

$$(1) \quad A \vec{x} = \vec{b} \Rightarrow \vec{x} = \underbrace{A^{-1} \vec{b}}_{?} \quad O(n^3)$$

$$(2) \quad A = LU$$

$$\begin{aligned} (A) & \rightarrow U \\ & L \rightarrow \begin{pmatrix} 1 & 0 \\ l_{ij} & 1 \end{pmatrix} \end{aligned} \quad \begin{aligned} L(U\vec{x}) &= \vec{b} \\ \vec{y} &= U\vec{x} \\ L\vec{y} &= \vec{b} \\ U\vec{x} &= \vec{y} \end{aligned} \quad \begin{aligned} & \Rightarrow O(n^2) \\ A \rightarrow LU \Rightarrow O(n^3) \end{aligned}$$

(1) Elementary matrices $I \rightarrow$ doesn't do anything

$$\begin{aligned} (i) \quad & \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ -7 & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 3 & \\ & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \end{aligned}$$

$R_2 \rightarrow R_2 + R_1$
 $R_3 \rightarrow R_3 + R_1$

(ii) Effect of elem op.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a+2b & b & c \\ d+2e & e & f \\ g+2h & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{pmatrix}$$

iii) Matrices are actions / functions.

$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{\text{Sub. 7 of 1st col} \\ \text{from 1st col}}} \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{\text{add 7 of 2nd col} \\ \text{to 1st col.}}} \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \xrightarrow{\substack{\text{Sub } 2R_2 \text{ by } R_2 \\ \text{div. } R_2 \text{ by } 2.}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i) m. R_2 by 2
 ii) $R_3 \rightarrow R_3 + 2R_2$

$A = LU$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{L_2} \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix} \xrightarrow{L_1} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{Sub. } 2R_2 \text{ by } R_2 \\ R_3}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{L_2 L_1}_L A = U \quad \boxed{L^{-1} = L^T}$$

$$A = LU \quad \hookrightarrow \text{how to get } L^{-1}$$

$$L = L_1 L_2$$

$$\underline{\underline{L^{-1} = L_2^{-1} L_1^{-1}}}$$

Compute the LU decomposition of the following matrix:

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

This is the same matrix for which we computed its inverse using Gaussian Elimination. Recall that the row matrix operations used during the process were:

- $R_2 = R_2 - 0.75R_1$, $R_3 = R_3 - 0.5R_1$, $R_4 = R_4 - 0.25R_1$
- $R_3 = R_3 - 0.85714R_2$, $R_4 = R_4 - 0.71429R_2$
- $R_4 = R_4 - 0.83333R_3$

The factors are stored in matrix L , and we get

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 1 & 0 & 0 \\ 0.5 & 0.85714 & 1 & 0 \\ 0.25 & 0.71429 & 0.83333 & 1 \end{pmatrix}$$

The matrix computed by using Gaussian Elimination is an upper triangular matrix:

$$U = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1.75 & 1.5 & 1.25 \\ 0 & 0 & 1.71429 & 1.42857 \\ 0 & 0 & 0 & 1.66667 \end{pmatrix}$$

$$AB = I$$

Let us try to compute the inverse of A in the case of an arbitrary 3×3 matrix, so that we get an idea of what the general process should be.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that we can split the product in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So we could solve (by Gaussian Elimination) each one of the systems of equations. Notice that what we want to find are the values of b_{ij} . So those are our unknowns.

Instead of repeating the same process three times for three different vectors, we can do all in just one step by using an augmented matrix that includes all the columns of the identity matrix.

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right)$$

and then use Gaussian Elimination to obtain an upper triangular matrix on the left side, so after several row matrix operations, the matrix will be transformed into the following matrix (we show each column as a vector)

7. Find $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 2 \end{pmatrix}$.

Solution: We need to use Gaussian elimination to reduce

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 & 0 & 1 \end{array} \right) \xrightarrow{III-2I} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{I-3II, III+II} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array} \right) \xrightarrow{I+2III, II-III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array} \right)$$

Thus the inverse is $\begin{pmatrix} -3 & -1 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$.