We have. $F = \int d^dx \left[\frac{k}{2} (\partial_i m_j)(\partial_i m_j) + \frac{r}{2} (m)^2 - u (m)^4 \right]$

for an n-vector model.

1.) Equilibrium magnetization. Let $\vec{m} = m_0 \hat{e}_1$ along some particular direction, with m_0 constant. Then: $(m)^2 = m_0^2$, $(m)^4 = m_0^4$

$$\frac{\partial f}{\partial m_0} = \frac{r}{\cancel{2}} \cancel{2} m_0 + 4u m_0^{\cancel{3}\,2} = 0 \qquad \boxed{\begin{array}{l} m_0 = 0 \\ m_0 = \pm\sqrt{-r/4u}\,. \end{array}}$$

2. Consider fluctuations: $m_i(x) = (m_0 + \eta^\ell(x))\hat{e}_1 + \sum_{j\geq 2}^{n} \eta_j^t(x)\hat{e}_j$

$(x$ in d dimensions$)$

Substitute this expression into F, and keep second order.

- $(\partial_i m_j)(\partial_i m_j) = (\nabla \eta^\ell)^2 + (\nabla \eta^t)^2$ using orthogonality of $\{\hat{e}_i\}$.

- $m^2 = m_0^2 + \eta^\ell(x)^2 + 2m_0\eta^\ell(x) + \underbrace{\sum_j \eta_j^t(x)^2}_{\|\vec{\eta}^t\|^2}$

- $m^4 = m_0^4 + 4m_0^3\eta^\ell(x) + \cancel{6m_0^2(\eta^\ell)^2}$
$\qquad + 6m_0^2\eta^\ell(x)^2 + 2m_0^2(\vec{\eta}^t)^2$

To this quadratic order the energy is: (V is the d-dimensional volume)

$$F = \left(\frac{r}{2}m_0^2 + u m_0^4\right)V + \int d^dx \left[\frac{k}{2}(\nabla\eta^\ell(x))^2 + \frac{r + 12m_0^2 u}{2}\eta^\ell(x)^2\right]$$
$$+ \int d^dx\left[\frac{k}{2}(\nabla\vec{\eta}^t(x))^2 + \frac{r + 4m_0^2 u}{2}(\vec{\eta}^t)^2\right]$$

3. The fluctuations $\langle \eta^\alpha(k)\,\eta^\beta(-k)\rangle$ trivially follow:

$$\langle \eta^\alpha(k)\eta^\beta(-k)\rangle = \frac{k_B T\,\delta_{\alpha\beta}}{K\left(q^2+\xi_\alpha^{-2}\right)}$$

There are two correlation lengths: transverse and longitudinal. They can be written explicitly by comparing with the free energy in (2):

- $$K\xi_l^{-2} = r + 12u\,m_0^2 = \begin{cases} r & T>T_c \quad (r>0) \\ r + \cancel{12}^{\,3}\cancel{u}\left(\dfrac{-r}{\cancel{4u}}\right) = -2r & (r<0,\ T<T_c) \end{cases}$$

- $$K\xi_t^{-2} = r + 4u\,m_0^2 = \begin{cases} r & (r>0,\ T>T_c) \\ r + \cancel{4u}\left(-\dfrac{r}{\cancel{4u}}\right) = 0 & T<T_c\,! \end{cases}$$

4. Longitudinal fluctuations are standard, with the correlation length showing a mean field divergence ($\nu = 1/2$) at the transition point $r=0$.

Transverse fluctuations have the same correlation length as longitudinal fluctuations in the isotropic phase ($r>0$), as one would expect as there is no broken symmetry, and hence no difference between longitudinal and transverse.

For $r<0$ on the other hand, $\langle |\eta^t|^2(k)\rangle \sim 1/q^2$, the divergence expected for a broken symmetry variable. Once one picks the $\hat{e}_1$ direction, there is no restoring force for transverse fluctuations (to quadratic order) and fluctuations diverge as $1/q^2$. This is an example of behavior after a ~~spon~~ symmetry is spontaneously broken.