Practice horners rule

1) $P(x) = 2x^4 - 3x^2 + 3x - 4$

$x_0 = -2$ to Newton-horners to approx a root.

Sol:

| | | | | | |
|---|---|---|---|---|---|
| | 2 | -3 | 3 | -4 | } ⇒ wrong |
| -2 | | -4 | 14 | -34 | |
| | 2 | -7 | 17 | | |

With carefully all coefficients.

| | | | | | |
|---|---|---|---|---|---|
| | 2 | 0 | -3 | 3 | -4 |
| -2 | | -4 | 8 | -10 | 14 |
| | 2 | -4 | 5 | -7 | (10) ← remainder ⇒ $P(-2)$ |

We want $P'(-2)$

| | | | | | |
|---|---|---|---|---|---|
| | 2 | -4 | 5 | -7 | |
| -2 | | 0 | -4 | 16 | -42 |
| | 2 | -8 | 21 | (-49) | ⇒ $P'(-2)$ |

$$\Rightarrow x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = -2 - \frac{10}{-49} \simeq -1.796.$$

Repeat

| | | | | | |
|---|---|---|---|---|---|
| | 2 | 0 | -3 | 3 | -4 |
| -1.796 | | -3.592 | 6.451 | -6.197 | 5.742 |
| | 2 | -3.592 | 3.451 | -3.197 | 1.742 ⇒ $P(x_1)$ |
| | | -3.592 | 12.902 | -27.358 | |
| | 2 | -7.184 | 16.353 | -32.565 | |

$$x_2 = -1.796 - \frac{1.742}{-32.565} \simeq -1.7425$$

Smly $x_3 = -1.73897$ & an actual zero $= -1.73896$

Note $Q(x)$ depends on approx being used.

## Understanding horner's rule

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$$

$$= c_n (x-r)^n + c_{n-1} (x-r)^{n-1} + \cdots c_1 (x-r) + c_0$$

Taylor expansion

we know $c_k = \frac{P^k(r)}{k!}$ ⇒ Inefficient

Applying horner's rule

$$q(x) = \frac{P(x) - P(r)}{x-r} = c_n (x-r)^{n-1} + c_{n-1} (x-r)^{n-2} + \cdots + c_1 + 0$$

repeat.

**Eg:-** $P(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$

Find taylor expansion about $r = 3$.

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | −4 | 7 | −5 | 2 |
| | 0 | 3 | −3 | 12 | 21 |
| 3 | 1 | −1 | 4 | 7 | (23) |
| | 0 | 3 | 6 | 30 | |
| 3 | 1 | 2 | 10 | (37) | |
| | 0 | 3 | 15 | | |
| | 1 | 5 | (25) | | |
| | | 3 | | | |
| | (1) | (8) | | | |

$$(x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23.$$

② $x^3 - 9x^2 + 12 = 0$ use newton to find roots ⟹ Exercise $(1.244, 8.847, -1.091)$

Understanding
Theory / Method

$P_n(x) = Q(x)(x - r_0) + r$

$P_n(r_0) = r$

Synthetic division

$P_n(x) = a_0 + a_1 x + a_2 x^2 \dots a_4 x^4$

$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$P_n(x) = a_0 + \left(a_1 + \left(\left(a_2 + \left(a_3 + a_4 x\right) x\right) x\right)\right) x$

$\alpha_4 = a_4$ ; $\alpha_3 = a_3 + \alpha_4 r_0$ ; $\alpha_2 = a_2 + r_3 r_0$ ; $\alpha_1 = a_1 + \alpha_2 r_0$ ; $\alpha_0 = a_0 + \alpha_1 r_0$

Synth. division

| | | $a_4$ | $a_3$ | $a_2$ | $a_1$ | $a_0$ |
|---|---|---|---|---|---|---|
| $r_0$ | | | $a_4 r_0$ | $\alpha_3 r_0$ | $\alpha_2 r_0$ | $\alpha_1 r_0$ |
| | | $a_4$ | $a_3 + a_4 r_0$ ($\alpha_3$) | $a_2 + \alpha_3 r_0$ ($\alpha_2$) | $\alpha_2 r_0 + a_1$ ($\alpha_1$) | $a_0 + \alpha_1 r_0$ |

Algorithm (Horner)

$\alpha = a_4$

for $i = n-1$ down to 0

$\alpha = a_i + \alpha r_0$

return $\alpha_0$

Comp-derivative !

$P_n(x) = Q(x)(x - r_0) + r$

we want $P_n'(x) \to P_n'(r_0)$.

$P_n'(x) = Q'(x)(x - r_0) + Q(x)$

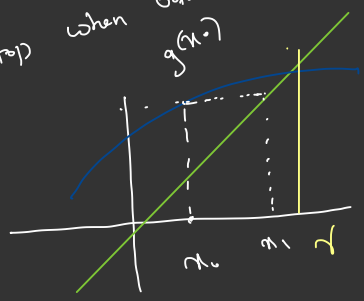
$P_n'(r_0) = Q(r_0)$

# Fixed-point iteration

Just enough theory / intuition

- $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = g(x_0)$ ⟹ Stop when both are ≈ (equal)




## Final thm

a) If $g'(x) < 1$ for all $x$ in an interval containing root $[r-\delta, r+\delta]$

& $x_0 \in [r-\delta, r+\delta]$

then seq. $x_{k+1} = g(x_k)$ converges to $r$.

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⇒ Fixed point ⇒ where the value of the func. doesn't change

why imp? $g(p) = p$.

$f(x) = 0$ find root. (or $f(p) = 0$)

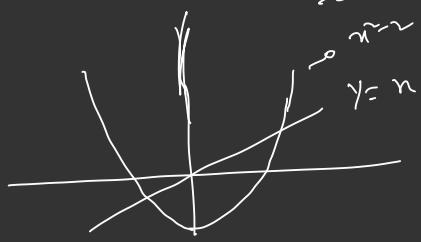
$x - f(x) = g(x)$ or as $g(x) = x + 3f(x)$

Yes comfortable with finding fixed points :

(1) Det any fixed points of $g(x) = x^2 - 2$

$g(p) = p = 0 \quad p = p^2 - 2$

$\Rightarrow p^2 - p - 2 = (p+1)(p-2)$ $\Rightarrow$ 2 fixed points.




$\Rightarrow x^3 + 4x^2 - 10 = 0$ has root in $[1,2]$

change to this form
$x = g(x)$

start at 1.5

(a) $x = x - x^3 - 4x^2 + 10$

| $x_k$ | $g(x)$ $x_{k+1}$ | error |
|---|---|---|
| 1.5 | -0.875 | |
| | 6.732 | |
| | -469.7 | |
| | $1.03 \times 10^8$ | |

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

| $x_k$ | $g(x)$ $x_{k+1}$ | error |
|---|---|---|
| 1.5 | 0.8165 | |
| | 2.9969 | |
| | $(-8.65)^{1/2}$ | |

(c) $x = x - \dfrac{x^3+4x^2-10}{3x^2+8x} = g(x)$

| $x_k$ | $x_{k+1}$ | error |
|---|---|---|
| 1.5 | 1.37... | |
| | 1.36... | |
| | 1.36... | |
| | 1.365230013 | |

!!

Que: How to find a good form?

(d) $x = \left(\dfrac{10}{4+x}\right)^{1/2}$.

$g(x)$

| $x_k$ | $x_{k+1}$ | error |
|---|---|---|
| 1.5 | 1.34... | |
| | 1.3673 | |
| | 1.3649 | |
| | 1.365... | |
| 12 | 1.365230013 | 0014 |

Let's revisit (a.)

$g_1(x) = x - x^3 - 4x^2 + 10$

$g_1(1) = 6$

$g_1(2) = -12$

$g_1'(x) = 1 - 3x^2 - 8x = 0 \quad |g_1'(x)| > 1$ for all $x \in [1,2]$

Now, for $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

$$g'(x) = \left| \frac{-5}{\sqrt{10}\,(4+x)^{3/2}} \right| \leq 0.15 \quad \forall\, x \in (1,2]$$

$m = \max\left(|g'(x)| \;\middle|\; x \in [r-\delta, r+\delta]\right)$

$\Rightarrow e_{k+1} \leq m\, e_k$ ⟵ as small as possible is good

Some more practice

① $x^3 + 6x^2 = 8$ ⟶ $[1,2]$ start at 1.5

② $x = x^3 + 6x^2 + x - 8$

③ $x = \sqrt{\frac{8}{x+6}}$

} ⇒ which is better ⟶ 1.0 641

or $g'(x) = \sqrt{2}\,/\,(x+1)^{3/2} < 1$

$g'(x) = 3x^2 + 12x + 1$

(No convergence)

# Fixed point method (basic examples)

$x^2 - x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = 1.618, -0.618033$

(1) $x^2 = x + 1 \qquad x_0 = 1$

$x_{i+1} = 1 + 1/x_i$

$x_1 = 1 + 1 = 2$

$x_2 = 1 + \frac{1}{2} = 1.5$

$x_3 = 1 + 1/1.5 = 1.666$

$\Rightarrow$

(2) $x_{i+1} = x_i^2 - 1 \qquad x_0 = 1$

$x_1 = 0$

$x_2 = -1$ } oscillate.

$(1, 2]$

(3) $x(x-1) = 1$

$x = \frac{1}{x-1} \qquad x_0 = 0$

$x_1 = -1$

$x_2 = -1/2$

$x_3 = -0.6$

$x_4 = \vdots$

$x_5 = -0.625 \ldots$

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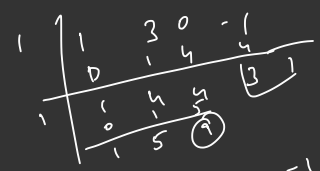
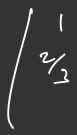
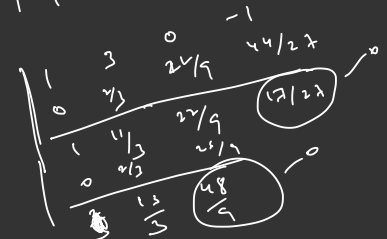
(1) $x^3 + 3x^2 - 1 \quad \Rightarrow x_0 = 1$



$\Rightarrow x_1 = 1 - \frac{3}{9} = 2/3$





$= 0 \quad \frac{2}{3}$

$= 0$

$= \frac{2}{3} - \frac{17/27}{48/9}$

$= \frac{2}{3} - \frac{17}{48} \times \frac{9}{3}$ ... 0.5

If you do it with deflation all the way down

$P_0 = 1 \Rightarrow P_5 = 0.53209$

$P_0 = -1 \Rightarrow P_3 = -0.652270$

$P_0 = -3 \Rightarrow P_3 = -2.87939$