(practice horners rule)

1) $P(x) = 2x^4 - 3x^2 + 3x - 4$

$x_0 = -2$ & Newton-horners to approx a root.

Sol:

$$\left.\begin{array}{r|rrrr} -2 & 2 & -3 & 3 & -4 \\ & & -4 & 14 & -34 \\ \hline & 2 & -7 & 17 & \end{array}\right\} \Rightarrow \text{wrong}$$

write carefully all coefficients.

$$\begin{array}{r|rrrrr} -2 & 2 & 0 & -3 & 3 & -4 \\ & 0 & -4 & 8 & -10 & 14 \\ \hline & 2 & -4 & 5 & -7 & 10 \end{array} \Rightarrow \text{remainder} \quad P(-2)$$

We want $P'(-2)$

$$\begin{array}{r|rrrr} -2 & 2 & -4 & 5 & -7 \\ & 0 & -4 & 16 & -42 \\ \hline & 2 & -8 & 21 & \boxed{-49} \end{array} \Rightarrow P'(-2)$$

$$\Rightarrow x_1 = x_0 - \frac{P(-2)}{P'(-2)} = -2 - \frac{10}{-49} \approx -1.796.$$

Repeat

$$\begin{array}{r|rrrrr} -1.796 & 2 & 0 & -3 & 3 & -4 \\ & & -3.592 & 6.451 & -6.197 & 5.742 \\ \hline & 2 & -3.592 & 3.451 & -3.197 & 1.742 \\ & & -3.592 & 12.902 & -29.368 & \\ \hline & 2 & -7.184 & 16.353 & -32.565 & \end{array} \quad 1.742 = P(x_1)$$

$$x_2 = -1.796 - \frac{1.742}{-32.565} \approx -1.7425$$

Smly $x_3 = -1.73897$ & an actual zero $\Rightarrow -1.73896$

Note $Q(x) \Rightarrow$ depends on approx being used.

Understanding horner's rule

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_0$$
$$= c_n (x-r)^n + c_{n-1} (x-r)^{n-1} + \dots c_1 (x-r) + c_0$$

Taylor expansion

we know $c_k = \dfrac{p^k(r)}{k!}$ → inefficient

Applying horner's rule

if $q(x) = \dfrac{p(x) - p(r)}{x - r} = c_n (x-r)^{n-1} + c_{n-1} (x-r)^{n-2} + \dots + c_1$

$c_1 = 0$

repeat.

Eg:- $P(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$

find taylor expansion about $r = 3$.

3	1	-4	7	-5	2
	0	3	-3	12	21
3	1	-1	4	7	23
	0	3	6	30	
3	1	2	10	37	
	0	3	15		
	1	5	25		
		3			
	1	8			

$$(x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23.$$

(2) $x^3 - 9x^2 + 12$ use newton's to find roots $\Rightarrow$ Exercise

$(1.244, 8.847, -1.091)$

understanding theory / method

$P_n(x) = Q(x)(x - r_0) + r$

$P_n(r_0) = \text{"} r \text{"}$

Synthetic division

$P_n(x) = a_0 + a_1 x + a_2 x^2 \dots a_4 x^4$

$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$P_n(x) = a_0 + (a_1 + ((a_2 + (a_3 + \underbrace{a_4 x}_{\alpha_4}) x) x)) x$

r_0

$\alpha_4 = a_4$

$\alpha_3 = a_3 + \alpha_4 r_0$

$\alpha_2 = a_2 + \alpha_3 r_0$

$\alpha_1 = a_1 + \alpha_2 r_0$

$\alpha_0 = a_0 + \alpha_1 r_0$

Synth. division

	a_4	a_3	a_2	a_1	a_0
r_0		$a_4 r_0$	$\alpha_3 r_0$	$\alpha_2 r_0$	$\alpha_1 r_0$
	a_4	$\underbrace{a_3 + a_4 r_0}_{\alpha_3}$	$\underbrace{a_2 + \alpha_3 r_0}_{\alpha_2}$	$\underbrace{\alpha_2 r_0 + a_1}_{\alpha_1}$	$a_0 + \alpha_1 r_0$

Algorithm (Horner)

$\alpha = a_n$

for $i = n-1$ down to 0

$\alpha = a_i + \alpha r_0$

return α_0

Comp - derivative !

$P_n(x) = Q(x)(x - r_0) + r$

we want $P_n'(x) \rightarrow P_n'(r_0)$.

$P_n'(x) = Q'(x)(x - r_0) + Q(x)$

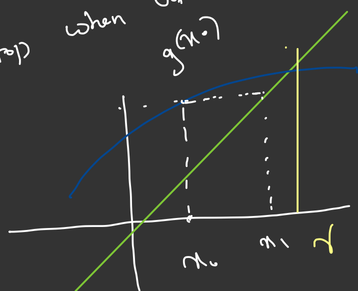
$P_n'(r_0) = Q(r_0)$

Fixed-point iteration

Just enough theory / intuition

- $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = g(x_0) \Rightarrow$$ stop when both are $\approx$ (equal) $g(x_0)$



Final Thm

a) If $g'(x) < 1$ for all x in an interval containing root $[r-\delta, r+\delta]$

& $x_0 \in [r-\delta, r+\delta]$

then seq. $x_{k+1} = g(x_k)$ converges to r.

$\Rightarrow$ Fixed point $\Rightarrow$ where the value of the func doesn't change

why imp? $g(p) = p$.

$f(x) \Rightarrow$ find root. (let $f(p) = 0$)

$x - f(x) = g(x)$ or as $g(x) = x + 3f(x)$

Get comfortable with finding fixed points :

① Det any fixed points to $g(x) = x^2 - 2$

$$g(p) = p \Rightarrow p = p^2 - 2$$

$$\Rightarrow p^2 - p - 2 = (p+1)(p-2) \Rightarrow \text{2 fixed points.}$$

Ex 2

$\Rightarrow x^3 + 4x^2 - 10 = 0$ has root in $[1,2]$

change to this form $x = g(x)$

$\boxed{\text{\& start at } 1.5}$

a. $x = x - x^3 - 4x^2 + 10$

x_k	x_{k+1} & $g(x)$	error
1.5	-0.875	
	6.732	
	-469.7	
	1.03×10^8	

b. $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

x_k	x_{k+1} & $g(x)$	error
1.5	0.8165	
	2.9969	
	$(-8.65)^{1/2}$	

(c) $x = x - \dfrac{x^3+4x^2-10}{3x^2+8x} \Rightarrow$

x_k	x_{k+1} ($g(x)$)	error
1.5	1.37...	
	1.36-...	
	1.36--	
	1.365230013	

!!

Que: How to find a good form?

(d) $x = \left(\dfrac{10}{4+x} \right)^{1/2}.$

x_k	x_{k+1} ($g(x)$)	error
1.5	1.34-	
	1.3673	
	1.3649	
	1.365--	
12	⋮	
	1.365230013 (0014)	

let's revisit (a.)

$g_1(x) = x - x^3 - 4x^2 + 10$

$g_1'(x) = 1 - 3x^2 - 8x \Rightarrow |g_1'(x)| > 1$ for all $x \in [1,2]$

$g_1(1) = 6$

$g_1(2) = -12$

Now, for $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

$$g'(x) = \left| \frac{-5}{\sqrt{10}\,(4+x)^{3/2}} \right| \le 0.15 \quad \forall\, x \in [1,2]$$

$m = \max\left(|g'(x)| \;\middle|\; x \in [r-\delta, r+\delta]\right)$

&

$e_{k+1} \le m\, e_k$

as small derivative as possible is good

Some more practice

① $x^3 + 6x^2 - 8$ → $[1,2]$ start at 1.5

$\boxed{1.0641}$

② $x = x^3 + 6x^2 + x - 8$ } ⇒ which is better

③ $x = \sqrt{\frac{8}{x+6}}$

&

$g'(x) = \sqrt{2} \,/\, (x+6)^{3/2} < 1$

$g'(x) = 3x^2 + 12x + 1$

(No convergence)