STAT 135 CONCEPTS OF STATISTICS
QUIZ 2, LAB 101

July 15, 2021

Instructions: *You have 35 minutes to complete the quiz and upload it on bCourses. This quiz is open book and you may use a calculator, but all work must be shown in order to receive full credit.*

Problem 1 (5 points total)**.** Let $X_1, \ldots, X_n$ be i.i.d random variable with probability density function

$$f(\theta x^{\theta-1}) = \begin{cases} \theta x^{\theta-1}, & \text{if } 0 \le x \le 1, \\ 0, \text{otherwise}, \end{cases}$$

in which $\theta \in [0, 1]$ is unknown.

(1) What is the MLE $\hat{\theta}_{MLE}$ of θ? Is $\hat{\theta}_{MLE}$ sufficient?
(2) Write the asymptotic Normal distribution of $\hat{\theta}_{MLE}$.
(3) Given $\hat{\theta}_{MLE} = 1.5$ and $n = 110$. Give a 95% bootstrap confidence interval for θ.
(4) Calculate $E(\hat{\theta}_{MLE})$ and $\text{Var}(\hat{\theta}_{MLE})$, and compare the variance to the CR lower bound. Which has lower value?
(*Hint*: (i)$E(aX) = aE(X)$, $\text{Var}(aX) = a^2\text{Var}(X)$; (ii) $-\log X_i \overset{\text{iid}}{\sim} \text{Gamma}(1, \theta)$; (iii) If $W \sim \text{Gamma}(\alpha, \beta)$, $E\left(\frac{1}{W}\right) = \frac{\beta}{\alpha-1}$ and $\text{Var}\left(\frac{1}{W}\right) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$.)

$\boxed{\text{Solution}}$ (1) The log-likelihood is

$$l(\theta) = n\log\theta + (\theta - 1)\sum_{i=1}^{n} \log X_i$$

Taking the derivation to get

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{-\sum_{i=1}^{n} \log X_i}$$

Since the joint likelihood $L(\theta) = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1} = \theta^n e^{(\theta-1)\sum_{i=1}^{n}\log X_i}$ we know by Fisher-Neyman factorization theorem that $\hat{\theta}_{MLE}$ is sufficient for θ.

(2) First we calculate the Fisher information

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log f\right) = -E\left[\frac{\partial^2}{\partial\theta^2}(\log\theta + (\theta-1)\log X)\right]$$

$= \frac{1}{\theta^2}$.

Thus, $\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \frac{1}{I(\theta)}) = N(0, \theta^2)$.

(3) The 95% bootstrap confidence interval is

$$\hat{\theta}_{MLE} \pm z_{\alpha/2} \frac{\hat{\theta}_{MLE}}{\sqrt{n}} = 1.5 \pm 1.96 \times \frac{1.5}{\sqrt{110}}$$

$$= [1.22, 1.78].$$

(4) Since $-\log X_i \overset{iid}{\sim} \text{Gamma}(1, \theta)$,

$$-\sum_{i=1}^{n} \log X_i \sim \text{Gamma}(n, \theta).$$

$$\Rightarrow E(\hat{\theta}_{MLE}) = n \times \frac{\theta}{n-1} = \frac{n\theta}{n-1}$$

$$var(\hat{\theta}_{MLE}) = n^2 \times \frac{\theta^2}{(n-1)^2(n-2)} = \frac{n^2\theta^2}{(n-1)^2(n-2)}.$$

The CR bound for $\hat{\theta}_{MLE}$ is

$$\frac{\left\{\left[\frac{n\theta}{n-1}\right]'\right\}^2}{n I(\theta)} = \frac{n^2}{(n-1)^2} \times \frac{\theta^2}{n}$$

$$= \frac{n\theta^2}{(n-1)^2}.$$

Since $\frac{n^2\theta^2}{(n-1)^2(n-2)} \Big/ \frac{n\theta^2}{(n-1)^2} = \frac{n}{n-2} > 1$,

the variance of $\hat{\theta}_{MLE}$ is strictly lower than the CR lower bound.