STAT 135 CONCEPTS OF STATISTICS
QUIZ 1, LAB 102

July 1, 2021

Instructions: You have 35 minutes to complete the quiz and upload it on bCourses. This quiz is open book and you may use a calculator, but all work must be shown in order to receive full credit.

Problem 1 (2 points). Consider a population consisting of 5 values: $2, 7, 4, 3, 10$, and suppose we draw a sample of size 2 from this population. Consider the proportion of *sample* values that are even. Find the sampling distribution of this statistic. Is it an unbiased estimator of the proportion of the population values that are even?

Solution. List out the possible samples and corresponding proportions as follows:

Sample	Proportion	Probability
{2,7}	1/2	1/10
{2,4}	1	1/10
{2,3}	1/2	1/10
{2,10}	1	1/10
{7,4}	1/2	1/10
{7,3}	0	1/10
{7,10}	1/2	1/10
{4,3}	1/2	1/10
{4,10}	1	1/10
{3,10}	1/2	1/10

The distribution of the sample proportion $\hat{p}$ is given by

$$\mathbb{P}[\hat{p} = 0] = 1/10$$
$$\mathbb{P}[\hat{p} = 1/2] = 6/10$$
$$\mathbb{P}[\hat{p} = 1] = 3/10$$

which implies

$$\mathbb{E}[\hat{p}] = 0+(1/2)*(6/10)+1*3/10 = 3/5.$$

Hence, $\hat{p}$ is an unbiased estimate of the proportion of population values that are even.

Problem 2 (1 points). Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. discrete random variables with probability mass function, or PMF, given by

$$\mathbb{P}[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Here, $\lambda > 0$ is an unknown parameter. What is the Method of Moments estimate of the parameter λ, $\hat{\lambda}_{MM}$, as a function of $X_1, X_2, \ldots, X_n$?

Solution (Short). Using the attached sheet, we can read off

$$\mu_1 = \mathbb{E}[X] = \lambda.$$

Hence, $\hat{\lambda}_{MM} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Solution (Long). For those who don't look at the sheet, direct calculation gives us

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\mathbb{P}[X=k] \\
&= \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!} \\
&= e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\
&= e^{-\lambda}\lambda\sum_{k=0}^{\infty}\frac{\lambda^{k}}{k!} \\
&= e^{-\lambda}\lambda e^{\lambda} \\
&= \lambda.
\end{aligned}$$

Problem 3 (2 points). Consider the setup of the previous problem for $n = 100$. Suppose we compute $\hat{\lambda}_{MM} = 0.25$. Compute the bootstrap 95% confidence interval for λ and compare with the corresponding interval which uses a conservative estimate for the standard error, assuming we know that the true parameter $\lambda \in [0, 1]$.

Solution. By linearity we know that $\mathbb{E}[\hat{\lambda}_{MM}] = \lambda$. Hence by the central limit theorem we know that

$$\sqrt{n}(\hat{\lambda}_{MM} - \lambda) \to \mathcal{N}(0, \sigma^2_{\hat{\lambda}_{MM}}),$$

where

$$\sigma_{\hat{\lambda}_{MM}} = \sqrt{\frac{\lambda}{n}}.$$

Using our known bounds on the values of λ, we can conservatively bound the standard error by $\sigma_{\hat{\lambda}_{MM}} \le \frac{1}{\sqrt{n}}$.

Thus, the bootstrap 95% confidence interval for λ is given by

$$\hat{\lambda}_{MM} \pm 1.96\sqrt{\frac{\hat{\lambda}_{MM}}{100}} = 0.25 \pm 1.96 \times \sqrt{\frac{0.25}{100}},$$

and a more conservative confidence interval is given by

$$0.25 \pm 1.96 \times \sqrt{\frac{1}{100}}.$$