**CS 323: Numerical Analysis** **Rutgers: Spring 2022**

# Homework #1

Deadline: February 10

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**Problem 1.** Find the real root of $x^2 = 0.7$ using 3 iterations of the bisection method with $a = 0.5$, $b = 2$.

**Solution.** We apply the bisection method to $f(x) = x^2 - 0.7$:

| iteration | a | b | c=(a+b)/2 | f(a) | f(c) | f(a)f(c) | action |
|---|---|---|---|---|---|---|---|
| 0 | 0.5 | 2 | 1.25 | -0.45 | 0.8625 | -0.38813 | b <-- c |
| 1 | 0.5 | 1.25 | 0.875 | -0.45 | 0.065625 | -0.02953 | b <-- c |
| 2 | 0.5 | 0.875 | 0.6875 | -0.45 | -0.22734 | 0.102305 | a <-- c |
| 3 | 0.6875 | 0.875 | 0.78125 | | | | |

After 3 iterations, we've computed a root of 0.78125. We note that $0.78125^2 - 0.7 = -0.08965$.

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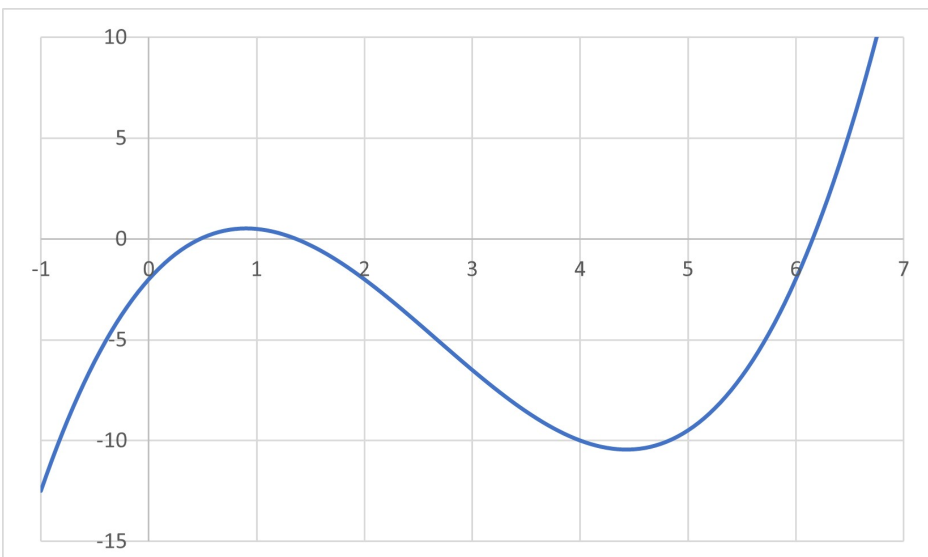
**Problem 2.** Find **all** real roots of $f(x) = -2 + 6x - 4x^2 + 0.5x^3$ using Newton's Method with $\epsilon = 0.01$.

**Solution.** First we plot $f(x)$ to get an idea where the real roots are:




Based on this plot, we run Newton's Method on $f(x)$ with initial values $x_0 \in \{0, 2, 6\}$. We apply differentiation rules to get $f'(x) = 6 - 8x + 1.5x^2$ and compute $x_1 = x_0 + f(x_0)/f'(x_0)$:

| iteration | x0 | f(x0) | f'(x0) | x1 | e=\|x1-x0\| |
|---|---|---|---|---|---|
| 1 | 0 | -2 | 6 | 0.333333 | 0.333333 |
| 2 | 0.333333 | -0.42593 | 3.5 | 0.455026 | 0.121693 |
| 3 | 0.455026 | -0.05093 | 2.670362 | 0.474099 | 0.019073 |
| 4 | 0.474099 | -0.0012 | 2.544361 | 0.474572 | 0.000473 |

| iteration | x0 | f(x0) | f'(x0) | x1 | e=\|x1-x0\| |
|---|---|---|---|---|---|
| 1 | 2 | -2 | -4 | 1.5 | 0.5 |
| 2 | 1.5 | -0.3125 | -2.625 | 1.380952 | 0.119048 |
| 3 | 1.380952 | -0.02565 | -2.18707 | 1.369227 | 0.011726 |
| 4 | 1.369227 | -0.00027 | -2.14164 | 1.369102 | 0.000124 |

| iteration | x0 | f(x0) | f'(x0) | x1 | e=\|x1-x0\| |
|---|---|---|---|---|---|
| 1 | 6 | -2 | 12 | 6.166667 | 0.166667 |
| 2 | 6.166667 | 0.141204 | 13.70833 | 6.156366 | 0.010301 |
| 3 | 6.156366 | 0.000556 | 13.60034 | 6.156325 | 4.09E-05 |

Our computed (real) roots are 0.47, 1.37, and 6.16. Since this polynomial is of degree 3, we know we have found all 3 real roots.

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**Problem 3.** The sum of 2 numbers is 20. If we add to each number its square root, the product of both sums is 155.55. Find the two numbers with $\epsilon = 10^{-4}$.

**Solution.** Let $x$ and $y$ be 2 numbers such that $x + y = 20$ and $(x + \sqrt{x})(y + \sqrt{y}) = 155.55$. We limit our search to $x$ and $y$ in the open interval $(0, 20)$ so that the square roots are real numbers and we can avoid dividing by zero. The first equation implies $y = 20 - x$, and we make this substitution for $y$ in the second equation to get $(x + \sqrt{x})((20 - x) + \sqrt{20 - x}) = 155.55$. It is tedious to calculate the first derivative of the left-hand side of this equation, so we attempt fixed-point iteration and write the second equation in the form $x = g(x)$:

$$
\begin{aligned}
(x + \sqrt{x})((20 - x) + \sqrt{20 - x}) = 155.55 &\iff x \cdot (20 - x + \sqrt{20 - x}) + \sqrt{x} \cdot (20 - x + \sqrt{20 - x}) = 155.55 \\
&\iff x = \frac{155.55}{20 - x + \sqrt{20 - x}} - \sqrt{x} \doteq g(x)
\end{aligned}
$$

We run fixed point iteration with $x_0 = 10$, setting $x_{k+1} = g(x_k)$ in each iteration $k$:

| k | x_k | x_k+1 | e |
|---|---|---|---|
| 0 | 10 | 8.655586 | 1.344414 |
| 1 | 8.655586 | 7.630561 | 1.025025 |
| 2 | 7.630561 | 7.02901 | 0.601551 |
| 3 | 7.02901 | 6.734794 | 0.294216 |
| 4 | 6.734794 | 6.604993 | 0.129801 |
| 5 | 6.604993 | 6.550514 | 0.054479 |
| 6 | 6.550514 | 6.528145 | 0.022369 |
| 7 | 6.528145 | 6.519044 | 0.009101 |
| 8 | 6.519044 | 6.515355 | 0.003689 |
| 9 | 6.515355 | 6.513862 | 0.001493 |
| 10 | 6.513862 | 6.513259 | 0.000604 |
| 11 | 6.513259 | 6.513014 | 0.000244 |
| 12 | 6.513014 | 6.512916 | 9.87E-05 |

So we see that fixed point iteration converges, yielding $x = 6.5129$ and $y = 20 - 6.5129 = 13.4871$.

**Problem 4.** The following equation is used to compute monthly payments on a mortgage:

$$A = \frac{P}{i}\left(1 - (1+i)^{-n}\right)$$

Where $A$ is the total mortgage amount, $P$ is the monthly payment, $i$ is the monthly interest rate, and $n$ is the number of months.

Suppose that a client wants an \$800,000.00 mortgage to be paid in 30 years but he can pay no more than \$7,000.00 each month. What is the highest monthly interest rate that he would be able to pay?

**Solution.** The highest monthly interest rate the client would be able to pay is the value $i$ that satisfies the given equation. This is equivalent to finding a positive root of the function

$$\begin{aligned} f(i) &= P(1-(1+i)^{-n}) - A \cdot i \\ &= \$7{,}000(1-(1+i)^{-12\cdot 30}) - \$800{,}000 \cdot i \end{aligned}$$

We use Newton's method. The first derivative of $f$ is given by

$$\begin{aligned} f'(i) &= P \cdot n(1+i)^{-n-1} - A \\ &= \$7{,}000 \cdot 360(1+i)^{-361} - \$800{,}000. \end{aligned}$$

To calculate an initial value $i_0$, we note that the *simple* interest rate, i.e. with no compounding of interest, would be $A/(P \cdot n) = 800/(7 \cdot 360) \approx 32\%$, and that an equivalent compounding interest rate would be $i$ that satisfies $1 + 32\% \cdot 360 = (1+i)^{360}$. So we solve for $i_0 = 116.2^{1/360} - 1 = 1.33\%$. We use $\epsilon = 10^{-4}$, based on the convention that interest is quoted in basis points, i.e. hundredths of a percentage point:

| iteration | i0 | f(i0) | f'(i0) | i1 | e=\|i1-i0\| |
|---|---|---|---|---|---|
| 1 | 1.33% | -3.70017 | -778.622 | 0.85% | 0.004752 |
| 2 | 0.85% | -0.1651 | -683.327 | 0.83% | 0.000242 |
| 3 | 0.83% | -0.00125 | -672.787 | 0.83% | 1.87E-06 |

So we find 0.83% is the highest monthly interest rate the client would be able to pay.

**Problem 5.** Enumerate all elements in $f_l(2, 2, -1, 1)$.

**Solution.** We list all base-2 values with 2-digit mantissas and exponents ranging between -1 and 1:

$$\begin{array}{lll} \pm\, 0.10 \times 10^{-1} & \pm\, 0.10 \times 10^{0} & \pm\, 0.10 \times 10^{1} \\ \pm\, 0.11 \times 10^{-1} & \pm\, 0.11 \times 10^{0} & \pm\, 0.11 \times 10^{1} \\ 0.00 \times 10^{0} & & \end{array}$$

**Problem 6.** Use the bisection method to find a root of $x^3 - 7x^2 + 14x - 6 = 0$ in $[1, 3.2]$ with $\epsilon = 10^{-2}$.

**Solution.** We apply the bisection method to $f(x) = x^3 - 7x^2 + 14x - 6$:

| iteration | a | b | c=(a+b)/2 | f(a) | f(c) | f(a)f(c) | action | b-a |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3.2 | 2.1 | 2 | 1.791 | 3.582 | a <-- c | 2.2 |
| 1 | 2.1 | 3.2 | 2.65 | 1.791 | 0.552125 | 0.988856 | a <-- c | 1.1 |
| 2 | 2.65 | 3.2 | 2.925 | 0.552125 | 0.085828 | 0.047388 | a <-- c | 0.55 |
| 3 | 2.925 | 3.2 | 3.0625 | 0.085828 | -0.05444 | -0.00467 | b <-- c | 0.275 |
| 4 | 2.925 | 3.0625 | 2.99375 | 0.085828 | 0.006328 | 0.000543 | a <-- c | 0.1375 |
| 5 | 2.99375 | 3.0625 | 3.028125 | 0.006328 | -0.02652 | -0.00017 | b <-- c | 0.06875 |
| 6 | 2.99375 | 3.028125 | 3.010938 | 0.006328 | -0.0107 | -6.8E-05 | b <-- c | 0.034375 |
| 7 | 2.99375 | 3.010938 | 3.002344 | 0.006328 | -0.00233 | -1.5E-05 | b <-- c | 0.017187 |
| 8 | 2.99375 | 3.002344 | $\boxed{2.998047}$ | | | | | 0.008594 |

After 8 iterations, we've computed a root of 3.00.

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**Problem 7.** Given the polynomial $P(x) = x^4 + 5x^3 - 9x^2 - 85x - 136$

a. Use Newton's method with Horner to find a root with $\epsilon = 10^{-5}$, starting from $x_0 = -4$.

b. If $x_r$ is the solution found before, find the polynomial $P_1(x)$ obtained by dividing the original polynomial by $x - x_r$.

c. Again use Newton's method with Horner to find a root of $P_1(x)$.

d. Verify that the root found is also a root of $P(x)$.

**Solution.** We proceed:

a. We visualize Horner's algorithm as synthetic division with $r = -4$ and coefficients $\{a_4, \ldots, a_0\} = \{1, 5, -9, -85, -136\}$, and find a root $x_r = -4.12311$:

Iteration 1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| r--> | -4 | 1 | 5 | -9 | -85 | -136 | |
| | | | -4 | -4 | 52 | 132 | |
| alphas--> | | 1 | 1 | -13 | -33 | -4 | <-- f(-4) |
| | | | -4 | 12 | 4 | | |
| betas--> | | 1 | -3 | -1 | -29 | | <-- f'(-4) |
| | x1 = | 1 - (-4) / (-29) = | | -4.13793 | | | |
| | error = | | | 0.137931 | | | |

Iteration 2

| | | | | | | |
|---|---|---|---|---|---|---|
| -4.13793 | 1 | 5 | -9 | -85 | -136 | |
| | | -4.13793 | -3.56718 | 52.00213 | 136.5429 | |
| | 1 | 0.862069 | -12.5672 | -32.9979 | 0.542902 | |
| | | -4.13793 | 13.55529 | -4.08873 | | |
| | 1 | -3.27586 | 0.988109 | -37.0866 | | |
| x1 = | -4.12329 | | | | | |
| error = | 0.014639 | | | | | |

Iteration 3

| | | | | | |
|---|---|---|---|---|---|
| -4.12329 | 1 | 5 | -9 | -85 | -136 |
| | | -4.12329 | -3.61492 | 52.01501 | 136.0067 |
| | 1 | 0.876708 | -12.6149 | -32.985 | 0.00675 |
| | | -4.12329 | 13.38662 | -3.18192 | |
| | 1 | -3.24658 | 0.771695 | -36.1669 | |
| x1 = -4.12311 | | | | | |
| error = 0.000187 | | | | | |

Iteration 4

| | | | | | |
|---|---|---|---|---|---|
| -4.12311 | 1 | 5 | -9 | -85 | -136 |
| | | -4.12311 | -3.61553 | 52.01515 | 136 |
| | 1 | 0.876894 | -12.6155 | -32.9848 | 1.09E-06 |
| | | -4.12311 | 13.38447 | -3.17044 | |
| | 1 | -3.24621 | 0.768944 | -36.1553 | |
| x1 = -4.12311 | | | | | |
| error = 3E-08 | < 10^-5 | | | | |

b. Immediately from the visualization of Iteration 4 above we have $P_1(x) = x^3 + 0.87689x^2 - 12.6155x - 32.9848$.

c. Again we visualize Horner's algorithm as synthetic division. Based on a plot of the polynomial (not shown), we can see that we'll have trouble with convergence if we start with $r = -4$. So we use $r = 4$ instead and find a root $x_{r1} = 4.12311$:

Iteration 1

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 0.876894 | -12.6155 | -32.9848 |
| | | 4 | 19.50758 | 27.5682 |
| | 1 | 4.876894 | 6.892049 | -5.41665 |
| | | 4 | 35.50758 | |
| | 1 | 8.876894 | 42.39963 | |

x1 = 4.127752
error = 0.127752

Iteration 2

| | | | | |
|---|---|---|---|---|
| 4.127752 | 1 | 0.876894 | -12.6155 | -32.9848 |
| | | 4.127752 | 20.65794 | 33.19709 |
| | 1 | 5.004647 | 8.042413 | 0.212244 |
| | | 4.127752 | 37.69628 | |
| | 1 | 9.132399 | 45.73869 | |

x1 = 4.123112
error = 0.00464

Iteration 3

| | | | | |
|---|---|---|---|---|
| 4.123112 | 1 | 0.876894 | -12.6155 | -32.9848 |
| | | 4.123112 | 20.61559 | 32.98513 |
| | 1 | 5.000006 | 8.000057 | 0.000285 |
| | | 4.123112 | 37.61564 | |
| | 1 | 9.123118 | 45.61569 | |

x1 = 4.123106
error = 6.26E-06 < 10^-5

d. We calculate:

$$\begin{aligned} P(4.12311) &= 4.12311^4 + 4.12311x^3 \cdots - 136 \\ &= 0.001645 \end{aligned}$$

So this is reasonably close to zero, though it is about 10 times larger than $P(-4.12311) = 1.58 \times 10^{-4}$.

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**Problem 8.** Use Newton's Method to find a solution of the equation $e^{6x} + 3(\ln 2)^2 e^{2x} - e^{4x} \ln 8 - (\ln 2)^3 = 0$ with error tolerance $10^{-5}$, and that is in the interval $-1 \leq x \leq 0$.

**Solution.** Call the left-hand side of this equation $f(x)$. We run Newton's Method with an initial value $x_0 = 0.5$ and apply differentiation rules to get $f'(x) = 6e^{6x} + 6(\ln 2)^2 e^{2x} - 4e^{4x} \ln 8$:

| iteration | x0 | f(x0) | f'(x0) | x1 | e=\|x1-x0\| |
|---|---|---|---|---|---|
| 1 | -0.5 | -0.03441 | 0.233528 | -0.35264 | 0.147362 |
| 2 | -0.35264 | -0.0079 | 0.117578 | -0.28544 | 0.067202 |
| 3 | -0.28544 | -0.0021 | 0.055645 | -0.24765 | 0.03779 |
| 4 | -0.24765 | -0.00059 | 0.025649 | -0.22474 | 0.022907 |
| 5 | -0.22474 | -0.00017 | 0.011658 | -0.21032 | 0.014418 |
| 6 | -0.21032 | -4.9E-05 | 0.005256 | -0.20105 | 0.009271 |
| 7 | -0.20105 | -1.4E-05 | 0.002357 | -0.19501 | 0.006039 |
| 8 | -0.19501 | -4.2E-06 | 0.001054 | -0.19105 | 0.003965 |
| 9 | -0.19105 | -1.2E-06 | 0.00047 | -0.18843 | 0.002617 |
| 10 | -0.18843 | -3.6E-07 | 0.00021 | -0.1867 | 0.001734 |
| 11 | -0.1867 | -1.1E-07 | 9.33E-05 | -0.18555 | 0.001151 |
| 12 | -0.18555 | -3.2E-08 | 4.15E-05 | -0.18478 | 0.000765 |
| 13 | -0.18478 | -9.4E-09 | 1.85E-05 | -0.18427 | 0.000509 |
| 14 | -0.18427 | -2.8E-09 | 8.21E-06 | -0.18393 | 0.000339 |
| 15 | -0.18393 | -8.2E-10 | 3.65E-06 | -0.18371 | 0.000226 |
| 16 | -0.18371 | -2.4E-10 | 1.62E-06 | -0.18356 | 0.00015 |
| 17 | -0.18356 | -7.2E-11 | 7.21E-07 | -0.18346 | 0.0001 |
| 18 | -0.18346 | -2.1E-11 | 3.21E-07 | -0.18339 | 6.68E-05 |
| 19 | -0.18339 | -6.3E-12 | 1.43E-07 | -0.18335 | 4.45E-05 |
| 20 | -0.18335 | -1.9E-12 | 6.33E-08 | -0.18332 | 2.97E-05 |
| 21 | -0.18332 | -5.6E-13 | 2.82E-08 | -0.1833 | 1.98E-05 |
| 22 | -0.1833 | -1.7E-13 | 1.25E-08 | -0.18328 | 1.32E-05 |
| 23 | -0.18328 | -4.9E-14 | 5.56E-09 | $\boxed{-0.18327}$ | 8.83E-06 |

So we compute a root of -0.18327. The convergence is much slower than we've seen for polynomials. If we plot the function on the interval $[-1, 0]$ (not shown), we can see why: the slope of $f(x)$ becomes very flat near the root, which "slows down" Newton's method.