Chapter 7

Special Random Processes

Many random phenomena in physical problems including 'noise' are well approximated by a special class of random process, namely Gaussian random process. A number of processes such as the Wiener process and the shot-noise process can be approximated, as per central limit theorem, by a Gaussian process. Moreover the output of a linear system in which the input is a weighted sum of a large number of independent samples of a random process tends to approach a Gaussian process. Gaussian processes play an important role in the theory and analysis of random phenomena, because they are good approximations to the observations and multivariate Gaussian distributions are analytically simple.

One of the most important uses of the Gaussian process is to model and analyse the effects of thermal noise in electronic circuits used in communication systems. Individual circuits contain resistors, inductors and capacitors as well as semiconductor devices. The resistors and semiconductor elements contain charged particles (free electrons) subjected to random motion due to thermal energy. The random motion of charged particles causes fluctuations in the current waveforms or information bearing signals that flow through these components. These fluctuations are called thermal noise, which are of sufficient strength to disturb a weak signal and to make the recognition of signals a difficult task. Models of thermal noise are used to identify and minimise the effects of noise in signal recognition.

Definition of a Gaussian Process

A real valued random process $\{X(t)\}$ is called a Gaussian process or normal process, if the random variables $X(t_1)$, $X(t_2)$, ..., $X(t_n)$ are jointly normal for every $n = 1, 2, \ldots$ and for any set of t_i's.

The nth order density of a Gaussian process is given by $f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$

$$= \frac{1}{(2\pi)^{n/2}\,|\Lambda|^{1/2}} \exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n} |\Lambda|_{ij}\,(x_i - \mu_i)(x_j - \mu_j)\right]$$

where $\mu_i = E\{X(t_i)\}$ and Λ is the nth order square matrix (λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$ (1)

Note *Gaussian process is completely specified by the first and second order moments, viz., means and covariances (variances).*

Note *When we consider the first order density of a Gaussian process,*

$$\Lambda = (\lambda_{11}) = [cov\{X(t_1), X(t_1)\}]$$
$$= [Var\{X(t_1)\}] = (\sigma_1^2)$$

$$\therefore \quad |\Lambda| = \sigma_1^2 \text{ and } |\Lambda|_{11} = 1$$

$$\therefore \quad f(x_1, t_1) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\{-(x_1 - \mu_1)^2/2\sigma_1^2\}$$

Note *When we consider the second order density of a Gaussian process,*

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 \\ r_{21}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\therefore \quad |\Lambda| = \sigma_1^2\sigma_2^2(1 - r^2), \text{ where } r_{12} = r_{21} = r$$

$$|\Lambda|_{11} = \sigma_2^2, \; |\Lambda|_{12} = -r\sigma_1\sigma_2, \; |\Lambda|_{21} = -r\sigma_1\sigma_2, \; |\Lambda|_{22} = \sigma_1^2$$

$$\therefore f(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2(1-r^2)}\{\sigma_2^2(x_1-\mu_1)^2 - 2r\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\}\right]$$

$$\text{i.e., } f(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2r(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}\right]$$

which we have made use of in many problems earlier.

Properties

1. If a Gaussian process is wide-sense stationary, it is also strict-sense stationary. (MU — Apr. 96; BDU — Apr. 96)

... and autocorrelation function of the output.

Poisson Process

There are many practical situations where the random times of occurrences of some specific events are of primary interest. For example, we may want to study the times at which components fail in a large system or the times at which jobs enter the queue in a computer system or the times of arrival of phone calls at an exchange or the times of emission of electrons from the cathode of a vacuum tube. In these examples, our main interest may not be the event itself but the sequence of random time instants at which the events occur. An ensemble of discrete sets of points from the time domain called a ***point process*** is used to model and analyse phenomena such as the ones mentioned above. An independent increments point process, i.e., a point process with the property that the number of occurrences in any finite collection of nonoverlapping time intervals are independent RVs, leads to a Poisson process.

Definition: If $X(t)$ represents the number of occurrences of a certain event in $(0, t)$, then the discrete random process $\{X(t)\}$ is called the Poisson process, provided the following postulates are satisfied:

(i) $P[1 \text{ occurrence in } (t, t+\Delta t)] = \lambda \Delta t + 0(\Delta t)$
(ii) $P[0 \text{ occurrence in } (t, t+\Delta t)] = 1 - \lambda \Delta t + 0(\Delta t)$
(iii) $P[2 \text{ or more occurrences in } (t, t+\Delta t) = 0(\Delta t)$
(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.
(v) The probability that the event occurs a specified number of times in $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Probability Law for the Poisson Process $\{X(t)\}$

Let λ be the number of occurrences of the event in unit time.

Let $\quad P_n(t) = P\{X(t) = n\}$

$$\begin{aligned}
\therefore \quad P_n(t+\Delta t) &= P\{X(t+\Delta t) = n\} \\
&= P\{(n-1) \text{ calls in } (0, t) \text{ and } 1 \text{ call in } (t, t+\Delta t)\} \\
&\quad + P\{n \text{ calls in } (0, t) \text{ and no call in } (t, t+\Delta t)\} \\
&= P_{n-1}(t)\, \lambda \Delta t + P_n(t)\,(1 - \lambda \Delta t) \text{ (by the postulates)}
\end{aligned}$$

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda \{P_{n-1}(t) - P_n(t)\}$$

Taking the limits as $\Delta t \to 0$

$$\frac{d}{dt} P_n(t) = \lambda \{P_{n-1}(t) - P_n(t)\} \quad (1)$$

Let the solution of the equation (1) be

$$P_n(t) = \frac{(\lambda t)^n}{\lfloor n} f(t) \quad (2)$$

Differentiating (2) with respect to t,

$$P'_n(t) = \frac{\lambda^n}{\lfloor n} \{n t^{n-1} f(t) + t^n f'(t)\} \quad (3)$$

Using (2) and (3) in (1),

$$\frac{\lambda^n}{\lfloor n} t^n f'(t) = -\lambda \frac{(\lambda t)^n}{\lfloor n} f(t)$$

i.e., $\quad f'(t) = -\lambda f(t)$

$$\therefore \quad f(t) = k e^{-\lambda t} \quad (4)$$

$$\begin{aligned}
\text{From (2), } \quad f(0) = P_0(0) &= P\{X(0) = 0\} \\
&= P\{\text{no event occurs in } (0, 0)\} \\
&= 1 \quad (5)
\end{aligned}$$

Using (5) in (4), we get $k = 1$ and hence

$$f(t) = e^{-\lambda t} \quad (6)$$

Using (6) in (2),

$$P_n(t) = P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{\lfloor n}, n = 0, 1, 2, \ldots, \infty$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

Note *We have assumed that the rate of occurrence of the event λ is a constant, but it can be function of t also. When λ is a constant, the process is called a* **homogeneous Poisson process**. *Unless specified otherwise, the Poisson process will be assumed homogeneous.*

Second-Order Probability Function of a Homogeneous Poisson Process

$$P[X(t_1) = n_1, X(t_2) = n_2]$$

$$= P[X(t_1) = n_1]\, P[X(t_2) = n_2/X(t_1) = n_1], \quad t_2 > t_1$$

$= P[X(t_1) = n_1]\, P$[the event occurs $(n_2 - n_1)$ times in the interval of length $(t_2 - t_1)$]

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{\lfloor n_1} \cdot \frac{e^{-\lambda(t_2 - t_1)}\{\lambda(t_2 - t_1)\}^{n_2 - n_1}}{\lfloor n_2 - n_1}, \text{ if } n_2 \geq n_1$$

$$= \begin{cases} \dfrac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}}{\lfloor n_1 \cdot \lfloor n_2 - n_1}, & n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

Proceeding similarly, we can get the third-order probability function as

$$P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]$$

$$= \begin{cases} \dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2 - t_1)^{n_2 - n_1}(t_3 - t_2)^{n_3 - n_2}}{\lfloor n_1 \lfloor n_2 - n_1 \lfloor n_3 - n_2}, & n_3 \geq n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

Mean and Autocorrelation of the Poisson Process

The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

$$\therefore \quad E\{X(t)\} = \text{Var}\{X(t)\} = \lambda t \qquad (1)$$

$$\therefore \quad E\{X^2(t)\} = \lambda t + \lambda^2 t^2$$

$$R_{xx}(t_1, t_2) = E\{X(t_1)\, X(t_2)\}$$

$$= E[X(t_1)\{X(t_2) - X(t_1) + X(t_1)\}]$$

$$= E[X(t_1)\{X(t_2) - X(t_1)\}] + E\{X^2(t_1)\}$$

$$= E[X(t_1)]\, E[X(t_2) - X(t_1)] + E\{X^2(t_1)\},$$

since $\{X(t)\}$ is a process of independent increments.

$$= \lambda t_1 [\lambda (t_2 - t_1)] + \lambda t_1 + \lambda^2 t_1^2, \text{ if } t_2 \geq t_1 \qquad \text{[by (1)]}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1, \text{ if } t_2 \geq t_1$$

or $$R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min (t_1, t_2)$$

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E\{X(t_1)\} E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1, \text{ if } t_2 \geq t_1$$

or $$= \min (t_1, t_2)$$

$$r_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{\operatorname{var}\{X(t_1)\} \operatorname{var}\{X(t_2)\}}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda t_1 \lambda t_2}} = \sqrt{\frac{t_1}{t_2}}, \text{ if } t_2 \geq t_1$$

Note *Poisson process is not a stationary process.*

Properties of Poisson Process

1. The Poisson process is a Markov process.

Proof

Consider $P[X(t_3) = n_3/X(t_2) = n_2, X(t_1) = n_1]$

$$= \frac{P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda (t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{\lfloor n_3 - n_2}$$

[refer to the second-and third-order probability functions of the Poisson process]

$$= P[X(t_3) = n_3/X(t_2) = n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1$, $X(t_2) = n_2$ depends only on the most recent value $X(t_2) = n_2$.

That is, the Poisson process possesses the Markov property. Hence the result.

2. **Additive property:** Sum of two independent Poisson processes is a Poisson process.

Proof

We have already derived in Chapter IV the characteristic function of a Poisson distribution with parameter λ as $e^{-\lambda (1 - e^{i\omega})}$

Therefore, the characteristic functions of $X_1(t)$ and $X_2(t)$ are given by

$$\phi_{X_1(t)}(\omega) = e^{-\lambda_1 t(1 - e^{i\omega})} \text{ and } \phi_{X_2(t)}(\omega) = e^{-\lambda_2 t(1 - e^{i\omega})}$$

Since $X_1(t)$ and $X_2(t)$ are independent,

$$\phi_{X_1(t)+X_2(t)}(\omega) = \phi_{X_1(t)}(\omega)\,\phi_{X_2(t)}(\omega)$$
$$= e^{-(\lambda_1+\lambda_2)t\,(1-e^{i\omega})}$$

which is the characteristic function of Poisson distribution with parameter $(\lambda_1 + \lambda_2)t$.

Therefore, $\{X_1(t) + X_2(t)\}$ is a Poisson process.

Alternative proof

Let $\quad X(t) = X_1(t) + X_2(t).$

$$P\{X(t) = n\} = \sum_{r=0}^{n} P\{X_1(t) = r\}\, P\{X_2(t) = n - r\}$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{\lfloor r} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{\lfloor n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \frac{1}{\lfloor n} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \left[(\lambda_1+\lambda_2)t\right]^n / \lfloor n$$

Therefore, $X_1(t) + X_2(t)\}$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)t$.

Note *The additive property holds good for any number of independent Poisson processes.*

3. Difference of two independent Poisson processes is not a Poisson process.

Proof

Let $\quad X(t) = X_1(t) - X_2(t)$

$$E\{X(t)\} = E\{X_1(t)\} - E\{X_2(t)\}$$
$$= (\lambda_1 - \lambda_2)t$$
$$E\{X^2(t)\} = E\{X_1^2(t)\} + E\{X_2^2(t)\} - 2E\{X_1(t)\}\,E(X_2(t)\}$$

(by independence)

$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2(\lambda_1 t)(\lambda_2 t)$$
$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$
$$\neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Recall that $E\{X^2(t)\}$ for a Poisson process $\{X(t)\}$ with parameter λ is given by $E\{X^2(t)\} = \lambda t + \lambda^2 t^2$.

Therefore, $\{X_1(t) - X_2(t)\}$ is not a Poisson process.

4. The interarrival time of a Poisson process, i.e., the interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution with mean $1/\lambda$.

Proof

Let two consecutive occurrences of the event be E_i and E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i and E_{i+1}. T is a continuous RV.

$$\begin{aligned} P(T > t) &= P\{E_{i+1} \text{ did not occur in } (t_i, t_i + t)\} \\ &= P\{\text{No event occurs in an interval of length } t\} \\ &= P\{X(t) = 0\} \\ &= e^{-\lambda t} \end{aligned}$$

Therefore, the cdf of T is given by

$$F(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

Therefore, the pdf of T is given by

$$f(t) = \lambda \ e^{-\lambda t} \quad (t \ge 0)$$

which is an exponential distribution with mean $1/\lambda$.

5. If the number of occurrences of an event E is an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrence of E has a constant probability p of being recorded and the recordings are independent of each other, then the number $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter λp.

Proof

$$\begin{aligned} P\{N(t) = n\} &= \sum_{r=0}^{\infty} P\{E \text{ occurs } (n+r) \text{ times in } t \text{ and } n \text{ of them are recorded}\} \\ &= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^{n+r}}{\lfloor n+r} (n+r)C_n p^n q^r, \quad q = 1 - p \\ &= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^{n+r}}{\lfloor n+r} \frac{\lfloor n+r}{\lfloor n \lfloor r} p^n q^r \\ &= \frac{e^{-\lambda t} (\lambda p t)^n}{\lfloor n} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{\lfloor r} \\ &= \frac{e^{-\lambda t} (\lambda p t)^n}{\lfloor n} e^{\lambda q t} \\ &= \frac{e^{-\lambda p t} (\lambda p t)^n}{\lfloor n}. \end{aligned}$$

Worked Example 7(B)

Example 1

Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute; find the probability that during a time interval of 2 min (i) exactly 4 customers arrive and (ii) more than 4 customers arrive.

(MU — Apr. 96)

Mean of the Poisson process $= \lambda t$
Mean arrival rate = mean number of arrivals per minute (unit time) $= \lambda$
Given $\lambda = 3$

$$P\{X(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{\lfloor k}$$

$$\therefore P\{X(2) = 4\} = \frac{e^{-6}6^4}{\lfloor 4} = 0.133$$

$$P\{X(2) > 4\} = 1 - [P\{X(2) = 0\} + P\{X(2) = 1\} + P\{X(2) = 2\} + P\{X(2) = 3\} + P\{X(2) = 4\}]$$

$$= 1 - \sum_{k=0}^{4} e^{-6}6^k/\lfloor k$$

$$= 0.715$$

Example 2

A machine goes out of order, whenever a component fails. The failure of this part follows a Poisson process with a mean rate of 1 per week. Find the probability that 2 weeks have elapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks.

(i) Here the unit time is 1 week.

Mean failure rate = mean number of failures in a week $= \lambda = 1$.

P{no failures in the 2 weeks since last failure}

$$= P\{X(2) = 0\}$$

$$= \frac{e^{-2\lambda}(2\lambda)^0}{\lfloor 0} = e^{-2} = 0.135$$

(ii) There are only 5 spare parts and the machine should not go out of order in the next 10 weeks.

$$P\{\text{for this event}\} = P\{X(10) \le 5\}$$

$$= \sum_{k=0}^{5} \frac{e^{-10}10^k}{\lfloor k}$$

$$= 0.068$$

Example 3

If $\{N_1(t)\}$ and $\{N_2(t)\}$ are 2 independent Poisson processes with parameters λ_1 and λ_2 respectively, show that

$P[N_1(t) = k/\{N_1(t) + N_2(t) = n\}] = nC_k\ p^k\ q^{n-k}$, where

$P = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and $q = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$.

Required conditional probability

$$= \frac{P[\{N_1(t) = k\} \cap \{N_1(t) + N_2(t) = n\}]}{P\{N_1(t) + N_2(t) = n\}}$$

$$= \frac{P[\{N_1(t) = k\} \cap \{N_2(t) = n-k\}]}{P\{N_1(t) + N_2(t) = n\}}$$

$$= \frac{\dfrac{e^{-\lambda_1 t}(\lambda_1 t)^k}{\lfloor k} \times \dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{\lfloor n-k}}{\dfrac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{\lfloor n}}$$

(by indepence and additive property)

$$= \frac{\lfloor n}{\lfloor k\ \lfloor n-k} \frac{(\lambda_1 t)^k (\lambda_2 t)^{n-k}}{\{(\lambda_1+\lambda_2)t\}^n}$$

$$= nC_k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= nC_k\ p^k\ q^{n-k}$$

Example 4

If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is (i) more than 1 min, (ii) between 1 min and 2 min and (iii) 4 min or less.

Refer to Property 4 of Poisson processes.

The interval T between 2 consecutive arrivals follows an exponential distribution with parameter $\lambda = 2$.

(i) $P(T > 1) = \int_1^{\infty} 2e^{-2t}\,dt = e^{-2} = 0.135$

(ii) $P(1 < T < 2) = \int_1^{2} 2e^{-2t}\,dt = e^{-2} - e^{-4} = 0.117$

(iii) $P(T \leq 4) = \int_0^{4} 2e^{-2t}\,dt = 1 - e^{-8} = 0.999$

Example 5

A radioactive source emits particles at a rate of 5 per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4-min period.

Refer to Property 5 of Poisson processes. The number of recorded particles $N(t)$ follows a Poisson process with parameter λp.

Here $\lambda = 5$ and $p = 0.6$.

$$\therefore \quad P\{N(t) = k\} = \frac{e^{-3t}(3t)^k}{\lfloor k}$$

$$\therefore \quad P\{N(4) = 10\} = \frac{e^{-12}(12)^{10}}{\lfloor 10}$$

$$= 0.104$$

Markov Process

Another interesting model of a random process is the one in which the value of the random process depends only upon the most recent previous value and is independent of all values in the more distant past. Such a model is called a Markov model and is often described by saying that a Markov process is one in which the future value is independent of the past values, given the present value. Models in which the future depends only upon the present are common among electrical engineering models.

Consider the experiment of tossing a fair coin a number of times. The possible outcomes at each trial are two—'head' with probability 1/2 and 'tail' with probability 1/2. If we denote the outcome of the nth toss, which is a RV, by X_n and the outcomes 'head' and 'tail' by 1 and 0 respectively, then

$$P\{X_n = 1\} = \frac{1}{2} \text{ and } P\{X_n = 0\} = \frac{1}{2};\ n = 1, 2, \ldots$$

Thus we have a sequence of independent RVs $X_1, X_2, \ldots$, since the trials are independent and hence the outcome of the nth trial does not depend in any way on the previous trials.

Consider now the RV that represents the total number of heads in the first n trials and is given by $S_n = X_1 + \ldots + X_n$. The possible values of S_n are 0, 1, 2, ..., n. If $S_n = k (k = 0, 1, \ldots, n)$, then the RV S_{n+1} $(= S_n + X_{n+1})$ can assume only 2 possible values, namely $k + 1$ [if the $(n + 1)$th trial results in a head] and k [if the $(n + 1)$th trial results in a tail].

Thus $$P\{S_{n+1} = k + 1/S_n = k\} = \frac{1}{2}$$

$$P\{S_{n+1} = k/S_n = k\} = \frac{1}{2}$$

These probabilities are not at all affected by the values of the RVs $S_1, S_2, \ldots, S_{n-1}$. Also the conditional probability of S_{n+1} given S_n depends on the value of S_n and not on the manner in which the value of S_n was reached. This is a simple example of a Markov chain. Random processes $\{X(t)\}$ (with Markov property) which take discrete values, whether t is discrete or continuous, are called ***Markov chains***. Poisson process, discussed earlier, is a continuous time Markov chain. In this section, we will discuss discrete time Markov chains.

Definition of a Markov Chain

If, for all n, $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}\}$, then the process $\{X_n\}$, $n = 0, 1, \ldots$, is called a Markov chain.

$(a_1, a_2, \ldots, a_n, \ldots)$ are called the states of the Markov chain. The conditional probability $P\{X_n = a_j/X_{n-1} = a_i\}$ is called the ***one-step transition probability*** from state a_i to state a_j at the nth step (trial) and is denoted by $p_{ij}(n-1, n)$.

If the one-step transition probability does not depend on the step, i.e., $p_{ij}(n-1, n) = p_{ij}(m-1, m)$ the Markov chain is called a ***homogeneous Markov chain*** or the chain is said to have stationary transition probabilities. The use of the word 'stationary' does not imply a stationary random sequence.

When the Markov chain is homogeneous, the one-step transition probability is denoted by p_{ij}. The matrix $P = \{p_{ij}\}$ is called (one-step) ***transition probability matrix***, shortly, tpm.

Note *The tpm of a Markov chain is a stochastic matrix, since $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all i, i.e., the sum of all the elements of any row of the tpm is 1. This is obvious because the transition from state a_i to any one of the states (including a_i itself) is a certain event.*

The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, i.e., $P\{X_n = a_j/X_0 = a_i\}$ is called the ***n-step transition probability*** and denoted by $p_{ij}(n)$.

Note $p_{ij}^{(1)} = p_{ij}$.

Let us consider an example in which we explain how the tpm is formed for a Markov chain. Assume that a man is at an integral point of the x-axis between the origin and the point $x = 3$. He takes a unit step either to the right with probability 0.7 or to the left with probability 0.3, unless he is at the origin when he takes a step to the right to reach $x = 1$ or he is at the point $x = 3$, when he takes a step to the left to reach $x = 2$. The chain is called 'Random walk with reflecting barriers'. The tpm is given below:

$$
\begin{array}{cc}
 & \text{States of } X_n \\
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\text{States of } X_{n-1} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

> **Note** p_{23} = *the element in the 2nd row, 3rd column of this tpm = 0.7. This means that, if the process is at state 2 at step $(n-1)$, the probability that it moves to state 3 at step $n = 0.7$, where n is any positive integer.*

Definition: If the probability that the process is in state a_i is p_i $(i = 1, 2, \ldots, k)$ at any arbitrary step, then the row vector $p = (p_1, p_2, \ldots, p_k)$ is called ***the probability distribution of the process*** at that time. In particular, $p^{(0)} = \{p_1^{(0)}, p_2^{(0)}, \ldots, p_k^{(0)}\}$ is the initial probability distribution.

[**Remark:** The transition probability matrix together with the initial probability distribution completely specifies a Markov chain $\{X_n\}$. In the example given above, let us assume that the initial probability distribution of the chain is $p^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

i.e., $\quad P\{X_0 = i\} = 1/4,\ i = 0, 1, 2, 3$

Then we have, for the example given above,

$$P\{X_1 = 2/X_0 = 1\} = 0.7; \quad P\{X_2 = 1/X_1 = 2\} = 0.3,$$

$$\begin{aligned}P\{X_2 = 1, X_1 = 2/X_0 = 1\} &= P\{X_2 = 1/X_1 = 2\} \times P\{X_1 = 2/X_0 = 1\}\\ &= 0.3 \times 0.7 = 0.21 \qquad (1)\end{aligned}$$

$$\begin{aligned}P\{X_2 = 1, X_1 = 2, X_0 = 1\} &= P\{X_0 = 1\} \times P\{X_2 = 1, X_1 = 2/X_0 = 1\}\\ &= 1/4 \times 0.21 = 0.0525 \quad \text{[by (1)]} \qquad (2)\end{aligned}$$

$$\begin{aligned}P\{X_3 = 3, X_2 = 1, X_1 = 2, X_0 = 1\} &= P\{X_2 = 1, X_1 = 2, X_0 = 1\}\\ &\qquad \times P\{X_3 = 3/X_2 = 1, X_1 = 2, X_0 = 1\}\\ &= 0.0525\, P\{X_3 = 3/X_2 = 1\} \quad \text{(Markov property) [by (2)]}\\ &= 0.0525 \times 0 = 0\end{aligned}$$

Chapman–Kolmogorov Theorem

If P is the tpm of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n.

i.e., $\quad = [p_{ij}^{(n)}] = [p_{ij}]^n$

Proof

$$P_{ij}^{(2)} = P\{X_2 = j/X_0 = i\}, \text{ since the chain is homogeneous.}$$

The state j can be reached from the state i in 2 steps through some intermediate state k.

Now $\quad p_{ij}^{(2)} = P\{X_2 = j/X_0 = i\} = P\{X_2 = j, X_1 = k/X_0 = i\}$

$$= P\{X_2 = j/X_1 = k, X_0 = i\}\ P\{X_1 = k/X_0 = i\}$$

Classification of States of a Markov Chain

If $p_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be ***irreducible***. The tpm of an irreducible chain is an irreducible matrix. Otherwise, the chain is said to be ***nonirreducible*** or ***reducible***.

State i of a Markov chain is called a ***return state***, if $p_{ii}^{(n)} > 0$ for some $n > 1$.

The ***period*** d_i of a return state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$, i.e., $d_i = \text{GCD}\{m: p_{ii}^{(m)} > 0\}$. State i is said to be ***periodic*** with period d_i if $d_i > 1$ and ***aperiodic*** if $d_i = 1$.

Obviously state i is aperiodic if $p_{ii} \neq 0$. The probability that the chain returns to state i, having started from state i, for the first time at the *nth* step (or after n transitions) is denoted by $f_{ii}^{(n)}$ and *called the* ***first return time probability*** or *the* ***recurrence time probability***. $\{n, f_{ii}^{(n)}\}$, $n = 1, 2, 3, \ldots$, is the distribution of recurrence times of the state i.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called *the* ***mean recurrence time*** of the state i.

A state i is said to be ***persistent*** or ***recurrent*** if the return to state i is certain, i.e., if $F_{ii} = 1$. The state i is said to be ***transient*** if the return to state i is uncertain, i.e., if $F_{ii} < 1$. The state i is said to be ***nonnull persistent*** if its mean recurrence time μ_{ii} is finite and ***null persistent***, if $\mu_{ii} = \infty$.

A nonnull persistent and aperiodic state is called ***ergodic***.

We give below two theorems, without proof, which will be helpful to classify the states of a Markov chain.

Example 1

The transition probability matrix of a Markov chain $\{X_n\}, n = 1, 2, 3, \ldots$, having 3 states 1, 2 and 3 is

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

and the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$.
Find (i) $P\{X_2 = 3\}$ and (ii) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$.

$$P^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

$$\text{(i)} \quad P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3/X_0 = i\} \times P\{X_0 = i\}$$

$$= p_{13}^{(2)} P\{(X_0 = 1) + p_{23}^{(2)} P(X_0 = 2) + p_{33}^{(2)} P(X_0 = 3)$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$

$$= 0.182 + 0.068 + 0.029$$

$$= 0.279$$

$$\text{(ii)} \quad P\{X_1 = 3/X_0 = 2\} = p_{23} = 0.2 \qquad (1)$$

$$P\{X_1 = 3, X_0 = 2\} = P\{X_1 = 3/X_0 = 2\} \times P\{X_0 = 2\}$$

$$= 0.2 \times 0.2 = 0.04 \text{ [by (1)]} \qquad (2)$$

$$P\{X_2 = 3, X_1 = 3, X_0 = 2\} = P\{X_2 = 3/X_1 = 3, X_0 = 2\} \times P\{X_1 = 3, X_0 = 2\}$$

$$= P\{X_2 = 3/X_1 = 3\} \times P\{X_1 = 3, X_0 = 2\}$$

(by Markov property)

$$= 0.3 \times 0.04 \text{ [by (2)]}$$

$$= 0.012 \qquad (3)$$

$$P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2/X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$\times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2/X_2 = 3\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

(by Markov property)

$= 0.4 \times 0.012$ [by (3)]

$= 0.0048$

Example 2

A fair die is tossed repeatedly. If X_n denotes the maximum of the numbers occuring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$.

Find also P^2 and $P(X_2 = 6)$

State space: $\{1, 2, 3, 4, 5, 6\}$

The tpm is formed using the following analysis.

Let X_n = the maximum of the numbers occurring in the first n trials = 3, say

Then $X_{n+1} = 3$, if the $(n+1)$th trial results in 1, 2 or 3

$= 4$, if the $(n+1)$th trial results in 4

$= 5$, if the $(n+1)$th trial results in 5

$= 6$, if the $(n+1)$th trial results in 6

$$\therefore P\{X_{n+1} = 3/X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$P\{X_{n+1} = i/X_n = 3\} = \frac{1}{6}, \text{ when } i = 4, 5, 6$$

Therefore, the transition probability matrix of the chain is

$$P = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^2 = \frac{1}{36}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

Initial state probability distribution is $p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ since all the values 1, 2, ..., 6 are equally likely.

$$P\{X_2 = 6\} = \sum_{i=1}^{6} P\{X_2 = 6/X_0 = i\} \times P\{X_0 = i\}$$

$$= \frac{1}{6} \sum_{i=1}^{6} p_{i6}^{(2)}$$

$$= \frac{1}{6} \times \frac{1}{36} \times (11 + 11 + 11 + 11 + 11 + 36)$$

$$= \frac{91}{216}$$

Example 3

A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. Find (i) the probability that he takes a train on the third day and (ii) the probability that he drives to work in the long run.

The travel pattern is a Markov chain, with state space = (train, car)

The tpm of the chain is

$$P = \begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

The initial state probability distribution is $p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$,

since P(travelling by car) = P(getting 6 in the toss of the die)

$$= \frac{1}{6}$$

and P(travelling by train) $= \frac{5}{6}$

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on the third day) $= \frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π, $\pi P = \pi$

i.e., $$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

i.e., $$\frac{1}{2}\pi_2 = \pi_1 \qquad (1)$$

and $$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \qquad (2)$$

Equations (1) and (2) are one and the same.

Therefore, consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

Solving, $\pi_1 = \frac{1}{3}$ and $\pi_2 = \frac{2}{3}$

$\therefore P\{\text{the man travels by car in the long run}\} = \frac{2}{3}$.

$$= \frac{a\left(1+r^m\right)}{1+(a-b)\,r^m}, \text{ where } b = 1-a$$

Example 5

A gambler has Rs 2/-. He bets Re. 1 at a time and wins Re. 1 with probability 1/2. He stops playing if he loses Rs 2 or wins Rs. 4 (a) What is the tpm of the related Markov chain? (b) What is the probability that he has lost his money at the end of 5 plays? (c) What is the probability that the game lasts more than 7 plays?

Let X_n represent the amount with the player at the end of the nth round of the play.

State space of $\{X_n\} = (0, 1, 2, 3, 4, 5, 6)$, as the game ends, if the player loses all the money ($X_n = 0$) or wins Rs. 4, i.e., has Rs. 6 ($X_n = 6$). The tpm of the Markov chain is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note *This is called a random walk with* ***absorbing barriers*** *at 0 and 6, since the chain cannot come out of the states 0 and 6, once it has entered them.*

The initial probability distribution of $\{X_n\}$ is $p(0) = (0, 0, 1, 0, 0, 0, 0)$, as the player has got Rs. 2/- to start with.

$$p^{(1)} = p^{(0)}\,P = (0,\ 1/2,\ 0,\ 1/2.\ 0,\ 0,\ 0)$$
$$p^{(2)} = p^{(1)}\,P = (1/4,\ 0,\ 1/2,\ 0,\ 1/4,\ 0,\ 0)$$
$$p^{(3)} = p^{(2)}\,P = (1/4,\ 1/4,\ 0,\ 3/8,\ 0,\ 1/8,\ 0)$$
$$p^{(4)} = p^{(3)}\,P = (3/8,\ 0,\ 5/16,\ 0,\ 1/4,\ 0,\ 1/16)$$
$$p^{(5)} = p^{(4)}\,P = (3/8,\ 5/32,\ 0,\ 9/32,\ 0,\ 1/8,\ 1/16)$$

$P\{$the man has lost his money at the end of 5 plays$\}$

$= P\{X_5 = 0\}$ = the entry corresponding to state 0 in $p^{(5)}$

$= 3/8$

Again $p^{(6)} = p^{(5)}\,P = \left(\frac{29}{64}, 0, \frac{7}{32}, 0, \frac{13}{64}, 0, \frac{1}{8}\right)$

$p^{(7)} = p^{(6)}\,P = \left(\frac{29}{64}, \frac{7}{64}, 0, \frac{27}{128}, 0, \frac{13}{128}, \frac{1}{8}\right)$

$P\{$the game lasts more than 7 rounds$\} = P\{$the system is neither in state 0 nor in 6 at the end of the seventh round$\}$

Example 7

Find the nature of the states of the Markov chain with the tpm

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}; \; P^3 = P$$

$\therefore \qquad P^4 = P^2$

and so on. In general, $P^{2n} = P^2,\ P^{2n+1} = P$

We note that $\qquad p_{00}^{(2)} > 0,\ p_{01}^{(1)} > 0,\ p_{02}^{(2)} > 0$

$$p_{10}^{(1)} > 0,\ p_{11}^{(2)} > 0,\ p_{12}^{(1)} > 0$$

$$p_{20}^{(2)} > 0,\ p_{21}^{(1)} > 0,\ p_{22}^{(2)} > 0$$

Therefore, the Markov chain is irreducible.

Also $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \ldots > 0$, for all i, all the states of the chain are periodic, with period 2.

Since the chain is finite and irreducible, all its states are nonnull persistent. All states are not ergodic.

Example 8

Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states. (MKU — Nov. 96)

The transition probability matrix of the process $\{X_n\}$ is given below:

$$\text{State of } X_{n-1} \begin{array}{c} \\ A \\ B \\ C \end{array} \overset{\text{State of } X_n}{\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}} \equiv P, \text{ say}$$

States of X_n depend only on states of X_{n-1}, but not on states of $X_{n-2}, X_{n-3}, \ldots$, or earlier states. Therefore, $\{X_n\}$ is a Markov chain.

Now $\qquad P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix};\ P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

$p_{11}^{(3)} > 0,\ p_{13}^{(2)} > 0,\ p_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ p_{33}^{(2)} > 0$ and all other $p_{ij}^{(1)} > 0$. Therefore, the chain is irreducible.

$$P^4 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix};\ P^5 = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix};\ P^6 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 3/8 & 1/2 \\ 1/8 & 3/8 & 3/8 \end{pmatrix}$$

and so on.

We note that $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(5)}; p_{ii}^{(6)}$ etc are > 0 for $i = 2, 3$, and GCD of $2, 3, 5, 6, \ldots = 1$.

Therefore, the states 2 and 3 (i.e., B and C) are periodic with period 1. i.e., aperiodic.

We note that $p_{11}^{(3)}, p_{11}^{(5)}, p_{11}^{(6)}$ etc. are > 0 and GCD of $3, 5, 6, \ldots = 1$

Therefore, the state 1 (i.e., state A) is periodic with p...
Since the chain is finite and irreducible, all its states are...
Moreover all the states are ergodic.

If the frequency

Exercise 7(C)

$d\omega =$

Part A (Short answer questions)

1. Define a Markov process.
2. Define a Markov chain and give an example of a Markov chain.
3. Prove that the Poisson process is a Markov process.
4. When is a Markov chain called homogeneous?
5. When is a homogeneous Markov chain said to be regular?
6. Define transition probability matrix of a Markov chain.
7. What is a stochastic matrix? When is it said to be regular?
8. Prove that the tpm of a Markov chain is a stochastic matrix.
9. Define n-step transition probability in a Markov chain.
10. State Chapman-Kolmogorov theorem.
11. What do you mean by probability distribution of a Markov chain?
12. When is a Markov chain completely specified?
13. What is meant by steady-state distribution of a Markov chain?
14. Write down the relation satisfied by the steady-state distribution and the tpm of a regular Markov chain.
15. If the tpm of a Markov chain is $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ find the steady-state distribution of the chain.
16. When is a Markov chain said to be irreducible or ergodic?
17. Prove that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ is the tpm of an irreducible Markov chain.
18. What do you mean by an absorbing Markov chain. Give an example.
19. If the initial state probability distribution of a Markov chain is $p^{(0)} = \left(\frac{5}{6}, \frac{1}{6}\right)$ and the tpm of the chain is $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$, find the probability distribution of the chain after 2 steps.

Part B

20. The tpm of a Markov chain with three states 0, 1, 2 is

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$