STAT 135 CONCEPTS OF STATISTICS
HOMEWORK 4

Assigned July 14, 2021, due July 22, 2021

This homework pertains to materials covered in Lecture 9 and 10. The assignment can be typed or handwritten, with your name on the document, and **with properly labeled input code and computer output for those problems that require it**. To obtain full credit, please write clearly and show your reasoning. If you choose to collaborate, the write-up should be your own. Please show your work! Upload the file to the Week 3 Assignment on bCourses.

Note in this homework, we use the following abbreviations: Uniformly most powerful (UMP) test, likelihood ratio test (LRT).

Problem 1. Suppose $X_1, \ldots, X_{30}$ are independently sampled from $N(\mu, 1)$. Consider test

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu < \mu_0.$$

(1) Derive the LRT with significance level $\alpha = 0.05$;
(2) Calculate the power function of the test in (1) and visualize in `R`;
(3) Use the power function to prove that the test in (1) is of size $\alpha = 0.05$ for testing

$$H_0 : \mu \geq \mu_0 \text{ versus } H_1 : \mu < \mu_0.$$

Problem 2. In each of the following situations, calculate the p-value of the observed data.

(1) For testing $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$, $X = 4$ is observed, where $X \sim \text{Poisson}(\lambda)$.
(2) For testing $H_0 : \lambda \leq 2$ versus $H_1 : \lambda > 2$, $X_1 = 2$, $X_2 = 4$ and $X_3 = 7$ ares observed, where $X_i \sim \text{Poisson}(\lambda)$ independently.

Problem 3. Let $X_1, \ldots, X_n$ be i.i.d $\text{U}(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test with a rejection region

$$R = \{X_{(1)} \geq k \text{ or } X_{(n)} \geq 1\},$$

in which k is constant to be determined by a pre-specified significance level.

(1) Determine k so that the test will have significance level α;
(2) Derive the power function of the test in (1);
(3) Prove that the test is a UMP test of size α.
(4) Determine n and k such that the test in (1) has significance level $\alpha = 0.1$ and its power function is at least 0.8 of $\theta > 1$.

Problem 4. Consider the i.i.d random variables $X_1, \dots, X_n$ from Bernoulli(p). We want to test

$$H_0 : p = 0.45 \text{ versus } H_1 : p \neq 0.45.$$

(1) Derive the expression for $-2 \log \lambda(\mathbf{X}_n)$.
(2) As in Example 6 of Lecture 10, simulate the sampling distribution of $-2 \log \lambda(\mathbf{X}_n)$ and compare it to the χ^2 approximation.
(3) If $n = 30$ and $\bar{X}_{30} = 0.463$, would you reject H_0 at significance level $\alpha = 0.05$? What about $\alpha = 0.01$?

Problem 5. Suppose $X_1, \dots, X_n$ are independently sampled from Pareto(θ, ν) with pdf

$$f(x|\theta) = \begin{cases} \frac{\theta \nu^\theta}{x^{\theta+1}}, & \text{if } x \geq \nu, \\ 0, & \text{otherwise.} \end{cases}$$

in which $\theta > 0$ and $\nu > 0$. We are again interested in testing

$$H_0 : \theta = 1, \ \nu \text{ unknown}, \qquad \text{versus} \qquad H_1 : \theta \neq 1, \ \nu \text{ unknown}.$$

(1) Use your results from Homework 3 or Example 7 of Lecture 10 and express $-2 \log \lambda(\mathbf{X}_n)$ in terms of

$$T(\mathbf{X}_n) = \log \left[\frac{\prod_{i=1}^n X_i}{X_{(1)}^n} \right],$$

in which $X_{(1)}$ is the sample minimum.
(2) Here are 30 i.i.d sample from this population:

$$\{3.832,\ 9.750,\ 2.868,\ 3.532,\ 44.750,\ 2.569$$
$$6.341,\ 4.847,\ 257.054,\ 2.391,\ 107.406,\ 7.190$$
$$3.711,\ 2.641,\ 3.779,\ 2.731,\ 3.656,\ 9.636$$
$$3.193,\ 10.727,\ 2.380,\ 8.507,\ 34.811,\ 5.664$$
$$2.317,\ 3.878,\ 6.578,\ 2.355,\ 2.401,\ 5.270\}$$

Test whether $\theta = 1$ using the Wilk's theorem.

Problem 6. Explain how to modify the t test on Page 17 of Lecture 10 to test $H_0 : \mu_X = \mu_Y + \Delta$ versus $H_1 : \mu_X \neq \mu_Y + \Delta$ where Δ is specified.

Problem 7. A study was done to compare the performances of engine bearings made of different compounds (McCool 1979). Ten bearings of each type were tested. The following table gives the times until failure (in units of millions of cycles):

(1) Assume Normal populations with equal variance and test the hypothesis that there is no difference between the two types of bearings.
(2) Test the same hypothesis using a non-parametric method.
(3) Which of the methods – that of part (1) or that of part (2) – do you think is better in this case?

Type I	Type II
3.03	3.19
5.53	4.26
5.60	4.47
9.30	4.53
9.92	4.67
12.51	4.69
12.95	12.78
15.21	6.79
16.04	9.37
16.84	12.75

(4) Estimate the probability that a type I bearing will outlast a type II bearing.

Problem 8. The assumption of equal variances, which was made in Theorem A & B of Lecture 10, is not always tenable. Let's look at a scenario where you actually have a good idea of what the variances are like for the two Normal populations. Say $X_1, \ldots, X_m \sim N(\mu_X, \sigma_X^2)$, and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$; all of the random variables are all independent. The parameters σ_X^2 and σ_Y^2 are known, whereas μ_X and μ_Y are unknown. We are going to test the equality of the means of two Normal populations:

$$H_0 : \mu_X = \mu_Y \text{ versus } H_1 : \mu_X \neq \mu_Y.$$

We now design a test based on the statistic $W = \bar{X}_n - \bar{Y}_m$.

(1) What is the sampling distribution of W?
(2) Consider the hypothesis test with the rejection region

$$R = \{|W| > k\}.$$

Which value of k achieves a pre-specified significance level of α? Express the result in terms of the known parameters σ_X^2, σ_Y^2, m, n, and α.

Problem 9. Now we don't assume that we know the variances for the two Normal populations in advance. If $X_1, \ldots, X_m \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$, we still have

$$\bar{X}_n - \bar{Y}_m \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

It seems natural to estimate σ_X^2 by S_1^2 and σ_Y^2 by S_2^2, in which $S_1^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $S_2^2 = (m-1)^{-1}\sum_{i=1}^m (Y_i - \bar{Y}_m)^2$. The asymptotic results indeed support this conjecture:

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_X - \mu_Y)}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \xrightarrow{d} t_\nu, \quad n \to \infty, \tag{1}$$

in which the degree of freedom of the t distribution $\nu \approx \min\{n-1, m-1\}$.

(1) Use this result to derive rejection regions for all three tests shown on Page 17 of Lecture 10 without assuming $\sigma_X^2 = \sigma_Y^2$.
(2) Perform the hypothesis testing again at significance level $\alpha = 0.05$ for the Byzantine church wood example on Page 19 of Lecture 10 using the rejection region you derived in (1). Do you reach a different conclusion? What is the p-value?

Problem 10. Independent random samples of 17 sophomores and 13 juniors attending a large university yield the following data on grade point averages

$$\begin{aligned}\mathbf{X}_{\text{sophomores}} = \{&3.04,\ 2.92,\ 2.86,\ 1.71,\ 3.60,\ 2.60\\ &3.49,\ 3.30,\ 2.28,\ 3.11,\ 2.88,\ 3.13\\ &2.82,\ 2.13,\ 2.11,\ 3.03,\ 3.27\}\end{aligned}$$

$$\begin{aligned}\mathbf{X}_{\text{juniors}} = \{&2.56,\ 3.47,\ 2.65,\ 2.77,\ 3.26\\ &3.00,\ 2.70,\ 3.20,\ 3.39,\ 3.00\\ &3.19,\ 2.58,\ 2.98\}\end{aligned}$$

(1) Apply the limit (1) to obtain a 95% confidence interval for $\mu_s - \mu_j$, where μ_s and μ_j are the mean GPAs for the sophomores and juniors respectively.
(2) At the $\alpha = 0.05$ significance level, do the data provide sufficient evidence to conclude that the mean GPAs of sophomores and juniors at the university differ?