Dirac Fermions

Probability of occupied/unoccupied states:-

For a gas of non interacting fermions, we derived in class the partition function:

$$Z_{FD} = \prod_k \left(1 + e^{\beta(\mu - \epsilon_k)}\right)$$

where k are the single particle states of energy ϵ_k.

The probability of any state ν is:

$$P_\nu = \frac{1}{Z} e^{-\beta E_\nu + \beta\mu N_\nu}$$

with: $\begin{cases} E_\nu = \sum_k n_k \epsilon_k \\ N_\nu = \sum_k n_k \end{cases}$

Hence:

$$P_\nu = \frac{1}{Z} \prod_k e^{n_k(-\beta\epsilon_k + \beta\mu)}$$

and therefore the probability of a state k (they are to be treated as independent in the grand canonical setting)

$$P_k = \frac{e^{n_k \beta(\mu - \epsilon_k)}}{\sum_{n_k} e^{\beta(\mu - \epsilon_k) n_k}}$$

the same factorization done in class to compute the partition function.

Since $n_k = 0, 1$ for a gas of Fermions, we have:

$$P_k(n_k = 0) = \frac{1}{1 + e^{\beta(\mu - \epsilon_k)}} \tag{1}$$

$$P_k(n_k = 1) = \frac{e^{\beta(\mu - \epsilon_k)}}{1 + e^{\beta(\mu - \epsilon_k)}} \tag{2}$$

[recall $\langle n_k \rangle_{FD} = \dfrac{1}{1 + e^{\beta(\epsilon_k - \mu)}}$]

~~DIRAC FERMIONS~~

~~Probability of occupied/unoccupied states:~~

~~For a gas of non in~~

We now consider the probability of occupation of $\epsilon_k = \mu + \delta$ and $\epsilon_k = \mu - \delta$

$\langle n_k^+ \rangle$ $$p(\epsilon_k = \mu + \delta, n_k = 1) = \frac{e^{\beta(\mu - \mu - \delta)}}{1 + e^{\beta(\mu - \delta - \mu)}} = \frac{e^{-\beta\delta}}{1 + e^{-\beta\delta}} = \boxed{\frac{1}{1 + e^{\beta\delta}}} = \langle n_k^+(\mu+\delta)\rangle$$

$$p(\epsilon_k = \mu - \delta, n_k = 0) = \frac{1}{1 + e^{\beta(\mu - \mu + \delta)}} = \frac{1}{1 + e^{\beta\delta}}$$

$$\langle n_k^- \rangle = \frac{1}{e^{-\beta\delta} + 1} = \frac{e^{\beta\delta}}{1 + e^{\beta\delta}}$$

$$\Rightarrow \quad p(\epsilon_k = \mu + \delta, n_k = 1) = p(\epsilon_k = \mu - \delta, n_k = 0)$$

occupied state at $\epsilon_k = \mu + \delta$ — unoccupied state at $\epsilon_k = \mu - \delta$

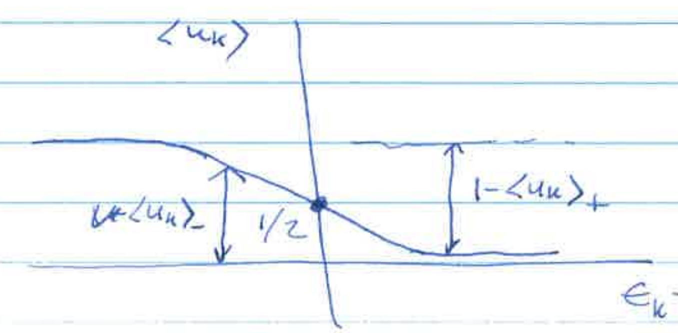
Same conclusion follows directly from occupation numbers:

- $$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} = \frac{e^{\beta(\mu - \epsilon_k)}}{1 + e^{\beta(\mu - \epsilon_k)}} \qquad \text{Same as Eq. (2)}$$

- $$1 - \langle n_k \rangle = 1 - \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} = \frac{e^{\beta(\epsilon_k - \mu)}}{1 + e^{\beta(\epsilon_k - \mu)}} = \frac{1}{1 + e^{\beta(\mu - \epsilon_k)}}$$

Same as Eq. (1)

Graphically:



are equal

The function $\langle n_k \rangle$ is symmetric around $\epsilon = \mu$:

$$\langle n_k \rangle = \frac{1}{1 + e^{\beta(\epsilon_k - \mu)}} \qquad \text{Take } \epsilon_k = \mu + \delta$$

$$\langle n_k \rangle = \frac{1}{1 + e^{\beta\delta}} \qquad \langle n_k \rangle = \frac{1}{2} \text{ at } \delta = 0$$

and there is symmetry of $(+\epsilon)$ and $(-\epsilon)$ states:

(See graph on page 2)

$$\langle n_k \rangle_- = 1 - \langle n_k \rangle_+$$

$$\frac{1}{1 + e^{\beta(-\epsilon-\mu)}} = 1 - \frac{1}{1 + e^{\beta(\epsilon-\mu)}} = \frac{\cancel{1} + e^{\beta(\epsilon-\mu)} - \cancel{1}}{1 + e^{\beta(\epsilon-\mu)}}$$

$$\Rightarrow \quad 1 + \cancel{e^{\beta(\epsilon-\mu)}} = e^{\beta(\epsilon-\mu)}\left(1 + e^{-\beta(\epsilon+\mu)}\right)$$

$$= \cancel{e^{\beta(\epsilon-\mu)}} + e^{-2\mu\beta} \quad \Rightarrow \quad 1 = e^{-2\mu\beta} \text{ or } \boxed{\mu = 0}$$

for any temperature.

(c) <u>Energy of an excitation at finite T</u>

An excitation means a $(-)$ particle leaves the state and occupies a $(+)$ state. Hence:

$$\langle E \rangle (T) \cancel{- \langle E \rangle (T=0)} = \underset{\text{Spin}}{2} \sum_k \Big[\underbrace{\langle n(k) \rangle_+}_{\text{prob. of occupied + state}} \epsilon_+(k) + \underbrace{\langle n(k) \rangle_-}_{\text{prob. of unoccupied (−) state.}}) \epsilon_-(k) \Big]$$

$$= 2 \sum \left[\langle n_k^+ \rangle \epsilon_k^+ - \epsilon_k^+ \left(1 - \langle n_k^+ \rangle\right) \right]$$

Given that $1 - \langle n_k \rangle_- = \langle n_k \rangle_+$

$$= 2\cancel{V}\int \quad 2\sum\left(2\langle n_k^+ \rangle \epsilon_k^+ - \epsilon_k^+\right)$$

$$\Rightarrow \langle E \rangle (T) - \langle E \rangle (T=0) = 2 \sum_k \Big[2 \langle n(k) \rangle_+ \epsilon_+(k) \Big] + 2\sum_k \epsilon_k^-$$

$$2V\int \frac{d^3k}{(2\pi)^3} \epsilon_-(k)$$

$$= 4 V \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon_+(k)}{1 + e^{\beta\epsilon_+(k)}}$$

with the usual transformation of sums into integrals.

$$\langle E \rangle = 2\sum_k \left[\langle u_k^+ \rangle \epsilon_k^+ + \langle u_k^- \rangle \epsilon_k^- \right]$$

$$= 2\sum_k \left[\langle u_k^+ \rangle \epsilon_k^+ + \left(1 - \langle u_k^+ \rangle\right) \epsilon_k^- \right]$$

$$= 2\sum_k \left[\langle u_k^+ \rangle \epsilon_k^+ - \left(1 - \langle u_k^+ \rangle\right) \epsilon_k^+ \right] =$$

$$= 2\sum \left(2\langle u_k^+ \rangle \epsilon_k^+ - \epsilon_k^+ \right) = 2\sum 2\langle u_k^+ \rangle \epsilon_k^+ + 2\sum \epsilon_k^-$$

$$\boxed{\langle E \rangle - 2\sum_k \epsilon_k^- = 4\sum \langle u_k^+ \rangle \epsilon_k^+} \quad E_0 = 2\sum \epsilon_k^-$$

Backwards:

$\cancel{\langle E \rangle - E(0)}$

$$\langle E \rangle - E(0) = 2\sum_k \left[\langle u_k^+ \rangle \epsilon_k^+ + \langle u_k^- \rangle \epsilon_k^- \right] - 2\sum \epsilon_k^-$$

$$= 2\sum \left[\langle u_k^+ \rangle \epsilon_k^+ + \langle u_k^- \rangle \epsilon_k^- - \epsilon_k^- \right]$$

$$= 2\sum \left[\langle u_k^+ \rangle \epsilon_k^+ - \left(1 - \langle u_k^- \rangle\right) \epsilon_k^- \right]$$

$$\cancel{= 2\sum \left[\langle u_k \rangle \epsilon_k^+ - \epsilon_k + \langle u_k^- \rangle \epsilon_k \right.}$$

$$= 2\sum \left[\langle u_k^+ \rangle \epsilon_k^+ - \langle u_k^+ \rangle \epsilon_k^- \right]$$

$$= 2\sum \left[\langle u_k^+ \rangle \epsilon_k^+ + \langle u_k^+ \rangle \epsilon_k^+ \right]$$

$$\langle E \rangle = 2 \sum_k \left[\langle n_k^+ \rangle \epsilon_k^+ + \langle n_k^- \rangle \epsilon_k^- \right]$$

$$\langle n_k^- \rangle = \left(1 - \langle n_k^+ \rangle \right)$$

$$\langle E \rangle = 2 \sum \left[\langle n_k^+ \rangle \epsilon_k^+ + \left(1 - \langle n_k^+ \rangle \right) \epsilon_k^- \right]$$

$$= 2 \sum \left[\langle n_k^+ \rangle \epsilon_k^+ - \left(1 - \langle n_k^+ \rangle \right) \epsilon_k^+ \right]$$

$$\cancel{\langle E \rangle = 2 \sum \langle n_k^+ \rangle \epsilon_k^+ +}$$

$$\langle E \rangle - E(T_0) = 2 \sum_k \left[\langle n_k^+ \rangle \epsilon_k^+ + \langle n_k^- \rangle \epsilon_k^- - \epsilon_k^- \right]$$

$$= 2 \sum_k \left[\langle n_k^+ \rangle \epsilon_k^+ + \left(\langle n_k^- \rangle - 1 \right) \epsilon_k^- \right]$$

$$= 2 \sum_k \left[\langle n_k^+ \rangle \epsilon_k^+ - \underbrace{\left(1 - \langle n_k^- \rangle \right)}_{\langle n_k^+ \rangle} \epsilon_k^+ \right.$$

(d) Evaluate the integral for small k:

$$\epsilon_+(k) = \sqrt{m^2c^4 + \hbar^2c^2k^2} = \sqrt{m^2c^4\left(1+\frac{\hbar^2c^2k^2}{m^2c^4}\right)} = mc^2\left(1+\frac{\hbar^2c^2k^2}{2m^2c^4}\right)$$

$$\Rightarrow \boxed{\epsilon_+(k) = mc^2 + \frac{\hbar^2k^2}{2m}} \quad \text{Gap at } k=0.$$

$$\langle E\rangle - E_0 = 4V\int\frac{d^3k}{(2\pi)^3}\frac{mc^2+\hbar^2k^2/2m}{1+e^{\beta(mc^2+\hbar^2k^2/2m)}} =$$

neglect 1 in denominator and retain only mc^2 in numerator

$$= \frac{4Vmc^2}{(2\pi)^3}\int d^3k\, e^{-\beta(mc^2+\hbar^2k^2/2m)} =$$

$$= \frac{4Vmc^2e^{-\beta mc^2}}{(2\pi)^3}\int d^3k\, e^{-\beta\frac{\hbar^2k^2}{2m}} =$$

$$= \frac{4Vmc^2e^{-\beta mc^2}}{\cancel{(2\pi)^3}\,(2\pi)^2}\cancel{4\pi}\int_0^\infty k^2dk\, e^{-\beta\frac{\hbar^2k^2}{2m}}$$

$$\cancel{\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right)}$$

$$\cancel{= \frac{2!\sqrt{\pi}}{4\,1!} = \frac{2\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}}$$

Tables: $$\int_0^\infty x^2e^{-ax^2}dx = \frac{\Gamma(3/2)}{2a^{3/2}} = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Exponent: $$\frac{\beta\hbar^2}{2m} = \frac{h^2}{2m\,4\pi^2\cdot k_BT} = \frac{1}{4\pi}\lambda^2$$

$\Gamma(3/2) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$$\Rightarrow \int_0^\infty k^2dk\, e^{-\frac{\lambda^2}{4\pi}\cdot k^2} = \frac{\sqrt{\pi}}{4}\frac{1}{(\lambda^2/4\pi)^{3/2}} = \frac{(4\pi)^{3/2}\sqrt{\pi}}{4}\frac{1}{\lambda^3}$$

$$= \frac{2\pi^2}{\lambda^3}$$

$$\Rightarrow \langle E\rangle - E_0 = \frac{4Vmc^2e^{-\beta mc^2}}{\cancel{2}\cancel{\pi}\,\pi}\frac{\cancel{2}\pi^{\cancel{2}}}{\lambda^3} = \frac{4V}{\lambda^3}mc^2e^{-\beta mc^2}$$