1.4 SOLUTION OF LINEAR SYSTEM BY GAUSSIAN ELIMINATION AND GAUSS-JORDON METHODS

There are two types of methods to solve simultaneous linear algebraic equations with many unknowns.

1. Direct method	2. Indirect method (or) Iterative method
(i) Gauss elimination method	(i) Gauss Jacobic method
(ii) Gauss Jordon method	(ii) Gauss Seidel method
(iii) Crout's method	

1.4.1 Gaussian Elimination method

Gauss elimination method is a direct method which consists of transforming the given system of simultaneous equations to an equivalent upper triangular system. From this system the required solution can be obtained by the method of back substitution.

Consider the n linear equations in n unknowns, viz.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \qquad \ldots (1) \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to

$$AX = B \qquad \ldots (2)$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix [A, B] to upper triangular matrix.

$$[A, B] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \quad \ldots (3)$$

Now, multiply the first row of (3), (if $a_{11} \neq 0$) by $-\frac{a_{i1}}{a_{11}}$ and add to the ith row of [A, B], where $i = 2, 3, \ldots, n$. By this, all elements in the first column of [A, B] except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right) \quad \ldots (4)$$

Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\frac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row ($i = 3, 4, \ldots n$). Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} \cdots a_{1n} & & b_1 \\ 0 & b_{22} & b_{23} \cdots b_{2n} & & c_2 \\ 0 & 0 & c_{33} \cdots c_{3n} & & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & c_{n3} \cdots c_{nn} & & d_n \end{array}\right) \quad \ldots (5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get [A, B] after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \dots & \dots & b_{2n} & c_2 \\ 0 & 0 & c_{33} & c_{34} & \dots & c_{3n} & d_3 \\ \vdots & \vdots & \vdots & \dots & \dots & & \vdots \\ 0 & 0 & 0 & \dots & \dots & \alpha_{nn} & K_n \end{array}\right)$$

From (6), the given system of linear equations is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ b_{22}x_2 + b_{23}x_3 + \dots + b_{2n}x_n &= c_2 \\ c_{33}x_3 + \dots + c_{3n}x_n &= d_3 \\ \dots\dots\dots\dots\dots\dots & \\ \alpha_{nn}x_n &= K_n \end{aligned}$$

Taking from the bottom to top of these equations, we solve for $x_n = \frac{K_n}{\alpha_{nn}}$. Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method, we solve for $x_n, x_{n-1}, x_{n-2}, \dots, x_2, x_1$

1.4.2 Gauss-Jordan elimination method

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system AX = B is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular matrix, but also lower triangular matrix by making all elements above the leading diagonal of A also are zeros.

Note : In this method, the values are get immediately without using the process of back substitution.

1. **Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method.** [M.U. Oct., 1999]

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Solution : (i) Gauss elimination method

The given system is equivalent to

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

$$A\,X = B$$

Here $[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$

Now, we will make the matrix A as a upper triangular.

Fix the first row, change 2 and 3 row with row 1

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \begin{matrix} R_2 \Leftrightarrow 5R_2 - R_1 \\ R_3 \Leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

Fix 1 and 2 row, change 3 row with 2nd row.

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad \ldots (1)\ R_3 \Leftrightarrow 52R_3 + 34R_2$$

This is an upper triangular matrix

From (1) we get [by back substitution]

$$3780\, z = 11340$$

$$z = 3$$

$$52y - 28z = -188$$

$$52y - 28\,(3) = -188$$

$$y = -2$$

$$10x - 2y + 3z = 23$$

$$10x - 2\,(-2) + 3\,(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x + 13 = 23$$

$$10x = 10$$

$$x = 1$$

Hence the solution is $x = 1, y = -2, z = 3$

(ii) Gauss-Jordan method

Take the equation (1)

$$(A, B) \sim \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{array}\right]$$

Now, we will make the matrix A a diagonal matrix.

Fix the third row and change 2nd row and first row

$$\sim \left[\begin{array}{ccc|c} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow 1260R_1 - R_3 \\ R_2 \leftrightarrow 135\,R_2 + R_3 \end{array}$$

Fix the 2 and 3 row change 1 row with 2nd row

$$\sim \left[\begin{array}{ccc|c} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{array}\right] R_1 \leftrightarrow 7020R_1 + 2520\,R$$

$$(\text{i.e.,}) \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\therefore x = 1, y = -2, z = 3$$

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2. **Solve the following system of equations by Gauss-Jordan method.**

$5x + 4y = 15$, $3x + 7y = 12$ [Anna, May 1999]

Solution : The given system is equivalent to $\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

$$A\,X = B$$

$$[A, B] = \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 3 & 7 & 12 \end{array}\right]$$

Now, we will make the matrix A is a diagonal matrix.

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 0 & 23 & 15 \end{array}\right] \quad R_2 \leftrightarrow 5R_2 - 3R_1$$

$$\sim \left[\begin{array}{cc|c} 115 & 0 & 285 \\ 0 & 23 & 15 \end{array}\right] \quad R_1 \leftrightarrow 23R_1 - 4R_2$$

$$\therefore\ 115x = 285$$

$$x = \frac{57}{23} = 2.4783 \text{ [correct to four decimal places]}$$

$$23y = 15$$

$$y = \frac{15}{23} = 0.6522 \text{ [correct to four decimal places]}$$

Solve the system of equations by Gauss-elimination method.

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

olution : The given system is equivalent to

$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -5 \\ -6 \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 4 & 29 & 4 & -29 \\ 0 & 4 & 4 & 19 & -34 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 5R_3 - R_1 \\ R_2 \leftrightarrow 5R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 0 & 970 & 120 & -1210 \\ 0 & 0 & 120 & 630 & -1380 \end{array}\right] \begin{array}{l} R_3 \leftrightarrow 34R_3 - 4R_2 \\ R_4 \leftrightarrow 34R_4 - 4R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 12 & 63 & -138 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow \dfrac{R_2}{2} \\ R_3 \leftrightarrow \dfrac{R_3}{10} \\ R_4 \leftrightarrow \dfrac{R_4}{10} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 0 & 5967 & -11934 \end{array}\right] \quad R_4 \leftrightarrow 97R_4 - 12R_3$$

(i.e.,) $5967\, x_4 = -11934$

$$x_4 = -2$$

$$97x_3 + 12x_4 = -121$$

$$97x_3 + 12(-2) = -121$$

$$97x_3 - 24 = -121$$

$$97x_3 = -121 + 24$$

$$97x_3 = -97$$

$$x_3 = -1$$

$$17x_2 + 2x_3 + 2x_4 = 28$$

$$17x_2 + 2(-1) + 2(-2) = 28$$

$$17x_2 - 2 - 4 = 28$$

$$17x_2 - 6 = 28$$

$$17x_2 = 28 + 6$$

$$17x_2 = 34$$

$$
\begin{aligned}
5x_1 + x_2 + x_3 + x_4 &= 4 \\
5x_1 + 2 + (-1) + (-2) &= 4 \\
5x_1 - 1 &= 4 \\
5x_1 &= 5 \\
x_1 &= 1
\end{aligned}
$$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = -2$.

4. Using Gauss-Elimination method, solve the system.

$3.15x - 1.96y + 3.85z = 12.95$

$2.13x + 5.12y - 2.89z = -8.61$

$5.92x + 3.05y + 2.15z = 6.88$ [M.K.U. 198

Solution : The given system is equivalent to

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$\text{(i.e.,) } [A, B] = \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array}\right]$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 21.2107 & -16.0195 & -54.992 \end{bmatrix} \begin{array}{l} R_2 \Leftrightarrow 3.15R_2 - 2.13R_1 \\ R_3 \Leftrightarrow 3.15R_3 - 5.92R_1 \end{array}$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 0 & 41.7892 & 43.8398 \end{bmatrix} R_3 \Leftrightarrow 20.3028R_3 - 21.2107R_2$$

$41.7892\, z = 43.8398$

$z = 1.049$ [correct to 3 decimal places]

$20.3028y - 17.304z = -54.705$

$20.3028y - (17.304)(1.049) = -54.705$

$20.3028y - 18.1519 = -54.705$

$20.3028y = -54.705 + 18.1519$

$= -36.5531$

$$y = -1.8004$$

$$= -1.800 \text{ [correct to 3 decimal places}$$

$$3.15x - 1.96y + 3.85z = 12.95$$

$$3.15x - 1.96(-1.8) + 3.85(1.049) = 12.95$$

$$3.15x + 3.528 + 4.03865 = 12.95$$

$$3.15x + 7.56665 = 12.95$$

$$3.15x = 12.95 - 7.56665$$

$$= 5.38335$$

$$x = 1.709$$

e the solution is $x = 1.709$, $y = -1.800$, $z = 1.049$

Using the Gauss-Jordan method solve the following equations

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

[A.U. Nov./Dec. 2004]

Solution : Interchanging the first and the last equation then

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right]$$

Fix the pivot element row and make the other elements zero in the pivot element column.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 10R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 + 9R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - 6.125\, R_3 \\ R_1 \leftrightarrow R_2 + 1.125\, R_3 \end{array}$$

$\therefore\ x = 1,\ y = 1,\ z = 1$

Solve x + 3y + 3z = 16, x + 4y + 3z = 18, x + 3y + 4z = 19 by Gauss-Jordan method. **[A.U. April/May 2005]**

Solution : Given :

$$x + 3y + 3z = 16$$

$$x + 4y + 3z = 18$$

$$x + 3y + 4z = 19$$

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_1 - 3R_3$$

Hence $x = 1$, $y = 2$, $z = 3$