Disturbance Compensating Model Predictive Control With Application to Ship Heading Control - A Report

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1 Introduction

The paper begins by introducing the problem of ship heading control or course keeping as a representative control problem for marine applications. Traditional ship heading controllers do not consider the rudder saturation problem and motion in other axes due to a large yaw velocity, causing undesirable behavior. Typical nonlinear control methods only take in these constraints via trial and error tuning instead of taking them explicitly in the design process.

In other control methodologies, certain strategies, such as Model Predictive Control (MPC)[8] and the reference governor [2], stand out for their explicit handling of input and state constraints. Reference [10] delves into managing rudder saturation within the MPC framework for tracking marine vessels, while [7] focuses on utilizing MPC for roll reduction in heading control. [7] doesn't explicitly address state constraints like yaw rate and roll angle. The path following with input (rudder) and state (roll) constraints is achieved via MPC in [4].

MPC, a sophisticated control technique, integrates optimization with feedback to handle systems facing constraints on inputs and states [8]. It utilizes an explicit model and the current measured or estimated state as a starting point to forecast how a plant will respond in the future. At each sampling interval, MPC determines the control action by solving an online finite-horizon open-loop optimal control problem. Moreover, due to its natural compatibility with multi-variable systems, MPC adeptly manages under-actuated or over-actuated issues by consolidating objectives into a unified function. A key factor behind MPC's success in industrial applications is its ability to enforce various types of constraints on the process [9]. Nevertheless, there are times when the optimization problem MPC tackles becomes infeasible due to model mismatches and/or disturbances. To illustrate, disturbances like waves can render a standard MPC ship heading controller infeasible at certain time steps.

In addressing feasibility challenges within MPC applications amid disturbances and model uncertainties, like steering a ship through unpredictable waves, numerous studies on robust MPC have been pursued. These efforts have resulted in extensive publications in the literature ([8], [5], and references therein).

Typical robust MPC approaches often consider bounded disturbances [11] - [6], assuming they are confined to a compact set and allowed to take values within that set. However, this assumption neglects knowledge of disturbance dynamics and may yield conservative results when such knowledge is available. Additionally, these robust MPC algorithms are typically computationally intensive.

To mitigate conservativeness and enhance efficiency, this paper draws inspiration from [1] and proposes a computationally efficient two-step algorithm for handling disturbances by leveraging disturbance information. The paper states that the contributions lie in the following aspects

1) A novel two-step disturbance compensating MPC (DC MPC) algorithm is proposed to achieve state constraint satisfaction and successive feasibility for linear systems with environmental disturbances. The theoretical proof is also provided.

2) Compared to the standard MPC, the proposed MPC controller is designed to achieve good system performance with low additional computational effort by leveraging "measurable" disturbances.

3) The proposed DC-MPC algorithm is successfully applied to ship heading control in wave fields to satisfy the yaw velocity constraints.

2 Problem Statement

The paper considers a LTI system with disturbances and presents the state space equation

$$x(k+1) = Ax(k) + Bu(k) + w(k), w \in w \tag{1}$$

where $x \in R^{n_o}$ is the system state, $u \in R^{n_i}$ is the control, and $w \in R^{n_o}$ is an unknown disturbance in the set W. A and B are system matrices which describe the system.

The paper also introduces the standard MPC optimization problem. To understand the optimization problem, we need to understand optimal control.

Optimal control is based on optimizing a performance metric regarding the control and the predictive state. It allows us to specify a dynamic model and the desired outcomes and apply constraints.

Model Predictive Control is an optimal control technique which optimizes an objective, which involves minimizing a cost function which defines the optimality criteria.

The standard MPC optimization problem $P'(x(k)))$ is given in the paper as

$$\underset{u(\cdot|k)}{min} \sum_{j=1}^{N_p} [x(k+j|k)^T Q x(k+j|k) + u(k+j-1|k)^T R u(k+j-1|k)] \tag{2}$$

subject to the constraints

$$\begin{aligned}
&x(k+j+1|k) = Ax(k+j|k) + Bu(k+j|k) \\
&x(k|k) = x(k) \\
&Cx(k+j+1|k) \leq D \\
&Su(k+j|k) \leq T
\end{aligned}$$

We look into the optimization problem more closely. The Q and R matrices let us determine how much we penalise bad performance and input effort respectively and shape the responses. In most optimization functions, we prefer $x^T Q x$ instead of $||Qx||$ (norm 2) as it magnifies the bigger errors. The odd orders do not gives us the full picture as their minimum does not minimization of the error as the summation would actually subtract negative cost values from the overall cost. This quadratic stage cost, combined with the linear model remains quadratic which can be solved computationally as they are convex and have a definite minimum value.

The term N_p also shows us the finite horizon aspect of the MPC, we do not solve the optimization problem for the infinite horizon problem, we solve for a finite horizon and redo the optimization at the next time step.

The constraint $x(k+j+1|k) = Ax(k+j|k) + Bu(k+j|k)$ tells us that the next state must be according to the dynamical model. $x(k|k) = x(k)$ tells us about the initial state.

The constraints $Cx(k+j+1|k) \leq D$ and $Su(k+j|k) \leq T$ are the state and input constraints. It gives us control over undesirable states and inputs and involve the C,D,S and T matrices.

If the optimization problem is feasible then the solution is denoted by the optimal control action sequence $\{u''^{*}(k|k), u''^{*}(k+1|k), \ldots, u''^{*}(k+N_p-1|k)\}$ accordingly the predicted optimal states are $\{x''^{*}(k+1|k), x''^{*}(k+2|k), \ldots, x''^{*}(k+N_p|k)\}$

According to the standard MPC approach, we use the first vector in this optimal sequence as our control action

$$u(k) = u''^{*}(k|k) \tag{3}$$

After applying the control action, we measure the state at the next time step and redo the optimization problem with that measured state as our initial state $P'(x(k+1)))$.

The paper now presents the problem, with disturbances $(w \neq 0)$**, even if the optimization problem** $P'(x(k)))$ **is feasible at time-step** k**, the feasibility of the MPC problem is not guaranteed at** $k+1$**. More specifically**

$$Cx(k+1) \leq D \tag{4}$$

cannot be guaranteed with $u(k) = u''^{*}(k|k)$**. We cannot guarantee a desirable state in the presence of a disturbance.**

The goal of the paper is to ensure repeated feasibility of the MPC Optimization Problem. If $Cx(k) \leq D$ **is satisfied then the paper wants to guarantee that** $Cx(k+1) \leq D$ **can be satisfied.**

The goal of the design is to make the response of the system in the presence of disturbances as close as possible to the response without disturbances.

3 Modified MPC

Considering the state space equation at time $k-1$ From the reference cited in the paper, the disturbance can be estimated the difference between the measured and expected output, it can be estimated by the following equation if the state and control are measurable [1]

$$\hat{w}(k-1) = x(k) - \hat{x}(k) \tag{5}$$

$$\hat{w}(k-1) = x(k) - Ax(k-1) - Bu(k-1) \tag{6}$$

Assumption 1 The paper now makes an assumption if the sampling time T_s is small or if the disturbance changes slowly in time, the disturbance at time step k, $w(k)$, can be estimated by

$$w(k) = \hat{w}(k-1) + \varepsilon \tag{7}$$

where $\varepsilon \in V$ and $V \subset W$. ε can also capture small model mismatches between k and $k-1$.

When the sampling time T_s is small, the disturbance variance between k and $k-1$ is very small, hence the bounds on V are much tighter than on W.

The paper does make a remark that this assumption is valid for applications where computational resources is not an issue and fast sampling can be implemented. The authors comment on their computer (2 GHz CPU with 1 GB RAM) giving satisfactory results with a sampling time of $0.5s$ for a 80 step prediction horizon.

The ship heading control requires a controller capable of controlling the heading during the dominant period of ocean waves which is $6-10s$ With their controller having a horizon of $80*0.5s = 40s$, their setup seems capable of handling the application's requirement.

With this assumption, a modified MPC approach is presented and referred to M-MPC where the disturbance information is directly utilized in the optimization problem $P''(x(k)), \hat{w}(k-1))$.

$$\min_{u(\cdot|k)} \sum_{j=1}^{N_p} [x(k+j|k)^T Q x(k+j|k) + u(k+j-1|k)^T R u(k+j-1|k)] \tag{8}$$

subject to the constraints

$$\begin{aligned}
&x(k|k) = x(k)\\
&x(k+1|k) = Ax(k|k) + Bu(k|k) + \hat{w}(k)\\
&x(k+j+1|k) = Ax(k+j|k) + Bu(k+j|k), j = 1, ..., N_p - 1\\
&Cx(k+j+1|k) \leq D, j = 0, 1, ..., N_p - 1\\
&Su(k+j|k) \leq Tj = 0, 1, ..., N_p - 1
\end{aligned}$$

In this model the first element of the optimal state sequence $x(k+1|k)$ is computed with the dynamical constraints

$$x(k+1|k) = Ax(k|k) + Bu(k|k) + \hat{w}(k) \tag{9}$$

and every element after that $x(k+j+1|k) = Ax(k+j|k)+Bu(k+j|k), j = 1, ..., N_p - 1$ is in accordance to the standard MPC. It modifies the linear model by considering the next step disturbance to predict future states, and the first element of the optimal control sequence of $P''(x(k)), \hat{w}(k-1))$ is implemented as the control input to the system.

Using the M-MPC scheme, the state constraints can be satisfied, however the paper reports that the system performance is not satisfactory for the ship heading control application. The simulation results with the M-MPC confirms the same (shown in the simulation section).

4 Disturbance Compensating MPC

The Disturbance Compensating MPC (DC-MPC) approach is proposed to not only satisfy the state constraints in the presence of disturbance, but to also retain the same performance level achieved by the ship heading control system in calm water (without disturbance).

The DC-MPC approach consists of roughly 4 steps taken to to calculate disturbance and disturbance compensation and then to solve the optimization problem and implement the control to the system.

Step 1 At time step k, we calculate the disturbance $\hat{w}(k-1)$ in the previous time step $k-1$ using the earlier approach in M-MPC, the measured values $x(k), x(k-1)$ and $u(k-1)$.

$$\hat{w}(k-1) = x(k) - \hat{x}(k) \tag{10}$$

$$\hat{w}(k-1) = x(k) - Ax(k-1) - Bu(k-1) \tag{11}$$

Step 2 We now diverge from the method taken in the M-MPC of using the disturbance in the dynamical constraints and instead introduce the disturbance compensation control element Δu.

We first consider the ideal state at $k+1$ with no disturbance $x(k+1)$

$$x_{ideal}(k+1) = Ax(k) + Bu(k) \tag{12}$$

subject to the constraints

$$Cx_{ideal}(k+1) \leq D \tag{13}$$

$$Su(k) \leq T$$

We then consider the state at $k+1$ with disturbance $x(k+1)$

$$x_{real}(k+1) = Ax(k) + Bu(k) + w(k) \tag{14}$$

We know according to our assumptions in M-MPC,

$$w(k) = \hat{w}(k-1) + \varepsilon \tag{15}$$

So we rewrite our equation as

$$x_{real}(k+1) = Ax(k) + Bu(k) + \hat{w}(k-1) + \varepsilon \tag{16}$$

The paper now proposes the core element of the DC-MPC scheme, a disturbance compensatory element Δu as a part of the control input that will nullify the effect of the disturbance $\hat{w}(k-1) + \varepsilon$. Using it in our previous equation, we get

$$x_{real}(k+1) = Ax(k) + B(u(k) + \Delta u) + \hat{w}(k-1) + \varepsilon \tag{17}$$

subject to the constraints

$$\begin{aligned} &Cx_{real}(k+1) \leq D \\ &Su(k) \leq T \end{aligned}$$

This should ideally behave like our system at the ideal state with the constraints on x. Taking both the constraints

$$Cx_{real}(k+1) \leq D \tag{18}$$

$$Cx_{ideal}(k+1) \leq D \tag{19}$$

Subtracting the 2 above equations, we get

$$Cx_{real}(k+1) - Cx_{ideal}(k+1) \leq 0 \tag{20}$$

$$C(Ax(k) + B(u(k) + \Delta u) + \hat{w}(k-1) + \varepsilon) - C(Ax(k) + Bu(k)) \leq 0 \tag{21}$$

Cancelling out the required terms, we get

$$C(B(\Delta u) + \hat{w}(k-1) + \varepsilon) \leq 0 \tag{22}$$

$$CB\Delta u + C\hat{w}(k-1) + C\varepsilon \leq 0 \tag{23}$$

The goal is to minimize this equation to attain the inequality, leading to the compensation of disturbance and performance of the ideal system. Our optimization problem needs to optimize to a scalar and we can use the norm of the equation to extract the optimal disturbance compensation while reducing to a scalar. The paper proposes to ignore ε as minimizing the problem will lead to $x_{real}(k+1) = x_{ideal}(k+1) + \varepsilon$ which means the state with disturbances is made to be almost ideal without disturbances.

Therefore, we minimize the norm of the above equation to extract the optimal disturbance compensation element. This is the low dimensional optimization problem $P_\Delta(\hat{w}(k-1))$

$$\min_{\Delta u \in R^{n_i}} \|CB\Delta u + C\hat{w}(k-1)\| \tag{24}$$

subject to

$$CB\Delta u \leq -C\hat{w}(k-1) - E \tag{25}$$

$$S\Delta u \leq T \tag{26}$$

Looking at the first constraint, we understand that it is a rewritten version of our previous inequality $CB\Delta u + C\hat{w}(k-1) + C\varepsilon \leq 0$ where the term $E = max(C\varepsilon)$ where $\varepsilon \in V$. The second constraint makes this new disturbance compensation control input obey the same constraints as the original control input $u(k)$.

Step 3 In the third step, a modified optimization problem $P(x(k), \Delta u^*)$ is formulated to incorporate the optimal disturbance compensation (Δu^*) into the MPC framework. The optimization problem $P(x(k), \Delta u^*)$ is written as follows:

$$\min_{u(\cdot|k)} \sum_{j=1}^{N_p} \big[x(k+j \mid k)^T Q x(k+j \mid k) \\ + u(k+j-1 \mid k)^T R u(k+j-1 \mid k) \big]$$

The goal here was to minimize the above cost function subjected to the following constraints.

$$\begin{aligned}
& x(k \mid k) = x(k) \\
& x(k+j+1 \mid k) = Ax(k+j \mid k) + Bu(k+j \mid k) \\
& Cx(k+j+1 \mid k) \leq D, \quad j = 0, 1, \ldots, N_p - 1 \\
& Su(k \mid k) \leq T - S\Delta u^* \\
& Su(k+j \mid k) \leq T, \quad j = 1, \ldots, N_p - 1.
\end{aligned}$$

Looking at the constraints, most of them look similar to the standard MPC scheme, except $Su(k \mid k) \leq T - S\Delta u^*$ which takes the optimal disturbance compensation control Δu into consideration while computing the constraints for $u(k)$ to not violate the constraint. The goal here is to combine the optimal control input with the optimal disturbance compensation control and use that as an input, so the sum of those control inputs should not violate the constraints on the control input. This is computed as

$$\begin{aligned}
& S(u(k \mid k) + \Delta u^*) \leq T \\
& Su(k \mid k) \leq T - S\Delta u^*
\end{aligned}$$

Step 4 Like we mentioned earlier, the goal here is to combine the optimal control input with the optimal disturbance compensation control and use that as an input to the system. So we implement the following control to the system:

$$u(k) = u^*(k \mid k) + \Delta u^* \tag{27}$$

Proposition 1 : The paper states that if the optimization problems $\mathcal{P}_\Delta(w(k-1))$ and $\mathcal{P}\left(x(k), \Delta u^*\right)$ are both feasible, the constraint

with the state $Cx(k+1) \leq D$, can always be guaranteed if the control $u(k) = u^*(k \mid k) + \Delta u^*$ is applied to the linear system.

Proof: If the optimization problems $\mathcal{P}_\Delta(w(k-1))$ and $\mathcal{P}\left(x(k), \Delta u^*\right)$ are feasible, from our previous derivations we have that the corresponding optimal solution should satisfy the following constraints:

$$\begin{aligned}
&CB\Delta u^* \leq -Cw(k-1) - E \\
&Cx^*(k+1 \mid k) \leq D \\
&Su^*(k \mid k) \leq T - S\Delta u^*.
\end{aligned}$$

When we compute $\mathcal{P}\left(x(k), \Delta u^*\right)$, it follows that $S\left(u^*(k \mid k) + \Delta u^*\right) \leq T$, thus, the input constraints $Su(k) \leq T$ are satisfied.

Applying the control $u(k) = u^*(k \mid k) + \Delta u^*$ and with our state with disturbance observation, the state $x(k+1)$ is given by

$$\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + w(k) \\
&= Ax(k) + B\left(u^*(k \mid k) + \Delta u^*\right) + w(k) \\
&= x^*(k+1 \mid k) + B\Delta u^* + \hat{w}(k-1) + \varepsilon.
\end{aligned}$$

Notice that inequalities $CB\Delta u^* \leq -Cw(k-1) - E$ and $(Cx^*(k+1 \mid k) \leq D$ are already satisfied. Adding each side of the above equations, we have

$$CB\Delta u^* + Cx^*(k+1 \mid k) \leq -C\hat{w}(k-1) - E + D \tag{28}$$

Rearranging the terms, we get

$$Cx^*(k+1 \mid k) + CB\Delta u^* + C\hat{w}(k-1) + E \leq D \tag{29}$$

Since $E = \max(C\varepsilon), C\varepsilon \leq E$, then

$$\begin{gathered}
Cx^*(k+1 \mid k) + CB\Delta u^* + C\hat{w}(k-1) + E \leq D \\
C(x^*(k+1 \mid k) + B\Delta u^* + \hat{w}(k-1) + \varepsilon) \leq D \\
C(x(k+1)) \leq D
\end{gathered}$$

Therefore, the state constraints $Cx(k+1) \leq D$ are satisfied

5 Remarks

The paper makes a few remarks and comparisons to the standard MPC scheme regarding computational effort and resiliency to disturbance.

Remark 1: The paper highlights that the computational effort needed for DC-MPC closely mirrors that of standard MPC. In the Step 3, both DC-MPC and MPC involves solving a quadratic programming problem with the same structure. The DC-MPC model addresses an optimization problem in Step 2 with a dimension of n_i, where n_i represents the control input dimension.

This low-dimensional optimization problem, in contrast to the $N_p \times n_i$ dimension of the quadratic programming problem in Step 3, incurs minimal additional computational cost. The proposed DC-MPC model demonstrates significantly lower computational complexity compared to the robust MPC model algorithm (M-MPC) due to the convoluted constraints. [8] [5]

Remark 2: The DC-MPC model sets itself apart from the M-MPC model by minimization of the cost function $\mathcal{P}_\Delta(\hat{w}(k-1))$. While addressing the optimization problem $\mathcal{P}_\Delta(\hat{w}(k-1))$,the objective is to bring the system's response as close as possible to a scenario without disturbances.

Specifically, achieving the condition $\|CB\Delta u + C\hat{w}(k-1)\| = 0$ ensures $x(k+1) = x^*(k+1 \mid k) + \varepsilon$. This shows that, in the presence of disturbances, the system states adjust to closely resemble the desired states without disturbances.

Remark 3: The feasibility of $P_\Delta(\hat{w}(k-1))$ is heavily dependent on the characteristics of the matrix CB, which plays an important role in controlling the influence on constrained states, and the magnitude of the disturbance. In cases where the disturbance is too large or CB is ill-conditioned, compensating for the disturbance may surpass the input limits, leading to the in-feasibility of $P_\Delta(\hat{w}(k-1))$, which ensures the feasibility relies in the compensation falling within the input constraints.

Applying this concept to ship heading control, where the rudder angle holds significant control authority over the constrained rate (yaw rate), the feasibility of $P_\Delta(\hat{w}(k-1))$ is guaranteed unless there is an unexpectedly high wave disturbance.

Remark 4: The assumption regarding the feasibility of $\mathcal{P}\left(x(k), \Delta u^{*}\right)$ may not hold for certain systems under specific conditions. Notably, when one of the constrained states is the direct integral of another state, ensuring repeated feasibility becomes challenging even if the initial state constraints are initially satisfied.

For instance, in a dynamical positioning system with position constraints, having the ship's initial position within the feasible region does not guarantee the ongoing feasibility of $\mathcal{P}\left(x(k), \Delta u^{*}\right)$. Specifically, when the ship is positioned on the boundary of the feasible region with a high speed pointing outside that region, $\mathcal{P}\left(x(k), \Delta u^{*}\right)$ may become infeasible.

These challenges are inherent in many MPC applications and are not unique to the proposed DC-MPC scheme. Typically, addressing this issue involves defining an invariant set and constraining the state within that set.

6 Application of DC-MPC in Ship Heading Control

Ship Heading Control Introduction

The paper now introduces the application of Ship heading control, also known as course-keeping, is a primary function of autopilots. The goal is to steer the ship's actual heading angle (ψ) towards the desired heading angle (ψ_d), typically assumed to be constant [1].

$$\psi \rightarrow \psi_d$$

Here, $\dot{\psi} = r$, where r denotes the yaw rate.

The paper borrows the model and the characteristics of the ship from the paper 'Guidance and Control of Ocean Vehicles' [1].

The Nomoto model, which is widely employed for ship heading control design, focuses on the yaw rate (r) as the single degree of freedom (DOF) for ship dynamics and the rudder angle (δ) as the control input.

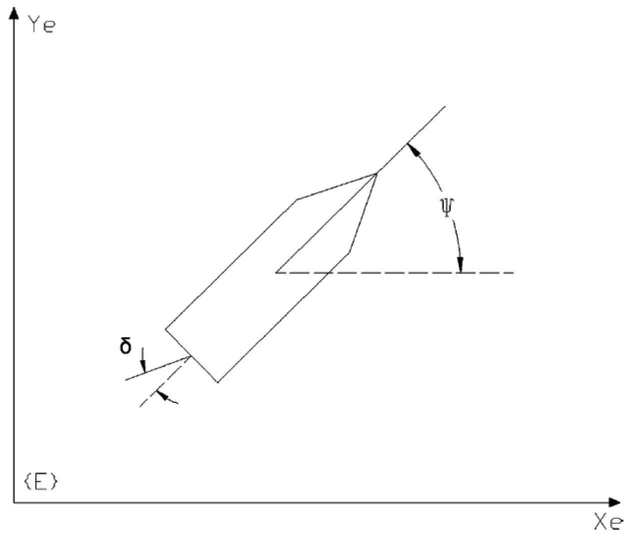

Fig. 1. Definition of ship heading and rudder angle.

This paper adopts the Nomoto model, neglecting other ship degrees of freedom. Using the example of the container ship S175 [1], the continuous-time linear dynamics are given by:

$$\dot{x} = A_c x + B_c \delta + w$$

with $x = [r, \psi]^T$ and

$$A_c = \begin{bmatrix} -0.1068 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0028 \\ 0 \end{bmatrix}$$

Given a specific sampling time T_s, the discrete-time system matrices are derived for this application with $T_s = 0.5$ s as follows:

We first consider the continuous standard state space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

we know that the matrix exponential is

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$$

and by premultiplying the previous expression we get

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

which we recognize as

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

and by integrating..

$$\begin{aligned} e^{-\mathbf{A}t}\mathbf{x}(t) - e^{0}\mathbf{x}(0) &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned}$$

Discretising the above expression keeping u is constant during each time-step.

$$\mathbf{x}[k] = \mathbf{x}(kT)$$

$$\mathbf{x}[k] = e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}[k+1] = e^{\mathbf{A}(k+1)T}\mathbf{x}(0) + \int_0^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\left[e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right] + \int_{kT}^{(k+1)T} e^{\mathbf{A}(kT+T-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We recognize the bracketed expression as $\mathbf{x}[k]$, and the second term can be simplified by substituting with the function $v(\tau) = kT + T - \tau$. Note that $d\tau = -dv$. We also assume that $\mathbf{u}$ is constant during the integral, which in turn yields

$$\begin{aligned} \mathbf{x}[k+1] &= e^{\mathbf{A}T}\mathbf{x}[k] - \left(\int_{v(kT)}^{v((k+1)T)} e^{\mathbf{A}v}dv\right)\mathbf{B}\mathbf{u}[k] \\ &= e^{\mathbf{A}T}\mathbf{x}[k] - \left(\int_T^0 e^{\mathbf{A}v}dv\right)\mathbf{B}\mathbf{u}[k] \\ &= e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_0^T e^{\mathbf{A}v}dv\right)\mathbf{B}\mathbf{u}[k] \\ &= e^{\mathbf{A}T}\mathbf{x}[k] + \mathbf{A}^{-1}\left(e^{\mathbf{A}T} - \mathbf{I}\right)\mathbf{B}\mathbf{u}[k] \end{aligned}$$

Therefore we have,

$$
\begin{aligned}
A_d &= e^{\mathbf{A}T} \\
B_d &= \mathbf{A}^{-1}\left(e^{\mathbf{A}T} - \mathbf{I}\right)\mathbf{B}
\end{aligned}
$$

Calculating our discrete time matrices A_d and B_d from the above derived equations

$$
A_d = \begin{bmatrix} 0.9480 & 0 \\ 0.4869 & 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0014 \\ 0.0003 \end{bmatrix}
$$

Rudder saturation and yaw rate limits are considered, leading to matrices C, D, S, and T as follows:

$$
\begin{aligned}
C &= \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.006 \\ 0.006 \end{bmatrix} \\
S &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad T = \begin{bmatrix} 35\pi/180 \\ 35\pi/180 \end{bmatrix}
\end{aligned}
$$

Feasibility Considerations

The feasibility of $\mathcal{P}_\Delta(\hat{w}(k-1))$ relies on both the disturbance magnitude and the control authority of the rudder input on the yaw rate. For a container ship S175 operating in a sea state 5 wave field, the maximum disturbance value Cw is $[0.0005, -0.0005]^T$ with a sampling time $T_s = 0.5$ s. In this case, the maximum disturbance variation is $E = [0.00005, 0.00005]^T$, ensuring the satisfaction of constraints (17) and (18) for $\mathcal{P}_\Delta(\hat{w}(k-1))$.

The feasibility of $\mathcal{P}\left(x(k), \Delta u^*\right)$ is assured if the initial yaw rate constraints are satisfied since the yaw rate is directly controlled by the input and is not the direct integral of other states.

DC-MPC Implementation

To demonstrate the effectiveness of DC-MPC, it is first implemented on a linear model subjected to sinusoidal and constant disturbances. The resulting performance is then evaluated in comparison to standard and M-MPC schemes.

Following this, the DC-MPC scheme is applied in the paper to the original nonlinear system operating in wave fields to validate its performance in the paper, but this report cannot cover the nonlinear case due to inability to access nonlinear test-bed.

7 Simulation Results

For simulation results, please checkout the hosted Python Notebook Google Colab Noteook

Linear Model With Constant and Sinusoidal Disturbances
The paper considers two disturbances, constant (-0.0015) and sinusoidal (0.001 sin(0.08 t)). In this report, we consider two disturbances constant (-0.0015) and sinusoidal (0.0004 sin(0.08 t)). Our reasoning will be explained in detail in the following paragraphs.

These disturbances mimic the first-order and second-order wave disturbance effects [3]. The paper enforces the rudder constraints $|\delta| \leq 35°$ and the yaw rate constraints $|r| \leq 0.006 rad/s$ $(0.34°/s)$

The standard MPC scheme is first studied by simulations, with the results summarized in Fig. 2. In this simulation, $T_s = 0.5s, N_p = 80, Q = \{1000, 800\}, R = 1$.

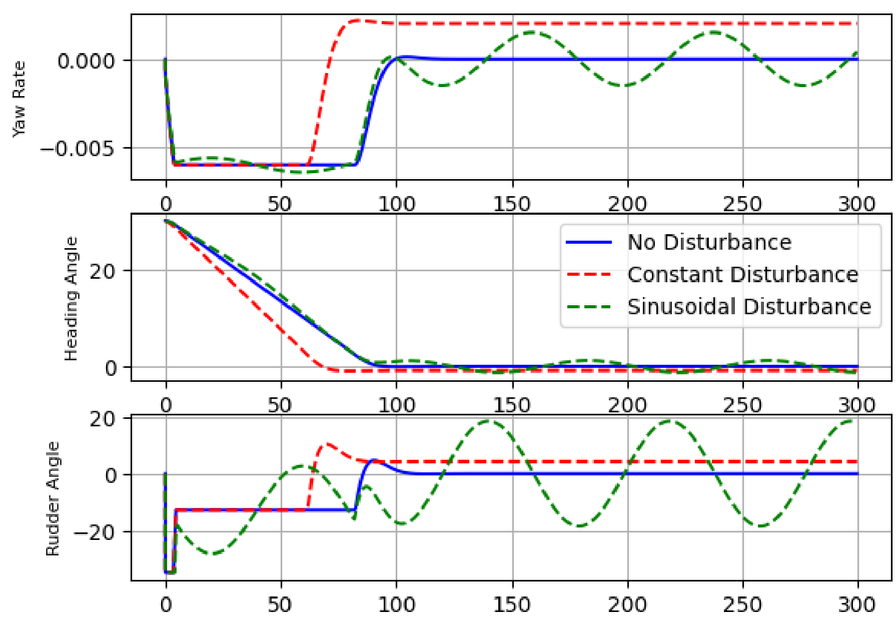

Fig. 2. Standard MPC ship heading controller simulations with and without

disturbances.

The authors have chosen the parameters to achieve good performance in calm water. Our simulations correlate with the simulations that the paper has published in accordance to the standard MPC's performance. Fig. 2 shows that although the standard MPC scheme achieves good performance in calm water in terms of meeting constraints and achieving desired heading, the performance of the standard MPC in the presence of disturbances is not satisfactory. First, the yaw constraint violations are observed with both constant and sinusoidal disturbances. Second, a steady-state error exists in the constant disturbance case, while heading angle oscillations are observed with the sinusoidal disturbance.

The authors state that to obtain the results shown in Fig. 2, it occurred in many time steps that the optimization problem of the standard MPC has no solution for the given state constraints. At these time steps, the yaw rate constraints are temporarily removed to avoid the breakdown of the MPC controller so that the simulation can continue, and the hard constraint on the input is imposed on the rudder. We instead lowered the amplitude of the sinusoidal wave, and showed the system's yaw rate instability with decreased amplitude (0.0004 sin (0.08 t)). Our simulation outputs highlight the same performance of the standard MPC controller.

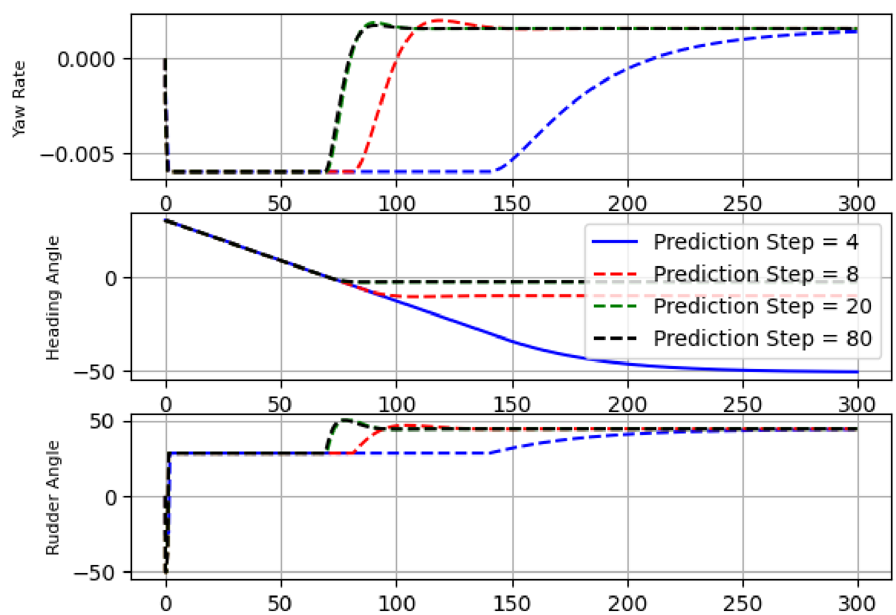

Fig. 3. Simulations of the M-MPC ship heading controller with constant disturbances for different prediction horizons.

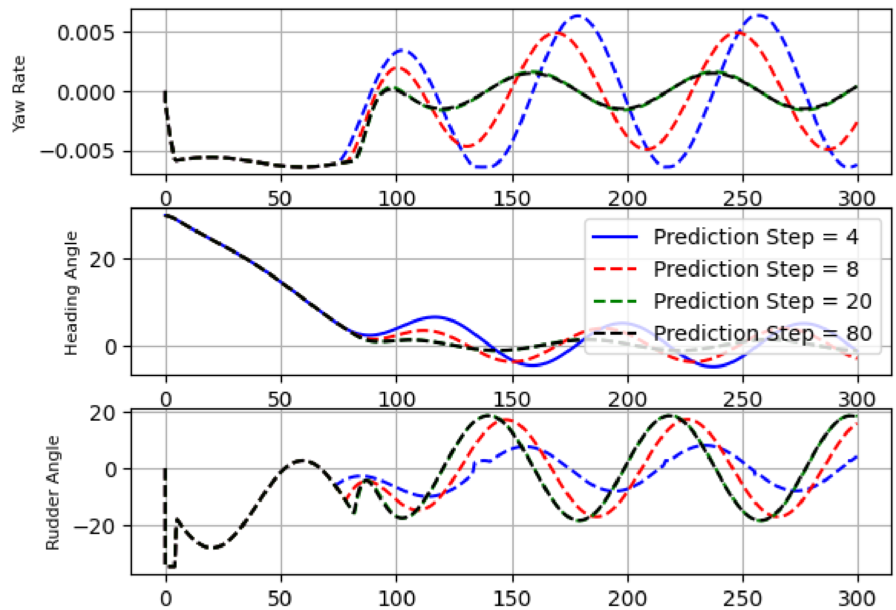

Fig. 4. Simulations of the M-MPC ship heading controller with sinusoidal disturbances for different prediction horizons.

The M-MPC and DC-MPC are also implemented with different prediction horizons to study their performance with constant and sinusoidal disturbances (4,8,20,80). The simulations of the M-MPC are summarized in Figs. 3 and 4, while those of the DC-MPC are shown in Figs. 5 and 6, for constant and sinusoidal disturbances, respectively. In these simulations, the controller parameters are kept the same as the standard MPC. The paper states that although the yaw constraints are successfully enforced by the M-MPC for all prediction horizons, the tracking performance to match the heading angle is not satisfactory, particularly with those having short prediction horizons. We've found this to be the case as well, additionally the yaw rate takes longer to stabilize in shorter prediction horizons. Longer prediction horizon results in better performance for both constant and sinusoidal disturbances. However, the steady-state error cannot be completely eliminated.

The paper proposes that the DC-MPC has the capability to satisfy the state (yaw) constraints for all prediction horizons with both constant and sinusoidal disturbances and that the DC-MPC scheme can eliminate the steady-state error with the constant disturbance for all prediction horizons. We've found that there are no yaw rate constraints, however when the prediction horizon is low (4) we see a very uncharacteristic steady state error, which we must further investigate. We can correlate with the paper's claims of largely reduces the heading angle oscillations with the sinusoidal disturbance compared with the standard MPC and M-MPC cases. With longer prediction horizons, the DC-MPC scheme achieves better performance in terms of faster heading angle tracking and less heading angle oscillations.

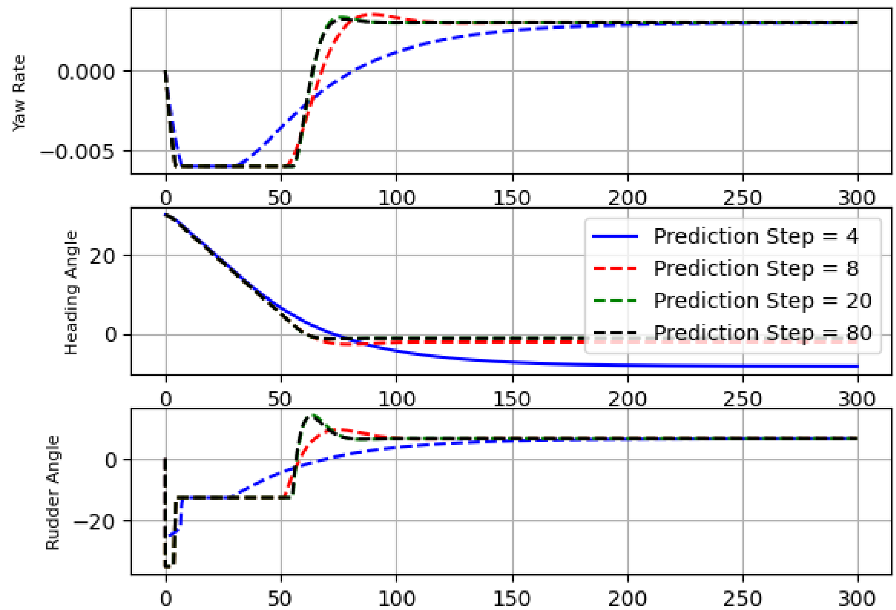

Fig. 5. Simulations of the DC-MPC ship heading controller with constant disturbances for different prediction horizons.

For the case of constant disturbance in Fig. 5, the rudder angle to compensate the disturbance is about 30 . This result indicates that the maximum disturbance magnitude the system (with a maximum 35 rudder angle) can handle to avoid state constraint violation is about 0.0016. The amplitude of heading angle oscillations with the DC-MPC is around 0.2 , while that with the M-MPC is around 0.9. Furthermore, the capability of the DC-MPC scheme to track the system response without disturbances is illustrated (also see Figs. 5 and 6), which is discussed in Remark 3.

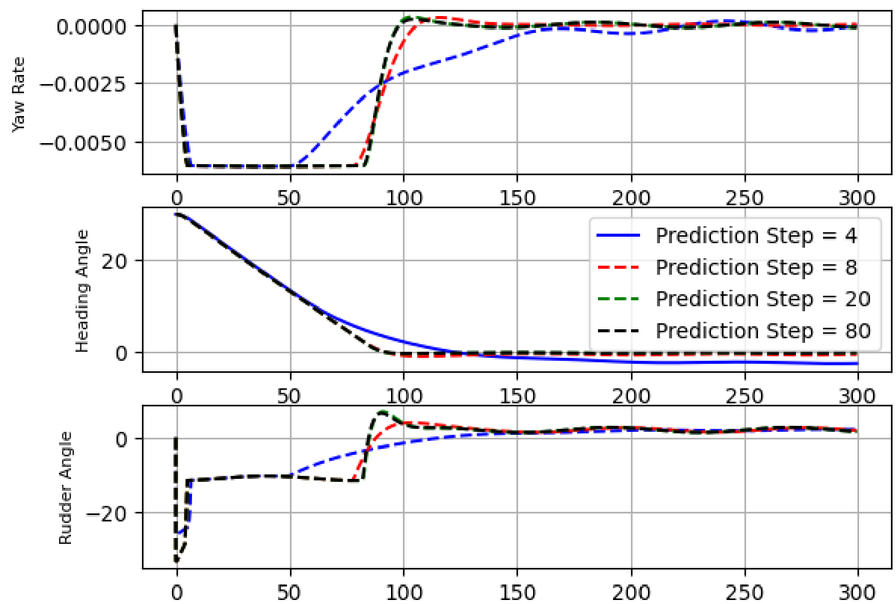

Fig. 6. Simulations of the DC-MPC ship heading controller with sinusoidal disturbances for different prediction horizons.

The different approaches adopted by the M-MPC and DC-MPC lead to the performance differences. The M-MPC scheme minimizes the cost function based on the predictions of the nominal system (considering only the disturbance in one time step); thus, the mismatch of the nominal system and real system results in the steady-state error (constant disturbance) or state oscillations (sinusoidal disturbance).

In contrast, the DC-MPC scheme is trying to track the desired no-disturbance performance (minimize the distance between the actual states and the predicted optimal states without disturbance), which results in steady-state error elimination and state oscillations reduction. The DC-MPC algorithm has the potential to be applied to other motion control problems with environmental disturbances, such as flight, automobile. and robotics controls, since in these,cases the system response without disturbances is always designed to be desirable.

The initial course-changing speed for the DC-MPC is slower than the standard MPC without yaw constraints, while the final convergence speeds for both cases are similar. However, the yaw velocity constraint violations were observed in the M-MPC case. As explained in Remark 2, the computational time for DC-MPC and M-MPC schemes are very similar because the DC-MPC scheme just requires to solve an additional 1-D optimization problem in this case.

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To further validate its performance, the authors evaluate DC-MPC scheme and compare it with the M-MPC scheme on the numerical test-bed developed in [17]. Unfortunately, we do not have access to the test-bed as it's behind a substantial paywall and could not find the data-set anywhere. The simulation output for the nonlinear case has been explained in Fig. 8 and 9. In the simulations, sea state 5 is used, and the initial ship heading angle with respect to the wave heading angle is 0. The same sampling time and predictive horizon are used. The DC-MPC can successfully enforce the yaw rate constraints

8 Conclusion

In this report, we've gone through the paper proposing the DC-MPC scheme, developed to compensate the disturbance and to address the optimization problem constraint infeasibility. First, a basic disturbance estimation and compensating method was discussed as a modification of the standard MPC (M-MPC) Then, the theoretical analysis was performed to show that DC-MPC can satisfy state constraints and achieve target performance under a given set of assumptions.

The DC-MPC scheme was applied to ship heading control on a linear model, and compared with standard and M-MPC. The simulations show that the DC-MPC can eliminate the drawbacks of standard MPC, thereby satisfying the state constraints, eliminating the steady-state error, and reducing the state and control oscillations. The simulation results also show that better performance are achieved by the use of DC-MPC scheme over the M-MPC scheme.

A point of analysis involves comparing the conservativeness of DC-MPC with robust MPC, especially considering that the latter doesn't explore specific disturbance properties beyond robustness. The authors state that this direction will be further explored, extending the investigation to a broader class of systems.

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