process.

A random variable (RV) is a rule (or function) that assigns a real number to every outcome of a random experiment, while a random process is a rule (or function) that assigns a time function to every outcome of a random experiment. For example, consider the random experiment of tossing a die at $t = 0$ and observing the number on the top face. The sample space of this experiment consists of the outcomes $\{1, 2, 3, \cdots, 6\}$. For each outcome of the experiment, let us arbitrarily assign a function of time t $(0 \leq t < \infty)$ in the following manner.

Outcome:	1	2	3	4	5	6
Function of time:	$x_1(t) = -4$	$x_2(t) = -2$	$x_3(t) = 2$	$x_4(t) = 4$	$x_5(t) = -t/2$	$x_6(t) = t/2$

The set of functions $\{x_1(t), x_2(t), \cdots, x_6(t)\}$ represents a random process.

Definition: A random process is a collection (or ensemble) of RVs $\{X(s, t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set).

The set of possible values of any individual member of the random process is called *state space*. Any individual member itself is called a *sample function* or a realisation of the process.

Average Values of Random Processes

As in the case of RVs random processes can be described in terms of averages or expected values, mostly derived from the first and second-order distributions of $\{X(t)\}$. *Mean* of the process $\{X(t)\}$ is the expected value of a typical member $X(t)$ of the process.

i.e., $\mu(t) = E\{X(t)\}$

Autocorrelation of the process $\{X(t)\}$, denoted by $R_{xx}(t_1, t_2)$ or $R_x(t_1, t_2)$ or $R(t_1, t_2)$, is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process.

i.e., $R(t_1, t_2) = E\{X(t_1) \times X(t_2)\}$

Autocovariance of the process $\{X(t)\}$, denoted by $C_{xx}(t_1, t_2)$ or $C_x(t_1, t_2)$ or $C(t_1, t_2)$, is defined as

$$C(t_1, t_2) = E[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}]$$
$$= R(t_1, t_2) - \mu(t_1) \times \mu(t_2)$$

Correlation coefficient of the process $\{X(t)\}$, denoted by $\rho_{xx}(t_1, t_2)$ or $\rho(t_1, t_2)$, is defined as

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1) \times C(t_2, t_2)}}$$

where $C(t_1, t_1)$ is the variance of $X(t_1)$.

When we deal with 2 or more random processes, we can use joint distribution functions or averages to describe the relationship between them.

Cross-correlation of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{xy}(t_1, t_2) = E\{X(t_1) \times Y(t_2)\}.$$

Cross-covariance of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1) \times \mu_y(t_2)$$

Cross correlation coefficient of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$\rho_{xy}(t_1, t_2) = \frac{C_{xy}(t_1, t_2)}{\sqrt{C_{xx}(t_1, t_1) \times C_{yy}(t_2, t_2)}}$$

Stationarity

A random process is called a *strongly stationary process* or *strict sense stationary process* (abbreviated as SSS process), if all its finite dimensional distributions are invariant under translation of time parameter.

That is, if the joint distribution (and hence the joint density) of $X(t_1), X(t_2), \cdots, X(t_n)$ is the same as that of $X(t_1 + h), X(t_2 + h), \cdots, X(t_n + h)$ for all $t_1, t_2, \cdots, t_n$ and $h(>0)$ and for all $n \geq 1$, then the random process $\{X(t)\}$ is called a SSS process. If the definition given above holds good for $n = 1, 2, \cdots, k$ only and not for $n > k$, then the process is called *kth order stationary*.

Note *If a random process is a SSS process, as per the definition, its first-order densities must be invariant under translation of time, i.e., the densities of $X(t)$ and $X(t+h)$ are the same, i.e., $f(x, t) = f(x, t+h)$.*

This is possible only if $f(x, t)$ is independent of t.

Therefore, first-order densities (and hence distribution function) of a SSS process are independent of time.

As a consequence, $E\{X(t)\}$ is also independent of t.

i.e., $$E\{X(t)\} = \mu = \text{a constant}$$

Also the second-order densities must be invariant under translation of time, i.e., the joint pdf of $\{X(t_1), X(t_2)\}$ is the same as that of $\{X(t_1+h), X(t_2+h)\}$.

i.e., $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h)$

This is possible only if $f(x_1, x_2, t_1, t_2)$ is function of $t = t_1 - t_2$.

Therefore, second-order densities (and hence distribution functions) of a SSS process are functions of $\tau = t_1 - t_2$.

As a consequence, $R(t_1, t_2) = E\{X(t_1)' X(t_2)\}$ is also a function of $\tau = t_1 - t_2$.

It is pointed out that if $E\{X(t)\}$ is a constant and $R(t_1, t_2)$ is a function of $(t_1 - t_2)$, the random process $\{X(t)\}$ need not be a SSS process.

The definition of strict sense stationarity can be extended as follows.

Two real-valued random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be *jointly stationary in the strict sense*, if the joint distribution of $X(t)$ and $Y(t)$ are invariant under translation of time.

The complex random process $\{Z(t)\}$, where $Z(t) = X(t) + iY(t)$, is said to be a SSS process if $\{X(t)\}$ and $\{Y(t)\}$ are jointly stationary in the strict sense.

Wide-sense stationarity: A random process $\{X(t)\}$ with finite first- and second-order moments is called a *weakly stationary process* or *covariance stationary process* or *wide-sense stationary process* (abbreviated as WSS process), if its mean is a constant and the autocorrelation depends only on the time difference.

i.e., if $\quad E\{X(t)\} = \mu$ and

$$E\{X(t) \times X(t-\tau)\} = R(\tau)$$

(**Note:** From the definitions given above, it is clear that a SSS process with finite first-and second-order moments is a WSS process, while a WSS process need not be a SSS process.)

A random process that is not stationary in any sense is called an *evolutionary process*.

Two random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be *jointly stationary in the wide sense*, if each process is individually a WSS process and $R_{xy}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only.

Example of a SSS Process

Let X_n denote the presence or absence of a pulse at the nth time instant in a digital communication system or digital data processing system.

systems. ... several applications in communication

Worked Example 6(A)

Example 1

Examine whether the Poisson process $\{X(t)\}$, given by the probability law $P\{X(t) = r\} = e^{-\lambda t}(\lambda t)^r/\underline{|r}$, $\{r = 0, 1, 2, \cdots\}$, is covariance stationary.

The probability distribution of $X(t)$ is a Poisson distribution with parameter λt.

$$\therefore \quad E\{X(t)\} = \lambda t \neq \text{a constant}$$

Therefore, the Poisson process is not covariance stationary.

Example 2

The process $\{X(t)\}$ whose probability distribution under certain conditions is given by

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \; n = 1, 2, \cdots$$

$$= \frac{at}{1+at}, \; n = 0$$

Show that it is not stationary.

(MU — Apr. 96)

The probability distribution of $X(t)$ is

$X(t) = n$:	0	1	2	3	$\cdots$
p_n:	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	$\cdots$

$$E\{X(t)\} = \sum_{n=0}^{\infty} np_n$$

$$= \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\{1 + 2\alpha + 3\alpha^2 + \cdots\}, \text{ where } \alpha = \frac{at}{1+at}$$

$$= \frac{1}{(1+at)^2}(1-\alpha)^{-2} = \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$$E\{X^2(t)\} = \sum_{n=0}^{\infty} n^2 p_n = \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$= \frac{1}{(1+at)^2}\left[\frac{2}{\left(1-\frac{at}{1+at}\right)^3} - \frac{1}{\left(1-\frac{at}{1+at}\right)^2}\right]$$

$$= 1 + 2at$$

$\therefore$ $\text{Var}\{X(t)\} = 2at$

If $\{X(t)\}$ is a stationary process, $E\{X(t)\}$ and $\text{Var}\{X(t)\}$ are constants. Since $\text{Var}\{X(t)\}$ is a function of t, the given process is not stationary.

Note

When $\{X(t)\}$ is a stationary process, $R\{t_1, t_2\} = E\{X(t_1)\, X(t_2)\}$ is function of $(t_1 - t_2)$.

$\therefore$ $E\{X^2(t)\}$ *is a constant*

Also $E\{X(t)\}$ *is a constant*

$\therefore$ $Var\{X(t)\}$ *is a constant*

Example 3

Show that the random process $X(t) = A\cos(\omega_0 t + \theta)$ is wide-sense stationary, if A and ω_0 are constants and θ is a uniformly distributed RV in $(0, 2\pi)$. (BDU — Apr. 97)

Since θ is uniformly distributed in $(0, 2\pi)$

$$f_\theta(\theta) = \frac{1}{2\pi}, 0 < \theta < 2\pi$$

$$E\{X(t)\} = E\{A\cos(\omega_0 t + \theta)\}$$

$$= A\int_0^{2\pi} \frac{1}{2\pi}\cos(\omega_0 t + \theta)d\theta$$

$$\left[\text{since } E\{g(\theta)\} = \int g(\theta) f_\theta(\theta) d\theta\right]$$

$$= \frac{A}{2\pi}\{\sin(2\pi + \omega_0 t) - \sin\omega_0 t\}$$
$$= 0 = a \text{ constant}$$

$$E\{X(t_1)\,X(t_2)\} = E\{A^2\cos(\omega_0 t_1 + \theta) \times \cos(\omega_0 t_2 + \theta)\}$$
$$= \frac{A^2}{2}\,E\{\cos[(t_1 + t_2)\omega_0 + 2\theta] + \cos[\omega_0(t_1 - t_2)]\}$$
$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\,[\cos\{(t_1 + t_2)\omega_0 + 2\theta\} + \cos\{(t_1 - t_2)\,\omega_0\}]\,d\theta$$
$$= \frac{A^2}{2}\cos\omega_0(t_1 - t_2)$$

i.e., $R(t_1, t_2)$ = a function of $(t_1 - t_2)$

Therefore, $\{X(t)\}$ is a WSS process.

Example 4

Given a RV Y with characteristic function

$$\phi(\omega) = E\{e^{i\omega Y}\}$$
$$= E\{\cos\omega Y + i\sin\omega Y\}$$

and a random process defined by $X(t) = \cos(\lambda t + Y)$, show that $\{X(t)\}$ is stationary in the wide sense

if $\phi(1) = \phi(2) = 0$ (MSU — Apr. 96)

$$E\{X(t)\} = E\{\cos(\lambda t + Y)\}$$
$$= \cos\lambda t \times E(\cos Y) - \sin\lambda t \times E(\sin Y) \tag{1}$$

Given $\phi(1) = 0$

i.e., $E\{\cos Y + i\sin Y\} = 0$

$\therefore$ $E(\cos Y) = 0 = E(\sin Y)$ (2)

Using (2) in (1), we get $E\{X(t)\} = 0$ (3)

$$E\{X(t_1) \times X(t_2)\} = E\{\cos(\lambda t_1 + Y) \times \cos(\lambda t_2 + Y)\}$$
$$= \cos\lambda t_1\cos\lambda t_2\,E(\cos^2 Y) + \sin\lambda t_1\sin\lambda t_2\,E(\sin^2 Y)$$
$$- \sin\lambda(t_1 + t_2)\,E(\sin Y\cos Y)$$
$$= \cos\lambda t_1\cos\lambda t_2\,E\left(\frac{1}{2} + \frac{1}{2}\cos 2Y\right) + \sin\lambda t_1\sin\lambda t_2$$
$$\cdot E\left(\frac{1}{2} - \frac{1}{2}\cos 2Y\right) - \frac{1}{2}\sin\lambda(t_1 + t_2)\,E(\sin 2Y) \tag{4}$$

Given: $\phi(2) = 0$

i.e., $E\{\cos 2Y + i\sin 2Y\} = 0$ (5)

$\therefore$ $E(\cos 2Y) = 0 = E(\sin 2Y)$

Using (5) in (4), we get

$$R(t_1, t_2) = E\{X(t_1) \times X(t_2)\} = \frac{1}{2}\{\cos \lambda t_1 \cos \lambda t_2 + \sin \lambda t_1 \sin \lambda t_2\}$$

$$= \frac{1}{2} \cos \lambda (t_1 - t_2)$$

From (3) and (6), it follows that $\{X(t)\}$ is a WSS process.

Example 6

If $\{X(t)\}$ is a wide-sense stationary process with autocorrelation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second-order moment of the RV $X(8) - X(5)$.

(BDU — Nov. 96)

Second moment of $X(8) - X(5)$ is given by

$$E[\{X(8) - X(5)\}^2] = E\{X^2(8)\} + E\{X^2(5)\} - 2E\{X(8)\, X(5)\} \quad (1)$$

Given: $\quad R(\tau) = A\,e^{-\alpha|\tau|}$

i.e., $\quad R(t_1, t_2) = A\,e^{-\alpha|t_1 - t_2|}$

$\therefore \quad E\{X^2(t)\} = R(t, t) = A$

$\therefore \quad E\{X^2(8)\} = E\{X^2(5)\} = A \quad (2)$

Also $\quad E\{X(8) \times X(5)\} = R(8, 5) = Ae^{-3\alpha} \quad (3)$

Using (2) and (3) in (1), we get

$$E[\{X(8) - X(5)\}^2] = 2A(1 - e^{-3\alpha})$$

Example 7

Show that the process $X(t) = A \cos \lambda t + B \sin \lambda t$ (where A and B are RVs) i wide-sense stationary, if

(i) $E(A) = E(B) = 0$,

(ii) $E(A^2) = E(B^2)$ and (iii) $E(AB) = 0$

(MSU — Apr. 96)

$$E\{X(t)\} = \cos \lambda t \times E(A) + \sin \lambda t \times E(B) \quad (1)$$

If $\{X(t)\}$ is to be a WSS process, $E\{X(t)\}$ must be a constant (i.e., independent of t).

In (1), if $E(A)$ and $E(B)$ are any constants other than zero, $E\{X(t)\}$ will be a function of t.

$\therefore \quad E(A) = E(B) = 0$

$$R(t_1, t_2) = E\{X(t_1) \times X(t_2)\}$$
$$= E\{(A \cos \lambda t_1 + B \sin \lambda t_1)(A \cos \lambda t_2 + B \sin \lambda t_2)\}$$
$$= E(A^2) \cos \lambda t_1 \cos \lambda t_2 + E(B^2) \sin \lambda t_1 \sin \lambda t_2 + E(AB) \sin \lambda(t_1 + t_2) \quad (2)$$

If $\{X(t)\}$ is to be a WSS process, $R(t_1, t_2)$ must be a function of $(t_1 - t_2)$.

$\therefore$ In (2), $\quad E(AB) = 0$ and $E(A^2) = E(B^2) = k$

Then $\quad R(t_1, t_2) = k \cos \lambda(t_1 - t_2)$

Autocorrelation Function and its Properties

Definition: If the process $\{X(t)\}$ is stationary either in the strict sense or in wide sense, then $E\{X(t)\} X(t-\tau)\}$ is a function of τ, denoted by $R_{xx}(\tau)$ or $R(\tau)$ or $R_x(\tau)$. This function $R(\tau)$ is called the autocorrelation function of the process $\{X(t)\}$.

Properties of R(τ)

1. $R(t)$ is an even function of τ

Proof

$$R(\tau) = E\{X(t) \times X(t-\tau)\}$$

$$\therefore \quad R(-\tau) = E\{X(t) \times X(t+\tau)\}$$

$$= E\{X(t+\tau) \times X(t)\}$$

$$= R(\tau)$$

Therefore, $R(\tau)$ is an even function of τ.

2. $R(\tau)$ is maximum at $\tau = 0$

i.e., $$|R(\tau)| \leq R(0)$$

Proof

Cauchy-schwarz inequality is

$$\{E(XY)\}^2 \le E(X^2) \times E(Y^2)$$

Put $X = X(t)$ and $Y = X(t - \tau)$

Then $[E(X(t) \times X(t - \tau)\}]^2 \le E\{X^2(t)\} \times E\{X^2(t - \tau)\}$

i.e., $\{R(t)\}^2 \le [E\{X^2(t)\}]^2$

[since $E\{X(t)\}$ and Var $\{X(t)\}$ are constant for a stationary process]

i.e., $\{R(\tau)\}^2 \le \{R(0)\}^2$

Taking square-root on both sides

$$|R(\tau)| \le R(0) \qquad \text{[since } R(0) = E\{X^2(t)\} \text{ is positive]}$$

3. If the autocorrelation function $R(t)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, it is continuous at every other point.

Proof

Consider $E[\{X(t) - X(t - \tau)\}^2]$

$= E\{X^2(t)\} + E\{X^2(t - \tau)\} - 2E\{X(t) \times X(t - \tau)\}$

$= R(0 + R(0) - 2R(\tau)$

$= 2[R(0) - R(\tau)]$ (1)

Since $R(\tau)$ is continuous at $\tau = 0$, $\lim\limits_{\tau \to 0} R(\tau) = R(0)$

i.e, $\lim\limits_{\tau \to 0}$ {R.S. of (1)} $= 0$

$\therefore$ $\lim\limits_{\tau \to 0}$ {L.S. of (1)} $= 0$

$\therefore$ $\lim\limits_{\tau \to 0} \{X(t - \tau)\} = X(t)$

i.e. $X(t)$ is continuous for all t (2)

Consider $R(\tau + h) - R(\tau)$

$= E[X(t) \times X\{t - (\tau + h)\}] - E[X(t) \times X(t - \tau)]$

$= E[X(t) \{X(t - \tau - h) - X(t - \tau)\}]$ (3)

Now $\lim\limits_{h \to 0} [X\{(t - \tau) - h\} - X(t - \tau)] = 0$, by (2)

$\therefore$ $\lim\limits_{h \to 0}$ {R.S. of (3)} $= 0$

$\therefore$ $\lim\limits_{h \to 0}$ {L.S. of (3) $= 0$

i.e., $\lim\limits_{h \to 0} \{R(\tau + h)\} = R(\tau)$

i.e., $R(\tau)$ is continuous for all τ

4. If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then $\lim\limits_{\tau \to \infty} R(\tau) = \mu_x^2$, provided the limit exists.

Proof

$$R(\tau) = E\{X(t) \times X(t - \tau)\}$$

When τ is very large, $X(t)$ and $X(t-\tau)$ are two sample functions (members) of the process $\{X(t)\}$ observed at a very long interval of time.

Therefore, $X(t)$ and $X(t-\tau)$ tend to become independent [$X(t)$ and $X(t-\tau)$ may be dependent, when $X(t)$ contains a periodic component, which is not true].

$$\therefore \quad \lim_{\tau \to \infty} \{R(\tau)\} = E\{X(t)\} \times E\{X(t-\tau)\}$$

$$= \mu_x^2 \qquad \text{[since } E\{X(t)\} \text{ is a constant]}$$

i.e.,
$$\mu_x = \sqrt{\lim_{\tau \to \infty} R(\tau)}$$

Cross-Correlation Function and Its Properties

Definition: If the processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary, then $E\{X(t) \times Y(t-\tau)\}$ is a function of τ, denoted by $R_{xy}(\tau)$. This function $R_{xy}(\tau)$ is called the cross-correlation function of the processes $\{X(t)\}$ and $\{Y(t)\}$.

We give below the properties of $R_{xy}(\tau)$ without proof. Proofs of these properties are left as exercises to the reader.

Properties

1. $R_{yx}(\tau) = R_{xy}(-\tau)$
2. $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0) \times R_{yy}(0)}$

   This means that the maximum of $R_{xy}(\tau)$ can occur anywhere, but it cannot exceed $\sqrt{R_{xx}(0) \times R_{yy}(0)}$.
3. $|R_{xy}(\tau)| \le 1/2\ \{R_{xx}(0) + R_{yy}(0)\}$
4. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal, then $R_{xy}(\tau) = 0$
5. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, then $R_{xy}(\tau) = \mu_x \times \mu_y$

Ergodicity

When we wish to take a measurement of a variable quantity in the laboratory, we usually obtain multiple measurements of the variable and average them to reduce measurement errors. If the value of the variable being measured is constant and errors are due to disturbances (noise) or due to the instability of the measuring instrument, then averaging is, in fact, a valid and useful technique. 'Time averaging' is an extension of this concept, which is used in the estimation of various statistics of random processes.

We normally use ensemble averages (or statistical averages) such as the mean and autocorrelation function for characterising random processes. To estimate ensemble averages, one has to compute a weighted average over all the member functions of the random process.

For example, the ensemble mean of a discrete random process $\{X(t)\}$ is computed by the formula $\mu_x = \sum_i x_i p_i$. If we have access only to a single sample function of the process, then we use its *time-average* to estimate the ensemble averages of the process.

Definition: If $\{X(t)]$ is a random process, then $\frac{1}{2T}\int_{-T}^{T} X(t)\, dt$ is called the *time-average* of $\{X(t)\}$ over $(-T, T)$ and denoted by $\overline{X}_T$.

In general, ensemble averages and time averages are not equal except for a very special class of random processes called *ergodic processes*. The concept of ergodicity deals with the equality of time averages and ensemble averages.

Definition: A random process $\{X(t)\}$ is said to be ergodic, if its ensemble averages are equal to appropriate time averages.

This definition implies that, with probability 1, any ensemble average of $\{X(t)\}$ can be determined from a single sample function of $\{X(t)\}$.

Note *Ergodicity is a stronger condition than stationarity and hence all random processes that are stationary are not ergodic. Moreover, ergodicity is usually defined with respect to one or more ensemble averages (such as mean and autocorrelation function) as discussed below and a process may be ergodic with respect to one ensemble average but not others.*