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Assignment 2 - Theoretical Part

a) Discretization Matrix A

$$- \Delta u(x,y) + k^2 . u(x,y) = f(x,y) \quad -(1) \quad (x,y) \in \Omega$$

$\Omega = [0,2] \times [0,1]$ with $k = 2\pi$

& $f(x,y) = 4\pi^2 \sin(2\pi x) \sinh(2\pi y)$

Discretization of Elliptic PDE using Finite Difference scheme with central difference scheme (for 2nd order consistency)

$$-\left[\frac{u(x-1,y) - 2u(x,y) + u(x+1,y)}{hx^2}\right] - \left[\frac{u(x,y-1) - 2u(x,y) + u(x,y+1)}{hy^2}\right]$$

$$+ k^2 u(x,y) = f(x,y)$$

$$\Rightarrow \left(\frac{2}{hx^2} + \frac{2}{hy^2} + k^2\right) u(x,y) - \frac{1}{hx^2}\left(u(x-1,y) + u(x+1,y)\right)$$

$$-\frac{1}{hy^2}\left(u(x,y-1) + u(x,y+1)\right) = f(x,y)$$

Let $\alpha = \frac{2}{hx^2} + \frac{2}{hy^2} + k^2$ with $k = 2\pi$

$\beta = \frac{-1}{hx^2}$, $\gamma = \frac{-1}{hy^2}$

Applying this discretization to each unknown node in domain & assembling the matrix we have,

$$\begin{bmatrix} \alpha & \beta & 0 & \gamma & 0 & \cdots & \cdots & 0 \\ \beta & \alpha & \beta & 0 & \gamma & 0 & \cdots & 0 \\ 0 & \beta & \alpha & 0 & 0 & \gamma & 0 \cdots & 0 \\ \gamma & 0 & 0 & \alpha & \beta & 0 & \gamma \cdots & 0 \\ 0 & \gamma & \ddots & \beta & \alpha & \beta & & \\ \vdots & 0 & \ddots & 0 & \beta & \alpha & \ddots & \\ \vdots & \vdots & 0 & & \ddots & & \ddots & \\ \vdots & \vdots & & & & \ddots & & \end{bmatrix} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{x-2} \\ u_{x-1} \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_{x-2} \\ f_{x-1} \end{bmatrix} + \begin{bmatrix} D\cdot B\cdot C \\ \\ \\ \\ \end{bmatrix}}_{\tilde{f} := f+g}$$

$$\text{matrix} = A \in \mathbb{R}^{(N_x-1)(N_x-1)\times(N_y-1)(N_y-1)}$$

It is a sparse matrix. To find whether it is strictly diagonally dominant we compare the magnitude of coeffis of matrix entries.

Magnitude of diagonal entries $|\alpha| = \frac{2}{h_x^2} + \frac{2}{h_y^2} + 4\pi^2$

Mag. of non-diagonal entries $|\beta| + |\gamma| = \frac{2}{h_x^2} + \frac{2}{h_y^2}$

So

$$|\alpha| > |\beta| + |\gamma|$$

$$\frac{2}{h_x^2} + \frac{2}{h_y^2} + k^2 > \frac{2}{h_x^2} + \frac{2}{h_y^2} \quad \text{for } k > 0 \text{ (here } 2\pi)$$

Thus Matrix A is strictly diagonally dominant. □

b) Iterative methods converge if and only if spectral radius of strictly diagonally dominant matrix is less than 1.

i.e. $\rho(A) < 1$.

$$\rho(A) \leq \max_i \left\{ \frac{\sum_{i \neq j} |a_{ij}|}{|a_{ii}|} \right\} \leq \frac{\left(\left|\frac{-2}{hx^2}\right| + \left|\frac{-2}{hy^2}\right| \right)}{\left(\left| \frac{2}{hx^2} + \frac{2}{hy^2} + k^2 \right| \right)}$$

$$= \frac{\frac{2}{hx^2} + \frac{2}{hy^2}}{\frac{2}{hx^2} + \frac{2}{hy^2} + k^2} < 1 \quad \forall k > 0$$

$\therefore \rho(A) < 1$.

So above Jacobi as well as Gauss Seidel Algorithms converge with a bound as,

$$\rho(A) \leq \frac{1}{1 + \left[\frac{k^2}{\left(\frac{2}{hx^2} + \frac{2}{hy^2} \right)} \right]}$$

$\square$