③ Binomial distribution

a)
$$p_n = \binom{N}{n} q^n (1-q)^{N-n}$$

N number of observations
q probability of an individual outcome.
n: number of "successes", (outcomes).

$$G(k) = \sum_{n=0}^{N} e^{ikn} p_n =$$

$$= \sum_n e^{ikn} \binom{N}{n} q^n (1-q)^{N-n} =$$

$$= \sum_{n=0}^{N} \binom{N}{n} (q e^{ik})^n (1-q)^{N-n}$$

We now use $(a+b)^N = \sum_{n=1}^{N} \binom{N}{n} a^n b^{N-n}$

$$\Rightarrow \boxed{G(k) = (q e^{ik} + 1 - q)^N}$$ Characteristic function of binomial.

Define $\lambda = Nq$ and take limit $\left.\begin{matrix} N \to \infty \\ q \to 0 \end{matrix}\right\} \lambda$ finite.

c) Poisson distribution

$$p_n = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$G(k) = \sum_{n=0} e^{ikn} p_n = \sum_n e^{ikn} \frac{\lambda^n}{n!} e^{-\lambda} = \sum_{n=0} \underbrace{\frac{(e^{ik}\lambda)^n}{n!}}_{} e^{-\lambda}$$

$$= e^{-\lambda} (e^{\lambda e^{ik}})$$

b) We now know $\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x$

For the binomial:

$$\cancel{G(k) = \left(\frac{\lambda}{N}e^{ik}+1-\frac{\lambda}{N}\right)^N} =$$

~~negligible as $N\to\infty$, λ finite.~~

$$\cancel{= \left(1+\frac{\lambda e^{ik}}{N}\right)^N = e^{\lambda e^{ik}}}$$

For binomial: $G(k) = \left(\frac{\lambda}{N}e^{ik}+1-\frac{\lambda}{N}\right)^N =$

$$= \left(\frac{\lambda e^{ik}-\lambda}{N}+1\right)^N \to e^{-\lambda+\lambda e^{ik}} \quad \text{as we had.}$$

d) Start from $p_n = \frac{1}{2^N}\binom{N}{n}$ as the binomial distribution.

Let us find the maximum. Take $\ln p_n$ to help with Stirling

$$\ln N! = N\ln N - N$$

$$\ln p_n = -N\ln 2 + \ln\frac{N!}{n!\,(N-n)!} =$$

$$= -N\ln 2 + \left(N\ln N - N\right) - \left(n\ln n - n\right) - \left((N-n)\ln(N-n) - (N-n)\right)$$

We now look at the maximum with respect to n:

$$\frac{d\ln p_n}{dn} = -\left(\ln n + \cancel{n.\frac{1}{n}} \cancel{-1}\right) - \left(-\ln(N-n) + \cancel{(N-n)\frac{1}{N-n}(-1)} + 1\right)$$

$$= -\ln n + \ln(N-n) = \ln\frac{(N-n)}{n} = 0 \Rightarrow N-n=n \Rightarrow \boxed{n = N/2}$$

So the maximum of p_n for large N occurs at $n = \frac{N}{2}$.

- Next order is

$$\left.\frac{d^2 \ln p_n}{dn^2}\right)_{n=\frac{N}{2}} = \frac{d}{dn}\Big(\ln(N-n) - \ln n\Big)\Big|_{\frac{N}{2}} =$$

$$= \frac{1}{N-n}(-1) - \frac{1}{n} = -\frac{1}{N-n} - \frac{1}{n} =$$

$$= -\frac{1}{N-\frac{N}{2}} - \frac{2}{N} = -\frac{2}{2N-N} - \frac{2}{N} = -\frac{4}{N}$$

- Higher order. We have

$$\frac{d^2 \ln p_n}{dn} = -\frac{\cancel{n} + N - \cancel{n}}{(N-n)n} = -\frac{N}{n(N-n)}$$

$$\frac{d^3 \ln p_n}{dn^3} = +N\left[\frac{N-n+n(-1)}{n^2(N-n)^2}\right] = \frac{N(N-2n)}{n^2(N-n)^2}$$

$$\therefore \left(\frac{d^3 \ln p_n}{dn^3}\right)_{n=\frac{N}{2}} = \frac{N\left(N-2\frac{N}{2}\right)}{\left(\frac{N}{2}\right)^2\left(N-\frac{N}{2}\right)^2} = 0.$$

One can continue ... one would get a Gaussian so far:

$$\cancel{\ln p_n = p_{\frac{N}{2}} + d\ln}\quad \ln p_n = \ln p_{\frac{N}{2}} + \underbrace{\left(\frac{d \ln p_n}{dn}\right)_{n=\frac{N}{2}}}_{=0}\left(n-\frac{N}{2}\right) +$$

$$+ \frac{1}{2}\underbrace{\left(\frac{d^2 \ln p_n}{dn^2}\right)_{n=\frac{N}{2}}}_{-\frac{4}{N}}\left(n-\frac{N}{2}\right)^2 + \cdots$$

which is:

$$p_n = p_{\frac{N}{2}} \cdot \exp\left[-\frac{(n-N/2)^2}{N/4}\right]$$