## Characteristics of Infinite Capacity, Multiple Server Poisson Queue Model II [M/M/s): (∞/FIFO) model], When $\lambda_n = \lambda$ for all $n(\lambda < s\mu)$

1. *Values of $P_0$ and $P_n$* :

or $$P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}} \tag{6}$$

2. *Average number of customers in the queue or average queue length*

$$L_q = E(N_q) = E(N-s) = \sum_{n=s}^{\infty} (n-s)P_n$$

$$= \sum_{x=0}^{\infty} x\,P_{x+s}$$

$$= \sum_{x=0}^{\infty} x \times \frac{1}{s!\,s^x}\left(\frac{\lambda}{\mu}\right)^{s+x} \cdot P_0$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0 \sum_{x=0}^{\infty} x\left(\frac{\lambda}{\mu s}\right)^x$$

$$= \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda}{\mu s} \cdot P_0 \frac{1}{\left(1-\frac{\lambda}{\mu s}\right)^2}$$

$$= \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 \tag{7}$$

3. *Average number of customers in the system*
   By Little's formula (iv),

$$E(N_s) = E(N_q) + \frac{\lambda}{\mu}$$

$$= \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 + \frac{\lambda}{\mu} \tag{8}$$

Result (8) can also be directly derived by using the definition $E(N_s) = \sum_{n=0}^{\infty} n\,P_n$.

4. *Average time a customer has to spend in the system*
By Little's formula (i)

$$E(W_s) = \frac{1}{\lambda} E(N_s)$$

$$= \frac{1}{\mu} + \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_0 \tag{9}$$

5. *Average time a customer has to spend in the queue*
By Little's formula (ii),

$$E(W_q) = \frac{1}{\lambda} E(N_q)$$

$$= \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \cdot P_0 \tag{10}$$

6. *Probability that an arrival has to wait*
Required probability = Probability that there are $s$ or more customers in the system

i.e., $$P(W_s > 0) = P(N \geq s)$$

$$= \sum_{n=s}^{\infty} P_n = \sum_{n=s}^{\infty} \frac{1}{s! s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \cdot P_0 \sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^{n-s}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \tag{11}$$

7. *Probability that an arrival enters the service without waiting*
Required probability
$= 1 - P(\text{an arrival has to wait})$

$$= 1 - \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)} \tag{12}$$

8. *Mean waiting time in the queue for those who actually wait.*

$$E(W_q / W_s > 0) = \frac{E(W_q)}{P(W_s > 0)}$$

$$= \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 \times \frac{s!\left(1-\frac{\lambda}{\mu s}\right)}{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0} \quad \text{[using (10) and (11)]}$$

$$= \frac{1}{\mu s\left(1-\frac{\lambda}{\mu s}\right)} = \frac{1}{\mu s - \lambda} \tag{13}$$

9. *Probability that there will be someone waiting*
Required probability $= P(N \geq s+1)$

$$= \sum_{n=s+1}^{\infty} P_n = \sum_{n=s}^{\infty} P_n - P(N = s)$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)} - \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!} \quad \text{[using (10) and (5)]}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!} \cdot \frac{\left(\frac{\lambda}{\mu s}\right)}{1-\frac{\lambda}{\mu s}} \tag{14}$$

10. *Average number of customers (in non-empty queues), who have to actually wait.*

$$L_w = E(N_q / N_q \geq 1)$$
$$= E(N_q)/P(N \geq s)$$

$$= \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1} \cdot P_0}{\left(1-\frac{\lambda}{\mu s}\right)^2} \cdot \frac{s!\left(1-\frac{\lambda}{\mu s}\right)}{\left(\frac{\lambda}{\mu}\right)^s P_0}$$

$= 0.299$ h or 18 min for bad weather

## Example 7

There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour.

(a) What fraction of the time all the typists will be busy?

(b) What is the average number of letters waiting to be typed?

(c) What is the average time a letter has to spend for waiting and for being typed?

(d) What is the probability that a letter will take longer than 20 min waiting to be typed and being typed?

$$\lambda = 15/\text{hour}; \ \mu = 6/\text{hour}; \ s = 3.$$

Hence this is a problem in multiple server [$(M/M/s)$: $(\infty/FIFO)$] model, i.e., model II.

(a) $P$(all the typists are busy) $= P(N \geq 3)$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^3 \cdot P_0}{3!\left(1 - \frac{\lambda}{3\mu}\right)} \quad \text{[by formula (11) of model II]}$$

$$= \frac{(2.5)^3 P_0}{6 \times \left(1 - \frac{2.5}{3}\right)} \quad (1)$$

Now
$$P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \cdot \left(\frac{\lambda}{\mu}\right)^s\right\}}$$

[by formula (6) of model II]

$$= \frac{1}{\left\{1 + 2\cdot5 + \frac{1}{2} \times (2\cdot5)^2\right\} + \left\{\frac{1}{6 \times \left(1 - \frac{5}{6}\right)} \times (2\cdot5)^3\right\}}$$

$$= \frac{1}{22.25} = 0.0449 \qquad (2)$$

Using (2) in (1), we have $P(N \geq 3) = 0.7016$.

Hence the fraction of the time all the typists will be busy = 0.7016.

(b) $$E(N_q) = \frac{1}{s.s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1} \cdot P_0}{\left(1 - \frac{\lambda}{\mu s}\right)^2}$$ [by formula (7) of model II]

$$= \frac{1}{3 \times 6} \times \frac{(2.5)^4}{\left(1 - \frac{2.5}{3}\right)^2} \times 0.0449 = 3.5078$$

(c) $E(W) = \frac{1}{\lambda} E(N)$ [by Little's formula (i)]

$$= \frac{1}{\lambda}\left\{E\left(N_q\right) + \frac{\lambda}{\mu}\right\},$$ [by Little's formula (iv)]

$$= \frac{1}{15}\{3.5078 + 2.5\} = 0.4005 \text{ h}$$

or 24 min, nearly

(d) $$P(W > t) = e^{-\mu t}\left\{1 + \frac{\left(\frac{\lambda}{\mu}\right)^s \left[1 - e^{-\mu t\left(s - 1 - \frac{\lambda}{\mu}\right)}\right] P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)\left(s - 1 - \frac{\lambda}{\mu}\right)}\right\}$$

(This formula has not been derived; it may be assumed.)

$$\therefore P\left(W > \frac{1}{3}\right) = e^{-6 \times \frac{1}{3}} = \left[1 + \frac{(2.5)^3\left\{1 - e^{(-2 \times -0.5)}\right\} \times 0.0449}{6\left(1 - \frac{2.5}{3}\right)(-0.5)}\right]$$

$$= e^{-2}\left[1 + \frac{0.7016\,(1 - e)}{(-0.5)}\right]$$

$$= 0.4616$$

## Example 8

Given an average arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rate of 22 customers per hour or at one of two channels in parallel with mean service rate of 11 customers per hour for each of the two channels. Assume both queues to be of Poisson type.

For the single channel service,

$$\lambda = 20/\text{hour and } \mu = 22/\text{hour.}$$

$$E(W) = \frac{1}{\mu - \lambda} \qquad \text{[by formula (7) of model I]}$$

$$= \frac{1}{2}\ \text{h}$$

For the two channel service,

$$\lambda = 20/\text{hour and } \mu = 11/\text{hour.}$$

$$E(W) = \frac{1}{\mu} + \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \times P_0$$

[by formula (9) of model II]

$$= \frac{1}{11} + \frac{1}{11 \times 2 \times 2} \times \frac{\left(\frac{20}{11}\right)^2}{\left(1 - \frac{20}{22}\right)^2} \times P_0$$

$$= 0.0909 + 9.0909 \times P_0 \qquad (1)$$

Now $$P_0^{-1} = \left\{ \sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n \right\} + \left\{ \frac{1}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \cdot \left(\frac{\lambda}{\mu}\right)^s \right\}$$

[by formula (6) of model II]

$$= 1 + \frac{20}{11} + \frac{1}{2 \times \frac{1}{11}} \times \left(\frac{20}{11}\right)^2$$

$$= 21$$

$$\therefore \qquad P_0 = 0.0476 \qquad (2)$$

Using (2) in (1), we have

$$E(W) = 0.5236 \text{ h}$$

As the average waiting time in single channel service is less than that in two channel service, the customer has to prefer the former.

## Example 12

A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.

(a) What is the probability that an arrival would have to wait in line?

(b) Find the average waiting time, average time spent in the system and the average number of cars in the system.

(c) For what percentage of time would a pump be idle on an average?

$$s = 4,\ \lambda = 30/\text{hour},\ \mu = 10/\text{hour}$$

(a) $P$(an arrival has to wait) $= P(W > 0)$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s \cdot P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \qquad \text{[by formula (11) of model II]}$$

$$= \frac{3^4 \times P_0}{24 \times \left(1 - \frac{3}{4}\right)} = 13.5 \times P_0 \qquad (1)$$

$$P_0 = \left[\left\{\sum_{n=0}^{s-1} \frac{1}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)}\right]^{-1} \qquad \text{[by formula (6) of model II]}$$

$$= \left[\left(1 + 3 + \frac{1}{2} \times 9 + \frac{1}{6} \times 27\right) + \frac{3^4}{24 \times \left(1 - \frac{3}{4}\right)}\right]^{-1}$$

$$= 0.0377$$

Using (2) in (1), $P(W > 0) = 0.5090$

(b) $E(W_q) = \frac{1}{\mu} \cdot \frac{1}{s \times s!} \times \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \times P_0$ [by formula (10) of model II]

$$= \frac{1}{10 \times 4 \times 24} \times \frac{3^4}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 = 0.0509 \text{ h}$$

or $\qquad$ 3.05 min

$E(W_s) = \frac{1}{\mu} + E(W_q)$ [by formulas (9) and (10) of model II]

$$= 6 + 3.05 = 9.05 \text{ min}$$

$E(N) = \frac{1}{s \times s!} \times \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} \times P_0 + \frac{\lambda}{\mu}$ [by formula (8) of model II]

$$= \frac{1}{4 \times 24} \times \frac{3^5}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 + 3$$

$$= 4.53 \text{ cars}$$

(c) The fraction of time when the pumps are busy = traffic intensity $= \frac{\lambda}{\mu s} = \frac{3}{4}$

$\therefore$ The fraction of time when the pumps are idle $= \frac{1}{4}$

Therefore, required percentage = 25%

## Example 11

A supermarket has two girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 min and if people arrive in Poisson fashion at the rate of 10 per hour,

(a) what is the probability that a customer has to wait for service?
(b) what is the expected percentage of idle time for each girl?
(c) if the customer has to wait in the queue, what is the expected length of his waiting time?

$$s = 2,\ \lambda = \frac{1}{6} \text{ per minute},\ \mu = \frac{1}{4} \text{ per minute}$$

(a) $P$(a customer has to wait for service)

$$= P(N \geq 2) = 1 - P_0 - P_1 \tag{1}$$

$$P_0 = \left[ \left\{ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n \right\} + \left\{ \frac{\left( \frac{\lambda}{\mu} \right)^s}{s! \left( 1 - \frac{\lambda}{\mu s} \right)} \right\} \right]^{-1}$$

[by formula (6) of model II]

$$= \left[ 1 + \frac{2}{3} + \frac{\left( \frac{2}{3} \right)^2}{2 \times \left( 1 - \frac{1}{3} \right)} \right]^{-1} = \frac{1}{2} \tag{2}$$

$$P_1 = \frac{\lambda}{\mu} \cdot P_0,$$ [by formula (4) of model II]

$$= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \tag{3}$$

Using (2) and (3) in (1), we have

$$P(N \geq 2) = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) Fraction of time when the girls are busy $= \dfrac{\lambda}{\mu s} = \dfrac{1}{3}$

$\therefore$ Fraction of time when the girls are idle $= \dfrac{2}{3}$

$\therefore$ Expected percentage of idle time for each girl $= \dfrac{2}{3} \times 100$

$$= 67$$

(c) $E(W_q / W_s > 0) = \dfrac{1}{\mu s - \lambda}$ [by formula (13) of model II]

$$= \frac{1}{\frac{1}{4} \times 2 - \frac{1}{6}} = 3 \text{ min}$$

# Characteristics of Finite Capacity, Single Server Poisson Queue Model III [(M/M/1): (k/FIFO) Model]

Now $$\sum_{n=0}^{k} P_n = 1$$

i.e., $$P_0 \sum_{n=0}^{k} \left(\frac{\lambda}{\mu}\right)^n = 1$$

i.e., $$P_0 \frac{\left\{1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right\}}{1-\frac{\lambda}{\mu}} = 1,$$

which is valid even for $\lambda > \mu$

$$\therefore P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, \text{ if } \quad \lambda \neq \mu & (4) \\[2ex] \dfrac{1}{k+1}, \text{ if } \lambda = \mu, \text{ since } \lim\limits_{\frac{\lambda}{\mu}\to 1} \left\{\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right\} = \dfrac{1}{k+1} & (5) \end{cases}$$

$$\therefore P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \cdot \left[\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right], \text{ if } \lambda \neq \mu & (6) \\[2ex] \dfrac{1}{k+1}, \text{ if } \lambda = \mu & (7) \end{cases}$$

2. *Average number of customers in the system*

$$E(N) = \sum_{n=0}^{k} nP_n = \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \cdot \sum_{n=0}^{k} n\left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \cdot \sum_{n=0}^{k} \frac{\mathrm{d}}{\mathrm{d}x}(x^n), \text{ where } x = \frac{\lambda}{\mu}$$

$$= \frac{\left(1-\frac{\lambda}{\mu}\right)\cdot\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \frac{d}{dx}\left(\frac{1-x^{k+1}}{1-x}\right)$$

$$= \frac{(1-x)\,x}{1-x^{k+1}}\cdot\left[\frac{(1-x)\left\{-(k+1)\,x^k\right\}+\left(1-x^{k+1}\right)}{(1-x)^2}\right]$$

$$= \frac{x\left[1-(k+1)\,x^k+k\,x^{k+1}\right]}{(1-x)\left(1-x^{k+1}\right)}$$

$$= \frac{x\left(1-x^{k+1}\right)-(k+1)(1-x)\,x^{k+1}}{(1-x)\left(1-x^{k+1}\right)}$$

$$= \frac{x}{1-x} - \frac{(k+1)\,x^{k+1}}{1-x^{k+1}}$$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, \text{ if } \lambda \neq \mu \tag{8}$$

and $$E(N) = \sum_{n=0}^{k} \frac{n}{k+1} = \frac{k}{2}, \text{ if } \lambda = \mu \tag{9}$$

3. *Average number of customers in the queue.*

$$E(N_q) = E(N-1) = \sum_{n=1}^{k} (n-1)P_n$$

$$= \sum_{n=0}^{k} n\,P_n - \sum_{n=1}^{k} P_n$$

$$= E(N) - (1-P_0) \tag{10}$$

As per Little's formula (iv),

$$E(N_q) = E(N) - \frac{\lambda}{\mu},$$

which is true when the average arrival rate is $\lambda$ throughout. Now we see that, in step (8), $1 - P_0 \neq \frac{\lambda}{\mu}$, because the average arrival rate is $\lambda$ as long as there is a vacancy in the queue and it is zero when the system is full.

Hence we define *the overall effective arrival rate*, denoted by $\lambda'$ or $\lambda_{eff}$, by u
step (8) and Little's formula as

$$\frac{\lambda'}{\mu} = 1 - P_0 \text{ or } \lambda' = \mu\,(1 - P_0)$$

Thus, step (8) can be rewritten as

$$E(N_q) = E(N) - \frac{\lambda'}{\mu}, \quad (\;$$

which is the modified Little's formula for this model.

4. *Average waiting times in the system and in the queue:*
   By the modified Little's formulas,

$$E(W_s) = \frac{1}{\lambda'}\,E(N) \quad (1$$

and $$E(W_q) = \frac{1}{\lambda'}\,E(N_q) \quad (14$$

where $\lambda'$ is the effective arrival rate, given by step (9).

## Example 13

In a single server queueing system with Poisson input and exponential service times, if the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 h and the maximum possible number of calling units in the system is 2, find $P_n(n \geq 0)$, average number of calling units in the system and in the queue and average waiting time in the system and in the queue.

The situation in this problem is one of finite capacity, single server Poisson queue models.

$$\lambda = 3, \mu = 4 \text{ and } k = 2$$

$$\text{As } \lambda \neq \mu,\ P_0 = \frac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (4) of model III]}$$

$$= \frac{1 - \dfrac{3}{4}}{1 - \left(\dfrac{3}{4}\right)^{3}} = \frac{16}{37} = 0.4324$$

$$\text{Since } \lambda \neq \mu,\ P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}}\right] \quad \text{[by formula (6) of model III]}$$

$$= (0.4324)\,(0.75)^n$$

$$E(N) = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (8) of model IV]}$$

$$= \frac{3}{4-3} - \frac{3 \times \left(\dfrac{3}{4}\right)^3}{1 - \left(\dfrac{3}{4}\right)^3} = 3 - \frac{81}{37} = \frac{30}{37} \simeq 0.8 \text{ calling unit}$$

$$E(N_q) = E(N) - (1 - P_0) \quad \text{[by formula (10) of model III]}$$

$$= \frac{30}{37} - \left(1 - \frac{16}{37}\right) = \frac{9}{37} = 0.24 \text{ calling unit}$$

$$E(W_s) = \frac{1}{\lambda'} E(N) \quad \text{[by formula (13) of model III]}$$

where $\lambda' = \mu(1 - P_0)$, [by formual (11) of model III]

$$= 4\left(1 - \frac{16}{37}\right) = \frac{84}{37}$$

$$\therefore \quad E(W_s) = \frac{37}{84} \times \frac{30}{37} = \frac{5}{14} \text{ h or } 21.4 \text{ min}$$

$$E(W_q) = \frac{1}{\lambda'} E(N_q) \quad \text{[by formula (14) of model III]}$$

$$= \frac{37}{84} \times \frac{9}{37} = \frac{3}{28} \text{ h or } 6.4 \text{ min}$$

## Example 14

The local one-person barber shop can accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair-cut). Customers arrive according to a Poisson distribution with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (Exponential service time).

(a) What percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is the expected number of customers waiting for a hair-cut?

(d) How much time can a customer expect to spend in the barber shop?

$$\lambda = 5, \mu = 4, k = 5$$

(a) $P(\text{the barber is idle}) = P(N = 0)$

$$= P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (4) of model III]}$$

$$= \frac{1 - \frac{5}{4}}{1 - \left(\frac{5}{4}\right)^6} = 0.0888$$

$\therefore$ Percentage of time when the barber is idle $\simeq$ 9.

(b) $P(\text{a customer is turned away}) = P(N > 5)$

$$= \left(\frac{\lambda}{\mu}\right)^5 \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right] \text{ [by formula (6) of model III]}$$

$$= \left(\frac{5}{4}\right)^5 \left[\frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6}\right]$$

$$= \frac{3125}{11529} = 0.2711$$

Therefore, $0.2711 \times$ potential customers are turned away.

(c) $E(N_q) = E(N) - (1 - P_0)$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} - (1-P_0),$$

[by formulas (6) and (10) of model III]

$$= \left\{-5 - \frac{6\times\left(\frac{5}{4}\right)^6}{1-\left(\frac{5}{4}\right)^6}\right\} - (1 - 0.0888)$$

$$= \frac{6\times\frac{15625}{4096}}{\frac{11529}{4096}} - 5.9112 \simeq 2.2 \text{ customers}$$

(d) $E(W) = \frac{1}{\lambda'} E(N)$ [by formula (13) of model III]

$$= \frac{1}{\mu(1-P_0)} \times E(N) = \frac{3.1317}{3.6448} \simeq 0.8592 \text{ h}$$

or 51.5 min

## Example 15

At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals and exponential service distribution, find the probabilities for the numbers of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results get modified?

(i) $\lambda = 6$ per hour, $\mu = 6$ per hour, $k = 2 + 1 = 3$

$$\text{Since } \lambda = \mu,\ P_0 = \frac{1}{k+1}$$

$$= \frac{1}{4} \quad \text{[by formula (5) of model III]}$$

$$P_n = \frac{1}{k+1} = \frac{1}{4} \quad \text{for } n = 1, 2, 3 \text{ [by formula (7) of model III]}$$

$$E(N) = \frac{k}{2} \quad \text{[by formula (9) of model III]}$$

$$= 1.5 \text{ trains}$$

$$E(W) = \frac{1}{\lambda'} E(N) \quad \text{[by formula (13) of model III]}$$

$$= \frac{1.5}{\mu(1-P_0)} = \frac{1.5}{6 \times \frac{3}{4}} = \frac{1}{3} \text{ h or 20 min}$$

(ii) $\lambda = 6;\ \mu = 12,\ k = 3$

$$\text{Since } \lambda \neq \mu,\ P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (4) of model III]}$$

$$= \frac{1-\frac{1}{2}}{1-\left(\frac{1}{2}\right)^4} = \frac{8}{15}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left\{\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right\} \quad \text{[by formula (6) of model III]}$$

$$= \frac{8}{15}\cdot\left(\frac{1}{2}\right)^n, \text{ for } n = 1, 2, 3.$$

$$E(N) = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (8) of model III]}$$

$$= 1 - \frac{4\times\left(\frac{1}{2}\right)^4}{1-\left(\frac{1}{2}\right)^4} = 1 - \frac{4}{15} = \frac{11}{15} \simeq 0.73 \text{ train}$$

$$E(W) = \frac{1}{\lambda'} E(N) \text{ by formula (13) of model III]}$$

$$= \frac{1}{\mu(1-P_0)} \times E(N)$$

$$= \frac{\frac{11}{15}}{12\left(1-\frac{8}{15}\right)} = \frac{11}{84} \text{ h or 7.9 min}$$

## Example 16

Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(a) Find the effective arrival rate at the clinic.

(b) What is the probability that an arriving patient will not wait?

(c) What is the expected waiting time until a patient is discharged from the clinic?

(a) $\lambda = 30$ per hour, $\mu = 20$ per hour, $k = 14 + 1 = 15$

$$\text{Since } \lambda \neq \mu,\ P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} \quad \text{[by formula (4) of model III]}$$

$$= \frac{1-\frac{3}{2}}{1-\left(\frac{3}{2}\right)^{16}} = 0.00076$$

Effective arrival rate $\lambda' = \mu\,(1 - P_0)$ [by formula (11) of model III]

$$= 20 \times (1 - 0.00076)$$

$$= 19.98 \text{ per hour}$$

(b) $P$(a patient will not wait)

$$= P_0 = 0.00076$$

(c) $$E(N) = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= -3 - \frac{16 \times \left(\frac{3}{2}\right)^{16}}{1 - \left(\frac{3}{2}\right)^{16}} = 13 \text{ patients nearly}$$

$$E(W) = \frac{E(N)}{\lambda'} = \frac{13}{19.98} = 0.65 \text{ h or } 39 \text{ min}$$