

**STAT 135 Concepts of Statistics**

Name: \_\_\_\_\_

Summer 2021 Lec 001

**Practice Midterm**

In class, 07/15/2021

**Time Limit:** 120 Minutes

*Instructions:* This exam is open book. It contains 3 pages (including this cover page) and 5 questions. Total of points is ~~71~~ 105

1. Start with the problems that you find the easiest. If you are stuck, move on to the next question.
2. Write clearly and show your work. Ambiguous or insufficient reasoning will be marked incorrect.
3. The bCourses page will be closed at 11:02 AM. Make sure you leave ample time to scan and submit your answers.
4. The abbreviations used in this exam include: probability density function (**pdf**), cumulative distribution function (**cdf**), Method of Moments (**MM**) estimator, Maximum likelihood estimator (**MLE**), Confidence interval (**CI**), Likelihood ratio test (**LRT**)

1. (34 points) Let  $X_1, \dots, X_n$  be i.i.d from a population with pdf

$$f(x|\theta) = \begin{cases} |x|/\theta^2, & \text{if } -\theta \leq x \leq \theta \\ 0, & \text{otherwise.} \end{cases}$$

in which  $\theta > 0$  is an unknown parameter.

- (a) (3 points) Find a MM estimator of  $\theta$ ;
- (b) (4 points) Find the MLE of  $\theta$ . Is it a sufficient statistic for  $\theta$ ? Explain why;
- (c) (5 points) \*(Bonus question) Denote the MLE by  $\hat{\theta}_{\text{MLE}}$ . Prove that  $\hat{\theta}_{\text{MLE}}$  can only take values in  $[0, \theta]$  and its the cdf satisfies

$$P(\hat{\theta}_{\text{MLE}} \leq t) = \left(\frac{t}{\theta}\right)^{2n}, \quad t \in [0, \theta].$$

- (d) (8 points) Show that  $Y = \frac{2n+1}{2n} \hat{\theta}_{\text{MLE}}$  is an unbiased estimator of  $\theta$ . Calculate  $\text{Var}(Y)$ .
- (e) (8 points) Use the cdf in (c) to explain why  $\hat{\theta}_{\text{MLE}}$  can not be approximated by a Normal distribution as  $n \rightarrow \infty$ . Is it contradictory to Theorem D of Lecture 4?

$$(a) \quad E X = \int_{-\theta}^{\theta} \frac{x |x|}{\theta^2} dx = 0.$$

$$E X^2 = \int_{-\theta}^{\theta} \frac{x^2 |x|}{\theta^2} dx = 2 \int_0^{\theta} \frac{x^3}{\theta^2} dx$$

$$= \frac{\theta^2}{2}.$$

Therefore, MM estimator of  $\theta$  is

$$\hat{\theta}_{MM} = \sqrt{\frac{2}{n} \sum_{i=1}^n x_i^2}.$$

$$(b) \quad \text{The joint likelihood } L(\theta) = \prod_{i=1}^n f(x_i | \theta)$$

$$= \frac{\prod_{i=1}^n |x_i|}{\theta^{2n}} \mathbb{1}(-\theta \leq x_{(1)} < x_{(n)} \leq \theta)$$

$$= \frac{\prod_{i=1}^n |x_i|}{\theta^{2n}} \mathbb{1}(\theta \geq \max\{-x_{(1)}, x_{(n)}\})$$

Thus,  $\hat{\theta}_{MLE} = \max\{-x_{(1)}, x_{(n)}\}$

By F-N factorization theorem,  $\hat{\theta}_{MLE}$  is sufficient.

(d) The pdf of  $\hat{\theta}_{MLE}$  is:

$$f(t) = 2n \frac{t^{2n-1}}{\theta^{2n}}, \quad t \in [0, \theta]$$

$$E(\hat{\theta}_{MLE}) = \int_0^{\theta} t f(t) dt = \int_0^{\theta} \frac{2n t^{2n}}{\theta^{2n}} dt$$

$$= \frac{2n}{2n+1} \theta,$$

$$E(Y) = \frac{2n+1}{2n} E(\hat{\theta}_{MLE}) = \theta. \quad \text{Therefore, } \hat{\theta}_{MLE} \text{ is unbiased.}$$

(d) Cont'd To calculate  $\text{var}(Y)$ , we calculate

$$\text{var}(\hat{\theta}_{MLE}) = E(\hat{\theta}_{MLE}^2) - [E(\hat{\theta}_{MLE})]^2$$

in which

$$E(\hat{\theta}_{MLE}^2) = \int_0^\theta 2n \frac{t^{2n+1}}{\theta^{2n}} dt = \frac{2n}{2n+2} \theta^2$$

Thus,

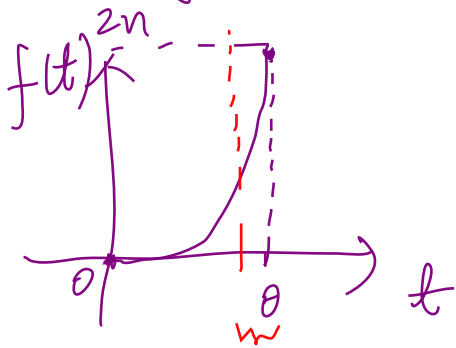
$$\text{var}(\hat{\theta}_{MLE}) = \frac{2n}{2n+2} \theta^2 - \left[ \frac{2n}{2n+1} \theta \right]^2 \quad \leftarrow$$

$$= \frac{2n \theta^2}{(2n+2)(2n+1)^2} \quad \leftarrow$$

and  $\text{var}(Y) = \left( \frac{2n+1}{2n} \right)^2 \text{var}(\hat{\theta}_{MLE}) = \frac{\theta^2}{2n(2n+1)}$

(e)  $F(t) = \left( \frac{t}{\theta} \right)^{2n}$ ,  $t \in [0, \theta]$

$$f(t) = 2n \frac{t^{2n-1}}{\theta^{2n}}, \quad t \in [0, \theta]$$



The density is not bell-shaped, which clearly is not converging in distribution to a Normal.

It is not contradictory because the pdf violates the assumptions needed to ensure the asymptotic normality of MLE:

- (1) Independence of support from  $\theta$ ;
- (2) Exchangability of integration & differentiation.

$$\begin{aligned}
 \text{le)} \quad \text{eff}(\gamma, \hat{\theta}_{MM}) &= \frac{\frac{\theta^2}{2n(2n+1)}}{\frac{\theta^2}{12n}} \\
 &= \frac{b}{2n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$


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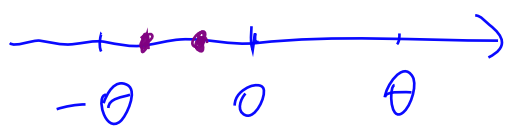
Side note:  $\sqrt{n}(\hat{\theta}_{MLE} - \theta) \rightarrow \mathcal{N}(0, \frac{b^2}{n})$

Rate of convergence for MLE:  $\sqrt{n}$   
 MM:  $\sqrt{n}$

In this case:

Rate of convergence for  $\gamma$ :  $\frac{1}{\sqrt{2n(2n+1)}}$   
 $\hat{\theta}_{MM}$ :  $\frac{1}{\sqrt{12n}}$

(c)  $\hat{\theta}_{MLE} = \max \{ -X_{(1)}, X_{(n)} \}$   
 Prove that  $\hat{\theta}_{MLE} \in [0, \theta]$   
 $- \theta \leq X_{(1)} \leq 0 \quad - \theta \leq X_{(n)} \leq 0$



① Both negative,  
 $-X_{(1)} > 0$

$\Rightarrow \hat{\theta}_{MLE} \in [0, \theta]$



②  $\max \{ -X_{(1)}, X_{(n)} \} \in [0, \theta]$

③ Both positive

$X_{(n)} > 0$

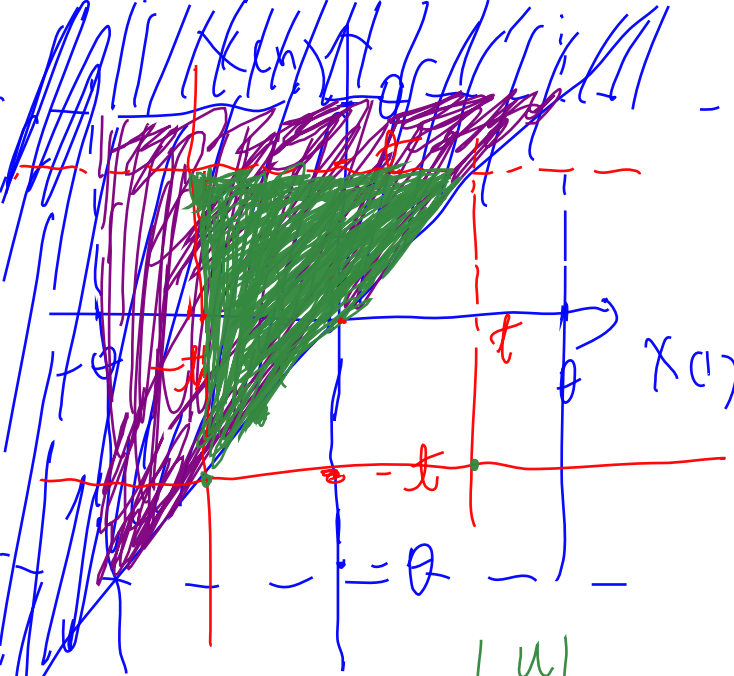
$\hat{\theta}_{MLE} \in [0, \theta] \in$

Prove  $P(\hat{\theta}_{MLE} \leq t) = \left( \frac{t}{\theta} \right)^{2n}$

$P(\hat{\theta}_{MLE} \leq t) = P(\max \{ -X_{(1)}, X_{(n)} \} \leq t)$   
 $= P(-X_{(1)} \leq t, X_{(n)} \leq t)$

$= P(\underbrace{-t \leq X_{(1)} < X_{(n)} \leq t}_{\text{range}})$

$V = X_{(1)}, U = X_{(n)}, f(u, v) = n(n-1) f(u) f(v) \times [F(u) - F(v)]^{n-2}, u \geq v.$



$$\int_{-t}^t \int_v^t f(u,v) du dv$$

$$= n(n-1) \int_{-t}^t \int_v^t f(u) f(v) [F(u) - F(v)]^{n-2} du dv$$

$$f(u) = \frac{|u|}{\theta^2}, \quad f(v) = \frac{|v|}{\theta^2}$$

$$F(u) = \frac{u|u|}{2\theta^2}, \quad F(v) = \frac{v|v|}{2\theta^2}$$

$$= \frac{n(n-1)}{\theta^{2n}} \int_{-t}^t \int_v^t |u| |v| \left( \frac{u|u|}{2} - \frac{v|v|}{2} \right)^{n-2} du dv$$

$$= \frac{n(n-1)}{\theta^{2n}} \int_{-t}^t \int_v^t |u| |v| \left( \frac{u|u|}{2} - \frac{v|v|}{2} \right)^{n-2} du dv$$

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$$= \frac{n(n-1)}{\theta^{2n}} \int_{-t}^t \int_v^t |u| |v| \left( \frac{u|u|}{2} - \frac{v|v|}{2} \right)^{n-2} du dv$$

- (f) (6 points) One can estimate  $\text{Var}(\hat{\theta}_{\text{MM}}) \approx \frac{\theta^2}{12n}$  for sufficiently large  $n$ . With this result, we can calculate the relative efficiency

$$\text{eff}(Y, \hat{\theta}_{\text{MM}}) = \frac{\text{Var}(Y)}{\text{Var}(\hat{\theta}_{\text{MM}})}.$$

Show that  $\text{eff}(Y, \hat{\theta}_{\text{MM}}) \rightarrow 0$  for large  $n$ , which shows the unbiased estimator  $Y$  is a much more efficient estimator than  $\hat{\theta}_{\text{MM}}$ .

2. (12 points) Suppose  $X_1, \dots, X_n$  are independently sampled from a Kumaraswamy distribution with parameter  $\beta$ . The probability density function of this distribution is

$$f(x|\beta) = \begin{cases} \beta(1-x)^{\beta-1}, & \text{if } x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

- (a) (6 points) Find the MM estimator  $\hat{\beta}_{\text{MM}}$  for  $\beta$ ;

(Hint:  $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = \text{Beta}(\alpha, \beta)$ .)

- (b) (6 points) Suppose a researcher collected 100 observations and the sample mean is 0.45. Calculate the MM estimate for  $\beta$ , and use the Delta Method to approximate the SE of your estimate  $\hat{\beta}_{\text{MM}}$ .

3. (16 points) Suppose  $X_1, \dots, X_n$  are i.i.d observations from a population with pmf

$$P(X = x|\theta) = \theta^x(1-\theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2}$$

- (a) (6 points) Find the MM estimator and MLE of  $\theta$ ;

- (b) (4 points) Try to find the MSE of each of the estimators (See Page 29 of Lecture 1 slides for the definition of MSE);

- (c) (6 points) Which estimator is preferred? Justify your choice.

4. (25 points) Let  $X_1, \dots, X_n$  be i.i.d with  $\text{beta}(\mu, 1)$  pdf and  $Y_1, \dots, Y_m$  be i.i.d with  $\text{beta}(\theta, 1)$ . Also assume that  $X$ 's and  $Y$ 's are independent of each other.

- (a) (8 points) Find an LRT of  $H_0 : \theta = \mu$  versus  $H_1 : \theta \neq \mu$ , and show that the LRT statistic is

$$T(\mathbf{X}_n, \mathbf{Y}_m) = \frac{\sum_{i=1}^n \log X_i}{\sum_{i=1}^n \log X_i + \sum_{i=1}^m \log Y_i}.$$

- (b) (5 points) Find the distribution of  $T$  when  $H_0$  is true, and show how to get a test of size  $\alpha = 0.05$ .

(Hint:  $-\log X_i \sim \text{Gamma}(1, 1/\mu)$  and  $-\log Y_i \sim \text{Gamma}(1, 1/\theta)$ . Also,  $W/(W+V) \sim \text{Beta}(n, m)$  if  $W \sim \text{Gamma}(n, 1/\mu)$ ,  $V \sim \text{Gamma}(m, 1/\mu)$  and  $W$  is independent of  $V$ .)

$$\begin{aligned}
 2. (a) E(X) &= \int_0^1 \beta x (1-x)^{\beta-1} dx \\
 &= \beta \int_0^1 x^{2-1} (1-x)^{\beta-1} dx = \beta \text{Beta}(2, \beta) \\
 &= \beta \frac{\Gamma(2) \Gamma(\beta)}{\Gamma(2+\beta)} = \beta \frac{\Gamma(2)}{(1+\beta) \Gamma(\beta)} \\
 &= \frac{1}{1+\beta}
 \end{aligned}$$

$$\Rightarrow \hat{\beta}_{MM} = \frac{1}{\bar{X}_n} - 1$$

$$(b) \bar{X}_{100} = 0.45$$

$$\hat{\beta}_{MM} = \frac{1}{0.45} - 1 = 1.222$$

$$\text{Denote } g(\mu) = \frac{1}{\mu} - 1$$

$$\text{By the Delta Method, } EX = \frac{1}{1+\beta} = \frac{\text{var}(X)}{(1+\beta)^2(2+\beta)}$$

$$\sqrt{n} (g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) N(0, b^2)$$

$$\text{Thus: } SE(\hat{\beta}_{MM}) = \sqrt{\text{var}(\hat{\beta}_{MM})} = \sqrt{\frac{\hat{\beta}_{MM} (1 + \hat{\beta}_{MM})^2}{n(2 + \hat{\beta}_{MM})}}$$

$$\begin{aligned}
 \text{var}(g(\bar{X}_n)) &= \text{var}\left(\frac{1}{\bar{X}_n} - 1\right) = \text{var}(\hat{\beta}_{MM}) \\
 &= \frac{[g'(\mu)]^2 b^2}{n} = \frac{\left[-\frac{1}{\mu}\right]^2 \frac{\beta}{(1+\beta)^2(2+\beta)}}{n} = \frac{\beta(1+\beta)^2}{n(2+\beta)}
 \end{aligned}$$



- (c) (4 points) Determine the threshold in the LRT rejection region so that the significance level  $\alpha = 0.05$ . You might need R to help compute the upper  $\alpha$  quantile of a  $\text{beta}(a, b)$  distribution: `qbeta(alpha, a, b, lower.tail=FALSE)`  $\leftarrow$  *qchisq*

- (d) (8 points) Suppose  $n = 13$  and  $m = 17$ . The samples collected are *qnorm*

$$\mathbf{X}_{13} = \{0.610, 0.344, 0.289, 0.700, 0.710, 0.266, 0.244, 0.919, 0.022, 0.006, 0.073, 0.849, 0.773\},$$

$$\mathbf{Y}_{17} = \{0.781, 0.479, 0.821, 0.766, 0.444, 0.443, 0.290, 0.862, 0.684, 0.151, 0.931, 0.753, 0.694, 0.121, 0.264, 0.731, 0.575\}.$$

Will you reject the null hypothesis? Report the  $p$ -value. You might need R to help compute the area under the curve of a  $\text{beta}(a, b)$  density: `pbeta(alpha, a, b, lower.tail=TRUE)` *qt*  
*qbeta*

5. (18 points) In a study of the effect of cigarette smoking on the carbon monoxide diffusing capacity (DL) of the lung, researchers found that current smokers had DL readings significantly lower than those of either ex-smokers or non-smokers. The carbon monoxide diffusing capacities for a random sample of  $n = 20$  current smokers are listed here: *pgamma*

$$\{103.768, 92.295, 100.615, 102.754, 88.602, 61.675, 88.017, 108.579, 73.003, 90.677, 71.210, 73.154, 123.086, 84.023, 82.115, 106.755, 91.052, 76.014, 89.222, 90.479\}$$

- (a) (6 points) Compute the sample mean and sample standard deviation of the above data; *pnorm*  
*pt*  
*pbeta*  
*pgamma*
- (b) (6 points) Do these data indicate that the mean DL reading for current smokers is significantly lower than 100, which is the average for nonsmokers? Use a one-sided hypothesis test, with  $\alpha = 0.01$ . Since  $n < 30$ , you will need to use exact Student's  $t$  distribution to find the rejection regions of the test.
- (c) (6 points) Calculate the 95% exact CI for population variance  $\sigma^2$ . Use this result and the duality between the CIs and hypothesis tests to make a decision about

$$H_0 : \sigma = 14.9 \text{ versus } H_1 : \sigma \neq 14.9.$$

Therefore, we fail to reject  $H_0$ .  $\rightarrow A = \{60 \in [83.59, 7]\}$

(b)  $H_0 : \mu = 100$  vs  $H_1 : \mu < 100$

$$R = \left\{ \frac{\sqrt{n} (\bar{X}_n - 100)}{S} < t_{n-1}(\alpha) \right\} \leftarrow qt$$

$$(c) \left[ \frac{n \hat{\sigma}_n^2}{\chi_{n-1}^2(\alpha/2)}, \frac{n \hat{\sigma}_n^2}{\chi_{n-1}^2(1-\alpha/2)} \right] = [8.3, 59.7]$$