# STAT 135 MIDTERM EXAMINATION A SOLUTIONS

July 20, 2021

**Problem 1.** Let $X_1, \ldots, X_n$ be i.i.d from a population with pdf

$$f(x|\theta) = \begin{cases} \frac{3x^2}{2\theta^3}, & \text{if } -\theta \le x \le \theta \\ 0, & \text{otherwise.} \end{cases}$$

in which $\theta > 0$ is an unknown parameter.

(a) (3 points) Find a MM estimator of $\theta$;

**Solution.** First we calculate the population moments:

$$E(X) = \int_{-\theta}^{\theta} \frac{3x^3}{2\theta^3} dx \overset{\text{odd function}}{=} 0,$$

$$E(X^2) = \int_{-\theta}^{\theta} \frac{3x^4}{2\theta^3} dx \overset{\text{even function}}{=} 2\int_{0}^{\theta} \frac{3x^4}{2\theta^3} dx = \frac{3\theta^2}{5}.$$

(*2 pts*)

Since the first moment does not contain any information about $\theta$, the MM estimator for $\theta$ is obtained using the second moment:

$$\hat{\theta}_{\text{MM}} = \sqrt{\frac{5}{3n}\sum_{i=1}^{n} X_i^2}.$$

(*1 pt*)

(b) (4 points) Find the MLE of $\theta$. Is it a sufficient statistic for $\theta$? Explain why;

**Solution.** The joint likelihood function can be written as

$$\begin{aligned} L(\theta|\mathbf{X}_n) &= \frac{3^n \prod_{i=1}^n X_i^2}{2^n \theta^{3n}} \mathbb{1}\{-\theta \le X_{(1)} \le X_{(n)} \le \theta\} \\ &= \frac{3^n \prod_{i=1}^n X_i^2}{2^n \theta^{3n}} \mathbb{1}\{\theta \ge \max\left(-X_{(1)}, X_{(n)}\right)\}. \end{aligned}$$

(*2 pts*)

Since $L(\theta|\mathbf{X}_n)$ is a strictly decreasing function of $\theta$, the maximum likelihood is achieved at $\hat{\theta}_{\text{MLE}} = \max\left(-X_{(1)}, X_{(n)}\right)$. (*1 pt*)

Note the joint likelihood can be factorized as

$$L(\theta|\mathbf{X}_n) = \frac{3^n \prod_{i=1}^n X_i^2}{2^n} \times \frac{1}{\theta^{3n}} \mathbb{1}\{\theta \ge \hat{\theta}_{\text{MLE}}\},$$

and $\hat{\theta}_{\text{MLE}}$ is sufficient by Neyman-Fisher factorization theorem. (*1 pt*)

(c) (5 points) *(Bonus question) Denote the MLE by $\hat{\theta}_{\mathrm{MLE}}$. Prove that $\hat{\theta}_{\mathrm{MLE}}$ can only take values in $[0,\theta]$ and its cdf satisfies

$$P(\hat{\theta}_{\mathrm{MLE}} \le t) = \left(\frac{t}{\theta}\right)^{3n}, \quad t \in [0,\theta].$$

**Solution.** Since both $-X_{(1)}$ and $X_{(n)}$ are in $[-\theta,\theta]$, we have $\hat{\theta}_{\mathrm{MLE}} \le \theta$. Also, because $X_{(1)} \le X_{(n)}$, there must be a positive value between $-X_{(1)}$ and $X_{(n)}$, which means $\hat{\theta}_{\mathrm{MLE}} \ge 0$.

Denote $U = X_{(1)}$ and $V = X_{(n)}$. The joint density between $U$ and $V$ is

$$f(u,v|\theta) = n(n-1)f(u|\theta)f(v|\theta)[F(v|\theta) - F(u|\theta)]^{n-2}, \quad v \ge u,$$

in which the cdf can be calculated as $F(u|\theta) = \frac{x^3}{2\theta^3} + \frac{1}{2}$.

Notice that

$$P(\hat{\theta}_{\mathrm{MLE}} \le t) = P\left[\max\left(-X_{(1)}, X_{(n)}\right) \le t\right] = P\left[-t \le -X_{(1)} \le X_{(n)} \le t\right],$$

from which we know the domain of integration w.r.t $(u,v)$; see Figure 1. Therefore,

$$\begin{aligned}
P(\hat{\theta}_{\mathrm{MLE}} \le t) &= \int_{-t}^{t}\int_{u}^{t} f(u,v)\,dv\,du \\
&= \frac{n(n-1)}{\theta^{3n}} \int_{-t}^{t}\int_{u}^{t} \frac{3u^2}{2}\cdot\frac{3v^2}{2}\cdot\left(\frac{v^3}{2}-\frac{u^3}{2}\right)^{n-2} dv\,du \\
&= \frac{n(n-1)}{\theta^{3n}} \int_{-t}^{t} \frac{3u^2}{2}\left[\int_{u}^{t}\left(\frac{v^3}{2}-\frac{u^3}{2}\right)^{n-2} d\frac{v^3}{2}\right] du \\
&= \frac{n(n-1)}{\theta^{3n}} \int_{-t}^{t} \frac{3u^2}{2}\left[\frac{1}{n-1}\left(\frac{t^3}{2}-\frac{u^3}{2}\right)^{n-1}\right] du \\
&= \frac{n(n-1)}{\theta^{3n}} \int_{-t}^{t} \left[\frac{1}{n-1}\left(\frac{t^3}{2}-\frac{u^3}{2}\right)^{n-1}\right] d\frac{u^3}{2} \\
&= \frac{n(n-1)}{\theta^{3n}} \cdot \frac{1}{n(n-1)}\left[\frac{t^3}{2}-\frac{(-t)^3}{2}\right]^{n} = \left(\frac{t}{\theta}\right)^{3n}.
\end{aligned}$$

(5 *pts*, no partial credits)

(d) (8 points) Show that $Y = \frac{3n+1}{3n}\hat{\theta}_{\mathrm{MLE}}$ is an unbiased estimator of $\theta$. Calculate $\mathrm{Var}(Y)$.

**Solution.** From (c), we know the pdf of $\hat{\theta}_{\mathrm{MLE}}$ is

$$f_{\mathrm{MLE}}(t) = \frac{3nt^{3n-1}}{\theta^{3n}}, \quad t \in [0,\theta].$$

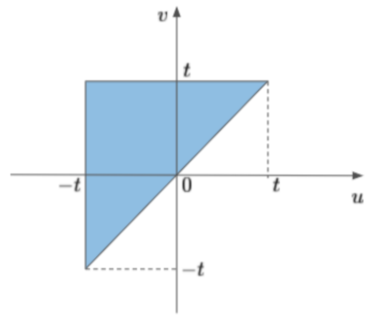


FIGURE 1. Domain of integration to calculate the cdf of $\hat{\theta}_{\text{MLE}}$.

Therefore,

$$E(\hat{\theta}_{\text{MLE}}) = \int_0^\theta \frac{3nt^{3n}}{\theta^{3n}} = \frac{3n\theta}{3n+1}, \qquad (\textit{1 pt})$$

$$E(\hat{\theta}^2_{\text{MLE}}) = \int_0^\theta \frac{3nt^{3n+1}}{\theta^{3n}} = \frac{3n\theta^2}{3n+2}, \qquad (\textit{1 pt})$$

$$\text{Var}(\hat{\theta}_{\text{MLE}}) = \frac{3n\theta^2}{3n+2} - \left(\frac{3n\theta}{3n+1}\right)^2 = \frac{3n\theta^2}{(3n+1)^2(3n+2)}. \qquad (\textit{2 pts})$$

Consequently, we have

$$E(Y) = \frac{3n+1}{3n} E(\hat{\theta}_{\text{MLE}}) = \theta, \qquad (\textit{1.5 pts})$$

$$\text{Var}(Y) = \left(\frac{3n+1}{3n}\right)^2 \text{Var}(\hat{\theta}_{\text{MLE}}) = \frac{\theta^2}{3n(3n+2)}, \qquad (\textit{1.5 pts})$$

from which we see that $Y$ is an unbiased estimator of $\theta$. (*1 pt*)

(e) (8 points) Use the cdf in (c) to explain why $\hat{\theta}_{\text{MLE}}$ can not be approximated by a Normal distribution as $n \to \infty$. Is it contradictory to Theorem D of Lecture 4?

**Solution.** Firstly, the density of $\hat{\theta}_{\text{MLE}}$ is not bell-shaped: it is strictly increasing from 0 to $\theta$, and as $n \to \infty$, the density becomes highly concentrated around $\theta$; see Figure 2. This behavior directly indicates the consistency of $\hat{\theta}_{\text{MLE}}$, whose rate of convergence $\sqrt{\text{Var}(\hat{\theta}_{\text{MLE}})} \approx 1/(3n)$ is much faster than $1/\sqrt{n}$ of the common asymptotic normality. (*3 pts*)

Secondly, the support of the pdf of $n^a\hat{\theta}_{\text{MLE}}$ with $a > 0$ is restricted in $[0, n^a\theta]$, whereas the density of a Normal variable is the real line. (*1 pt*)

It is not contradictory to Theorem D of Lecture 4 because some vital assumptions needed to ensure asymptotic normality are violated in this case:

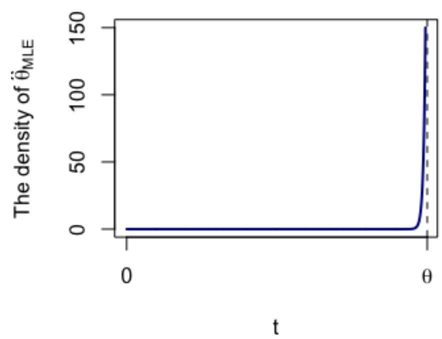


FIGURE 2. Density of of $\hat{\theta}_{\text{MLE}}$.

- The support of $f(x|\theta)$ is not independent of the model parameter $\theta$; *(2 pts)*
- There is no exchangeability between the integration and differentiation. *(2 pts)*

*(Assign full marks as long as the student's answer is sufficiently adequate.)*

(f) (6 points) One can estimate $\text{Var}(\hat{\theta}_{\text{MM}}) \approx \frac{\theta^2}{21n}$ for sufficiently large $n$. With this result, we can calculate the relative efficiency

$$\text{eff}(Y, \hat{\theta}_{\text{MM}}) = \frac{\text{Var}(Y)}{\text{Var}(\hat{\theta}_{\text{MM}})}.$$

Show that $\text{eff}(Y, \hat{\theta}_{\text{MM}}) \to 0$ for large $n$, which indicates the unbiased estimator $Y$ is a much more efficient estimator than $\hat{\theta}_{\text{MM}}$.

**Solution.** By definition and results in (d),

$$\begin{aligned} \text{eff}(Y, \hat{\theta}_{\text{MM}}) &= \frac{\theta^2/\{3n(3n+2)\}}{\theta^2/(21n)} && \textit{(2 pts)} \\ &= \frac{7}{3n+2} && \textit{(2 pts)} \\ &\to 0, && \textit{(2 pts)} \end{aligned}$$

which indeed shows that the unbiased estimator $Y$ is a much more efficient estimator than $\hat{\theta}_{\text{MM}}$.

**Problem 2.** Suppose that we observe two independent random samples: $X_1, \dots, X_n$ are i.i.d exponential($\theta$), and $Y_1, \dots, Y_m$ are i.i.d exponential($\mu$).

(a) (8 points) Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$, and show that the rejection region can be based solely on the statistic

$$T(\mathbf{X}_n, \mathbf{Y}_n) = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}.$$

**Solution.** The joint likelihood can be written as

$$L(\theta, \mu | \mathbf{X}_n, \mathbf{Y}_n) = \theta^n \exp\left(-\theta \sum_{i=1}^n X_i\right) \cdot \mu^m \exp\left(-\mu \sum_{i=1}^n Y_i\right). \quad \textit{(1 pt)}$$

The unrestricted parameter space $\Theta = \{\theta > 0, \mu > 0\}$. By the independence between $\mathbf{X}_n$ and $\mathbf{Y}_n$, the unrestricted maximum likelihood is attained when $L(\theta|\mathbf{X}_n)$ and $L(\mu|\mathbf{Y}_n)$ both attain their respective maximum likelihood. Thus,

$$\begin{aligned}\sup_{(\theta,\mu)\in\Theta} L(\theta, \mu|\mathbf{X}_n, \mathbf{Y}_n) &= L(1/\bar{X}_n, 1/\bar{Y}_m|\mathbf{X}_n, \mathbf{Y}_n) \\ &= \left(\frac{n}{\sum_{i=1}^n X_i}\right)^n \left(\frac{m}{\sum_{i=1}^m Y_i}\right)^m \exp(-n-m). \qquad \textit{(2 pts)}\end{aligned}$$

The restricted parameter space $\Theta_0 = \{\theta = \mu\}$. Under $H_0$, $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim}$ exponential($\theta$). Therefore, the restricted maximum likelihood is attained at $\hat{\theta}_{\text{Res}} = (m+n)/\left(\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i\right)$, and

$$\begin{aligned}\sup_{(\theta,\mu)\in\Theta_0} L(\theta, \mu|\mathbf{X}_n, \mathbf{Y}_n) &= L(\hat{\theta}_{\text{Res}}, \hat{\theta}_{\text{Res}}|\mathbf{X}_n, \mathbf{Y}_n) \\ &= \left(\frac{m+n}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}\right)^{m+n} \exp(-n-m). \qquad \textit{(2 pts)}\end{aligned}$$

The likelihood ratio can be calculated

$$\lambda(\mathbf{X}_n) = \frac{\sup_{(\theta,\mu)\in\Theta_0}}{\sup_{(\theta,\mu)\in\Theta}} = \frac{(m+n)^{m+n}}{n^n m^m} T^n(\mathbf{X}_n, \mathbf{Y}_m)\left(1 - T(\mathbf{X}_n, \mathbf{Y}_m)\right)^m, \quad \textit{(1 pt)}$$

which is a uni-modal function of $T = T(\mathbf{X}_n, \mathbf{Y}_m)$ which attains its maximum at $\frac{n}{n+m}$. Therefore, the rejection region is only dependent on $T$, which becomes the LRT statistic – $\lambda(\mathbf{X}_n, \mathbf{Y}_m) \leq c$ is equivalent to rejecting if $T \leq c_1$ or $T \geq c_2$, where $c_1 < \frac{n}{n+m} < c_2$. *(2 pts)*

(b) (5 points) Find the distribution of $T$ when $H_0$ is true.

**Solution.** Under $H_0$, $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m \overset{\text{iid}}{\sim}$ Gamma$(1, \theta)$. Thus,

$$\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta), \quad \sum_{i=1}^m Y_i \sim \text{Gamma}(m, \theta), \quad \sum_{i=1}^n X_i \perp\!\!\!\perp \sum_{i=1}^m Y_i,$$

*(3 pts)*

from which we know

$$T(\mathbf{X}_n, \mathbf{Y}_m) = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i} \sim \text{Beta}(n, m). \qquad \textit{(2 pts)}$$

(c) (6 points) Suppose $n = 14$ and $m = 16$. Determine the threshold in the LRT rejection region so that the significance level $\alpha = 0.05$. You might need `R` to help compute the upper $\alpha$ quantile of a Beta$(a, b)$ distribution: `qbeta(alpha, a, b, lower.tail=FALSE)`

**Solution.** To make sure the LRT has a significance level $\alpha = 0.05$, the rejection region should be

$$R = \{T \le \text{Beta}(1-\alpha/2; n, m) \text{ or } T \ge \text{Beta}(\alpha/2; n, m)\}, \qquad (\textit{3 pts})$$

in which Beta$(\alpha/2; n, m)$ is the upper $\alpha$ quantile of the Beta$(a, b)$ distribution.

Since $n = 14$ and $m = 16$, we can use `R` to obtain

$$\text{Beta}(1-\alpha/2; n, m) = 0.2945, \quad \text{Beta}(\alpha/2; n, m) = 0.6431. \qquad (\textit{3 pts})$$

(d) (6 points) A researcher was able to collect two samples from the two populations:

$$\mathbf{X}_{14} = \{0.691,\ 0.493,\ 0.749,\ 0.492,\ 0.343,\ 1.763,\ 0.339,\ 0.541,\ 0.372,\ 0.183,\ 1.306,\ 1.413,\ 0.640,\ 0.433\},$$

$$\mathbf{Y}_{16} = \{0.721,\ 4.547,\ 2.228,\ 0.247,\ 2.962,\ 0.518,\ 0.159,\ 1.030,\ 1.577,\ 0.432,\ 2.105,\ 2.689,\ 3.218,\ 0.350,\ 0.160,\ 1.144\}.$$

Will you reject the null hypothesis? Report the $p$-value. You might need `R` to help compute the area under the curve of a Beta$(a, b)$ density: `pbeta(alpha, a, b, lower.tail=TRUE)`

**Solution.** The LRT statistic is

$$T(\mathbf{X}_{14}, \mathbf{Y}_{16}) = \frac{9.758}{24.087} = 0.2883. \qquad (\textit{2 pts})$$

Recall that $p$-value is the probability of observing a statistic as extreme as 0.2883 under $H_0$. Since $0.2883 < n/(m + n) = 0.4667$, the observed statistic is larger than the mode of the uni-modal function in (1). Therefore, being as extreme as 0.2883 means (also see Figure 3)

$$\begin{aligned} p\text{-value} =& 2 * P\{T(\mathbf{X}_n, \mathbf{Y}_m) \le 0.2883 \mid H_0\} \\ =& 2\texttt{*pbeta(0.2883, 14, 16, lower.tail=TRUE)} \\ =& 0.0416 < 0.05. \qquad (\textit{2 pts}) \end{aligned}$$

Therefore, we reject $H_0$ at significance level 0.05, and conclude that there is enough evidence to support the alternative hypothesis $\theta \neq \mu$. (*You can also check whether* $T(\mathbf{X}_{14}, \mathbf{Y}_{16})$ *is in the rejection region in* (c).) (*2 pts*)

**Problem 3.** Imagine we have collected a random sample of 31 energy bars of a certain brand from a number of different stores to represent the population of energy bars available to the general consumer. Let's assume that the weights of protein content in the energy bars are normally distributed with $\mu$ and $\sigma^2$. The labels on the bars claim that each bar contains 20 grams of

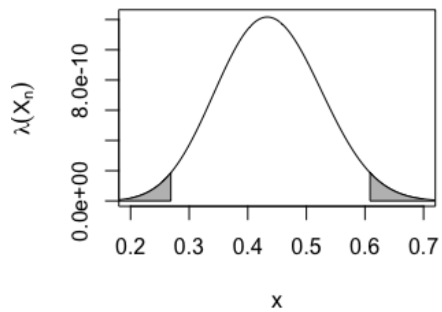


FIGURE 3. What it means to be as extreme as 0.6086678 under $H_0$.

protein. However, the following are what we observed from the 31 energy bars:

$$\{20.70,\ 27.46,\ 22.15,\ 19.85,\ 21.29,\ 24.75,\ 20.75,\ 22.91,\ 25.34,\ 20.33,\ 21.54,\ 21.08,\ 22.14,\ 19.56,\ 21.10,\ 18.04,\ 24.12,\ 19.95,\ 19.72,\ 18.28,\ 16.26,\ 17.46,\ 20.53,\ 22.12,\ 25.06,\ 22.44,\ 19.08,\ 19.88,\ 21.39,\ 22.33,\ 25.79\}$$

(1) (6 points) Calculate the exact 99% confidence intervals for the population mean $\mu$ and variance $\sigma^2$;

**Solution.** The sample mean and variance can be computed as

$$\bar{X}_n = 21.4, \quad S^2 = 6.46008, \quad \hat{\sigma}_n^2 = 6.25169. \qquad \textit{(1 pts)}$$

When $\alpha = 0.01$,

$$t_{30}(\alpha/2) = 2.75, \quad \chi^2_{30}(1-\alpha/2) = 13.787, \quad \chi^2_{30}(\alpha/2) = 53.672, \qquad \textit{(2 pts)}$$

the 99% exact confidence interval for $\mu$ is

$$\bar{X}_n \pm \frac{S}{\sqrt{n}} t_{30}(\alpha/2) = [20.145, 22.655], \qquad \textit{(1.5 pts)}$$

and the 99% exact confidence interval for $\sigma^2$ is

$$\left[\frac{n\hat{\sigma}_n^2}{\chi^2_{30}(\alpha/2)}, \frac{n\hat{\sigma}_n^2}{\chi^2_{30}(1-\alpha/2)}\right] = [3.611, 14.057]. \qquad \textit{(1.5 pts)}$$

(2) (6 points) Researchers want to know if the mean protein content for all energy bars is actually less than the advertised 20 grams. Construct a null and an alternative hypotheses. Test these hypothesis at significance level $\alpha = 0.05$;

**Solution.** The null and alternative hypotheses are

$$H_0 : \mu = 20 \text{ versus } H_1 : \mu < 20. \qquad (\textit{2 pts})$$

The LRT rejection region at significance level $\alpha = 0.05$ is

$$R = \left\{ \frac{\sqrt{n}\,(\bar{X}_n - 20)}{S} \leq -t_{30}(\alpha) \right\}, \qquad (\textit{2 pts})$$

in which $t_{30}(\alpha) = 1.6973$.

Since $\sqrt{n}\left(\bar{X}_n - 20\right)/S = 3.067$ which is not in the rejection region, we fail to reject $H_0$ and conclude that there is not enough evidence to show the protein content is less than the advertised amount. (*You can reach the same conclusion via calculating the p-value = 0.9977 which is not significant at all.*) (*2 pts*)

(3) (6 points) Calculate the threshold of the LRT rejection region at significance level $\alpha = 0.05$ for testing

$$H_0 : \sigma^2 = 6.5 \text{ versus } H_1 : \sigma^2 \neq 6.5.$$

Is the LRT uniformly most powerful in this case? Does the sample variance fall within the rejection region?

**Solution.** The LRT of the two-sided test should be

$$R = \left\{ \frac{(n-1)\,S^2}{6.5} \geq \chi^2_{n-1}(\alpha/2) \;\text{ or }\; \frac{(n-1)\,S^2}{6.5} \leq \chi^2_{n-1}(1-\alpha/2) \right\},$$

in which

$$\chi^2_{n-1}(1-\alpha/2) = 16.791, \quad \chi^2_{n-1}(\alpha/2) = 46.979 \qquad (\textit{2 pts})$$

when $\alpha = 0.05$.

Because $(n-1)S^2/6.5 = 29.816$, the sample is not in the rejection region, and hence we fail to reject $H_0$. (*2 pts*)

This LRT is not UMP because we are dealing with a two-sided test. In particular, a two-sided test would fail to give us a monotonic power function in the alternative parameter space, which violates the assumption of Karlin-Rubin. (*2 pts*)

**Problem 4.** Suppose $X_1, \ldots, X_n$ are independently sampled from Pareto$(\theta, 1)$ with pdf

$$f(x|\theta) = \begin{cases} \frac{\theta}{x^{\theta+1}}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

in which $\theta > 2$.

(a) (4 points) Calculate the first two population moments and find a MM estimator of $\theta$;

**Solution.** The first two population moments are

$$\mu = \int_1^\infty \frac{\theta}{x^\theta} dx = \frac{\theta}{\theta - 1}, \qquad (\textit{1 pt})$$
$$\mu_2 = \int_1^\infty \frac{\theta}{x^{\theta-1}} dx = \frac{\theta}{\theta - 2}. \qquad (\textit{1 pt})$$

A MM estimator of $\theta$ can be written as

$$\hat{\theta}_{\mathrm{MM}} = \frac{\bar{X}_n}{\bar{X}_n - 1}. \qquad (\textit{2 pts})$$

(b) (8 points) Calculate the Fisher information $I(\theta)$ and the CR lower bound for all unbiased estimators of $\psi(\theta) = \frac{\theta}{\theta-1}$. Does the sample mean $\bar{X}_n$ attain this lower bound?

**Solution.** By Lemma C of Lecture 4,

$$I(\theta) = -E_\theta \left[ \frac{\partial^2}{\partial \theta^2} \log f \right] = \frac{1}{\theta^2}. \qquad (\textit{2 pts})$$

Thus, the Cramer-Rao bound for all unbiased estimators of $\psi(\theta)$ is

$$\text{CR\_bound}[\psi(\theta)] = \frac{[\psi'(\theta)]^2}{nI(\theta)} = \frac{\theta^2}{n(\theta-1)^4}. \qquad (\textit{2 pts})$$

Compare it with the variance of $\bar{X}_n$:

$$\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{\theta/(\theta-2) - \theta^2/(\theta-1)^2}{n} = \frac{\theta}{n(\theta-2)(\theta-1)^2}, \qquad (\textit{2 pts})$$
$$\text{and } \frac{\mathrm{Var}(\bar{X}_n)}{\text{CR\_bound}[\psi(\theta)]} = \frac{(\theta-1)^2}{\theta(\theta-2)} = 1 + \frac{1}{\theta(\theta-2)} > 1. \qquad (\textit{2 pts})$$

Therefore, the sample mean $\bar{X}_n$ does not attain this lower bound

(c) (6 points) Denote the MM estimator in (a) as $\hat{\theta}_{\mathrm{MM}}$. Apply the Delta Method to obtain the limiting Normal distribution of $\hat{\theta}_{\mathrm{MM}}$;

**Solution.** Denote $g(x) = \frac{x}{x-1}$. Then $g(\mu) = \theta$, $g(\bar{X}_n) = \hat{\theta}_{\mathrm{MM}}$ and by the Delta method,

$$\sqrt{n}\{g(\bar{X}_n) - g(\mu)\} = \sqrt{n}\{\hat{\theta}_{\mathrm{MM}} - \theta\} \xrightarrow{d} N(0, [g'(\mu)]^2 \sigma^2), \qquad (\textit{3 pts})$$

in which $g'(\mu) = -1/(\mu-1)^2 = -(\theta-1)^2$, $\sigma^2 \overset{\text{From (b)}}{=} \frac{\theta}{(\theta-2)(\theta-1)^2}$, and

$$[g'(\mu)]^2 \sigma^2 = \frac{\theta(\theta-1)^2}{(\theta-2)}. \qquad (\textit{3 pts})$$

(d) (4 points) Similarly to Theorem D & Corollary D of Lecture 2, construct a 95% bootstrap confidence interval for $\theta$ using the asymptotic normality in (c);

**Solution.** When $\alpha = 0.05$, $z_{\alpha/2} = 1.96$, and the 95% bootstrap confidence interval for $\theta$ can be constructed as follows

$$\hat{\theta}_{\text{MM}} \pm 1.96 \times \sqrt{\frac{\hat{\theta}_{\text{MM}}(\hat{\theta}_{\text{MM}} - 1)^2}{n(\hat{\theta}_{\text{MM}} - 2)}}. \qquad \textit{(4 pts)}$$

(e) (6 points) Suppose a researcher collected 56 observations and the sample mean was 1.76. Calculate the values of the lower and upper bounds of the CI in (d). Apply the duality between CIs and hypothesis tests to make a decision about

$$H_0 : \theta = 2.3 \text{ versus } H_1 : \theta \neq 2.3.$$

Do you reject the null hypothesis at the significance level $\alpha = 0.05$?

**Solution.** Since $\bar{X}_n = 1.76$, we have

$$\hat{\theta}_{\text{MM}} = \frac{1.76}{1.76 - 1} = 2.3158. \qquad \textit{(1 pt)}$$

Plug into the CI in (d), and we obtain the 95% CI for $\theta$:

$$\hat{\theta}_{\text{MM}} \pm 1.96 \times \sqrt{\frac{\hat{\theta}_{\text{MM}}(\hat{\theta}_{\text{MM}} - 1)^2}{n(\hat{\theta}_{\text{MM}} - 2)}} = [1.383, 3.249]. \qquad \textit{(2 pts)}$$

By the duality between CIs and hypothesis tests, a acceptance region at significance level $\alpha = 0.05$ should be

$$A = \left\{ \theta_0 \in \left[ \hat{\theta}_{\text{MM}} \pm 1.96 \times \sqrt{\frac{\hat{\theta}_{\text{MM}}(\hat{\theta}_{\text{MM}} - 1)^2}{n(\hat{\theta}_{\text{MM}} - 2)}} \right] \right\}, \qquad \textit{(2 pts)}$$

in which $\theta_0$ is the null parameter. Since in our case $\theta_0 = 2.3 \in [1.383, 3.249]$, we accept $H_0$ or fail to reject the null hypothesis. *(1 pt)*