Lab 6 Solution

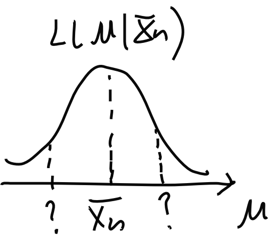
1. Likelihood Ratio Test

a) The likelihood function is

$$L(\mu \mid \underline{X}_n) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}$$

The LRT statistic should be

$$\lambda(\underline{X}_n) = \frac{\sup_{\mu\in\Theta_0} L(\mu \mid \underline{X}_n)}{\sup_{\mu\in\Theta} L(\mu \mid \underline{X}_n)} = \frac{L(\mu_0 \mid \underline{X}_n)}{\sup_{\mu\le\mu_0} L(\mu \mid \underline{X}_n)}$$



$$= \begin{cases} \dfrac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{X}_n)^2}}, & \text{if } \mu_0 \ge \bar{X}_n \\ 1, & \text{if } \mu_0 < \bar{X}_n \end{cases}$$

$$= \begin{cases} e^{-\frac{n}{2\sigma^2}(\bar{X}_n-\mu_0)^2}, & \text{if } \mu_0 \ge \bar{X}_n \\ 1, & \text{if } \mu_0 < \bar{X}_n \end{cases}$$

Therefore, the rejection region should be

$$R = \{\underbrace{\lambda(\underline{X}_n)}_{\in[0,1]} \le c\} = \left\{ e^{-\frac{n}{2\sigma^2}(\bar{X}_n-\mu_0)^2} \le c \text{ while } \mu_0 \ge \bar{X}_n \right\}$$

$$= \left\{ (\bar{X}_n-\mu_0)^2 \ge \frac{2\sigma^2 \log 1/c}{n} \right\} = \boxed{\{\bar{X}_n \le c'\}}$$

where $c' = \mu_0 - \sqrt{\frac{2\sigma^2 \log 1/c}{n}}$.

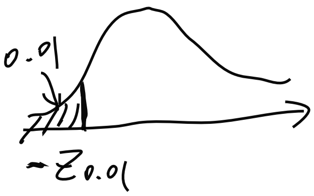
b) The significance level of this test is

$$\beta(\mu_0) = P\left(\underline{X}_n \in R \,\middle|\, \Theta_0\right) = P\left(\bar{X}_n \le c' \,\middle|\, \mu = \mu_0\right)$$

$$= P\left(\underbrace{\boxed{\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}}}_{\sim N(0,1)} \le \frac{c' - \mu_0}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0\right)$$

$\bar{X}_n \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ under Θ_0

$$= \Phi\left(\frac{c' - \mu_0}{\sigma/\sqrt{n}}\right) = 0.01$$



That is, we want

$$\frac{c' - \mu_0}{\sigma/\sqrt{n}} = -z_{0.01} \doteq -2.326$$

$$\Rightarrow c' = \mu_0 - 2.326 \frac{\sigma}{\sqrt{n}}.$$

c) Yes. By Neyman-Pearson Lemma, the LRT test

$$\delta(\underline{X}_n) = \mathbb{1}\left\{ \bar{X}_n \le \mu_0 - 2.326 \frac{\sigma}{\sqrt{n}} \right\}$$

is of significance level α, and thus it is the UMP test for any simple hypothesis test of the form

$$H_0: \mu = \mu_0 \quad \text{versus} \quad H_1: \mu = \mu_1 \quad (\mu_1 < \mu_0).$$

Equivalently, $\beta(\mu_1)$ is the largest possible power among all tests of size α for any $\mu_1 < \mu_0$.

Therefore, the LRT defined previously is the UMP test.

d) By definition,

$$\beta(\mu) = P(\bar{X}_n \in R \mid \mu) = P(\bar{X}_n \le c' \mid \mu)$$

$$= P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \le \frac{c' - \mu}{\sigma/\sqrt{n}} \,\middle|\, \mu\right)$$

$$= \Phi\left(\frac{c' - \mu}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{\mu_0 - 2.326\frac{\sigma}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}}\right)$$

$$= \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - 2.326\right).$$

e) The rejection region is

$$R = \left\{\bar{X}_n \le 1 - 2.326 \times \frac{1.5}{\sqrt{20}}\right\} = \left\{\bar{X}_n \le 0.2500228\right\}.$$

Since we observed $\bar{x}_n = 0.1$, the sample is in the rejection region and thus we reject the null hypothesis.

$$p\text{-value} = P(\bar{X}_n \le 0.1 \mid \mu_0)$$

$$= P\left(\frac{\bar{X}_n - 1}{1.5/\sqrt{20}} \le \frac{0.1 - 1}{1.5/\sqrt{20}}\right)$$

$$= \Phi\left(\frac{-0.9}{1.5/\sqrt{20}}\right) = \Phi(-2.683)$$

$$= 0.0036,$$

which is very significant.

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beta_func <- function(mu){
  mu_0 = 1
  sigma = 1
  n = 20
  res = pnorm((mu_0-mu)/(sigma/sqrt(n))-2.326)
  return(res)
}

curve(beta_func,-1,2, xlab=expression(mu), ylab = expression(beta(mu)))
abline(h=0.01, lty=2, col='red')
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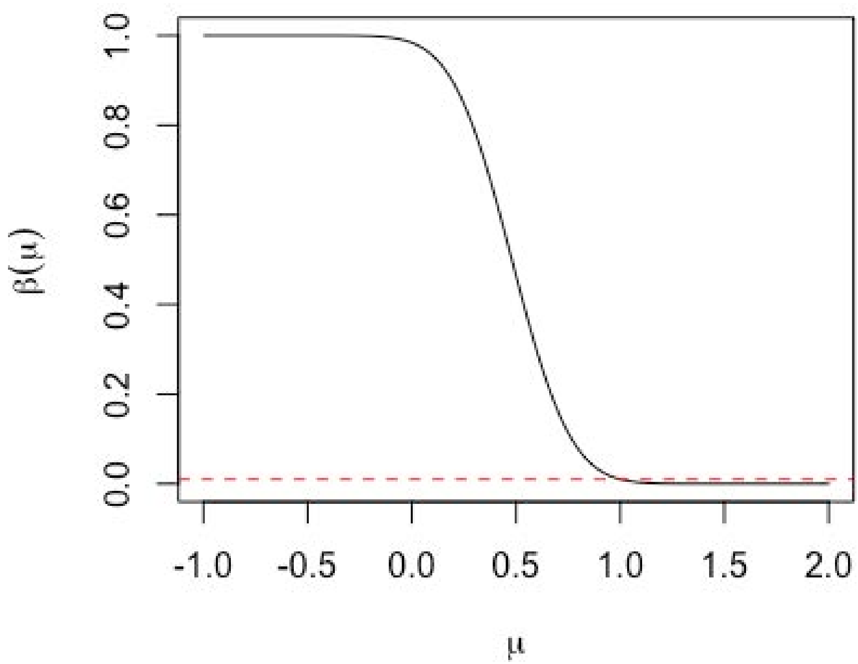

2. a) Likelihood ratio

$$\lambda(x) = \frac{L(\Theta_0 \mid x)}{L(\Theta_1 \mid x)} = \begin{cases} 0.2/0.1 = 2, & X = x_1 \\ 0.3/0.4 = 3/4, & X = x_2 \\ 0.3/0.1 = 1, & X = x_3 \\ 0.2/0.4 = 1/2, & X = x_4 \end{cases}$$

b) The Rejection region should be

$$\{\lambda(x) \le c\} = \begin{cases} \{x_1, x_2, x_3, x_4\} & \text{if } c \ge 2 \\ \{x_2, x_3, x_4\} & \text{if } 1 \le c < 2 \\ \{x_2, x_4\} & \text{if } 3/4 \le c < 1 \\ \{x_4\} & \text{if } 1/2 \le c < 3/4 \\ \phi & \text{if } 0 \le c < 1/2 \end{cases}$$

If we want the test to be of significance level $\alpha = 0.2$,

$$\phi(\lambda(x) \le c \mid \Theta_0) = 0.2$$

Note $P(X = x_4 \mid \Theta_0) = 0.2$.

Therefore, LRT should be

$$\delta = \mathbb{1}\{\lambda(x) \le c\} \quad \text{with } 1/2 \le c < 3/4$$

$$= \mathbb{1}\{X = x_4\}.$$

Similarly, if $\alpha = 0.5$, note $P(X = x_2 \text{ or } X = x_4)$

$= 0.3 + 0.2 = 0.5$.

Thus, LRT should be

$$\delta = \mathbb{1}\{\lambda(x) \le c\} \text{ with } 3/4 \le c < 1 = \boxed{\mathbb{1}\{X = x_2 \text{ or } x_4\}}$$