(a) Consider the following approximate model of a liquid. Interactions between molecules are neglected except that all molecules experience a constant binding energy $-\eta$ that confines its motion to a volume v_0 per particle, and fixed. Show that for a ~~larg~~ classical liquid of N indistinguishable molecules, the partition function is:

$$Z_L = \frac{1}{N_L!}\left[N_L v_0 \, e^{\beta\eta} Z_p\right]^{N_L}$$

where Z_p denotes the ideal gas contribution arising from the integral over momenta:

$$Z_p = \frac{1}{h^3}\int d^3p \, e^{-\beta p^2/2m}.$$

(b) ~~Ass. Assume now that this liquid coexists with its vapor (an ideal gas)~~. Calculate the chemical potential of the liquid μ_L and of the vapor μ_G, assumed an ideal gas

(c) At coexistence between liquid and vapor phases: $\mu_L = \mu_G$. Find the pressure (vapor pressure as it is called) as a function of temperature at coexistence.

(a) For a single particle $\quad Z_1 = \frac{1}{h^3}\int d^3r\, d^3p \, e^{-\beta p^2/2m} e^{\beta\eta}$

$$= \cancel{\frac{1}{h^3}} e^{\beta\eta}(N_L v_0) Z_p$$

$$\Rightarrow \boxed{Z_L = \frac{1}{N_L!}\left[e^{\beta\eta}(N_L v_0) Z_p\right]^{N_L}}$$

For the ideal gas $\eta = 0$ and $N_L v_0$ is V_G, the volume of the gas phase (not fixed).

$$\boxed{Z_G = \frac{1}{N_G!}\left[V_G Z_p\right]^{N_G}}$$

(b) $\mu = \dfrac{\partial F}{\partial N} = -k_B T \dfrac{\partial \ln Z}{\partial N}$

$\cancel{\mu_G = -k_B T}$ $\quad F_G = -k_B T \ln Z_G = -k_B T \left[N_G \ln (V_G Z_p) - N_G \ln N_G + N_G \right]$

$$\mu_G = -k_B T \left[\ln (V_G Z_p) - \ln N_G - \cancel{N_G} \frac{1}{\cancel{N_G}} \cancel{+1} \right]$$

$$\boxed{\mu_G = -k_B T \left[\ln (V_G Z_p) - \ln N_G \right]}$$

$$F_L = -k_B T \left[N_L \ln (v_0 Z_p) + N_L \beta \eta + \cancel{N_L \ln N_L} \cancel{- N_L \ln N_L} + N_L \right]$$

$$\boxed{\mu_L = -k_B T \left[\ln (v_0 Z_p) + \beta \eta + 1 \right]}$$

(c) At coexistence $\mu_L = \mu_G$

$$\ln (V_G \cancel{Z_p}) - \ln N_G = \ln (v_0 \cancel{Z_p}) + \beta \eta + 1$$

$$\ln \left(\frac{V_G}{N_G} \right) = \ln (v_0) + \beta \eta + 1 \Rightarrow \frac{V_G}{N_G} = e^{\ln v_0} e^{\beta \eta + 1} = v_0 e^{\beta \eta + 1}$$

Ideal gas $= \dfrac{P}{k_B T} = \dfrac{N_G}{V_G}$

$$\Rightarrow \boxed{\frac{P}{k_B T} = \frac{e^{-(\beta \eta + 1)}}{v_0}}$$

this is the vapor pressure as a function of temperature.