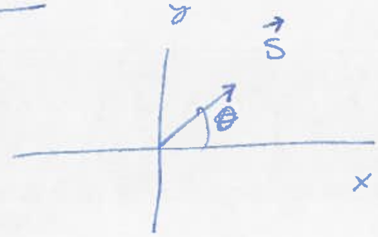
XY model in two dimensions

Consider a planar spin



~~$\vec{S} = S_0 (\cos\varphi,$~~

$\vec{S} = S_0 (\cos\theta, \sin\theta)$

or: $\psi = S_0 e^{i\theta}$

~~Due to an~~

The energy of a configuration is:

$$E = \frac{\rho_s}{2} \int d^2x \, (\nabla\theta)^2$$

The order parameter correlation function:

$$G(\vec{r}) = \langle \psi^*(0) \psi(\vec{r}) \rangle = S_0^2 \langle e^{-i\theta(0)} e^{i\theta(\vec{r})} \rangle =$$

$$= S_0^2 \langle e^{i(\theta(r) - \theta(0))} \rangle$$

Given the energy E, the variable θ is Gaussianly distributed.

Hence:

$$G(\vec{r}) = S_0^2 \, e^{-\frac{1}{2} \langle (\theta(r) - \theta(0))^2 \rangle}$$

Compute the exponent:

$$\langle (\theta(r) - \theta(0))^2 \rangle = 2 \langle \theta^2 \rangle - 2 \langle \theta(0) \theta(r) \rangle =$$

$$= 2 \left(\langle \theta^2 \rangle - \langle \theta(0) \theta(r) \rangle \right)$$

From the expression for the energy given:

$$\langle |\theta_k|^2 \rangle = \frac{k_B T}{\rho_s k^2} \rightarrow \langle \theta(0) \theta(\vec{r}) \rangle = \int \frac{d^2k}{(2\pi)^2} \frac{k_B T}{\rho_s k^2} e^{i\vec{k}\cdot\vec{r}}$$

$$\rightarrow \langle \theta^2 \rangle = \int \frac{d^2k}{(2\pi)^2} \frac{k_B T}{\rho_s k^2}$$

We find:

$$\langle (\theta(r)-\theta(0))^2 \rangle = \frac{2k_BT}{\rho_s} \int \frac{d^2k}{(2\pi)^2} \frac{1-e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Introduce polar coordinates:

$$\langle (\theta(0)-\theta(r))^2 \rangle = \frac{2k_BT}{\rho_s} \int \frac{k\,dk\,d\varphi}{(2\pi)^2} \frac{1-e^{ikr\cos\varphi}}{k^2}$$

$$= \frac{2k_BT}{\rho_s} \int \frac{k\,dk}{(2\pi)^2} \frac{1}{k^2} \int_0^{2\pi} d\varphi \left(1-e^{ikr\cos\varphi}\right)$$

Recalling that: $J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\varphi}\,d\varphi$

$$\langle (\theta(0)-\theta(r))^2 \rangle = \frac{2k_BT}{\rho_s} \int \frac{k\,dk}{(2\pi)^2} \frac{1}{k^2} \left(2\pi - 2\pi J_0(kr)\right)$$

$$= \frac{2k_BT}{(2\pi)\rho_s} \int_0^{\Lambda} \frac{dk}{k} \left(1-J_0(kr)\right)$$

We now approximate for large distances, so that $\Lambda r \gg 1$ and recall that $J_0(z) \sim \sqrt{\frac{2}{\pi z}}$ as $z \to \infty$

Hence:

$$\langle (\theta(0)-\theta(r))^2 \rangle = \frac{2k_BT}{(2\pi)\rho_s} \int_0^{\Lambda r} \frac{dx}{x} \left(1-J_0(x)\right) \qquad (x = kr)$$

$$= \frac{2k_BT}{(2\pi)\rho_s} \left\{ \underbrace{\int_0^1 \frac{dx}{x}(1-J_0(x))}_{\substack{\text{constant} \\ (J_0(x\to 0)\to 1)}} + \underbrace{\int_1^{\Lambda r} \frac{dx}{X}}_{\ln(\Lambda r)} - \underbrace{\int_1^{\Lambda r} \frac{J_0(x)}{x}\,dx}_{\text{negligible for } x \text{ large}} \right\}$$

Therefore $\langle (\theta(r) - \theta(0))^2 \rangle = \frac{k_B T}{\pi \rho_s} \ln(\Lambda r)$

and therefore $G(r) = S_0^2 \, e^{-\frac{k_B T}{2\pi\rho_s} \ln(\Lambda r)} =$

$$= S_0^2 \, e^{\ln(\Lambda r)^{-k_B T / 2\pi\rho_s}} = \boxed{\frac{S_0^2}{r^\eta} = G(r)}$$

$$\boxed{\eta = \frac{k_B T}{2\pi\rho_s}}$$

- No long range order: $\langle \delta\theta^2 \rangle$ increases with distance.
- Algebraic correlations with a temperature dependent exponent.