CS323 LECTURE NOTES - LECTURE 18

1 Numerical Integration

It is a well known fact that most integrals cannot be solved in closed form (finite sequence of additions, subtractions, multiplication, divisions), for example

$$\int_a^b e^{-x^2} dx$$

In this cases we can use numerical methods. The firs idea that comes to mind when trying to evaluate an integral is to use its definition using Riemann's sums, in which the integration interval $[a, b]$ is divided in n sub-intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ and the following summation is computed

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \quad \xi_i \in [x_{i-1}, x_i]$$

This method requires the computation of n rectangles and its precision depends on the choice of ξ_i. Its computation requires n subtractions, n products and n additions. Since products are computationally more expensive than sums, we would like to have other methods that are easy to program and do not require the computation of many products. A plus would be to find error bounds that are reasonable and easy to compute.

You might imagine that since we have talked about interpolating polynomials before, we will use Lagrange polynomials for numerical integration, where we will compute the integral of the interpolating polynomial instead of the function itself.

1.1 Trapezoidal Rule

In this first method we will use a Lagrange polynomial of degree one to approximate the integral

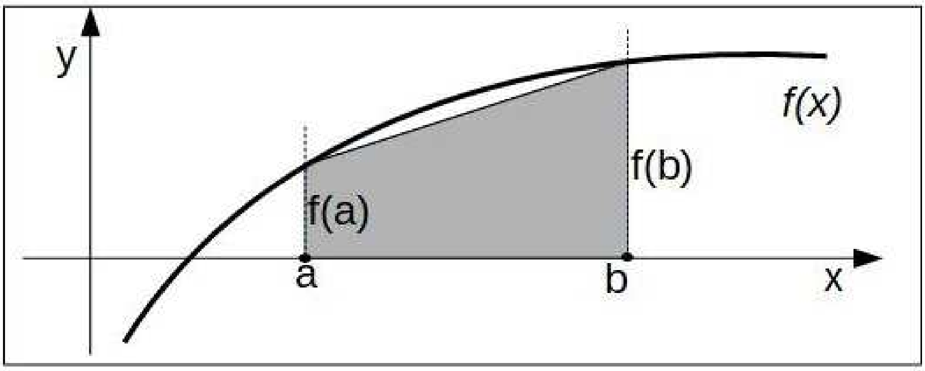
$$\int_a^b f(x)dx$$

As we said before, the first degree Lagrange polynomial that goes through the points $(a, f(a)), (b, f(b))$ is:

$$\begin{aligned}
P_1(x) &= \frac{(x-b)}{(a-b)} f(a) + \frac{(x-a)}{(b-a)} f(b) \\
&= \frac{1}{b-a}\left((x-a)f(b) - (x-b)f(a)\right) \\
&= \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right) x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right)
\end{aligned}$$

If we integrate this polynomial we get

$$\begin{aligned}
\int_a^b f(x)\underset{.}{\mathrm{x}} &\approx \int_a^b P_1(x)\underset{.}{\mathrm{x}} \\
&= \int_a^b \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right) x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right) \underset{.}{\mathrm{x}} \\
&= \frac{1}{2}\left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right) x^2\Big|_a^b - \left(\frac{bf(a)}{a-b} + \frac{af(b)}{b-a}\right) x\Big|_a^b \\
&= \frac{1}{2}(b-a)(f(a)+f(b))
\end{aligned}$$



Notice that this formula can be obtained also if we observe that the first degree Lagrange polynomial is a line that goes through the points $(a, f(a)), (b, f(b))$. So the area under the curve corresponds to a trapezoid with height $b-a$, base 1 equal to $f(a)$, and base 2 equal to $f(b)$, therefore its area is equal to

$$A = \frac{1}{2}(b-a)(f(a)+f(b))$$

This is the reason why this formula is called the *Trapezoidal rule*:

$$\int_a^b f(x)dx \approx \frac{1}{2}(b-a)(f(a)+f(b))$$

Example

Approximate the value of the following integral using the *trapezoidal rule*

$$\int_1^2 e^x dx$$

Solution:

$$\int_1^2 e^x dx \approx \frac{1}{2}(1)(e^1+e^2) = 5.05367$$

The actual solution is 4.670774, therefore the error=0.382896

1.2 Composite Trapezoidal Rule

As we did in the case of Riemann's sums, we will subdivide the interval $[a, b]$ into n sub-intervals of equal length $h = \frac{b-a}{n}$

If we now approximate the integral of $f(x)$ in each segment using the trapezoidal rule, we have the following formula for each sub-interval i

$$\begin{aligned} \int_{x_{i-1}}^{x_i} f(x)\underset{.}{x} &\approx \frac{1}{2}(x_i - x_{i-1})(f(x_i)+f(x_{i-1})) \\ &= \frac{1}{2}h(f(x_i)+f(x_{i-1})) \end{aligned}$$

where $h = x_i - x_{i-1}$ is the length of each sub-interval.

If we compute the integral over the entire interval $[a,b]$ where $x_0 = a$ and $x_n = b$ we get

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\underset{\cdot}{x} \\
&\approx \sum_{i=1}^{n} \frac{1}{2} h(f(x_{i-1}) + f(x_i)) \\
&= \frac{1}{2} h\left(f(x_0) + f(x_1) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)\right)
\end{aligned}$$

Notice that the terms $f(x_0)$ and $f(x_n)$ appear only once, but the other terms appear twice, we have
The ***Composite Trapezoidal Rule***:

$$\int_a^b f(x)dx \approx \frac{1}{2} h(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i))$$

where $h = \frac{b-a}{n}$, $x_i = a + ih$

1.2.1 Analysis

Assuming that we already have all the values of $f(x_i)$ previously computed , the computation of a definite integral using this method requires only $n+1$ additions and only two products, which is a great improvement over using Riemann's sums.

Example

Approximate the value of the following integral using the *composite trapezoidal rule* with $n = 8$

$$\int_1^2 e^x dx$$

Since $n = 8$, then $h = \frac{2-1}{8} = 0.125$, the values of x_i can be obtained as $x_i = 1 + 0.125i$, $i = 0 \ldots 8$. Notice that there are 8 intervals and 9 points. Writing the values on a table we have:

i	x_i	$f(x_i)$
0	1	2.718282
1	1.125	3.080217
2	1.25	3.490343
3	1.375	3.955077
4	1.5	4.481689
5	1.625	5.078419
6	1.75	5.754603
7	1.875	6.520819
8	2	7.389056

$$\int_1^2 e^x dx \approx \frac{1}{2}(0.125)(2.718282 + 7.389056 + 2(32.361167)) = 4.67685$$

Which is a much better approximation than the one obtained before. The error in this case is 0.006