CS323 LECTURE NOTES - LECTURE 9

1 Applications of Gaussian Elimination

1.1 Determinants by Gaussian Elimination

A different and more efficient way to compute the determinant of a matrix A is to use row matrix operations on the original matrix to transform it into an upper triangular matrix. We can recall the following results from linear algebra:

- The value of the determinant of A does not change if we use $R_i = cR_i$, or $R_j = R_j + cR_i$
- The value of the determinant of A will change sign if we use $R_i \leftrightarrow R_j$

Notice also that using minors (starting from the bottom left element of the matrix) on a triangular matrix we can see that the determinant can be easily computed as just the product of the main diagonal.

This takes us back to Gaussian Elimination, since we can use it to transform A into an upper triangular matrix, being careful of keeping track of the row swaps. Then use the resulting matrix to compute the determinant just by multiplying all the elements on its main diagonal.

Algorithm: determinant

Input: A: $n \times n$ matrix

sign $\leftarrow 1$

for $i = 1$ to $n - 1$

    if $a_{ii} == 0$

        if there exists a row $k > i$, such that $a_{ki} \neq 0$ then

            $R_k \leftrightarrow R_i$

            sign $\leftarrow$ -sign

        else

            return 0

    else

        for $j = i + 1$ to n

            $R_j = R_j - \frac{a_{ji}}{a_{ii}} R_i$

$det = sign$;

for $i = 1$ to n

    $det = det * a_{ii}$

return det

Example:

Compute the determinant of

$$A = \begin{pmatrix} 2 & 6 & 4 \\ 3 & 13 & 8 \\ -1 & -1 & -2 \end{pmatrix}$$

After each iteration we get the following matrices:

i	A	sign
1	$\begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 2 \\ 0 & 2 & 0 \end{pmatrix}$	1
2	$\begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix}$	1

Notice that the final matrix is an upper triangular matrix, therefore the determinant is

$$\det(A) = (1)(2)(4)(-1) = -8$$

Where the first 1 corresponds to the value of the variable $sign$.

1.2 Inverse by Gaussian Elimination

Given a $n \times n$ matrix A, its inverse A^{-1} is a matrix such that

$$AA^{-1} = A^{-1}A = I$$

All the entries in the identity matrix I are 0 except for the main diagonal, which contains only 1's

$$I_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The identity matrix has the property that

$$IB = B$$

Where B is any $n \times m$ matrix.

The inverse of a matrix can be used to solve a system of linear equations:

$$\begin{aligned} A\bar{x} &= \bar{b} \\ \Rightarrow A^{-1}A\bar{x} &= A^{-1}\bar{b} \\ \Rightarrow I\bar{x} &= A^{-1}\bar{b} \\ \Rightarrow x &= A^{-1}\bar{b} \end{aligned}$$

Suppose that B is the inverse of A, so we have that

$$AB = I$$

Let us try to compute the inverse of A in the case of an arbitrary 3×3 matrix, so that we get an idea of what the general process should be.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that we can split the product in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So we could solve (by Gaussian Elimination) each one of the systems of equations. Notice that what we want to find are the values of b_ij. So those are our unknowns.

Instead of repeating the same process three times for thee different vectors, we can do all in just one step by using an augmented matrix that includes all the columns of the identity matrix.

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

and then use Gaussian Elimination to obtain an upper triangular matrix on the left side, so after several row matrix operations, the matrix will be transformed into the following matrix (we show each column as a vector)

$$\left(\text{U} \,\middle|\, \bar{c_1} \quad \bar{c_2} \quad \bar{c_3} \right)$$

Where U represents the upper triangular matrix and c_1, c_2, and c_3 are the columns obtained form the identity matrix.

The system of equations represented by each column can be solved using backward substitution in order to obtain the inverse of A.

Example:

Compute the inverse of

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

We start by using row matrix operation to transform the matrix augmented with the inverse into an upper triangular matrix (on the left hand side):

$$A = \left(\begin{array}{cccc|cccc} 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 4 & 3 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right)$$

After operations: $R_2 = R_2 - 0.75R_1$, $R_3 = R_3 - 0.5R_1$, $R_4 = R_4 - 0.25R_1$ we get

$$\sim \left(\begin{array}{cccc|cccc} 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & -0.75 & 1 & 0 & 0 \\ 0 & 1.5 & 3 & 2.5 & -0.5 & 0 & 1 & 0 \\ 0 & 1.25 & 2.5 & 3.75 & -0.25 & 0 & 0 & 1 \end{array}\right)$$

After operations: $R_3 = R_3 - 0.85714R_2$, $R_4 = R_4 - 0.71429R_2$ we get

$$\sim \left(\begin{array}{cccc|cccc} 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & -0.75 & 1 & 0 & 0 \\ 0 & 0 & 1.71429 & 1.42857 & 0.14286 & -0.85714 & 1 & 0 \\ 0 & 0 & 1.42857 & 2.85714 & 0.28571 & -0.71429 & 0 & 1 \end{array}\right)$$

After operation: $R_4 = R_4 - 0.833333R_2$ we get

$$\sim \left(\begin{array}{cccc|cccc} 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & -0.75 & 1 & 0 & 0 \\ 0 & 0 & 1.71429 & 1.42857 & 0.14286 & -0.85714 & 1 & 0 \\ 0 & 0 & 0 & 1.66667 & 0.16667 & 0 & -0.83333 & 1 \end{array}\right)$$

We now solve the following tree systems of equations using backward substitution:

$$\left(\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & 1.75 & 1.5 & 1.25 & -0.75 \\ 0 & 0 & 1.71429 & 1.42857 & 0.14286 \\ 0 & 0 & 0 & 1.66667 & 0.16667 \end{array}\right) \longrightarrow \begin{pmatrix} 0.6 \\ -0.5 \\ 0 \\ 0.1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & 1 \\ 0 & 0 & 1.71429 & 1.42857 & -0.85714 \\ 0 & 0 & 0 & 1.66667 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} -0.5 \\ 1 \\ -0.5 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & 0 \\ 0 & 0 & 1.71429 & 1.42857 & 1 \\ 0 & 0 & 0 & 1.66667 & -0.833333 \end{array}\right) \longrightarrow \begin{pmatrix} 0 \\ -0.5 \\ 1 \\ -0.5 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 0 \\ 0 & 1.75 & 1.5 & 1.25 & 0 \\ 0 & 0 & 1.71429 & 1.42857 & 0 \\ 0 & 0 & 0 & 1.66667 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} 0.1 \\ 0 \\ -0.5 \\ 0.6 \end{pmatrix}$$

Each solution represents one column of the inverse:

$$A^{-1} = \begin{pmatrix} 0.6 & -0.5 & 0 & 0.1 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0.1 & 0 & -0.5 & 0.6 \end{pmatrix}$$

1.3 LU Decomposition by Gaussian Elimination

Suppose that we were able to factor a matrix A into a lower triangular matrix L and an upper triangular matrix U such that the product was equal to A, i.e.

$$A = LU$$

We could use this decomposition to solve a system of equations $A\bar{x} = \bar{b}$ in the following way:

$$\begin{aligned} A\bar{x} &= \bar{b} \\ (LU)\bar{x} &= \bar{b} \\ L(U\bar{x}) &= \bar{b} \end{aligned}$$

If we let $\bar{y} = U\bar{x}$, the system can be written as two systems:

$$\begin{aligned} L\bar{y} &= \bar{b} \\ U\bar{x} &= \bar{y} \end{aligned}$$

Since we know $\bar{b}$ and L we can compute $\bar{y}$ by using forward substitution. Once we have $\bar{y}$ we can compute $\bar{x}$ by backward substitution. Notice that to solve both systems of equations would only require $O(n^2)$ operations. Of course the problem is to compute the LU factorization in the first place.

Notice that Gaussian Elimination already produces an upper triangular matrix U. It can be shown (linear algebra) that the factors used to multiply each row in the Gaussian Elimination algorithm, can be stored in a lower triangular matrix L and that

$$A = LU$$

providing the desired decomposition. We only need to add one instruction to our Gaussian Elimination algorithm (version 1.0) noticing that we require the main diagonal of L to consists of 1's.

Algorithm: Gaussian Elimination 1.0 (LU decomposition)
Input: Augmented matrix a_{ij} $(1 \leq i \leq n,\ \ 1 \leq j \leq n+1)$

Let L be equal to the identity matrix
for $j = 1$ to $n-1$
  for $i = j+1$ to n
    // Store the factors in l_{ji} of L $n+1)$
    $R_i = R_i - \left(\dfrac{a_{i,j}}{a_{j,j}}\right) R_j$
    $l_{ij} = \frac{a_{i,j}}{a_{j,j}}$

Example

Compute the LU decomposition of the following matrix:

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

This is the same matrix for which we computed its inverse using Gaussian Elimination. Recall that the row matrix operations used during the process were:

- $R_2 = R_2 - 0.75R_1$, $R_3 = R_3 - 0.5R_1$, $R_4 = R_4 - 0.25R_1$
- $R_3 = R_3 - 0.85714R_2$, $R_4 = R_4 - 0.71429R_2$
- $R_4 = R_4 - 0.833333R_2$

The factors are stored in matrix L, and we get

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 1 & 0 & 0 \\ 0.5 & 0.85714 & 1 & 0 \\ 0.25 & 0.71429 & 0.83333 & 1 \end{pmatrix}$$

The matrix computed by using Gaussian Elimination is an upper triangular matrix:

$$U = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1.75 & 1.5 & 1.25 \\ 0 & 0 & 1.71429 & 1.42857 \\ 0 & 0 & 0 & 1.66667 \end{pmatrix}$$

If we multiply them we get the original matrix A:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.75 & 1 & 0 & 0 \\ 0.5 & 0.85714 & 1 & 0 \\ 0.25 & 0.71429 & 0.83333 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1.75 & 1.5 & 1.25 \\ 0 & 0 & 1.71429 & 1.42857 \\ 0 & 0 & 0 & 1.66667 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$