**Obs.** Rewriting ...

We have $A^{-1}X = \frac{1}{\lambda}X$

If we use this equation, then *the above method yields the smallest eig...*

**Example 4.10.** *Determine the largest eigen value and the corresponding eigen vector of the matrix* $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

**Sol.** Let the initial approximation to the eigen vector corresponding to the largest eigen value of $A$ be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

So the first aporoximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

Thus the second aporoximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

$$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

**Example 4.11.** *Find the largest eigen value and the corresponding eigen vector of the matrix* $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ *using power method. Take* $[1, 0, 0]^T$ *as initial eigen vector.*

*(V.T.U., B.E., 2009)*

**Sol.** Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

So the first approximation to the eigen value is 2 and the corresponding eigen vector $X^{(1)} = [1, -0.5, 0]'$.

Hence $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

Repeating the above process, we get

$$AX^{(2)} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)}; \qquad AX^{(3)} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)};$$

$$AX^{(4)} = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)}; \quad AX^{(5)} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen-vector is $[0.74, -1, 0.67]'$.

**Example 4.12.** *Obtain by power method, the numerically dominant eigen value and eigen vector of the matrix.*

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}. \qquad \textit{(Anna, B. Tech., 2007)}$$

**Sol.** Let the initial approximation to the eigen vector be $X = [1, 1, 1]'$. Then

$$AX = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix} = -18\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

So the first approximation to eigen value is $-18$ and the corresponding eigen vector is $[-0.444, 0.222, 1]'$.

Now $$AX^{(1)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} -0.444 \\ 0.222 \\ 1 \end{bmatrix} = -10.548\begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$\therefore$ The second approximation to the eigen value is $-10.548$ and the eigen vector is $[1, -0.105, -0.736]'$.

Repeating the above process

$$AX^{(2)} = -18.948\begin{bmatrix} -0.930 \\ 0.361 \\ 1 \end{bmatrix} = \lambda^{3} X^{(3)}; AX^{(3)} = -18.394\begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = -19.698\begin{bmatrix} -0.995 \\ 0.462 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}; AX^{(5)} = -19.773\begin{bmatrix} 1 \\ -480 \\ -0.999 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

$$AX^{(6)} = -19.922\begin{bmatrix} -0.997 \\ 0.490 \\ 1 \end{bmatrix} = \lambda^{(7)}X^{(7)}; AX^{(7)} = -19.956\begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix} = \lambda^{(8)}X^{(8)}$$

Since $\lambda^{(7)} = \lambda^{(8)}$ and $X^{(7)} = X^{(8)}$ approximately, therefore the dominant eigen value and the corresponding eigen vector are given by

$$\lambda^{(8)} X^{(8)} = 19.956 \begin{bmatrix} -1 \\ 0.495 \\ 0.999 \end{bmatrix} \text{ i.e., } 20 \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}$$

Hence the dominant eigen value is 20 and eigen vector is $[-1, 0.5, 1]'$.

## PROBLEMS 4.2

1. Find the eigen values and eigen vectors of the matrices

(a) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

2. Find the latent roots and the latent vectors of the matrices

(a) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (*J.N.T.U., B. Tech., 2005*) (b) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (*Rohtak, B. Tech., 2012*)

(c) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*V.T.U., B.E., 2013*)

3. Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (*U.P.T.U., B. Tech., 2006*)

(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (*Anna, B. Tech., 2013*)

4. Using Gerschgorin circles, find the limits of the eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

## 4.12. JACOBI'S METHOD

Let $A$ be a given real symmetric matrix. Its eigen values are real and there exists a real orthogonal matrix $B$ such that $B^{-1} AB$ is a diagonal matrix $D$. Jacobi's method consists of diagonalising $A$ by applying a series of orthogonal transformations $B_1, B_2, ..., B_r$ such that their product $B$ satisfies the equation $B^{-1} AB = D$.

For this purpose, we choose the numerically largest non-diagonal element $a_{ij}$ and form a $2 \times 2$ submatrix $A_1 = \begin{bmatrix} a_{ij} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$.

where $a_{ij} = a_{ji}$, which can easily be diagonalised.

Consider an orthogonal matrix $B_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ so that $B_1^{-1} = B_1'$.

Then $\quad B^{-1}A_1B_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} a_{ii}\cos^2\theta + a_{jj}\sin^2\theta + a_{ij}\sin 2\theta, & a_{ij}\cos 2\theta + \frac{1}{2}(a_{jj} - a_{ii})\sin 2\theta \\ a_{ij}\cos 2\theta + \frac{1}{2}(a_{jj} - a_{ii})\sin 2\theta, & a_{ii}\sin^2\theta + a_{jj}\cos^2\theta - a_{ij}\sin 2\theta \end{bmatrix} \quad ...(1)$$

Now this matrix will reduce to the diagonal form, if $a_{ij}\cos 2\theta + \frac{1}{2}(a_{jj} - a_{ii})\sin 2\theta = 0$

*i.e.* if $$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} \quad ...(2)$$

This equation gives four values of $\theta$, but to get the least possible rotation, we choose $-\pi/4 \le \theta \le \pi/4$.

Thus (1) reduces to a diagonal matrix.

As a next step, the largest non-diagonal element (in magnitude) in the new rotated matrix is found and the above procedure is repeated using the orthogonal matrix $B_2$.

In this way, a series of such transformations are performed so as to annihiliate the non-diagonal elements. After making $r$ transformations, we obtain

$$B_r^{-1} B_{r-1}^{-1} \ldots B_1^{-1} A B_1 \ldots\ldots B_{r-1} B_r = B^{-1} AB$$

As $r \to \infty$, $B^{-1} AB$ approaches a diagonal matrix whose diagonal elements are the eigen values of $A$.

Also the corresponding columns of $B = B_1 B_2 \ldots\ldots B_r$, are the eigen vectors of $A$.

**Example 4.13.** *Using Jacobi's method, find all the eigen values and the eigen vectors [of] the matrix*

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}.$$

**Sol.** Here the largest non-diagonal element is $a_{13} = a_{31} = 2$. Also $a_{11} = 1$ and $a_{33} = 1$.

$$\therefore \qquad \tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 2}{1 - 1} \to \infty$$

$$2\theta = \pi/2 \quad \text{or} \quad \theta = \pi/4$$

Then
$$B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \text{ and } B_1^{-1} = B'$$

$\therefore$ The first transformation gives

$$D_1 = B_1^{-1} A B_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now the largest non-diagonal element is $a_{12} = a_{21} = 2$. Also $a_{11} = 3$ and $a_{22} = 3$.

$$\therefore \qquad \tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{0} \to \infty.$$

$$2\theta = \pi/2 \quad \text{or} \quad \theta = \pi/4.$$

Then
$$B_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The second transformation gives

$$B_2^{-1} D_1 B_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence the eigen values of the given matrix are 5, 1, – 1 and the corresponding eigen-[v]ectors are the columns of

$$B = B_1 B_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

**Note.** A disadvantage of Jacobi's method is that the element annihiliated by a transformation, may [n]ot remain zero during the subsequent transformations. Given's su... [z]eros already formed. *But instead of leading* ... disturb

**Example 4.14.** *Obtain using Jacobi's method, all the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.25 \\ 0.5 & 0.25 & 2 \end{bmatrix}$$

**Sol.** Here the largest non-diagonal element is $a_{12} = 1$.
Also $a_{11} = 1$, $a_{22} = 1$.

$$\therefore \qquad \tan 2\theta = \frac{2a_{12}}{a_{11} - a_{12}} = \frac{2 \times 1}{0} \to \infty.$$

$$i.e., \qquad 2\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{4}.$$

Then
$$B_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B_1^{-1} = B_1'$$

$\therefore$ The first transformation is

$$D_1 = B_1^{-1} A B_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/4 \\ 1/2 & 1/4 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 3\sqrt{2}/8 \\ 0 & 0 & -\sqrt{2}/8 \\ 3\sqrt{2}/8 & -\sqrt{2}/8 & 2 \end{bmatrix}$$

Now the largest non-diagonal element of $+ D_1$ is $a_{13} = 3\sqrt{2}/8$. Also $\alpha_{11} = 2$, $\alpha_{33} = 2$.

$$\therefore \qquad \tan 2\theta = \frac{2\alpha_{13}}{\alpha_{11} - \alpha_{33}} \to \infty, \; i.e., \; 2\theta = \frac{\pi}{2} \text{ or } \theta = \frac{\pi}{4}.$$

Then
$$B_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$\therefore$ The second transformation gives

$$D_2 = B_2^{-1} D_1 B_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 3\sqrt{2}/8 \\ 0 & 0 & -\sqrt{2}/8 \\ 3\sqrt{2}/8 & -\sqrt{2}/8 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2.530 & -0.125 & 0 \\ -0.125 & 0 & -0.125 \\ 0 & -0.125 & 1.47 \end{bmatrix}$$

Repeating the above steps, we obtain

$$B_3 = \begin{bmatrix} 0.998 & 0.049 & 0 \\ -0.049 & 0.998 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$D_3 = B_3^{-1} D_2 B_3 = \begin{bmatrix} 2.536 & -0.000 & 0.006 \\ -0.000 & -0.006 & -0.125 \\ 0.006 & -0.125 & 1.469 \end{bmatrix}$$

Hence the eigen values of $A$ are 2.536, – 0.006, 1.469 approximately and th

sponding eigen vectors are the columns of $B = B_1 B_2 B_3 = \begin{bmatrix} 0.531 & -0.721 & -0.444 \\ 0.461 & 0.686 & -0.562 \\ 0.710 & 0.094 & 0.698 \end{bmatrix}$

## 4.13 GIVEN'S METHOD