① $A\bar{x} = \bar{b} \Rightarrow \underset{?}{\bar{x}} = \underset{?}{\underbrace{A^{-1}\bar{b}}}$ $O(n^2)$

② $A = LU$

$(A) \rightarrow U$

$L \rightarrow \begin{pmatrix} 1 & & 0 \\ l_{ij} & 1 & \\ & & 1 \end{pmatrix}$

$L(\underbrace{U\bar{x}}_{y}) = \bar{b}$

$L\bar{y} = \bar{b}$

$U\bar{x} = \bar{y}$

$\Rightarrow O(n^2)$

$A \to LU \Rightarrow O(n^3)$

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① Elementary matrices $\quad I \rightarrow$ doesn't do anything

(i) $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ & 1 & \\ & -7 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ & 3 & \\ & 3 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$ $\Rightarrow$

$R_j \rightarrow R_i + R_j$

$R_2 \rightarrow R_2 + R_1$

(ii) Effect of elem. oper.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a+2b & b & c \\ d+2e & e & f \\ g+2h & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{pmatrix}$$

iii) Matrices are actions / functions.

$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to$$ Sub. 7 of 2nd col from 1st col

$\Downarrow$

" inv "

add 7 of 2nd col to 1st col $$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(i) m. R_2 by 2

ii) $R_3 \to R_3 + 2R_2$

$\to$ Sub $2R_2$ for R_3

div. R_2 by 2.

$\underbrace{A = LU}$

$$\overset{l_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \overset{l_1}{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \Rightarrow \overset{U}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix}}$$

!!

Sub. $2R_2$ for R_3

$$\to \begin{pmatrix} 1 & 2 & 3 \\ & -3 & -6 \\ & 0 & 1 \end{pmatrix}$$

$$\overbrace{l_2 \, l_1}^{l} A = U$$

$$\boxed{l^{-1} = L}$$

$A = LU$ $\to$ how to get " L "

$l = l_2 l_1$

$\underline{\underline{L = l_1^{-1} l_2^{-1}}}$

Compute the LU decomposition of the following matrix:

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

This is the same matrix for which we computed its inverse using Gaussian Elimination. Recall that the row matrix operations used during the process were:

- $R_2 = R_2 - 0.75R_1$, $R_3 = R_3 - 0.5R_1$, $R_4 = R_4 - 0.25R_1$
- $R_3 = R_3 - 0.85714R_2$, $R_4 = R_4 - 0.71429R_2$
- $R_4 = R_4 - 0.833333R_2$

The factors are stored in matrix L, and we get

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 1 & 0 & 0 \\ 0.5 & 0.85714 & 1 & 0 \\ 0.25 & 0.71429 & 0.83333 & 1 \end{pmatrix}$$

The matrix computed by using Gaussian Elimination is an upper triangular matrix:

$$U = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1.75 & 1.5 & 1.25 \\ 0 & 0 & 1.71429 & 1.42857 \\ 0 & 0 & 0 & 1.66667 \end{pmatrix}$$

$$AB = I$$

Let us try to compute the inverse of A in the case of an arbitrary 3×3 matrix, so that we get an idea of what the general process should be.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that we can split the product in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So we could solve (by Gaussian Elimination) each one of the systems of equations. Notice that what we want to find are the values of b_ij. So those are our unknowns.

Instead of repeating the same process three times for thee different vectors, we can do all in just one step by using an augmented matrix that includes all the columns of the identity matrix.

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

and then use Gaussian Elimination to obtain an upper triangular matrix on the left side, so after several row matrix operations, the matrix will be transformed into the following matrix (we show each column as a vector)

7. Find $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 2 \end{pmatrix}$.

Solution: We need to use Gaussian elimination to reduce

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{III-2I} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I-3II, III+II} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \xrightarrow{I+2III, II-III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

Thus the inverse is $\begin{pmatrix} -3 & -1 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$.

Find an LU decomposition for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Once again, we begin by using Gaussian Elimination. We take $R_2 - 4R_1 \to R_2$ to get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix} \tag{5}$$

We now take $R_3 - 7R_1 \to R_3$ to get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \tag{6}$$

→ non invertible

Lastly we take $R_3 - 2R_2 \to R_3$ to obtain our upper triangular matrix U:

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Our corresponding lower triangular matrix L will once again have 1's along the main diagonal, and the entries underneath the main diagonal are obtained from the corresponding inverse operations. Thus:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \tag{8}$$

Therefore an LU decomposition for A is:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = LU \tag{9}$$

Note in this particular example that the third row of U is all zeroes. This implies that A itself is noninvertible.