Equilibrium Excess Returns $= 3\%$ p.m. $= \pi$
2. Variance of Implied Equilibrium Excess Return $= 1.2\%^2 = S$
3. Predicted Excess Returns $= 2\%$ p.m. $= Q$. (absolute view)
4. Uncertainity about the view $= 0.25\% = \Omega$.
5. $\tau = 1$, $P = 1$

$$E(r) - r_f = \left[(\tau S)^{-1} + P^T \Omega^{-1} P\right]^{-1} \left[(\tau S)^{-1}\pi + P^T \Omega^{-1} Q\right]$$

$$= \left[(0.012)^{-1} + (0.0025)^{-1}\right]^{-1} \left[(0.012)^{-1}(-0.03) + (0.0025)^{-1}(0.02)\right]$$

$$= \left[83.33 + 400\right]^{-1} \left[83.33\,(-0.03) + 400\,(0.02)\right]$$

$$= 0.00207\left[-2.5 + 8\right] = 0.11 = 1.1\% \text{ p.m.}$$

↑ estimate of our expected Returns.

If we use a higher value of 'Ω', our estimate is going to go down towards the implied excess returns. In this case if we use the implied equilibrium excess returns, we would place lesser weight on this asset, inspite of believing that the asset may offer a better return.

Now that our view is incorporated and we have a higher return estimate, we may actually place more weight on this asset in our portfolio, which would make sense.

1st term $= 0.00207$

$w_1 = 83.33$

$w_2 = 400$

$\therefore \; 0.00207 \times (83.33) + 0.00207 \times (400)$

$= 0.172 + 0.828$

$= \underline{\underline{1}} \Rightarrow$ Sum of weight $= 1$