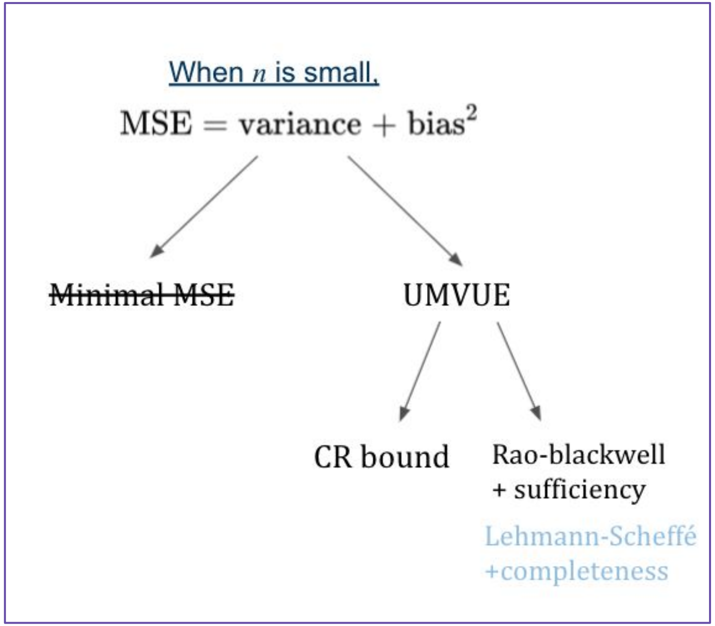
Exact distribution of $\bar{X}_n$ and $\hat{\sigma}_n^2$ under $N(\mu, \sigma^2)$

8.5.3 of Rice

07/06/2021

Berkeley
UNIVERSITY OF CALIFORNIA

In the previous lecture,



When n is small,

$$\text{MSE} = \text{variance} + \text{bias}^2$$

~~Minimal MSE~~

UMVUE

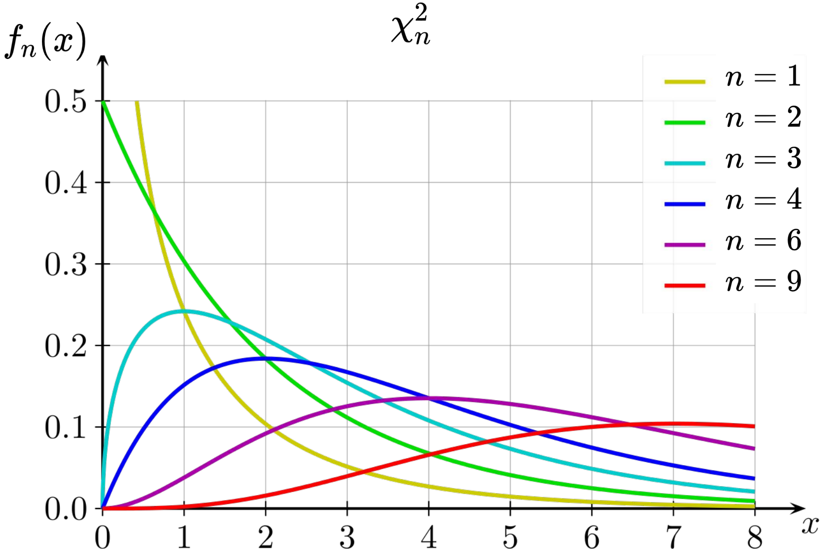
CR bound

Rao-blackwell
+ sufficiency

Lehmann-Scheffé
+completeness

- CR lower bound and UMVUE.
  - The estimator with minimum variance among **all** unbiased estimators.
- Sufficient statistics
  - $U(-\theta, \theta)$
  - $\mathbf{X}_n \,|\, T(\mathbf{X}_n)$ is difficult to deal with.
  - Fisher-Neyman Factorization Theorem.
  - Exponential family.
- Rao-blackwell Theorem
  - Construct a better estimator using a sufficient statistic.

Exact sampling distribution under $N(\mu, \sigma^2)$



- $\chi^2_n = Z_1^2 + Z_2^2 + \ldots + Z_n^2.$
- $\chi^2_n \sim \text{Gamma}(\frac{n}{2}, \frac{1}{2})$
- Given two independent random variables $T_n \sim \chi^2_n$ and $S_m \sim \chi^2_m$, prove $T_n + S_m \sim \chi^2_{m+n}$.

Exact sampling distribution under $N(\mu, \sigma^2)$

$E\bar{X}_n = \mu$

$var\,\bar{X}_n = \frac{\sigma^2}{n}$

Theorem G. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Then $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, $\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2/\sigma^2 \sim \chi^2_{n-1}$ and they are *independent* of each other.

Proof*. Recap: 1. Orthogonal matrix $Q^TQ = QQ^T = I$.

2. $X \sim N(\vec{\mu}, I)$, $Y = QX \sim N(Q\vec{\mu}, Q\Sigma Q^T)$.

3. For any non-zero vector $\vec{v}$, we can construct an orthogonal matrix Q such as the first row of Q is $\vec{v}$. ⇐ Gram-Schmidt Theorem.

under IID assumption,

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N\left[\mu\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \sigma^2 I\right].$$

The orthogonal matrix we want to construct is

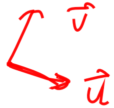
$$Q = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \cdots & \bullet \end{pmatrix}$$

and $Q^TQ = QQ^T = I$

$$Y = Q\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N\left(Q\mu\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \underset{QQ^T}{\sigma^2 Q I Q^T}\right) = N\left(\mu Q\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \sigma^2 I\right)$$

Cont'd : $\underline{Y} = Q\underline{X} = N\left(\mu Q\underbrace{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}}, \sigma^2 I\right) = N\left(\begin{pmatrix}\sqrt{n}\mu\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}, \sigma^2 I\right).$

$\frac{1}{\sqrt{n}}(1, \cdots 1)$



Recall $Q\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}}\\ \vdots & & \vdots\\ \vdots & & \vdots\end{pmatrix}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}\sqrt{n}\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$

$\underline{Y} = Q\underline{X} = \begin{pmatrix}\frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}}\\ \vdots & \ddots & \vdots\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}\frac{x_1+\cdots+x_n}{\sqrt{n}}\\ Y_2\\ Y_3\\ \vdots\\ Y_n\end{pmatrix} = \begin{pmatrix}\sqrt{n}\bar{X}_n\\ Y_2\\ \vdots\\ Y_n\end{pmatrix}$

$\sqrt{n}\bar{X}_n$ and $\underbrace{(Y_2, \cdots Y_n)}$ are indep.

$\Downarrow$

$\bar{X}_n$ and $\underbrace{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}$ are also indep.

$\underline{Y}^T\underline{Y} = (\sqrt{n}\bar{X}_n, Y_2, \cdots Y_n)\begin{pmatrix}\sqrt{n}\bar{X}_n\\ Y_2\\ \vdots\\ Y_n\end{pmatrix} = \underbrace{n\bar{X}_n^2} + Y_2^2 + \cdots + Y_n^2$

$\parallel$ $\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$(Q\underline{X})^T(Q\underline{X}) = \underline{X}^T\underbrace{Q^TQ}_{\overset{\parallel}{I}}\underline{X} = \underline{X}^T\underline{X} = x_1^2 + \cdots + x_n^2 = \sum X_i^2$

$\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ //

$\Rightarrow \boxed{\sum X_i^2 - n\bar{X}_n^2} = \underbrace{Y_2^2 + \cdots + Y_n^2} \sim \sigma^2\chi^2_{n-1}.$

since $\begin{pmatrix}Y_2\\ \vdots\\ Y_n\end{pmatrix} \sim N\left(\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}, \sigma^2 \underline{I_{n-1}}\right).$

$\frac{\sum(X_i - \bar{X}_n)^2}{\sigma^2} \sim \chi^2_{n-1} \perp\!\!\!\perp \bar{X}_n$ ■

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \leftarrow$ MLE, MM

Exact sampling distribution under $N(\mu, \sigma^2)$

Definition. A student *t* distributed r.v. with df=n can be generated using independent $Z \sim N(0,1)$ and $U \sim \chi^2_{\underset{\uparrow}{n}}$:

$$\frac{Z}{\sqrt{U/n}} \sim t_{\underset{\uparrow}{n}}.$$

$$\boxed{\Rightarrow \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t_{n-1}.}$$

$\uparrow$

$\bar{X}_n \perp\!\!\!\perp S^2$

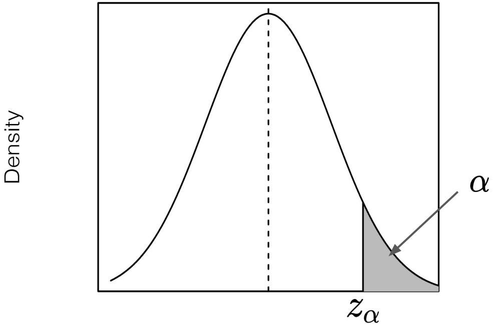
$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

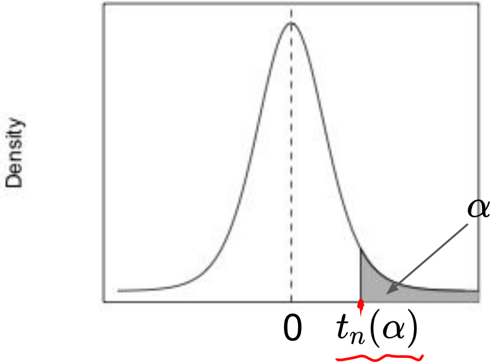
$$\frac{\bar{X}_n - \mu}{\cancel{\sigma}/\sqrt{n}} \Big/ \sqrt{\frac{\cancel{(n-1)}S^2}{\cancel{\sigma^2}} \Big/ \cancel{(n-1)}} = \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

z_α

Normal distribution



Student's t distribution

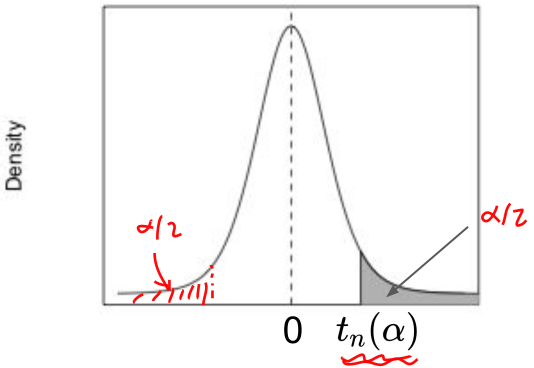


$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n}{2})}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

Exact confidence intervals under $N(\mu, \sigma^2)$

Theorem A. Suppose $X_1, \ldots, X_n$ are i.i.d $N(\mu, \sigma^2)$. Then

$$P\left(-t_{n-1}(\alpha/2) \leq \frac{\sqrt{n}(\overline{X}_n - \mu)}{S} \leq t_{n-1}(\alpha/2)\right) = 1 - \alpha$$



$(1-\alpha) \times 100\,\%$ exact CI for μ :

$$P\left(\overline{X}_n - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2) \leq \mu \leq \overline{X}_n + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)\right) = 1 - \alpha$$

$t_{n-1}(\alpha/2)$

```
qt(p = alpha, df = n, lower.tail = FALSE)
```

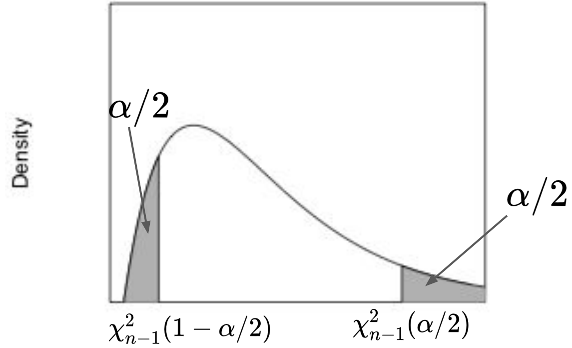
$\alpha/2$ $n-1$

$$\frac{n \sigma_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

Exact confidence intervals under $N(\mu, \sigma^2)$

Theorem B. Suppose $X_1, \ldots, X_n$ are i.i.d $N(\mu, \sigma^2)$. Then

$$P\left(\underbrace{\chi^2_{n-1}(1-\alpha/2) \leq \frac{n\hat{\sigma}_n^2}{\sigma^2} \leq \chi^2_{n-1}(\alpha/2)}\right) = 1-\alpha$$



$(1-\alpha) \times 100\,\%$ exact CI for σ^2 :

$$P\left(\frac{n\hat{\sigma}_n^2}{\chi^2_{n-1}(\alpha/2)} \leq \sigma^2 \leq \frac{n\hat{\sigma}_n^2}{\chi^2_{n-1}(1-\alpha/2)}\right) = 1-\alpha$$

```
qchisq(p = alpha/2, df = n-1, lower.tail = FALSE)
qchisq(p = 1-alpha/2, df = n-1, lower.tail = FALSE)
```

Exact confidence intervals under $N(\mu, \sigma^2)$

Example 1. Let's say we observed 12 i.i.d Normal samples:

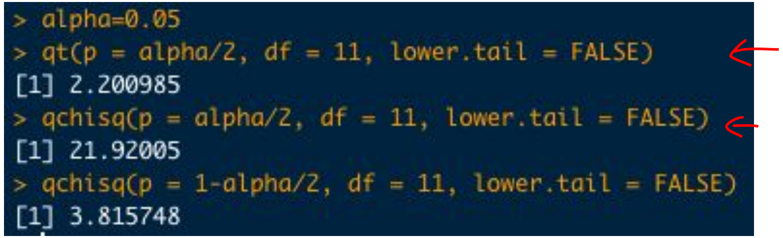
{6.749, 6.658, 3.966, 8.359, 4.043, 5.245, 3.375, 6.621, 5.216, 10.945, 2.260, 2.015}

Find 95% exact confidence intervals for μ and σ^2.

$$\bar{X}_{12} = 5.454$$

$$\frac{1}{n-1}\Sigma(X_i-\bar{X}_n)^2 = \quad S^2 = 6.703$$

$$\frac{1}{n}\Sigma(X_i-\bar{X}_n)^2 \quad \underline{\hat{\sigma}_n^2 = 6.145}$$



```
> alpha=0.05
> qt(p = alpha/2, df = 11, lower.tail = FALSE)
[1] 2.200985
> qchisq(p = alpha/2, df = 11, lower.tail = FALSE)
[1] 21.92005
> qchisq(p = 1-alpha/2, df = 11, lower.tail = FALSE)
[1] 3.815748
```

95% CI for μ:

$$\bar{X}_n \pm \frac{S}{\sqrt{n}} t_{n-1}\left(\frac{\alpha}{2}\right)$$

$$= 5.454 \pm \sqrt{\frac{6.703}{12}}\, \underline{t_{11}\left(\frac{0.05}{2}\right)} = 5.454 \pm \sqrt{\frac{6.703}{12}} \times 2.2009$$

$$= [3.900, 7.190].$$

95% CI for σ^2:

$$\left[\frac{n\hat{\sigma}_n^2}{\chi^2_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}_n^2}{\chi^2_{n-1}(1-\alpha/2)}\right] = \left[\frac{12 \times 6.145}{21.92005}, \frac{12 \times 6.145}{3.815748}\right]$$

$$= [3.364, 19.325].$$

Bootstrap confidence intervals under $N(\mu, \sigma^2)$

Example 1 *cont'd*. Let's say we observed 12 i.i.d Normal samples:

{6.749, 6.658, 3.966, 8.359, 4.043, 5.245, 3.375, 6.621, 5.216, 10.945, 2.260, 2.015}

Find 95% bootstrap confidence intervals for μ and σ^2 .

$$z_{\alpha/4} = z_{0.05/4} = 2.241.$$

$$\bar{X}_{12} = 5.454$$

$$S^2 = 6.703 \leftarrow$$

$$\hat{\sigma}_n^2 = 6.145$$

$$> \cancel{t_{n-1}(\alpha/2)}$$

$1-\alpha/2/2$

$$\bar{X}_n \pm \frac{z_{\alpha/4}}{\sqrt{nI(\mu)}} = 5.454 \pm \frac{2.241}{\sqrt{12 \times 1/\sigma^2}} \overset{\text{plug in}}{=} 5.454 \pm 2.241\sqrt{\frac{6.703}{12}}$$

$$\hat{\sigma}_n \pm \frac{z_{\alpha/4}}{\sqrt{nI(\sigma)}} = \left(\sqrt{6.145} \pm \frac{2.241}{\sqrt{12 \times 2/\sigma^2}}\right)^2 = [3.941, 7.149] = [3.900, 7.190]$$

σ^2 ↓

$$I(\mu, \sigma) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}$$

$$\overset{\text{plug in}}{=} \left(\sqrt{6.145} \pm \frac{2.241\sqrt{6.145}}{\sqrt{12 \times 2}}\right)^2 = [1.345, 3.613]^2 = [1.809, 3.053].$$

Bootstrap confidence intervals under $N(\mu, \sigma^2)$

Example 1 *cont'd*. Let's say we observed 12 i.i.d Normal samples:

{6.749, 6.658, 3.966, 8.359, 4.043, 5.245, 3.375, 6.621, 5.216, 10.945, 2.260, 2.015}

Find 95% bootstrap confidence intervals for μ and σ^2 .

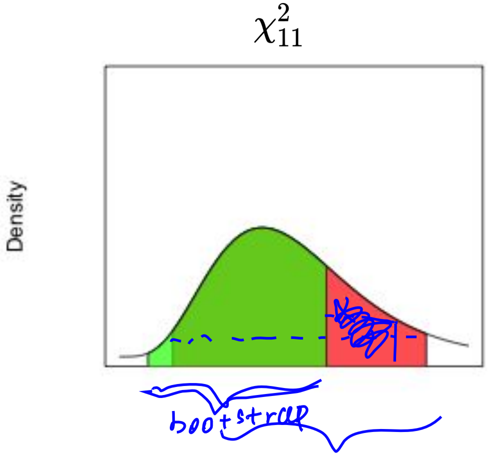
$(1-\alpha) \times 100\,\%$ exact CI for σ^2 :

[3.364, 16.325]

$(1-\alpha) \times 100\,\%$ bootstrap CI for σ^2 :

[1.809, 13.053]

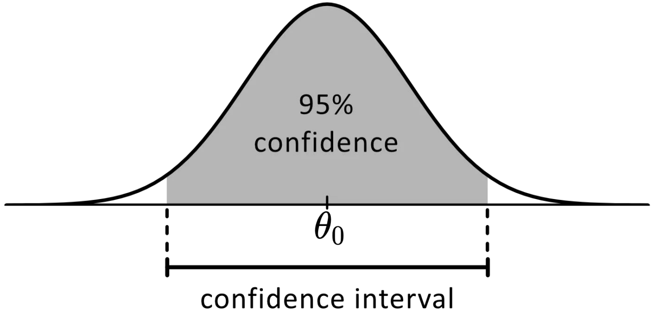
Seemingly more accurate CI



exact → more centered around the mode.

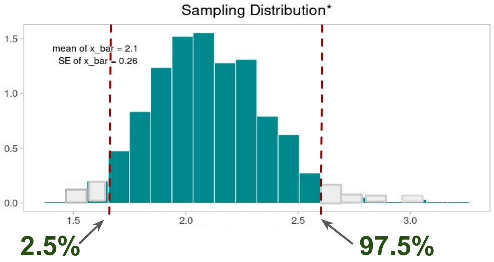
Short summary of CI

Sampling distribution of an estimator



- $\bar{X}_n$ and $\hat{\sigma}_n^2$ for population mean and variance;
- MM estimators;
- Maximum likelihood estimators.

1. *Exact sampling distribution;*
   e.g. MLE for $U(-\theta, \theta)$,
   $\bar{X}_n$ *for* $\mathrm{Gamma}(\alpha, \beta)$ *and* $N(\mu, \sigma^2)$
   $\hat{\sigma}_n^2$ *for* $N(\mu, \sigma^2)$
2. *Asymptotic normality (Lecture 5);*
   $\bar{X}_n,\ g(\bar{X}_n)$, *MLE*
3. *Parametric or non-parametric bootstrapping (Page 10, Lecture 3 & Lab 3).*



Hypothesis testing

9.2 of Rice

07/06/2021

Hypothesis testing

Definition. A hypothesis is a statement about a population parameter.

Income population: $\mu > 50,000$

Youtube A/B testing: $\underline{\alpha_1/\beta_1} < \underline{\alpha_2/\beta_2}$

$\alpha/\beta = EX$ with $X \sim \text{Gamma}(\alpha, \beta)$

Definition. The goal of testing hypotheses is to decide, *based on a sample from the population*, which of two complementary hypotheses is true:

H_0 : null hypothesis ⟷ H_1 : alternative hypothesis

- ➔ *De facto*
- ➔ *Accepted fact*
- ➔ *Popular conception*
- ➔ *Ineffective*

- ➔ *Surprising*
- ➔ *Unorthodox*
- ➔ *Effective*

We are trying to **reject** a conventional idea to establish something new.

Hypothesis testing

Galileo Galilei

H_0: Geocentrism (Earth is the center of the universe) ⟷ H_1: Heliocentrism (Earth rotating daily and revolving around the sun)

Courtroom trial

H_0: The defendant is not guilty ⟷ H_1: The defendant is guilty

The hypothesis of innocence is rejected only when an error is very unlikely.

Efficacy of vaccines

H_0: The new vaccine is not effective ⟷ H_1: The new vaccine is effective

The hypothesis of ineffectiveness is rejected only when there is strong evidence that the vaccine prevents infection.

Type I error

Definition. The H_0 is rejected when it is actually true.

The occurrence of type I error should be controlled to be rare.

Courtroom trial

H_0: The defendant is not guilty ⟷ H_1: The defendant is guilty

The conviction of an innocent defendant.

Efficacy of vaccines

H_0: The new vaccine is not effective ⟷ H_1: The new vaccine is effective

The approval of an ineffective vaccine.

Definition. The *significance level* of test $\alpha = \mathrm{P}(\text{Type I error})$.

Type II error

Definition. The H_0 is not rejected when H_1 is actually true.

The consequence of type II error is less severe.

Courtroom trial

H_0: The defendant is not guilty ⟷ H_1: The defendant is guilty

Acquitting a person who committed the crime.

Efficacy of vaccines

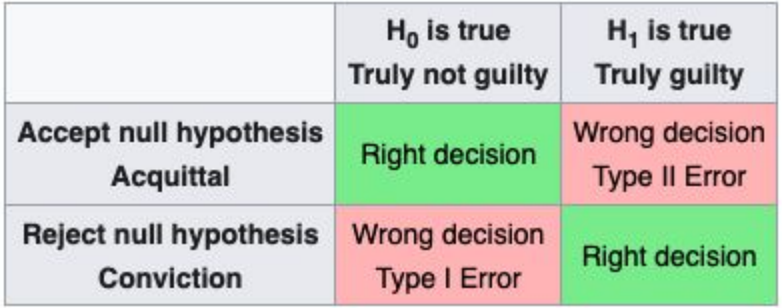
H_0: The new vaccine is not effective ⟷ H_1: The new vaccine is effective

Discarding a vaccine which is life-saving.

Definition. The *power* of test $\beta = \mathrm{P}(\text{Correctly reject } H_0) \;=\; 1 - \mathrm{P}(\text{Type II error})$

The asymmetric nature of HT

To make the testing result more convincing, we should make it more difficult to reject the null hypothesis H_0.



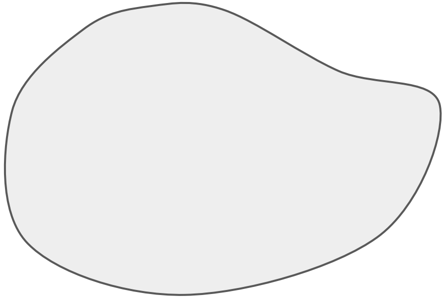
	H_0 is true Truly not guilty	H_1 is true Truly guilty
Accept null hypothesis **Acquittal**	Right decision	Wrong decision Type II Error
Reject null hypothesis **Conviction**	Wrong decision Type I Error	Right decision

Mis-stating the hypotheses will muddy the rest of the testing process.

Test statistic & rejection region

For notational simplicity, denote $\mathbf{X}_n = (X_1, \ldots, X_n)$.

Definition. A statistic $T(\mathbf{X}_n)$ is *the test statistic* if it is used to decide whether to reject H_0.



All possible values of $T(\mathbf{X}_n)$

Definition. *Rejection region* is the subset of test statistic values that leads to the rejection of H_0.

Test statistic & rejection region

For notational simplicity, denote $\mathbf{X}_n = (X_1, \ldots, X_n)$.

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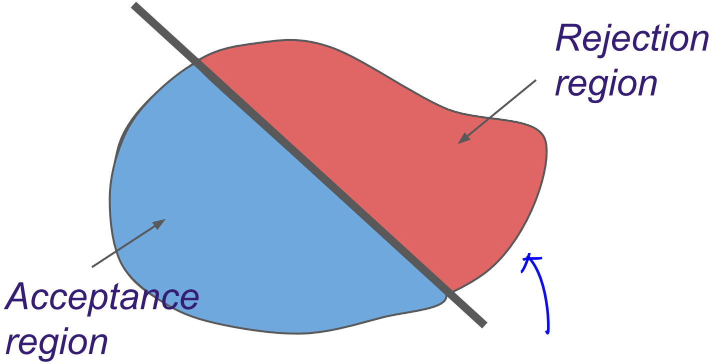


All possible values of $T(\mathbf{X}_n)$

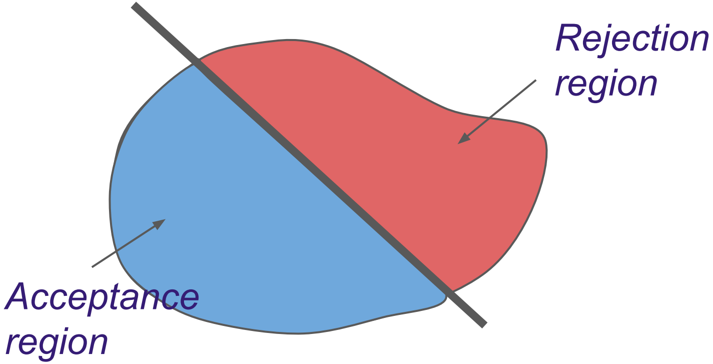
Definition. *Rejection region* is the subset of test statistic values that leads to the rejection of H_0.

⇩ To mitigate type I error

$$R = \{\text{Unlikely } T(\mathbf{X}_n) \text{ values under } H_0\}$$

How to find rejection region?

$$R = \{\text{Unlikely } T(\mathbf{X}_n) \text{ values under } H_0\}$$



All possible values of $T(\mathbf{X}_n)$

μ, p

$$H_0 : \mu = \overset{H_0}{5'} \quad \leftrightarrow \quad H_1 : \mu = \overset{H_1}{6'}$$

$$R = \{\bar{X}_n : \bar{X}_n > 5'9''\}$$

← simple HT

$$H_0 : p = 0.5 \quad \leftrightarrow \quad H_1 : p < 0.5$$

$$R = \{\hat{p}_n : \hat{p}_n < 0.45\}$$

← composite HT

$$\hat{p}_n = \frac{\#\text{ of left-handed obs}}{\#\text{ of all obs}}$$

$\max\{x_1, \dots x_n\}$ $\sup[\dots]$

Likelihood ratio test (LRT)

$$\frac{\sup_{\Theta_0} L(\theta|X_n)}{\sup_{\Theta_1} L(\theta|X_n)}$$

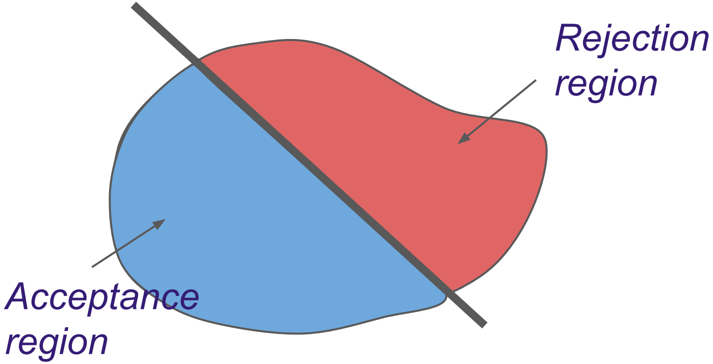
$\sup \Theta \geq \sup \Theta_0$

$$R = \{\text{Unlikely } T(\mathbf{X}_n) \text{ values under } H_0\}$$

Denote $\Theta = \Theta_0 \cup \Theta_1$.

$$H_0 : \theta \in \Theta_0 \quad \leftrightarrow \quad H_1 : \theta \in \Theta_1$$

$$L(\theta \,|\, \mathbf{X}_n) = \prod_{i=1}^{n} f(X_i \,|\, \theta) =$$ *The likelihood of observing* $X_1, \dots, X_n$ *under* θ.



All possible values of $\lambda(\mathbf{X}_n)$

Definition. A *likelihood ratio test statistic* is defined as

$$\lambda(\mathbf{X}_n) = \frac{\sup_{\Theta_0} L(\theta \,|\, \mathbf{X}_n)}{\sup_{\Theta} L(\theta \,|\, \mathbf{X}_n)}$$

← restricted

← unrestricted

The rejection region of a LRT should be

$$\{\lambda(\mathbf{X}_n) \leq c\}$$

in which $0 \leq c \leq 1$.

$\lambda(X_n) \downarrow$, H_0 is less likely.

Likelihood ratio test (LRT)

$R = \{ \hat{p}_n > c \}$

$\frac{1}{\max\{a,b\}} = \min\left(\frac{1}{a}, \frac{1}{b}\right)$

Example 2. Let $X_1, \dots, X_n$ be i.i.d $\text{Bernoulli}(p)$. Consider testing

$$H_0 : p = 0.49 \quad \leftrightarrow \quad H_1 : p = 0.51. \in [0,1]$$

Find the LRT statistic and a rejection region.

$$\lambda(\mathbf{X}_n) = \frac{\sup_{\Theta_0} L(\theta \,|\, \mathbf{X}_n)}{\sup_{\Theta} L(\theta \,|\, \mathbf{X}_n)}$$

Solution. $L(p|\mathbf{X}_n) = p^{\Sigma X_i}(1-p)^{n-\Sigma X_i}$

$$\lambda(\mathbf{X}_n) = \frac{\sup_{\Theta_0} L(p|\mathbf{X}_n)}{\sup_{\Theta} L(p|\mathbf{X}_n)} = \frac{L(0.49|\mathbf{X}_n)}{\max\{L(0.49|\mathbf{X}_n), L(0.51|\mathbf{X}_n)\}}$$

$$= \frac{0.49^{\Sigma X_i}\, 0.51^{n-\Sigma X_i}}{\max\{0.49^{\Sigma X_i}\, 0.51^{n-\Sigma X_i},\ 0.51^{\Sigma X_i}\, 0.49^{n-\Sigma X_i}\}} = \frac{1}{\max\left\{1, \left(\frac{0.51}{0.49}\right)^{\Sigma X_i}\left(\frac{0.49}{0.51}\right)^{n-\Sigma X_i}\right\}}$$

$$= \frac{1}{\max\left\{1, \left(\frac{0.51}{0.49}\right)^{2\Sigma X_i - n}\right\}} = \min\left\{1, \left(\frac{0.49}{0.51}\right)^{2\Sigma X_i - n}\right\}$$ ← LRT statistic

By definition of LRT,

$c \in [0,1]$

$$R = \{ \underbrace{\lambda(\bar{X}_n) \le c} \} = \left\{ \underbrace{\min\left\{1, \left(\frac{0.49}{0.51}\right)^{2\Sigma x_i - n}\right\}} \overset{\le}{} \le c \right\}$$

$$= \left\{ \left(\frac{0.49}{0.51}\right)^{2\Sigma x_i - n} \le c \right\}$$

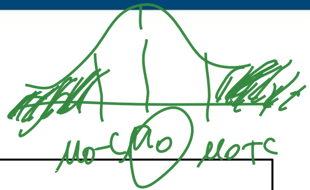
$$= \left\{ \underbrace{(2\Sigma x_i - n)}_{\cancel{<0} \; >0} \underbrace{\log \frac{0.49}{0.51}}_{<0} \le \underbrace{\log c}_{\le 0} \right\}$$

$$= \left\{ 2\Sigma x_i - n \ge \log c \Big/ \log \frac{0.49}{0.51} \right\} = \{ \underbrace{\bar{X}_n \ge c'} \}$$

where $c' = \dfrac{n + \log c \Big/ \log \frac{0.49}{0.51}}{2n}$

Likelihood ratio test (LRT)

$$\Big\{ \underbrace{|\bar{X}_n - \mu_0| \ge c} \Big\}$$



Example 2. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, 1)$. Consider testing

$$\underline{H_0 : \mu = \boxed{\mu_0}} \quad \leftrightarrow \quad \underline{H_1 : \mu \neq \mu_0}.$$

Find the LRT statistic and a rejection region.

$$\lambda(\mathbf{X}_n) = \frac{\sup_{\Theta_0} L(\theta \,|\, \mathbf{X}_n)}{\sup_{\Theta} L(\theta \,|\, \mathbf{X}_n)}$$

$\Theta_0 = \{ \mu_0 \}$ ←

$\Theta = \Theta_0 \cup \Theta_1 = \mathbb{R}$,

Solution.

$$L(\mu \,|\, \mathbf{X}_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2}}$$

$$\rightarrow \underbrace{\sup_{\Theta} L(\mu \,|\, \mathbf{X}_n)} = L(\underset{\overset{\downarrow}{\bar{X}_n}}{\hat{\mu}_{MLE}} \,|\, \mathbf{X}_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} \frac{(X_i-\bar{X}_n)^2}{2}}$$

$$\rightarrow \sup_{\Theta_0} L(\mu \,|\, \mathbf{X}_n) = L(\mu_0 \,|\, \mathbf{X}_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} \frac{(X_i-\mu_0)^2}{2}}$$

Thus $\lambda(\underline{X}_n) = \dfrac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2}}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}{2}}} = e^{-\frac{1}{2}\left\{\underbrace{\sum_{i=1}^{n}(x_i-\mu_0)^2}_{\uparrow} - \underbrace{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}_{\uparrow}\right\}}$

$$= e^{-\frac{1}{2}\left\{\cancel{\sum_{i=1}^{n} x_i^2} + \underline{\underline{n\mu_0^2}} - \underbrace{2\mu_0\sum_{i=1}^{n} x_i}_{=2n\mu_0\bar{x}_n} - \cancel{\sum_{i=1}^{n} x_i^2} - \underbrace{n\bar{x}_n^2 + 2\bar{x}_n \overset{n\bar{x}_n}{\boxed{\sum x_i}}}_{=n\bar{x}_n^2}\right\}}$$

$$= e^{-\frac{n}{2}\left\{\underbrace{\mu_0^2 - 2\mu_0\bar{x}_n + \bar{x}_n^2}\right\}}$$

$$= e^{-\frac{n}{2}(\bar{x}_n-\mu_0)^2}$$

$c\uparrow$ easier to reject H_0

$c\downarrow$ $\quad c=0 \quad n\to$

By definition of LRT, $\underline{R = \{\lambda(\underline{X}_n) \le c\}} = \left\{e^{-\frac{n}{2}(\bar{X}_n-\mu_0)^2} \le c\right\}$

$$= \left\{-\frac{n}{2}(\bar{X}_n-\mu_0)^2 \le \log c\right\} = \left\{(\bar{X}_n-\mu_0)^2 \ge \frac{2\log 1/c}{n}\right\}$$

$= \{|\bar{X}_n-\mu_0| \ge c'\}$ where $c' = \sqrt{\dfrac{2\log 1/c}{n}}$.

Tomorrow ...

- Prescribe a significance level α for a test; ✓
- P-value; ✓
- Uniformly most powerful test. ✓