Correlation function. ~~Kar~~ Kardar, Statistical Mechanics of Fields, p. 38

Consider the correlation function:

$$\hat{G}(q) = \frac{(k_BT/K)}{q^2+\xi^{-2}}$$

We define:

$$G(x) = \int \frac{d^dq}{(2\pi)^d}\, e^{i\vec{q}\cdot\vec{x}}\, \hat{G}(q)$$

, the real space correlation function in d dimensions.

$$\nabla^2 G(x) = \int \frac{d^dq}{(2\pi)^d}\,(-q^2)\,\frac{(k_BT/K)}{q^2+\xi^{-2}}\, e^{i\vec{q}\vec{x}}$$

$$= -\int \frac{d^dq}{(2\pi)^d}\left[1 - \frac{\xi^{-2}}{q^2+\xi^{-2}}\right](k_BT/K)\, e^{i\vec{q}\cdot\vec{x}}$$

$$\frac{q^2 + \cancel{\xi^{-2} - \xi^{-2}}}{q^2+\xi^{-2}}$$

$$= -\,k_BT/K \int \frac{d^dq}{(2\pi)^d}\, e^{i\vec{q}\cdot\vec{x}} + \int \frac{d^dq}{(2\pi)^d}\,\frac{k_BT/K\;\xi^{-2}}{q^2+\xi^{-2}}\, e^{i\vec{q}\cdot\vec{x}}$$

$$= -\frac{k_BT}{K}\,\delta(\vec{x}) + \xi^{-2}\, G(x).$$

Define: $G(x) = -\dfrac{k_BT}{K}\, I(x)$

$$\cancel{-}\,\cancel{\frac{k_BT}{K}}\,\nabla^2 I(x) = \cancel{-}\,\cancel{\frac{k_BT}{K}}\,\delta(x)\ \cancel{-}\ \xi^{-2}\,\cancel{\frac{k_BT}{K}}\, I(x)$$

$$\nabla^2 I(x) = \delta(x) + \frac{I(x)}{\xi^2}$$ Helmholtz equation

If a problem has spherical symmetry, then in d dimensions: $\nabla^2 = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}$ ← Give this

Show that at long distances:

$$I(r) \sim \frac{e^{-r/\xi}}{r^p}$$ and determine p.

$$\frac{dI}{dr} = \frac{1}{r^p}\left(-\frac{1}{\xi}\right)e^{-r/\xi} + (-p)\frac{e^{-r/\xi}}{r^{p+1}}$$

$$= -\frac{1}{\xi}I - \frac{p}{r}I = -\left(\frac{p}{r} + \frac{1}{\xi}\right)I$$

$$\cancel{\frac{d^2I}{dr^2} = -\frac{1}{\xi}\left(-\frac{1}{\xi}I - \frac{pI}{r}\right)}$$

$$\frac{d^2I}{dr^2} = -\left(-\frac{p}{r^2}\right)I + \left(\frac{p}{r} + \frac{1}{\xi}\right)^2 I$$

$$= \frac{p}{r^2}I + \frac{p^2}{r^2}I + \frac{2p}{r\xi}I + \frac{1}{\xi^2}I$$

$$= \left(\frac{p(p+1)}{r^2} + \frac{2p}{r\xi} + \frac{1}{\xi^2}\right)I$$

$\delta(x)$ at long distances

Hence: Substituting

$$\left(\frac{p(p+1)}{r^2} + \frac{2p}{r\xi} + \frac{1}{\xi^2}\right)I - \frac{d-1}{r}\left(\frac{p}{r} + \frac{1}{\xi}\right)I = 0 + \frac{1}{\xi^2}I$$

$$\Rightarrow \frac{p(p+1)}{r^2} + \frac{2p}{r\xi} - \frac{(d-1)p}{r^2} - \frac{d-1}{r\xi} = 0$$

[For large r and ξ, but $1/\xi \ll 1$ $\quad \frac{p(p+1)}{r^2} - \frac{(d-1)p}{r^2} = 0 \Rightarrow p = $ …]

If I am really asking at long distance. Then the $1/r^2$ terms are negligible. Hence:

$$\frac{2p}{r\xi} - \frac{d-1}{r\xi} = 0 \Rightarrow \boxed{p = (d-1)/2}$$

Take $G(r) = e^{-r/\xi} u(r)$

$$\Rightarrow u'' + u'\left(-\frac{2}{\xi} + \frac{d-1}{r}\right) + u\left(-\frac{d-1}{\xi r}\right) = 0$$

Solution $u(x) = x^{-p} \sum_{n=0}^{\infty} a_n x^n$

Substitute and

$$a_0\left[(-p)(-p-1) - p(d-1)\right] = 0$$

$$a_0 p\left[p+1-d+1\right] = 0 \quad \begin{array}{l} \rightarrow p = 0 \\ \rightarrow p = d-2 \end{array}$$

- For $d > 2$ $\quad p = d-2$
- For $d = 2$, $p = 0$ is a double root.

According to Frobenius Theorem, the two linearly independent solutions are:

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$$

$$y_2(x) = y_1(x) \ln|x| + \sum_{n=1}^{\infty} b_n x^n$$

Since y_1 is regular, the dominant contribution is $\ln|x|$ in $d=2$

$$\Rightarrow G(r) = \begin{cases} e^{-r/\xi} & d=1 \\ \ln(r)\, e^{-r/\xi} & \text{for } d=2 \\ \dfrac{e^{-r/\xi}}{r^{d-2}} & d>2 \end{cases}$$