Graph Theory

INTRODUCTION

Graphs are discrete structures consisting of vertices and edges that connect these vertices. Depending on the type and number of edges that can connect a pair of vertices, there are many kinds of different graphs. The graph models can be used to represent almost every problem involving discrete arrangement of objects, where we are not concerned with their internal properties but with their inter-relationship. Eventhough Graph theory is an old subject, one of the reasons for the recent interest in it is its applicability in many diverse fields such as computer science, physical sciences, electrical and communication engineering and economics.

In this section, we shall define a graph as an abstract mathematical system and also represent graphs diagrammatically. Then we shall discuss some of the basic concepts and theorems of graph theory.

BASIC DEFINITIONS

A *graph* $G = (V, E)$ consists of a non-empty set V, called the set of *vertices* (*nodes, points*) and a set E of ordered or unordered pairs of elements of V, called the set of *edges*, such that there is a mapping from the set E to the set of ordered or unordered pairs of elements of V.

If an edge $e \in E$ is associated with an ordered pair (u, v) or an unordered pair (u, v), where $u, v \in V$, then e is said to *connect* or *join* the nodes u and v. The edge e that connects the nodes u and v is said to be *incident* on each of the nodes. The pair of nodes that are connected by an edge are called *adjacent nodes*.

A node of a graph which is not adjacent to any other node (viz., which is not connected by an edge to any other node) is called an *isolated node*. A graph containing only isolated nodes (viz. no edges) is called a *null graph*.

If in graph $G = (V, E)$, each edge $e \in E$ is associated with an ordered pair of vertices, then G is called a *directed graph* or *digraph*. If each edge is associated with an unordered pair of vertices, then G is called an *undirected* graph.

Note When a graph is represented diagrammatically, the vertex set is represented as a set of points in plane and an edge is represented by a line segment or an arc (not necessarily straight) joining the two vertices incident with it. In the diagram of a digraph, each edge $e = (u, v)$ is represented by means of an arrow or directed curve drawn from the initial point u to the terminal point v as in the Figs 7.1.

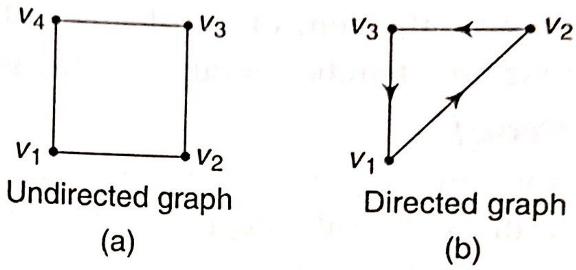


Fig. 7.1

An edge of a graph that joins a vertex to itself is called a *loop*. The direction of a loop is not significant, as the initial and terminal nodes are one and the same.

If, in a directed or undirected graph, certain pairs of veritices are joined by more than one edge, such edges are called *parallel* edges. In the case of directed edges, the two possible edges between a pair of vertices which are opposite in direction are considered distinct.

A graph, in which there is only one edge between a pair of vertices, is called a *simple graph*.

A graph which contains some parallel edges is called a *multigraph*.

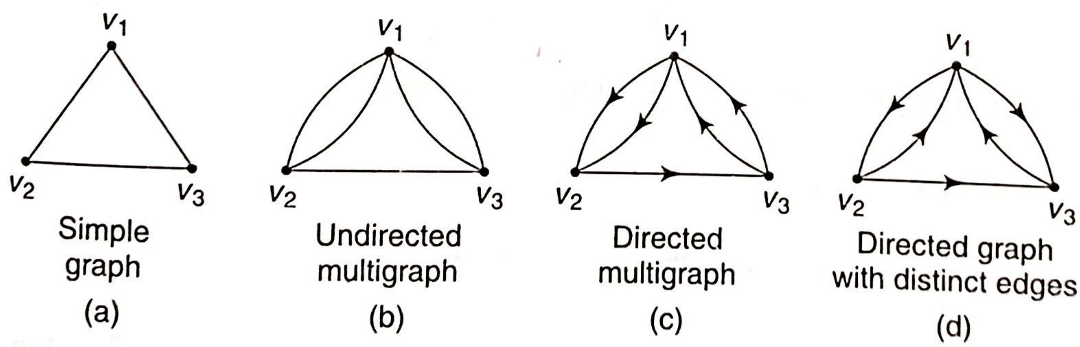


Fig. 7.2

A graph in which loops and parallel edges are allowed is called a *pseudograph*. Graphs in which a number (weight) is assigned to each edge are called *weighted graphs*.

DEGREE OF A VERTEX

The degree of a vertex in an undirected graph is the number of edges incident with it, with the exception that a loop at a vertex contributes twice to the degree of that vertex. The degree of a vertex v is denoted by deg (v). Clearly the degree of an isolated vertex is zero. If the degree of a vertex is one, it is called a *pendant* vertex.

For example, let consider the graph in Fig. 7.3.

$\deg(v_1) = 2$, $\deg(v_2) = \deg(v_3) = \deg(v_5) = 4$,

$\deg(v_4) = 1$, $\deg(v_6) = 3$, $\deg(v_7) = 0$.

We note that v_4 is a pendant vertex and v_7 is an isolated vertex.

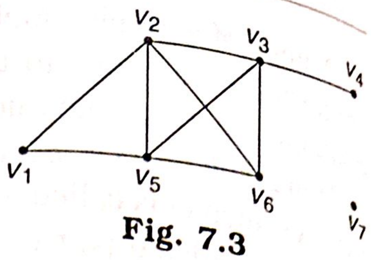


Fig. 7.3

Theorem (The Handshaking theorem)

If $G = (V, E)$ is an undirected graph with e edges, then $\sum_i \deg(v_i) = 2e$.

Viz., the sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the e edges contribute $(2e)$ to the sum of the degrees of the verities

viz., $\sum_i \deg(v_i) = 2e$.

Theorem

The number of vertices of odd degree in an undirected graph is even.

Proof

Let $G = (V, E)$ be the undirected graph.

Let V_1 and V_2 be the sets of vertices of G of even and odd degrees respectively. Then, by the previous theorem,

$$2e = \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) \tag{1}$$

since each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even.

As the L.H.S. of (1) is even, we get

$$\sum_{v_j \in V_2} \deg(v_j) \text{ is even.}$$

Since each $\deg(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ or in V_2 is even, i.e., the number of vertices of odd degree is even.

Definitions

In a directed graph, the number of edges with v as their terminal vertex (viz., the number of edges that converge at v) is called the *in-degree of* v and is denoted as $\deg^-(v)$.

The number of edges with v as their initial vertex, (viz., the number of edges that emanate from v) is called the *out-degree of* v and is denoted as $\deg^+(v)$.

A vertex with zero in degree is called a *source* and a vertex with zero out-degree is called a *sink*.

Let us consider the following directed graph.

We note that $\deg^-(a) = 3$, $\deg^-(b) = 1$, $\deg^-(c) = 2$, $\deg^-(d) = 1$ and $\deg^+(a) = 1$, $\deg^+(b) = 2$, $\deg^+(c) = 1$, $\deg^+(d) = 3$.

Also we note that $\Sigma \deg^-(v) = \Sigma \deg^+(v) =$ the number of edges $= 7$.

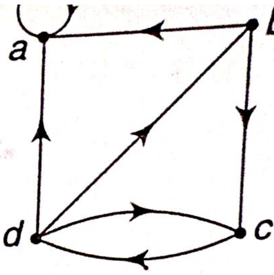


Fig. 7.4

This property is true for any directed graph

$$G = (V, E), \text{ viz., } \sum_{v \in V} \deg^-(v)$$

$$= \sum_{v \in V} \deg^+(v) = e.$$

This is obvious, because each edge of the graph converges at one vertex and emanates from one vertex and hence contributes 1 each to the sum of the in-degrees and to the sum of the out-degrees.

SOME SPECIAL SIMPLE GRAPHS

Complete graph

A simple graph, in which there is exactly one edge between each pair of distinct vertices, is called a *complete graph*.

The complete graph on n vertices is denoted by K_n. Figure 7.5 shows the graphs K_1 through K_6.

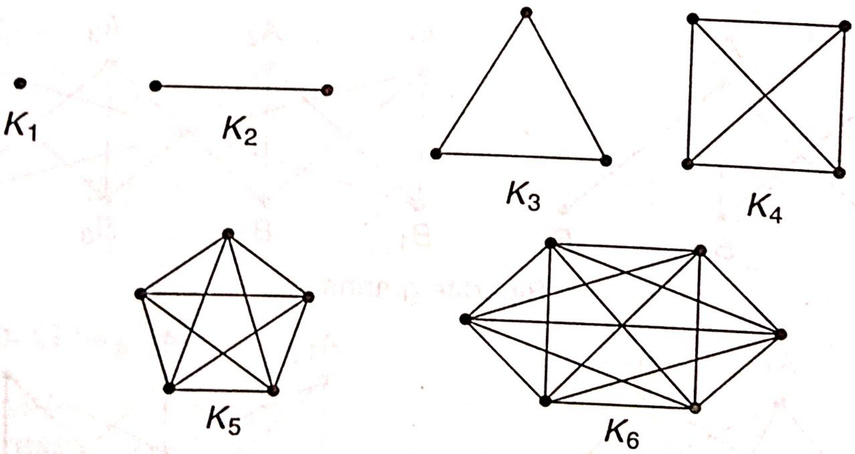


Fig. 7.5

Note The number of edges in K_n is nC_2 or $\frac{n(n-1)}{2}$. Hence, the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Regular graph

If every vertex of a simple graph has the same degree, then the graph is called a *regular graph*. If every vertex in a regular graph has degree n, then the graph is called *n-regular*. Figure 7.6 shows some 2-regular and 3-regular graphs.

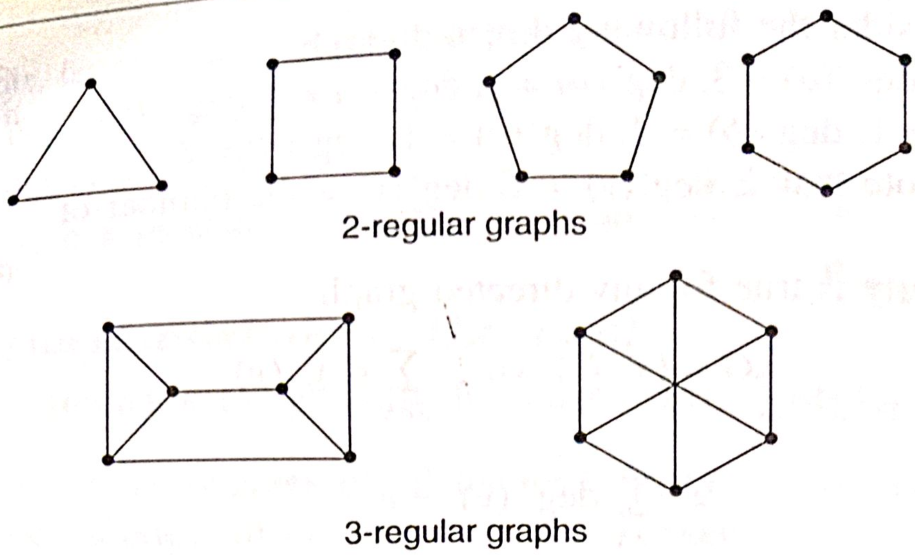


Fig. 7.6

Bipartite graph

If the vertex set V of a simple graph $G = (V, E)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2), then G is called a *bipartite graph.*

If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a *completely bipartite graph.* If V_1 contains m vertices and V_2 contains n vertices, the completely bipartite graph is denoted by $K_{m,n}$. Figure 7.7 shows some bipartite and some completely bipartite graphs.

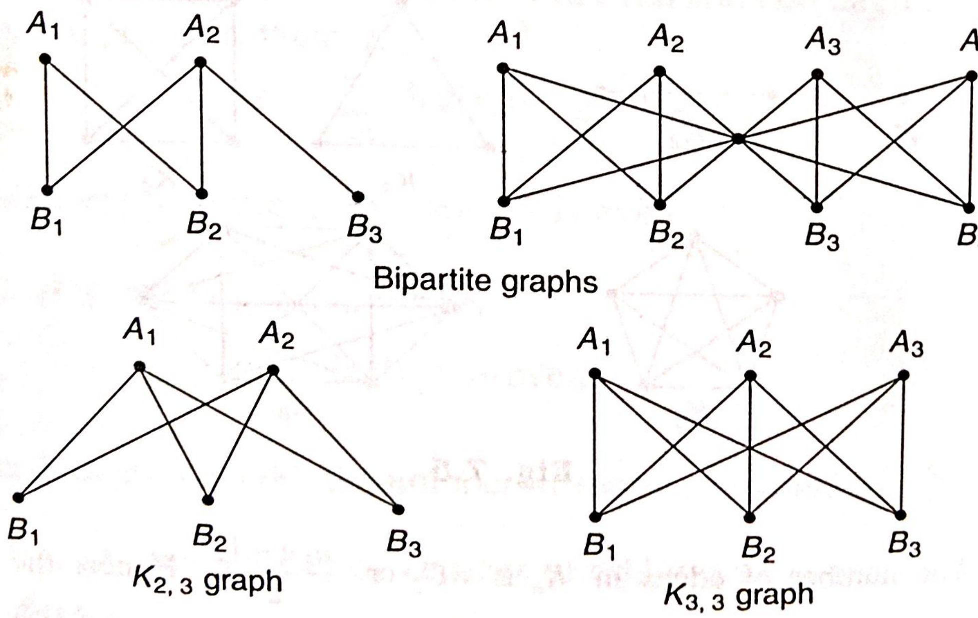


Fig. 7.7

Subgraphs

A graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

If $V' \subset V$ and $E' \subset E$, then H is called a *proper subgraph* of G.

If $V' = V$, then H is called a *spanning subgraph of G.* A spanning subgraph of G need not contain all its edges.

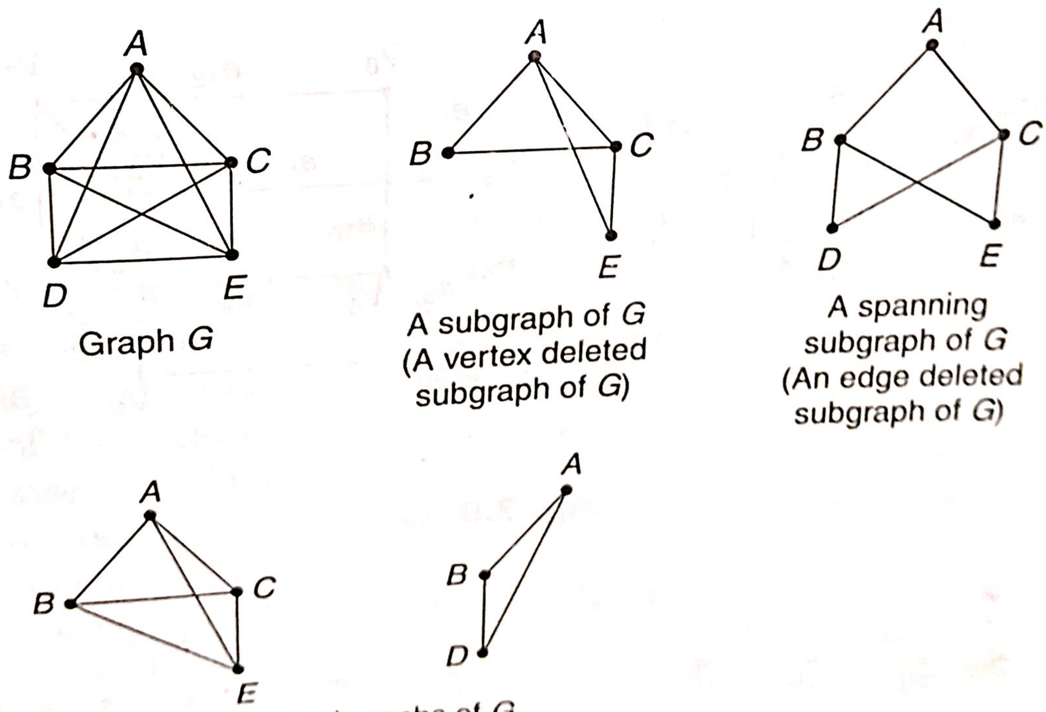
Any subgraph of a graph G can be obtained by removing certain vertices and edges from G. It is to be noted that the removal of an edge does not go with the removal of its adjacent vertices, whereas the removal of a vertex goes with the removal of any edge incident on it.

If we delete a subset U of V and all the edges incident on the elements of U from a graph $G = (V, E)$, then the subgraph $(G - U)$ is called a *vertex deleted subgraph of G*.

If we delete a subset F of E from a graph $G(V, E)$, then the subgraph $(G - F)$ is called an *edge deleted subgraph of G*.

A subgraph $H = (V', E')$ of $G = (V, E)$, where $V' \subseteq V$ and E' consists of only those edges that are incident on the elements of V', is called an *induced subgraph of G*.

Figure 7.8 shows different subgraphs of a given graph G.



Graph G

A subgraph of G (A vertex deleted subgraph of G)

A spanning subgraph of G (An edge deleted subgraph of G)

Induced subgraphs of G

Fig. 7.8

Isomorphic Graphs

Two graphs G_1 and G_2 are said to be *isomorphic* to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

viz., a graph $G_1 = (V_1, E_1)$ is isomorphic to the graph $G_2 = (V_2, E_2)$, if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and between the edge sets E_1 and E_2 in such a way that if e_1 is incident on u_1 and v_1 in G_1, then the corresponding edge e_2 in G_2 is incident on u_2 and v_2 which correspond to u_1 and v_1 respectively. Such a correspondence is called *graph isomorphism*. Figure 7.9 shows pairs of isomorphic graphs.

From Fig. 7.9, we observe that isomorphic graphs have (i) the same number of vertices, (ii) the same number of edges and (iii) the corresponding vertices with the same degree. This property is called an *invariant* with respect to isomorphic graphs. If any of these conditions is not satisfied in two graphs, they cannot be isomorphic. However, these conditions are not sufficient for graph isomorphism, as seen from the following Fig. 7.10.

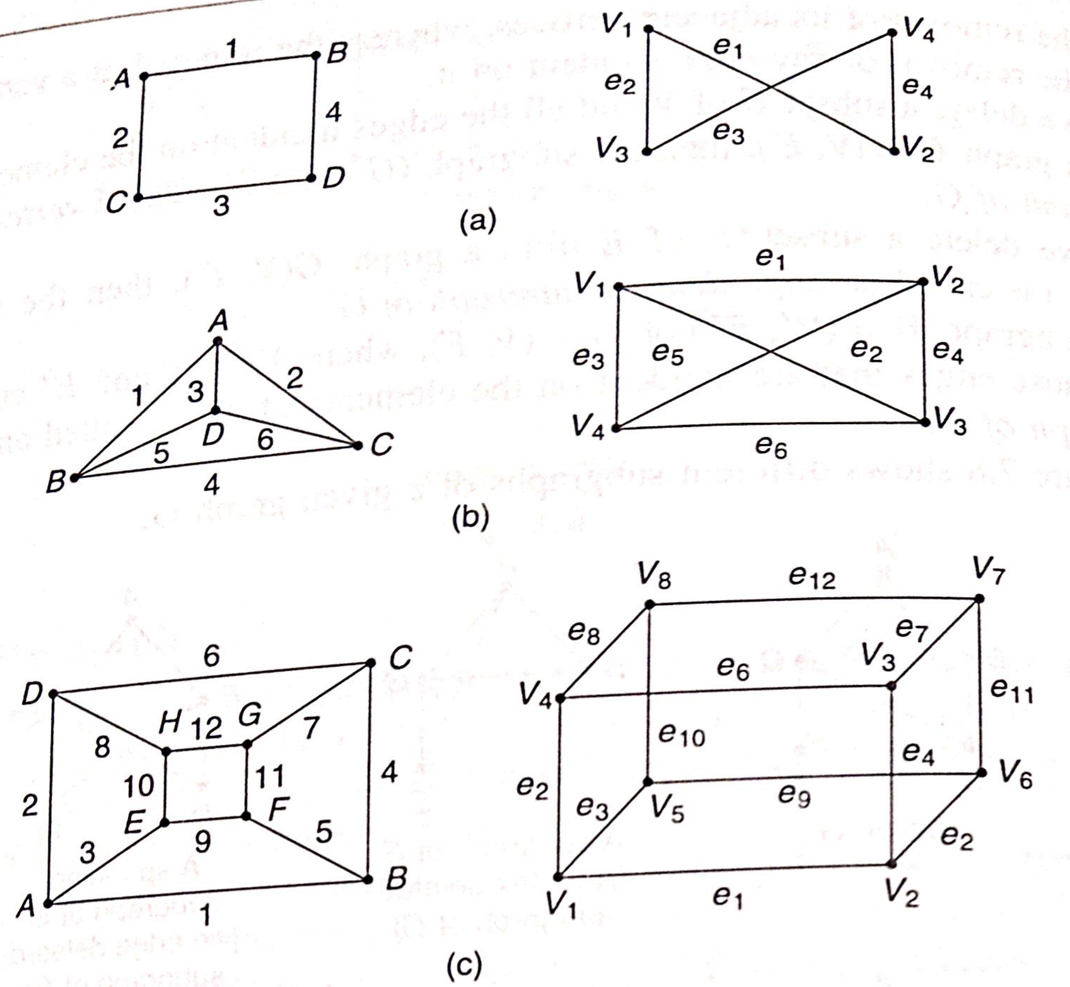


Fig. 7.9

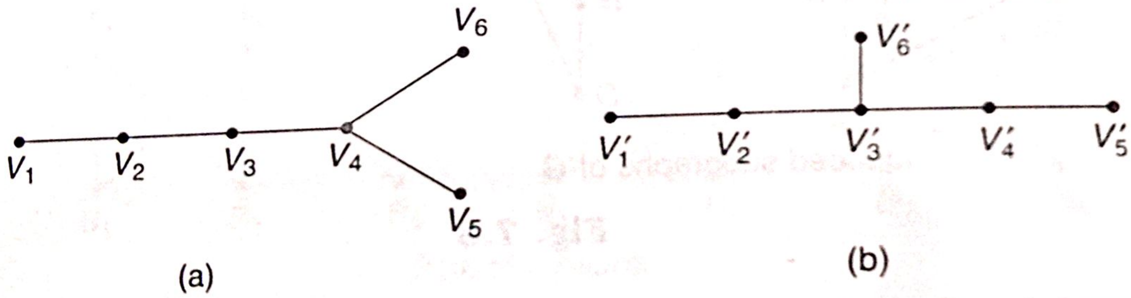


Fig. 7.10

There are 6 vertices and 5 edges in both the graphs.

There are 3 vertices namely V_1, V_5, V_6 (V_1', V_5', V_6') each of degree 1; 2 vertices namely V_2, V_3 (V_2', V_4') each of degree 2; 1 vertex namely V_4 (V_3') of degree 3.

Thus, all the three conditions are satisfied, but the two graphs 7.10 (a) and (b) are not isomorphic, since the vertices V_2 and V_3 are adjacent in (a) whereas the corresponding vertices V_2' and V_4' are not adjacent.

To determine whether two graphs are isomorphic, it will be easier to consider their matrix representations. Two types of matrices commonly used to represent graphs will be discussed in the following section.

MATRIX REPRESENTATION OF GRAPHS

When G is a simple graph with n vertices $v_1, v_2, \ldots v_n$, the matrix A(or A_G) = $[a_{ij}]$,

$$\text{where, } a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

is called the *adjacency matrix* of G.

For example, if G is the graph given in Fig. 7.11, then the adjacency matrix A is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

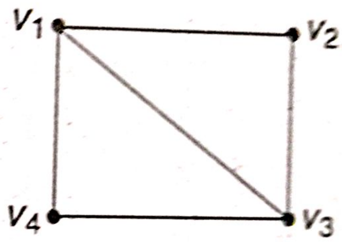


Fig. 7.11

The following basic properties of an adjacency matrix are obvious:

1. Since a simple graph has no loops, each diagonal entry of A, viz., $a_{ij} = 0$, for $i = 1, 2, \ldots n$.
2. The adjacency matrix of simple graph is symmetric, viz., $a_{ij} = a_{ji}$, since both of these entries are 1 when v_i and v_j are adjacent and both are 0 otherwise. Conversely, given any symmetric zero-one matrix A which contains only 0's on its diagonal, there exists a simple graph G whose adjacency matrix is A.
3. $\deg(v_i)$ is equal to the number of 1's in the i^{th} row or i^{th} column.

Note A pseudograph (viz., an undirected graph with loops and parallel edges) ... a loop at the

Definition

If $G = (V, E)$ is an undirected graph with n vertices $v_1, v_2, \ldots v_n$ and m edges $e_1, e_2, \ldots, e_m$, then the $(n \times m)$ matrix $B = [b_{ij}]$,

$$\text{where } b_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident on } v_i \\ 0, & \text{otherwise} \end{cases}$$

is called *the incidence matrix* of G.

For example, the incidence matrix of the graph shown in Fig. 7.14 is given alongside.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

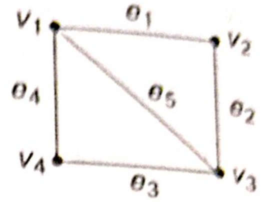


Fig. 7.14

The following basic properties of an incidence matrix are obvious:

1. Each column of B contains exactly two unit entries.
2. A row with all 0 entries corresponds to an isolated vertex.
3. A row with a single unit entry corresponds to a pendant vertex.
4. $\deg(v_i)$ is equal to the number of 1's in the i^{th} row.

Note Incidence matrices can also be used to represent pseudographs. Parallel edges are represented in the incidence matrix using columns with identical entries, since these edges are incident on the same pair of vertices. Loop is represented by a column with exactly one unit entry, corresponding to the concerned vertex. For example, the incidence matrix of the graph in Fig. 7.15 is given alongside.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

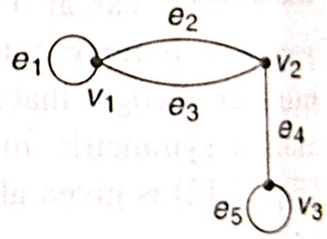


Fig. 7.15

Isomorphism and Adjacency Matrices

We state two theorems (without proof) which will help us to prove that two labeled graphs are isomorphic:

Theorem 1

Two graphs are isomorphic, if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

Theorem 2

Two labeled graphs G_1 and G_2 with adjacency matrices A_1 and A_2 respectively are isomorphic, if and only if, there exists a permutation matrix P such that $PA_1P^T = A_2$.

Note A matrix whose rows are the rows of the unit matrix, but not necessarily in their natural order, is called a *permutation matrix*.

For example, let us consider the two graphs shown in Fig. 7.16.

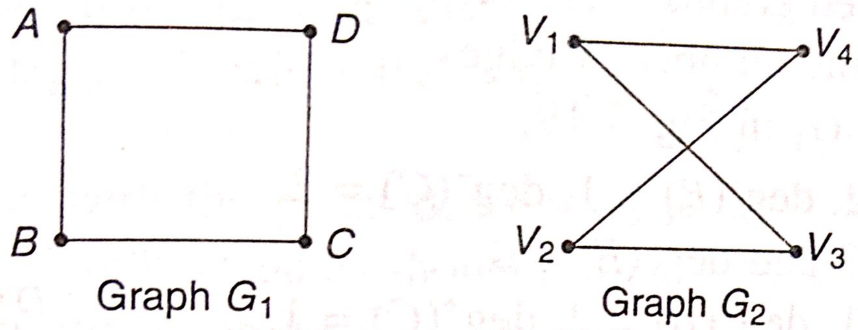


Fig. 7.16

Now $$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

If we assume that $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we can see that $PA_1P^T = A_2$. Hence, the two graphs G_1 and G_2 are isomorphic such that $A \to V_1$, $B \to V_3$, $C \to V_2$ and $D \to V_4$.

WORKED EXAMPLES 7(A)

Example 7.1 Find the number of vertices, the number of edges and the degree of each vertex in the following undirected graphs. Verify also the handshaking theorem in each case.

(i) For the graph G_1 in Fig. 7.17,

the number of vertices = 6

the number of edges = 9

$\deg(A) = 2$, $\deg(B) = 4$, $\deg(C) = 4$,

$\deg(D) = 3$, $\deg(E) = 4$, $\deg(F) = 1$

Now $\Sigma \deg(A) = 2 + 4 + 4 + 3 + 4 + 1 = 18$

$= 2 \times 9 = 2 \times$ no. of edges.

Hence, the theorem is true.

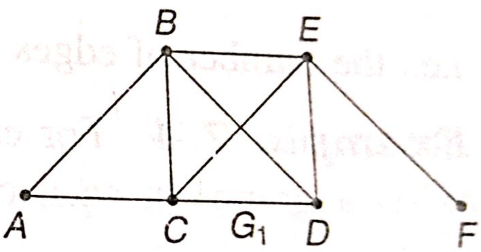


Fig. 7.17

(ii) For the graph G_2 in Fig. 7.18,

the number of vertices = 5

the number of edges = 13

deg $(A) = 6$, deg $(B) = 6$, deg $(C) = 6$,

deg $(D) = 5$, deg $(E) = 3$

Obviously, Σ deg $A = 2 \times$ no. of edges.

Hence, the theorem is verified.

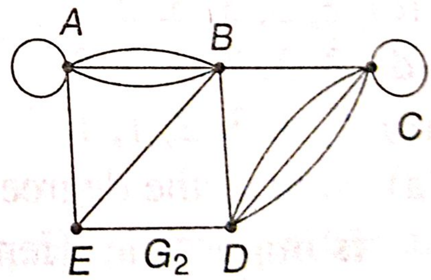


Fig. 7.18

Example 7.2 Find the in-degree and out-degree of each vertex of each of the following directed graphs. Also verify that the sum of the in-degrees (or the out-degres) equals the number of edges.

(i) For the graph G_1 in Fig. 7.19,

$\deg^-(A) = 2$, $\deg^-(B) = 1$, $\deg^-(C) = 2$,
$\deg^-(D) = 3$ and $\deg^-(E) = 0$
$\deg^+(A) = 1$, $\deg^+(B) = 2$, $\deg^+(C) = 1$,
$\deg^+(D) = 1$ and $\deg^+(E) = 3$

We see that $\Sigma\deg^-(A) = \Sigma\deg^+(A) = 8$
= the no. of edges of G_1.

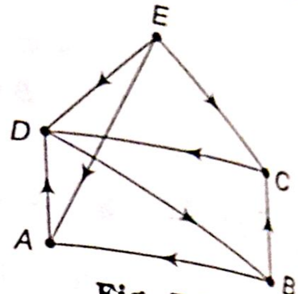


Fig. 7.19

(ii) For the graph G_2 in Fig. 7.20

$\deg^-(A) = 5$, $\deg^+(A) = 2$
$\deg^-(B) = 3$, $\deg^+(B) = 3$
$\deg^-(C) = 1$, $\deg^+(C) = 6$
$\deg^-(D) = 4$, $\deg^+(D) = 2$

We see that $\Sigma \deg^-(A) = \Sigma \deg^+(A) = 13$
= the no. of edges of G_2.

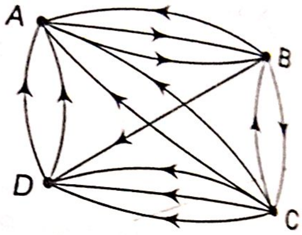


Fig. 7.20

Example 7.3 If all the vertices of an undirected graph are each of odd degree k, show that the number of edges of the graph is a multiple of k.

Since the number of vertices of odd degree in an undirected graph is even, let it be $2n$. Let the number of edges be n_e.

Then by the hand-shaking theorem,

$$\sum_{i=1}^{2n} \deg(v_i) = 2n_e$$

i.e.,

$$\sum_{i=1}^{2n} k = 2n_e \text{ or } 2nk = 2n_e$$

$\therefore$

$$n_e = nk$$

i.e., the number of edges is a multiple of k.

Example 7.4 For each of the following degree sequences, find if there exists a graph. In each case, either draw a graph or explain why no graphs exists.

(a) 4, 4, 4, 3, 2
(b) 5, 5, 4, 3, 2, 1
(c) 3, 3, 3, 3, 2
(d) 3, 3, 3, 3, 3, 3
(e) 5, 4, 3, 2, 1, 1

(a) Sum of the degrees of all the vertices = 17, which is an odd number. This is impossible. Hence, no graph exists with the given degree sequence.

(b) There are 6 vertices. Hence, a vertex of degree 5 in the graph must be adjacent to all other vertices.

As there are 2 vertices each of degree 5, all other vertices should be of degree at least 2. But the given degree sequence contains a 1. Hence, no graph is possible with the given degree sequence.

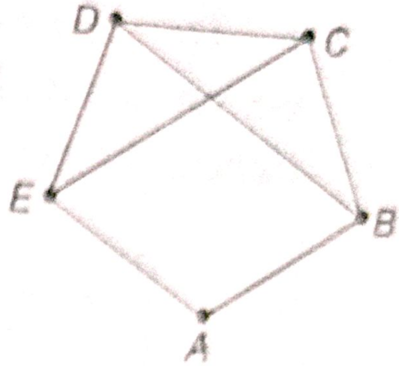


Fig. 7.21

(c) A simple graph with the given degree sequence is possible, as shown in Fig. 7.21. The vertices B, C, D, E are of degree 3, while the vertex A is of degree 2.

(d) A simple graph with the given description is not possible. Only a multigraph as shown in Fig. 7.22 is possible with the given degree sequence.

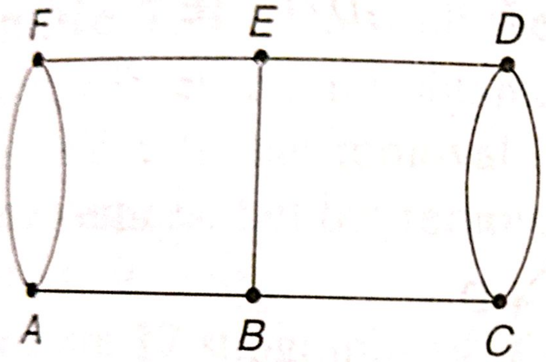


Fig. 7.22

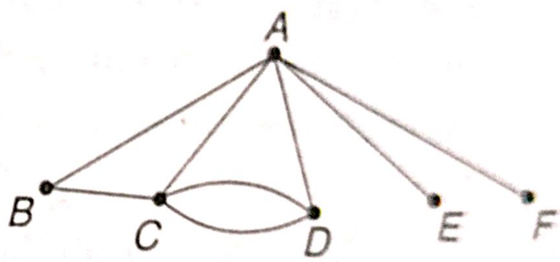


Fig. 7.23

(e) Only a multigraph as shown in Fig. 7.23 is possible with the given degree sequence.

The degrees of A, C, D, B, E and F are respectively 5, 4, 3, 2, 1, 1.

Example 7.7 Prove that the number of edges in a bipartite graph with n vertices is at most $\left(\frac{n^2}{2}\right)$.

Let the vertex set be partitioned into the subsets V_1 and V_2. Let V_1 contain x vertices. Then V_2 contains $(n - x)$ vertices.

The largest number of edges of the graph can be obtained, when each of the x vertices in V_1 is connected to each of the $(n - x)$ vertices in V_2.

$\therefore$ The largest number of edges, $f(x) = x(n - x)$, is a function of x.

Now we have to find the value of x for which $f(x)$ is maximum.

By calculus, $f'(x) = n - 2x$ and $f''(x) = -2$

$f'(x) = 0$, when $x = \frac{n}{2}$ and $f''\left(\frac{n}{2}\right) < 0$.

Hence, $f(x)$ is maximum, when $x = \frac{n}{2}$.

$\therefore$ Maximum number of edges required

$$= f\left(\frac{n}{2}\right) = \frac{n^2}{2}.$$