Uniformly most powerful tests

9.2.3 of Rice

07/08/2021

In the previous lecture,

- Power function $\beta(\theta) = P(\mathbf{X}_n \in R \,|\, \theta)$.
  - $$\beta(\theta) = \begin{cases} \text{Type I error}, & \theta \in \Theta_0, \\ 1 - \text{Type II error}, & \theta \in \Theta_1. \end{cases}$$
  - Ideally, power function is 0 for $\theta \in \Theta_0$ and 1 for $\theta \in \Theta_1$.
  - When n is sufficiently large, Type I and II error can be managed at the same time.
  - When n is limited, Type I and II error are <u>inversely proportional</u>.
- Uniformly most powerful test:
  - Maximizing power among tests with $\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$.
  - Neyman-Pearson Lemma: LRT with $\beta(\theta_0) = \alpha$ is UMP for simple hypotheses testing.
- Perform hypothesis testing with real data:
  - Specify significance level α and calculate the rejection region;
  - Calculate p-value and compare it with α.

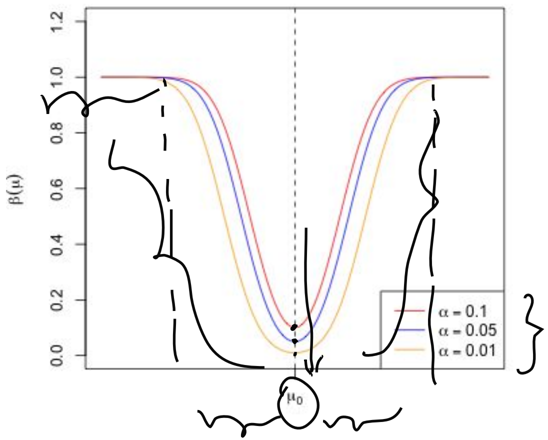
No free lunch

$$H_0 : \underbrace{\mu = \mu_0}_{\text{singleton}} \quad \leftrightarrow \quad H_1 : \mu \neq \mu_0.$$

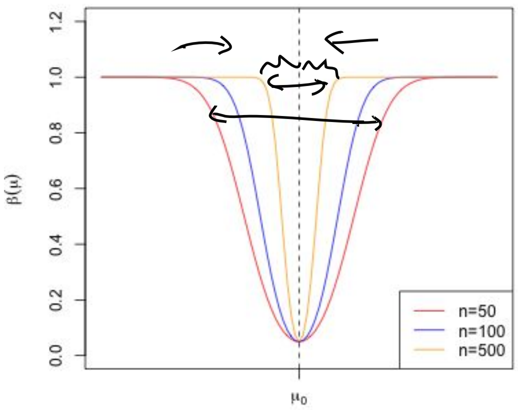
$\beta(\mu_0)$

$= P(\text{Type I error})$

n=50 fixed



α =0.05 fixed



α

$\beta(\theta) = P(R \mid \theta)$

UMP tests for one-sided hypothesis

Example 4. Let $X_1, \ldots, X_{20}$ be i.i.d $N(\mu, 1)$. Consider two tests and find the LRT rejection region.

$$H_0: \mu = 1 \quad \leftrightarrow \quad H_1: \mu = 2.$$

$$H_0: \mu = 1 \quad \leftrightarrow \quad H_1: \mu > 1.$$

$$R = \{\bar{X}_n \geq c'\}$$

→ Denote the LRT as δ:

→ LRT $\quad \delta(\bar{X}_n) = \mathbb{1}\{\bar{X}_n \geq c'\}$

We have to prove that for any other test $\delta'(\bar{X}_n) = \mathbb{1}(\bar{X}_n \in R')$ of size α, i.e.

$\sup_{\theta \in \Theta_0} \beta'(\theta) \leq \alpha$, $\quad \underbrace{\beta(\theta)}_{\delta} \geq \underbrace{\beta'(\theta)}_{\delta'}$ for any $\theta \in \Theta_1$.

First, let's consider $\mu_1 > 1$ and the following test:

→ $H_0: \mu = 1$ vs. $H_1: \mu = \mu_1$. $\Rightarrow R = \{\bar{X}_n \geq c'\}$

Therefore, δ and δ' are also the size α tests for the previous hypotheses.

By Neyman-Pearson lemma, $1 - P_\delta(\text{Type II error}) \geq 1 - P_{\delta'}(\text{Type II error})$

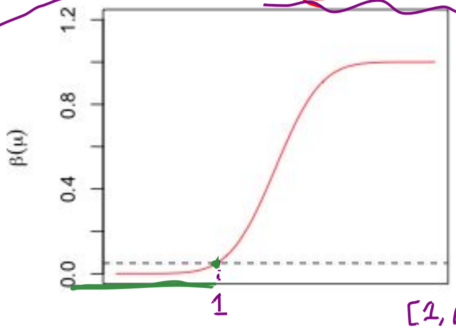
$\beta(\mu_1) \geq \beta'(\mu_1) \quad \Leftarrow \quad \beta(\mu_1) = P(R \mid \mu = \mu_1) \quad \beta'(\mu_1) = P(R' \mid \mu = \mu_1)$

Under μ, $\bar{X}_n \sim N(\mu, \frac{1}{n})$

$\beta(\mu) = P(R \mid \mu) = P(\bar{X}_n \ge c' \mid \mu) = P\left(\frac{\bar{X}_n - \mu}{1/\sqrt{n}} \ge \frac{c'-\mu}{1/\sqrt{n}}\right)$ $\quad Z \sim N(0,1)$

UMP tests for one-sided hypothesis

$$\beta(\mu) = 1 - \Phi\left(\frac{c' - \mu}{1/\sqrt{n}}\right)$$



Strictly increasing

$\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha$

$[1, \infty)$

$\mathbb{R}$

$$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu = 2.$$

$$\Downarrow$$

$$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu > 1.$$

$\Rightarrow \beta(\theta_0) = \alpha$

$\sup_{\mu \le 1} \beta(\mu) = \beta(1) = \alpha$

$\Downarrow$ ***Karlin-Rubin****

R / R^c

The LRT with significance level α for $H_0: \mu = 1$ vs $H_1: \mu > 1$

$$H_0 : \mu \le 1 \quad \leftrightarrow \quad H_1 : \mu > 1.$$

is of size α for $H_0: \mu \le 1$ vs. $H_1: \mu > 1$

$\sup_{\mu \le 1} \beta(\theta) \le \alpha \Rightarrow \sup_{\mu = 1} \beta(\theta) \le \alpha$ $\quad \| \; \beta(1)$

$\beta(\mu) \ge \beta'(\mu)$ for $\mu > 1$

LRT is uniformly most powerful!

R' / R^c

By Neyman-Pearson lemma, δ & δ' are still size α test for $H_0: \mu = 1$ vs $H_1: \mu = \mu_1$

$\beta(\mu) \ge \beta'(\mu) \Longleftarrow$

UMP tests for one-sided hypothesis

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0).$$

$$\Downarrow$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$

$$\Downarrow$$ ***Karlin-Rubin***

$$H_0 : \theta \leq \theta_0 \quad \leftrightarrow \quad H_1 : \theta > \theta_0.$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta = \theta_1 \quad (\theta_1 < \theta_0).$$

$$\Downarrow$$

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

$$\Downarrow$$ ***Karlin-Rubin***

$$H_0 : \theta \geq \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

LRT is uniformly most powerful!

~~UMP tests~~ for two-sided hypothesis

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta \neq \theta_0.$$

$$H_0 : \theta_1 \leq \theta \leq \theta_2 \quad \leftrightarrow \quad H_1 : \theta > \theta_2 \text{ or } \theta < \theta_1.$$

In general, UMP tests do not exist for two-sided hypothesis.

Likelihood ratio tests under $N(\mu, \sigma^2)$

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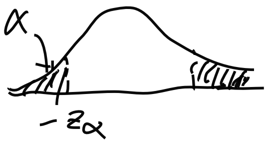
significance level : under $H_0: \mu = 1$,

σ^2 known

$\beta(1) = P(\text{Type I error}) = P(R \mid \mu = 1) = \alpha$. $\bar{X}_n \sim N\left(1, \frac{\sigma^2}{n}\right)$

Example 4 *cont'd*. Let $X_1, \ldots, X_{20}$ be i.i.d $N(\mu, \sigma^2)$. Consider the following tests and find the LRT rejection region. Calculate threshold c' such that significance level is equal to α.

$H_0: \mu = 1 \quad \leftrightarrow \quad H_1: \mu > 1. \quad \Longleftrightarrow \quad R = \{\bar{X}_n \geq c'\} = \left\{\bar{X}_n \geq 1 + z_\alpha \frac{\sigma}{\sqrt{n}}\right\}$ $\Leftarrow$ α



$P(\bar{X}_n \geq c' \mid \mu = 1) = P\left(\frac{\bar{X}_n - 1}{\sigma/\sqrt{n}} \geq \frac{c' - 1}{\sigma/\sqrt{n}} \,\middle|\, \mu = 1\right) = \alpha$.

By the definition of z_α, $\frac{c'-1}{\sigma/\sqrt{n}} = z_\alpha$ which gives $c' = 1 + z_\alpha \frac{\sigma}{\sqrt{n}}$.

$H_0: \mu = 1 \quad \leftrightarrow \quad H_1: \mu < 1. \quad \Longleftrightarrow \quad R = \{\bar{X}_n \leq c'\} = \left\{\bar{X}_n \leq 1 - z_\alpha \frac{\sigma}{\sqrt{n}}\right\}$ $\Leftarrow$

$P(\bar{X}_n \leq c' \mid \mu = 1) = P\left(\frac{\bar{X}_n - 1}{\sigma/\sqrt{n}} \leq \frac{c' - 1}{\sigma/\sqrt{n}} \,\middle|\, \mu = 1\right) = \alpha$

Thus, $\frac{c'-1}{\sigma/\sqrt{n}} = -z_\alpha$ which gives $c' = 1 - z_\alpha \frac{\sigma}{\sqrt{n}}$.

$H_0: \mu = 1 \quad \leftrightarrow \quad H_1: \mu \neq 1. \quad \Longleftrightarrow \quad R = \{|\bar{X}_n - 1| \geq c'\} = \left\{|\bar{X}_n - 1| \geq z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right\}$. $\Leftarrow$

$P(|\bar{X}_n - 1| \geq c' \mid \mu = 1) = P\left(\left|\frac{\bar{X}_n - 1}{\sigma/\sqrt{n}}\right| \geq \frac{c'}{\sigma/\sqrt{n}} \,\middle|\, \mu = 1\right)$

Thus, $\frac{c'}{\sigma/\sqrt{n}} = z_{\frac{\alpha}{2}}$ which gives $c' = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

σ^2 known

Example 4 *cont'd*. Let $X_1, \ldots, X_{20}$ be i.i.d $N(\mu, \sigma^2)$. Consider the following tests and find the LRT rejection region. Calculate threshold c' such that significance level is equal to α.

$$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu > 1. \quad \Longleftrightarrow \quad R = \{\bar{X}_n \geq c'\}$$

$$H_0 : \mu \leq 1 \quad \leftrightarrow \quad H_1 : \mu > 1.$$

Karlin-Rubin

$$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu < 1. \quad \Longleftrightarrow \quad R = \{\bar{X}_n \leq c'\}$$

$$H_0 : \mu \geq 1 \quad \leftrightarrow \quad H_1 : \mu < 1.$$

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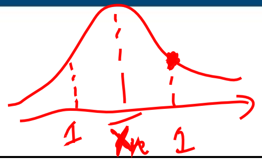
LRT for simpler null hypothesis is also UMP for composite null hypothesis.

σ^2 unknown



Example 5. Let $X_1, \ldots, X_{20}$ be i.i.d $N(\mu, \sigma^2)$. Consider the following tests and find the LRT rejection region. Calculate threshold c' such that significance level is equal to α.

$$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu > 1. \qquad \Longleftrightarrow \qquad R \overset{?}{=} \{\bar{X}_n \geq c'\}$$

Solution : $L(\theta \mid \underline{X}_n) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2} =$

$$\lambda(\underline{X}_n) = \frac{\sup_{\Theta_0} L(\theta \mid \underline{X}_n)}{\sup_{\Theta} L(\theta \mid \underline{X}_n)} \begin{matrix} \leftarrow \text{restricted} \\ \leftarrow \text{unrestricted} \end{matrix}$$

$\Theta_0 = \{\mu = 1,\ \sigma^2 > 0\}$

$\Theta = \Theta_0 \cup \Theta_1$

$= \{\cancel{\mu \in \mathbb{R}},\ \sigma^2 > 0\}$

$\mu \geq 1$

If $\bar{X}_n \geq 1$

$$\sup_{\Theta} L(\theta \mid \underline{X}_n) = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}_n^2}}\right)^n e^{-\frac{\cancel{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}{2\,\frac{1}{n}\cancel{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}} = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}_n^2}}\right)^n e^{-\frac{n}{2}}$$

$\hat{\mu}_{MLE} = \bar{X}_n$

$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

If $\bar{X}_n < 1$, $\sup_{\Theta} L(\theta \mid \underline{X}_n) = \sup_{\Theta_0} L(\theta \mid \underline{X}_n)$

σ^2 unknown

$$\sup_{\{\mu=1,\,\sigma^2>0\}} L(\theta \mid \underline{X}_n) = \sup_{\{\mu=1,\,\sigma^2>0\}} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-1)^2}$$

$$l(\sigma^2) = -\frac{n}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-1)^2 \xrightarrow{\frac{\partial l}{\partial \sigma^2}=0} \hat{\sigma}^2_{RES\text{-}MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i-1)^2$$

$$\Rightarrow \sup_{\Theta_0} L(\theta \mid \underline{X}_n) = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2_{RES\text{-}MLE}}}\right)^n e^{-\frac{\cancel{\sum_{i=1}^{n}(X_i-1)^2}}{2\frac{1}{n}\cancel{\sum_{i=1}^{n}(X_i-1)^2}}}$$

$$= \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2_{Res\text{-}MLE}}}\right)^n e^{-\frac{n}{2}}$$

$$\Rightarrow \lambda(\underline{X}_n) = \begin{cases} \left(\sqrt{\dfrac{\hat{\sigma}_n^2}{\hat{\sigma}^2_{RES\text{-}MLE}}}\right)^n, & \text{if } \bar{X}_n \geq 1 \\ 1, & \text{if } \bar{X}_n < 1 \end{cases}$$

Rejection region =

$$R = \{\lambda(\underline{X}_n) \leq c\}.$$

$$\lambda(\bar{X}_n) \le c \iff \left(\sqrt{\frac{\hat{\sigma}_n^2}{\hat{\sigma}^2_{Res\text{-}MLE}}}\right)^n \le c \quad \text{while } \bar{X}_n \ge 1$$

$$\iff \frac{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}{\cancel{\frac{1}{n}}\sum_{i=1}^{n}(X_i-1)^2} \le c^{\frac{2}{n}} \quad \text{while } \bar{X}_n \ge 1$$

$$= \frac{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}{\sum_{i=1}^{n}(X_i-\bar{X}_n+\bar{X}_n-1)^2}$$

$$= \frac{\sum(X_i-\bar{X}_n)^2}{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 + n(\bar{X}_n-1)^2 + 2(\bar{X}_n-1)\underbrace{\sum_{i=1}^{n}(X_i-\bar{X}_n)}_{=0}}$$

$$= \frac{1}{1+\frac{n(\bar{X}_n-1)^2}{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}} \le c^{\frac{2}{n}} \quad \text{while } \bar{X}_n \ge 1$$

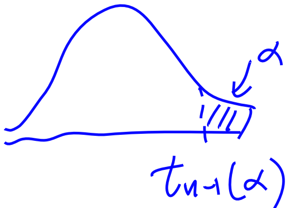
$$\iff \frac{n(\bar{X}_n-1)^2}{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2} \ge c^{-\frac{2}{n}} - 1 \quad \text{while } \bar{X}_n \ge 1$$

$$\iff \frac{\sqrt{n}(\bar{X}_n-1)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}} \ge c'$$

$$c' = \sqrt{(c^{-\frac{2}{n}}-1)(n-1)}$$

Under $\mu=1$, $X_1, \dots, X_n \overset{iid}{\sim} N(1, \sigma^2)$

Thus, by Theorem A, $\frac{\sqrt{n}(\bar{X}_n-1)}{S} \sim t_{n-1}$



$$R = \left\{ \boxed{\frac{\sqrt{n}(\bar{X}_n-1)}{\sqrt{S^2}} = \frac{\sqrt{n}(\bar{X}_n-1)}{S} \ge c'} \right\}$$

Calculate significance level:

$$P(R \mid \mu=1) = P\left(\underbrace{\frac{\sqrt{n}(\bar{X}_n-1)}{S}}_{t_{n-1}} \ge c' \,\middle|\, \mu=1\right) \stackrel{\downarrow}{=} \alpha \Leftarrow c' = t_{n-1}(\alpha)$$

σ^2 unknown

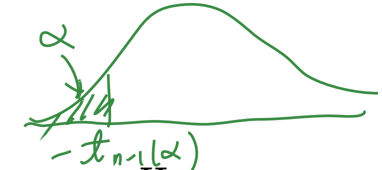
Under $\Theta_0: \mu = \mu_0$, $\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \sim t_{n-1}$

Example 5 *cont'd*. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Consider the following tests and find the LRT rejection region. Calculate threshold c' such that significance level is equal to α.

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu > \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \geq c' \right\}$$

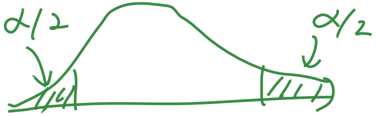
$$\alpha = P\left(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \geq c' \,\middle|\, \mu = \mu_0 \right) \Rightarrow c' = t_{n-1}(\alpha).$$

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu < \mu_0. \quad \Longleftrightarrow \quad R = \underbrace{\left\{ \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \leq c' \right\}}$$



$$\Rightarrow c' = t_{n-1}(1-\alpha) = -t_{n-1}(\alpha)$$

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu \neq \mu_0. \quad \Longleftrightarrow \quad R = \underbrace{\left\{ \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \right| \geq c' \right\}}$$



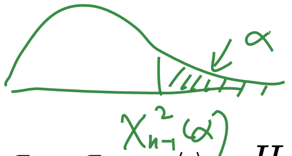
$$\Rightarrow c' = t_{n-1}(\alpha/2).$$

Under Θ_0, $\frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$

σ^2 unknown

Example 5 *cont'd*. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Consider the following tests and find the LRT rejection region. Calculate threshold c' such that significance level is equal to α.

$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma > \sigma_0 \,. \quad \Longleftrightarrow \quad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \geq c' \right\}$$



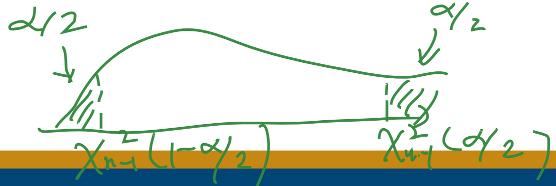
$c' = \chi^2_{n-1}(\alpha)$

$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma < \sigma_0 \,. \quad \Longleftrightarrow \quad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \leq c' \right\}$$

$\Rightarrow \quad c' = \chi^2_{n-1}(1-\alpha)$

$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma \neq \sigma_0 \,. \quad \Longleftrightarrow \quad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \geq c_1 \text{ or } \frac{(n-1)S^2}{\sigma_0^2} \leq c_2 \right\}$$

$c_1 = \chi^2_{n-1}(\alpha/2)$, $c_2 = \chi^2_{n-1}(1-\alpha/2)$.



Duality between CIs and hypothesis tests

9.3 of Rice

07/08/2021

Exact CI and LRT for μ

set of μ

$\downarrow$

$(1-\alpha) \times 100\,\%$ exact CI for μ :

$$P\left(\underbrace{\overline{X}_n - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2) \leq \mu \leq \overline{X}_n + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)}\right) = 1-\alpha$$

$CI : \left[\bar{X}_n - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2),\ \bar{X}_n + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)\right]$

LRT with significance level α :

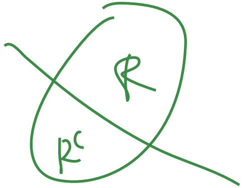
set of $\bar{X}_n$

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \right| \geq t_{n-1}(\alpha/2) \right\}$$

Rejection region

$\bar{X}_n \in R$

$\downarrow$



$$A = R^c = \left\{ \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \right| < t_{n-1}(\alpha/2) \right\}$$

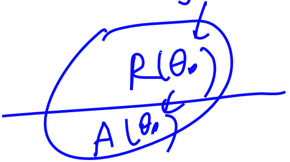
$$= \left\{ t_{n-1}(\alpha/2) < \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} < t_{n-1}(\alpha/2) \right\} = \left\{ \bar{X}_n - t_{n-1}(\alpha/2)\frac{S}{\sqrt{n}} < \mu_0 < \bar{X}_n + t_{n-1}(\alpha/2)\frac{S}{\sqrt{n}} \right\}$$

Inverting the test statistic

Denote Θ as the set of all possible θ values.

Theorem C. For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a significance level α test of $H_0: \theta = \theta_0$. Define $C(\mathbf{X}_n) = \{\theta_0 : \mathbf{X}_n \in A(\theta_0)\}$.
Then $C(\mathbf{X}_n)$ is a $(1-\alpha) \times 100\%$ confidence region for θ.

Proof. By definition of confidence Region, the confidence level for $C(X_n)$ is



$$P(\theta_0 \in C(X_n) \mid \theta = \theta_0) = P(X_n \in A(\theta_0) \mid \theta = \theta_0)$$

$$= 1 - \underbrace{P(X_n \in R(\theta_0) \mid \theta = \theta_0)}_{=\alpha} = 1 - \alpha.$$

Every hypothesis test corresponds to a CI, and vice versa.

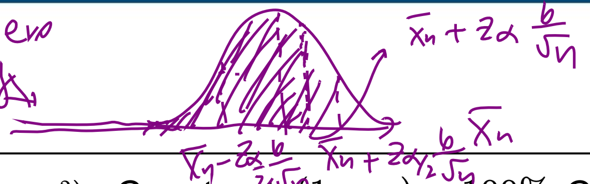
Theorem C'. Suppose that $C(\mathbf{X}_n)$ is a $(1-\alpha) \times 100\%$ confidence region for θ, that is, for every θ_0,

$$P(\theta_0 \in C(\mathbf{X}_n) \mid \theta = \theta_0) = 1 - \alpha.$$

Then the acceptance region of a level α test of $H_0: \theta = \theta_0$ is

$$A(\theta_0) = \{\mathbf{X}_n : \theta_0 \in C(\mathbf{X}_n)\}.$$

σ^2 known



Example 4 *cont'd*. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Construct $(1-\alpha) \times 100\%$ CI for μ.

$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu \neq 1. \quad \Longleftrightarrow \quad \overset{R(\mu_0)}{R} = \left\{ \left| \bar{X}_n - \mu_0 \right| \geq z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right\}$

$z_\alpha < z_{\alpha/2}$

$A(\mu_0) = \left\{ |\bar{X}_n - \mu_0| < z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right\}$

$= \left\{ \bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu_0 < \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right\}$

$\bar{X}_n$

$\Rightarrow \quad C(\bar{X}_n) = \left\{ \bar{X}_n - z_{\alpha/2}\sigma/\sqrt{n} < \mu < \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right\} = \bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu < 1. \quad \Longleftrightarrow \quad R = \left\{ \bar{X}_n \leq \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$

$A(\mu_0) = \left\{ \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}} > \mu_0 \right\}$

$\Rightarrow \quad C(\bar{X}_n) = \left(-\infty, \bar{X}_n + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$

$H_0 : \mu = 1 \quad \leftrightarrow \quad H_1 : \mu > 1. \quad \Longleftrightarrow \quad R = \left\{ \bar{X}_n \geq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$

$\Rightarrow \quad C(\bar{X}_n) = \left(\bar{X}_n - z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right)$

σ^2 unknown

Example 5 cont'd. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Construct $(1-\alpha) \times 100\%$ CI for μ.

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu > \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \geq t_{n-1}(\alpha) \right\}$$

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu < \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \leq -t_{n-1}(\alpha) \right\}$$

This is left as an exercise.

$$H_0 : \mu = \mu_0 \quad \leftrightarrow \quad H_1 : \mu \neq \mu_0. \quad \Longleftrightarrow \quad R = \left\{ \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} \right| \geq t_{n-1}(\alpha/2) \right\}$$

σ^2 unknown

Example 5 *cont'd*. Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Construct $(1-\alpha)\times 100\%$ CI for σ^2.

$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma > \sigma_0 \,. \iff \qquad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \geq \chi^2_{n-1}(\alpha) \right\}$$

$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma < \sigma_0 \,. \iff \qquad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \leq \chi^2_{n-1}(1-\alpha) \right\}$$

This is left as an exercise.

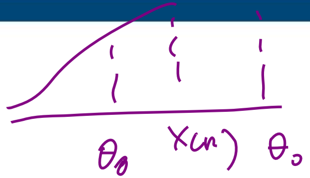
$$H_0 : \sigma = \sigma_0 \quad \leftrightarrow \quad H_1 : \sigma \neq \sigma_0 \,. \iff \qquad R = \left\{ \frac{(n-1)S^2}{\sigma_0^2} \geq \chi^2_{n-1}(\alpha/2) \ \text{ or } \ \frac{(n-1)S^2}{\sigma_0^2} \leq \chi^2_{n-1}(1-\alpha/2) \right\}$$

Generalized LRT

9.4 of Rice

07/08/2021

$\lambda(\mathbf{X}_n)$ might be difficult to deal with.



Example 6. Let $X_1, \ldots, X_n$ be i.i.d $U(0, \theta)$. Consider

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

Solution: $L(\theta \mid \mathbf{X}_n) = \left(\frac{1}{\theta}\right)^n \mathbb{1}\{0 < x_1, \cdots x_n < \theta\}$

$= \left(\frac{1}{\theta}\right)^n \mathbb{1}\{\theta > X_{(n)}\}.$

$\Theta_0 = \{\theta = \theta_0\}$

$\Theta = \{\theta \le \theta_0\}$

$$\lambda(\mathbf{X}_n) = \frac{\sup_{\Theta_0} L(\theta \mid \mathbf{X}_n)}{\sup_{\Theta} L(\theta \mid \mathbf{X}_n)} = \frac{\left(\frac{1}{\theta_0}\right)^n \mathbb{1}(\theta_0 > X_{(n)})}{\left[\frac{1}{X_{(n)}}\right]^n \mathbb{1}\{\theta_0 > X_{(n)}\} + \left[\frac{1}{\theta_0}\right]^n \mathbb{1}\{\theta_0 \le X_{(n)}\}}$$

$$= \begin{cases} 0, & \theta_0 \le X_{(n)} \\ \left(\frac{X_{(n)}}{\theta_0}\right)^n, & \theta_0 > X_{(n)} \end{cases}$$

Rejection region

$$\{\lambda(\mathbf{X}_n) \le c\} = \left\{\left(\frac{X_{(n)}}{\theta_0}\right)^n \le c\right\}.$$

Wilk's theorem

Denote $\Theta = \Theta_0 \cup \Theta_1$.

Theorem D. Let $X_1, \ldots, X_n$ be i.i.d $f(x \mid \theta)$, and the LRT test statistic for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ is $\lambda(\mathbf{X}_n)$. Under Θ_0,

$$-2 \log \lambda(\mathbf{X}_n) \xrightarrow{d} \chi^2_\nu \quad \text{as} \quad n \to \infty,$$

in which $\nu = \dim(\Theta) - \dim(\Theta_0)$.

Note: dim($\boldsymbol{\Theta}$) = the number of free parameters in $\boldsymbol{\Theta}$.

If $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, we want to check

$$H_0 : \theta_1 = \theta_2 = \theta_3 \text{ and } \theta_4 = \theta_5 \quad \text{versus} \quad H_1 : H_0 \text{ is not true.}$$

$\lambda(\mathbf{X}_n)$ might be difficult to deal with.

Example 6 *cont'd* . Let $X_1, \ldots, X_n$ be i.i.d $U(0, \theta)$. Consider

$$H_0 : \theta = \theta_0 \quad \leftrightarrow \quad H_1 : \theta < \theta_0.$$

Next Tuesday ...

Tests for two independent samples:

- Comparing means of two normal populations with same/different variances.
- Comparing means of paired samples