Midterm review

Point estimation & Hypothesis testing

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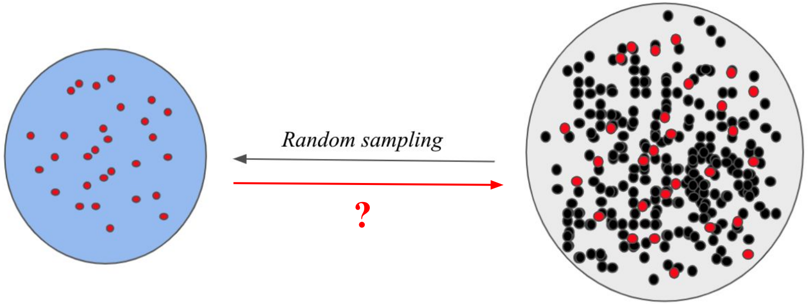
Berkeley
UNIVERSITY OF CALIFORNIA

Population vs. samples

How to learn from samples about $f(x)$?



- **Point estimate**: a particular value $\hat{\theta}$ that best approximates the parameter of interest.
  - MLE; MM estimators;
  - Sufficiency; Estimation error quantification.
- **Interval estimate**: an interval $[\theta - a,\ \theta + b]$ that would contain the true parameter θ with a certain degree of *confidence*.
- **Hypothesis testing**: whether to reject a hypothesis.
  - $\theta > a$?
  - $\theta < a$?

Formulate the research problem

↓ $Population\ f(x)$

Make propositions about $f(x)$

↓ $Parametric\ f(x \mid \theta)$

Survey design & Data collection

↓ $X_1, \ldots X_n$

Parameter estimation of θ

↓

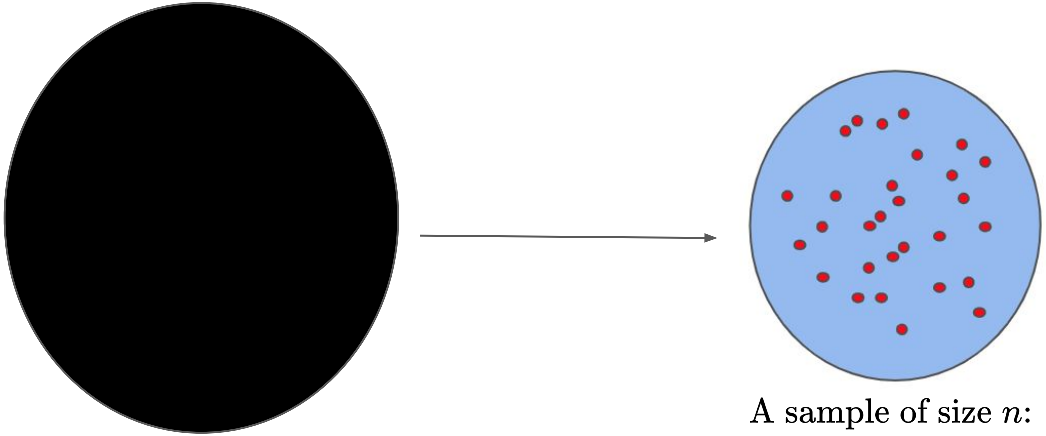
Conclusion

- **Regression**: A special parametric model

$$f(x \mid \beta, \sigma^2) = \underbrace{\mathbf{z}\beta}_{\textbf{Explanatory variables}} + N(0, \sigma^2)$$

IID assumption

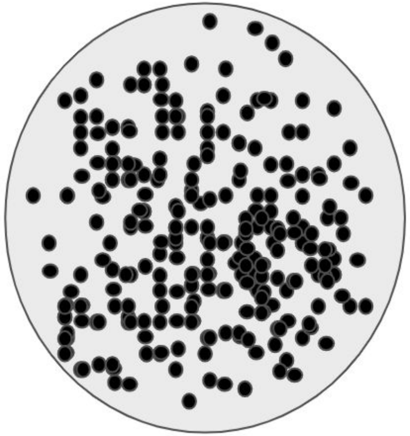
If independently drawn, each sample X_l will be identically and independently distributed (i.i.d) under the true $f(x)$.



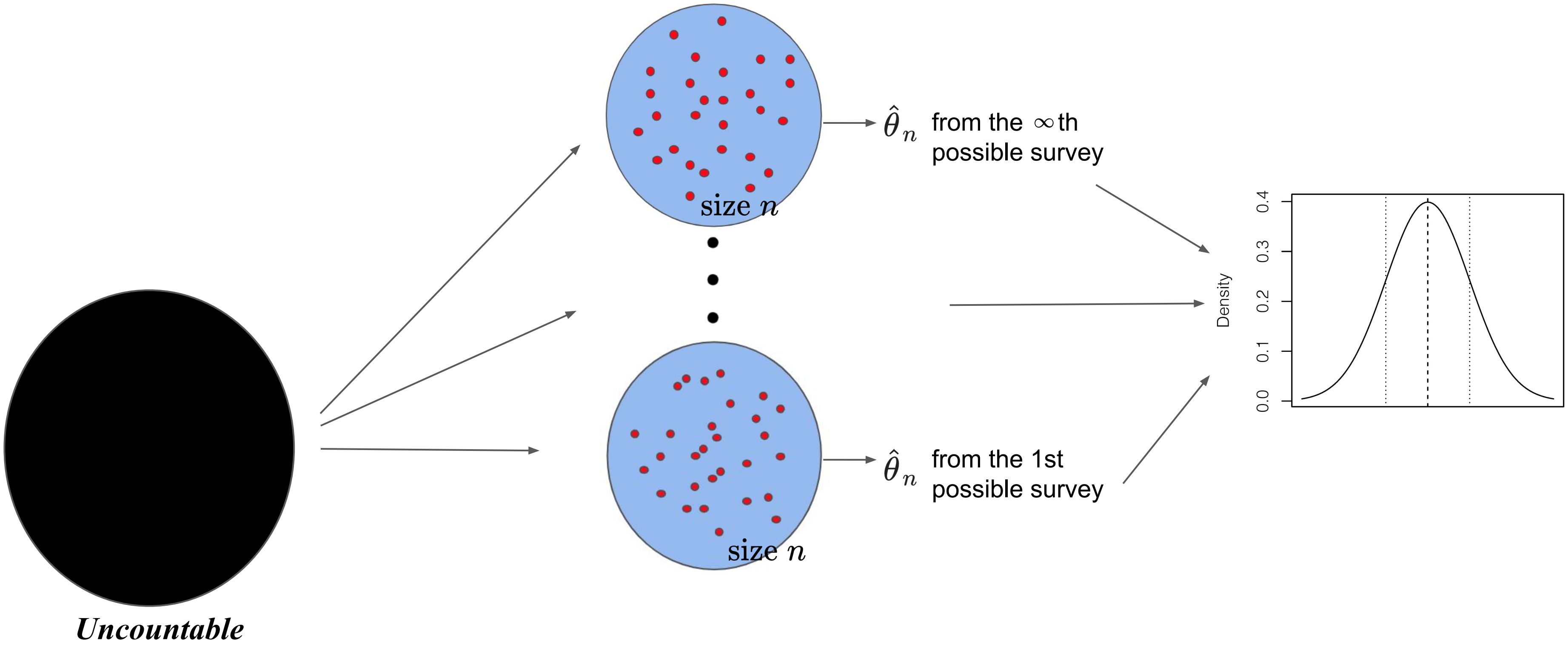
Uncountable

A sample of size n: $X_1, \ldots, X_n$

For finite population, I.I.D can ***only*** *be assumed if sampled* ***with*** *replacement.*

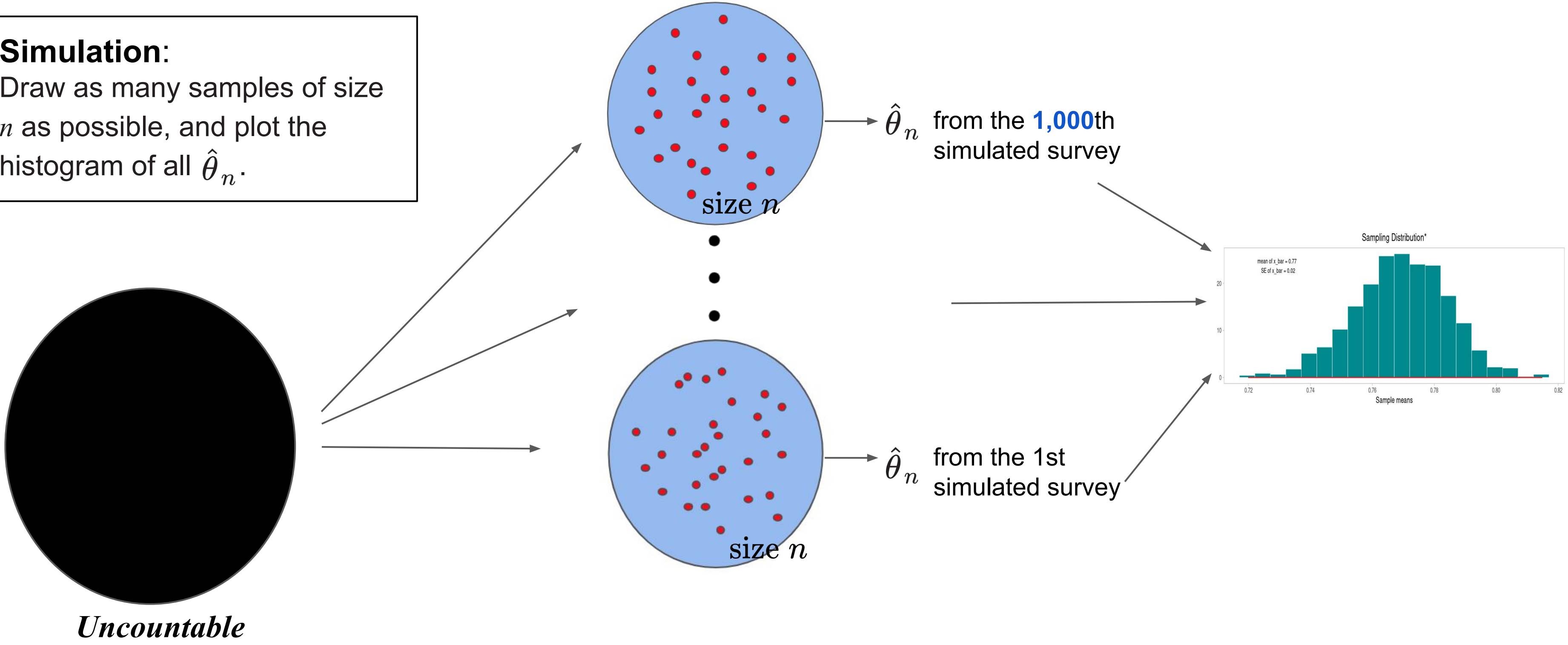


Sampling distribution



Bootstrapping

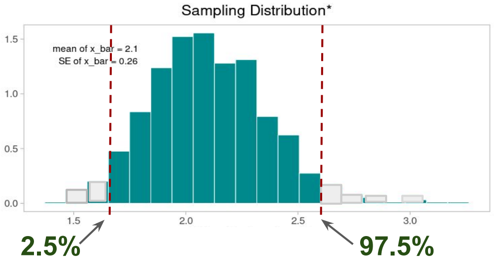
Simulation:
Draw as many samples of size n as possible, and plot the histogram of all $\hat{\theta}_n$.



Sampling distribution

- $\bar{X}_n$ and $\hat{\sigma}_n^2$ for population mean and variance;
- MM estimators;
- Maximum likelihood estimators.

1. *Exact sampling distribution;*
   e.g. MLE for $U(-\theta, \theta)$,
   $\bar{X}_n$ *for* $\mathrm{Gamma}(\alpha, \beta)$ *and* $N(\mu, \sigma^2)$
   $\hat{\sigma}_n^2$ *for* $N(\mu, \sigma^2)$
2. *Asymptotic normality;*
   $\bar{X}_n$, $g(\bar{X}_n)$, *MLE*
3. *Parametric or non-parametric bootstrapping (Page 10, Lecture 3 & Lab 3).*



Confidence interval

$$\hat{\theta}_n \pm \mathrm{Distr}_{\alpha/2} * \mathrm{SE}(\hat{\theta}_n)$$

The confidence levels represents theoretical long-run frequency (i.e., the proportion) of confidence intervals that contain the true θ.

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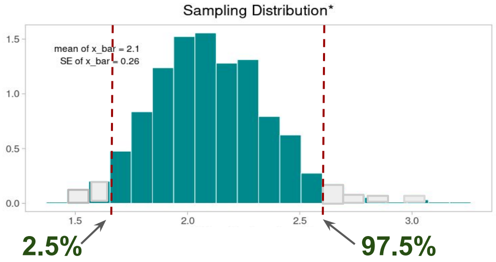


Evaluating estimators

- $\bar{X}_n$ and $\hat{\sigma}_n^2$
- MM
- MLE

When n is small,

$$\text{MSE} = \text{variance} + \text{bias}^2$$

~~Minimal MSE~~

UMVUE

CR bound

Rao-blackwell
+ sufficiency

Lehmann-Scheffé
+completeness

When n is sufficiently large,

Consistency

Asymptotic normality

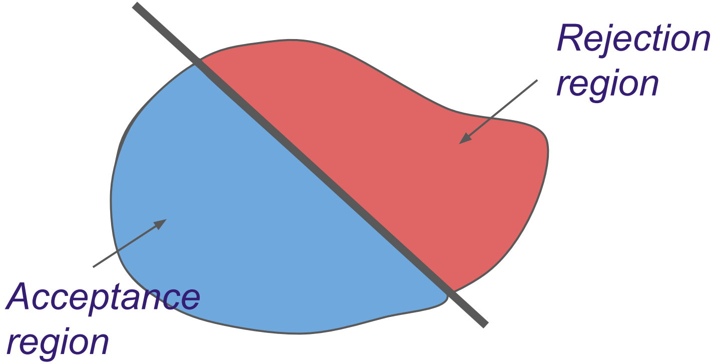
Hypothesis testing

$$H_0 : \theta \in \Theta_0 \quad \leftrightarrow \quad H_1 : \theta \in \Theta_1$$

$$R = \{\text{Unlikely } T(\mathbf{X}_n) \text{ values under } H_0\}$$

formulate ⇐ LRT

$$R = \{\lambda(\mathbf{X}_n) \le c\}$$



All possible values of $T(\mathbf{X}_n)$

Strategy: *Maximizing power among tests with* $\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha$.

	H_0 is true	H_1 is true
Reject H_0	Type I error	Correct decision
Fail to reject H_0	Correct decision	Type II error

UMP tests

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0).$$

$$\Downarrow$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$

$$\Downarrow$$ ***Karlin-Rubin***

$$H_0 : \theta \leq \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta = \theta_1 \quad (\theta_1 < \theta_0).$$

$$\Downarrow$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

$$\Downarrow$$ ***Karlin-Rubin***

$$H_0 : \theta \geq \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

LRT is uniformly most powerful!

Perform HT

Perform hypothesis testing with real data:

- Specify significance level α and calculate the rejection region;
- Calculate p-value and compare it with α.

p**-value**. Under H_0, the probability of observing a result *at least as extreme as* the test statistic.

Duality between CIs and HTs

$(1-\alpha)\times 100\,\%$ exact CI for μ :

$$P\left(\overline{X}_n - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2) \le \mu \le \overline{X}_n + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)\right) = 1-\alpha$$

LRT with significance level α :

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \right| \ge t_{n-1}(\alpha/2) \right\}$$