CS323 LECTURE NOTES - LECTURE 6

1 Fixed Point Iteration

1.1 Introduction

Previously we discussed Newton's Method, which is based on the formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The right hand side of this formula can be seen as another function $g(x)$, and so the method could be rewritten as

$$x_1 = g(x_0) \quad \text{where} \quad g(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In other words, we actually are solving the problem of finding a root r of the equation $r = g(r)$ where we provide a an initial guess x_0 and compute a sequence of values $x_1, x_2, \ldots$

$$\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ x_3 &= g(x_2) \\ x_4 &= g(x_3) \\ &\vdots \end{aligned}$$

and we stop when we exceed the maximum number of iterations allowed or the error $|x_k - x_{k-1}| < \epsilon$, where ϵ is the desired error tolerance.

In doing this, we hope that the sequence $x_1, x_2, \ldots$ will converge a root r of $x = g(x)$.

There are several ways in which the problem of finding a root of $f(x) = 0$ can be transformed into a *Fixed Point Iteration* problem. A very simple one is to add x to both sides of the equation

starting with	$f(x) = 0$
add x on both sides	$f(x) + x = x$
rewrite	$x = g(x)$ where $g(x) = f(x) + x$

Example 1

Solve $\cos x - x = 0$ given $x_0 = 1$ and $\epsilon = 10^{-4}$
We start by adding x to both sides, so we get $g(x) = \cos x$, which is our fixed point problem. Now we compute the sequence

k	x_k	$x_{k+1} = \cos x_k$	$\lvert x_1 - x_0 \rvert$
0	1.0000	0.5403	0.45970
1	0.5403	0.8576	0.31725
2	0.8576	0.6543	0.20326
3	0.6543	0.7935	0.13919
4	0.7935	0.7014	0.09211
5	0.7014	0.7640	0.06259
6	0.7640	0.7221	0.04186
7	0.7221	0.7504	0.02832
8	0.7504	0.7314	0.01901
9	0.7314	0.7442	0.01283
10	0.7442	0.7356	0.00863
11	0.7356	0.7414	0.00582
12	0.7414	0.7375	0.00392
13	0.7375	0.7401	0.00264
14	0.7401	0.7384	0.00178
15	0.7384	0.7396	0.00120
16	0.7396	0.7388	0.00081
17	0.7388	0.7393	0.00054
18	0.7393	0.7389	0.00037
19	0.7389	0.7392	0.00025
20	0.7392	0.7390	0.00017
21	0.7390	0.7391	0.00011
22	0.7391	0.7391	0.00008

As we can see the sequence converges, and the solution 0.7391 with the desired tolerance is found in 22 iterations.

Example 2

Solve $(2x-1)e^{-x}=0$
Using the same trick as before, we add x to both sides of the equation and we get $(2x-1)e^{-x}+x=x$, so

$$g(x)=(2x-1)e^{-x}+x$$

the root r is clearly $1/2$ since the exponential can never be 0.
Therefore, we will try to solve the *fixed point iteration* problem using an initial guess $x_0=0.49$.
Now we compute the sequence of x_k values

k	x_k	$x_{k+1}=\cos x_k$	$\lvert x_1-x_0\rvert$
0	0.4900	0.4777	0.01225
1	0.4777	0.4501	0.02760
2	0.4501	0.3866	0.06357
3	0.3866	0.2325	0.15411
4	0.2325	-0.1916	0.42408
5	-0.1916	-1.8669	1.67534
6	-1.8669	-32.4883	30.62134

It is enough to compute a few iterations to notice that the error is increasing. So, the sequence clearly does not converge.

1.2 Geometric Interpretation

Figure 1

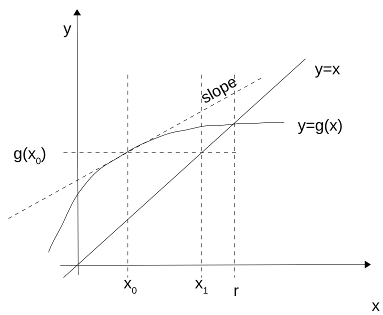
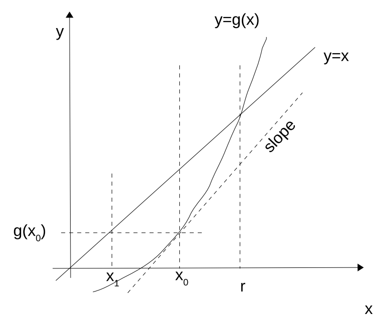


Figure 2



Consider the two cases shown in figures 1 and 2. In figure 1, r represents the point where the line $y = x$ and the function $y = g(x)$ intersect, i.e. it is the root. The original guess is given by x_0, notice that the new point x_1 computed based on $g(x_0)$ is closer to r than the original point, therefore it is going to converge.

On the other hand, in the case of figure 2, the new point x_1 is farther away from the root r than the original guess x_0, which suggests that the sequence is going to diverge.

The difference between both cases is the slope of the line tangent to $g(x_0)$, in the case of figure 1, the slope is < 1 which makes the new point closer to the solution, but in the case of figure 2, the slope is > 1 which makes the new point farther away from the root r.

The analysis of both figures suggests that the convergence of the algorithm must have some relation to the derivative of $g(x)$. So, we will keep this in mind when analyzing this more formally.

1.3 Convergence Analysis

To determine if it converges or diverges, we can use the error computed in two successive iterations (e_k, e_{k+1}), if the error is decremented after each iteration, it converges, otherwise it diverges.

$$e_k = |r - x_k|$$
$$e_{k+1} = |r - x_{k+1}|$$

Since the absolute error is given by the absolute value of the difference of the actual value and the approximation. We know that the actual value is r such that $r = g(r)$.

Recall also that $x_{k+1} = g(x_k)$, so we have that

$$e_{k+1} = |g(r) - g(x_k)|$$

As we said, we want to somehow link the convergence of the fixed point iteration method to the derivative. So the following theorem from calculus comes to mind.

Mean Value Theorem. If $f(x)$ is a continuous differentiable function in the interval $[a, b]$ then $\exists\ \xi \in [a, b]$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

i.e. $f(b) - f(a) = f'(\xi)(b - a)$

Now we can apply this theorem to our case, so we have

$$e_{k+1} = |g(r) - g(x_k)| = |g'(\xi)||r - x_k|$$

where ξ is in between r and x_k.

We can see that $|r - x_k|$ is just the absolute error e_k, so we get

$$e_{k+1} = |g'(\xi)|e_k$$

This is what we were looking for since it shows a relationship between the error in iteration k and the error in iteration $k+1$. For the method to converge we want to error to be smaller with each successive iteration, therefore we need for $e_{k-1} < e_k$, and the only way in which that happens is if

$$|g'(\xi)| < 1$$

Let us write the convergence criterion formally:

Given $g(x) \in \mathbb{R} \to \mathbb{R}$ such that $r \in \mathbb{R}$ satisfies $r = g(r)$.
If $|g'(x)| < 1$ for all x in an interval containing the root ($[r-\delta, r+\delta]$) and if $x_0 \in [r - \delta, r + \delta]$
Then the sequence $x_{k+1} = g(x_k)$ converges to r

In order to make things simple when analyzing the convergence of function we can compute the maximum value of $|g'(x)|$ in the interval in which we are interested.

$$m = \max\{|g'(x)|\,|x \in [r-\delta, r+\delta]\,\}$$

Notice that because of the convergence criterion $m < 1$, and since m is the maximum of $|g'(x)|$ then $|g'(x)| \leq m$, so we have

$$e_{k+1} = |g'(\xi)|e_k \leq m\ e_k$$

1.4 Examples (again)

Let us go back to our examples and analyze them based on the convergence criteria the we found:
Example 1 We had $g(x) = \cos x$, so $g'(x) = -\sin x$, so

$$|g'(x)| = |\sin x|$$

Since we know that $r \approx 0.7\ldots$ and we were given $x_0 = 1$, then we can use the interval $[0, 1]$ for the purpose of convergence analysis. Since $\sin x$ is an increasing function in this interval, its maximum is given by $|\sin x|$ in the left hand side of the interval, i.e.

$$m = \max\{|g'(x)|\,|x \in [0,1]\,\} = |\sin 1| = 0.8414\ldots < 1$$

Therefore it converges, since $m < 1$

Example 2 We had $g(x) = (2x-1)e^{-x} + x$, so

$$|g'(x)| = |-(2x-1)e^{-x} + 2e^{-x} + 1|$$

The interval that we need to use includes $r = 0.5$. For the method to converge we need $|g'(x)| < 1$ for every x in an interval including r. Notice that

$$|g'(0.5)| = |2e^{-0.5} + 1| > 1$$

Therefore it will not converge (we already knew that).

1.5 Fixed Point Iteration Algorithm

The algorithm is similar to Newton's, it only requires us to keep track of the error tolerance and the number of iterations as exit conditions for the repeat statement

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Fixed Point Iteration Algorithm
Input: $g : \mathbb{R} \to \mathbb{R}, x_0 \in \mathbb{R}, \epsilon > 0, N \in \mathbb{Z}^+$

$count = 0$
repeat
    $x_1 = g(x_0)$
    $e = |x_1 - x_0|$
    $x_0 = x_1$
    $count + +$
until $e < \epsilon$ or $count > N$
if $count > N$ return error
return x_1

1.6 Convergence and Number of Iterations

We want to answer the question of how fast the algorithm converges as a function of the number of iterations. This will help us find out the approximate number of iterations that it might require for the algorithm to converge. So, we are looking for a relationship between the error in iteration k and the error that the algorithm makes during the first iteration.

$$e_k = |r - x_k| \leq (\text{something including k})|x_1 - x_0|$$

Assume that the algorithm converges (we are studying the rate of convergence after all), so

$$|g'(x)| \leq m < 1 \;\; \forall x \in [r - \delta, r + \delta]$$

Remember that

$$e_k = |g'(x)|e_{k-1} \leq m\; e_{k-1}$$

We can use this formula for all the previous iterations:

$$\begin{gathered} e_k \leq m\; e_{k-1} \\ e_{k-1} \leq m\; e_{k-2} \\ e_{k-2} \leq m\; e_{k-3} \\ \vdots \\ e_1 \leq m\; e_0 \end{gathered}$$

Substituting the each formula in the previous one, we have

$$e_k \leq m\; e_{k-1} \leq m^2\; e_{k-2} \leq \ldots \leq m^k\; e_0$$

So $e_k \leq m^k e_0$, where $e_0 = |r - x_0|$. Still this formula is not yet in the form that we want since we need to have $|x_1 - x_0|$ instead or $|r - x_0|$ because we don't know the value of r. Let us try to write e_0 as a function of $|x_1 - x_0|$ by using a simple trick (add and subtract x_1), so we get

$$\begin{aligned} e_0 &= |r - x_0| \\ &= |r - x_1 + x_1 - x_0| \qquad \text{added } \; 0 = x_1 - x_1 \\ &= |(r - x_1) + (x_1 - x_0)| \\ &\leq |r - x_1| + |x_1 - x_0| \qquad \text{using Triangle Inequality} \end{aligned}$$

Since we know that $|r - x_1| = e_1 \leq m\, e_0$ we get

$$e_0 \leq m\, e_0 + |x_1 - x_0|$$

and solving for e_0, knowing that $1 - m > 0$ because $m < 1$, we have

$$e_0 \leq \frac{|x_1 - x_0|}{1 - m}$$

Substituting in e_k we get

$$e_k \leq \frac{m^k}{1 - m}|x_1 - x_0|$$

This formula can be used to obtain an estimate of the number of iterations k needed to get an error $e_k \approx \epsilon$, since all the values are known except for k