... in the above three equations gives $x = 5, y = 4, z = -7$.

(5) Factorization method*. This method is based on the fact that every square matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, *i.e.* if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \neq 0, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \neq 0, \text{ etc.}$$

Also such a factorisation if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as $AX = B$ ...(1)

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $A = LU$, ...(2)

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Then (1) becomes $LUX = B$ ...(3)

Writing $UX = V$, ...(4), (3) becomes $LV = B$

which is equivalent to the equations $v_1 = b_1,\ l_{21}v_1 + v_2 = b_2,\ l_{31}v_1 + l_{32}v_2 + v_3 = b_3$

Solving these for v_1, v_2, v_3, we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1,\ u_{22}x_2 + u_{23}x_3 = v_2,\ u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by *back-substitution*.

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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*Another name given to this decomposition is Triangularisation method.

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(i) $u_{11} = a_{11}$, $\quad u_{12} = a_{12}$, $\quad u_{13} = a_{13}$

(ii) $l_{21}u_{11} = a_{21}$ $\quad$ or $\quad l_{21} = a_{21}/a_{11}$; $\quad l_{31}u_{11} = a_{31}$ $\quad$ or $\quad l_{31} = a_{31}/a_{11}$

(iii) $l_{21}u_{12} + u_{22} = a_{22}$ $\quad$ or $\quad u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}$ $\quad$ or $\quad u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}} a_{13}$

(iv) $l_{31}u_{12} + l_{32}u_{22} = a_{32}$ $\quad$ or $\quad l_{32} = \dfrac{1}{u_{22}}\left[a_{32} - \dfrac{a_{31}}{a_{11}} a_{12}\right]$

(v) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order :

(i) First row of U, $\qquad$ (ii) First column of L,

(iii) Second row of U, $\qquad$ (iv) Second column of L,

(v) Third row of U.

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and for finding the inverse of a matrix. The number of operations involved in terms of multiplications for a system of 10 equations by this method is about 110 as compared 333 operations of the Gauss method. Among the direct methods, factorization method is also preferred as the software for computers.

Example 3.23. *Apply factorization method to solve the equations :*

$$3x + 2y + 7z = 4 \; ; \; 2x + 3y + z = 5 \; ; \; 3x + 4y + z = 7.$$

Sol. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (i.e. A),

so that

(i) R_1 of U : $u_{11} = 3$, $\quad u_{12} = 2$, $\quad u_{13} = 7$.

(ii) C_1 of L : $l_{21}u_{11} = 2$, $\quad \therefore \; l_{21} = 2/3$,

$l_{31}u_{11} = 3$, $\quad \therefore \; l_{31} = 1$.

(iii) R_2 of U : $l_{21}u_{12} + u_{22} = 3$, $\quad \therefore \; u_{22} = 5/3$,

$l_{21}u_{13} + u_{23} = 1$, $\quad \therefore \; u_{23} = -11/3$.

(iv) C_2 of L : $l_{31}u_{12} + l_{32}u_{22} = 4$ $\quad \therefore \; l_{32} = 6/5$.

(v) R_3 of U : $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$ $\quad \therefore \; u_{33} = -8/5$.

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

Solving this system, we have $v_1 = 4$,

$$\frac{2}{3}v_1 + v_2 = 5 \qquad \text{or} \qquad v_2 = \frac{7}{3}$$

$$v_1 + \frac{6}{5}v_2 + v_3 = 7 \qquad \text{or} \qquad v_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e. $\qquad 3x + 2y + 7z = 4, \quad \frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}, \quad -\frac{8}{5}z = \frac{1}{5}$

By back-substitution, we have

$$z = -1/8,\ y = 9/8 \quad \text{and} \quad x = 7/8.$$

Example 3.24. *Solve the equations $10x - 7y + 3z + 5u = 6$; $-6x + 8y - z - 4u = 5$; $3x + y + 4z + 11u = 2$; $5x - 9y - 2z + 4u = 7$ by Factorization method.* (*cf. Example 3.19*)

Sol. Let
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} \quad (i.e.\ A)$$

so that

(*i*) R_1 of U : $u_{11} = 10, u_{12} = -7, u_{13} = 3, u_{14} = 5$

(*ii*) C_1 of L : $l_{21} = -0.6, l_{31} = 0.3, l_{41} = 0.5$

(*iii*) R_2 of U : $u_{22} = 3.8, u_{23} = 0.8, u_{24} = -1$

(*iv*) C_2 of L : $l_{32} = 0.81579, l_{42} = -1.44737$

(*v*) R_3 of U : $u_{33} = 2.44737, u_{34} = 10.31579$

(*vi*) C_3 of L : $l_{43} = -0.95699$

(*vii*) R_4 of U : $u_{44} = 9.92474$

Thus
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81577 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6,\ v_2 = 8.6,\ v_3 = -6.81579,\ v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e.

$$10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6,$$

$$2.44737z + 10.31579u = -6.81579, \; u = 1.$$

By back-substitution, we get

$$u = 1, z = -7, y = 4, x = 5.$$