Fluctuations of occupation numbers.-

The grand canonical partition functions are:

$$Z_{BE} = \prod_k \frac{1}{1 - e^{\beta(\mu - \epsilon_k)}}$$

$$Z_{FD} = \prod_k \left(1 + e^{\beta(\mu-\epsilon_k)}\right)$$

$$Z_{MB} = \sum_{N=0}^{\infty} e^{\beta\mu N} \cdot \frac{1}{N!} (Z_1)^N = \exp\left(e^{\beta\mu} Z_1\right)$$

$$Z_1 = \sum_k e^{-\beta\epsilon_k}$$

(a) Show that the average occupation number is:

$$\langle n_j \rangle = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_j}$$

BE: $$\langle n_j \rangle = +\frac{1}{\beta}\frac{\partial}{\partial \epsilon_j}\sum_k \ln\left(1 - e^{\beta(\mu-\epsilon_k)}\right) =$$

$$= \frac{1}{\cancel{\beta}}\frac{-\cancel{\beta}}{1 - e^{\beta(\mu-\epsilon_k)}} = \frac{1}{e^{\beta(\mu-\epsilon_k)} - 1}$$

FD: $$\langle n_j \rangle = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_j}\sum_k \ln\left(1 + e^{\beta(\mu-\epsilon_k)}\right) =$$

$$= \cancel{-}\frac{1}{\cancel{\beta}}\frac{\cancel{-}\cancel{\beta}}{1 + e^{\beta(\mu-\epsilon_k)}} = \frac{1}{e^{\beta(\mu-\epsilon_k)} + 1}$$

MB : $\langle n_j \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_j} e^{\beta\mu} \sum_k e^{-\beta \epsilon_k} =$

$= \cancel{-}\frac{1}{\cancel{\beta}} e^{\beta\mu} e^{-\beta\epsilon_j} \cancel{(-\beta)} = e^{-\beta(\epsilon_j - \mu)}$ ✓

(1) In general:

$$Z = \prod_k \sum_{n_k=0}^{\infty} \left(e^{\beta(\mu-\epsilon_k)}\right)^{n_k}$$

$$\cancel{-\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_j}} = \cancel{-\frac{1}{\beta}\frac{\partial}{\partial \epsilon_j} \sum_k \sum_{n_k=0}^{\infty} \ln\left(e^{\beta(\mu-\epsilon_k)}\right)^{n_k}} =$$

$$= \cancel{-\frac{1}{\beta}\frac{\partial}{\partial \epsilon_j} \sum_{n_j=0}^{\infty} \ln\left(e^{\beta(\mu-\epsilon_j)}\right)^{n_j}} =$$

$$= \cancel{-\frac{1}{\beta}} =$$

$$-\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_j} = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_j} \sum_k \ln\left(\sum_{n_k=0}^{\infty} e^{\beta(\mu-\epsilon_k)}\right)^{n_k} =$$

$$= -\frac{1}{\beta}\frac{\partial \ln \sum_{n_j=0}^{\infty} \left(e^{\beta(\mu-\epsilon_k)}\right)^{n_j}}{\partial \epsilon_j} =$$

$$= \cancel{-\frac{1}{\beta}} = \cancel{-}\frac{1}{\cancel{\beta}} \frac{\sum_{n_j} (\cancel{-\beta} n_j) e^{\beta(\mu-\epsilon_j)}}{\sum_{n_j=0}^{\infty} e^{\beta(\mu-\epsilon_j) n_j}}$$

$$\Rightarrow -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_j} = \frac{\sum_{n_j=0}^{\infty} n_j e^{\beta(\mu-\epsilon_j) n_j}}{\sum_{n_j=0}^{\infty} e^{\beta(\mu-\epsilon_j) n_j}} = \langle n_j \rangle$$

(2)

Now: $\langle n_j^2\rangle - \langle n_j\rangle^2 = \left(-\frac{1}{\beta}\frac{\partial}{\partial\epsilon_j}\right)^2 \ln Z =$

$$= +\frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j^2}\ln Z = \frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j^2}\sum_k \ln\left(\sum_{n_k=0}^{\infty} e^{\beta(\mu-\epsilon_k)n_k}\right) =$$

$$= \frac{1}{\beta^2}\frac{\partial}{\partial\epsilon_j}\frac{\sum_j (-\beta)n_j\, e^{\beta(\mu-\epsilon_j)n_j}}{\sum_{n_j} e^{\beta(\mu-\epsilon_j)n_j}} =$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_j}\frac{\sum n_j\, e^{\beta(\mu-\epsilon_j)n_j}}{\sum_{n_j} e^{\beta(\mu-\epsilon_j)n_j}} =$$

$$= -\frac{1}{\beta}\frac{\left[\sum n_j e^{\beta(\mu-\epsilon_j)n_j}\cdot(-\beta n_j)\right]\cancel{\sum e^{\beta(\mu-\epsilon_j)n_j}}\Big] - \left(\sum n_j e^{()}\right)\left(\sum(-\beta n_j)e^{()}\right)}{\left(\sum e^{\beta(\mu-\epsilon_j)n_j}\right)^2}$$

$$= \frac{\sum n_j^2 e^{\beta(\mu-\epsilon_j)n_j}}{\sum e^{\beta(\mu-\epsilon_j)n_j}} - \frac{\left(\sum n_j e^{\beta(\mu-\epsilon_j)n_j}\right)^2}{\left(\sum e^{\beta(\mu-\epsilon_j)n_j}\right)^2}$$

$$= \langle n_j^2\rangle - \langle n_j\rangle^2. \checkmark$$

(i)

BE: $\langle n_j^2\rangle - \cancel{\langle n_j\rangle^2} = \frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j}\ln\prod_k \frac{1}{1-e^{\beta(\mu-\epsilon_k)}} =$

$$= \frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j^2}\ln\frac{1}{1-e^{\beta(\mu-\epsilon_j)}} =$$

$$= \cancel{\frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j}(1-e^{\beta(\mu-\epsilon_j)})} = -\frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_j^2}\ln\left(1-e^{+\beta(\mu-\epsilon_j)}\right)$$

$$= +\frac{1}{\beta^2}\frac{\partial}{\partial\epsilon_j}\frac{-e^{\beta(\mu-\epsilon_j)}(\cancel{-}\beta)}{1-e^{\beta(\mu-\epsilon_j)}} = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_j}\frac{e^{\beta(\mu-\epsilon_j)}}{1-e^{\beta(\mu-\epsilon_j)}}$$

$$= -\frac{1}{\beta}\,\frac{e^{\beta(\mu-\epsilon_j)}\cancel{(-\beta)(1-e^{\beta(\mu-\epsilon_j)})} - e^{\beta(\mu-\epsilon_j)}(-1)\cancel{(1-e^{\beta(\mu-\epsilon_j)})}\cancel{(-\beta)}}{(1-e^{\beta(\mu-\epsilon_j)})^2}$$

$$= \frac{e^{\beta(\mu-\epsilon_j)} + e^{\beta(\mu-\epsilon_j)}}{(1-e^{\beta(\mu-\epsilon_j)})}$$

$$= +\frac{1}{\beta}\,\frac{e^{()}(-\beta)(1-e^{()}) + e^{\beta()}\cancel{(-1)}e^{()}\cancel{(-\beta)}}{(1-e^{()})^2}$$

$$= \frac{e^{()}}{(1-e^{()})} + \frac{(e^{()})^2}{(1-e^{()})^2}$$

Now $\langle n_j \rangle = \dfrac{1}{e^{()}-1}$

$$\Rightarrow \frac{\langle n_j^2\rangle - \langle n_j\rangle^2}{\langle n_j\rangle^2} = \cancel{(e^{()}-1)^2}\cdot\frac{e^{()}(1-e^{()}) + e^{()}e^{()}}{\cancel{(1-e^{()})^2}}$$

$$= e^{()} - \cancel{e^{()}e^{()}} + \cancel{e^{()}e^{()}} = e^{\beta(\mu-\epsilon_j)}$$

Now ∴ $\dfrac{1}{\langle n_j\rangle} = e^{()} - 1 \Rightarrow e^{()} = \dfrac{1}{\langle n_j\rangle} + 1$ ✓