Q4) Bi - secant method

$f(a) \cdot f(b) < 0$

i) $(a, f(a))$ $(b, f(b))$ $\underbrace{\quad}_{\text{"}\frac{a+b}{2}\text{"}}$ $\Rightarrow$ Σ iterate

ii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

iii)

(a) $\Rightarrow \frac{y - f(b)}{x - b} = \frac{f(b) - f(c)}{b - a}$

(b) set $y = 0$

$\Rightarrow \frac{-f(b)}{c - b} = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \frac{(a-b) f(b)}{f(b) - f(a)} \cancel{+} b = c$

repeat

$c = b - \frac{f(b)(b-c)}{f(b) - f(c)}$

if $f(c) \cdot f(b) < 0$

$b = c$

else

$a = c$

$\epsilon = |b - a|$

$\vdots$

return b.

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(1) Interpolation

$x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$

$\boxed{f(x) = 1/x}$ $\Rightarrow$ approx. (I)

(1) $= f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0)$

(2) $P(x) = \sum_{0}^{2} f(x_k) L(x_k)$ $L_k = \frac{\prod_{i \neq k}(x - x_i)}{(x_k - x_i)}$

Jacobi

1) $3x_1 - x_2 + x_3 = 1$

$3x_1 + 6x_2 + 2x_3 = 0$

$3x_1 + 3x_2 + 7x_3 = 4$

$\boxed{x^0 = 0}$

$x_1 = \frac{1 + x_2 - x_3}{3} = 1/3$

$x_2 = \frac{-3x_1 - 2x_3}{6} = 0$

$x_3 = \frac{4 - 3x_1 - 3x_2}{7} = 4/7$

$0.14, -0.35, 0.42$

Sidel

$x^* = (1.25, -1.33, 0.2)$

Jacobi

$(L + D + U)\bar{x} = \bar{b}$

$D\bar{x} = \bar{b} - (L+U)\bar{x}$

$\bar{x}^{k+1} = D^{-1}b - D^{-1}(L+U)\bar{x}^{(k)}$

$\bar{x} = D^{-1}b - D^{-1}(L+U)\bar{x}$

$\underbrace{\bar{x} - x^{k+1}}_{\bar{e}_{k+1}} = -D^{-1}(L+U)(\bar{x} - \bar{x}^{k})$

$= -D^{-1}(L+U)\bar{e}_k.$

$\|\bar{e}_{k+1}\| \leq \|D^{-1}\| \, \|L+U\| \, \bar{e}_k$

Theorem

the polynomial of degree n that goes exactly through $n+1$ points is unique.

Proof

We know that a given n-degree polynomial has exactly n roots. Suppose that there are two distinct n-degree polynomials $P_n(x)$ and $Q_n(x)$ that agree on the points $x_1, x_2, \ldots, x_{n+1}$, i.e.

$$P_{n-1}(x_i) = Q_{n-1}(x_i) \quad \forall i = 1, \ldots n+1$$

Let us define the following polynomial

$$R_n(x) = P_n(x) - Q_n(x)$$

This polynomial clearly satisfies

$$R_n(x_i) = 0 \quad \forall i = 1, \ldots, n+1$$

but we know from the **Fundamental Theorem of Algebra** that the only n-degree polynomial with $n+1$ roots is the 0 polynomial. Therefore,

$$R_n(x) = 0; \;\; \forall x \in \mathbb{R}$$

Example 2 Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$
\begin{aligned}
E_1:&\quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2:&\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25,\\
E_3:&\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4:&\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}
$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Solution We saw in Example 1 that the Jacobi method for this system has the form

$$
\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5},\\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11},\\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10},\\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}
$$

Hence we have

$$
T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}.
$$

Algorithm 7.1 implements the Jacobi iterative technique.

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4),$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4),$$

and

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75).$$

Also, $f(x_0) = f(2) = 1/2$, $f(x_1) = f(2.75) = 4/11$, and $f(x_2) = f(4) = 1/4$, so

$$\begin{aligned} P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\ &= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\ &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}. \end{aligned}$$

(b) An approximation to $f(3) = 1/3$ (see Figure 3.6) is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned}
10x_1 - x_2 + 2x_3 \quad &= 6,\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25,\\
2x_1 - x_2 + 10x_3 - x_4 &= -11,\\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}.
\end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

An Example of divergence

(1) $x_1 - 5x_2 = -4$

$7x_1 - x_2 = 6$

$x_1 = -4 + 5x_2$

$x_2 = -6 + 7x_1$

(0, 0)

($x_1 = 1$, $x_2 = 1$)

1st

$\begin{pmatrix} -4 \\ -6 \end{pmatrix}$

n	0	1	2	3	4
x_1	0	-4	-34	-174	-1244
x_2	0	-6	-34	-244	-1244

Gauss Seidel

n	0	1	2	3	4
x_1	0	-4	-174	-6124	-214,374
x_2	0	-34	-1224	-42874	-1,500,624

Strictly diagonal yes

$$3x_1 - x_2 = -4$$

$$2x_1 + 5x_2 = 2$$

no

$$4x_1 + 2x_2 - x_3 = -1$$

$$x_1 + 2x_3 = -4$$

$$3x_1 - 5x_2 + x_3 = 3$$

Balancing

$$x_1 - 5x_2 = -4$$

$$7x_1 - x_2 = 6$$

interchange rows

$$7x_1 - 1x_2 = 6$$

$$1x_1 - 5 = -4$$

$$x_1 = \frac{6}{7} + \frac{1}{7}x_2$$

$$x_2 = \frac{4}{5} + \frac{1}{5}x_1$$

x_1	0.0	0.8571	0.9959	0.9911
x_2	0.0	0.9714	0.9992.	1.0

<u>but</u> if

$-4x_1 + 5x_2 = 1$ $\Rightarrow$ not strictly diagonal

$x_1 + 2x_2 = 3$ and both converge

EXAMPLE 2 Write out the cardinal polynomials appropriate to the problem of interpolating the following table, and give the Lagrange form of the interpolating polynomial:

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	-1	7

Solution Using Equation (2), we have

$$\ell_0(x) = \frac{\left(x - \frac{1}{4}\right)(x - 1)}{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{3} - 1\right)} = -18\left(x - \frac{1}{4}\right)(x - 1)$$

$$\ell_1(x) = \frac{\left(x - \frac{1}{3}\right)(x - 1)}{\left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4} - 1\right)} = 16\left(x - \frac{1}{3}\right)(x - 1)$$

$$\ell_2(x) = \frac{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)} = 2\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Therefore, the interpolating polynomial in Lagrange's form is

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x - 1) - 16\left(x - \frac{1}{3}\right)(x - 1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right) \quad \blacksquare$$

Existence of Interpolating Polynomial

In Example 2, we found the Lagrange form of the interpolating polynomial:

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x - 1) - 16\left(x - \frac{1}{3}\right)(x - 1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

It can be simplified to

$$p_2(x) = -\frac{79}{6} + \frac{349}{6}x - 38x^2$$

Now, we learn that this polynomial can be written in another form called the nested Newton form:

$$p_2(x) = 2 + \left(x - \frac{1}{3}\right)\left[36 + \left(x - \frac{1}{4}\right)(-38)\right]$$

It involves the fewest arithmetic operations and is recommended for evaluating $p_2(x)$. It cannot be overemphasized that the Newton and Lagrange forms are just two different derivations for precisely the same polynomial. The Newton form has the advantage of easy extensibility to accommodate additional data points.

EXAMPLE 3 Using the Newton algorithm, find the interpolating polynomial of least degree for this table:

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

Solution In the construction, five successive polynomials appear; these are labeled p_0, p_1, p_2, p_3, and p_4. The polynomial p_0 is defined to be

Polynomials p_0, p_1, p_2, p_3, p_4

$$p_0(x) = -5$$

The polynomial p_1 has the form

$$p_1(x) = p_0(x) + c(x - x_0) = -5 + c(x - 0)$$

The interpolation condition placed on p_1 is that $p_1(1) = -3$. Therefore, we have $-5 + c(1 - 0) = -3$. Hence, $c = 2$, and p_1 is

$$p_1(x) = -5 + 2x$$

The polynomial p_2 has the form

$$p_2(x) = p_1(x) + c(x - x_0)(x - x_1) = -5 + 2x + cx(x - 1)$$

The interpolation condition placed on p_2 is that $p_2(-1) = -15$. Hence, we have $-5 + 2(-1) + c(-1)(-1 - 1) = -15$. This yields $c = -4$, so

$$p_2(x) = -5 + 2x - 4x(x - 1)$$

The remaining steps for $p_3(x)$ are similar. The final result is the Newton form of the interpolating polynomial:

$$p_4(x) = -5 + 2x - 4x(x - 1) + 8x(x - 1)(x + 1) + 3x(x - 1)(x + 1)(x - 2) \quad ■$$