Stat 135
7/13/2021

Lab 7 Solution

1. Likelihood Ratio Test and Prior Probabilities

The i.i.d random variables X_1 and X_2 are $Poisson(\theta)$; that is, they have probability distribution $P(X_i = k) = e^{-\theta}\frac{\theta^k}{k!}, k = 0, 1, 2, \ldots$. We consider the two (simple) hypotheses $H_0 : \theta = 1$ and $H_1 : \theta = 2$. Define: $f(k_1, k_2|H_i) = P(X_1 = k_1, X_2 = k_2|H_i)$ wih $i = 0, 1$.
Suppose we accept H_0 if

$$\frac{f(X_1, X_2|H_0)}{f(X_1, X_2|H_1)} > 2$$

and reject H_0 otherwise.
(a) Show that the ratio above depends only on the statistic $S = X_1 + X_2$. What is the acceptance region R_0 of the above test? That is, for which values of S do we accept H_0?
(b) Find the significance level α of this test.
(c) Find the power β of this test.
Solution: (a)

$$\Lambda = \frac{f(X_1, X_2|H_0)}{f(X_1, X_2|H_1)} = \frac{f(X_1|H_0)\,f(X_2|H_0)}{f(X_1|H_1)\,f(X_2|H_1)} = \frac{e^{-1}\frac{1^{X_1}}{X_1!}\,e^{-1}\frac{1^{X_2}}{X_2!}}{e^{-2}\frac{2^{X_1}}{X_2!}\,e^{-2}\frac{2^{X_2}}{X_2!}} = e^{2}\frac{1}{2^{X_1+X_2}}$$

So: $\Lambda = \frac{e^2}{2^S}$. $\Lambda > 2 \Leftrightarrow \frac{e^2}{2^S} > 2 \Leftrightarrow 2^S < \frac{1}{2}e^2 \Leftrightarrow S < \log_2\left(\frac{e^2}{2}\right)$

$\Leftrightarrow S < 2\log_2 e - 1 \simeq 1.88$, so $R_0 = \{0, 1\}$

(b) $\alpha = P(\text{Type I error}) = P(\text{reject } H_0|H_0)$

$= P(S \geq 2|H_0)$:

Remember: if $Y \sim Poisson(\lambda)$ and $Z \sim Poisson(\mu)$ are independent then $Y + Z \sim Poisson(\lambda + \mu)$

so, under the assumption H_0, $S \sim Poisson(1+1) = Poisson(2)$.

$\Rightarrow \alpha = P(S \geq 2|H_0) = 1 - P(S = 0 \text{ or } 1|H_0) =$

$= 1 - e^{-2}\frac{2^0}{0!} - e^{-2}\frac{2^1}{1!} = \boxed{1 - 3e^{-2}}$

(c) $\beta = 1 - P(\text{accept } H_0|H_1) =$ (under H_1, $S \sim Poisson(4)$)

$= 1 - P(S = 0|H_1) - P(S = 1|H_1) = 1 - e^{-4}\frac{4^0}{0!} - e^{-4}\frac{4^1}{1!}$

$= \boxed{1 - 5e^{-4}}$ ∎

(d) Discuss with your neighbor in what sense are confidence intervals and hypothesis tests dual concepts. This refers to section 9.3 in Rice.
Solution: A 95% CI tell you whether you can reject the null hypothesis at a 5% level of significance depending whether or not the 95% CI contains the null parameter. Conversely whether you reject the null hypothesis or not in a hypothesis test tells you whether the corresponding 95%CI contains the null parameter. It goes both ways.
Put another way, you reject the null hypothesis at a 5% level of significance exactly when the your test statistic is outside of your 95% CI.

2. General Review (Short Answer Questions)

2.1. Give the precise definition of the p-value of a hypothesis test

Solution:
For a given sample $X_1, X_2, \ldots, X_n$ it is the smallest value of the significance level which makes us reject H_0 for such sample.

2.2. Is the p-value of a test a random variable? If so, why? If not, why not?

Solution: It depends on the sample $X_1, \ldots, X_n$, so it is a random variable.

2.3. Suppose that the null hypothesis H_0 is simple, while the alternative hypothesis H_1 is composite. When is a hypothesis test "uniformly most powerful?

Solution:
It is uniformly most powerful when it is the most powerful test for any simple alternative hypothesis that is contained in H_1.

2.4. State the Neyman-Pearson Lemma as precisely as you can remember.

Solution:
For simple hypotheses, among all hypothesis tests with significance level α, the one that has maximum power is the likelihood ratio test.

2.5. **True or False** (Explain your answer precisely)
If the p-value is 0.03, the corresponding test will reject at the significance level 0.02.

Solution:
FALSE. The p-value is the smallest α at which H_0 is rejected, so it will not reject at $\alpha = 0.2$ when the p-value is 0.3.

2.6. Let λ be a generalized ratio statistic. Under smoothness conditions on the probability density functions involved, the null distribution of $-2\log\lambda$ tends to what distribution?

Solution: It will tend to a chi-squared distribution with df $= \dim \Theta - \dim \Theta_0$, $n \to \infty$

2.7. **True or False** (Explain your answer precisely)
The likelihood ration statistic is always less than or equal to 1.

Solution:
TRUE.

$$\lambda(\mathcal{X}_n) = \frac{\max_{\theta\in\Theta_0} L(\theta)}{\max_{\theta\in\Theta} L(\theta)}$$

and since $\Theta_0 \in \Theta$, the numerator will be upper bounded by the denominator.