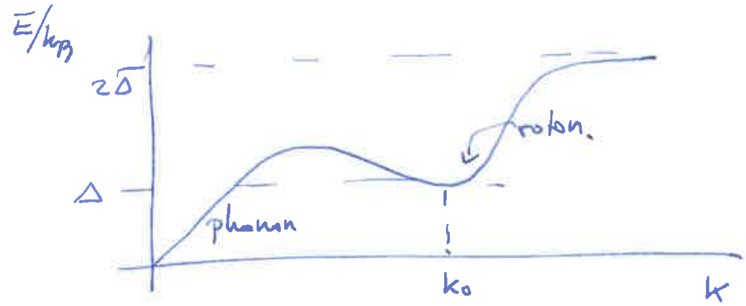
Excitations in ^4He



The measured energy spectrum is:

- Phonon part of the spectrum:

$$\epsilon_{ph} = \hbar c k \qquad c = 239\,m/s \quad (\text{speed of sound})$$

. Pathria 207
. Huang 11.6

- Near the roton minimum:

$$\epsilon_{rt} = \Delta + \frac{\hbar^2 (k-k_0)^2}{2\sigma} \qquad \Delta = 8.65\,K \quad k_0 = 1.92\,\mathring{A}^{-1} \quad \sigma = 0.16\,m.$$

(a) Phonon specific heat.

$$\langle E_{ph} \rangle = \int d^3k \frac{V}{(2\pi)^3} \frac{\hbar c k}{e^{\beta \epsilon_k} - 1} = \frac{\hbar c V}{2\pi^2} \int_0^\infty k^2 dk \frac{k}{e^{\beta \hbar c k} - 1}$$

Same as Debye model. No cut-off for number of normal modes.

Define: $x = \beta \hbar c k$

$$\Rightarrow \langle E_{ph} \rangle = \frac{\hbar c V}{2\pi^2} \int_0^\infty \frac{(x/\beta\hbar c)^3 \, dx/\beta\hbar c}{e^x - 1} =$$

$$= \frac{\hbar c V}{2\pi^2} \frac{1}{(\beta \hbar c)^4} \underbrace{\int_0^\infty \frac{x^3 dx}{e^x - 1}}_{\pi^4/15} = \frac{\pi^2 \hbar c V}{30} \frac{(k_B T)^4}{(\hbar c)^4}$$

$$= \frac{\pi^2}{(\hbar c)^3} \frac{V}{30} (k_B T)^4$$

Same as for solid. $\rightarrow C_{ph} \sim T^3$.

(b) For rotons.

$$\langle E_{rot} \rangle = \frac{V}{2\pi} \int_0^\infty k^2 dk \frac{\Delta + \hbar^2 (k-k_0)^2/2\sigma}{e^{\beta(\Delta + \hbar^2 (k-k_0)^2/2\sigma)} - 1}$$

Assume low temperatures, $\beta \to \infty$

$$\Rightarrow \langle E_{rot} \rangle \simeq \frac{V}{2\pi} \int_0^\infty k^2 dk \left(\Delta + \frac{\hbar^2 (k-k_0)^2}{2\sigma}\right) e^{-\beta\Delta} e^{-\beta\hbar^2 (k-k_0)^2/2\sigma}$$

$$\langle E_{rot} \rangle \simeq \frac{V}{2\pi} e^{-\beta\Delta} \int_0^\infty k^2 dk \left(\Delta + \frac{\hbar^2 (k-k_0)^2}{2\sigma}\right) e^{-\beta\hbar^2 (k-k_0)^2/2\sigma}$$

For large β, the integral highly peaked around k_0.

Neglect $\Delta + \frac{\hbar^2 (k-k_0)^2}{2\sigma} \simeq \Delta$

$$\simeq \frac{V}{2\pi} \Delta e^{-\beta\Delta} \int_0^\infty k^2 dk\, e^{-\beta\hbar^2 (k-k_0)^2/2\sigma}$$

Now define: $x^2 = \beta\hbar^2 (k-k_0)^2/2\sigma$ $\quad \frac{2\sigma x^2}{\beta\hbar^2} = (k-k_0)^2 \rightarrow k = k_0 + \sqrt{\frac{2\sigma}{\beta\hbar^2}}\, x$

$$dk = \sqrt{\frac{2\sigma}{\beta\hbar^2}} \cdot dx$$

$$\Rightarrow \langle E_{rot} \rangle \simeq \frac{V\Delta}{2\pi} e^{-\beta\Delta} \int_0^\infty \left(k_0 + \sqrt{\frac{2\sigma}{\beta\hbar^2}}\, x\right) e^{-x^2} \sqrt{\frac{2\sigma}{\beta\hbar^2}}\, dx$$

$$= \frac{V\Delta}{2\pi} e^{-\beta\Delta} \sqrt{\frac{2\sigma}{\beta\hbar^2}}\, k_0^2 \int_{-\frac{\beta\hbar^2 k_0^2}{2\sigma}}^\infty dx\, e^{-x^2} \left(1 + \sqrt{\frac{2\sigma}{\beta\hbar^2}} \frac{x}{k_0}\right)$$

peaked around $x=0$.

linear term does not contribute around $x=0$

Neglect and extend to $-\infty, \infty$

and $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$

$$\Rightarrow \boxed{\langle E_{rot} \rangle = \frac{V\Delta}{2\pi} k_0^2 e^{-\beta\Delta} \frac{\sqrt{2\pi\sigma k_B T}}{\hbar}}$$

and the specific heat vanishes exponentially.