STAT 135 CONCEPTS OF STATISTICS
HOMEWORK 4

Assigned July 26, 2021, due July 22, 2021

This homework pertains to materials covered in Lecture 9 and 10. The assignment can be typed or handwritten, with your name on the document, and **with properly labeled input code and computer output for those problems that require it**. To obtain full credit, please write clearly and show your reasoning. If you choose to collaborate, the write-up should be your own. Please show your work! Upload the file to the Week 4 Assignment on bCourses.

Note in this homework, we use the following abbreviations: Uniformly most powerful (UMP) test, likelihood ratio test (LRT).

Problem 1. Suppose $X_1, \dots, X_{30}$ are independently sampled from $N(\mu, 1)$. Consider test

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu < \mu_0.$$

(1) Derive the LRT with significance level $\alpha = 0.05$;
(2) Calculate the power function of the test in (1) and visualize in `R`;
(3) Use the power function to prove that the test in (1) is of size $\alpha = 0.05$ for testing

$$H_0 : \mu \geq \mu_0 \text{ versus } H_1 : \mu < \mu_0.$$

Solution.

(1) The likelihood function is

$$L(\mu|\mathbf{X}_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(X_i - \mu)^2\right\}.$$

We have seen repeatedly that the unrestricted maximum of $L(\mu|\mathbf{X}_n)$ is attained at $\hat{\mu} = \bar{X}_n$. Now that $\Theta = \{\mu \leq \mu_0\}$, we have

$$\lambda(\mathbf{X}_n) = \frac{L(\mu_0|\mathbf{X}_n)}{\sup_{\mu\in\Theta} L(\mu|\mathbf{X}_n)} = \begin{cases} \dfrac{L(\mu_0|\mathbf{X}_n)}{L(\bar{X}_n|\mathbf{X}_n)}, & \text{if } \bar{X}_n \leq \mu_0, \\ 1, & \text{if } \bar{X}_n > \mu_0, \end{cases}$$

$$= \begin{cases} \exp\left\{-\dfrac{n}{2}(\bar{X}_n - \mu_0)^2\right\}, & \text{if } \bar{X}_n \leq \mu_0, \\ 1, & \text{if } \bar{X}_n > \mu_0. \end{cases}$$

Thus, the rejection region should be

$$R = \{\lambda(\mathbf{X}_n) \leq c\} = \{(\bar{X}_n - \mu_0)^2 \geq c' \text{ while } \bar{X}_n \leq \mu_0\} = \{\mu_0 - \bar{X}_n \geq c''\}.$$

To ensure $\alpha = 0.05$,

$$P(R|\mu=\mu_0) = P\left(\sqrt{n}(\bar{X}_n - \mu_0) \le -\sqrt{n}c''|\mu=\mu_0\right) = 0.05,$$

which requires $\sqrt{n}c'' = z_{0.05} = 1.64$ and

$$R = \left\{\bar{X}_n - \mu_0 \le -\frac{1.64}{\sqrt{n}}\right\}.$$

(2) The power can be calculated as follows

$$\beta(\mu) = P(R|\mu) = P\left[\sqrt{n}(\bar{X}_n - \mu) \le \sqrt{n}(\mu_0-\mu) - 1.64\right] = \Phi(\sqrt{n}(\mu_0-\mu)-1.64).$$

This function can be visualized in `R` as follows:

```
# mu_diff = mu-mu_0
# n = sample size
n = 20
func <- function(mu_diff) pnorm(-sqrt(n)*mu_diff-1.64)
curve(func, -2,1, xlab=expression(mu),
    ylab=expression(beta(mu)), xaxt='n', col='blue')

n = 50
func <- function(mu_diff) pnorm(-sqrt(n)*mu_diff-1.64)
curve(func, -2,1, add=TRUE, col='red')

n = 100
func <- function(mu_diff) pnorm(-sqrt(n)*mu_diff-1.64)
curve(func, -2,1, add=TRUE, col='green')

axis(1, at=0,label=expression(mu[0]))
abline(h=0.05, lty=2)
legend('topright', col=c('blue','red','green'), lty=1,
    legend = c('n=20', 'n=50', 'n=100'))
```

(3) From Figure 1, we see that $\beta(\mu)$ is a strictly decreasing function of μ no matter what the sample size n is. Denote the test in (1) by δ_n. Then δ_n is of significance level 0.05, i.e. $\sup_{\mu=\mu_0}\beta(\mu) = \beta(\mu_0) = 0.05$.

For the test in (3) where the null is also composite, the size of δ_n is

$$\text{size}(\delta_n) = \sup_{\mu\ge\mu_0}\beta(\mu) \overset{\text{monotonicity}}{=} \beta(\mu_0) = 0.05.$$

which ends the proof.

Problem 2. In each of the following situations, calculate the p-value of the observed data.

(1) For testing $H_0 : \lambda \le 1$ versus $H_1 : \lambda > 1$, $X = 4$ is observed, where $X \sim \text{Poisson}(\lambda)$.

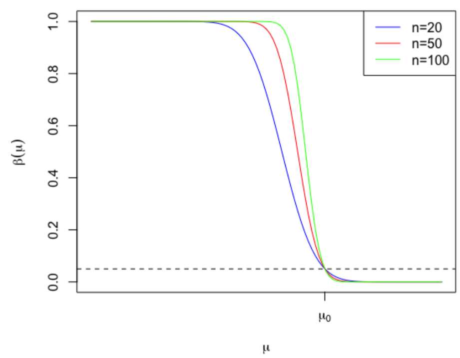

FIGURE 1. The curves of $\beta(\mu)$ with different sample sizes

(2) For testing $H_0 : \lambda \leq 2$ versus $H_1 : \lambda > 2$, $X_1 = 2$, $X_2 = 4$ and $X_3 = 7$ ares observed, where $X_i \sim \text{Poisson}(\lambda)$ independently.

Solution. First we derive the LRT of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda > \lambda_0$. The joint likelihood of $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ is

$$L(\lambda|\mathbf{X}_n) = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n X_i!}.$$

We know that the unrestricted maximum of $L(\lambda|\mathbf{X}_n)$ is attained at $\hat{\lambda} = \bar{X}_n$. Thus,

$$\lambda(\mathbf{X}_n) = \frac{L(\lambda_0|\mathbf{X}_n)}{\sup_{\lambda \geq \lambda_0} L(\lambda|\mathbf{X}_n)} = \begin{cases} \dfrac{L(\lambda_0|\mathbf{X}_n)}{L(\bar{X}_n|\mathbf{X}_n)}, & \text{if } \lambda_0 \leq \bar{X}_n, \\ 1, & \text{if } \lambda_0 > \bar{X}_n, \end{cases}$$

$$\begin{cases} \exp\{-n\lambda_0 + n\bar{X}_n \log \lambda_0 + n\bar{X}_n - n\bar{X}_n \log \bar{X}_n\}, & \text{if } \lambda_0 \leq \bar{X}_n, \\ 1, & \text{if } \lambda_0 > \bar{X}_n. \end{cases}$$

Therefore, the rejection region should be

$$R = \{\lambda(\mathbf{X}_n) \leq c\} = \{\bar{X}_n \log \lambda_0 + \bar{X}_n - \bar{X}_n \log \bar{X}_n \leq c' \text{ while } \lambda_0 \leq \bar{X}_n\}.$$

Note that $f(t) = t\log\lambda_0 + t - t\log t$ is strictly increasing in $(0, \lambda_0)$ and strictly decreasing in (λ_0, ∞). Thus, the above rejection region can be further simplified

$$R = \{\bar{X}_n \geq c''\} \text{ with } c'' > \lambda_0.$$

Similarly to Problem 1, the power function $\beta(\lambda) = P(R|\lambda)$ is a strictly increasing function of λ (*Intuitively, higher rate λ leads to higher probability of observing a sample mean greater than some threshold c''*). Therefore, Karlin-Rubin is applicable in this case.

(1) For one sample $X = 4$,

$$p\text{-value} = \sup_{\lambda' \le 1} P(X \ge 4|\lambda = \lambda') \overset{\text{monotonicity}}{=} P(X \ge 4|\lambda = 1)$$
$$= 1 - \sum_{i=0}^{3} P(X = i|\lambda = 1) = 0.01898816.$$

(2) For three samples, $X_1 + X_2 + X_3 \sim \text{Poisson}(3 * 2)$ under $\lambda = 2$. Therefore,

$$p\text{-value} = \sup_{\lambda' \le 2} P(X_1 + X_2 + X_3 \ge 2 + 4 + 7|\lambda = \lambda')$$
$$\overset{\text{monotonicity}}{=} P(X_1 + X_2 + X_3 \ge 13|\lambda = 2) = 1 - P(X_1 + X_2 + X_3 \le 12|\lambda = 2)$$

=`1-ppois(12, lambda=3*2, lower.tail=TRUE)` $= 0.008827484$.

Problem 3. Let $X_1, \dots, X_n$ be i.i.d $\text{U}(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test with a rejection region

$$R = \{X_{(1)} \ge k \text{ or } X_{(n)} \ge 1\},$$

in which k is constant to be determined by a pre-specified significance level.

(1) Determine k so that the test will have significance level α;
(2) Derive the power function of the test in (1);
(3) Prove that the test is a UMP test of size α.
(4) Determine n and k such that the test in (1) has significance level $\alpha = 0.1$ and its power function is at least 0.8 of $\theta > 1$.

Solution. Denote $U = X_{(1)}$ and $V = X(n)$. We first derive the joint density of (U, V):

$$f(u, v) = n(n-1)(v-u)^{n-2}, \quad \theta < u < v < \theta + 1.$$

(1) By definition, the significance level of this test is

$$P(R|\theta = 0) = P(X_{(1)} \ge k \text{ or } X_{(n)} \ge 1|\theta = 0) = P(X_{(1)} \ge k|\theta = 0)$$
$$= P^n(X \ge k|\theta = 0) = (1-k)^n.$$

Thus, $k = 1 - \alpha^{1/n}$.

(2) By definition, the power function is

$$\beta(\theta) = P(R|\theta) = P(X_{(1)} \ge 1 - \alpha^{1/n} \text{ or } X_{(n)} \ge 1|\theta)$$
$$= P(X_{(1)} \ge 1 - \alpha^{1/n}|\theta) + P(X_{(n)} \ge 1|\theta) - P(X_{(1)} \ge 1 - \alpha^{1/n} \text{ and } X_{(n)} \ge 1|\theta).$$

The domain of integration in the above joint probability is illustrated in Figure 2. Notice that we have an additional restriction that $\theta < u < v < \theta + 1$. Therefore, we need to integrate in different cases:

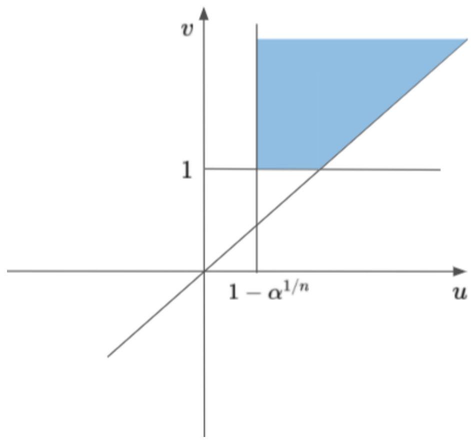

FIGURE 2. The domain of integration to calculate = $P(X_{(1)} \geq 1-\alpha^{1/n}$ and $X_{(n)} \geq 1|\theta)$.

- If $1 < \theta \leq 1-\alpha^{1/n}$,

$$P(X_{(1)} \geq 1-\alpha^{1/n} \text{ and } X_{(n)} \geq 1|\theta) = \int_1^{\theta+1}\int_{1-\alpha^{1/n}}^{v} f(u,v)dudv = (\theta+\alpha^{1/n})^n - \alpha,$$
$$P(X_{(1)} \geq 1-\alpha^{1/n}|\theta) = (\theta+\alpha^{1/n})^n,$$
$$P(X_{(n)} \geq 1|\theta) = 1-(1-\theta)^n.$$

  Thus, $\beta(\theta) = \alpha + 1 - (1-\theta)^n$.
- If $1-\alpha^{1/n} < \theta \leq 1$,

$$P(X_{(1)} \geq 1-\alpha^{1/n} \text{ and } X_{(n)} \geq 1|\theta) = \int_1^{\theta+1}\int_{\theta}^{v} f(u,v)dudv = 1-(1-\theta)^n,$$
$$P(X_{(1)} \geq 1-\alpha^{1/n}|\theta) = 1,$$
$$P(X_{(n)} \geq 1|\theta) = 1-(1-\theta)^n.$$

  Thus, $\beta(\theta) = 1$.
- If $\theta > 1$, $\beta(\theta) = P(X_{(1)} \geq 1-\alpha^{1/n}$ or $X_{(n)} \geq 1|\theta) = 1$.

(3) Let's derive the rejection region of the LRT. The joint likelihood is

$$L(\theta|\mathbf{X}_n) = \mathbb{1}\{\theta < X_{(1)} < X_{(n)} < \theta+1\}.$$

Thus,

$$\sup_{\theta\geq 0} L(\theta|\mathbf{X}_n) = \begin{cases} 0, & \text{if } X_{(1)} \leq 0, \\ 1, & \text{if } X_{(1)} > 0. \end{cases}$$
$$L(0|\mathbf{X}_n) = \begin{cases} 0, & \text{if } X_{(n)} \geq 1, \\ 1, & \text{if } X_{(n)} < 1, \end{cases}$$

which gives the likelihood ratio

$$\lambda(\mathbf{X}_n) = \frac{L(0|\mathbf{X}_n)}{\sup_{\theta\geq 1} L(\theta|\mathbf{X}_n)} = \begin{cases} 0, & \text{if } X_{(n)} \geq 1, X_{(1)} > 0 \\ 1, & \text{if } X_{(n)} < 1, X_{(1)} > 0 \\ \infty, & \text{otherwise .} \end{cases}$$

Therefore, having $\lambda(\mathbf{X}_n) \leq 1$ requires $X_{(n)} \geq 1$ and $X_{(1)} \geq 0$ or $X_{(n)} < 1$ and $X_{(1)} > 0$. This is equivalent to require $X_{(n)} \geq 1$ or $X_{(1)} \geq k$ for a small k, which means the test is UMP of size α.

(4) Since $\beta(\theta) = 1$ for $\theta = 1$, any k and n that satisfies $k = 1 - 0.1^{1/n}$ will meet the conditions.

Problem 4. Consider the i.i.d random variables $X_1, \ldots, X_n$ from Bernoulli(p). We want to test

$$H_0 : p = 0.45 \text{ versus } H_1 : p \neq 0.45.$$

(1) Derive the expression for $-2\log\lambda(\mathbf{X}_n)$.
(2) As in Example 6 of Lecture 10, simulate the sampling distribution of $-2\log\lambda(\mathbf{X}_n)$ and compare it to the χ^2 approximation.
(3) If $n = 30$ and $\bar{X}_{30} = 0.463$, would you reject H_0 at significance level $\alpha = 0.05$? What about $\alpha = 0.01$?

Solution.

(1) The likelihood ratio is

$$\lambda(\mathbf{X}_n) = \frac{0.45^{\sum_{i=1}^n X_i} 0.55^{n-\sum_{i=1}^n X_i}}{\bar{X}_n^{\sum_{i=1}^n X_i}(1-\bar{X}_n)^{n-\sum_{i=1}^n X_i}},$$

and

$$-2\log\lambda(\mathbf{X}_n) = 2n\bar{X}_n \log\frac{\bar{X}_n}{0.45} + 2n(1-\bar{X}_n)\log\frac{1-\bar{X}_n}{0.55}.$$

(2) Simulate the sampling distribution under H_0 and compare it to χ^2 in R:

```
# Calculate -2log lik_ratio from one sample with 30
    observations.
neg_two_log_lambda <- function(n=30){
    samples <- rbinom(n, size = 1, prob = 0.45)
    xbar <- mean(samples)
    res <-
        2*n*xbar*log(xbar/0.45)+2*n*(1-xbar)*log((1-xbar)/0.55)
    return(res)
}

# We will calculate -2log lik_ratio from 20,000 samples
Res <- replicate(20000, neg_two_log_lambda())
hist(Res, breaks=50, freq = FALSE,
    xlab=expression(lambda(X[n])),main='')
```

```
# Plot chisq density with df=1
chisq_density <- function(x) dchisq(x, df=1)
curve(chisq_density, 0, 20, col='red', add=TRUE)
```

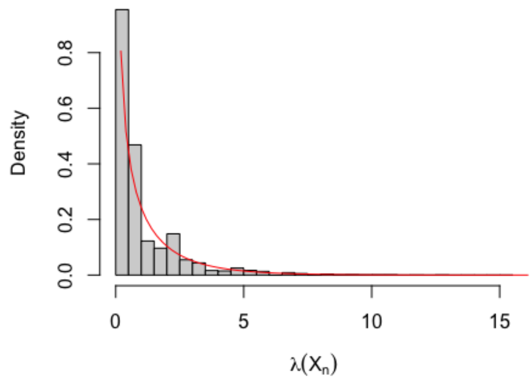

FIGURE 3. The sampling distribution of $-2\log\lambda(\mathbf{X}_n)$ compared to the density of χ^2_1.

From Figure 3, we see that the sampling distribution of $-2\log\lambda(\mathbf{X}_n)$ is well approximated by the density of χ^2_1.

(3) If $n = 30$ and $\bar{X}_{30} = 0.463$, $-2\log\lambda(\mathbf{X}_n) = 0.02045139$. The rejection region at $\alpha = 0.05$ is $R = \{-2\log\lambda(\mathbf{X}_n) \geq \chi^2_1(\alpha) = 3.841459\}$. The rejection region at $\alpha = 0.01$ is $R = \{-2\log\lambda(\mathbf{X}_n) \geq \chi^2_1(\alpha) = 6.634897\}$. Thus, the sample is not in either of the rejection regions and we fail to reject the null hypothesis at both significance levels.

Problem 5. Suppose $X_1, \ldots, X_n$ are independently sampled from Pareto(θ, ν) with pdf

$$f(x|\theta) = \begin{cases} \frac{\theta\nu^\theta}{x^{\theta+1}}, & \text{if } x \geq \nu, \\ 0, & \text{otherwise.} \end{cases}$$

in which $\theta > 0$ and $\nu > 0$. We are again interested in testing

$$H_0 : \theta = 1,\ \nu \text{ unknown}, \qquad \text{versus} \qquad H_1 : \theta \neq 1,\ \nu \text{ unknown}.$$

(1) Use your results from Homework 3 or Example 7 of Lecture 10 and express $-2\log\lambda(\mathbf{X}_n)$ in terms of

$$T(\mathbf{X}_n) = \log\left[\frac{\prod_{i=1}^n X_i}{X_{(1)}^n}\right],$$

in which $X_{(1)}$ is the sample minimum.

(2) Here are 30 i.i.d sample from this population:

$$\{3.832,\ 9.750,\ 2.868,\ 3.532,\ 44.750,\ 2.569,$$
$$6.341,\ 4.847,\ 257.054,\ 2.391,\ 107.406,\ 7.190,$$
$$3.711,\ 2.641,\ 3.779,\ 2.731,\ 3.656,\ 9.636,$$
$$3.193,\ 10.727,\ 2.380,\ 8.507,\ 34.811,\ 5.664,$$
$$2.317,\ 3.878,\ 6.578,\ 2.355,\ 2.401,\ 5.270\}$$

Test whether $\theta = 1$ using the Wilk's theorem.

Solution.

(1) From Homework 3,

$$\lambda(\mathbf{X}_n) = n^{-n}e^n \frac{T^n(\mathbf{X}_n)}{e^{T(\mathbf{X}_n)}}.$$

Therefore,

$$-2\log\lambda(\mathbf{X}_n) = 2n\log n - 2n - 2n\log T(\mathbf{X}_n) + 2T(\mathbf{X}_n).$$

(2) We can calculate from the data that $n = 30$, $T(\mathbf{X}_n) = 29.44627$ and $-2\log\lambda(\mathbf{X}_n) = 0.0103481$. The rejection region at $\alpha = 0.05$ is $R = \{-2\log\lambda(\mathbf{X}_n) \geq \chi_1^2(\alpha) = 3.841459\}$. Therefore, the sample is not in the rejection region and we fail to reject H_0.

Problem 6. Explain how to modify the t test on Page 17 of Lecture 10 to test $H_0 : \mu_X = \mu_Y + \Delta$ versus $H_1 : \mu_X \neq \mu_Y + \Delta$ where Δ is specified.

Solution. Under this H_0, we have

$$\frac{\bar{X}_n - \bar{Y}_m - \Delta}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}.$$

Therefore, we only need to replace $\frac{\bar{X}_n - \bar{Y}_m}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$ in the rejection regions on Page 17 with $\frac{\bar{X}_n - \bar{Y}_m - \Delta}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$.

Problem 7. A study was done to compare the performances of engine bearings made of different compounds (McCool 1979). Ten bearings of each type were tested. The following table gives the times until failure (in units of millions of cycles):

(1) Assume Normal populations with equal variance and test the hypothesis that there is no difference between the two types of bearings.
(2) Test the same hypothesis using a non-parametric method.
(3) Which of the methods – that of part (1) or that of part (2) – do you think is better in this case?
(4) Estimate the probability that a type I bearing will outlast a type II bearing.

Type I	Type II
3.03	3.19
5.53	4.26
5.60	4.47
9.30	4.53
9.92	4.67
12.51	4.69
12.95	12.78
15.21	6.79
16.04	9.37
16.84	12.75

Solution.

(1) The rejection region at $\alpha = 0.05$ is

$$R = \left\{ \frac{|\bar{X}_{10} - \bar{Y}_{10}|}{S_p\sqrt{\frac{1}{10} + \frac{1}{10}}} \geq t_{10+10-2}\,(\alpha/2) = 2.101 \right\}.$$

From the data set, we have $\frac{|\bar{X}_{10} - \bar{Y}_{10}|}{S_p\sqrt{\frac{1}{10} + \frac{1}{10}}} = 0.7781$, which is not in the rejection region. Therefore, we fail to reject H_0.

(2) The pooled ranks for each column are Therefore, the rank sums are

Type I	Type II
1	2
8	3
9	4
11	5
13	6
14	7
17	16
18	10
19	12
20	15

$R_1 = 130,\ R_2 = 80$ and

$$U_1 = nm + n(n+1)/2 - R_1 = 25$$
$$U_2 = nm + m(m+1)/2 - R_2 = 75.$$

Thus, $U = 25$. Looking up the table of critical values, the rejection region at $\alpha = 0.05$ is $R = \{U \leq 23\}$. Therefore, the sample is not in the rejection region and we fail to reject H_0.

(3) Method (2) is better because it does not assume normality and equal variance.
(4) From Page 44 of Rice, we estimate the probability that type I outlasts type II by

$$\hat{\pi} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{i=1}^{m} Z_{ij} = 0.75, \text{ where } Z_{ij} = \begin{cases} 1, & \text{if } X_i > Y_j \\ 0, & \text{otherwise.} \end{cases}$$

Problem 8. The assumption of equal variances, which was made in Theorem A & B of Lecture 10, is not always tenable. Let's look at a scenario where you actually have a good idea of what the variances are like for the two Normal populations. Say $X_1, \ldots, X_n \sim N(\mu_X, \sigma_X^2)$, and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$; all of the random variables are all independent. The parameters σ_X^2 and σ_Y^2 are known, whereas μ_X and μ_Y are unknown. We are going to test the equality of the means of two Normal populations:

$$H_0 : \mu_X = \mu_Y \text{ versus } H_1 : \mu_X \neq \mu_Y.$$

We now design a test based on the statistic $W = \bar{X}_n - \bar{Y}_m$.

(1) What is the sampling distribution of W?
(2) Consider the hypothesis test with the rejection region

$$R = \{|W| > k\}.$$

Which value of k achieves a pre-specified significance level of α? Express the result in terms of the known parameters σ_X^2, σ_Y^2, m, n, and α.

Solution.

(1) The sampling distribution of W is $N(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m})$.
(2) By definition, the significance level is

$$\begin{aligned} P(R|\mu_X = \mu_Y) &= P\left(|\bar{X}_n - \bar{Y}_m| > k|\mu_X = \mu_Y\right) \\ &= P\left(\frac{|\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}| > \frac{k}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}\middle| \mu_X = \mu_Y\right) = \alpha. \end{aligned}$$

Therefore,

$$\frac{k}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} = z_{\alpha/2},$$

that is, $k = z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$.

Problem 9. Now we don't assume that we know the variances for the two Normal populations in advance. If $X_1, \ldots, X_m \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim N(\mu_Y, \sigma_Y^2)$, we still have

$$\bar{X}_n - \bar{Y}_m \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

It seems natural to estimate σ_X^2 by S_1^2 and σ_Y^2 by S_2^2, in which $S_1^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $S_2^2 = (m-1)^{-1}\sum_{i=1}^m (Y_i - \bar{Y}_m)^2$. The asymptotic results indeed support this conjecture:

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_X - \mu_Y)}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \xrightarrow{d} t_\nu, \quad n \to \infty, \tag{1}$$

in which the degree of freedom of the t distribution $\nu \approx \min\{n-1, m-1\}$.

(1) Use this result to derive rejection regions for all three tests shown on Page 17 of Lecture 10 without assuming $\sigma_X^2 = \sigma_Y^2$.
(2) Perform the hypothesis testing again at significance level $\alpha = 0.05$ for the Byzantine church wood example on Page 19 of Lecture 10 using the rejection region you derived in (1). Do you reach a different conclusion? What is the p-value?

Solution.

(1) For testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$,

$$R = \left\{ \frac{\left|\bar{X}_n - \bar{Y}_m\right|}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \geq t_\nu\left(\alpha/2\right) \right\}.$$

For testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$,

$$R = \left\{ \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \geq t_\nu\left(\alpha\right) \right\}.$$

For testing $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X < \mu_Y$,

$$R = \left\{ \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \leq -t_\nu\left(\alpha\right) \right\}.$$

(2) Since $t_\nu(\alpha/2) = t_9(0.05/2) = 2.262$ and

$$\frac{\left|\bar{X}_n - \bar{Y}_m\right|}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} = 1.459884.$$

Therefore, the sample is not in the rejection region and we fail to reject H_0, which is the same conclusion as what we derived in class using the equal variance assumption.

Also, p-value = `2*pt(1.459884, 9, lower.tail=FALSE)`=0.1783.

Problem 10. Independent random samples of 17 sophomores and 13 juniors attending a large university yield the following data on grade point averages

$$\mathbf{X}_{\text{sophomores}} = \{3.04,\ 2.92,\ 2.86,\ 1.71,\ 3.60,\ 2.60,\ 3.49,\ 3.30,\ 2.28,\ 3.11,\ 2.88,\ 3.13,\ 2.82,\ 2.13,\ 2.11,\ 3.03,\ 3.27\}$$

$$\begin{aligned}\mathbf{X}_{\text{juniors}} = \{&2.56,\ 3.47,\ 2.65,\ 2.77,\ 3.26,\\ &3.00,\ 2.70,\ 3.20,\ 3.39,\ 3.00,\\ &3.19,\ 2.58,\ 2.98\}\end{aligned}$$

(1) Apply the limit (1) to obtain a 95% confidence interval for $\mu_s - \mu_j$, where μ_s and μ_j are the mean GPAs for the sophomores and juniors respectively.
(2) At the $\alpha = 0.05$ significance level, do the data provide sufficient evidence to conclude that the mean GPAs of sophomores and juniors at the university differ?

Solution.

(1) From the data, we can calculate $\bar{X}_n = 2.84$, $\bar{Y}_m = 2.980769$, $S_1^2 = 0.270225$, $S_2^2 = 0.09564103$ and $t_\nu(\alpha/2) = t_{13}(0.05/2) = 2.1604$. By the asymptotic t distribution in Problem 9, we know the 95% CI can be obtained as

$$\bar{X}_n - \bar{Y}_m \pm t_\nu\left(\alpha/2\right)\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}} = [-0.46197, 0.18043].$$

(2) By the duality between the CIs and hypothesis tests, the acceptance region for this test is

$$A = \left\{0 \in \left[\bar{X}_n - \bar{Y}_m \pm t_\nu\left(\alpha/2\right)\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}\right]\right\}.$$

In this case, $0 \in [-0.46197, 0.18043]$, and we accept H_0 and conclude that the data do not provide sufficient evidence that the mean GPAs of sophomores and juniors at the university differ.