Legrange interpolation

1) (x_0, y_0) (x_1, y_1) (x_2, y_2) $\Rightarrow$ No reuse.

$$y_0(L_0) + y_1(L_1) + y_2(L_2)$$

$$L_0 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$\Rightarrow y_0 L_0 + y_1 L_1'$$

$$L_1' = \frac{(x-x_0)}{(x_1-x_0)}$$

$L_1' \neq L_1$ $\Rightarrow$ calculate from scratch.

2) Neville's method

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$\Rightarrow P_{i\,j-1}$

$P_{i+1,j}$

$\Rightarrow$ claim

$$P_{i\,j-1}\frac{x-x_j}{x_i-x_j} + P_{i+1\,j}\frac{(x-x_i)}{x_j-x_i}$$

$$P_{ij} = P_{i+1,j}\frac{(x-x_i)}{x_j-x_i} + P_{i\,j-1}\frac{(x-x_j)}{x_i-x_j}$$

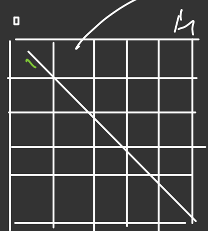
$$P_{ij} = \begin{cases} y_i & x = x_i \\ y_j & x = x_j \\ y_K & x_K \quad i<K<j \end{cases}$$

$$\frac{x_K-x_i}{x_j-x_i} + \frac{x_K-x_j}{x_i-x_j} = \frac{\cancel{x_K - x_i - x_K + x_j}}{(x_j-x_i)} = 1$$

Base case

$$P_{ii} = y_i$$

$\Rightarrow P_{01} = P_{11}, P_{00}$



Algorithm:

Input $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n), x$

for $k = 0$ to n

$\quad P_{kk} = y_k$

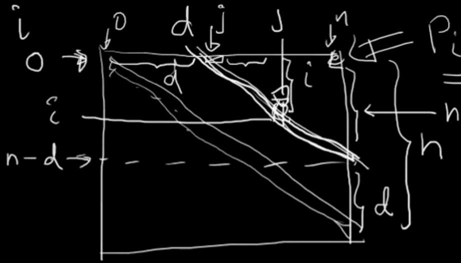
for $d = 1$ to n

$\quad$ for $i = 0$ to $n-d$

$\quad\quad j = d + i$

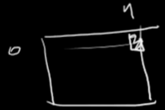
$$P_{ij} = \frac{P_{i+1,j}(x - x_i) - P_{i,j-1}(x - x_j)}{x_j - x_i}$$

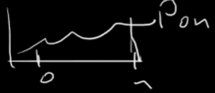
return P_{0n}



$P_{kk} = y_k$

$j = d + i$





1) $f(x) = \frac{1}{\sqrt{x}}$; $x = 81$

(16) (64) (100)

	0	1	2
(16) 0	0.25		
(64) 1		.125	
(100) 2			0.1

16	—	0.25
64	—	.125
100	—	0.1

$P_{01} = 0.08072$

.1058

0.1058

2) $x = 3$?

x.	y.
1	-6
2	2
4	12

-6		
	2	
		12

$\frac{24}{3}$

7

$\frac{(x_1 - x) P_{00} + x - x_0 \, P_{11}}{x_1 - x_0} = \frac{6+4}{1}$

$= 10$

Newton's method

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

EXAMPLE 3 Using the Newton algorithm, find the interpolating polynomial of least degree for this table:

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

Polynomials p_0, p_1, p_2, p_3, p_4

Solution In the construction, five successive polynomials appear; these are labeled p_0, p_1, p_2, p_3, and p_4. The polynomial p_0 is defined to be

$$p_0(x) = -5$$

The polynomial p_1 has the form

$$p_1(x) = p_0(x) + c(x - x_0) = -5 + c(x - 0)$$

The interpolation condition placed on p_1 is that $p_1(1) = -3$. Therefore, we have $-5 + c(1 - 0) = -3$. Hence, $c = 2$, and p_1 is

$$p_1(x) = -5 + 2x$$

The polynomial p_2 has the form

$$p_2(x) = p_1(x) + c(x - x_0)(x - x_1) = -5 + 2x + cx(x - 1)$$

The interpolation condition placed on p_2 is that $p_2(-1) = -15$. Hence, we have $-5 + 2(-1) + c(-1)(-1 - 1) = -15$. This yields $c = -4$, so

$$p_2(x) = -5 + 2x - 4x(x - 1)$$

The remaining steps for $p_3(x)$ are similar. The final result is the Newton form of the interpolating polynomial:

$$p_4(x) = -5 + 2x - 4x(x - 1) + 8x(x - 1)(x + 1) + 3x(x - 1)(x + 1)(x - 2) \quad ■$$

Derivation:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$

$$P_n(x_0) = a_0 = y_0 = 0 \qquad f(x_0)$$

$$P_n(x_1) = a_0 + a_1(x - x_0)$$

$$= a_0 + a_1(x_1 - x_0) = y_1$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, x_1)$$

$$f(x_0) = y_0, \quad f(x_n) = y_n$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$a_2 = f[x_0, x_1, x_2]$$

$$a_3 = f(x_0, x_1, x_2, x_3]$$

Ex:

x	y
1	-6
2	2
4	12

$x = 3$?

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$f(x_0) = a_0 = -6$$

$$f(x_0, x_1) = a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 8 \; ; \quad f(x_1, x_2) = 5$$

$$f(x_0, x_1, x_2) = \frac{f(x_2, x_1) - f(x_1, x_0)}{x_2 - x_1} = -1$$

$$f(x) = -6 + 8(x-1) - (x^2 - 3x + 2)$$

$$= -x^2 + 11x - 16$$

$(3, -10)$ add one more point

$\left((x-1)(x-2)(x-4)\right) f(x_0 \ldots x_3)$ ✓ 0

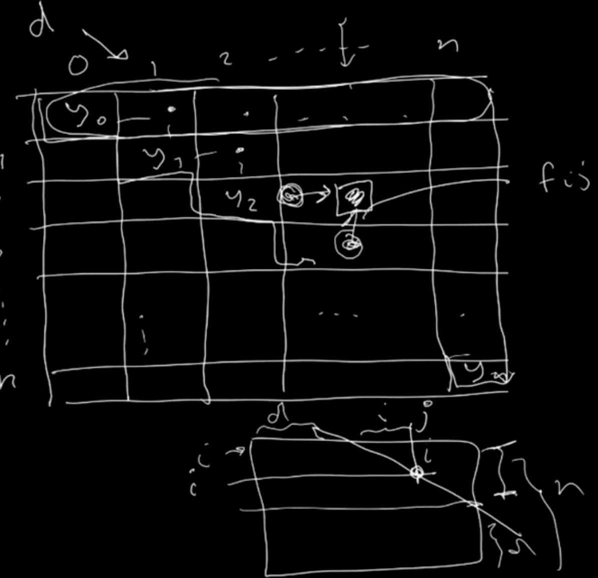
$$f(x_3) = -10$$

$$f(x_2, x_3) = \frac{-10-12}{3-4} = 22$$

$$f(x_1, x_2, x_3) = \frac{22-5}{3-2} = 17$$

$$a_3 = \frac{17+1}{3-1} = 9 \;//$$

$$-x^2 + 11x - 16 + 9(x^3 - 7x^2 + 14x - 8).$$



Algorithm Divided Diff

Input $(x_0, y_0) \dots (x_n, y_n)$

Output: $a_0, \dots a_n$

// notice: $f_{ij} = f[x_i, \dots x_j]$

for $i = 0$ to n

  $f_{ii} = y_i$

for $d = 1$ to n

  for $i = 0$ to $n - d$

    $j = i + d$

    $$f_{ij} = \frac{f_{i+1,j} - f_{i,j-1}}{x_j - x_i}$$