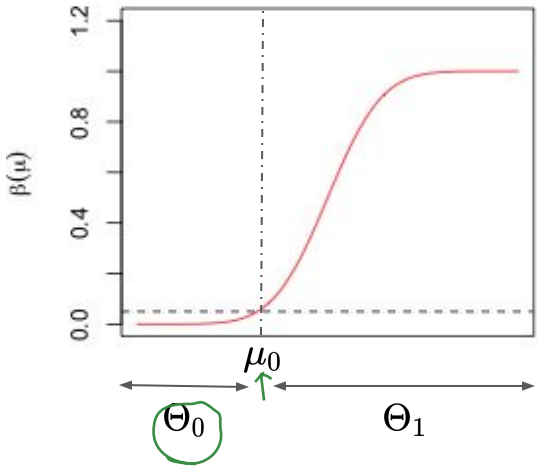
Generalized LRT

Examples

07/13/2021

In the previous lecture,

$$H_0 : \mu \leq \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0.$$



- LRT is uniformly most powerful:
  - Neyman-Pearson Lemma:
    $$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0).$$
  - Proof in class:
    $$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$
  - Karlin-Rubin Theorem under monotonicity:
    $$H_0 : \theta \leq \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$
- LRT under $N(\mu, \sigma^2)$:
  - σ^2 known or unknown;
  - Duality between CIs and hypothesis tests
- Generalized LRT:
  - When the exact sampling distribution of $\lambda(\mathbf{X}_n)$ is hard to obtain;
  - Under H_0, Wilk's theorem guarantees
    $$-2 \log \lambda(\mathbf{X}_n) \xrightarrow{d} \chi^2_\nu \quad \text{as} \quad n \to \infty.$$

Generalized LRT example

$p(x=x\mid\lambda) = \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}$ $\quad \hat{\lambda}_{MLE} = \bar{X}_n$

Example 6. Let $X_1, \ldots, X_{25}$ be i.i.d $Poisson(\lambda)$. Consider

$$H_0 : \lambda = 5 \quad \leftrightarrow \quad H_1 : \lambda \neq 5.$$

$\{\lambda(\bar{x}_n) \le c\}$

Solution: $L(\lambda \mid \bar{x}_n) = \dfrac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^n x_i!}$ $\quad e^{\sum x_i \log 5}$ $\quad H_0 = \{\lambda = 5\}$

$H = H_0 \cup H_1 = \mathbb{R}^+$

$$\lambda(\bar{x}_n) = \frac{\sup_{H_0} L(\lambda \mid \bar{x}_n)}{\sup_{H} L(\lambda \mid \bar{x}_n)} = \frac{e^{-5n}\, 5^{\sum x_i} \Big/ \prod_{i=1}^n x_i!}{e^{-n\bar{x}_n}\, \bar{x}_n^{\sum x_i} \Big/ \prod_{i=1}^n x_i!} = e^{n(\bar{x}_n - 5)} e^{(\sum x_i)(\log 5 - \log \bar{x}_n)}$$

unrestricted $\qquad e^{(\sum x_i)\log \bar{x}_n}$ $\qquad$ only depends on $\bar{x}_n$.

$$= e^{n(\bar{x}_n - 5) + n\bar{x}_n(\log 5 - \log \bar{x}_n)}$$

$$-2\log\lambda(\bar{x}_n) = -2\left[n(\bar{x}_n - 5) + n\bar{x}_n(\log 5 - \log \bar{x}_n)\right] \text{ d} \chi^2_1.$$

Generalized LRT example

Example 6. Let $X_1, \ldots, X_{25}$ be i.i.d $Poisson(\lambda)$. Consider

$$H_0 : \lambda = 5 \quad \leftrightarrow \quad H_1 : \lambda \neq 5.$$

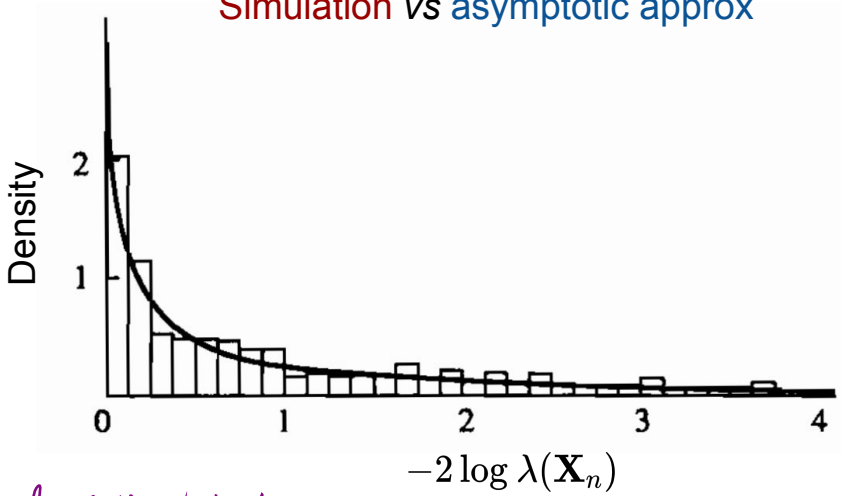
$$-2\log\lambda(\mathbf{X}_n) = -2n\left[(\bar{X}_n - 5) + \bar{X}_n \log\left(5/\bar{X}_n\right)\right] \xrightarrow{d} \chi^2_1 \text{ under } H_0.$$

① Generate 10,000 samples, each sample has 25 iid ~~Poisson(λ)~~ Poisson(5).

② For each sample, we calculate the $\bar{X}_n$ and $-2\log\lambda(\bar{X}_n)$.

③ Plot the histogram using all the likelihood ratio values.

Simulation *vs* asymptotic approx



Generalized LRT example

Example 6. Let $X_1, \ldots, X_{25}$ be i.i.d $Poisson(\lambda)$. Consider

$$H_0 : \lambda = 5 \quad \leftrightarrow \quad H_1 : \lambda \neq 5.$$

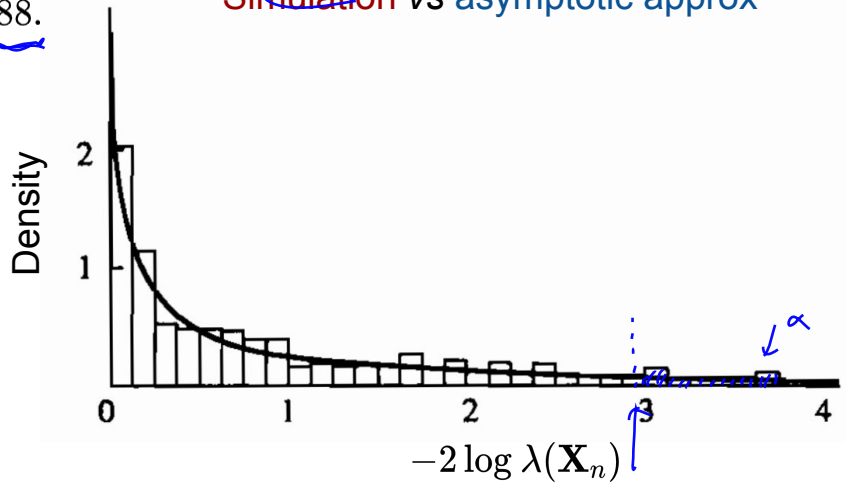
If we observe $\bar{X}_{25} = 5.93$, do we reject H_0?

$$-2 \log \lambda(\mathbf{X}_n) = -50[(5.93 - 5) + 5.93 \cdot \log(5/5.93)] = 4.0788.$$

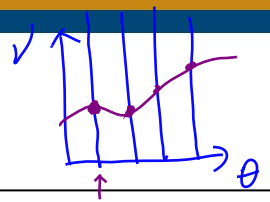
Upper α quantile	0.20	0.10	0.05	0.01
Simulated	1.630	2.726	3.744	6.304
χ^2	1.642	2.706	3.841	6.635

Reject Reject Reject Fail to Reject

Simulation *vs* asymptotic approx



Generalized LRT example



$\in f(\theta)$ profiling out ν

Example 7 (*HW3 Prob 6*). Let $X_1, \ldots, X_n$ be i.i.d $Pareto(\theta, \nu)$ with ν being unknown. Consider

$$H_0 : \theta = 1 \quad \leftrightarrow \quad H_1 : \theta \neq 1.$$

Derive the expression for $-2 \log \lambda(\mathbf{X}_n)$.

$$f(x|\theta) = \begin{cases} \frac{\theta \nu^\theta}{x^{\theta+1}}, & \text{if } x \geq \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Solution. $L(\theta, \nu \mid \mathbf{X}_n) = \frac{\theta^n \nu^{n\theta}}{\left(\prod_{i=1}^n x_i\right)^{\theta+1}} \mathbb{1}\{x_{(1)} \geq \nu\}$

$H_0 = \{\theta = 1, \nu > 0\}$, $H = \{\theta > 0, \nu > 0\}$

$\sup_{H_0} L(\theta, \nu \mid \mathbf{X}_n) = \sup_{\nu > 0} \frac{\nu^n}{\left(\prod_{i=1}^n x_i\right)^2} \mathbb{1}\{x_{(1)} \geq \nu\} = \frac{x_{(1)}^n}{\left(\prod_{i=1}^n x_i\right)^2}$ (at $\nu = x_{(1)}$)

$\ell(\theta, \nu \mid \mathbf{X}_n) = \begin{cases} n \log\theta + n\theta \log\nu - (\theta+1) \sum_{i=1}^n \log x_i & \text{if } \nu \leq x_{(1)} \\ -\infty, & \text{if } \nu > x_{(1)} \end{cases}$

$\hat{\theta}_{MLE} = e^{\frac{\sum_{i=1}^n \log \frac{x_{(1)}}{x_i}}{n}}$ if $\nu \leq x_{(1)}$

$= \frac{\sum \log(x_i / x_{(1)})}{n}$

$\hat{\theta}_{MLE} = \frac{\sum \log x_i - n \log x_{(1)}}{n}$

Fix θ, $\max_{\nu \leq x_{(1)}} \ell(\theta, \nu \mid \mathbf{X}_n) = n \log\theta + n\theta \log x_{(1)} - (\theta+1) \sum_{i=1}^n \log x_i = f(\theta)$

$\max_{\theta > 0} f(\theta) = \sup_{H} L(\theta, \nu \mid \mathbf{X}_n) = \left[\frac{n}{\log \prod_{i=1}^n x_i / x_{(1)}}\right]^n \left[\frac{\prod_{i=1}^n x_{(1)}}{\prod_{i=1}^n x_i}\right]^{\hat{\theta}_{MLE}} \prod_{i=1}^n x_i = \left[\frac{\log (x_i / x_{(1)})}{n}\right]^n$

$\Leftarrow \frac{\partial f(\theta)}{\partial \theta} = \frac{n}{\theta} + n \log x_{(1)} - \sum_{i=1}^n \log x_i = 0$

Generalized LRT example

$\hat{\theta}_{MLE} = \dfrac{n}{\sum_{i=1}^{n} \log X_i / X_{(1)}}$

Example 7 (*HW3 Prob 6*). Let $X_1, \ldots, X_n$ be i.i.d $Pareto(\theta, \nu)$ with ν being unknown. Consider

$$H_0 : \theta = 1 \quad \leftrightarrow \quad H_1 : \theta \neq 1.$$

Derive the expression for $-2 \log \lambda(\mathbf{X}_n)$. $\Rightarrow \chi^2_1$

Solution cont'd.

$$\sup_{\Theta} L(\theta, \nu \mid \underline{X}_n) = \left[\frac{n}{\log \prod_{i=1}^{n} \frac{X_i}{X_{(1)}}} \right]^n e^{\hat{\theta}_{MLE} \sum_{i=1}^{n} \log (X_i / X_{(1)})} \Big/ \prod_{i=1}^{n} X_i$$

$$= \left[\frac{n}{\log \prod_{i=1}^{n} \frac{X_i}{X_{(1)}}} \right]^n e^n \Big/ \prod_{i=1}^{n} X_i$$

Sampling distribution is difficult.

Denote $T(\underline{X}_n) = \log \left(\prod_{i=1}^{n} \frac{X_i}{X_{(1)}} \right)$

$$\lambda(\underline{X}_n) = \frac{X_{(1)}^n / \left(\prod_{i=1}^{n} X_i \right)}{\left[\frac{n}{\log \prod_{i=1}^{n} \frac{X_i}{X_{(1)}}} \right]^n e^n / \prod_{i=1}^{n} X_i} = n^n e^n \prod_{i=1}^{n} \frac{X_{(1)}}{X_i} \times \left[\log \prod_{i=1}^{n} \frac{X_i}{X_{(1)}} \right]^n$$

$$= n^n e^n \underbrace{e^{-T(\underline{X}_n)}}_{\Leftarrow f(t) = e^{-t} t^n} \times T(\underline{X}_n)^n$$

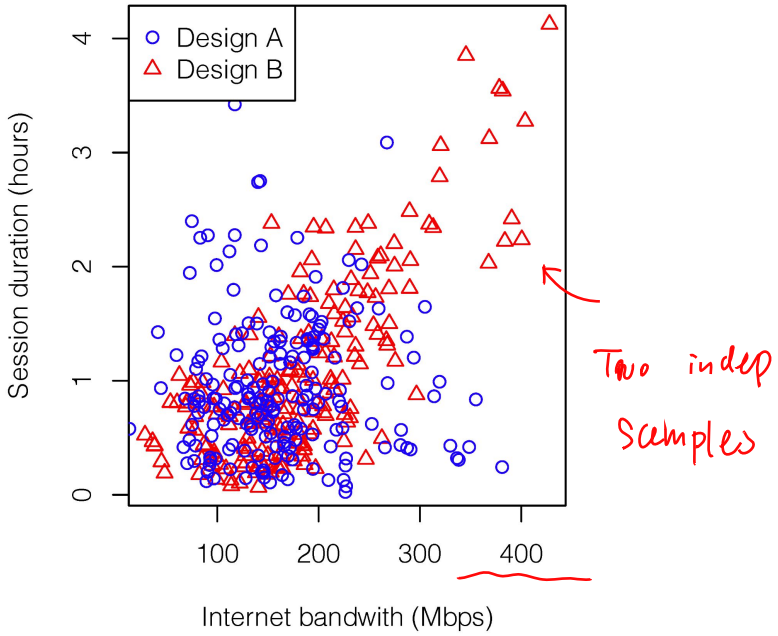
$$R = \{ \lambda(\underline{X}_n) \le c \} = \{ T(\underline{X}_n) \le t_1 \text{ or } T(\underline{X}_n) \ge t_2 \}$$

Two independent samples (*Post-midterm*)

11.2 of Rice

07/13/2021

Youtube A/B testing

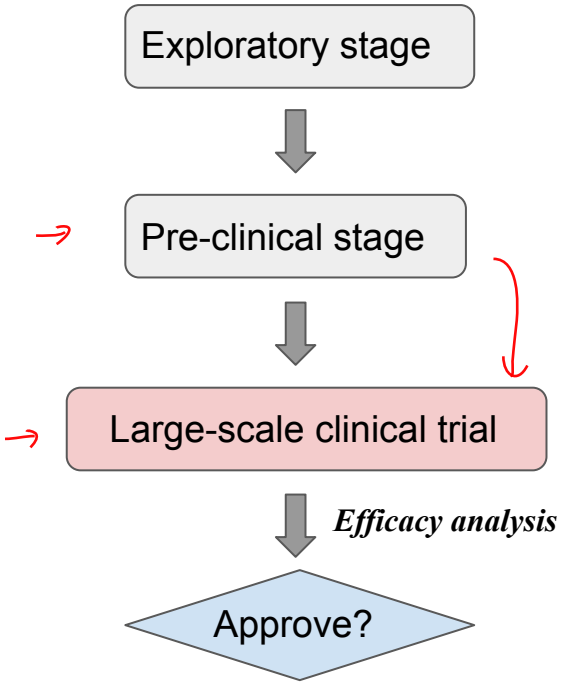


Two indep samples

- Seems like Design B has more users with fast Internet.
- Was the A/B testing fully randomized?
- Denote mean bandwidth in Design A by μ_A, and similarly denote μ_B. Can we test

$$H_0 : \mu_A = \mu_B \quad \text{versus} \quad H_1 : \mu_A < \mu_B \ ?$$

Vaccine Testing and Approval



Large, randomized, double-blind, placebo-controlled trial:

	Pfizer / *BNT162b2*	Placebo
SARS-CoV-2 infected	9	169
No infection	21,711	21,559
Total	21,720	21,728

* *Age ≥ 16, infections observed with onset at least 7 days after the second dose;*
* *Phase 3 data from Pfizer.com*

$$H_0 : p_{\text{pfizer}} = p_{\text{placebo}} \quad \text{versus} \quad H_1 : p_{\text{pfizer}} < p_{\text{placebo}}.$$

$$H_0 : p_{\text{pfizer}} \geq p_{\text{placebo}}/3 \quad \text{versus} \quad H_1 : p_{\text{pfizer}} < p_{\text{placebo}}/3.$$

Independent samples under two Normal populations

Example 1. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2)$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown.
Consider the hypotheses

$$H_0 : \mu_X = \mu_Y \quad \text{versus} \quad H_1 : \mu_X \neq \mu_Y.$$

Derive the LRT statistic and its rejection region.

Solution. The joint likelihood is : $L(\mu_X, \mu_Y, \sigma^2 | \underline{X}_n, \underline{Y}_n) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{m+n} e^{-\frac{\sum_{i=1}^n (x_i-\mu_X)^2}{2\sigma^2} - \frac{\sum (Y_i-\mu_Y)^2}{2\sigma^2}}$

$H_0 = \{ \mu_X = \mu_Y , \sigma^2 > 0 \}$, $H = \{ \mu_X \in \mathbb{R}, \mu_Y \in \mathbb{R}, \sigma^2 > 0 \}$

To calculate the unrestricted maximum likelihood $\sup_H L(\mu_X, \mu_Y, \sigma^2 | \underline{X}_n, \underline{Y}_n)$:

$$\ell(\mu_X, \mu_Y, \sigma^2 | \underline{X}_n, \underline{Y}_n) = -\frac{m+n}{2}\log(2\pi\sigma^2) - \frac{\sum(x_i-\mu_X)^2 + \sum(Y_i-\mu_Y)^2}{2\sigma^2}$$

$$\begin{cases} \frac{\partial \ell}{\partial \mu_X} = \frac{\sum(x_i-\mu_X)}{\sigma^2} = 0 \Rightarrow \hat{\mu}_X = \overline{X}_n \\ \frac{\partial \ell}{\partial \mu_Y} = \frac{\sum(Y_i-\mu_Y)}{\sigma^2} = 0 \Rightarrow \hat{\mu}_Y = \overline{Y}_m \\ \frac{\partial \ell}{\partial \sigma^2} = -\frac{m+n}{2}\cdot\frac{2\sigma}{\sigma^2} + \frac{\sum(x_i-\mu_X)^2 + \sum(Y_i-\mu_Y)^2}{\sigma^3} \Rightarrow \hat{\sigma}^2_p = \frac{\sum(x_i-\overline{X}_n)^2 + \sum(Y_i-\overline{Y}_m)^2}{m+n} \end{cases}$$

pooled variance estimate

$\sum(x_i-\hat{\mu})^2+\sum(y_i-\hat{\mu})^2 = \sum(x_i-\bar{x}_n)^2+n(\bar{x}_n-\hat{\mu})+\sum(y_i-\bar{y}_m)^2+m(\bar{y}_m-\hat{\mu})$

$x_i-\bar{x}_n+\bar{x}_n-\hat{\mu}$ $\quad = (m+n)\hat{\sigma}_p^2 +$

Independent samples under two Normal populations

Solution cont'd.

$$\sup_{H} L(\mu_x,\mu_y,\sigma^2 \mid \underline{x}_n,\underline{y}_m) = L(\hat{\mu}_x,\hat{\mu}_y,\hat{\sigma}_p^2 \mid \underline{x}_n,\underline{y}_m)$$

$$= \left(\frac{1}{\sqrt{2\pi\hat{\sigma}_p^2}}\right)^{m+n} e^{-\frac{\sum(x_i-\bar{x}_n)^2+\sum(y_i-\bar{y}_m)^2}{2\hat{\sigma}_p^2}} = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}_p^2}}\right)^{m+n} e^{-\frac{m+n}{2}}$$

To calculate the restricted maximum likelihood $\sup_{H_0} L(\mu_x,\mu_y,\sigma^2 \mid \underline{x}_n,\underline{y}_m)$:

$H_0: \mu_x=\mu_y$, $x_1,\dots x_n \sim N(\mu_x,\sigma^2)$, $y_1,\dots y_m \sim N(\mu_x,\sigma^2)$

$$\sup_{H_0} L(\mu_x,\mu_y,\sigma^2 \mid \underline{x}_n,\underline{y}_m) = L(\hat{\mu},\hat{\mu},\hat{\sigma}^2)$$

$\hat{\mu} = \frac{\sum x_i+\sum y_i}{m+n}$, $\hat{\sigma}^2 = \frac{\sum(x_i-\hat{\mu})^2+\sum(y_i-\hat{\mu})^2}{m+n}$

$$= \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^{m+n} e^{-\frac{\sum(x_i-\hat{\mu})^2+\sum(y_i-\hat{\mu})^2}{\hat{\sigma}^2}} = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^{m+n} e^{-\frac{m+n}{2}}$$

$$\lambda(\underline{x}_n,\underline{y}_m) = \left(\frac{\hat{\sigma}_p^2}{\hat{\sigma}^2}\right)^{\frac{m+n}{2}} = \left(\frac{\sum(x_i-\bar{x}_n)^2+\sum(y_i-\bar{y}_m)^2}{\sum(x_i-\hat{\mu})^2+\sum(y_i-\hat{\mu})^2}\right)^{\frac{m+n}{2}}$$

From the previous page,

$$\sum (x_i-\hat{\mu})^2 + \sum (y_i-\hat{\mu}) = (m+n)\hat{\sigma}_p^2 + n(\bar{x}_n-\hat{\mu})^2 + m(\bar{y}_m-\hat{\mu})^2$$

$$n\left(\bar{x}_n - \frac{n\bar{x}_n+m\bar{y}_m}{n+m}\right)^2 = m\left[\frac{n(\bar{x}_n-\bar{y}_m)}{n+m}\right]^2$$

$$n\left(\frac{m(\bar{x}_n-\bar{y}_m)}{n+m}\right)^2$$

$$= (m+n)\hat{\sigma}_p^2 + \frac{mn}{(n+m)^2}(n+m)(\bar{x}_n-\bar{y}_n)^2$$

$$= (m+n)\hat{\sigma}_p^2 + \frac{mn}{m+n}(\bar{x}_n-\bar{y}_m)^2$$

$$\lambda(\bar{x}_n, \bar{y}_n) = \left[\frac{(m+n)\hat{\sigma}_p^2}{(m+n)\hat{\sigma}_p^2 + \frac{mn}{m+n}(\bar{x}_n-\bar{y}_m)^2}\right]^{\frac{m+n}{2}} = \left[\frac{1}{1+\frac{mn}{(m+n)^2}\frac{(\bar{x}_n-\bar{y}_n)^2}{\hat{\sigma}_p^2}}\right]^{\frac{m+n}{2}}$$

$$R = \{\lambda(\bar{x}_n, \bar{y}_n) \le c\} = \left\{\frac{mn}{(m+n)^2}\frac{(\bar{x}_n-\bar{y}_m)^2}{\hat{\sigma}_p^2} \ge c'\right\}$$

$$\left\{\frac{|\bar{x}_n-\bar{y}_m|}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \ge c''\right\} = \left\{\sqrt{\frac{mn}{m+n}}\cdot\frac{|\bar{x}_n-\bar{y}_m|}{\sqrt{\sum(x_i-\bar{x}_n)^2+\sum(y_i-\bar{y}_n)^2}} \ge \sqrt{c'}\right\}$$

$\sqrt{\frac{1}{m}+\frac{1}{n}}$

$\to (n+m-2)s_p^2$

← consistent with our intuition,

Independent samples under two Normal populations

Theorem A. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2), Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown. Then

$$\sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2 + \sum_{i=1}^{m} \left(Y_i - \bar{Y}_m\right)^2 \sim \sigma^2 \chi^2_{m+n-2},$$

and it is mutually independent from $\bar{X}_n$ and $\bar{Y}_m$.

Proof

indep $\{$ $X_1, \ldots X_n$ iid $N(\mu_X, \sigma^2)$

$Y_1, \ldots Y_m$ iid $N(\mu_Y, \sigma^2)$

$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \sim \chi^2_{n-1}$ indep from $\bar{X}_n$

$\frac{1}{\sigma^2}\sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2 \sim \chi^2_{m-1}$ indep from $\bar{Y}_m$

$\frac{\sum(X_i - \bar{X}_n)^2 + \sum(Y_i - \bar{Y}_m)^2}{\sigma^2} \sim \chi^2_{n-1} + \chi^2_{m-1} = \chi^2_{n+m-2}$

$\perp\!\!\!\perp \bar{X}_n \perp\!\!\!\perp \bar{Y}_m$.

Independent samples under two Normal populations

Theorem B. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2)$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown. Then

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}.$$

$$S_p^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + \sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2}{n+m-2}$$

Independent samples under two Normal populations

Example 1 *cont'd*. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2)$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown. Consider the hypotheses

$$H_0 : \mu_X = \mu_Y \quad \text{versus} \quad H_1 : \mu_X \neq \mu_Y.$$

Derive the test with significance level α.

$$\frac{|\bar{X}_n - \bar{Y}_m|}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}}$$

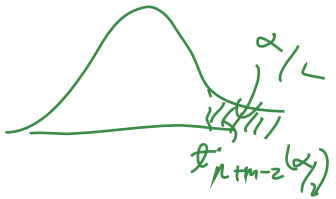
Solution. We derived the rejection region: $R = \left\{ \sqrt{\frac{mn}{m+n}} \frac{|\bar{X}_n - \bar{Y}_m|}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2 + \sum_{i=1}^m (Y_i - \bar{Y}_m)^2}} \geq c'' \right\}$.

Under H_0, $\mu_X - \mu_Y = 0$ and $\dfrac{\bar{X}_n - \bar{Y}_m}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}$

To get the significance level α:

$$\alpha = P(R \mid H_0) = P\left(\frac{|\bar{X}_n - \bar{Y}_m|}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \geq c'' \;\middle|\; \mu_X = \mu_Y \right)$$

$$\Rightarrow c'' = t_{n+m-2}(\alpha/2),$$



Independent samples under two Normal populations

Corollary B. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2)$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown.
Then $(1-\alpha) \times 100\,\%$ exact CI for $\mu_X - \mu_Y$ is

$$(\bar{X}_n - \bar{Y}_m) \pm t_{n+m-2}(\alpha/2) \cdot S_p \sqrt{\frac{1}{n} + \frac{1}{m}}.$$

$\in$

$1-\alpha$

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_X - \mu_Y)}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}$$

$-t_{n+m-2}(\alpha/2)$ $\quad t_{n+m-2}(\alpha/2)$

$$1-\alpha = P\left(\left| \frac{(\bar{X}_n - \bar{Y}_m) - (\mu_X - \mu_Y)}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \right| \le t_{n+m-2}(\alpha/2) \right)$$

$$= P\left[(\bar{X}_n - \bar{Y}_m) - t_{n+m-2}(\alpha/2) \cdot S_p \sqrt{\frac{1}{n} + \frac{1}{m}} \le \mu_X - \mu_Y \le (\bar{X}_n - \bar{Y}_m) + t_{n+m-2}(\alpha/2)\, S_p \sqrt{\frac{1}{n} + \frac{1}{m}} \right]$$

Independent samples under two Normal populations

Example 1 *cont'd*. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma^2)$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma^2)$ and the variance is unknown. The LRTs with significance level α for the following hypotheses can be derived.

$$H_0 : \mu_X = \mu_Y \quad \leftrightarrow \quad H_1 : \mu_X \neq \mu_Y. \quad \Longleftrightarrow \quad R = \left\{ \frac{|\bar{X}_n - \bar{Y}_m|}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \geq t_{n+m-2}(\alpha/2) \right\}$$

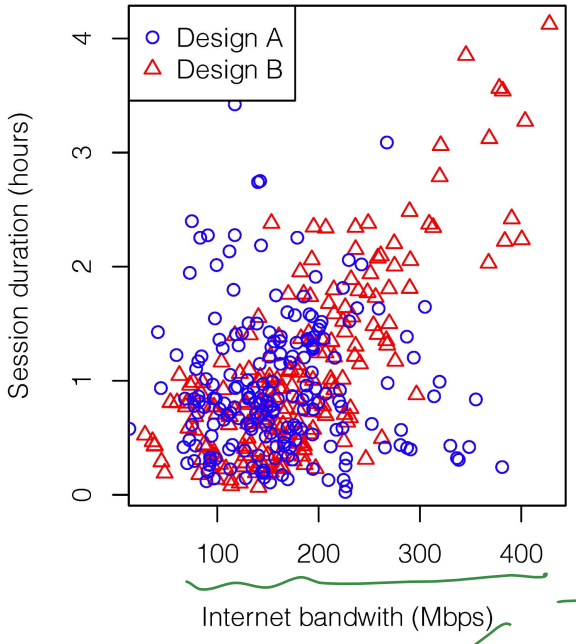
$$A = \left\{ (\bar{X}_n - \bar{Y}_m) - t_{n+m-2}(\alpha/2) \cdot S_p\sqrt{\tfrac{1}{n}+\tfrac{1}{m}} \leq \mu_X - \mu_Y \leq (\bar{X}_n - \bar{Y}_m) + t_{n+m-2}(\alpha/2)\, S_p\sqrt{\tfrac{1}{n}+\tfrac{1}{m}} \right\}$$

$$= \left\{ \frac{|\bar{X}_n - \bar{Y}_m|}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \leq t_{n+m-2}(\alpha/2) \right\}$$

$$H_0 : \mu_X = \mu_Y \leftrightarrow \quad H_1 : \mu_X > \mu_Y. \quad \Longleftrightarrow \quad R = \left\{ \frac{\bar{X}_n - \bar{Y}_m}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \geq t_{n+m-2}(\alpha) \right\} \quad \in$$

$$H_0 : \mu_X = \mu_Y \quad \leftrightarrow \quad H_1 : \mu_X < \mu_Y. \quad \Longleftrightarrow \quad R = \left\{ \frac{\bar{X}_n - \bar{Y}_m}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \leq -t_{n+m-2}(\alpha) \right\} \quad \in$$

Youtube A/B testing



$H_0 : \mu_A = \mu_B$ versus $H_1 : \mu_A < \mu_B$?

Assume $X_1, \ldots, X_{200} \sim N(\mu_A, \sigma^2)$, $Y_1, \ldots, Y_{200} \sim N(\mu_B, \sigma^2)$.

The LRT rejection region is $R = \left\{ \dfrac{\bar{X}_{200} - \bar{Y}_{200}}{S_p\sqrt{\frac{1}{200} + \frac{1}{200}}} \le -t_{398}(\alpha) \right\}$.

```
> mean(A_bandwidth)
[1] 3.108194
> mean(B_bandwidth)
[1] 3.384364
> {sum((A_bandwidth-mean(A_bandwidth))^2) + sum((B_bandwidth-mean(B_bandwidth))^2)}/398
[1] 1.633449
> qt(0.05, df=398, lower.tail=FALSE)
[1] 1.648691
```

Conclusion: We reject H_0, and conclude that there is enough evidence to suggest that $\mu_A < \mu_B$.

Byzantine church wood

p-value $= P\left(\dfrac{|\bar{X}_n - \bar{Y}_m|}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \geq 1.2899 \,\middle|\, H_0\right) = 2 * pt(-1.2899, df=21)$

$= 0.21109$

> 0.05

Example 2. Samples of wood were obtained from the core and periphery of a Byzantine church. The date of the wood were determined.

$\alpha = 0.05$

$\bar{X}_n = 1249.86$ $\bar{Y}_m = 1261.33$

$S_p^2 = 433.13$

Core		Periphery	
1294	1251	1284	1274
1279	1248	1272	1264
1274	1240	1256	1256
1264	1232	1254	1250
1263	1220	1242	
1254	1218		
1251	1210		

$n = 14$ $m = 9$

$H_0 : \mu_C = \mu_P$ versus $H_1 : \mu_C \neq \mu_P$?

The rejection region is $R = \left\{ \dfrac{|\bar{X}_n - \bar{Y}_m|}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \geq t_{n+m-2}(\alpha/2) \right\}$.

$t_{n+m-2}(\alpha/2) = t_{21}(0.05/2) = 2.0796.$

$\dfrac{|\bar{X}_n - \bar{Y}_m|}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} = \dfrac{|1249.86 - 1261.33|}{\sqrt{433.13}\sqrt{\frac{1}{14}+\frac{1}{9}}} = 1.2899$

< 2.0796

We fail to reject H_0, and conclude that there is not enough evidence to support the alternative hypothesis.

Tomorrow ...

- Non-parametric approach - Mann-Whitney test
- Comparing **paired** samples