$$= \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7}$$

Example 7

A random variable X has the following probability distribution.

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) the value of K, (ii) $P(1.5 < X < 4.5/X > 2)$ and (iii) the smallest value λ for which $P(X \le \lambda) > 1/2$.

$$\Sigma\, p(x) = 1$$

$\therefore$ $$10K^2 + 9K = 1$$

i.e., $$(10K-1)(K+1) = 0$$

$\therefore$ $$K = \frac{1}{10} \text{ or } -1.$$

The value $K = -1$ makes some values of $p(x)$ negative, which is meaningless

$$\therefore \quad K = \frac{1}{10}$$

The actual distribution is given below:

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(i) $P(1.5 < X < 4.5 / X > 2) = P(A/B)$, say

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P[(1 \cdot 5 < X < 4 \cdot 5) \cap (X > 2)]}{P(X > 2)}$$

$$= \frac{P(X=3) + P(X=4)}{\sum_{r=3}^{7}(X=r)} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(ii) By trials, $P(X \le 0) = 0$; $P(X \le 1) = \frac{1}{10}$; $P(X \le 2) = \frac{3}{10}$

$$P(X \le 3) = \frac{5}{10};\ P(X \le 4) = \frac{8}{10}$$

Therefore, the smallest value of λ satisfying the condition $P(X \le \lambda) > 1/2$ is 4.

Example 8

$$\text{If } p(x) = \begin{cases} x\,e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

(a) show that $p(x)$ is a pdf (of a continuous RV X.)
(b) find its distribution function $P(x)$. (BU — Nov. 96)
(a) If $p(x)$ is to be a pdf, $p(x) \ge 0$ and

$$\int_{R_X} p(x)\, dx = 1$$

Obviously, $p(x) = xe^{-x^2/2} \ge 0$, when $x \ge 0$

$$\text{Now } \int_0^\infty p(x)dx = \int_0^\infty xe^{-x^2/2}\, dx = \int_0^\infty e^{-t}\, dt \text{ (putting } t = x^2/2)$$

$$= 1$$

$\therefore$ $p(x)$ is a legitimate pdf of a RV X.

$$F(x) = P(X \le x) = \int_0^x f(x)dx$$

$\therefore$ $F(x) = 0$, when $x < 0$

and $F(x) = \int_0^X xe^{-x^2/2}\, dx = 1 - e^{-x^2/2}$, when $x \ge 0$.

Example 9

If the density function of a continuous RV X is given by

$$\begin{aligned} f(x) &= ax, & 0 \le x \le 1 \\ &= a, & 1 \le x \le 2 \\ &= 3a - ax, & 2 \le x \le 3 \\ &= 0, & \text{elsewhere} \end{aligned}$$

(i) find the value of a
(ii) find the cdf of X
(iii) If x_1, x_2 and x_3 are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5?

(i) Since $f(x)$ is a pdf, $\int_{R_X} f(x)\mathrm{d}x = 1$

i.e., $$\int_0^3 f(x)\mathrm{d}x = 1$$

i.e., $$\int_0^1 ax\mathrm{d}x + \int_1^2 a\mathrm{d}x + \int_2^3 (3a - ax)\mathrm{d}x = 1$$

i.e., $$2a = 1$$

∴ $$a = \frac{1}{2}$$

(ii) $F(x) = P(X \le x) = 0$, when $x < 0$

$$F(x) = \int_0^x \frac{x}{2}\,\mathrm{d}x = \frac{x^4}{4}, \text{ when } 0 \le x \le 1$$

$$= \int_0^1 \frac{x}{2}\,\mathrm{d}x + \int_1^X \frac{1}{2}\,\mathrm{d}x = \frac{x}{2} - \frac{1}{4} \text{ when } 1 \le x \le 2$$

$$= \int_0^1 \frac{x}{2}\,\mathrm{d}x + \int_1^2 \frac{1}{2}\,\mathrm{d}x + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)\mathrm{d}x = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}, \text{ when } 2 \le x \le 3$$

$$= 1, \text{ when } x > 3$$

(iii) $$p(X > 1.5) = \int_{1.5}^3 f(x)\,\mathrm{d}x$$

$$= \int_{1.5}^2 \frac{1}{2}\,\mathrm{d}x + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)\mathrm{d}x = \frac{1}{2}$$

Choosing an X and observing its value can be considered as a trial and $(X > 1.5)$ can be considered a success.

∴ $$p = 1/2,\ q = 1/2$$

As we choose 3 independent observations of X, $n = 3$.

By Bernoulli's theorem,

P(exactly one value > 1.5)

$$= P(1 \text{ success}) = 3C_1 \times (p)^1 \times (q)^2 = \frac{3}{8}$$

Example 10

A continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$. (MU — Apr. 96)

By the property of pdf,

$\int_{R_x} f(x)\,dx = 1$. X takes values between 2 and 5.

$$\therefore \quad \int_2^5 k(1 + x)dx = 1$$

$$\text{i.e.,} \quad \frac{27}{2} k = 1$$

$$\therefore \quad k = \frac{2}{27}$$

Now $\quad p(X < 4) = p(2 < X < 4) = \int_2^4 k(1 + x)dx = \dfrac{16}{27}$

Example 11

A continuous RV X has a pdf $f(x) = kx^2e^{-x}$; x 1 0. Find k, mean and variance.

(MKU — Apr. 97)

By the property of pdf,

$$\int_0^\infty kx^2e^{-x}dx = 1$$

$$\text{i.e.,} \quad 2k = 1$$

$$\therefore \quad k = \frac{1}{2}$$

Mean of X is defined as

$$E(X) = \int_{R_x} xf(x)dx$$

(refer to Chapter 4)

Variance of X is defined as

$$V(X) = E(X^2) - \{E(X)\}^2,$$

where $E(X^2) = \int_{R_x} x^2 f(x)dx$ (refer to Chapter 4)

$$\therefore \quad E(X) = \frac{1}{2}\int_0^\infty x^3e^{-x}dx$$

$$= \frac{1}{2}[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})]_0^\infty$$

$$= 3$$

$$E(X^2) = \frac{1}{2}\int_0^{\infty} x^4 e^{-x} \mathrm{d}x$$

$$= \frac{1}{2}\left[x^4(-e^{-x}) - 4x^3(e^{-x}) + 12x^2(-e^{-x}) - 24x(e^{-x}) + 24(-e^{-x})\right]_0^{\infty}$$

$$= 12$$

$$\therefore \quad V(X) = E(X^2) - \{E(X)\}^2 = 3$$

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Example 13

A continuous RV has a pdf $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that

(i) $P(X \le a) = P(X > a)$ and

Find

and cd

(ii) $P(X > b) = 0.05$ (BDU — Nov. 96)

(i) $P(X \le a) = P(X > a)$

$$\therefore \int_0^a 3x^2 dx = \int_a^1 3x^2 dx$$

i.e., $$a^3 = 1 - a^3$$

i.e., $$a^3 = \frac{1}{2}$$

$\therefore a = 0.7937$

(ii) $P(X > b) = 0.05$

$$\int_b^1 3x^2 dx = 0.05$$

i.e., $$b^3 = 95$$

$\therefore$ $$b = 0.9830$$

Example 14

The distribution function of a RV X is given by $F(x) = 1 - (1 + x)e^{-x}$, $x \ge 0$. Find the density function, mean and variance of X. (MKU — Nov. 96)

By the property of $F(x)$, the pdf $f(x)$ is given by $f(x) = F'(x)$ at points of continuity of $F(x)$.

The given cdf is continuous for $x \ge 0$.

$\therefore$ $$f(x) = (1 + x)e^{-x} - e^{-x} = xe^{-x}, x \ge 0$$

$$E(X) = \int_0^\infty x^2 e^{-x} dx = 2$$

$E(x) = \int x f(x)\,dx.$

$$E(X^2) = \int_0^\infty x^3 e^{-x} dx = 6$$

$$V(X) = E(X^2) - [E(X)]^2 = 2$$

Example 15

The cdf of a continuous RV X is given by

$$F(x) = 0, x < 0$$
$$= x^2, 0 \le x < \frac{1}{2}$$
$$= 1 - \frac{3}{25}(3 - x)^2, \frac{1}{2} \le x < 3$$
$$= 1, x \ge 3$$

Find the pdf of X and evaluate $P(|X| \le 1)$ and $P(\frac{1}{3} \le X < 4)$ using both the pdf and cdf.

The points $x = 0$, 1/2 and 3 are points of continuity

$$\therefore \quad f(x) = 0, \; x < 0$$

$$= 2x, \; 0 \le x < \frac{1}{2}$$

$$= \frac{6}{25}(3 - x), \; \frac{1}{2} \le x < 3$$

$$= 0, \; x \ge 3$$

Although the points $x = 1/2, 3$ are points of discontinuity for $f(x)$, we may assume that $f\left(\frac{1}{2}\right) = \frac{3}{5}$ and $f(3) = 0$.

$$P(|X| \le 1) = p(-1 \le x \le 1)$$

$$= \int_{-1}^{1} f(x)dx = \int_{0}^{1/2} 2x dx + \int_{1/2}^{1} \frac{6}{25}(3 - x)\,dx \text{ (using property of pdf)}$$

$$= \frac{13}{25}$$

If we use property of cdf

$$P(|X| \le 1) = P(-1 \le x \le 1) = F(1) - F(-1) = \frac{13}{25}$$

If we use the property of pdf

$$P(1/3 \le X < 4) = \int_{1/3}^{1/2} 2x dx + \int_{1/2}^{3} \frac{6}{25}(3 - x)dx = \frac{8}{9}$$

If we use the property of cdf

$$P(1/3 \le X < 4) = F(4) - F\left(\frac{1}{3}\right)$$

$$= 1 - \frac{1}{9} = \frac{8}{9}$$

Example 16

If the RV k is uniformly distributed over (0, 5) what is the probability that the