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Econ 142 Problem Set 2

1(a) $E[(\bar{Y}-\mu)^2]$

$= E[(\frac{1}{N}\sum_i Y_i - \mu)^2]$

$= E[\frac{1}{N^2}\sum_i (Y_i-\mu)^2]$

$= \frac{1}{N^2} E[\sum (Y_i-\mu)^2]$

$= \frac{1}{N^2}\cdot N\sigma^2 = \frac{\sigma^2}{N}$ ■

$- E[\bar{Y}] = \frac{1}{n} E[\sum_{i=1}^{n} Y_i]$

$= \frac{1}{n} n\mu$

$= \mu$

$- E[(Y_0-\mu)^2] = var(Y_i)$

$= \sigma^2$

$E[(\bar{Y}-\mu)^2] = var(\bar{Y})$

$= var(\frac{1}{N}\sum_i Y_i)$

$= \frac{1}{N^2} var(\sum_i Y_i)$

$= \frac{1}{N^2}\cdot N\sigma^2$

$= \frac{\sigma^2}{N}$

$D = \frac{1}{N}\sum_i (Y_i-\bar{Y})^2$

1(b) $D = \frac{1}{N}\sum_i ((Y_i-\mu)+(\mu-\bar{Y}))^2$

$= \frac{1}{N}\sum_i ((Y_i-\mu)^2+(\mu-\bar{Y})^2+2(Y_i-\mu)(\mu-\bar{Y}))$

$= \frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2 + \frac{1}{N}\sum_i 2(Y_i-\mu)(\mu-\bar{Y})$

$= \frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2 + \frac{2}{N}\sum_i (Y_i-\mu)(\mu-\bar{Y})$

$= \frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2 + \frac{2}{N}\sum_i Y_i\mu - Y_i\bar{Y} - \mu^2 + \mu\bar{Y}$

$= \frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2$ ■

1(c) $D = \frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2$

$E[D] = E[\frac{1}{N}\sum_i (Y_i-\mu)^2 - \frac{1}{N}\sum_i (\bar{Y}-\mu)^2]$

$= E[\frac{1}{N}\sum_i (Y_i-\mu)^2] - E[\frac{1}{N}\sum_i (\bar{Y}-\mu)^2]$

$= \frac{1}{N} E[\sum_i (Y_i-\mu)^2] - \frac{1}{N} E[\sum_i (\bar{Y}-\mu)^2]$

$$E[D] = \frac{1}{N} E\left[\sum_i E\left[(Y_i - \mu)^2\right]\right] - \frac{1}{N} E\left[\sum_i E\left[(\bar{Y} - \mu)^2\right]\right]$$

$$= \frac{1}{N} \cdot N\sigma^2 - \frac{1}{N}\sigma^2$$

$$= \sigma^2 - \frac{\sigma^2}{N}$$

$$= \frac{N\sigma^2 - \sigma^2}{N} = \frac{N-1}{N}\sigma^2$$

(d) In (c), we show that

$$E[\theta] = E\left[\frac{1}{N}\sum_i (Y_i - \bar{Y})^2\right] = \frac{N-1}{N}\sigma^2.$$

$$\frac{1}{N-1} E\left[\sum_i (Y_i - \bar{Y})^2\right] = \frac{1}{N-1} E\left[\sum_i \left((Y_i - \mu) + (\mu - \bar{Y})\right)^2\right]$$

$$= \frac{1}{N-1} E\left[\sum_i (Y_i-\mu)^2 + (\mu - \bar{Y})^2 + 2(Y_i - \mu)(\mu - \bar{Y})\right]$$

$$= \frac{1}{N-1}\left[E\left[\sum_i (Y_i-\mu)^2\right] + E\left[\sum_i (\mu-\bar{Y})^2\right] + E\left[2(Y_i-\mu)(\mu-\bar{Y})\right]\right]$$

expectation of this is zero

$$= \frac{1}{N-1}\left[E\left[\sum_i (Y_i-\mu)^2\right] + E\left[\sum_i (\mu-\bar{Y})^2\right]\right]$$

$$= \frac{1}{N-1}\left[E\left[\sum_i E[(Y_i-\mu)^2]\right] + E\left[\sum_i E(\mu-\bar{Y})^2\right]\right]$$

$$= \frac{1}{N-1}\left[n\sigma^2 - \sigma^2\right] = \frac{\sigma^2(N-1)}{N-1} = \sigma^2$$

2(a) $Y_i = \alpha + \beta D_i + u_i$

optimization problem; minimize sum of squared errors

$\hat{u}_i = Y_i - \alpha - \beta D_i$

FOC_α : $\underset{\min}{u_i^2} = \sum_{i=0}^{n} (Y_i - \hat{\alpha} - \hat{\beta} D_i)^2$

$$\frac{\partial u_i^2}{\partial \alpha} = -2\sum_{i=0}^{n}(Y_i - \hat{\alpha} - \hat{\beta}D_i) = 0$$

$$\sum_{i=0}^{n}(Y_i - \hat{\alpha} - \hat{\beta}D_i) = 0$$

$$-N\hat{\alpha} = -\sum_{i=0}^{n} Y_i + \beta\sum_{i=0}^{n} D_i$$

$$-N\hat{\alpha} = -\sum_{i=0}^{n} Y_i$$ (since only effect of dummy off)

$$\hat{\alpha} = \frac{1}{N_0}\sum_{i\in 0} Y_i$$

FOC_β : $$\frac{\partial u_i^2}{\partial \beta} = -2\sum_{i=0}^{n} D_i(Y_i - \hat{\alpha} - \hat{\beta}D_i) = 0$$

$$\sum_{i=0}^{n} D_i(Y_i - \hat{\alpha} - \hat{\beta}D_i) = 0$$

$$\sum_{i=0}^{N} D_i(Y_i - \hat{\alpha} - \hat{\beta} D_i) = 0$$

$$\sum_{i=0}^{N} D_i Y_i - \hat{\alpha}\sum_{i=0}^{N} D_i - \hat{\beta}\sum_{i=0}^{N} D_i^2 = 0$$

$$-\hat{\alpha}\sum_{i=0}^{N} D_i - \hat{\beta}\sum_{i=0}^{N} D_i^2 = -\sum_{i=0}^{N} D_i Y_i$$

In this case, Dummies are turned on $(D_i = 1)$.

$$-N\hat{\alpha} - N\hat{\beta} = -\sum_{i=0}^{N} Y_i$$

$$N_1(\hat{\alpha} + \hat{\beta}) = \sum_{i=0}^{N} Y_i$$

$$\hat{\alpha} + \hat{\beta} = \frac{1}{N_1}\sum_{i \in 1} Y_i$$

(6) $Y_i = \alpha + \beta_1 D_{1i} + \beta_2 D_{2i} + u_i$

$$\hat{u}_i = Y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i}$$

FOC_{α}: $\sum_{i=0}^{N} (Y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i})^2$

$$\frac{\partial}{\partial \alpha} = -2\sum_{i=0}^{N} (Y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i}) = 0$$

$$= \sum_{i=0}^{N} (Y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i}) = 0$$

(both dummies are off, $D_{1i} = 0$, $D_{2i} = 0$)

$$= \sum_{i=0}^{N} Y_i - N_0\alpha - \sum_{i=0}^{N} \beta_1 D_{1i} - \sum_{i=0}^{N} \beta_2 D_{2i} = 0$$

$$= \sum_{i=0}^{N} Y_i - N_0\hat{\alpha} = 0$$

$$-N_0\hat{\alpha} = -\sum_{i=0}^{N} Y_i$$

$$\hat{\alpha} = \frac{1}{N_0}\sum_{i \in 0} Y_i$$

$FOC_{\hat{\beta}_1}$: $\sum_{i=0}^{N} (Y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i} - \hat{\beta}_2 D_{2i})^2$

$$\frac{\partial}{\partial \beta_1} = -2\sum_{i=0}^{N} D_{1i}(Y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i} - \hat{\beta}_2 D_{2i}) = 0$$

$$= \sum_{i=0}^{N} D_{1i} Y_i - \hat{\alpha}\sum_{i=0}^{N} D_{1i} - \hat{\beta}_1 \sum_{i=0}^{N} D_{1i}^2 - \hat{\beta}_2 \sum_{i=0}^{N} D_{1i} D_{2i} = 0$$

In this case, $D_{1i}=1$, $D_{2i}=0$

$$\Rightarrow \sum_{i=0}^{N} Y_i - N\hat{\alpha} - N\hat{\beta}_1 = 0$$

$$N_1(\hat{\alpha}+\hat{\beta}_1) = \sum_{i=0}^{N} Y_i$$

$$\hat{\alpha}+\hat{\beta}_1 = \frac{1}{N_1}\sum_{i=0}^{N} Y_i$$

$$FOC_{\hat{\beta}_2}: \sum_{i=0}^{N} (Y_i - \alpha - \beta_1 D_{1i} - \beta_2 D_{2i})^2$$

$$\Rightarrow \frac{\partial}{\partial \hat{\beta}_2} = -2\sum_{i=0}^{N} D_{2i}(Y_i - \hat{\alpha} - \hat{\beta}_1 D_{1i} - \hat{\beta}_2 D_{2i}) = 0$$

$$= \sum_{i=0}^{N} D_{2i} Y_i - \hat{\alpha}\sum_{i=0}^{N} D_{2i} - \hat{\beta}_1 \sum_{i=0}^{N} D_{1i} D_{2i} - \hat{\beta}_2 \sum_{i=0}^{N} D_{2i}^2 = 0$$

In this case, $D_{2i}=1$, $D_{1i}=0$

$$= \sum_{i=0}^{N} Y_i - N_2\hat{\alpha} - N_2\hat{\beta}_2 = 0$$

$$N_2(\hat{\alpha}+\hat{\beta}_2) = \sum_{i=0}^{N} Y_i$$

$$\hat{\alpha}+\hat{\beta}_2 = \frac{1}{N_2}\sum_{i=0}^{N} Y_i$$

3. model:

$Y_i = X_i \beta^* + u_i$ old auxiliary: $X_{ji} = X'_{(\sim j)i}\pi + \varepsilon_i$

new auxiliary: $Y_i = X'_{(\sim j)i}\lambda + \phi_i$

$$FOC_\lambda: E[X'_{(\sim j)i}\phi_i] = 0$$

$$\Rightarrow E[\phi_i X_{ni}] = 0$$

unless $n=j$.

$$FOC_{\beta^*}: E[X_i u_i] = 0$$

$$\Rightarrow E[u_i \varepsilon_i] = E[u_i(X_{ji} - X'_{(\sim j)i}\pi)] = 0$$

$$FOC_\pi: E[X'_{(\sim j)i}\varepsilon_i] = 0$$

$$\Rightarrow E[\varepsilon_i X_{ni}] = 0$$

unless $n=j$.

$E[\epsilon_i x_{ji}] = E[\epsilon_i(x'_{(\sim j)i}\pi + \epsilon_i)]$ invoke factor π

$= E[\epsilon_i^2]$

so,

$E[\epsilon_i \phi_i] = \beta^*_j E[\epsilon_i^2]$

$$\beta^*_j = E[\epsilon_i \phi_i] (E[\epsilon_i^2])^{-1}$$