

Practice
Horner's rule

1) $P(x) = 2x^4 - 3x^2 + 3x - 4$

$x_0 = -2$ \rightarrow Newton-Horner to approx a root.

Sol:

$$-2 \left| \begin{array}{cccc} 2 & -3 & 3 & -4 \\ & -4 & 14 & -34 \\ \hline & 2 & -1 & 17 \end{array} \right\} \rightarrow \text{wrong}$$

Write carefully all coefficients.

$$-2 \left| \begin{array}{ccccc} 2 & 0 & -3 & 3 & -4 \\ & 0 & -4 & 8 & -10 & 14 \\ \hline & 2 & -4 & 5 & -7 & 10 \end{array} \right\} \rightarrow \text{recommended } P(-2)$$

We want $P'(-2)$

$$-2 \left| \begin{array}{cccc} 2 & -4 & 5 & -7 \\ & 0 & -4 & 16 & -42 \\ \hline & 2 & -8 & 21 & -49 \end{array} \right\} = 0 \quad P'(-2)$$

$$\Rightarrow x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = -2 - \frac{10}{-49} \approx -1.796.$$

Repeat

$$-1.796 \left| \begin{array}{ccccc} 2 & 0 & -3 & 3 & -4 \\ & -3.592 & 6.451 & -6.197 & 5.742 \\ \hline & 2 & -3.592 & 3.451 & -3.197 & 1.742 \\ & & -3.592 & 12.902 & -29.369 \\ \hline & & 2 & -7.184 & 16.353 & -32.565 \end{array} \right\} = P(x_1)$$

$$x_2 = -1.796 - \frac{1.742}{-32.565} \approx -1.7425$$

simly $x_3 = -1.73897$ \approx an actual zero $= 0$ -1.73896

Note $Q(n)$ depends on approx being used.

Understanding horner's rule

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$= c_n (x-r)^n + c_{n-1} (x-r)^{n-1} + \dots + c_1 (x-r) + c_0$$

Taylor expansion \Rightarrow

we know $c_k = \frac{P^k(r)}{k!}$ \Rightarrow inefficient

Applying horner's rule

$$q(x) = \frac{P(x) - P(r)}{x-r} = c_n (x-r)^{n-1} + c_{n-1} (x-r)^{n-2} + \dots + c_1$$

\Downarrow
repeat

eg:-

$$P(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$$

find Taylor expansion about $r=3$.

3	1	-4	7	-5	2
	0	3	-3	12	21
3	1	1	4	7	23
	0	3	6	30	
3	1	2	10	37	
	0	3	15		
	1	5	25		
		3			
	①	⑧			

$$(x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23$$

② $x^3 - 9x^2 + 12$ use ^{roots} ^{find} ^{to} ^{roots} \Rightarrow Exercise $(1.244, 8.847, -1.091)$

understanding
theory / method

$$P_n(x) = Q(x)(x - r_0) + Y \quad \left| \quad \begin{array}{l} \text{Synthetic division} \\ P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \end{array} \right.$$

$$P_n(r_0) = "Y" = 0$$

$$P_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

$$P_n(x) = a_0 + (a_1 + ((a_2 + (a_3 + a_4x)x)x)x)x$$

\searrow a_4

$$\begin{array}{l} a_4 = a_4 \\ a_3 = a_3 + a_4 r_0 \end{array} \quad \left| \quad \begin{array}{l} a_2 = a_2 + a_3 r_0 \\ a_1 = a_1 + a_2 r_0 \end{array} \right. \quad \left. \begin{array}{l} \\ \end{array} \right\} a_0 = a_0 + a_1 r_0$$

Synthetic division

	a_4	a_3	a_2	a_1	a_0
r_0		$a_4 r_0$	$a_3 r_0$	$a_2 r_0$	$a_1 r_0$
	a_4	$a_3 + a_4 r_0$	$a_2 + a_3 r_0$	$a_1 + a_2 r_0$	$a_0 + a_1 r_0$
		$\xrightarrow{a_3}$	$\xrightarrow{a_2}$	$\xrightarrow{a_1}$	

Algorithm (Horner)

$a = a_n$
for $i = n-1$ down to 0
 $a = a_i + a r_0$
return a

Comp - division :

$$P_n(x) = Q(x)(x - r_0) + Y$$

we want $P'_n(x) = Q'(x)(x - r_0) + P'_n(r_0)$

$$P'_n(x) = Q'(x)(x - r_0) + Q(n)$$

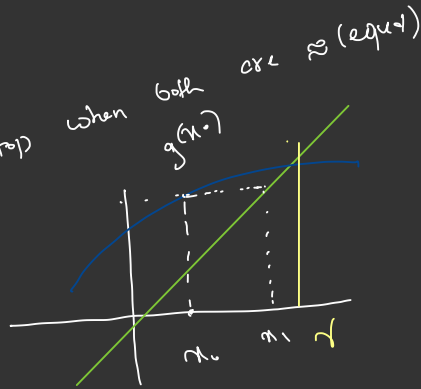
$$P'_n(r_0) = Q(r_0)$$

Fixed-point iteration
 Just enough theory/intuition

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = g(x_0)$$

Stop



Final thm

a) If $|g'(x)| < 1$ for all x in an interval containing root $[x-\delta, x+\delta]$
 $\exists x_0 \in [x-\delta, x+\delta]$
 then seq. $x_{k+1} = g(x_k)$ converges to x .

\Rightarrow Fixed point \Rightarrow where the value of the func. doesn't change

Why imp? $g(p) = p$

$f(p) = 0$ (we want $f(p) = 0$)

$f(x) = 0$ find root. as $g(x) = x - f(x)/f'(x)$

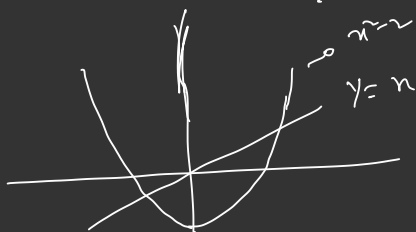
$$x - f(x)/f'(x) = g(x)$$

yes compatible with finding fixed points:

(1) Det any fixed points to $g(n) = n^2 - 2$

\Rightarrow 2 fixed points $p = n_k$.

$$g(p) = p \Rightarrow p = p^2 - 2 \Rightarrow p^2 - p - 2 = (p+1)(p-2)$$



Ex 2
 $\Rightarrow x^3 + 4x^2 - 10 = 0$ has root in $[1, 2]$

change to this form

$$x = g(x)$$

Start at 1.5

(a) $x = \sqrt[3]{10 - 4x^2}$

x_k	$g(x_k)$
1.5	-0.875
	6.732
	-467.7
	1.03×10^8

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

x_k	$g_2(x_k)$
1.5	0.8165
	2.9969
	$(-8.65)^{1/2}$

$$c) \quad x = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \quad \Rightarrow$$

!!
0

Ques: how to find a good form?

x_k	x_{k+1}	$g(x)$
1.5	1.37...	<u>error</u>
	1.36...	
	1.36...	
	1.365230013	

$$d) \quad x = \left(\frac{10}{4+x} \right)^{1/2}$$

x_k	x_{k+1}	$g(x)$
1.5	1.34...	<u>error</u>
	1.3673	
	1.3649	
	1.365...	
1.2	1.365231013	0.014

let's revisit (a)

$$g_1(x) = x - x^3 - 4x^2 + 10$$

$$g_1(1) = 6$$

$$g_1(2) = -12$$

$$g_1'(x) = 1 - 3x^2 - 8x = 0 \quad |g_1'(u)| > 1 \quad \forall u \in [1, 2]$$

Now, for $g_2(x) = \left(\frac{10}{4+x} \right)^{1/2}$

$$g_2'(x) = \left| \frac{-5}{\sqrt{10} (4+x)^{3/2}} \right| \leq 0.15 \quad \forall x \in [1, 2]$$

$$m = \max (|g_2'(x)| \mid x \in [x-s, x+s])$$

$$\frac{e}{2} \leq m e_k \Rightarrow$$

as small derivative
as possible is good

Some more practice

①

$$x^3 + 6x^2 - 8$$

[1, 2]

start at 1.5

②

$$x = x^3 + 6x^2 + x - 8$$

which is better

③

$$x = \sqrt{\frac{8}{x+6}}$$

$$g_3(x) =$$

$$\sqrt{2} / (x+1)^{3/2} < 1$$

$$g_3'(x) = 3x^2 + 12x + 1$$

(No convergence)

1.0641

Fixed point method (basic examples)

$$x^2 - x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 1.618, -0.6180337$$

④

$$x^2 = x + 1 \quad x_0 = 1$$

$$x_{i+1} = 1 + 1/x_i$$

$$x_1 = 1 + 1 = 2$$

$$x_2 = 1 + \frac{1}{2} = 1.5$$

$$x_3 = 1 + \frac{1}{1.5} = 1.666$$

⑤

$$x_{i+1} = \tilde{x}_i - 1 \quad x_0 = 1$$

$$\left. \begin{array}{l} x_1 = 0 \\ x_2 = -1 \end{array} \right\} \text{oscillates.}$$

[1, 2]

⑥

$$x(x-1) = 1$$

$$x = \frac{1}{x-1} \quad x_0 = 0$$

$$x_1 = -1$$

$$x_2 = -1/2$$

$$x_3 = -0.6$$

$$x_4 = -0.618$$

$$x_5 = -0.618$$

⑦ $x^3 + 3x^2 - 1 \quad x_0 = 1$

$$\begin{array}{cccc|c} 1 & 1 & 3 & 0 & -1 \\ & 0 & 1 & 4 & 4 \\ & 1 & 0 & 1 & 5 \\ \hline & 0 & 5 & 5 & 2 \end{array}$$

$$\Rightarrow \pi_1 = 1 - \frac{3}{9} = \frac{2}{3}$$

$$\left| \begin{array}{c} 1 \\ 2/3 \end{array} \right|$$

$$\frac{47}{23} \approx 2.04$$

$$\Rightarrow \frac{2}{3} \left| \begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ & 4/3 & 24/9 & 44/27 & 0 \\ & 1 & 11/3 & 22/9 & 0 \\ & 0 & 4/3 & 28/9 & 0 \\ \hline & 1 & 13/3 & 48/9 & 0 \end{array} \right|$$

$$\Rightarrow$$

$$= \frac{2}{3} - \frac{17/27}{48/9} = \frac{2}{3} - \frac{17}{48 \cdot 3} \approx 0.5$$