① Error in Polynomial Interpolation $x_0, x_1, \ldots x_n$

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{n+1}(\xi)$$

⇒ it should exist..

↳ can't estimate as we don't know f.

$w(x) = \prod_{i=0}^{n} (x - x_i)$ is made small

if $\Delta x = \frac{a+b}{n}$? $[a, b]$

$x_i = a + i\Delta x$

let $a = -1$, $b = 1$ $i = 0, 1, 2 \ldots 4$

choice 1 ⇒ $x_i = -1 + 0.5i$

$w(x) = (x-1)(x+0.5)(x-0)(x-0.5)(x-1)$

max $[-1, 1] \Rightarrow \underline{\underline{0.11}}$

choice 2 ⇒ $x_0 = -0.95$, $x_1 = -0.58$

$x_2 = 0$, $x_3 = 0.58$

$x_4 = 0.95$

max = $\underline{\underline{0.06}}$

Recall ⇒ Langrange polynomial → unique

but can be gene. via divided diff

$P_n = a_0 + a_1(x - x_0) + a_2(\;)\ldots$

$a_0 = f(x_0)$

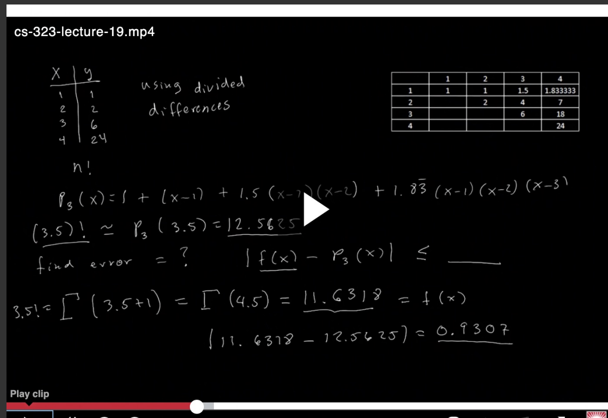
$f[x_1, x_0] = a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ ..

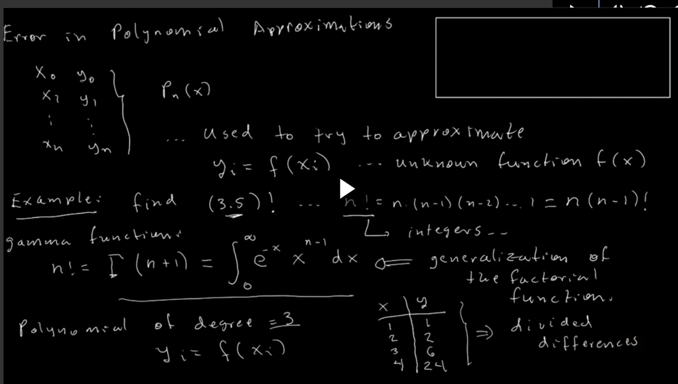
$$a_K = f[x_0 \ldots x_k] = \frac{f[x_1 \ldots x_k] - f[x_0 \ldots x_{k-1}]}{x_k - x_0}$$

	x	y
x_0	1	1
x_1	2	2
x_2	3	6
x_3	4	24

$$\Rightarrow 1 + 1(x) + f[x_0 . x_2](x-1)(x-2)$$
$$+ f[x_0 \dots x_3](x-1)(x-2)(x-3)$$

	x_0	x_1	x_2	x_3
x_0	1	1	1.5	1.83
x_1		2	4	7
x_2			6	18
x_3				24





Continuous Least Squares

$$\int_a^b (f(x) - P_n(x))^2\, dx = \int_a^b \left(f(x) - \sum_0^n a_k x^k\right)^2$$

$$\Rightarrow \frac{\partial E}{\partial a_j} = \int_a^b 2\left(f(x) - \sum_0^n a_k x^k\right)(-x^j) = 0$$

$$\Rightarrow \sum_{k=0}^{n} a_k \int_a^b x^{k+j} = \int_a^b f(x)\, x^j\, dx$$

$j=0$ $\quad a_0 \int 1\,dx + a_1 \int x\,dx + a_2 \int x^2 dx \ldots$

$a_0 \int x\,dx + \ldots$

$$\begin{pmatrix} \int 1\,dx & \int x\,dx & \ldots \\ \int x\,dx & \ldots & \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \int f(x)\,dx \\ \int x f(x)\,dx \end{pmatrix} \Rightarrow \text{ill-conditioned.}$$

Aside // Eg: Integ. by parts

$$\int x e^x\,dx = \int e^x - \int$$

$$\frac{\partial}{\partial x}\left[f(x).g(x)\right] = f'(x).g(x) + f(x).g'(x)$$

$$f(x).g(x) = \int f'(x).g(x) + \int f(x)\,g'(x)$$

$$\int f(x).g'(x)\,dx = f(x).g(x) - \int f'(x).g(x)\,dx$$

$$\Rightarrow x e^x - \int e^x = \int x e^x\,dx$$

$\int x \sin x\,dx$ $\quad -\cos x = g(x)$

$= x.-\cos x + \int \cos x$

$= -x\cos x + \int \cos x$

$-x\cos x + \sin x$

$\int_0^1 x \sin \pi x\,dx$

$= \left.-\frac{x\cos \pi x}{\pi}\right)_0^1 + \frac{1}{\pi}\left.\frac{\sin \pi x}{\pi}\right)_0^1$

$-\frac{x\cos \pi x}{\pi}$

$= \frac{1}{\pi}$ //.

$\sin \pi x \quad x \in (0, 1)$ $\Rightarrow \begin{pmatrix} & \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{matrix} \int \sin \pi x \\ \int x \sin \pi x \\ \int x^2 \sin \pi x \end{matrix}$

$h(x)$

⇒ continued in later page

Numerical integration

$\int_a^b f(x)$ → can't get anti-derivative / hard

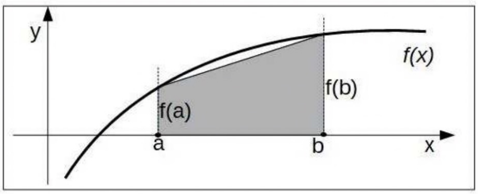
↳ which points you choose is imp?

Eg: $\int_0^2 f(x)$

	$(x+1)^{-1}$	e^x	$\sqrt{1+x^2}$
exact :	1.099	6.3	2.95
Trapezoid :	1.33	8.3	3.3
	1.11	6.4	2.96

$\frac{1}{2}(b-a)(f(a)+f(b))$

$\frac{b-a}{3}\left(f(a) + 4f(x_1) + f(a+1)\right)$



Notice that this formula can be obtained also if we observe that the first degree Lagrange polynomial is a line that goes through the points $(a, f(a)), (b, f(b))$. So the area under the curve corresponds to a trapezoid with height $b-a$, base 1 equal to $f(a)$, and base 2

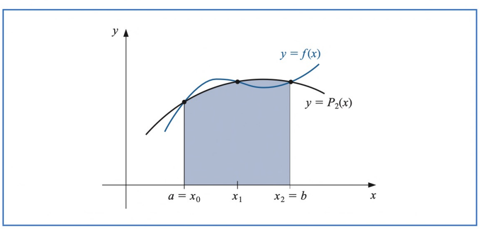
As we said before, the first degree Lagrange polynomial that goes through the points $(a, f(a)), (b, f(b))$ is:

$$\begin{aligned} P_1(x) &= \frac{(x-b)}{(a-b)}f(a) + \frac{(x-a)}{(b-a)}f(b) \\ &= \frac{1}{b-a}((x-a)f(b) - (x-b)f(a)) \\ &= \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right)x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right) \end{aligned}$$

If we integrate this polynomial we get

$$\begin{aligned} \int_a^b f(x)x &\approx \int_a^b P_1(x)x \\ &= \int_a^b \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right)x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right)x \\ &= \frac{1}{2}\left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right)x^2\Big|_a^b - \left(\frac{bf(a)}{a-b} + \frac{af(b)}{b-a}\right)x\Big|_a^b \\ &= \frac{1}{2}(b-a)(f(a)+f(b)) \end{aligned}$$

Figure 4.4



Therefore

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2}\left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\right]dx + \int_{x_0}^{x_2}\frac{(x-x_0)(x-x_1)(x-x_2)}{6}f^{(3)}(\xi(x))\,dx.$$

EXAMPLE 1 Find the least squares approximating polynomial of degree two for the function $f(x) = \sin \pi x$ on the interval $[0, 1]$. The normal equations for $P_2(x) = a_2 x^2 + a_1 x + a_0$ are given by:

$$a_0 \int_0^1 1\,dx + a_1 \int_0^1 x\,dx + a_2 \int_0^1 x^2\,dx = \int_0^1 \sin \pi x\,dx,$$

$$a_0 \int_0^1 x\,dx + a_1 \int_0^1 x^2\,dx + a_2 \int_0^1 x^3\,dx = \int_0^1 x \sin \pi x\,dx,$$

$$a_0 \int_0^1 x^2\,dx + a_1 \int_0^1 x^3\,dx + a_2 \int_0^1 x^4\,dx = \int_0^1 x^2 \sin \pi x\,dx.$$

Performing the integration yields

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{\pi}, \qquad \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{\pi},$$

$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{\pi^2 - 4}{\pi^3}.$$

The three equations in three unknowns can be solved to obtain

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.050465 \qquad \text{and} \qquad a_1 = -a_2 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251.$$

Consequently, the least squares polynomial approximation of degree two for $f(x) = \sin \pi x$ on $[0, 1]$ is $P_2(x) = -4.12251x^2 + 4.12251x - 0.050465$. (See Figure 8.6.)