4.2. ERROR IN POLYNOMIAL INTERPOLATION

For a given function $f(x)$ defined on an interval $[a, b]$, let $P_n(x)$ denote the polynomial of degree $\leq n$ interpolating $f(x)$ at $n+1$ points $x_0, x_1, \ldots, x_n$ in $[a, b]$

$$P_n(x) = \sum_{j=0}^{n} f(x_j) L_j(x) \tag{4.40}$$

In this section, we consider carefully the error in polynomial interpolation, giving more precise information on its behavior as x and n vary. We begin with a formula for the error.

Theorem 4.2.1 Let $n \geq 0$, let $f(x)$ have $n+1$ continuous derivatives on $[a, b]$, and let $x_0, x_1, \ldots, x_n$ be distinct node points in $[a, b]$. Then

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(c_x) \tag{4.41}$$

for $a \leq x \leq b$, where c_x is an unknown point between the minimum and maximum of $x_0, x_1, \ldots, x_n$, and x.

We omit the proof, since it does not contain any useful additional information regarding interpolation. A sketch of the case $n = 1$, linear interpolation, is given in Problem 18.

Example 4.2.2 Take $f(x) = e^x$ on $[0, 1]$, and consider the error in linear interpolation to $f(x)$ using nodes x_0 and x_1 satisfying $0 \leq x_0 \leq x_1 \leq 1$. From (4.41),

$$e^x - P_1(x) = \frac{(x - x_0)(x - x_1)}{2} e^{c_x} \tag{4.42}$$

for some c_x between the minimum and maximum of x_0, x_1, and x. For this example, assume $x_0 < x < x_1$. Then we note the interpolation error is negative, and we write

$$e^x - P_1(x) = -\frac{(x_1 - x)(x - x_0)}{2} e^{c_x}$$

This shows that the error is approximately a quadratic polynomial with roots at x_0 and x_1, provided that e^{c_x} is approximately constant for $x_0 < x < x_1$ (which is approximately true if $[x_0, x_1]$ is a short interval). Since $x_0 \le c_x \le x_1$, we have the upper and lower bounds

$$\frac{(x_1 - x)(x - x_0)}{2} e^{x_0} \le \left| e^x - P_1(x) \right| \le \frac{(x_1 - x)(x - x_0)}{2} e^{x_1}$$

To obtain a bound independent of x, use

$$\max_{x_0 \le x \le x_1} \frac{(x_1 - x)(x - x_0)}{2} = \frac{h^2}{8}, \qquad h = x_1 - x_0 \tag{4.43}$$

This follows easily by noting that $(x_1 - x)(x - x_0)$ is a quadratic with roots at x_0 and x_1 and thus its maximum value occurs midway between the roots. Substituting $x = (x_0 + x_1)/2$ yields the value $h^2/8$.

Noting that $e^{x_1} \le e$ on $[0, 1]$, we have the bound

$$\left| e^x - P_1(x) \right| \le \frac{h^2 e}{8}, \qquad 0 \le x_0 \le x \le x_1 \le 1 \tag{4.44}$$

independent of x, x_0, and x_1. Recall Example 4.1.2 of Section 4.1. With $x = 0.826$ and $h = 0.01$, we have

$$\left| e^x - P_1(x) \right| \le \frac{(0.01)^2(2.72)}{8} = 0.0000340 \tag{4.45}$$

The actual error is -0.0000276, which satisfies this bound. ■

4.2.2 Behavior of the Error

When we consider the error formula (4.41) or (4.53), the polynomial

$$\Psi_n(x) = (x - x_0) \cdots (x - x_n) \tag{4.55}$$

is the most important quantity in determining the behavior of the error. We will examine its behavior for $x_0 \leq x \leq x_n$ when the node points $x_0, \ldots, x_n$ are evenly spaced.

For larger values of n, say, $n \geq 5$, the values of $\Psi_n(x)$ change greatly through the interval $x_0 \leq x \leq x_n$. The values in $[x_0, x_1]$ and $[x_{n-1}, x_n]$ become much larger than the values in the middle of $[x_0, x_n]$. This can be proved theoretically, but we only suggest the result by looking at the graph of $\Psi_n(x)$ when $n = 6$. This is given in Figure 4.5; note the relatively larger values in $[x_0, x_1]$ and $[x_5, x_6]$ as compared with the values in $[x_2, x_4]$. As n increases, this disparity also increases.

When considering $\Psi_n(x)$ as a part of the error formula (4.41) or (4.53) for $f(x) - P_n(x)$, these remarks imply that the interpolation error at x is likely to be smaller when it is near the middle of the node points. In practical interpolation problems, high-degree polynomial interpolation with evenly spaced nodes is seldom used because of these difficulties. However, we will learn in Section 4.6 that high-degree polynomial interpolation with a suitably chosen set of node points can be very useful in obtaining polynomial approximations to functions.

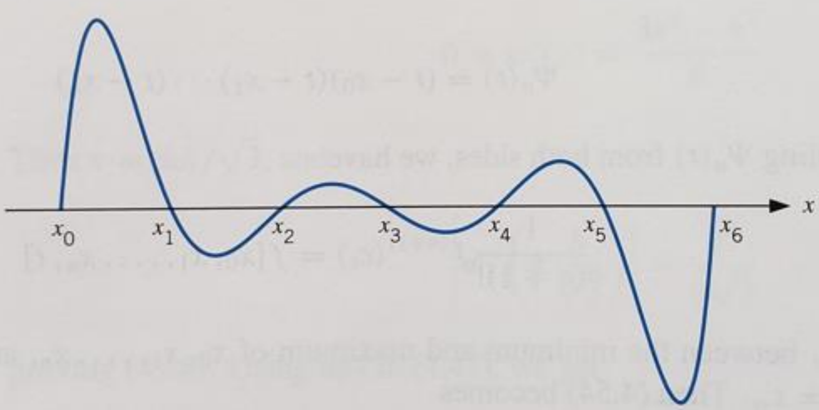


Figure 4.5. $y = \Psi_6(x)$

Least Squares Approximation

The previous section considered the problem of least squares approximation to fit a collection of data. The other approximation problem mentioned in the introduction concerns the approximation of functions.

Suppose $f \in C[a, b]$ and that a polynomial of degree at most n, P_n, is required that will minimize the error

$$\int_a^b [f(x) - P_n(x)]^2\, dx.$$

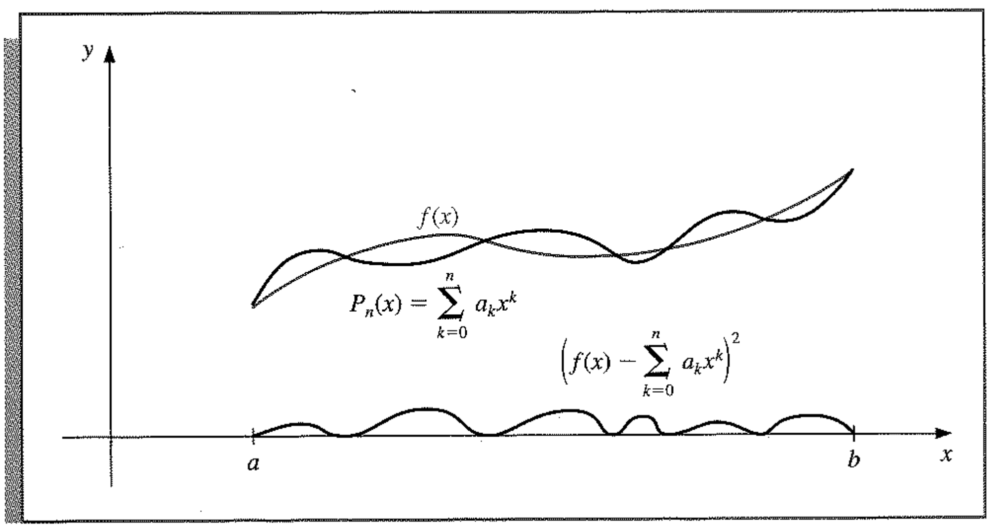
To determine a least squares approximating polynomial, that is, a polynomial to minimize this expression, let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k,$$

and define, as shown in Figure 8.5,

$$E(a_0, a_1, \ldots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k\right)^2 dx.$$

Figure 8.5



The problem is to find real coefficients $a_0, \ldots, a_n$ that will minimize E. A necessary condition for the numbers $a_0, \ldots, a_n$ to minimize E is that

$$\frac{\partial E}{\partial a_j} = 0, \qquad \text{for each } j = 0, 1, \ldots, n.$$

Since
$$E = \int_a^b [f(x)]^2\,dx - 2\sum_{k=0}^{n} a_k \int_a^b x^k f(x)\,dx + \int_a^b \left(\sum_{k=0}^{n} a_k x^k\right)^2 dx,$$

$$\frac{\partial E}{\partial a_j} = -2\int_a^b x^j f(x)\,dx + 2\sum_{k=0}^{n} a_k \int_a^b x^{j+k}\,dx.$$

Hence, to find P_n, the $(n + 1)$ linear **normal equations**

$$\textbf{(8.6)} \qquad \sum_{k=0}^{n} a_k \int_a^b x^{j+k}\,dx = \int_a^b x^j f(x)\,dx, \qquad j = 0, 1, \ldots, n,$$

must be solved for the $(n + 1)$ unknowns a_j. It can be shown that the normal equations always have a unique solution provided $f \in C[a, b]$ and $a \neq b$. (See Exercise 18.)

EXAMPLE 1 Find the least squares approximating polynomial of degree two for the function $f(x) = \sin \pi x$ on the interval $[0, 1]$. The normal equations for $P_2(x) = a_2 x^2 + a_1 x + a_0$ are given by:

$$a_0 \int_0^1 1\,dx + a_1 \int_0^1 x\,dx + a_2 \int_0^1 x^2\,dx = \int_0^1 \sin \pi x\,dx,$$

$$a_0 \int_0^1 x\,dx + a_1 \int_0^1 x^2\,dx + a_2 \int_0^1 x^3\,dx = \int_0^1 x \sin \pi x\,dx,$$

$$a_0 \int_0^1 x^2\,dx + a_1 \int_0^1 x^3\,dx + a_2 \int_0^1 x^4\,dx = \int_0^1 x^2 \sin \pi x\,dx.$$

Performing the integration yields

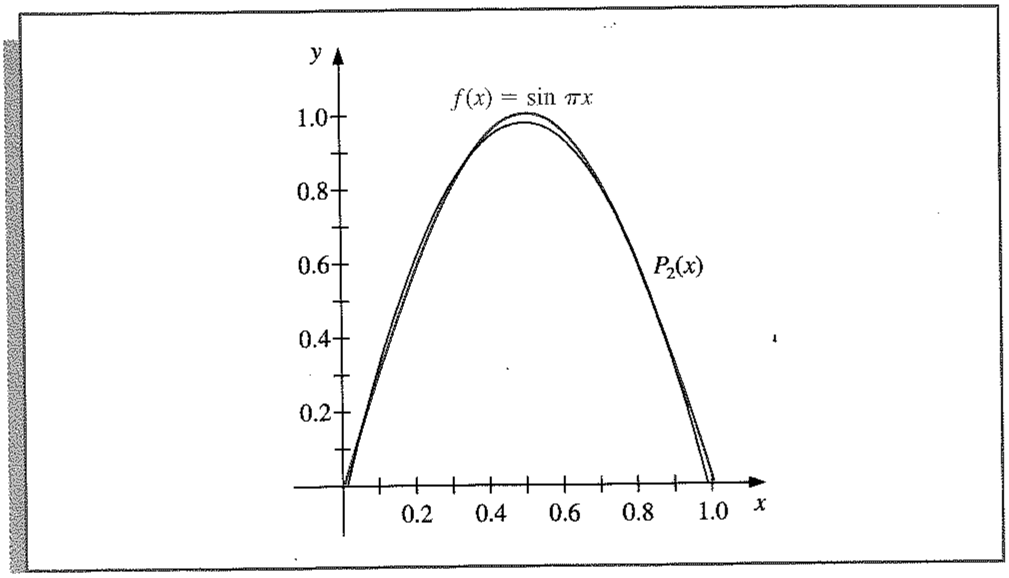
$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{\pi}, \qquad \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{\pi},$$
$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{\pi^2 - 4}{\pi^3}.$$

The three equations in three unknowns can be solved to obtain

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.050465 \qquad \text{and} \qquad a_1 = -a_2 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251.$$

Consequently, the least squares polynomial approximation of degree two for $f(x) = \sin \pi x$ on $[0, 1]$ is $P_2(x) = -4.12251x^2 + 4.12251x - 0.050465$. (See Figure 8.6.)

Figure 8.6



Example 1 illustrates the difficulty in obtaining a least squares polynomial approximation. An $(n + 1) \times (n + 1)$ linear system for the coefficients $a_0, \ldots, a_n$ of P_n must be solved. The coefficients in the linear system are of the form

$$\int_a^b x^{j+k}\,dx = \frac{b^{j+k+1} - a^{j+k+1}}{j + k + 1},$$

a linear system that does not have a convenient numerical solution. The matrix in the linear system is known as a **Hilbert matrix**. This ill-conditioned matrix is a classic example for demonstrating round-off error difficulties; no pivoting technique can be used satisfactorily. (See Exercise 6 of Section 7.4.) Another disadvantage is similar to the situation that occurred when the Lagrange polynomials were first introduced in Section 3.1. The calculations that were performed in obtaining the best nth-degree polynomial, P_n, do not lessen the amount of work required to obtain P_{n+1}, the polynomial of next higher degree.

From: Elementary Numerical Analysis by Atkinson & Han
Numerical Analysis by Burden & Faires