Lagrange interpolation

1) (x_0, y_0) $\quad (x_1, y_1)$ $\quad (x_2, y_2)$ $\Rightarrow$ No reuse.

$$y_0 (L_0) + y_1 (L_1) + y_2 (L_2)$$

$$L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$\Rightarrow y_0 L_0 + y_1 L_1'$$

$$L_1' = \frac{(x-x_0)}{(x_1-x_0)}$$

$\underbrace{L_1' \neq L_1}$ $\Rightarrow$ calculate from scratch.

2) Neville's method

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$\Rightarrow P_{i\,j-1}$

$P_{i+1,j}$

$\Rightarrow$ claim

$$P_{ij} = P_{i+1,j} \frac{(x-x_i)}{x_j - x_i} + P_{i\,j-1} \frac{(x-x_j)}{x_i - x_j}$$

$$P_{i\,j-1} \frac{x-x_j}{x_i - x_j} + P_{i+1\,j} \frac{(x-x_i)}{x_j - x_i}$$

$$P_{ij} = \begin{cases} y_i & x = x_i \\ y_j & x = x_j \\ y_k & x_k \quad i<k<j \end{cases}$$

$$\frac{x_k - x_i}{x_j - x_i} + \frac{x_k - x_j}{x_i - x_j}$$

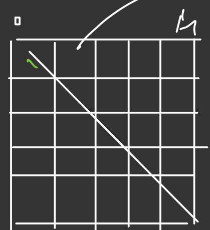
$$\frac{\cancel{x_k - x_i - x_k + x_j}}{(x_j - x_i)} = 1$$

Base case

$$P_{ii} = y_i$$

$P_{01} = P_{11}$

P_{00}



Aside Interpolation & Accuracy $x = 1.5$

~ sample

x	$f(x)$
1.0	0.76...
1.3	0.62...
1.6	0.45...
1.9	0.28...
2.2	0.11

degree ~

linear polynomial

1.3, 1.6

$$P_1(1.5) = \frac{1.5-1.6}{1.3-1.6} f(1.3) + \left(\frac{1.5-1.3}{1.6-1.3}\right) f(1.6)$$

$$\approx 0.5102$$

$P_2(1.5) = \approx 0.51128$

$\hat{P}_2(1.5) = \approx 0.5124$

P_3 (most accurate)

$\hat{P}_3$

$P_4 = 0.51182$

$1.5 = 0.5118277$

Algorithm:

Input $(x_0, y_0), (x_1, y_1) \cdots (x_n, y_n), X$

for $k = 0$ to n

$P_{kk} = y_k$

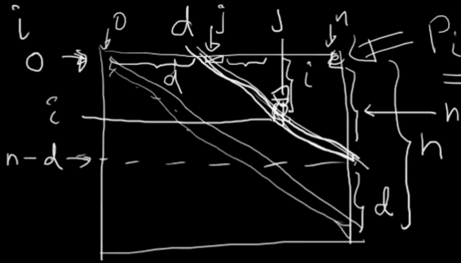
for $d = 1$ to n

for $i = 0$ to $n-d$

$j = d + i$

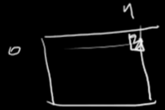
$$P_{ij} = \frac{P_{i+1,j}(X - x_i) - P_{i,j-1}(X - x_j)}{x_j - x_i}$$

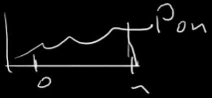
return P_{0n}



$P_{kk} = y_k$

$j = d + i$





1) $f(x) = \frac{1}{\sqrt{x}}$; $x = 81$

16	—	0.25	(16) 0
64	—	.125	(64) 1
100	—	0.1	(100) 2

(16) (64) (100)

	0	1	2
0	0.25		
1		.125	
2			0.1

$P_{01} = 0.08072$

.1058

0.1058

$$P_{ij} = \frac{P_{i+1,j}(x - x_i)}{x_j - x_i} + \frac{P_{i,j-1}(x - x_j)}{x_i - x_j}$$

$P_{01} = P_{11} \cdot \frac{}{} + \frac{}{} P_{00}$

$\frac{81-16}{64-16} \times .125 + \frac{81-64}{16-64} \times 0.25$

2)

x	y
1	-6
2	2
4	12

$x = 3$?

-6		
	2	
		12

$\frac{24}{3}$

7

$\frac{(x_1 - x) P_{00} + x - x_0 P_{11}}{x_1 - x_0} = \frac{6+4}{1}$

$= 10$

Newton's method

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

EXAMPLE 3 Using the Newton algorithm, find the interpolating polynomial of least degree for this table:

x	0	1	-1	2	-2
y	-5	-3	-15	39	-9

Polynomials p_0, p_1, p_2, p_3, p_4

Solution In the construction, five successive polynomials appear; these are labeled p_0, p_1, p_2, p_3, and p_4. The polynomial p_0 is defined to be

$$p_0(x) = -5$$

The polynomial p_1 has the form

$$p_1(x) = p_0(x) + c(x - x_0) = -5 + c(x - 0)$$

The interpolation condition placed on p_1 is that $p_1(1) = -3$. Therefore, we have $-5 + c(1 - 0) = -3$. Hence, $c = 2$, and p_1 is

$$p_1(x) = -5 + 2x$$

The polynomial p_2 has the form

$$p_2(x) = p_1(x) + c(x - x_0)(x - x_1) = -5 + 2x + cx(x - 1)$$

The interpolation condition placed on p_2 is that $p_2(-1) = -15$. Hence, we have $-5 + 2(-1) + c(-1)(-1 - 1) = -15$. This yields $c = -4$, so

$$p_2(x) = -5 + 2x - 4x(x - 1)$$

The remaining steps for $p_3(x)$ are similar. The final result is the Newton form of the interpolating polynomial:

$$p_4(x) = -5 + 2x - 4x(x - 1) + 8x(x - 1)(x + 1) + 3x(x - 1)(x + 1)(x - 2) \quad \blacksquare$$

Derivation:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + - -$$

$$P_n(x_0) = a_0 = y_0 \quad = 0 \quad f(x_0)$$

$$P_n(x_1) = a_0 + a_1(x - x_0)$$

$$= a_0 + a_1(x_1 - x_0) = y_1$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0\ x_1)$$

$$f(x_0) = y_0, \quad f(x_n) = y_n$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$a_2 = f[x_0, x_1, x_2]$$

$$a_3 = f(x_0, x_1, x_2, x_3]$$

Ex:

x	y
1	-6
2	2
4	12

$x = 3$?

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$f(x_0) = a_0 = -6$

$f(x_0, x_1) = a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = 8$; $f(x_1, x_2) = 5$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$= -1$

$$f(x) = -6 + 8(x-1)\,(x^2 - 3x + 2)$$

$$= -x^2 + 11x - 16$$

$(3, -10)$ add on more point

$$\big((x-1)(x-2)(x-4)\big)\, f(x_0 \dots x_3) \to 0$$

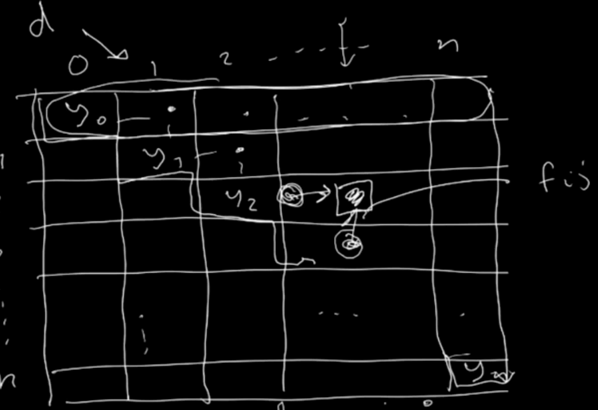
$$f(x_3) = -10$$

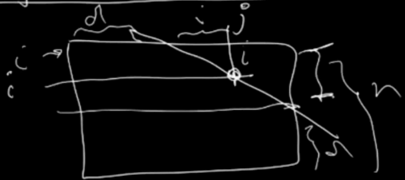
$$f(x_2, x_3) = \frac{-10 - 12}{3 - 4} = 22$$

$$f(x_1\, x_2\, x_3) = \frac{22 - 5}{3 - 2} = 17$$

$$a_3 = \frac{17 + 1}{3 - 1} = 9 \text{ //}$$

$$-x^2 + 11x - 16 + 9(x^3 - 7x^2 + 14x - 8).$$





Algorithm Divided Diff

Input $(x_0, y_0) \dots (x_n, y_n)$

Output: $a_0, \dots a_n$

// notice: $f_{ij} = f[x_i, \dots x_j]$

for $i = 0$ to n
  $f_{ii} = y_i$

for $d = 1$ to n
  for $i = 0$ to $n - d$
    $j = i + d$
    $$f_{ij} = \frac{f_{i+1,j} - f_{i,j-1}}{x_j - x_i}$$