Stat 135
8/5/2021

Lab 14 Solution

1. Multiple Linear Regression
Two objects are located on a line at points $0 < \beta_1 < \beta_2$. These points aren't precisely known. A surveyer makes the following measurements:
(i) He stands at the origin and measures the 2 distances from there to β_1, β_2. Let these measurements be denoted $y_1 = 10$, $y_2 = 11.5$.
(ii) He goes to β_1 and measures the distance from there to β_2. Call this distance $y_3 = 1$.

(a) Set up a linear model $y = X\beta + e$

$$\begin{bmatrix} 10 \\ 11.5 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

(b) Find the least square estimate of β_1, β_2
$\hat{\beta} = (X^T X)^{-1} X^T y$ where

$$(X^T X)^{-1} = (\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix})^{-1} = (\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix})^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$X^T y = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 11.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 12.5 \end{bmatrix}$$

Therefore

$$\hat{\beta} = (X^T X)^{-1} X^T y = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 12.5 \end{bmatrix} = \begin{bmatrix} 10.17 \\ 11.33 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

(c) Find the estimate of σ^2
Our estimate for σ^2 is $\frac{RSS}{n-k-1} = \frac{RSS}{3-1-1} = RSS$ where $RSS = (y - \hat{y})^T \cdot (y - \hat{y})$ and

$$y = \begin{bmatrix} 10 \\ 11.5 \\ 1 \end{bmatrix} \text{ and } \hat{y} = X\hat{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 10.17 \\ 11.33 \end{bmatrix} = \begin{bmatrix} 10.17 \\ 11.33 \\ 1.17 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -0.17 \\ 0.17 \\ 0.17 \end{bmatrix}$$

Therefore our estimate is $\hat{\sigma}^2 = RSS/1 = 3*0.17^2 = 0.0867$.

(d) Find $s_{\hat{\beta}_1}, s_{\hat{\beta}_2}$

$$Var(\hat{\beta}) = \sigma^2(X^TX)^{-1} \approx 0.0867 \cdot \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.0578 & 0.0289 \\ 0.0289 & 0.0578 \end{bmatrix}, \therefore \begin{bmatrix} \text{se}_{\hat{\beta}_1} \\ \text{se}_{\hat{\beta}_2} \end{bmatrix} = \begin{bmatrix} 0.2404 \\ 0.2404 \end{bmatrix}$$

(e) Estimate $\beta_1 - \beta_2$ and its standard error

An unbiased estimator would be $\hat{\beta}_1 - \hat{\beta}_2 = 10.1667\text{-}11.3333 = -1.167$.

$$Var(\hat{\beta}_1 - \hat{\beta}_2) = Var(\hat{\beta}_1) + Var(\hat{\beta}_2) - 2Cov(\hat{\beta}_1, \hat{\beta}_2)$$
$$= 0.0578 + 0.0578 - 2 * 0.0289 = 0.0578.$$

$$\therefore \text{se}_{\hat{\beta}_1 - \hat{\beta}_2} = 0.2404.$$

(f) Test the null $H_0 : \beta_1 = \beta_2$

$$t_{n-2} = t_1 = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2)}{\text{se}_{\hat{\beta}_1 - \hat{\beta}_2}} = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{\text{se}_{\hat{\beta}_1 - \hat{\beta}_2}}$$
$$= \frac{-1.167}{0.2404} = -4.854.$$

2*pt$(-4.854\,, df = 1) = 0.1293 > 0.05$

Therefore we do not reject the null.

2. Bayesian Statistic

   Consider a biased coin with probability of landing heads equal to θ. Also, let X be a random variable that is equal to 1 when the coin lands heads, and is otherwise equal to 0. In other words, we assume that $P(X = 1|\Theta = \theta) = \theta$, while $P(X = 0|\Theta = \theta) = 1 - \theta$. We consider the following prior for Θ :

$$f_\Theta(\theta) = \begin{cases} \theta e^\theta & \text{for } 0 \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the probability that a coin toss results in heads: i.e., compute $P(X = 1)$.
   Solution:

$$P(X=1) = \int_0^1 P(X=1|\Theta=\theta) f_\Theta(\theta) d\theta = \int_0^1 \theta * \theta e^\theta d\theta = \int_0^1 \theta^2 e^\theta d\theta$$

$$\text{By parts:} \quad = [\theta^2 e^\theta]_0^1 - 2\int_0^1 \theta e^\theta = [e^1 - 0] - 2 = e - 2 \approx 0.7183$$

   (b) Given that a coin toss results in heads, find the posterior density for Θ.
   Solution:

$$f_{\Theta|X}(\theta|X=1) = \frac{P(X=1|\Theta=\theta)f_\Theta(\theta)}{P(X=1)} = \frac{\theta^2 e^\theta}{e-2}$$

   for $\theta \in [0, 1]$, zero otherwise.

   (c) Given that the first toss resulted in heads, find the conditional probability of heads on the second toss. (*Hint:* this is tricky. Use the posterior from part (b) as the new prior!)

   Solution:
   The "new" prior is the function from part (b).

$$P(X_2=1|X_1=1) = \int_0^1 P(X=1|\Theta=\theta) f_{\Theta|X_1)}(\theta|X_1=1) d\theta = \int_0^1 \frac{\theta^3 e^\theta}{e-2} d\theta$$

$$\text{by parts:} \quad = \frac{1}{e-2}\{[\theta^3 e^\theta]_0^1 - 3\int_0^1 \theta^2 e^\theta d\theta\}$$

$$\text{from part (a):} \quad = \frac{1}{e-2}\{e - 3(e-2)\} = \frac{6-2e}{e-2} \approx 0.7844$$