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① error in polynomial interpolation  $x_0, x_1, \dots, x_n$

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi)$$

$\Rightarrow$  it should exist.  
 $\hookrightarrow$  can't estimate as we don't know  $f$ .

$$w(x) = \prod_{i=0}^n (x-x_i)$$

$\hookrightarrow$  much small

$$\Delta x = \frac{a+b}{n} \quad ? \quad [a, b]$$

$$x_i = a + i \Delta x \quad i = 0, 1, 2, \dots, n$$

let  $a = -1, b = 1$   
 $i = 0, 1, 2, \dots, n$

$$x_i = -1 + 0.5i$$

$$w(x) = (x-(-1))(x-0.5)(x-0)(x-0.5)(x-1)$$

choice 1  $\Rightarrow$

$$\max_{x \in [-1, 1]} |w(x)| = 0.11$$

choice 2  $\Rightarrow$

$$x_0 = -0.95, x_1 = -0.58$$

$$x_2 = 0, x_3 = 0.58$$

$$x_4 = 0.95$$

$$\hookrightarrow \max = 0.06$$

Recall  $\Rightarrow$  Lagrange polynomial  $\rightarrow$  unique  
 $\hookrightarrow$  but can be given via divided diff

$$P_n = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$

$$a_0 = f(x_0)$$

$$f[x_0, x_1] = a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_k = \frac{f[x_0, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

$n$	$y$
$n_0$	1
$n_1$	2
$n_2$	6
$n_3$	24

$\approx 1 +$

$$1(n) + f[n_0, n_1] (n-1)(n-2)$$

$$+ f[n_0, n_1, n_2] (n-1)(n-2)(n-3)$$

$\sum$

$n$	$y$	$h_1$	$h_2$	$h_3$
$n_0$	1	1	1.5	1.83
$n_1$	2		4	7
$n_2$	6		6	19
$n_3$	24			24

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X | y

using divided differences

	1	2	3	4
1	1	1	1.5	1.833333
2		2	4	7
3			6	19
4				24

$n!$

$$p_3(x) = f + (x-1) + 1.5(x-1)(x-2) + 1.83(x-1)(x-2)(x-3)$$

$$(3.5)! \approx p_3(3.5) = 12.5625$$

$$\text{find error} = ? \quad |f(x) - p_3(x)| \leq \underline{\hspace{2cm}}$$

$$3.5! = \Gamma(3.5+1) = \Gamma(4.5) = 11.63178 = f(x)$$

$$|11.63178 - 12.5625| = 0.9307$$

Play clip

Error in Polynomial Approximations

$$\begin{Bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{Bmatrix} \Rightarrow p_n(x)$$

... used to try to approximate

$$y_i = f(x_i) \quad \dots \text{unknown function } f(x)$$

Example: find  $(3.5)!$  ...  $n! = n(n-1)(n-2)\dots 1 = n(n-1)!$

gamma function:

$$n! = \Gamma(n+1) = \int_0^\infty e^{-x} x^{n-1} dx \quad \text{generalization of the factorial function.}$$

Polynomial of degree  $\leq 3$   
 $y_i = f(x_i)$

x	y
1	1
2	2
3	6
4	24

$\Rightarrow$  divided differences

Continuous Large Spans

$$\int_a^b (f(x) - p_n(x))^2 dx = \int_a^b \left( f(x) - \sum_{k=0}^n a_k x^k \right)^2 dx$$

$$\Rightarrow \frac{\partial E}{\partial a_j} = \int_a^b 2 \left( f(x) - \sum_{k=0}^n a_k x^k \right) (-x^j) dx = 0$$

$$\Rightarrow \sum_{k=0}^n a_k \int_a^b x^{k+j} dx = \int_a^b f(x) x^j dx$$

$$j \rightarrow a_0 \int 1 dx + a_1 \int x dx + a_2 \int x^2 dx \dots$$

$$a_0 \int n dx + \dots$$

$$b \begin{pmatrix} \int 1 dx & \int n dx & \dots \\ \int n dx & \dots & \dots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \int f(n) dx \\ \int n f(n) dx \end{pmatrix} = 0 \quad \text{ill-conditioned.}$$

Aside  
25/11 Interp. by prob

$$\int n e^n dx = \int e^n - 1$$

$$\frac{\partial}{\partial n} [f(n), g(n)] = f'(n) \cdot g(n) + f(n) \cdot g'(n)$$

$$f(n) \cdot g(n) = \int f'(n) \cdot g(n) + f(n) \cdot g'(n)$$

$$\int f(n) \cdot g(n) = \int f'(n) \cdot g(n) dx + \int f(n) \cdot g'(n) dx$$

$$-\frac{\cos \pi n}{\pi}$$

$$\int n e^n = \int e^n = \int n e^n dx$$

$$\int n \sin \pi n dx \quad \omega n = g(n) \\ = n \cos n + \int \cos n \\ = -n \cos n + \int \cos n \\ = -n \cos n \sin n$$

$$\int n \sin \pi n dx \\ = \left( -\frac{n \cos \pi n}{\pi} \right)' \\ = \left( \frac{1}{\pi} \right)''$$

$$h(n) \xrightarrow{\sin \pi n} x \in (0, \pi) \rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \int \sin \pi n \\ \int n \sin \pi n \\ \int n^2 \sin \pi n \end{pmatrix}$$

continued in last page

Numerical  
Integration

↳  $\int_a^b f(x) \approx$  can't get an-dex-ten / hand

which points you choose to imp?

Left  
Right

$(x+1)^{-1} \approx \int f(x)$

ex

$(x+1)^{-1}$

1.091

6.3

8.3

6.4

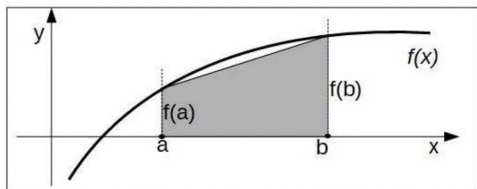
2.96

2.95

3.3

$\frac{1}{2}(b-a)(f(a)+f(b))$

$\frac{b-a}{2}(f(a)+f(b))$



Notice that this formula can be obtained also if we observe that the first degree Lagrange polynomial is a line that goes through the points  $(a, f(a))$ ,  $(b, f(b))$ . So the area under the curve corresponds to a trapezoid with height  $b-a$ , base 1 equal to  $f(a)$ , and base 2

As we said before, the first degree Lagrange polynomial that goes through the points  $(a, f(a))$ ,  $(b, f(b))$  is:

$$\begin{aligned} P_1(x) &= \frac{(x-b)}{(a-b)}f(a) + \frac{(x-a)}{(b-a)}f(b) \\ &= \frac{1}{b-a}((x-a)f(b) - (x-b)f(a)) \\ &= \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right)x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right) \end{aligned}$$

If we integrate this polynomial we get

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b P_1(x) dx \\ &= \int_a^b \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right)x - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right) dx \\ &= \frac{1}{2} \left(\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right) x^2 \Big|_a^b - \left(\frac{bf(b)}{a-b} + \frac{af(b)}{b-a}\right) x \Big|_a^b \\ &= \frac{1}{2}(b-a)(f(a) + f(b)) \end{aligned}$$

**EXAMPLE 1** Find the least squares approximating polynomial of degree two for the function  $f(x) = \sin \pi x$  on the interval  $[0, 1]$ . The normal equations for  $P_2(x) = a_2 x^2 + a_1 x + a_0$  are given by:

$$\begin{aligned} a_0 \int_0^1 1 \, dx + a_1 \int_0^1 x \, dx + a_2 \int_0^1 x^2 \, dx &= \int_0^1 \sin \pi x \, dx, \\ a_0 \int_0^1 x \, dx + a_1 \int_0^1 x^2 \, dx + a_2 \int_0^1 x^3 \, dx &= \int_0^1 x \sin \pi x \, dx, \\ a_0 \int_0^1 x^2 \, dx + a_1 \int_0^1 x^3 \, dx + a_2 \int_0^1 x^4 \, dx &= \int_0^1 x^2 \sin \pi x \, dx. \end{aligned}$$

Performing the integration yields

$$\begin{aligned} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 &= \frac{2}{\pi}, & \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 &= \frac{1}{\pi}, \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 &= \frac{\pi^2 - 4}{\pi^3}. \end{aligned}$$

The three equations in three unknowns can be solved to obtain

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.050465 \quad \text{and} \quad a_1 = -a_2 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251.$$

Consequently, the least squares polynomial approximation of degree two for  $f(x) = \sin \pi x$  on  $[0, 1]$  is  $P_2(x) = -4.12251x^2 + 4.12251x - 0.050465$ . (See Figure 8.6.)