CS323 LECTURE NOTES - LECTURE 8

1 Cramer's Rule

As we remember from linear algebra, there is another way of solving a system of linear equations. We can use Cramer's Rule, which requires us to compute determinants.

Recall that given the system of equations

$$A\bar{x} = \bar{b}$$

We can solve it by computing the determinant of A, as well as the determinants of the matrix A_i obtained by replacing column i by the column vector b. It can be shown (linear algebra course) that the solutions are:

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \forall x = 1, \ldots, n$$

This gives us an easy way to solve the system of equations, but we now need to find an efficient way to compute the determinant of a given matrix

1.1 Using minors

The first method that comes to mind when trying to find the determinant of a $n \times n$ matrix is to use minors. We must also keep in mind that given a problem there might be several different algorithms to solve it, and it is important to compare them to determine which one is the most appropriate for the particular case that we are trying to solve.

Given a $n \times n$ matrix A, the minor of i, j is equal to the determinant of size $(n-1) \times (n-1)$ obtained after eliminating from A row i and column j.

To compute the determinant of A by using minors, we pick a row and then compute the summation of the products of the elements of that row times its corresponding minor being careful to alternate signs as shown in the following formula.

$$|A| = \sum_{j=1}^{n} (-1)^{j+i} a_{ij} C_{ij}$$

where C_{ij} is the minor i, j or matrix A.
Example

Use minors to compute the determinant of the following matrix

$$A = \begin{pmatrix} 2 & 6 & 4 \\ 3 & 13 & 8 \\ -1 & -1 & -2 \end{pmatrix}$$

Using the formula given above (picking row number 1):

$$|A| = 2 \begin{vmatrix} 13 & 8 \\ -1 & -2 \end{vmatrix} - 6 \begin{vmatrix} 3 & 8 \\ -1 & -2 \end{vmatrix} + 4 \begin{vmatrix} 3 & 13 \\ -1 & -1 \end{vmatrix}$$

and the result is

$$|A| = -8$$

1.2 Analysis

Notice that to compute the determinant of a matrix of size n we must compute n determinants of size $n - 1$, and to compute each one of them, we must compute $n-1$ determinants of size $n-2$, and so on, until we get to compute a determinant of size 1 which is just a number. The total running time of this algorithm is clearly

$$T(n) \in O(n!)$$

It is an interesting exercise to compute the time it takes to compute a determinant of a 30×30 matrix by using this algorithm. If you assume a normal modern computer, it will take many times the age of the Universe to find the answer!