Perfection is achieved, not when there is nothing more to add,
but when there is nothing left to take away.
Antoine de Saint-Exupèry (1900–1944)

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Functional-Integral Representation of Quantum Field Theory

In Chapter 7 we have quantized various fields with the help of canonical commutation rules between field variables and their canonical conjugate field momenta. From these and the temporal behavior of the fields determined by the field equations of motion we have derived the Green functions of the theory. These contain all experimentally measurable informations on the quantum field theory. They can all be derived from functional derivatives of certain generating functionals. For a real scalar field this was typically an expectation value

$$Z[j] \equiv \langle 0|T[j]|0\rangle \tag{14.1}$$

where $T[j]$ was the time-ordered product in (7.827)

$$T[j] \equiv \hat{T} e^{i\int d^4x\, j(x)\phi(x)}. \tag{14.2}$$

Here the Green functions can all be obtained from functional derivatives of $Z[j]$ of the type (7.841).

For complex scalar fields, the corresponding generating functional is given by the expectation value (7.849). Now the Green functions can all be obtained from functional derivatives of the type (7.850).

In theoretical physics, Fourier transformations have always played an important role in yielding complementary insights into mathematical structures. Due to the conjugate appearance of fields $\phi(x)$ and sources $j(x)$ in expressions like (14.2), this is also true for generating functionals and constitutes a basis for the functional-integral formalism of quantum field theory.

14.1 Functional Fourier Transformations

An important observation is now that instead of calculating these generating functionals as done in Chapter 7 from a formalism of field theory, in which $\phi(x)$ is a field operator, they can also be derived from a functional Fourier transform of another functional $\tilde{Z}[\phi]$ that depends on a *classical* field $\phi(x)$:

$$Z[j] \equiv \int \mathcal{D}\phi \tilde{Z}[\phi]\, e^{i\int d^D x\, j(x)\phi(x)}. \tag{14.3}$$

The symbol $\int \mathcal{D}\phi(x)$ in this expression is called a *functional integral*.

The mathematics of functional integration is an own discipline that is presented in many textbooks [1, 2]. Functional integrals were first introduced in ordinary quantum mechanics by R.P. Feynman [3], who used them to express physical amplitudes without employing operators. The uncertainty relation that can be expressed by an equal-time commutation relation $[x(t), p(t)] = i\hbar$ between $x(t)$ and the conjugate variable $p(t)$ of quantum mechanics is the consequence of quantum fluctuations of the classical variables $x(t)$ and $p(t)$. In quantum mechanics, the functional integral is merely a *path integral* of a fluctuation variable $x(t)$. In field theory, there is a fluctuating path for each space point $\mathbf{x}$. Instead of a time-dependent variable $x(t)$ one deals with more general dynamical variables $\phi_{\mathbf{x}}(t) = \phi(\mathbf{x}, t) = \phi(x)$, one for each spacepoint $\mathbf{x}$. The path integrals over all $x(t)$ go over into functional integrals over all fluctuating fields $\phi(x)$.

Functional integrals may be defined most simply in a discretized approximation. Spacetime is grated into a fine spacetime lattice. For every spacetime coordinate x^μ we introduce a discrete lattice point close to it

$$x^\mu \to x_n^\mu \equiv n\epsilon, \quad n = 0, \ \pm 1, \ \pm 2, \ \pm 3, \tag{14.4}$$

where ϵ is a very small lattice spacing. Then we may approximate integrals by sums:

$$\int d^D x\, j(x)\phi(x) \approx \epsilon^D \sum_{\mathbf{n}} j(x_{\mathbf{n}})\phi(x_{\mathbf{n}}) \equiv \epsilon^D \sum_{\mathbf{n}} j_{\mathbf{n}}\phi_{\mathbf{n}} \tag{14.5}$$

where $\mathbf{n}$ is to be read as a D-dimensional index $(n_0, n_1, \ldots, n_D)$, one for each component of the spacetime vector x^μ. Now we define $\int \mathcal{D}\phi(x)$ as the infinite product of integrals over $\phi_{\mathbf{n}}$ at each point $x_{\mathbf{n}}$:

$$\int \mathcal{D}\phi(x) = \prod_{\mathbf{n}} \int \frac{d\phi_{\mathbf{n}}}{\sqrt{2\pi i/\epsilon^D}}. \tag{14.6}$$

Operations with functional integrals are very similar to those with ordinary integrals. For example, the Fourier transform of (14.3) can be inverted, by analogy with ordinary integrals, to obtain:

$$\tilde{Z}[\phi] \equiv \int \mathcal{D}j(x) Z[j] e^{-i\int d^D x\, j(x)\phi(x)}. \tag{14.7}$$

There are functional analogs of the Dirac δ-function:

$$\int \mathcal{D}j(x)\, e^{-i\int d^D x\, j(x)\phi(x)} = \delta[\phi], \tag{14.8}$$

$$\int \mathcal{D}\phi(x)\, e^{i\int d^D x\, j(x)\phi(x)} = \delta[j], \tag{14.9}$$

called δ-*functionals*. In the lattice approximation corresponding to (14.6), they are defined as infinite products of ordinary δ-functions

$$\delta[\phi] = \prod_{\mathbf{n}} \sqrt{2\pi i/\epsilon^D}\delta(\phi_{\mathbf{n}}), \qquad \delta[j] = \prod_{\mathbf{n}} \sqrt{2\pi i/\epsilon^D}\delta(j_{\mathbf{n}}). \tag{14.10}$$

They have the obvious property

$$\int \mathcal{D}\phi\, \delta[\phi] = 1, \qquad \int \mathcal{D}j\, \delta[j] = 1. \tag{14.11}$$

A commonly used notation for the measure (14.6) of functional integrals employs continuously infinite product of integrals which must be imagined as the continuum limit of the lattice product (14.6). In this notation one writes

$$\int \mathcal{D}\phi(x) = \prod_x \int \frac{d\phi(x)}{\sqrt{2\pi i}}, \qquad \int \mathcal{D}j(x) = \prod_x \int \frac{dj(x)}{\sqrt{2\pi i}}, \tag{14.12}$$

and the associated δ-functionals as

$$\delta[\phi] = \prod_x \sqrt{2\pi i}\,\delta(\phi(x)), \qquad \delta[j] = \prod_x \sqrt{2\pi i}\,\delta(j(x)). \tag{14.13}$$

14.2 Gaussian Functional Integral

Only very few functional integrals can be solved explicitly. The simplest nontrivial example is the Gaussian integral[1]

$$\int \mathcal{D}j(x) e^{-\frac{i}{2}\int d^D x\, d^D x'\, j(x) M(x,x') j(x')}. \tag{14.14}$$

In the discretized form, this can be written as

$$\left[\prod_{\mathbf{n}} \int \frac{dj_{\mathbf{n}}}{\sqrt{2\pi i/\epsilon^D}}\right] e^{-\frac{i}{2}\epsilon^{2D}\sum_{\mathbf{n},\mathbf{m}} j_{\mathbf{n}} M_{\mathbf{nm}} j_{\mathbf{m}}}. \tag{14.15}$$

We may assume M to be a real symmetric functional matrix, since its antisymmetric part would not contribute to (14.14). Such a matrix may be diagonalized by a rotation

$$j_{\mathbf{n}} \to j'_{\mathbf{n}} = R_{\mathbf{nm}} j_{\mathbf{m}}, \tag{14.16}$$

which leaves the measure of integration invariant

$$\frac{\partial\left(j_1, \ldots, j_{\mathbf{n}}\right)}{\partial\left(j'_1, \ldots, j'_{\mathbf{n}}\right)} = \det R^{-1} = 1. \tag{14.17}$$

[1]Mathematically speaking, integrals with an imaginary quadratic exponent are more accurately called *Fresnel integrals*, but field theorists do not make this distinction.

In the diagonal form, the multiple integral (14.15) factorizes into a product of Gaussian integrals, which are easily calculated:

$$\prod_{\mathbf{n}} \int \frac{dj'_{\mathbf{n}}}{\sqrt{2\pi i/\epsilon^D}} e^{-\frac{i}{2}\epsilon^{2D} \sum_{\mathbf{n}} j'_{\mathbf{n}} M_{\mathbf{n}} j'_{\mathbf{n}}} = \prod_{\mathbf{n}} \frac{1}{\sqrt{-\epsilon^D M_{\mathbf{n}}}} = \det{}^{-1/2}(-\epsilon^D M). \tag{14.18}$$

On the right-hand side we have used the fact that the product of diagonal values $M_{\mathbf{n}}$ is equal to the determinant of M. The final result

$$\left[\prod_{\mathbf{n}} \int \frac{dj_{\mathbf{n}}}{\sqrt{2\pi i/\epsilon^D}}\right] e^{-\frac{i}{2}\epsilon^{2D} \sum_{\mathbf{n},\mathbf{m}} j_{\mathbf{n}} M_{\mathbf{nm}} j_{\mathbf{m}}} = \det{}^{-1/2}(-\epsilon^D M) \tag{14.19}$$

is invariant under rotations, so that it holds also without diagonalizing the matrix.

This formula can be taken to the continuum limit of infinitely fine gratings $\epsilon \to 0$. Recall the well-known matrix formula

$$\det A = e^{\log \det A} = e^{\mathrm{tr} \log A}, \tag{14.20}$$

and expand $\mathrm{tr} \log A$ into power series as follows

$$\mathrm{tr} \log A = \mathrm{tr} \log\left[1 + (A-1)\right] = -\mathrm{tr} \sum_{k=1}^{\infty} \frac{(-)^k}{k} (A-1)^k. \tag{14.21}$$

The advantage of this expansion is that when approximating the functional matrix A by the discrete matrix $\epsilon^D M$, the traces of powers of $\epsilon^D M$ remain well-defined objects in the continuum limit $\epsilon \to 0$:

$$\begin{aligned}
\mathrm{tr}\Big(\epsilon^D M\Big) &= \epsilon^D \sum_{\mathbf{n}} M_{\mathbf{nn}} \to \int d^D x\, M(x,x) \equiv \mathrm{Tr}\, M, \\
\mathrm{tr}\Big(\epsilon^D M\Big)^2 &= \epsilon^{2D} \sum_{\mathbf{n},\mathbf{m}} M_{\mathbf{nm}} M_{\mathbf{mm}} \to \int d^D x\, d^D x'\, M(x,x') M(x',x) \equiv \mathrm{Tr}\, M^2, \\
&\ \ \vdots \ .
\end{aligned} \tag{14.22}$$

We therefore rewrite the right-hand side of (14.19) as $\exp[-(1/2)\mathrm{tr}\log(-\epsilon^D M)]$, and expand

$$\begin{aligned}
\mathrm{tr} \log(-\epsilon^D M) &= \mathrm{tr} \log\Big[1 + \Big(-\epsilon^D M - 1\Big)\Big] \\
&\xrightarrow[\epsilon \to 0]{} -\sum_{k=1}^{\infty} \frac{(-)^k}{k} \int d^D x_1 \cdots d^D x_k \Big[-M(x_1,x_2) - \delta^{(D)}(x_1 - x_2)\Big] \times \cdots \\
&\qquad \times \Big[-M(x_2,x_3) - \delta^{(D)}(x_2 - x_3)\Big] \Big[-M(x_k,x_1) - \delta^{(D)}(x_k - x_1)\Big].
\end{aligned} \tag{14.23}$$

The expansion on the right-hand side defines the trace of the logarithm of the functional matrix $-M(x,x')$, and will be denoted by $\mathrm{Tr}\,\log(-M)$. This, in turn, serves to define the *functional determinant* of $-M(x,x')$ by generalizing formula (14.20) to functional matrices:

$$\mathrm{Det}\,(-M) = e^{\log\det(-M)} = e^{\mathrm{Tr}\log(-M)}. \tag{14.24}$$

Thus we obtain for the functional integral (14.14) the result:

$$\int \mathcal{D}j(x) e^{-\frac{i}{2}\int d^Dx\, d^Dx'\, j(x)M(x,x')j(x')} = \mathrm{Det}^{-1/2}(-M) = e^{-\frac{1}{2}\mathrm{Tr}\log(-M)}. \tag{14.25}$$

This formula can be generalized to complex integration variables by separating the currents into real and imaginary parts, $j(x) = [j_1(x) + ij_2(x)]/\sqrt{2}$. Each integral gives the same functional determinant so that

$$\int \mathcal{D}j^*(x)\mathcal{D}j(x) e^{-i\int d^Dx\, d^Dx'\, j^*(x)M(x,x')j(x')} = e^{-\mathrm{Tr}\log M}. \tag{14.26}$$

Here $M(x,x')$ is an arbitrary *Hermitian* matrix, and the measure of integration for complex variables $j(x)$ is defined as the product of the measures for real and imaginary parts: $\mathcal{D}j^*(x)\mathcal{D}j(x) \equiv \mathcal{D}j_1(x)\mathcal{D}j_2(x)$.

14.3 Functional Formulation for Free Quantum Fields

Having calculated the Gaussian functional integrals (14.25) and (14.26) we are able to perform the functional integrations over the generating functional (14.3) to derive its Fourier transform $\tilde{Z}[\phi]$. First we shall do so only for the free-field generating functional (7.843), which we shall equip with a subscript 0 to emphasize the free situation:

$$Z_0[j] = e^{-\frac{1}{2}\int d^4y_1 d^4y_2\, j(y_1)G_0(y_1,y_2)j(y_2)}. \tag{14.27}$$

By writing $M(x,x')$ as

$$M(x,x') = -iG_0(x,x'), \tag{14.28}$$

the Gaussian functional integral (14.14) becomes

$$\int \mathcal{D}j(x) e^{-\frac{1}{2}\int d^Dx\, d^Dx'\, j(x)G_0(x,x')j(x')} = \mathrm{Det}^{-1/2}(-iG_0). \tag{14.29}$$

This result can immediately be extended to calculate the functional Fourier tranform of the generating functional $Z_0[j]$ defined in (14.3). Thus we want to form

$$\tilde{Z}_0[\phi] = \int \mathcal{D}j(x) e^{-\frac{1}{2}\int d^Dx\, d^Dx'\, j(x)G_0(x,x')j(x') + i\int j(x)\phi(x)}. \tag{14.30}$$

The extra term linear in $j(x)$ does not change the harmonic nature of the exponent. The integral can be reduced to the Gaussian form (14.29) by a simple quadratic completion process. For this manipulation it is useful to omit the spacetime indices, and use an obvious functional vector notation to rewrite (14.30) as

$$\tilde{Z}_0[\phi] = \int \mathcal{D}j e^{-\frac{i}{2}j^T\frac{1}{i}G_0 j + ij^T\phi}. \tag{14.31}$$