Hubbard - Stratonovich.

a) The energy needs to be extensive and $\sum_{i=1}^{N}\sum_{j=1}^{N} S_i S_j \sim O(N^2)$

Therefore J_∞ needs to be $O(1/N)$

b) Completing the square:

$$-\frac{Na}{2}y^2 + axy = -\frac{Na}{2}\left[y^2 - \frac{2}{N}xy + \frac{x^2}{N^2} - \frac{x^2}{N}\right] = -\frac{Na}{2}\left(y-\frac{x}{N}\right)^2 + \frac{a}{2N}x^2$$

Hence:

$$\int dy \sqrt{\frac{Na}{2\pi}}\, e^{-\frac{Na}{2}y^2 + axy} = e^{\frac{a}{2N}x^2}\int dy\sqrt{\frac{Na}{2\pi}}\, e^{-\frac{Na}{2}\left(y-\frac{x}{N}\right)^2} =$$

Let $z = y - x/N \qquad dz = dy$

$$= e^{\frac{a}{2N}x^2}\int dz\sqrt{\frac{Na}{2\pi}}\, e^{-\alpha z^2} \qquad \alpha = \frac{Na}{2}$$

Since: $\int dz\, e^{-\alpha z^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\Rightarrow \quad e^{\frac{a}{2N}x^2} = \int dy\sqrt{\frac{Na}{2\pi}}\, e^{-\frac{Na}{2}y^2 + axy}.$$

c) Let $x = \sum_{i=1}^{N} S_i$ so that the energy can be written: (Introduce $J_\infty = J/N$)

$$E = -\frac{J}{2N}\sum_i\sum_j S_i S_j - H\sum S_i = -\frac{J}{2N}x^2 - Hx$$

Hence the partition function:

$$Z = \sum_{S_1=-1}^{1}\sum_{S_2}\cdots\sum_{S_N} e^{-\beta E} = \sum\cdots\sum e^{\frac{\beta J}{2N}x^2 + \beta H x} =$$

$$= \int dy\sqrt{\frac{JN}{2\pi k_B T}}\, e^{-\frac{J\beta N}{2}y^2}\left(\sum\cdots\sum e^{\beta x(H+Jy)}\right)$$

$$= \int dy\sqrt{\frac{JN}{2\pi k_B T}}\, e^{-\frac{J\beta N}{2}y^2}\left(\sum_{S_i \text{ any } i} e^{\beta x(H+Jy)}\right)^N =$$

$$= \int dy \sqrt{\frac{JN}{2\pi k_B T}}\; e^{-\frac{J\beta N}{2}y^2}\; 2\cosh^N \beta\left(H + Jy\right)$$

Then:

$$Z = \int dy \sqrt{\frac{JN}{2\pi k_B T}} \cdot e^{-\frac{J\beta N}{2}y^2}\; e^{\ln 2\cosh^N \beta (H + Jy)} =$$

$$= \int dy \sqrt{\frac{JN}{2\pi k_B T}}\; e^{-N\beta \left[\frac{J}{2}y^2 - k_B T \ln\left(2\cosh\beta\left(H+Jy\right)\right)\right]}$$

Finally: $Z = \int dy \sqrt{\frac{JN}{2\pi k_B T}}\; e^{-\beta N L(y)}$ $\quad L(y) = \frac{J}{2}y^2 - k_B T \ln(2\cosh\beta \cdot (H+Jy))$

d) In the thermodynamic limit of $N \to \infty$ we approximate the integral by the value of the integrand at its (degenerate) maxima:

$$Z = \sum_\alpha e^{-\beta N L(y_\alpha)}$$

$$0 = \frac{\partial L(\alpha)}{\partial y} = Jy_\alpha - k_B T \frac{2\sinh\beta\left(H + Jy_\alpha\right)}{2\cosh\beta(H + Jy_\alpha)} \cdot \beta J = 0$$

$$\Rightarrow \boxed{y_\alpha = \tanh\beta\left(Jy_\alpha + H\right)}$$ Same expression found in mean field.

There are potentially three solutions.

If we restrict $y_\alpha = y_0$, one ~~maximum~~ maximum of probability; then:

$$m_0 = \frac{k_B T}{N}\frac{\partial \ln Z}{\partial H} = \cancel{M(y_0)} = y_0 \quad \text{since} \quad \ln Z = \ln e^{-\beta N L(y_0)} = -\beta N L(y_0)$$

$$\Rightarrow \frac{\partial \ln Z}{\partial H} = -\beta N \left[-\frac{1}{\beta} \frac{2\sinh\beta(H + Jy)}{2\cosh\beta(H+Jy)} \beta \right] =$$

$$= N\beta \tanh\beta\left(H + Jy\right)$$

$$\to m_0 = y_0.$$

f) y_0 is a function of H:

$$\frac{\partial y_0}{\partial H} = \sec^2\beta(H+Jy_0)\left[\beta+\beta J\,\frac{\partial y_0}{\partial H}\right] \quad \text{from} \quad y_0 = \tanh\beta\,(H+Jy_0)$$

$$\Rightarrow \frac{\partial y_0}{\partial H}\left[1-\beta J\,\mathrm{sech}^2\beta\,(H+Jy_0)\right] = \beta\,\mathrm{sech}^2\beta(H+Jy_0)$$

$$\Rightarrow \quad \chi = \frac{\partial y_0}{\partial H} = \beta\,\frac{\mathrm{sech}^2\beta J y_0}{1-\beta J\,\mathrm{sech}^2\beta J y_0}$$

which diverges as $y_0 \to 0$ at $\beta J = 1$, the critical point.