# STAT 135 Midterm B

1] (a) $f(x|\theta) = \begin{cases} \frac{5x^4}{2\theta^5} & -\theta \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$ $\quad \theta > 0$

$$\mu = E(X) = \int_{-\theta}^{\theta} x \frac{5x^4}{2\theta^5} dx = \frac{1}{2\theta^5} \int_{-\theta}^{\theta} 5x^5 dx$$

$$= \frac{1}{2\theta^5}\left[\int_{-\theta}^{\theta} 5x^5 dx\right] = \frac{1}{2\theta^5}\left[\left[\frac{5}{6}x^6\right]_{-\theta}^{\theta}\right] = \frac{1}{2\theta^5}\left[\frac{5}{6}\theta^6 - \frac{5}{6}(\theta)^6\right]$$
$$= 0$$

$$\mu_2 = E(X^2) = \int_{-\theta}^{\theta} x^2 \frac{5x^4}{2\theta^5} dx = \frac{1}{2\theta^5}\left[2\int_0^{\theta} 5x^6 dx\right] =$$

$$\frac{1}{2\theta^5}\left[2\left[\frac{5}{7}x^7\right]_0^{\theta}\right] = \frac{1}{\theta^5}\left[\frac{5}{7}\theta^7\right] = \frac{5}{7}\theta^2$$

$$\mu_2 = \frac{5}{7}\theta^2 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{5}{7}\hat{\theta}_{MM}^2$$

$$\hat{\theta}_{MM} = \sqrt{\frac{7}{5n}\sum_{i=1}^{n} x_i^2}$$

(b) $$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{5x_i^4}{2\theta^5} = \frac{\prod_{i=1}^{n} 5x_i^4}{2^n\theta^{5n}}$$

now $-\theta \le x \le \theta$ (boundary conds. from question.)

$$\Rightarrow -\theta \le X_{(1)} \le X_{(n)} \le \theta$$

$$\Rightarrow \hat{\theta}_{MLE} = \max\{-X_{(1)}, X_{(n)}\}$$

Let $\mathbb{1}\{\cdot\}$ be indicator function.

Then, $L(\theta) = f(X_n|\theta)$ can be written as

$$L(\theta) = f(X_n|\theta) = \frac{\prod_{i=1}^{n} 5x_i^4}{2^n\theta^{5n}} \cdot \mathbb{1}\{\theta \ge \max\{-X_{(1)}, X_{(n)}\}\}$$

Thus, let $h(X_n) = 1$, $g(T(X_n), \theta) =$ ____

By factorization theorem, $\hat{\theta}_{MLE}$ is thus a sufficient statistic for $\theta$.

(d) from (c), $P(\hat{\theta}_{MLE} \leq t) = \left(\frac{t}{\theta}\right)^{5n}$ $t \in [0, \theta]$

$\Rightarrow$ pdf for $\hat{\theta}_{MLE}$ is: $f(t) = 5n \frac{t^{5n-1}}{\theta^{5n}}$

$$E(\hat{\theta}_{MLE}] = \int_0^{\theta} t f(t)\,dt = \frac{5n}{\theta^{5n}} \int_0^{\theta} t^{5n}\,dt$$

$$= \frac{5n}{\theta^{5n}}\left[\frac{1}{(5n+1)} t^{5n+1}\right]_0^{\theta} = \frac{5n}{\theta^{5n}}\left[\frac{1}{5n+1}\theta^{5n+1}\right]$$

$$= \frac{5n}{5n+1}\theta$$

$$E[Y] = \frac{5n+1}{5n} E[\hat{\theta}_{MLE}] = \frac{5n+1}{5n}\left[\frac{5n}{5n+1}\theta\right] = \theta$$

$$bias(Y) = E[Y - \theta] = E(Y) - \theta = \theta - \theta = 0$$

Hence $Y$ is unbiased estimator of $\theta$.

$$Var(Y) = \left(\frac{5n+1}{5n}\right)^2 var(\hat{\theta}_{MLE})$$

$$Var(\hat{\theta}_{MLE}) = E(\hat{\theta}_{MLE}^2) - E(\hat{\theta}_{MLE})^2$$

$$E(\hat{\theta}_{MLE}^2) = \int_0^{\theta} t^2 f(t)\,dt = \frac{5n}{\theta^{5n}} \int_0^{\theta} t^{5n+1}\,dt$$

$$= \frac{5n}{\theta^{5n}}\left[\frac{1}{5n+2} t^{5n+2}\right]_0^{\theta} = \frac{5n}{\theta^{5n}}\left[\frac{\theta^{5n+2}}{5n+2}\right] = \frac{5n}{5n+2}\theta^2$$

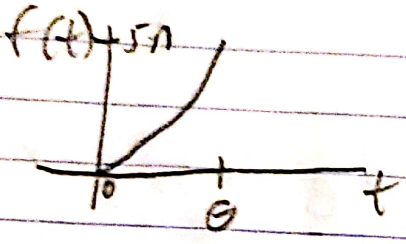
$$Var(\hat{\theta}_{MLE}) = \frac{5n}{5n+2}\theta^2 - \left(\frac{5n}{5n+1}\right)^2\theta^2 = \theta^2\left[\frac{5n}{5n+2} - \left(\frac{5n}{5n+1}\right)^2\right]$$

$$Var(Y) = \left(\frac{5n+1}{5n}\right)^2\left[\theta^2\left[\frac{5n}{5n+2} - \left(\frac{5n}{5n+1}\right)^2\right]\right]$$

$$= \left(\frac{5n+1}{5n}\right)^2\theta^2\left[\frac{5n}{5n+2} - \left(\frac{5n}{5n+1}\right)^2\right] = \frac{\theta^2}{5n(5n+2)}$$

(e) $F(t) = \left(\frac{t}{\theta}\right)^{5n}, \; t \in [0, \theta]$

$f(t) = 5n \frac{t^{5n-1}}{\theta^{5n}}, \; t \in [0, \theta]$




Density is not bellshaped, not converging to a Normal distribution.
Not contradictory from theorem D.
(·) pdf violates assumptions for asymptotic normality of MLE:
- independence of support from θ
$\{x : f(x|\theta) > 0\}$ not dependent on θ condition.
we see f(t) depends on θ.

(f) $\text{eff}(Y, \hat{\theta}_{MM}) = \frac{Var(Y)}{Var(\hat{\theta}_{MM})} = \frac{\frac{\theta^2}{5n(5n+2)}}{\frac{\theta^2}{45n}}$

$= \frac{45n}{5n(5n+2)} = \frac{9}{5n+2}$

as $n \to \infty$, $\text{eff}(Y, \hat{\theta}_{MM}) \to 0$

Hence, $\text{eff}(Y, \hat{\theta}_{MM}) \to 0$ for large N.

2] (a) $L(\mu, \theta \mid X_n, Y_m) = \left(\prod_{i=1}^{n} \theta e^{-\theta x_i}\right)\left(\prod_{i=1}^{m} \mu e^{-\mu y_i}\right)$

$= \left(\theta^n \prod_{i=1}^{n} e^{-\theta x_i}\right)\left(\mu^m \prod_{i=1}^{m} e^{-\mu y_i}\right)$

under $H_0: \theta = \mu$

$L(\theta) = \theta^{n+m}\left(\prod_{i=1}^{n} e^{-x_i} \prod_{i=1}^{m} e^{-y_i}\right)^{\theta}$

Hence, $\lambda(X_n, Y_m) = \dfrac{\hat{\theta}_0^{\,n+m}\left(\prod_{i=1}^{n} e^{-x_i} \prod e^{-y_i}\right)^{\hat{\theta}_0}}{\hat{\theta}^n \hat{\mu}^m \left(\prod_{i=1}^{n} e^{-\hat{\theta} x_i} \prod_{i=1}^{m} e^{-\hat{\mu} y_i}\right)}$

$= \left(\dfrac{\hat{\theta}_0^{\,n+m}}{\hat{\theta}^n \hat{\mu}^m}\right)\left(\prod_{i=1}^{n} e^{-x_i}\right)^{\hat{\theta}_0 - \hat{\theta}}\left(e^{-y_i}\right)^{\hat{\theta}_0 - \hat{\mu}}$

for $\hat{\theta}_0$ : $\ell(\theta) = n+m \log(\theta) + \theta \log\left(\prod_{i=1}^{n} e^{-x_i} \prod_{i=1}^{m} e^{-y_i}\right)$

$\dfrac{\partial \ell}{\partial \theta} = \dfrac{n+m}{\theta} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} y_i = 0$

$\dfrac{n+m}{\theta} = + \sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i$

$\hat{\theta}_0 = \dfrac{n+m}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i}$

For $\hat{\theta}, \hat{\mu}$: $\ell(\theta, \mu) = n \log(\theta) - \theta \sum_{i=1}^{n} x_i + m(\log(\mu)) - \mu \sum_{i=1}^{m} y_i$

$\dfrac{\partial \ell}{\partial \theta} = \dfrac{n}{\theta} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\theta} = \dfrac{n}{\sum_{i=1}^{n} x_i}$

$$\frac{\partial l}{\partial \mu} = \frac{m}{\mu} - \sum_{i=1}^{m} Y_i = 0 \Rightarrow \hat{\mu} = \frac{m}{\sum_{i=1}^{m} Y_i}$$

$$\lambda(X_n, Y_m) = \left(\frac{n+m}{n}\right)^n \left(\frac{n+m}{m}\right)^m \left(1 - \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i}\right)^n \left(\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i}\right)^m$$

let $T(X_n, Y_n) = \dfrac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i} \Rightarrow \lambda(X_n, Y_m)$ is a unimodal function of $T$.

$\Rightarrow R\{T \leq C_1 \text{ or } T \geq C_2\}$ is equivalent rejection region.

(6) when $H_0$ is true, $\theta = \mu$

numerator: $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

denominator: $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

$\sum_{i=1}^{m} Y_i \sim \text{Gamma}(m, \theta)$

from HW3, practice midterm $\Rightarrow$ $\frac{W}{W+V} \sim \text{Beta}(n, m)$ if $W \sim \text{Gamma}(n, 1/\mu)$

$V \sim \text{Gamma}(m, 1/\mu)$

$W$ indep of $V$

$\Rightarrow$ applies here as well, so let $T = \dfrac{\sum X_i}{\sum X_i + \sum Y_i}$

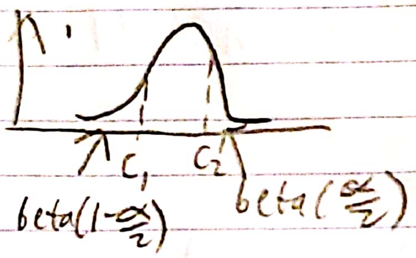
$\Rightarrow T \sim \text{Beta}(n, m)$

(c) $n=13, m=17$

$qbeta(\alpha, 13, 17, lower.tail=False) = 0.5825361$

$\uparrow$ $C_2$

$R = \{T \le C_1 \text{ or } T \ge C_2\} = \alpha$ $\quad C_1 + C_2$ are constants




Hence, threshold is 0.5825361.

(d) $R = \{T \le C_1 \text{ or } T \ge C_2\}$

For this data, $T(X_n, Y_n) = 0.5524846$

p-value $= P(T(X_n, Y_n) \le 0.5524846 \mid H_0)$

$= 2(1 - pbeta(0.5524846, 13, 17, lower.tail = True)$

p-value $= 0.1890958 > 0.05$

Hence, we fail to reject $H_0$ at significance level 0.05.

3 (a) $n = 57$, $\bar{X}_n = 70.4211$, $S = 9.9480$

assume pulse $\sim N(\mu, \sigma^2)$

For $\mu$; 99% exact CI:

$$70.4211 \pm \frac{9.9480}{\sqrt{57}} t_{56}(0.005)$$

$= [68.90758, 73.93462]$

for $\sigma^2$: 99% exact CI

$$\left[\frac{n\hat{\sigma}_n^2}{\chi_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}_n^2}{\chi_{n-1}(1-\frac{\alpha}{2})}\right]$$

$= [64.84229, 173.6162]$

(b) $H_0: \mu = 72$ $H_A: \mu < 72$

at significance level $\alpha = 0.05$.

95% exact CI for $\mu$: $\bar{x}_n \pm \frac{s}{\sqrt{n}} t_{56}(0.025)$

$[67.78154, 73.06066]$

theorem c of lecture 9 → using duality between confidence intervals and hypothesis tests, and seeing that 72 is in the confidence interval, we fail to reject the null hypothesis at significance level 0.05. (since $\mu$ is in acceptance region, which is constructed by confidence interval.)

(c) $-2\lambda(x_n) \xrightarrow{d} \chi_v^2$ as $n \to \infty$ by wilk's Thm

$v = \dim(\Theta) - \dim(\Theta_0)$

$= 1$

~~(a)~~)

p-value $= P(R \mid H_0) =$

$P(-2\log \lambda(X_n) \geq c' \mid \sigma^2 = 81)$

$c' = $ qchisq( 0.05 , 1, lower.tail = False)

$= 3.841459$

LRT is not uniformly most powerful in this case.

Sample variance not fall in rejection region since it is contained within a tighter confidence interval in (a).

4] (a) $\mu_1 = E(X) = \int_1^\infty x \frac{\theta}{x^{\theta+1}} dx = \theta \int_1^\infty \frac{x}{x^{\theta+1}} dx$

$\mu_2 = E(X^2) = \int_1^\infty x^2 \frac{\theta}{x^{\theta+1}} dx = \theta \int_1^\infty \frac{x^2}{x^{\theta+1}} dx$

(b) $L(\theta) = \prod_{i=1}^n \frac{\theta}{x_i^{\theta+1}} = \frac{\theta^n}{\prod_{i=1}^n x_i^{\theta+1}}$

$\ell(\theta) = n\log(\theta) - (\theta+1)\sum_{i=1}^n x_i$

$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i$ $\quad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2}$

$I(\theta) = \frac{n}{\theta^2}$

CR bound $= \frac{1}{nI(\theta)} = \frac{1}{n\left[\frac{n}{\theta^2}\right]} = \frac{1}{\frac{n^2}{\theta^2}} = \frac{\theta^2}{n^2}$

$\text{var}(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$ no, sample mean does not achieve lower bound.

(c) $\hat{\theta}_{MM}$ from (a).

$$\sqrt{n}\left(g(\hat{\theta}_{MM}) - g(\mu)\right) \xrightarrow{d} g'(\mu)\, N(0, \sigma^2)$$

$g(\mu) =$ plugging in $\mu$ into $\hat{\theta}_{MM}$

$g'(\mu) =$ derivative of that.

(d) CI 99% for $\theta$ would be

$$\hat{\theta}_{MM} \pm \frac{z_{0.01/2}\, \hat{\sigma}_n}{\sqrt{n}} \leftarrow$$ $\hat{\sigma}_n$ is sqrt of variance from part (c).