Before you turn this problem in, make sure everything runs as expected. First, **restart the kernel** (in the menubar, select Kernel → Restart) and then **run all cells** (in the menubar, select Cell → Run All).

Make sure you fill in any place that says `YOUR CODE HERE` or "YOUR ANSWER HERE", as well as your name and collaborators below:

```
In [1]: NAME = "Vinay Maruri"
        COLLABORATORS = "Jackson Le"
```

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Homework 3: Loss Minimization

Modeling, Estimation and Gradient Descent

Due Date: Tuesday 10/9, 11:59 PM

Course Policies

Here are some important course policies. These are also located at http://www.ds100.org/fa18/ (http://www.ds100.org/fa18/).

Collaboration Policy

Data science is a collaborative activity. While you may talk with others about the homework, we ask that you **write your solutions individually**. If you do discuss the assignments with others please **include their names** at the top of your solution.

This Assignment

In this homework, we explore modeling data, estimating optimal parameters and a numerical estimation method, gradient descent. These concepts are some of the fundamentals of data science and machine learning and will serve as the building blocks for future projects, classes, and work.

After this homework, you should feel comfortable with the following:

- Practice reasoning about a model
- Build some intuition for loss functions and how they behave
- Work through deriving the gradient of a loss with respect to model parameters
- Work through a basic version of gradient descent.

This homework is comprised of completing code, deriving analytic solutions, writing LaTex and visualizing loss.

Submission - IMPORTANT, PLEASE READ

For this assignment and future assignments (homework and projects) you will also submit your free response and plotting questions to gradescope. To do this, you can download as PDF (`File->Download As->PDF via Latex (.pdf)`). You are responsible for submitting and tagging your answers in gradescope. For each free response and plotting question, please include:

1. Relevant code used to generate the plot or inform your insights
2. The written free response or plot

We are doing this to make it easier on our graders and for you, in the case you need to submit a regrade request. Gradescope (as of now) is still better for manual grading.

Score breakdown

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Question	Points
Question 1a	1
Question 1b	1
Question 1c	1
Question 1d	1
Question 1e	1
Question 2a	2
Question 2b	1
Question 2c	1
Question 2d	1
Question 2e	1
Question 2f	1
Question 3a	1
Question 3b	3
Question 3c	2
Question 4a	3
Question 4b	1
Question 4c	1
Question 4d	1
Question 4e	1
Question 5a	2
Question 5b	4
Question 5c	0
Question 5d	0
Question 6a	3
Question 6b	3
Question 6c	3
Question 6d	3
Question 6e	3
Question 6f	3
Question 6g	3
Question 7a	1
Question 7b	1

Question	Points
Question 7c	1
Question 7d	1
Question 7e	0
Total	56

Getting Started

In [2]:
```
# Imports
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import csv
import re
import seaborn as sns

# Set some parameters
plt.rcParams['figure.figsize'] = (12, 9)
plt.rcParams['font.size'] = 16
np.set_printoptions(4)
```

In [3]:
```
# We will use plot_3d helper function to help us visualize gradient
from hw3_utils import plot_3d
```

Load Data

Load the data.csv file into a pandas dataframe.
Note that we are reading the data directly from the URL address.

In [4]:
```
# Run this cell to load our sample data
data = pd.read_csv("https://github.com/DS-100/fa18/raw/gh-pages/assets/dataset
s/hw3_data.csv", index_col=0)
data.head()
```

Out[4]:

	x	y
0	-5.000000	-7.672309
1	-4.966555	-7.779735
2	-4.933110	-7.995938
3	-4.899666	-8.197059
4	-4.866221	-8.183883

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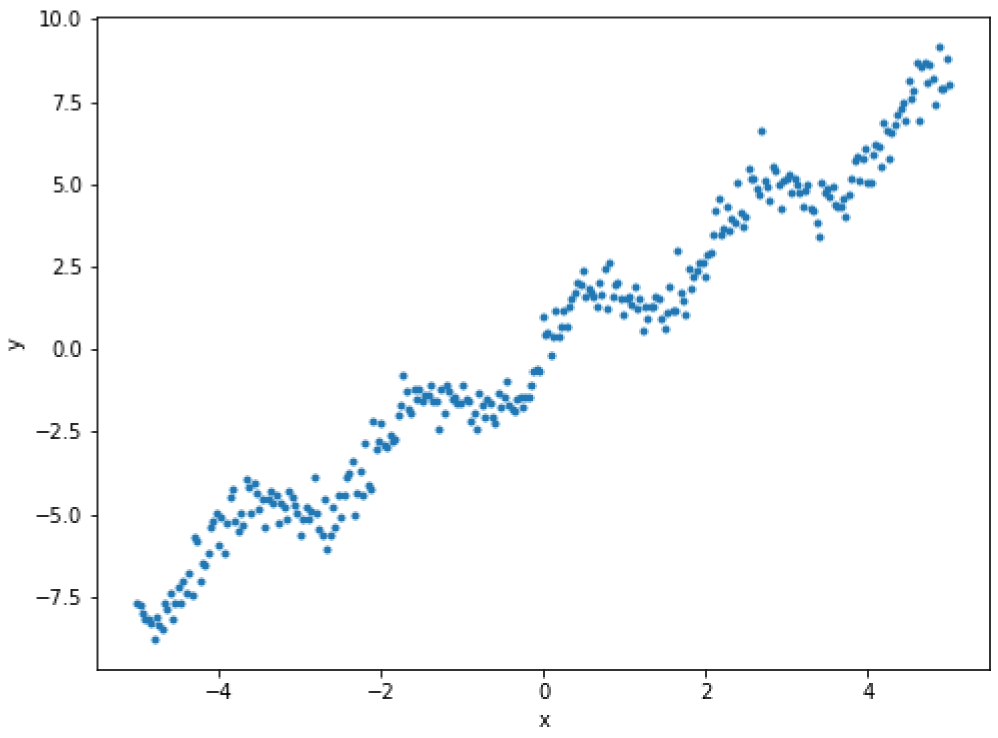
1: A Simple Model

Let's start by examining our data and creating a simple model that can represent this data.

Question 1

Question 1a

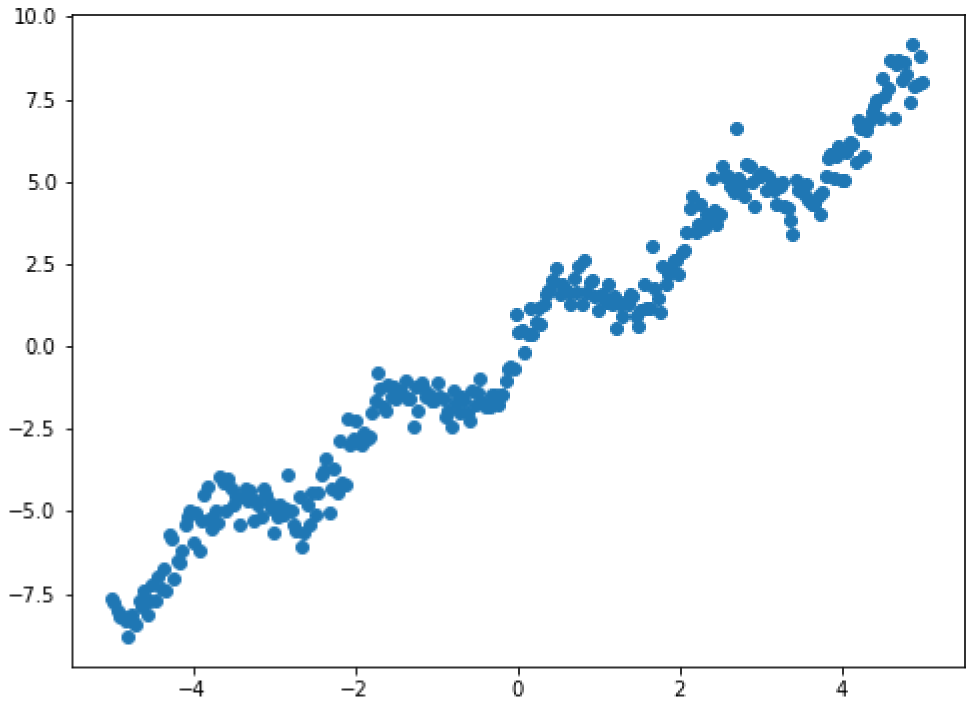
First, let's visualize the data in a scatter plot. After implementing the `scatter` function below, you should see something like this:



```
In [5]: def scatter(x, y):
            """
            Generate a scatter plot using x and y

            Keyword arguments:
            x -- the vector of values x
            y -- the vector of values y
            """
            plt.figure(figsize=(8, 6))
            plt.scatter(x, y)

        x = data['x']
        y = data['y']
        scatter(x,y)
```



Question 1b

Describe any significant observations about the distribution of the data. How can you describe the relationship between x and y?

I would describe the relationship between x and y as roughly linear. There are no clear signs of a non-linear relationship, as one cannot plot a cosine, sine, or other non-linear regression function onto this data with a good fit. There appear to be no outliers, there is no clear skew to the data, and the spread of the data is relatively tight, as the range of the data in the x and y direction is relatively narrow.

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Question 1c

The data looks roughly linear, with some extra noise. For now, let's assume that the data follows some underlying linear model. We define the underlying linear model that predicts the value y using the value x as: $f_{\theta^*}(x) = \theta^* \cdot x$

Since we cannot find the value of the population parameter θ^* exactly, we will assume that our dataset approximates our population and use our dataset to estimate θ^*. We denote our estimation with θ, our fitted estimation with $\hat{\theta}$, and our model as:

$$f_\theta(x) = \theta \cdot x$$

Based on this equation, define the linear model function `linear_model` below to estimate $\mathbf{y}$ (the y-values) given $\mathbf{x}$ (the x-values) and θ. This model is similar to the model you defined in Lab 5: Modeling and Estimation.

In [6]:
```
def linear_model(x, theta):
    """
    Returns the estimate of y given x and theta

    Keyword arguments:
    x -- the vector of values x
    theta -- the scalar theta
    """
    y = theta * x
    return y
```

In [7]:
```
assert linear_model(0, 1) == 0
assert linear_model(10, 10) == 100
assert np.sum(linear_model(np.array([3, 5]), 3)) == 24
assert linear_model(np.array([7, 8]), 4).mean() == 30
```

Question 1d

In class, we learned that the L^2 (or squared) loss function is smooth and continuous. Let's use L^2 loss to evaluate our estimate θ, which we will use later to identify an optimal θ, represented as $\hat{\theta}$. Define the L^2 loss function `l2_loss` below.

In [8]:
```
def l2_loss(y, y_hat):
    """
    Returns the average l2 loss given y and y_hat

    Keyword arguments:
    y -- the vector of true values y
    y_hat -- the vector of predicted values y_hat
    """
    return np.mean((y - y_hat)**2)
```

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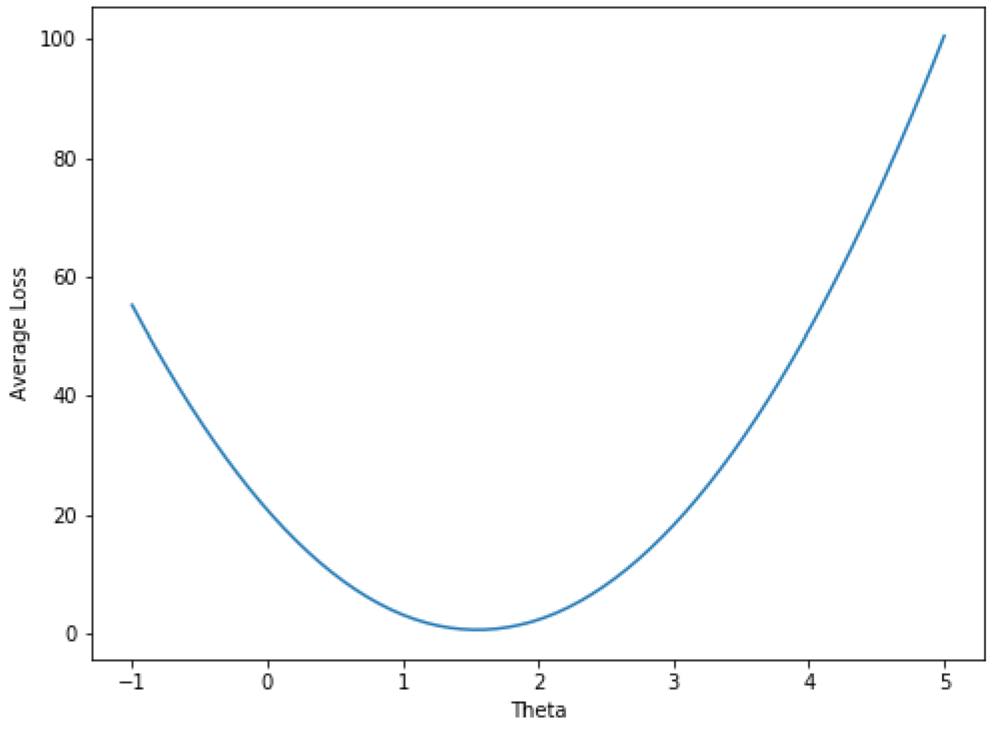
In [9]:
```
assert l2_loss(2, 1) == 1
assert l2_loss(2, 0) == 4
assert l2_loss(5, 1) == 16
assert l2_loss(np.array([5, 6]), np.array([1, 1])) == 20.5
assert l2_loss(np.array([1, 1, 1]), np.array([4, 1, 4])) == 6.0
```

Question 1e

First, visualize the L^2 loss as a function of θ, where several different values of θ are given. Be sure to label your axes properly. You plot should look something like this:



What looks like the optimal value, $\hat{\theta}$, based on the visualization? Set `theta_star_guess` to the value of θ that appears to minimize our loss.

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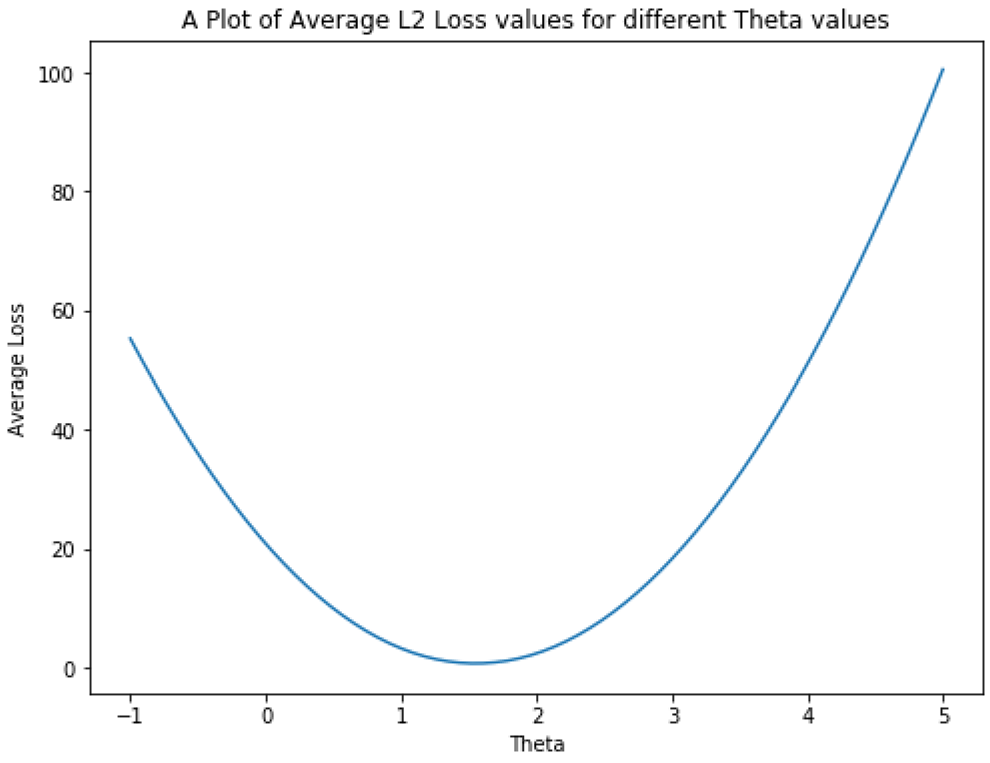
In [10]:
```
def visualize(x, y, thetas):
    """
    Plots the average l2 loss for given x, y as a function of theta.
    Use the functions you wrote for linear_model and l2_loss.

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    thetas -- an array containing different estimates of the scalar theta
    """
    avg_loss = [l2_loss(y, linear_model(x, theta)) for theta in thetas]
    plt.figure(figsize=(8,6))
    plt.plot(thetas, avg_loss)
    plt.xlabel('Theta')
    plt.ylabel('Average Loss')
    plt.title('A Plot of Average L2 Loss values for different Theta values')


thetas = np.linspace(-1, 5, 70)
visualize(x, y, thetas)

theta_star_guess = 1.5
```



In [11]:
```
assert l2_loss(3, 2) == 1
assert l2_loss(0, 10) == 100
assert 1 <= theta_star_guess <= 2
```

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2: Fitting our Simple Model

Now that we have defined a simple linear model and loss function, let's begin working on fitting our model to the data.

Question 2

Let's confirm our visual findings for optimal $\hat{\theta}$.

Question 2a

First, find the analytical solution for the optimal $\hat{\theta}$ for average L^2 loss. Write up your solution in the cell below using LaTex.

Hint: notice that we now have $\mathbf{x}$ and $\mathbf{y}$ instead of x and y. This means that when writing the loss function $L(\mathbf{x}, \mathbf{y}, \theta)$, you'll need to take the average of the squared losses for each $y_i, f_\theta(x_i)$ pair. For tips on getting started, see chapter chapter 10 (https://www.textbook.ds100.org/ch/10/modeling_loss_functions.html) of the textbook. Note that if you click "Open in DataHub", you can access the LaTeX source code of the book chapter, which you might find handy for typing up your work. Show your work, i.e. don't just write the answer.

Our loss function is defined as: L(θ, x, y) = $\frac{1}{n}\sum_{i=1}^{n}(y_i - x_i\theta)^2$. Expanding the summation: L(θ, x, y) = $\frac{1}{n}[(y_1 - x_1\theta)^2 + (y_2 - x_2\theta)^2 + \cdots + (y_n - x_n\theta)^2]$. Continuing the expansion: L(θ, x, y) = $\frac{1}{n}[(y_1^2 - 2y_1x_1\theta + x_1^2\theta^2) + (y_2^2 - 2y_2x_2\theta + x_2^2\theta^2) + \cdots + (y_n^2 - 2y_nx_n\theta + x_n^2\theta^2)]$. Now I take the partial derivative: $\frac{\partial L}{\partial \theta} = (\frac{-2y_1x_1}{n} + \frac{2x_1^2\theta}{n}) + (\frac{-2y_2x_2}{n} + \frac{2x_2^2\theta}{n}) + \cdots + (\frac{-2y_nx_n}{n} + \frac{2x_n^2\theta}{n})$. Now, I set the partial derivative equal to zero: 0 = $(\frac{-2y_1x_1}{n} + \frac{2x_1^2\theta}{n}) + (\frac{-2y_2x_2}{n} + \frac{2x_2^2\theta}{n}) + \cdots + (\frac{-2y_nx_n}{n} + \frac{2x_n^2\theta}{n})$. I then solve the equation: 0 = $\frac{2}{n}(-y_1x_1 + x_1^2\theta - y_2x_2 + x_2^2\theta - \cdots - y_nx_n + x_n^2\theta)$; $0 = (-y_1x_1 - y_2x_2 - \cdots - y_nx_n) + \hat{\theta}(x_1^2 + x_2^2 + \cdots + x_n^2)$; $(y_1x_1 + y_2x_2 + \cdots + y_nx_n) = \hat{\theta}(x_1^2 + x_2^2 + \cdots + x_n^2)$; $\hat{\theta} = \frac{y_1x_1 + y_2x_2 + \cdots + y_nx_n}{x_1^2 + x_2^2 + \cdots + x_n^2}$; as a summation: $\hat{\theta} = \frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}x_i^2}$

Question 2b

Now that we have the analytic solution for $\hat{\theta}$, implement the function `find_theta` that calculates the numerical value of $\hat{\theta}$ based on our data $\mathbf{x}$, $\mathbf{y}$.

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In [12]:
```
def find_theta(x, y):
    """
    Find optimal theta given x and y

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    """
    theta_opt = np.sum(x*y)/np.sum(x**2)
    return theta_opt
```

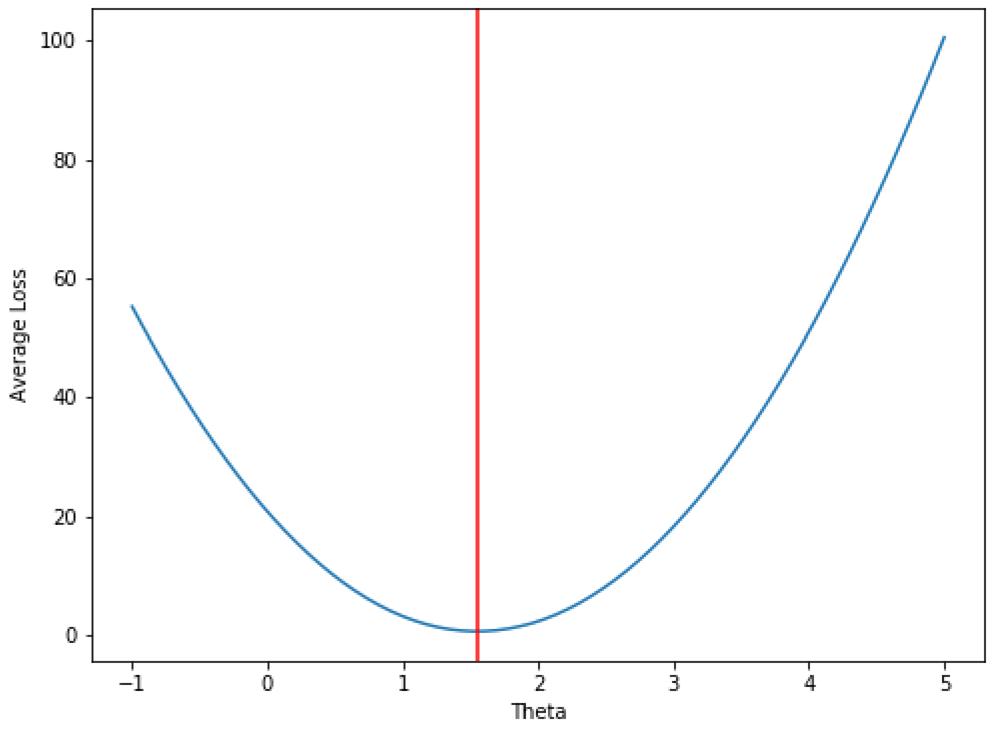
In [13]:
```
t_hat = find_theta(x, y)
print(f'theta_opt = {t_hat}')

assert 1.4 <= t_hat <= 1.6
```

```
theta_opt = 1.5502648085962225
```

Question 2c

Now, let's plot our loss function again using the `visualize` function. But this time, add a vertical line at the optimal value of theta (plot the line $x = \hat{\theta}$). Your plot should look something like this:



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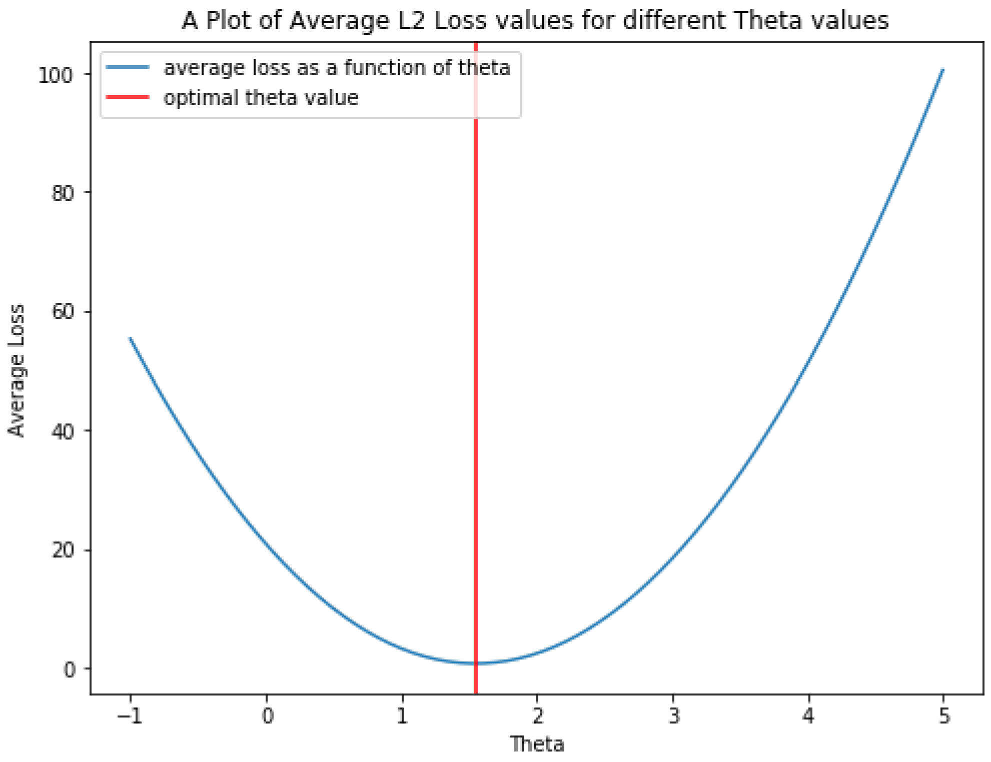
In [14]:
```
theta_opt = 1.5502648085962225
avg_loss = [l2_loss(y, linear_model(x, theta)) for theta in thetas]
plt.figure(figsize=(8,6))
plt.plot(thetas, avg_loss)
plt.axvline(x = theta_opt, color = 'red')
plt.xlabel('Theta')
plt.ylabel('Average Loss')
plt.title('A Plot of Average L2 Loss values for different Theta values')
plt.legend(labels = ['average loss as a function of theta', 'optimal theta val
ue'])
```

Out[14]:
```
<matplotlib.legend.Legend at 0x7f7e017db908>
```



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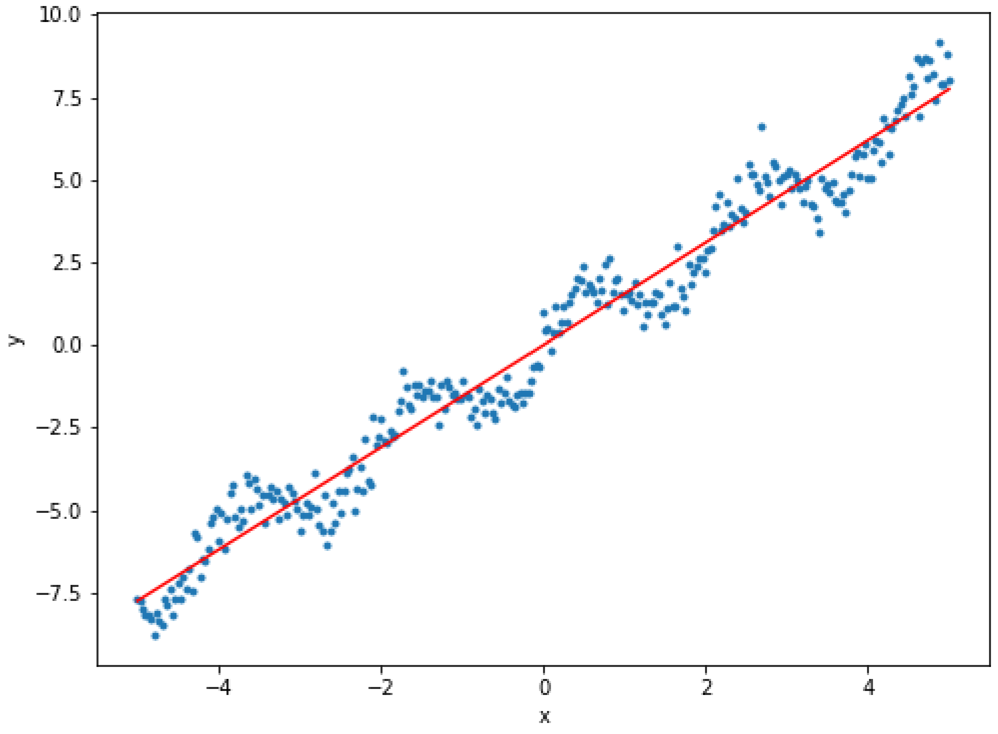
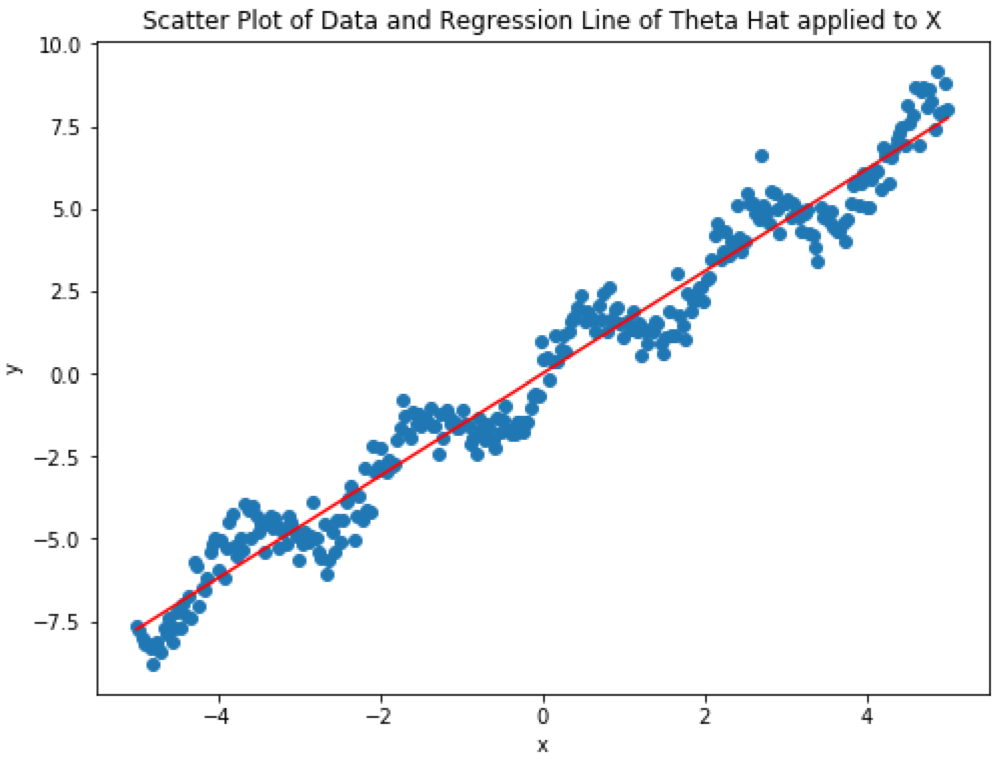
Question 2d

We now have an optimal value for θ that minimizes our loss. In the cell below, plot the scatter plot of the data from Question 1a (you can reuse the `scatter` function here). But this time, add the line $f_{\hat{\theta}}(x) = \hat{\theta} \cdot \mathbf{x}$ using the $\hat{\theta}$ you computed above. Your plot should look something like this:



```
In [15]: scatter(x,y)
         theta_opt = np.dot(1.5502648085962225, x)
         plt.plot(x, theta_opt, color = 'r')
         plt.xlabel('x')
         plt.ylabel('y')
         plt.title('Scatter Plot of Data and Regression Line of Theta Hat applied to X'
         )

Out[15]: Text(0.5,1,'Scatter Plot of Data and Regression Line of Theta Hat applied to
         X')
```



Question 2e

Great! It looks like our estimator $f_{\hat{\theta}}(x)$ is able to capture a lot of the data with a single parameter θ. Now let's try to remove the linear portion of our model from the data to see if we missed anything.

The remaining data is known as the residual, $\mathbf{r} = \mathbf{y} - \hat{\theta} \cdot \mathbf{x}$. Below, write a function to find the residual and plot the residuals corresponding to x in a scatter plot. Plot a horizontal line at $y = 0$ to assist visualization.

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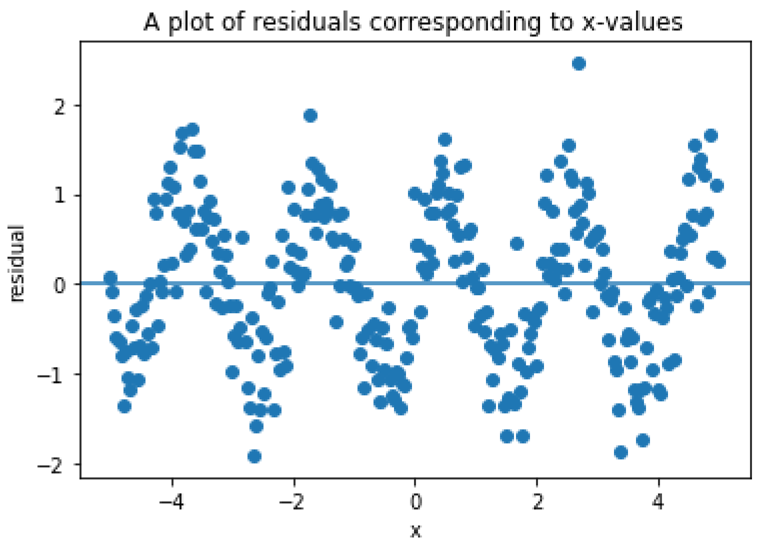
In [16]:
```
def visualize_residual(x, y):
    """
    Plot a scatter plot of the residuals, the remaining
    values after removing the linear model from our data.

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    """
    r = y - np.dot(1.5502648085962225, x)
    plt.scatter(x, r)
    plt.axhline(y=0)
    plt.ylabel('residual')
    plt.xlabel('x')
    plt.title('A plot of residuals corresponding to x-values')

visualize_residual(x, y)
```



Question 2f

What does the residual look like? Do you notice a relationship between x and r?

The graph of the residuals looks like a cosine or sine curve. You can clearly see that the function is periodic and resembles the shape of a cosine or sine curve. I don't see a correlation between x and r, but there is a relationship between x and r. Over a period of roughly 1 on the x axis, we notice that the residual seems to be the same. This is further shown with the y=0 curve. We can clearly see that multiple datapoints are on or very close to the curve, meaning that residual values are repeated for different x values and they repeat periodically. Hence, there is not a one to one relationship between x and r.

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3: Increasing Model Complexity

It looks like the remaining data is sinusoidal, meaning our original data follows a linear function and a sinusoidal function. Let's define a new model to address this discovery and find optimal parameters to best fit the data:

$$f_{\boldsymbol{\theta}}(x) = \theta_1 x + sin(\theta_2 x)$$

Now, our model is parameterized by both θ_1 and θ_2, or composed together, $\boldsymbol{\theta}$.

Note that a generalized sine function $a\sin(bx + c)$ has three parameters: amplitude scaling parameter a, frequency parameter b and phase shifting parameter c. Looking at the residual plot above, it looks like the residual is zero at x = 0, and the residual swings between -1 and 1. Thus, it seems reasonable to effectively set the scaling and phase shifting parameter (a and c in this case) to 1 and 0 respectively. While we could try to fit a and c, we're unlikely to get much benefit. When you're done with the homework, you can try adding a and c to our model and fitting these values to see if you can get a better loss.

Question 3a

As in Question 1, fill in the `sin_model` function that predicts $\mathbf{y}$ (the y-values) using $\mathbf{x}$ (the x-values), but this time based on our new equation.

Hint: Try to do this without using for loops. The `np.sin` function may help you.

In [17]:
```
def sin_model(x, theta_1, theta_2):
    """
    Predict the estimate of y given x, theta_1, theta_2

    Keyword arguments:
    x -- the vector of values x
    theta_1 -- the scalar value theta_1
    theta_2 -- the scalar value theta_2
    """
    y = np.dot(theta_1, x) + np.sin(np.dot(theta_2, x))
    return y
```

In [18]:
```
assert np.isclose(sin_model(1, 1, np.pi), 1.0000000000000002)
# Check that we accept x as arrays
assert len(sin_model(x, 2, 2)) > 1
```

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Question 3b

Use the average L^2 loss to compute $\frac{\partial L}{\partial \theta_1}, \frac{\partial L}{\partial \theta_2}$.

First, we will use LaTex to write $L(\mathbf{x}, \mathbf{y}, \theta_1, \theta_2)$, $\frac{\partial L}{\partial \theta_1}$, and $\frac{\partial L}{\partial \theta_2}$ given $\mathbf{x}$, $\mathbf{y}$, $\boldsymbol{\theta}$.

You don't need to write out the full derivation. Just the final expression is fine.

$L(x, y, \theta_1, \theta_2) = \frac{1}{n}\sum_{i=1}^{n}(y_i - (x_i\theta_1 + sin(\theta_2 x_i)))^2$; $\frac{\partial L}{\partial \theta_1} = \frac{-2}{n}\sum_{i=1}^{n}x_i(y_i - x_i\theta_1 - sin(\theta_2 x_i))$; $\frac{\partial L}{\partial \theta_2} = \frac{-2}{n}\sum_{i=1}^{n}x_i cos(\theta_2 x_i)(y_i - x_i\theta_1 - sin(\theta_2 x_i))$

Question 3c

Now, implement the functions `dt1` and `dt2`, which should compute $\frac{\partial L}{\partial \theta_1}$ and $\frac{\partial L}{\partial \theta_2}$ respectively. Use the formulas you wrote for $\frac{\partial L}{\partial \theta_1}$ and $\frac{\partial L}{\partial \theta_2}$ in the previous exercise. In the functions below, the parameter `theta` is a vector that looks like (θ_1, θ_2).

Note: To keep your code a bit more concise, be aware that `np.mean` does the same thing as `np.sum` divided by the length of the numpy array.

In [19]:
```
def dt1(x, y, theta):
    """
    Compute the numerical value of the partial of l2 loss with respect to thet
a_1

    Keyword arguments:
    x -- the vector of all x values
    y -- the vector of all y values
    theta -- the vector of values theta
    """
    return (-2/len(x))*np.sum(x*(y - np.dot(theta[0], x) -np.sin(theta[1]*x)))
```

In [20]:
```
def dt2(x, y, theta):
    """
    Compute the numerical value of the partial of l2 loss with respect to thet
a_2

    Keyword arguments:
    x -- the vector of all x values
    y -- the vector of all y values
    theta -- the vector of values theta
    """
    return (-2/len(x))*np.sum(x*np.cos(theta[1]*x)*(y - np.dot(x, theta[0]) -
np.sin(theta[1]*x)))
```

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In [21]:
```
# This function calls dt1 and dt2 and returns the gradient dt. It is already i
mplemented for you.
def dt(x, y, theta):
    """
    Returns the gradient of l2 loss with respect to vector theta

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    theta -- the vector of values theta
    """
    return np.array([dt1(x,y,theta), dt2(x,y,theta)])
```

In [22]:
```
assert np.isclose(dt1(x, y, [0, np.pi]), -25.376660670924529)
assert np.isclose(dt2(x, y, [0, np.pi]), 1.9427210155296564)
```

4: Gradient Descent

Now try to solve for the optimal $\hat{\theta}$ analytically...

Just kidding!

You can try but we don't recommend it. When finding an analytic solution becomes difficult or impossible, we resort to alternative optimization methods for finding an approximate solution.

Question 4

So let's try implementing a numerical optimization method: gradient descent!

Question 4a

Implement the `grad_desc` function that performs gradient descent for a finite number of iterations. This function takes in an array for $\mathbf{x}$ (`x`), an array for $\mathbf{y}$ (`y`), and an initial value for θ (`theta`). `alpha` will be the learning rate (or step size, whichever term you prefer). In this part, we'll use a static learning rate that is the same at every time step.

At each time step, use the gradient and `alpha` to update your current `theta`. Also at each time step, be sure to save the current `theta` in `theta_history`, along with the L^2 loss (computed with the current `theta`) in `loss_history`.

Hints:

- Write out the gradient update equation (1 step). What variables will you need for each gradient update? Of these variables, which ones do you already have, and which ones will you need to recompute at each time step?
- You may need a loop here to update `theta` several times
- Recall that the gradient descent update function follows the form:

$$\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \alpha\left(\nabla_{\boldsymbol{\theta}}\mathbf{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}^{(t)})\right)$$

In [23]:
```python
# Run me
def init_t():
    """Creates an initial theta [0, 0] of shape (2,) as a starting point for g
radient descent"""
    return np.zeros((2,))
```

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In [24]:
```
def grad_desc(x, y, theta, num_iter=20, alpha=0.1):
    """
    Run gradient descent update for a finite number of iterations and static l
earning rate

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    theta -- the vector of values theta to use at first iteration
    num_iter -- the max number of iterations
    alpha -- the learning rate (also called the step size)

    Return:
    theta -- the optimal value of theta after num_iter of gradient descent
    theta_history -- the series of theta values over each iteration of gradien
t descent
    loss_history -- the series of loss values over each iteration of gradient
 descent
    """
    theta_history = []
    loss_history = []
    i = 0
    while i < num_iter:
        theta = theta - alpha*(dt(x,y,theta))
        yhat = np.dot(theta[0], x) + np.sin(np.dot(theta[1], x))
        losstemp = l2_loss(y, yhat)
        theta_history.append(theta)
        loss_history.append(losstemp)
        i += 1
    return theta, theta_history, loss_history
```

In [25]:
```
t = init_t()
t_est, ts, loss = grad_desc(x, y, t, num_iter=20, alpha=0.1)

assert len(ts) == len(loss) == 20 # theta history and loss history are 20 item
s in them
assert ts[0].shape == (2,) # theta history contains theta values
assert np.isscalar(loss[0]) # loss history is a list of scalar values, not vec
tor

assert loss[1] - loss[-1] > 0 # loss is decreasing

assert np.allclose(np.sum(t_est), 4.5, atol=2e-1)  # theta_est should be close
 to our value
```

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Question 4b

Now, let's try using a decaying learning rate. Implement `grad_desc_decay` below, which performs gradient descent with a learning rate that decreases slightly with each time step. You should be able to copy most of your work from the previous part, but you'll need to tweak how you update `theta` at each time step.

By decaying learning rate, we mean instead of just a number α, the learning should be now $\frac{\alpha}{i+1}$ where i is the current number of iteration. (Why do we need to add '+ 1' in the denominator?)

In [26]:
```
def grad_desc_decay(x, y, theta, num_iter=20, alpha=0.1):
    """
    Run gradient descent update for a finite number of iterations and decaying
 learning rate

    Keyword arguments:
    x -- the vector of values x
    y -- the vector of values y
    theta -- the vector of values theta
    num_iter -- the max number of iterations
    alpha -- the learning rate

    Return:
    theta -- the optimal value of theta after num_iter of gradient descent
    theta_history -- the series of theta values over each iteration of gradien
t descent
    loss_history -- the series of loss values over each iteration of gradient
 descent
    """
    theta_history = []
    loss_history = []
    i = 0
    while i < num_iter:
        theta = theta - (alpha/(i+1))*(dt(x,y,theta))
        yhat = np.dot(theta[0], x) + np.sin(np.dot(theta[1], x))
        losstemp = l2_loss(y, yhat)
        theta_history.append(theta)
        loss_history.append(losstemp)
        i += 1
    return theta, theta_history, loss_history
```

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In [27]:
```
t = init_t()
t_est_decay, ts_decay, loss_decay = grad_desc_decay(x, y, t, num_iter=20, alph
a=0.1)

assert len(ts_decay) == len(loss_decay) == 20 # theta history and loss history
 are 20 items in them
assert ts_decay[0].shape == (2,) # theta history contains theta values
assert np.isscalar(loss[0]) # loss history should be a list of values, not vec
tor

assert loss_decay[1] - loss_decay[-1] > 0 # loss is decreasing

assert np.allclose(np.sum(t_est_decay), 4.5, atol=2e-1)  # theta_est should be
 close to our value
```

Question 4c

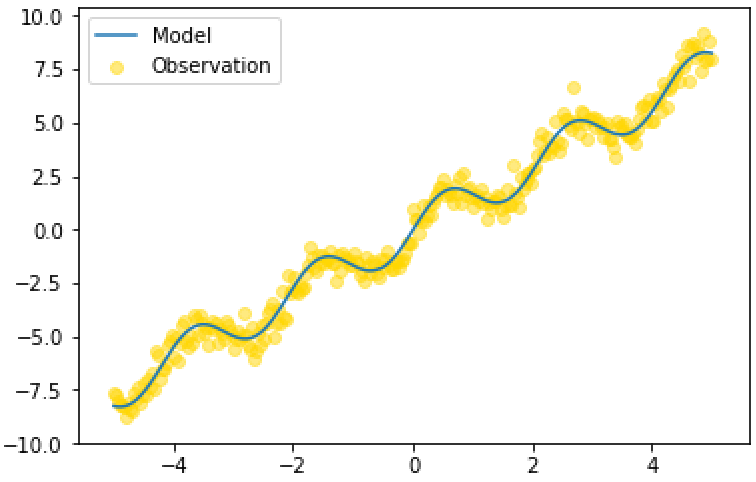
Let's visually inspect our results of running gradient descent to optimize θ. Plot our x-values with our model's predicted y-values over the original scatter plot. Did gradient descent successfully optimize θ?

In [28]:
```
# Run me
t = init_t()
t_est, ts, loss = grad_desc(x, y, t)

t = init_t()
t_est_decay, ts_decay, loss_decay = grad_desc_decay(x, y, t)
```

In [29]:
```
y_pred = sin_model(x, t_est[0], t_est[1])

plt.plot(x, y_pred, label='Model')
plt.scatter(x, y, alpha=0.5, label='Observation', color='gold')
plt.legend();
```



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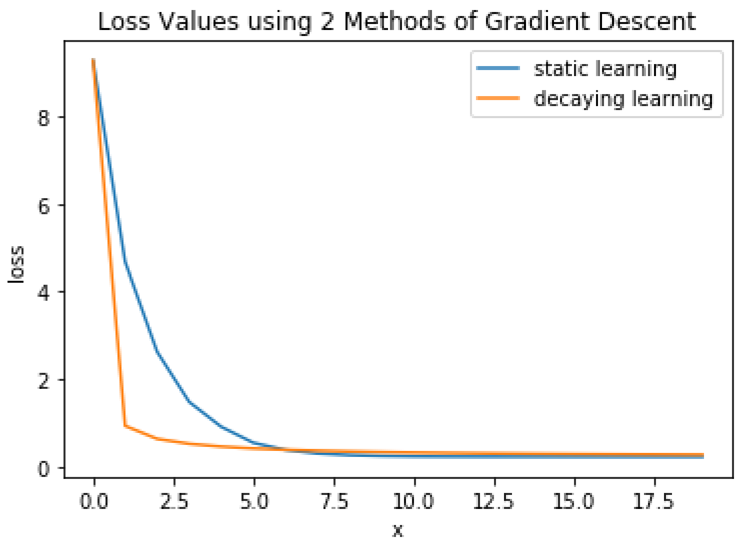
It appears that gradient descent successfully optimized theta. Our model is a very good fit of our observed data. The graph shows that the original scatter plot is more or less fitted by our model's predicted y-values over the x-values.

Question 4d

Let's compare our two gradient descent methods and see how they differ. Plot the loss values over each iteration of gradient descent for both static learning rate and decaying learning rate.

```
In [30]: plt.plot(loss) # Plot of loss history for static learning rate
         plt.plot(loss_decay) # Plot of loss history for decaying learning rate
         plt.xlabel('x')
         plt.ylabel('loss')
         plt.title('Loss Values using 2 Methods of Gradient Descent')
         plt.legend(labels = ['static learning', 'decaying learning'])

Out[30]: <matplotlib.legend.Legend at 0x7f7e01682470>
```



Question 4e

Compare and contrast the performance of the two gradient descent methods. Which method begins to converge more quickly?

It appears that the decaying learning method begins to converge more quickly. The static learning rate method takes almost 4 to 5 times as long to converge as the decaying learning rate method. Hence, the decaying learning rate method results in less loss incurred by model predictions, and optimizes model parameters better than the static learning rate method does.

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5: Visualizing Loss

Question 5:

Let's visualize our loss functions and gain some insight as to how gradient descent and stochastic gradient descent are optimizing our model parameters.

Question 5a:

In the previous plot is about the loss decrease over time, but what exactly is path the theta value? Run the following three cells.

In [31]:
```
# Run me
ts = np.array(ts).squeeze()
ts_decay = np.array(ts_decay).squeeze()
loss = np.array(loss)
loss_decay = np.array(loss_decay)
```

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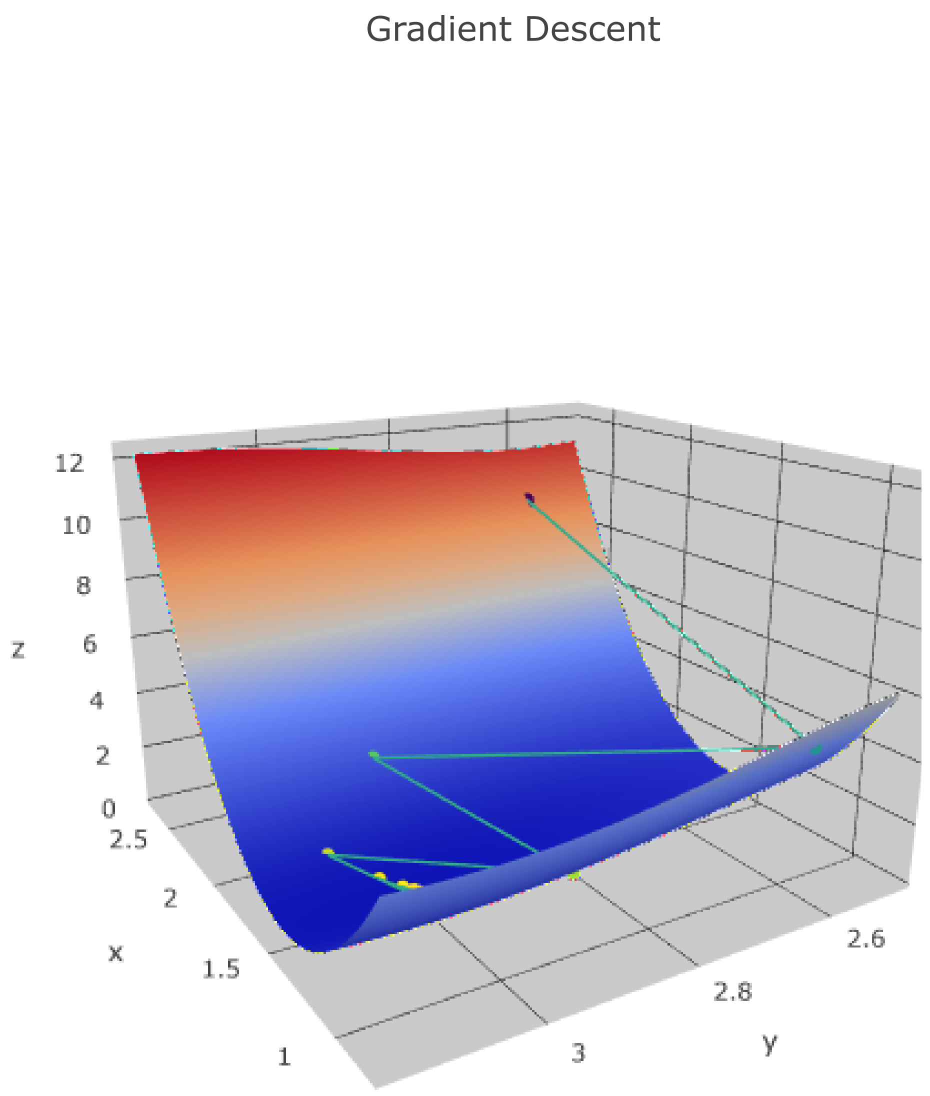
In [32]:
```
# Run me to see a 3D plot (gradient descent with static alpha)
plot_3d(ts[:, 0], ts[:, 1], loss, l2_loss, sin_model, x, y)
```



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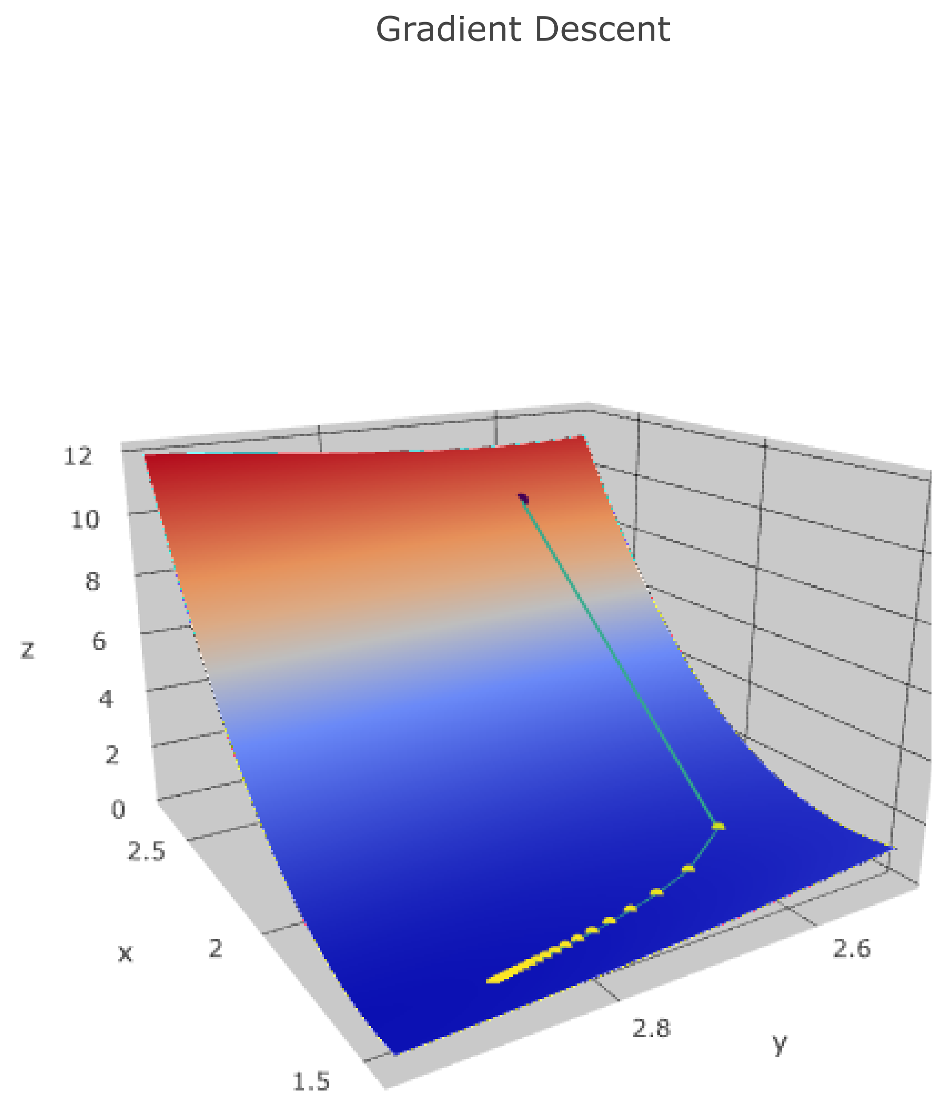
```
In [33]: # Run me to see another 3D plot (gradient descent with decaying alpha)
         plot_3d(ts_decay[:, 0], ts_decay[:, 1], loss_decay, l2_loss, sin_model, x, y)
```



In the following cell, write 1-2 sentences about the differences between using a static learning rate and a learning rate with decay for gradient descent. Use the loss history plot as well as the two 3D visualization to support your answer.

Our suspicion from question 4e is confirmed. As the 3D visualizations show, using a static learning rate for gradient descent results in slower convergence to a global minimum than using a learning rate with decay for gradient descent. As the loss history plots show, using a static learning rate for gradient descent results in more loss between our model's predictions and the actual values, whereas using a decaying learning rate results in less loss and faster convergence to a minimum possible loss between the model's prediction and the actual values, given our model's design.

Question 5b:

Another common way of visualizing 3D dynamics is with a *contour* plot.

Please refer to this notebook when you are working on the next question: Please refer to this notebook when you are working on the next question: http://www.ds100.org/fa18/assets/lectures/lec09/09-Models-and-Estimation-II.html (http://www.ds100.org/fa18/assets/lectures/lec09/09-Models-and-Estimation-II.html). Search the page for `go.Contour`.

In next question, fill in the necessary part to create a contour plot. Then run the following cells.

In [34]:
```
## Run me
import plotly
import plotly.graph_objs as go
plotly.offline.init_notebook_mode(connected=True)
```

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In [35]:
```
def contour_plot(title, theta_history, loss_function, model, x, y):
    """
    The function takes the following as argument:
        theta_history: a (N, 2) array of theta history
        loss: a list or array of loss value
        loss_function: for example, l2_loss
        model: for example, sin_model
        x: the original x input
        y: the original y output
    """
    theta_1_series = theta_history[:,0] # a list or array of theta_1 value
    theta_2_series = theta_history[:,1] # a list or array of theta_2 value


    thata_points = go.Scatter(name="Theta Values",
                              x= theta_1_series, #this line
                              y= theta_2_series, #this line
                              mode="lines+markers")

    ## In the following block of code, we generate the z value
    ## across a 2D grid
    t1_s = np.linspace(np.min(theta_1_series) - 0.1, np.max(theta_1_series) +
0.1)
    t2_s = np.linspace(np.min(theta_2_series) - 0.1, np.max(theta_2_series) +
0.1)

    x_s, y_s = np.meshgrid(t1_s, t2_s)
    data = np.stack([x_s.flatten(), y_s.flatten()]).T
    ls = []
    for t1, t2 in data:
        l = loss_function(model(x, t1, t2), y)
        ls.append(l)
    z = np.array(ls).reshape(50, 50)


    lr_loss_contours = go.Contour(x= t1_s,  #this line
                                  y= t2_s, #this line
                                  z= z,  #this line
                                  colorscale='Viridis', reversescale=True)


    plotly.offline.iplot(go.Figure(data=[lr_loss_contours, thata_points], layo
ut={'title': title}))
```

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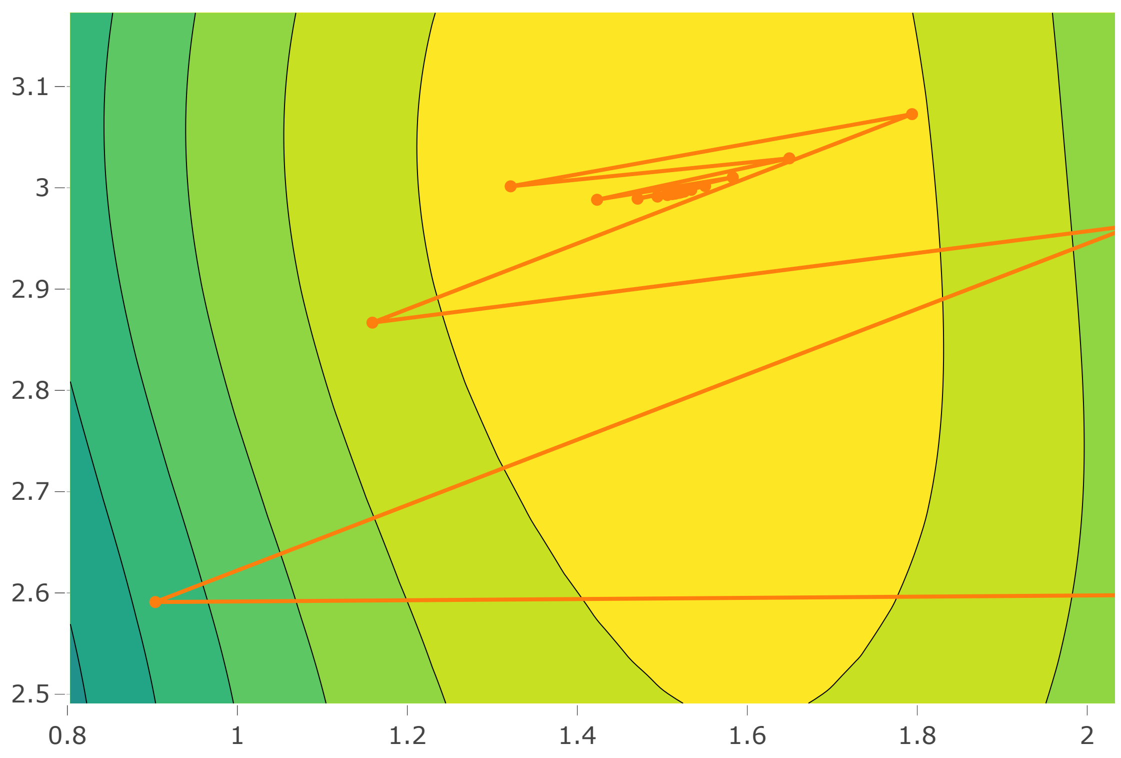
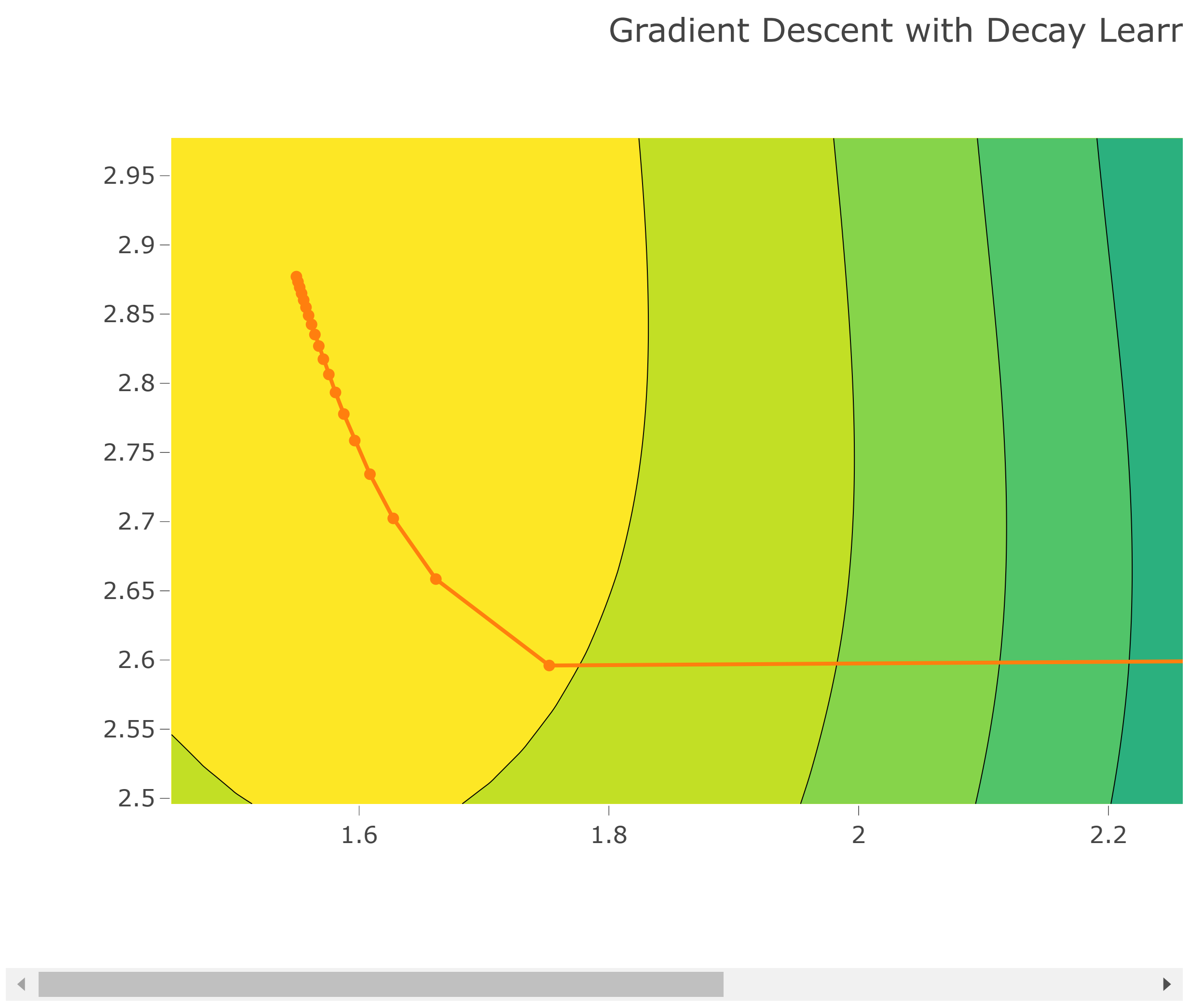
```
In [36]: # Run this
         contour_plot('Gradient Descent with Static Learning Rate', ts, l2_loss, sin_mo
         del, x, y)
```

Gradient Descent with Static Learn



In [37]:
```
## Run me
contour_plot('Gradient Descent with Decay Learning Rate', ts_decay, l2_loss, s
in_model, x, y)
```



In the following cells, write down the answer to the following questions:

- How do you interpret the two contour plots?
- Compare contour plot and 3D plot, what are the pros and cons of each?

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The two contour plots show that for a given x, y, and z value, we can estimate an optimized theta value. Interpreting the plots, we can see that the gradient descent model with a static learning rate takes many more steps to converge to an absolute optimized theta value (given the model), which is the global minimum in this plot. By contrast, the gradient descent model with a decaying learning rate seems to converge to the absolute optimized theta value (given the model) much more quickly, with less steps and no wild swings in the x value or on the contour plot.

Comparing the contour plot with the 3D plot, I can tell more easily that the gradient descent converges to a global minimum on the 3D plot because there is a more clearly defined visual bottom to the plot compared to the contour plot, where it is visually harder to interpret the concept of a minima since it only displays visuals horizontally in 2D. By contrast, the contour plot is easier to read than the 3D plot because you have a better spatial frame to read the plot from. You have less axes, and there is a uniform increasing and decreasing direction on the axes, which is not the case with the 3D plot.

How to Improve?

Question 5c (optional)

Try adding the two additional model parameters for phase and amplitude that we ignored (see 3a). What are the optimal phase and amplitude values for your four parameter model? Do you get a better loss?

YOUR ANSWER HERE

Question 5d (optional)

It looks like our basic two parameter model, a combination of a linear function and sinusoidal function, was able to almost perfectly fit our data. It turns out that many real world scenarios come from relatively simple models.

At the same time, the real world can be incredibly complex and a simple model wouldn't work so well. Consider the example below; it is neither linear, nor sinusoidal, nor quadratic.

Optional: Suggest how we could iteratively create a model to fit this data and how we might improve our results.

Extra optional: Try and build a model that fits this data.

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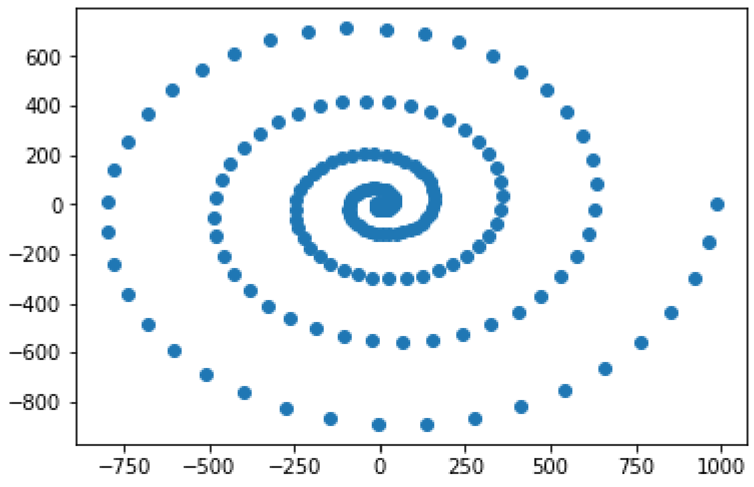
In [38]:
```
x = []
y = []
for t in np.linspace(0,10*np.pi, 200):
    r = ((t)**2)
    x.append(r*np.cos(t))
    y.append(r*np.sin(t))

plt.scatter(x,y)
plt.show()
```



YOUR ANSWER HERE

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6: Short Analytic Problems

Let's work through some problems to solidify the foundations of gradient descent. If these questions are hard, consider reviewing lecture and supplementary materials.

Question 6

Complete the problems below. **Show your work and solution in LaTeX**. Here are some useful examples of LaTex syntax:

Summation: $\sum_{i=1}^{n} a_i$

Exponent: a^2

Fraction: $\frac{a}{b}$

Multiplication: $a \cdot b$

Derivative: $\frac{\partial}{\partial a}$

Symbols: α, β, θ

Convexity

Question 6a

In lecture 8 (http://www.ds100.org/fa18/syllabus#lecture-week-5), we introduced the idea of a convex function. Let $h(x) = f(x) + g(x)$ where f, g are convex functions. Prove that h is convex.

Definition of Convex Function: Function f is convex if and only if: tf(a) + (1-t)f(b) $\leq$ f(ta + (1-t)b), $\forall a, \forall b, t \in [0, 1]$. (From Lecture 8 slides, Data 100, Fall 2018).

Since we know that f and g are convex functions, it follows from the definition of convex functions: (1) tf(a) + (1-t)f(b) $\leq$ f(ta + (1-t)b), $\forall a, \forall b, t \in [0, 1]$; (2) tg(a) + (1-t)g(b) $\leq$ g(ta + (1-t)b), $\forall a, \forall b, t \in [0, 1]$

Adding (1) and (2) together: tf(a) + tg(a) + (1-t)f(b) + (1-t)g(b) $\leq$ f(ta + (1-t)b) + g(ta + (1-t)b), $\forall a, \forall b, t \in [0, 1]$

Since h(x) is defined such that h(x) = f(x) + g(x), we can simplify this inequality to: th(a) + (1-t)h(b) $\leq$ h(ta) + (1-t)h(b), $\forall a, \forall b, t \in [0, 1]$

This further simplifies to: th(a) + (1-t)h(b) $\leq$ h(ta + (1-t)b), $\forall a, \forall b, t \in [0, 1]$

In other words, the definition of a convex function. Hence, h is a convex function.

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Mutlivariable/vector calculus mechanical problems

Question 6b

Show that the sum of the squared error

$$L(\mathbf{w}) = ||\mathbf{Xw} - \mathbf{y}||_2^2$$

can be expanded into

$$L(\mathbf{w}) = \mathbf{w}^T\mathbf{X}^T\mathbf{X}\ \mathbf{w} - 2\mathbf{y}^T\mathbf{X}\ \mathbf{w} + \mathbf{y}^T\mathbf{y}$$

using vector/matrix notation.

$$L(\mathbf{w}) = ||\mathbf{Xw} - \mathbf{y}||_2^2$$

= $(Xw - y)^T(Xw - y)$

= $(X^Tw^T - y^T)(Xw - y)$

= $X^Tw^TXw - X^Tw^Ty - y^TXw + y^Ty$ (Since the two middle terms are transposes of each other and are scalars, we can combine them)

= $w^TX^TXw - 2y^TXw + y^Ty$

Question 6c

Solve for the optimal $\mathbf{w}$, assuming $\mathbf{X}$ is full rank. Use the Matrix Derivative rules from lecture 11 (http://www.ds100.org/fa18/syllabus#lecture-week-6).

To solve for the optimal w, we take the gradient of L(w) and set it equal to the zero vector.

$\nabla_w(w^TX^TXw) - \nabla_w(2y^TXw) + \nabla_w(y^Ty) = \vec{0}$

Rule 1: $\nabla_\theta(A\theta) = A^T$; Rule 2: $\nabla_\theta(\theta^TA\theta) = A\theta + A^T\theta$ (From Lecture 11 slides, Data 100, Fall 2018)

Applying both rules, we take the gradient and get: $(X^TX + (X^TX)^T)w - 2yX^T + \vec{0} = \vec{0}$

Simplify: $(2X^TX)w = 2yX^T$

Simplify and I get the optimal w: $w = (X^TX)^{-1}(yX^T)$

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Question 6d

Repeat the steps above for ridge regression as described in lecture 12 (http://www.ds100.org/fa18/syllabus#lecture-week-6). Recall that ridge regression uses the following l2 regularized sum of squared error.

$$L(\mathbf{w}) = ||\mathbf{Xw} - \mathbf{y}||_2^2 + \lambda||\mathbf{w}||_2^2$$

Remark: Let I_s denote an identity matrix of arbitrary non-zero positive square dimensions.

$$L(\mathbf{w}) = ||\mathbf{Xw} - \mathbf{y}||_2^2 + \lambda||\mathbf{w}||_2^2$$

Expanding out the terms:

$$L(\mathbf{w}) = w^TX^TXw - 2y^TXw + y^Ty + \lambda w^Tw$$

Take the gradient of L(w) to solve for optimal w: $\nabla_w(w^TX^TXw) - \nabla_w(2y^TXw) + \nabla_w(y^Ty) + \nabla_w(\lambda w^Tw) = \vec{0}$

$(X^TX + (X^TX)^T)w - 2yX^T + \lambda w + \lambda^Tw = \vec{0}$

$(2X^TX)w - 2yX^T + 2\lambda w = \vec{0}$ (since lambda is a scalar)

$(X^TX)w + \lambda w = yX^T$

In order to solve the problem, we must multiply lambda by I_s, otherwise addition is not defined since X^TX is a square matrix of arbitrary non-zero positive dimension and lambda is a scalar. This preserves the properties of lambda by having only lambda on the diagonals and zero everywhere else in the matrix, still scaling w.

$w(X^TX + \lambda I_s) = yX^T$

We solve for w to yield the optimal w: $w = (X^TX + \lambda I_s)^{-1}(yX^T)$

Question 6e

Compare the analytic solutions of least squares and ridge regression. Why does ridge regression guarantee that we can find a unique solution? What are some of the tradeoffs (pros/cons) of using ridge regression?

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Ridge regression guarantees a unique solution, because we add the λI_s term to the solution, meaning that even if X^{T}X = 0 or if X^{T}X is non-invertible, then we can find the optimal w, and hence have a solution to the ridge regression that is unique.

The analytic solution to least squares does not guarantee a unique solution, because if X^{T}X = 0 or if X^{T}X is non-invertible, then we cannot solve for the optimal w, and hence there would be no solution to least squares in this case.

Some pros of using ridge regression are that it manages the bias/variance tradeoff well, and by adding an extra term to the model, it reduces the standard errors, thus resulting in a model that makes less biased predictions.

Some cons of using ridge regression are that it involves more model complexity (since there are more terms), and it results in more variance since the addition of the term penalizes more complicated models and favors simpler models, resulting in higher variance, but lower bias models.

Expectation and Variance

Question 6f

In lecture 10 (http://www.ds100.org/fa18/syllabus#lecture-week-6), we completed half of the proof for the linearity of expectation. Your task in this question is to complete the second half.

For reference, in lecture we showed that:

$$\mathrm{E}[aX + bY + c] = a\mathrm{E}[X] + \sum_{x \in \mathrm{X}} \sum_{y \in \mathrm{Y}} P(x, y)by + c$$

To complete this proof, prove that:

$$b\mathrm{E}[Y] = \sum_{x \in \mathrm{X}} \sum_{y \in \mathrm{Y}} P(x, y)by$$

Note: You cannot simply start with the given equation and use linearity of expectation. Start with the summation on the right side and manipulate it to get the left side.

Hint: What can we do with the order of the summations?

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Definition of expectation: $E[X] = \sum_{x \in X} xP(x)$ (Lecture 10 slides, Data 100, Fall 2018)

The right hand side of above states:

$aE[X] + \sum_{x \in X} \sum_{y \in Y} P(x, y)by + c$

I will manipulate this to prove the right hand side equals the left hand side.

= $aE[X] + \sum_{x \in X} \sum_{y \in Y} P(x|y)P(y)by + c$ (Using def of conditional probability in Lecture 10 slides)

= $aE[X] + b\sum_{y \in Y} P(y)y \sum_{x \in X} P(x|y) + c$ (Since we can switch order of summation without changing value of expression)

= $aE[X] + b\sum_{y \in Y} P(y)y + c$

= $aE[X] + bE[Y] + c$ (by definition of expectation)

= $E[aX + bY + c]$ (by reversing linearity of expectation)

Hence, by starting with the right hand side of the equation (and the summation on the right hand side), I have manipulated it to be equivalent to the left hand side. I have shown that the left hand side equals the right hand side. Thus, I have shown that $bE[Y] = \sum_{x \in X} \sum_{y \in Y} P(x, y)by$.

Question 6g

Prove that if two random variables X and Y are independent, then $Var(X - Y) = Var(X) + Var(Y)$.

Definition of Variance for a random variable: $Var(X) = E[X^2] - E[X]^2$ (Lecture 10 slides, Data 100, Fall 2018)

Remark: I treat (X-Y) as its own random variable. This is true because X-Y forms its own probability distribution.

$Var(X - Y) = E[(X - Y)^2] - E[X - Y]^2$

$= E[X^2 - 2XY + Y^2] - (E[X] - E[Y])^2$

$= E[X^2] - 2E[X]E[Y] + E[Y^2] - (E[X]^2 - 2E[X]E[Y] + E[Y]^2)$ (expansion and applying linearity)

$= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2)$

$= Var(X) + Var(Y)$

I have thus shown that if X and Y are independent, Var(X-Y) = Var(X) + Var(Y).

7: Quick Regex Problems

Here are some quick problems to review your knowledge of regular expressions.

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Question 7a

Write a regular expression to match the following strings without using the `|` operator.

1. **Match:** `abcdefg`
2. **Match:** `abcde`
3. **Match:** `abc`
4. **Skip:** `c abc`

```
In [39]: regxa = r"^abc*.*."
```

```
In [40]: assert ("|" not in regxa)
         assert (re.search(regxa, "abc").group() == "abc")
         assert (re.search(regxa, "abcde").group() == "abcde")
         assert (re.search(regxa, "abcdefg").group() == "abcdefg")
         assert (re.search(regxa, "c abc") is None)
```

Question 7b

Write a regular expression to match the following strings without using the `|` operator.

1. **Match:** `can`
2. **Match:** `man`
3. **Match:** `fan`
4. **Skip:** `dan`
5. **Skip:** `ran`
6. **Skip:** `pan`

```
In [41]: regxb = r"^[cmf][a][n]"
```

```
In [42]: assert ("|" not in regxb)
         assert (re.match(regxb, 'can').group() == "can")
         assert (re.match(regxb, 'fan').group() == "fan")
         assert (re.match(regxb, 'man').group() == "man")
         assert (re.match(regxb, 'dan') is None)
         assert (re.match(regxb, 'ran') is None)
         assert (re.match(regxb, 'pan') is None)
```

Question 7c:

Write a regular expression to extract and print the quantity and type of objects in a string. You may assume that a space separates quantity and type, ie. `"{quantity} {type}"`. See the example string below for more detail.

1. **Hint:** use `re.findall`
2. **Hint:** use `\d` for digits and one of either `*` or `+`.

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In [43]:
```
text_qc = "I've got 10 eggs that I stole from 20 gooses belonging to 30 giant
s."

res_qc = re.findall(r"[123]\d[\s][eg][goi][goa][ssn]\S?\S?", text_qc)

res_qc
```

Out[43]:
```
['10 eggs', '20 gooses', '30 giants']
```

In [44]:
```
assert res_qc == ['10 eggs', '20 gooses', '30 giants']
```

Question 7d:

Write a regular expression to replace at most 2 occurrences of space, comma, or dot with a colon.

Hint: use `re.sub(regex, "newtext", string, number_of_occurences)`

In [45]:
```
text_qd = 'Python Exercises, PHP exercises.'
res_qd = re.sub(r"(\s|[.]|[,])", ":", text_qd, 2)

res_qd
```

Out[45]:
```
'Python:Exercises: PHP exercises.'
```

In [46]:
```
assert res_qd == 'Python:Exercises: PHP exercises.'
```

Question 7e (optional):

Write a regular expression to replace all words that are not "`mushroom`" with "`badger`".

In [47]:
```
text_qe = 'this is a word mushroom mushroom'
res_qe = re.sub(r"\b(?!mushroom)\b\S+", "badger" , text_qe)
# Hint: https://www.regextester.com/94017

res_qe
```

Out[47]:
```
'badger badger badger badger mushroom mushroom'
```

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Submission - IMPORTANT, PLEASE READ

For this assignment and future assignments (homework and projects) you will also submit your free response and plotting questions to gradescope. To do this, you can download as PDF (`File->Download As->PDF via Latex (.pdf)`). You are responsible for submitting and tagging your answers in gradescope. For each free response and plotting question, please include:

1. Relevant code used to generate the plot or inform your insights
2. The written free response or plot

We are doing this to make it easier on our graders and for you, in the case you need to submit a regrade request. Gradescope (as of now) is still better for manual grading.

Submission

You're done!

Before submitting this assignment, ensure to:

1. Restart the Kernel (in the menubar, select Kernel->Restart & Run All)
2. Validate the notebook by clicking the "Validate" button

Finally, make sure to **submit** the assignment via the Assignments tab in Datahub

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