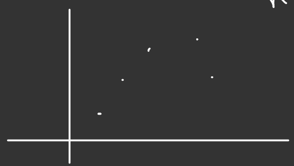
Cubic Splines

1.) 

$n+1$

Piece wise approx

I

$$1. \ a_i = f(x_i)$$

II

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & \dots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \dots & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 & \dots & 0 \\ 0 & 0 & 0 & h_3 & 2(h_3+h_4) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \frac{3}{h_2}(a_3-a_2) - \frac{3}{h_1}(a_2-a_1) \\ \frac{3}{h_3}(a_4-a_3) - \frac{3}{h_2}(a_3-a_2) \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

III

solving for d_i in (7)

$$d_i = \frac{1}{3h_i}(c_{i+1} - c_i) \quad \textbf{(11)}$$

now substitute d_i in (5) and solve for b_i

$$b_i = \frac{a_{i+1} - a_i}{h_i} - \frac{2c_i + c_{i+1}}{3} h_i \quad \textbf{(10)}$$

Simple example

Example 1 Construct a natural cubic spline that passes through the points $(1, 2)$, $(2, 3)$, and $(3, 5)$.

Solution This spline consists of two cubics. The first for the interval $[1, 2]$, denoted

$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3,$$

and the other for $[2, 3]$, denoted

$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3.$$

There are 8 constants to be determined, which requires 8 conditions. Four conditions come from the fact that the splines must agree with the data at the nodes. Hence

$$2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0, \quad 3 = f(2) = a_1, \quad \text{and}$$
$$5 = f(3) = a_1 + b_1 + c_1 + d_1.$$

Two more come from the fact that $S_0'(2) = S_1'(2)$ and $S_0''(2) = S_1''(2)$. These are

$$S_0'(2) = S_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1 \quad \text{and} \quad S_0''(2) = S_1''(2): \quad 2c_0 + 6d_0 = 2c_1$$

The final two come from the natural boundary conditions:

$$S_0''(1) = 0: \quad 2c_0 = 0 \quad \text{and} \quad S_1''(3) = 0: \quad 2c_1 + 6d_1 = 0.$$

Solving this system of equations gives the spline

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & \text{for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, & \text{for } x \in [2,3] \end{cases}$$

Example 2 At the beginning of Chapter 3 we gave some Taylor polynomials to approximate the exponential $f(x) = e^x$. Use the data points $(0, 1)$, $(1, e)$, $(2, e^2)$, and $(3, e^3)$ to form a natural spline $S(x)$ that approximates $f(x) = e^x$.

Solution We have $n = 3$, $h_0 = h_1 = h_2 = 1$, $a_0 = 1$, $a_1 = e$, $a_2 = e^2$, and $a_3 = e^3$. So the matrix A and the vectors **b** and **x** given in Theorem 3.11 have the forms

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The vector-matrix equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the system of equations

$$c_0 = 0,$$
$$c_0 + 4c_1 + c_2 = 3(e^2 - 2e + 1),$$
$$c_1 + 4c_2 + c_3 = 3(e^3 - 2e^2 + e),$$
$$c_3 = 0.$$

This system has the solution $c_0 = c_3 = 0$, and to 5 decimal places,

$$c_1 = \frac{1}{5}(-e^3 + 6e^2 - 9e + 4) \approx 0.75685, \quad \text{and} \quad c_2 = \frac{1}{5}(4e^3 - 9e^2 + 6e - 1) \approx 5.83007.$$

Solving for the remaining constants gives

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$$
$$= (e - 1) - \frac{1}{15}(-e^3 + 6e^2 - 9e + 4) \approx 1.46600,$$
$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$$
$$= (e^2 - e) - \frac{1}{15}(2e^3 + 3e^2 - 12e + 7) \approx 2.22285,$$
$$b_2 = \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(c_3 + 2c_2)$$
$$= (e^3 - e^2) - \frac{1}{15}(8e^3 - 18e^2 + 12e - 2) \approx 8.80977,$$
$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{15}(-e^3 + 6e^2 - 9e + 4) \approx 0.25228,$$
$$d_1 = \frac{1}{3h_1}(c_2 - c_1) = \frac{1}{3}(e^3 - 3e^2 + 3e - 1) \approx 1.69107,$$

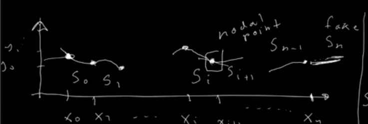
and

$$d_2 = \frac{1}{3h_2}(c_3 - c_1) = \frac{1}{15}(-4e^3 + 9e^2 - 6e + 1) \approx -1.94336.$$

The natural cubic spine is described piecewise by

$$S(x) = \begin{cases} 1 + 1.46600x + 0.25228x^3, & \text{for } x \in [0, 1], \\ 2.71828 + 2.22285(x-1) + 0.75685(x-1)^2 + 1.69107(x-1)^3, & \text{for } x \in [1, 2], \\ 7.38906 + 8.80977(x-2) + 5.83007(x-2)^2 - 1.94336(x-2)^3, & \text{for } x \in [2, 3]. \end{cases}$$

Derivation.



n polynomials:

$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$

4n unknowns

conditions:

① $S_i(x_i) = y_i \quad \forall i = 0 \ldots n$ $\boxed{n+1}$

② $S_i(x_{i+1}) = S_{i+1}(x_{i+1}) \quad \forall i = 1, \ldots n-1$ $\boxed{n-1}$

③ $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \quad \forall i = 1, \ldots n-1$ $\boxed{n-1}$

④ $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) \quad \forall i = 1, \ldots, n-1$ $\boxed{n-1}$

⑤ $S_0''(x_0) = 0 \quad ; \quad S_n''(x_n) = 0$ $\boxed{2}$

$4n - 2$

4n

$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$

$S_{i+1}(x) = a_{i+1} + b_{i+1}(x - x_{i+1}) + c_{i+1}(x - x_{i+1})^2 + d_{i+1}(x - x_{i+1})^3$

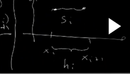
$S_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$

$S_{i+1}'(x) = b_{i+1} + 2c_{i+1}(x - x_{i+1}) + 3d_{i+1}(x - x_{i+1})^2$

$S_i''(x) = 2c_i + 6d_i(x - x_i)$

$S_{i+1}''(x) = 2c_{i+1} + 6d_{i+1}(x - x_{i+1})$

$h_i = x_{i+1} - x_i$



⑤ $S_0''(x_0) = 0$ $\qquad S_n''(x_n) = 0$

$c_0 = 0$ $\qquad c_n = 0$

① $S_i(x_i) = y_i$

$a_i = y_i$

② $S_{i+1}(x_{i+1}) = S_i(x_{i+1})$

$a_{i+1} = a_i + b_i h_i + c_i h_i^2 + d_i h_i^3$ $\boxed{5}$

③ $S_{i+1}'(x_{i+1}) = S_i'(x_{i+1})$

$b_{i+1} = b_i + 2c_i h_i + 3d_i h_i^2$ $\boxed{6}$

④ $S_{i+1}''(x_{i+1}) = S_i''(x_{i+1})$

$2c_{i+1} = 2c_i + 6d_i h_i$

$c_{i+1} = c_i + 3d_i h_i$ $\boxed{7}$

1)

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

It is helpful to write the equations as follows:

$$a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i x_i^0,$$

$$a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} y_i x_i^1,$$

$$\vdots$$

$$a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} y_i x_i^n.$$

Example 2 Fit the data in Table 8.3 with the discrete least squares polynomial of degree at most 2.

Solution For this problem, $n = 2, m = 5$, and the three normal equations are

$$\begin{aligned} 5a_0 + \quad 2.5a_1 + \quad 1.875a_2 &= 8.7680, \\ 2.5a_0 + \quad 1.875a_1 + 1.5625a_2 &= 5.4514, \\ 1.875a_0 + 1.5625a_1 + 1.3828a_2 &= 4.4015. \end{aligned}$$

Table 8.3

i	x_i	y_i
1	0	1.0000
2	0.25	1.2840
3	0.50	1.6487
4	0.75	2.1170
5	1.00	2.7183

To solve this system using Maple, we first define the equations

$eq1 := 5a0 + 2.5a1 + 1.875a2 = 8.7680$:
$eq2 := 2.5a0 + 1.875a1 + 1.5625a2 = 5.4514$:
$eq3 := 1.875a0 + 1.5625a1 + 1.3828a2 = 4.4015$

2)

.5

i	x_i	y_i	$\ln y_i$	x_i^2	$x_i \ln y_i$
1	1.00	5.10	1.629	1.0000	1.629
2	1.25	5.79	1.756	1.5625	2.195
3	1.50	6.53	1.876	2.2500	2.814
4	1.75	7.45	2.008	3.0625	3.514
5	2.00	8.46	2.135	4.0000	4.270
	7.50		9.404	11.875	14.422

$$a_0 = \frac{\sum_{i=1}^{m} x_i^2 \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i y_i \sum_{i=1}^{m} x_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

$$a_1 = \frac{m\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}.$$

If x_i is graphed with $\ln y_i$, the data appear to have a linear relation, so it is reasonable to assume an approximation of the form

$$y = be^{ax}, \quad \text{which implies that} \quad \ln y = \ln b + ax.$$

Extending the table and summing the appropriate columns gives the remaining data in Table 8.5.

Using the normal equations (8.1) and (8.2),

$$a = \frac{(5)(14.422) - (7.5)(9.404)}{(5)(11.875) - (7.5)^2} = 0.5056$$

and

$$\ln b = \frac{(11.875)(9.404) - (14.422)(7.5)}{(5)(11.875) - (7.5)^2} = 1.122.$$

With $\ln b = 1.122$ we have $b = e^{1.122} = 3.071$, and the approximation assumes the form

$$y = 3.071e^{0.5056x}.$$

$$P_n(x) = \sum_{j=0}^{n} a_j x^j$$

$$E(a_0, a_1, \dots, a_n) = \sum_{i=1}^{m} \left(P_n(x_i) - y_i\right)^2$$

$$= \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j - y_i \right)^2$$

$$\frac{\partial E}{\partial a_0} = 0, \quad \frac{\partial E}{\partial a_1} = 0 \quad \dots \quad \frac{\partial E}{\partial a_n} = 0$$

$$\frac{\partial a_j}{\partial a_k} = \begin{cases} 1 \\ 0 \end{cases}$$

$$\frac{\partial E}{\partial a_k} = \sum_{i=1}^{m} 2\left(\sum_{j=0}^{n} a_j x_i^j - y_i \right)\left(\sum_{j=0}^{n} x_i^j \frac{\partial a_j}{\partial a_k} \right)$$

$$P_n(x) = \sum_{j=0}^{n} a_j x^j$$

$$E(a_0, a_1, \dots, a_n) = \sum_{i=1}^{m} \left(P_n(x_i) - y_i\right)^2$$

$$\frac{\partial E}{\partial a_k} = \sum_{i=1}^{m} 2\left(\sum_{j=0}^{n} a_j x_i^j - y_i \right)\left(\sum_{j=0}^{n} x_i^j \frac{\partial a_j}{\partial a_k} \right)$$

$$\left(\sum_{j=0}^{n} x_i^j \frac{\partial a_j}{\partial a_k} \right) \longrightarrow x_i^k$$

$$\frac{\partial E}{\partial a_k} = 2 \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j - y_i \right) x_i^k = 0 \quad \dots \quad k = 0 \dots n$$

$$\sum_{i=1}^{m} \sum_{j=0}^{n} a_j x_i^j x_i^k - \sum_{i=1}^{m} x_i^k y_i = 0 \quad \forall k = 0 \dots n$$

$$m a_0 + \left(\sum_{i=1}^{m} x_i\right) a_1 + \left(\sum_{i=1}^{m} x_i^2\right) a_2 + \dots \left(\sum_{i=1}^{m} x_i^n\right) a_n = \sum_{i=1}^{m} y_i$$

$$\left(\sum_{i=1}^{m} x_i\right) a_0 + \left(\sum_{i=1}^{m} x_i^2\right) a_1 + \left(\sum_{i=1}^{m} x_i^3\right) a_2 + \dots \left(\sum_{i=1}^{m} x_i^{n+1}\right) a_n = \sum_{i=1}^{m} x_i y_i$$

$$\vdots$$

$$\begin{pmatrix} m & \sum x_i & \sum x_i^2 & \sum x_i^3 & \dots & \sum x_i^n \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \dots & \sum x_i^{n+1} \\ \vdots & & & & & \\ \sum x_i^n & \sum x_i^{n+1} & \sum x_i^{n+2} & \dots & \dots & \sum x_i^{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \vdots \\ \sum x_i^n y_i \end{pmatrix}$$

$$y_{pred} = x\theta$$

$$(x\theta - y)$$

$$y_1$$

$$\begin{pmatrix} 1 & x_1 & x_2 & \dots \\ \vdots & & & \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$

$$(\theta_0 + \theta_1 x_1 + \theta_2 x_2 \dots - y)$$

$$(y - X\theta)^T (y - X\theta)$$

$$(y^T - \theta^T X^T)(y - X\theta)$$

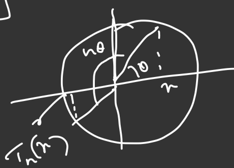
$$\frac{1}{2} \frac{\partial}{\partial \theta} \left(y^T y - y^T X \theta - \theta^T X^T y + \theta^T X^T X \theta \right)$$

$$-y^T X + \theta^T X^T X = 0$$

$$\theta = (X^T X)^{-1} X^T y$$

$$\frac{1}{2}(X\theta - y)^T (X\theta - y)$$

$T_n : [-1, 1] \to [-1, 1]$



$T_n(x) = \cos(n \cos^{-1} x)$

Zeros $\quad \theta = \cos^{-1} x \qquad k \in \mathbb{Z}$

$\cos(n\theta) = 0$

$\Rightarrow n\theta = \frac{\pi}{2} + k\pi$

$\theta = \frac{\pi + 2\pi k}{2n}$

1) $n = 0 \to$ no zeros

2) $n = 1 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow x = 0$

3) $n = 2 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

For $n \geq 0$ exactly "n" zeros.

$\theta = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \ldots \frac{(2n-1)\pi}{2n}$

max & min

$\cos n\theta = \pm 1$

$n\theta = k\pi$

$\theta = \frac{k\pi}{n} = 0, \frac{\pi}{2}, \frac{2\pi}{n} \ldots \pi$

So $|T_n(x)|$ assume max exactly $n+1$ times.

Using euler's formula:

$e^{i\theta} = \cos\theta + i\sin\theta$

& $\cos(\alpha + \beta) + i\sin(\alpha + \beta) = e^{i(\alpha+\beta)}$

$= e^{i\alpha} e^{i\beta}$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \cos\beta\sin\alpha)$

$= \cos\alpha\cos\beta$

demoivre's formula

$$\cos n\theta + i\sin n\theta = e^{in\theta}$$
$$= (e^{i\theta})^n$$
$$= (\cos\theta + i\sin\theta)^n$$
$$= \cos^n\theta + n i\cos^{n-1}\theta \cdot \sin\theta - \binom{n}{2}\cos^{n-2}\theta\,\sin^2\theta - \cdots$$

Setting real parts equal !

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\,\sin^2\theta + {}^nC_4\cos^{n-4}\theta\,\sin^4\theta \cdots$$
$$= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\,(1-\cos^2\theta) + \binom{n}{4}\cos^{n-4}\theta\,(1-\cos^2\theta)^2 + \cdots$$

$x = \cos\theta$

$$\Rightarrow T_n(x) = x^n - \binom{n}{2}x^{n-2}(1-x^2) + \cdots$$

Triple recursion formula

$$T_{n+1}(x) = \cos(n\theta+\theta)$$
$$= \cos n\theta \cdot \cos\theta - \sin(n\theta)\sin\theta$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$T_{n-1}(x) = \cos(n\theta - \theta)$$
$$= \cos n\theta\cos\theta + \sin(n\theta)\sin\theta$$

$$T_{n+1} + T_{n-1} = 2\cos n\theta \cdot \cos\theta$$
$$= 2x\,T_n(x).$$

$$\Rightarrow T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$T_n(x) = \cos(n\cos^{-1}x)$$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3 = 2x(2x^2-1) - x$

$= 4x^3 - 3x$

1) $T_n \to n$ distinct zeros

2) $|T_n(x)|$ takes its max 1 at exactly $n+1$ distinct points separ. by zeros

3) $T_{even}(x)$ is even function

4) $n>0$, leading coeff $\to 2^{n-1}$

Optimality of $\tilde{T}_n = \frac{T_n(x)}{2^{n-1}}$ monic (leading coefficient = 1)

$= x^n + c_{n-1} x^{n-1} - \cdots$

$\max_{-1 \le x \le 1} |\tilde{T}_n(x)| = \frac{1}{2^{n-1}}$ $\Rightarrow$ Thm: $\tilde{T}_n$ smallest max absolute value on $[-1,1]$

$\max_{-1 \le x \le 1} |P(x)| \ge \frac{1}{2^{n-1}}$

Proof

let $\max_{-1 \le x \le 1} |P(x)| < \frac{1}{2^{n-1}}$

Why useful?

$\tilde{T}_{n+1}(x) = (x - r_0) \cdots (x - r_n)$

$$k=0,1,\ldots,n \quad a_0 \sum \phi_0 \phi_0 \cdots$$

$$\begin{bmatrix} \sum \phi_0^2 & \sum \phi_0 \phi_1 & \cdots & \sum \phi_0 \phi_n \\ \sum \phi_1 \phi_0 & \sum \phi_1^2 & \sum \phi_1 \phi_2 & \sum \phi_1 \phi_n \\ \vdots & & & \\ \sum \phi_n \phi_0 & \sum \phi_n \phi_1 & \cdots & \sum \phi_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{pmatrix} \sum y_i \phi_0 \\ \sum y_i \phi_1 \\ \vdots \end{pmatrix}$$