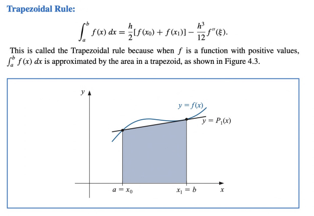
1)

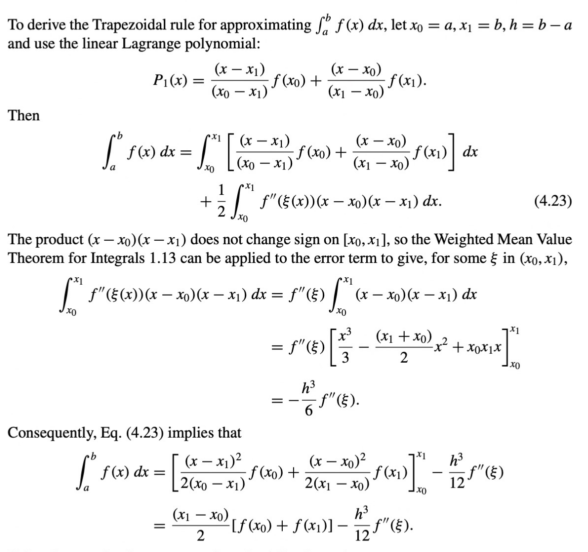


Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi).$$

This is called the Trapezoidal rule because when f is a function with positive values, $\int_a^b f(x)\,dx$ is approximated by the area in a trapezoid, as shown in Figure 4.3.

②



To derive the Trapezoidal rule for approximating $\int_a^b f(x)\,dx$, let $x_0 = a, x_1 = b, h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

Then

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] dx$$
$$+ \frac{1}{2} \int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)\,dx. \qquad (4.23)$$

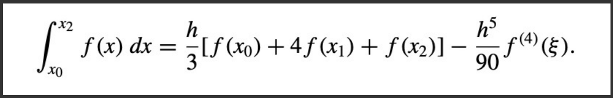
The product $(x - x_0)(x - x_1)$ does not change sign on $[x_0, x_1]$, so the Weighted Mean Value Theorem for Integrals 1.13 can be applied to the error term to give, for some ξ in (x_0, x_1),

$$\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)\,dx = f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1)\,dx$$
$$= f''(\xi) \left[\frac{x^3}{3} - \frac{(x_1 + x_0)}{2} x^2 + x_0 x_1 x \right]_{x_0}^{x_1}$$
$$= -\frac{h^3}{6} f''(\xi).$$

Consequently, Eq. (4.23) implies that

$$\int_a^b f(x)\,dx = \left[\frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1) \right]_{x_0}^{x_1} - \frac{h^3}{12} f''(\xi)$$
$$= \frac{(x_1 - x_0)}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi).$$

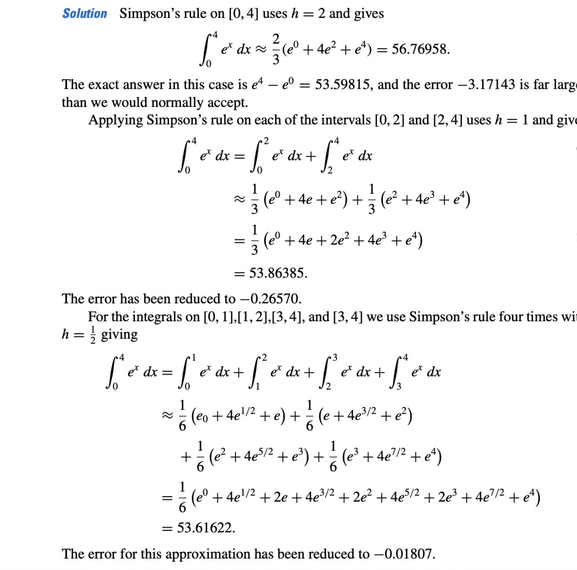
③ Simpson's rule



$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

④ we can divide these into more points & reduce error

⑤ Composite Simpson rule



Solution Simpson's rule on $[0, 4]$ uses $h = 2$ and gives

$$\int_0^4 e^x\,dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

The exact answer in this case is $e^4 - e^0 = 53.59815$, and the error -3.17143 is far larger than we would normally accept.

Applying Simpson's rule on each of the intervals $[0, 2]$ and $[2, 4]$ uses $h = 1$ and gives

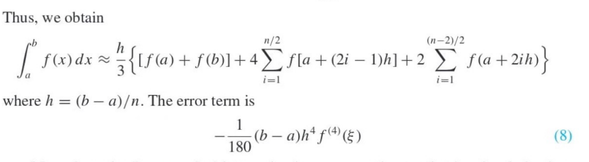
$$\int_0^4 e^x\,dx = \int_0^2 e^x\,dx + \int_2^4 e^x\,dx$$
$$\approx \frac{1}{3}\left(e^0 + 4e + e^2\right) + \frac{1}{3}\left(e^2 + 4e^3 + e^4\right)$$
$$= \frac{1}{3}\left(e^0 + 4e + 2e^2 + 4e^3 + e^4\right)$$
$$= 53.86385.$$

The error has been reduced to -0.26570.

For the integrals on $[0, 1]$,$[1, 2]$,$[3, 4]$, and $[3, 4]$ we use Simpson's rule four times with $h = \frac{1}{2}$ giving

$$\int_0^4 e^x\,dx = \int_0^1 e^x\,dx + \int_1^2 e^x\,dx + \int_2^3 e^x\,dx + \int_3^4 e^x\,dx$$
$$\approx \frac{1}{6}\left(e_0 + 4e^{1/2} + e\right) + \frac{1}{6}\left(e + 4e^{3/2} + e^2\right)$$
$$+ \frac{1}{6}\left(e^2 + 4e^{5/2} + e^3\right) + \frac{1}{6}\left(e^3 + 4e^{7/2} + e^4\right)$$
$$= \frac{1}{6}\left(e^0 + 4e^{1/2} + 2e + 4e^{3/2} + 2e^2 + 4e^{5/2} + 2e^3 + 4e^{7/2} + e^4\right)$$
$$= 53.61622.$$

The error for this approximation has been reduced to -0.01807. ■



Thus, we obtain

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left\{[f(a) + f(b)] + 4\sum_{i=1}^{n/2} f[a + (2i - 1)h] + 2\sum_{i=1}^{(n-2)/2} f(a + 2ih)\right\}$$

where $h = (b - a)/n$. The error term is

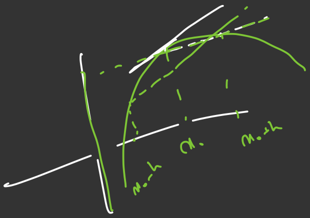
$$-\frac{1}{180}(b - a)h^4 f^{(4)}(\xi) \qquad (8)$$

⑥ Error

(i) $\frac{b-a}{12} h^2 \overset{\text{max}}{f''(z)}$

(ii) $\frac{b-a}{180} h^4 f^4(z)$

diff



① $f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$ error $\le \frac{h}{2}|f''(c)|$ $c \in [x_0, x]$

② $\frac{f(x_1+h) - f(x_1-h)}{2h}$ error $\le \frac{h^2}{6}|f^3(c)|$ $c \in x_1-h, x_1+h$

Ordi diff eq.

Eq. one/more derivatives of a function (unknown)

① $x' - x = e^t \Rightarrow x(t) = te^t + ce^t$

② $x'' + 9x = e^t \Rightarrow x(t) = c_1 \sin 3t + c_2 \cos 3t$

First order

$x' = f(x,t)$

$x(a) \Rightarrow$ known

---

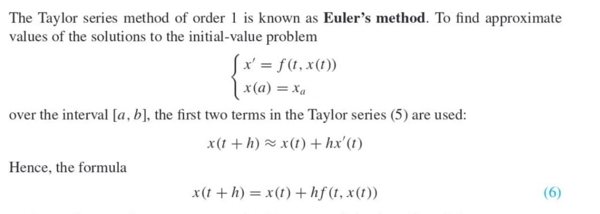
1) $x' = x+1$ $x(0) = 0$ $x = e^t - 1$ special methods but often no closed form solution

Ex

② $x' = e^{-\sqrt{t^2 - \sin t}} + \underbrace{\ln|\sin t + \tanh t^3|}$ $\Rightarrow$ No closed form.

Ex

Ex



The Taylor series method of order 1 is known as **Euler's method**. To find approximate values of the solutions to the initial-value problem

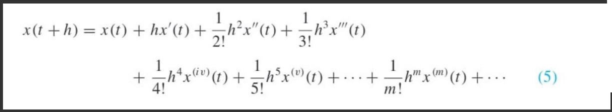
$$\begin{cases} x' = f(t, x(t)) \\ x(a) = x_a \end{cases}$$

over the interval $[a, b]$, the first two terms in the Taylor series (5) are used:

$$x(t+h) \approx x(t) + hx'(t)$$

Hence, the formula

$$x(t+h) = x(t) + hf(t, x(t)) \qquad (6)$$



$$x(t+h) = x(t) + hx'(t) + \frac{1}{2!}h^2x''(t) + \frac{1}{3!}h^3x'''(t) + \frac{1}{4!}h^4x^{(iv)}(t) + \frac{1}{5!}h^5x^{(v)}(t) + \cdots + \frac{1}{m!}h^mx^{(m)}(t) + \cdots \qquad (5)$$

① $x' = x - t^2 + 1 \qquad 0 \le t \le 2 \qquad x(-) = 0.5$

$t = 2$? $\quad h = 0.5$

$x(0) = 0.5$

$x(0.5) = 0.5 + (0.5)(x(0) - (0.0)^2 + 1) = 1.25$

$x(1) = 1.25 + 0.5(1.25 - (0.5)^2 + 1) = 2.25$

$x(1.5) = 3.375$

$x(2) = 4.4375$

$= (t+1)^2 - 0.5e^t.$