EXERCISE 7(A)

Part A: (Short answer questions)

1. Define simple graph, multigraph and pseudograph, with an example for each.
2. What do you mean by degree of a vertex? What are the degrees of an isolated vertex and a pendant vertex?
3. State and prove the hand-shaking theorem.
4. Define in-degree and out-degree of a vertex.
5. What is meant by source and sink in graph theory?
6. Define complete graph and give an example.
7. Draw K_5 and K_6.
8. Define regular graph. Can a regular graph be a complete graph?
9. Can a complete graph be a regular graph? Establish your answer by 2 examples.
10. Define n-regular graph. Give one example for each of 2-regular and 3-regular graphs.
11. Define a bipartite graph with an example.
12. In what way a completely bipartite graph differs from a bipartite graph?
13. Draw $K_{2,3}$ and $K_{3,3}$ graphs.

14. Define a subgraph and spanning subgraph.
15. What is an induced subgraph? Give an example.
16. Define graph isomorphism and give an example of two isomorphic graphs.
17. What is the invariant property of isomorphic graphs?
18. Give an example to show that the invariant conditions are not sufficient for graph isomorphism.

Represent the following graphs by adjacency matrices:

19.

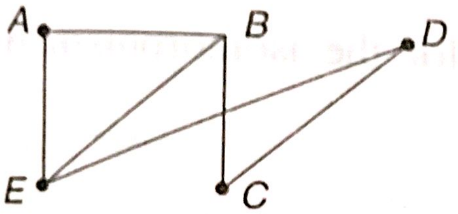


Fig. 7.31

20.

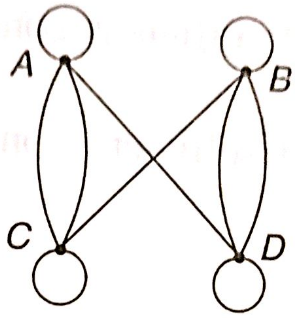


Fig. 7.32

21.

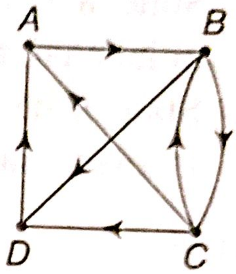


Fig. 7.33

Draw the graphs represented by the following adjacency matrices:

22. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Represent the following graphs by incidence graphs:

25.

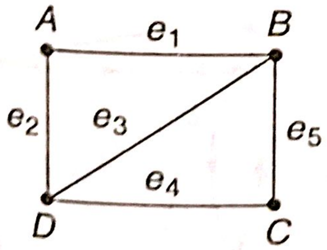


Fig. 7.34

26.

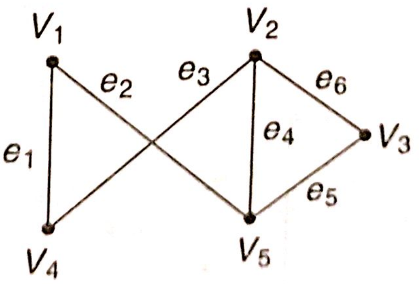


Fig. 7.35

27.

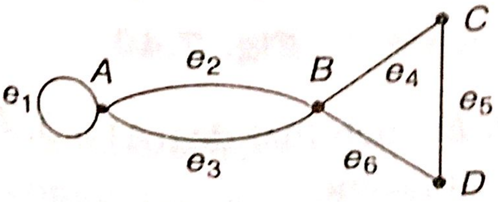
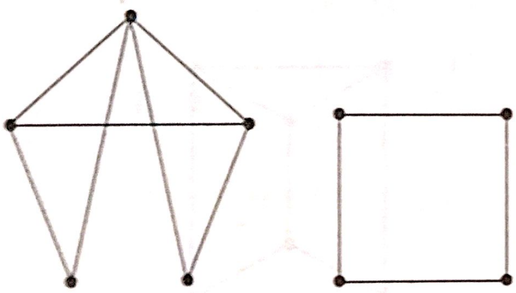


Fig. 7.36

idence matrices:

41. For each pair of graphs given in Figs 7.42(a) and 7.42(b), find whether or not the graph on the right is a subgraph of the one on the left. If it is, label the vertices of the subgraph and then use the same symbols to label the corresponding vertices of the main graph.

(i)



(ii)

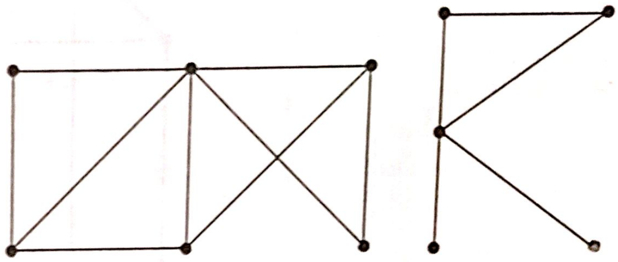


Fig. 7.42(a)

Fig. 7.42(b)

42. Examine whether the following pairs of graphs are isomorphic. If not isomorphic, give the reasons.

(i)

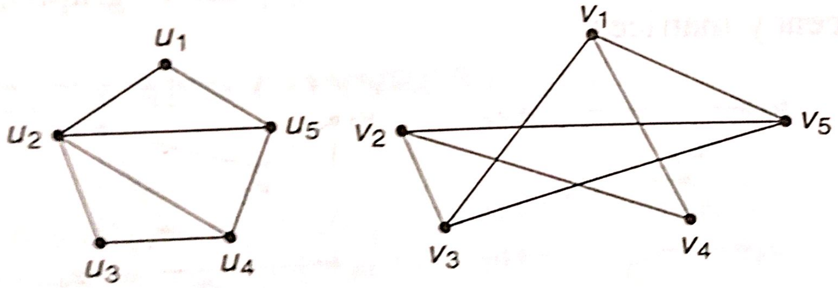


Fig. 7.43(a)

(ii)

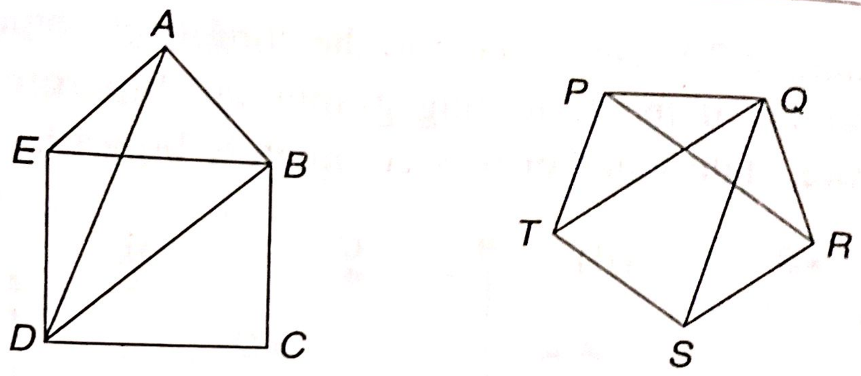


Fig. 7.43(b)

(iii)

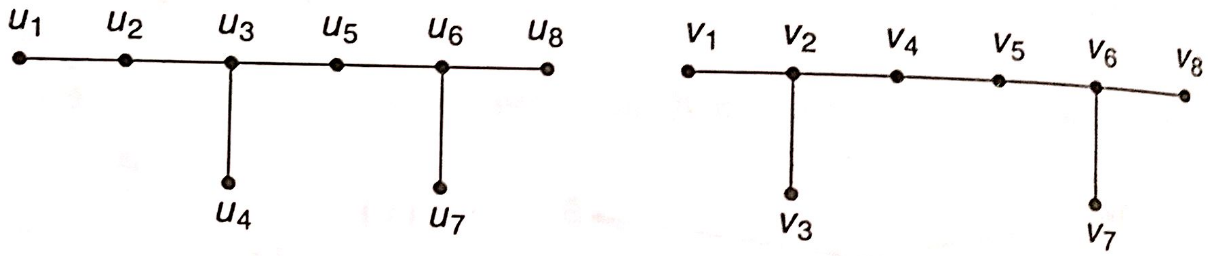


Fig. 7.43(c)

43. Examine whether the following pairs of graphs are isomorphic. If isomorphic, label the vertices of the two graphs to show that their adjacency matrices are the same.

(i)

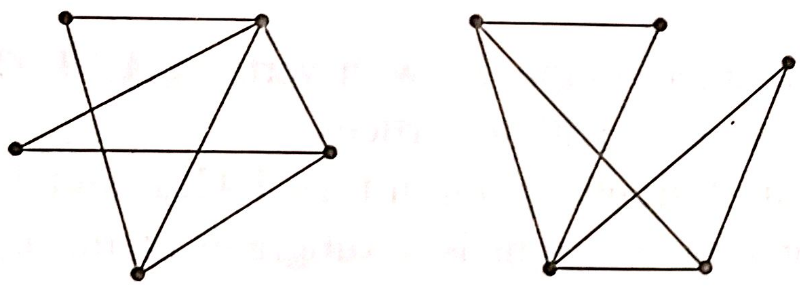


Fig. 7.44(a)

(ii)

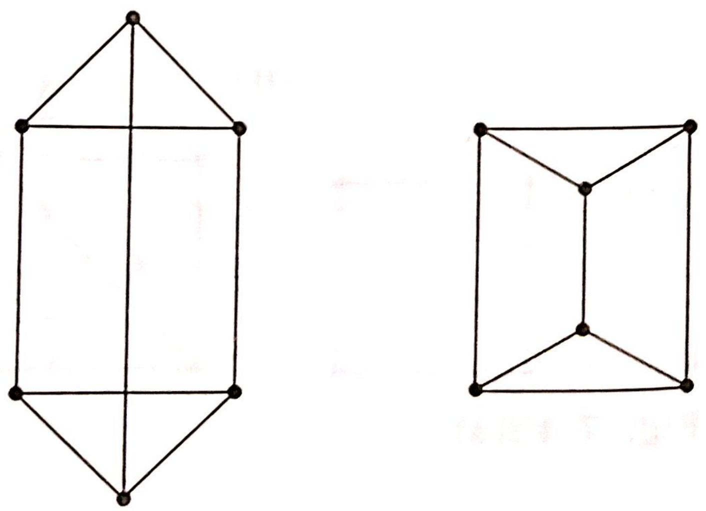


Fig. 7.44(b)

44. Establish the isomorphism of the following pairs of graphs, by considering

PATHS, CYCLES AND CONNECTIVITY

Definitions

A *path* in a graph is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

It the edges in a path are distinct, it is called a *simple path*.

In the graph given in Fig. 7.46, $V_1\ e_1\ V_2\ e_2\ V_3\ e_5\ V_1\ e_1\ V_2$ is a path, since it contains the e_1 twice.

$V_1\ e_4\ V_4\ e_6\ V_2\ e_2\ V_3\ e_7\ V_5$ is a simple path, as no edge appears more than once. The number of edges in a path (simple or general) is called the *length* of the path.

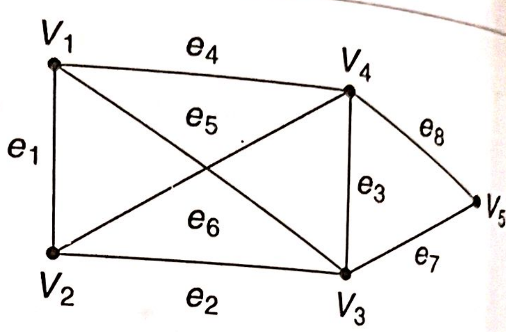


Fig. 7.46

The length of both the paths given above is 4. If the initial and final vertices of a path (of non-zero length) are the same, the path is called a *circuit* or *cycle.*

If the initial and final vertices of a simple path of non-zero length are the same, the simple path is called a simple *circuit* or a simple *cycle.*

In the graph given in Fig. 7.46, $V_1\ e_1\ V_2\ e_2\ V_3\ e_3\ V_4\ e_6\ V_2\ e_1\ V_1$ is a circuit of length 5 whereas $V_1\ e_5\ V_3\ e_7\ V_5\ e_8\ V_4\ e_4\ V_1$ is a simple circuit of length 4.

Connectedness in Undirected Graphs

Definition

An undirected graph is said to be connected if a path between every pair of distinct vertices of the graph.

A graph that is not connected is called *disconnected.*

In Fig. 7.47, G_1 and G_2 are connected, while G_3 is not connected.

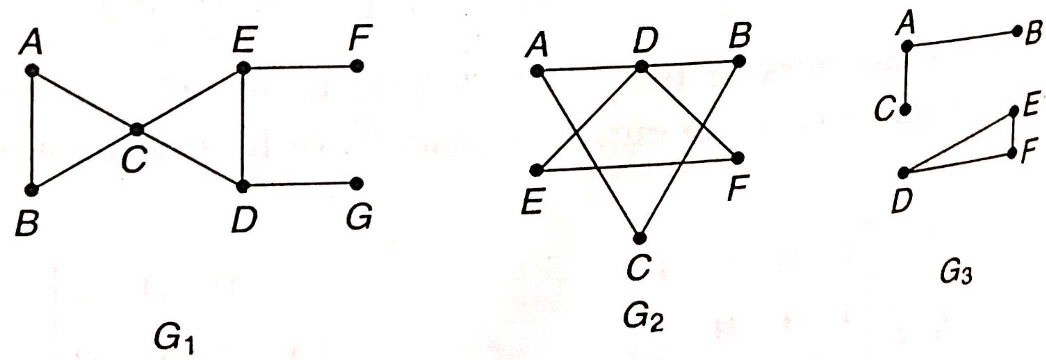


Fig. 7.47

connected subgraphs.

Theorem

If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof

Case (i) Let G be connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree. But we have already proved that the number of odd vertices is even. Clearly there is a path connecting v_1 and v_2, since G is connected.

Case (ii) Let G be disconnected.

Then the components of G are connected. Hence, v_1 and v_2 should belong to the same component of G.

Hence, there is a path between v_1 and v_2.

Theorem

The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\dfrac{(n-k)(n-k+1)}{2}$.

Proof

Let the number of vertices in the i^{th} component of G be n_i $(n_i \geq 1)$.

Then $$n_1 + n_2 + \cdots + n_k = n \text{ or } \sum_{i=1}^{k} n_i = n \quad (1)$$

Hence, $$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$\therefore$ $$\left\{\sum_{i=1}^{k} (n_i - 1)\right\}^2 = n^2 - 2n\,k + k^2$$

$$\ldots 1) = n^2 - 2n\,k + k^2 \quad (2)$$

$\therefore$ $$\left\{\sum_{i=1}^{k}(n_i-1)\right\}^2 = n^2 - 2n\,k + k^2$$

i.e., $$\sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i\neq j}(n_i-1)(n_j-1) = n^2 - 2n\,k + k^2 \qquad (2)$$

i.e., $$\sum_{i=1}^{k}(n_i-1)^2 \leq n^2 - 2n\,k + k^2$$

[$\because$ the second member in the L.S of (2) is ≥ 0, as each $n_i \geq 1$]

i.e., $$\sum_{i\neq 1}^{k}(n_i^2 - 2n_i + 1) \leq n^2 - 2n\,k + k^2$$

i.e., $$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2n\,k + k^2 + 2n - k \qquad (3)$$

Now the maximum number of edges in the i^{th} component of $G = \frac{1}{2}n_i(n_i - 1)$

$\therefore$ Maximum number of edges of G

$$= \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n, \text{ by (1)}$$

$$\leq \frac{1}{2}(n^2 - 2n\,k + k^2 + 2n - k) - \frac{1}{2}n, \text{ by (3)}$$

i.e., $$\leq \frac{1}{2}(n^2 - 2n\,k + k^2 + n - k)$$

i.e., $$\leq \frac{1}{2}\{(n - k)^2 + (n - k)\}$$

i.e., $$\leq \frac{1}{2}(n - k)(n - k + 1).$$

EULERIAN AND HAMILTONIAN GRAPHS

Definitions

A path of graph G is called an Eulerian path, if it includes each edge of G exactly once.

A circuit of a graph G is called an *Eulerian circuit*, if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an *Eulerian graph*.

For example, let us consider the graphs given in Fig. 7.51 and Fig. 7.52.

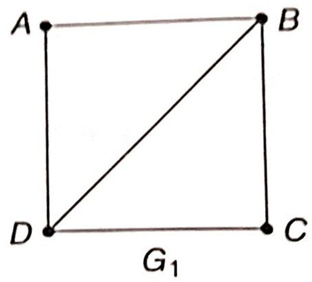


Fig. 7.51

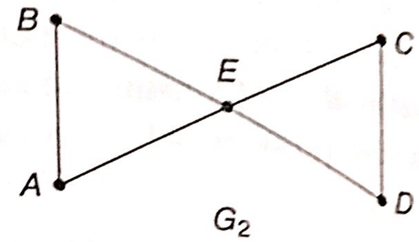


Fig. 7.52

Graph G_1 contains an Eulerian path between B and D namely, $B - D - C - B - A - D$, since it includes each of the edges exactly once.

Graph G_2 contains an Eulerian circuit, namely, $A - E - C - D - E - B - A$, since it includes each of the edges exactly once.

G_2 is an Euler graph, as it contains an Eulerian circuit.

The necessary and sufficient conditions for the existence of Euler circuits and Euler paths are given in two theorems, which we state below without proof.

Theorem 1

A connected graph contains an Euler circuit, if and only if each of its vertices is of even degree.

Theorem 2

A connected graph contains an Euler path, if and only if it has exactly two vertices of odd degree.

Note The Euler path will have the odd degree vertices as its end points.

In the graph G_1 given in Fig. 7.51, the vertices B and D are of degree 3 each. Hence, an Eulerian path existed between B and D.

In the graph G_2 [Fig. 7.52], all the vertices are of even degree. Hence, an Euler circuit existed.

Definitions

A path of a graph G is called a *Hamiltonian path*, if it includes each vertex of G exactly once.

A circuit of a graph G is called a *Hamiltonian circuit*, if it includes each vertex of G exactly once, except the starting and end vertices (which are one and the same) which appear twice.

A graph containing a Hamiltonian circuit is called a *Hamiltonian graph*.