Master thesis: documentation

Vinayak Gholap

1 Constitutive relations for coupled magneto-elastic material

Neo-Hookean hyperelastic material model for coupled magneto-elastic material:

$$\boxed{\Psi = \Psi(J, \mathbf{C}, \mathbb{H}) = \frac{\mu}{2}[\mathbf{C} : \mathbf{I} - \mathbf{I} : \mathbf{I} - 2 \ln J] + \frac{\lambda}{2}(\ln J)^2 - \frac{\mu_0 \mu_r}{2}[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]} \tag{1}$$

where J: Jacobian,
$\mathbf{C}$: Right Cauchy-Green deformation tensor,
$\mathbb{H}$: Applied magnetic vector field,
μ: Shear modulus,
λ: Lamé 1^{st} parameter,
μ_0: vacuum magnetic permeability,
μ_r: relative magnetic permeability of magneto-elastic material.

1.1 2^{nd} Piola-Kirchhoff stress

$$\begin{aligned}
\mathbf{S} &= 2\frac{\partial \Psi}{\partial \mathbf{C}} \\
&= 2\left[\frac{\mu}{2}\left\{\frac{\partial[\mathbf{C} : \mathbf{I}]}{\partial \mathbf{C}} - 2\frac{\partial \ln J}{\partial \mathbf{C}}\right\} + \frac{\lambda}{2}\frac{\partial(\ln J)^2}{\partial \mathbf{C}} - \frac{\mu_0\mu_r}{2}\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}}\right]
\end{aligned}$$

Side calculation:

$$\begin{aligned}
\frac{\partial[\mathbf{C} : \mathbf{I}]}{\partial \mathbf{C}} &= \mathbf{I} \\
\frac{\partial \ln J}{\partial \mathbf{C}} &= \frac{1}{J}\frac{\partial J}{\partial \mathbf{C}} \\
\frac{\partial(\ln J)^2}{\partial \mathbf{C}} &= 2\ \ln J\ \frac{\partial \ln J}{\partial \mathbf{C}} \\
\frac{\partial J}{\partial \mathbf{C}} &= \frac{1}{2}J\mathbf{C}^{-1} \quad \text{c.f. [see 1, page 46 Equation (3.124)]}
\end{aligned}$$

$$\mathbf{S} = \mu\mathbf{I} - \mu\mathbf{C}^{-1} + \lambda \ln J\mathbf{C}^{-1} - \mu_0\mu_r\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}}$$

Side calculation: Note $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ is symmetric $\implies$ $\mathbf{C}^{-1}$ is also symmetric

$$\begin{aligned}
\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}} &= \frac{\partial[JC^{-1}_{IJ}H_I H_J]}{\partial C_{KL}}\ \mathbf{E}_K \otimes \mathbf{E}_L \\
&= \frac{\partial J}{\partial C_{KL}}\ C^{-1}_{IJ}\ H_I\ H_J\ \mathbf{E}_K \otimes \mathbf{E}_L + J\ \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}\ H_I\ H_J\ \mathbf{E}_K \otimes \mathbf{E}_L \\
&= \frac{J}{2}\ C^{-1}_{KL}\ C^{-1}_{IJ}\ H_I\ H_J\ \mathbf{E}_K \otimes \mathbf{E}_L + J\ \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}\ H_I\ H_J\ \mathbf{E}_K \otimes \mathbf{E}_L
\end{aligned}$$

c.f. [see 1, page 519]:

$$\frac{\partial C^{-1}_{IJ}}{\partial C_{KL}} = -\frac{1}{2}[C^{-1}_{IK}\ C^{-1}_{LJ} + C^{-1}_{IL}\ C^{-1}_{KJ}]$$

$$
\begin{aligned}
\frac{\partial[J\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]}{\partial\mathbf{C}} &= \frac{J}{2}\ C^{-1}_{KL}\ C^{-1}_{IJ}\ H_I\ H_J\ \mathbf{E}_K\otimes\mathbf{E}_L + \frac{-J}{2}\left[C^{-1}_{IK}\ C^{-1}_{LJ}\ H_I\ H_J + C^{-1}_{IL}\ C^{-1}_{KJ}\ H_I\ H_J\right]\ \mathbf{E}_K\otimes\mathbf{E}_L\\
&= \frac{J}{2}\ C^{-1}_{IJ}\ H_I\ H_J\ C^{-1}_{KL}\ \mathbf{E}_K\otimes\mathbf{E}_L - \frac{J}{2}\left[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{LI}\ H_I\ C^{-1}_{KJ}\ H_J\right]\ \mathbf{E}_K\otimes\mathbf{E}_L\\
&= \frac{J}{2}\ C^{-1}_{IJ}\ H_I\ H_J\ C^{-1}_{KL}\ \mathbf{E}_K\otimes\mathbf{E}_L - J[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}\ \mathbf{E}_K\otimes\mathbf{E}_L\\
&= \frac{J}{2}[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\mathbf{C}^{-1} - J[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}
\end{aligned}
$$

$$
\boxed{\mathbf{S} = \mu\mathbf{I} - [\mu - \lambda\ln J]\mathbf{C}^{-1} - \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\mathbf{C}^{-1} + \mu_0\mu_r\ J\ [(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}}
$$

1.2 4^{th} order material elasticity tensor

$$
\begin{aligned}
\mathfrak{C} &= 2\frac{\partial\mathbf{S}}{\partial\mathbf{C}}\\
&= 2\frac{\partial S_{KL}}{\partial C_{MN}}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N\\
&= 2\left\{-\frac{\partial[\mu-\lambda\ln J]}{\partial\mathbf{C}}\mathbf{C}^{-1} - [\mu-\lambda\ln J]\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}} - \frac{\mu_0\mu_r}{2}\frac{\partial J}{\partial\mathbf{C}}[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\mathbf{C}^{-1}\right\}\\
&\quad + 2\left\{-\frac{\mu_0\mu_r}{2}\ J\ \frac{\partial[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]}{\partial\mathbf{C}}\mathbf{C}^{-1} - \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}}\right\}\\
&\quad + 2\left\{\mu_0\mu_r\frac{\partial J}{\partial\mathbf{C}}[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym} + \mu_0\mu_r\ J\ \frac{\partial[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}}{\partial\mathbf{C}}\right\}\\
&= 2\left\{\frac{\lambda}{J}\frac{\partial J}{\partial C_{MN}}\ C^{-1}_{KL} - [\mu-\lambda\ln J]\left\{\frac{-1}{2}\left(C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML}\right)\right\}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N\\
&\quad + 2\left\{-\frac{\mu_0\mu_r}{2}\frac{1}{2}\ J\ C^{-1}_{MN}\ [C^{-1}_{IJ}\ H_I\ H_J]C^{-1}_{KL} - \frac{\mu_0\mu_r}{2}J\ \frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}}C^{-1}_{KL}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N\\
&\quad + 2\left\{-\frac{\mu_0\mu_r}{2}\ J\ [C^{-1}_{IJ}\ H_I\ H_J]\left\{\frac{-1}{2}(C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML})\right\}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N\\
&\quad + 2\left\{\frac{\mu_0\mu_r}{2}\ J\ C^{-1}_{MN}[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym} + \mu_0\mu_r\ J\ \frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N\\
&= \lambda\ C^{-1}_{MN}\ C^{-1}_{KL} + [\mu-\lambda\ln J]\left(C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML}\right) - \frac{\mu_0\mu_r}{2}\ J\ C^{-1}_{MN}\ [C^{-1}_{IJ}\ H_I\ H_J]C^{-1}_{KL}\\
&\quad - \mu_0\mu_r J\ \frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}}\ C^{-1}_{KL} + \frac{\mu_0\mu_r}{2}\ J\ [C^{-1}_{IJ}\ H_I\ H_J](C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML})\\
&\quad + \mu_0\mu_r\ J\ C^{-1}_{MN}[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym} + 2\mu_0\mu_r\ J\ \frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N
\end{aligned}
$$

Side calculation 1:

$$\begin{aligned}
\frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}} &= \frac{\partial C^{-1}_{IJ}}{\partial C_{MN}}\ H_I\ H_J\ \mathbf{E}_M \otimes \mathbf{E}_N \\
&= \frac{-1}{2}[C^{-1}_{IM}\ C^{-1}_{NJ} + C^{-1}_{IN}\ C^{-1}_{MJ}]\ H_I\ H_J\ \mathbf{E}_M \otimes \mathbf{E}_N \\
&= \frac{-1}{2}[C^{-1}_{MI}\ H_I\ C^{-1}_{NJ}\ H_J + C^{-1}_{NI}\ H_I\ C^{-1}_{MJ}\ H_J]\ \mathbf{E}_M \otimes \mathbf{E}_N \\
&= -[C^{-1}_{MI}\ H_I\ C^{-1}_{NJ}\ H_J]^{sym}\ \mathbf{E}_M \otimes \mathbf{E}_N \\
&= -[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}
\end{aligned}$$

Side calculation 2:

$$\begin{aligned}
\frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}} &= \left[\frac{\partial C^{-1}_{KI}}{\partial C_{MN}}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{KI}\ H_I\ \frac{\partial C^{-1}_{LJ}}{\partial C_{MN}}\ H_J\right]\ \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_M \otimes \mathbf{E}_N \\
&= -\frac{1}{2}\left[(C^{-1}_{KM}\ C^{-1}_{NI} + C^{-1}_{KN}\ C^{-1}_{MI})\ H_I\ C^{-1}_{LJ}\ H_J\right]\ \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_M \otimes \mathbf{E}_N \\
&\quad -\frac{1}{2}\left[C^{-1}_{KI}\ H_I\ (C^{-1}_{LM}\ C^{-1}_{NJ} + C^{-1}_{LN}\ C^{-1}_{MJ})\ H_J\right]\ \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_M \otimes \mathbf{E}_N \\
&= -\frac{1}{2}\left[C^{-1}_{KM}\ C^{-1}_{NI}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{KN}\ C^{-1}_{MI}\ H_I\ C^{-1}_{LJ}\ H_J\right]\ \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_M \otimes \mathbf{E}_N \\
&\quad -\frac{1}{2}\left[C^{-1}_{KI}\ H_I\ C^{-1}_{LM}\ C^{-1}_{NJ}\ H_J + C^{-1}_{KI}\ H_I\ C^{-1}_{LN}\ C^{-1}_{MJ}\ H_J\right]\ \mathbf{E}_K \otimes \mathbf{E}_L \otimes \mathbf{E}_M \otimes \mathbf{E}_N \\
&= -\frac{1}{2}\left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})\right] \\
&\quad -\frac{1}{2}\left[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})\right] \\
&= -[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C} = &\ \lambda\ \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2[\mu - \lambda \ln J]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&- \frac{\mu_0 \mu_r}{2}\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}](\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}) + \mu_0 \mu_r\ J\ [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym} \otimes \mathbf{C}^{-1} \\
&- \mu_0 \mu_r\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} + \mu_0 \mu_r\ J\ (\mathbf{C}^{-1} \otimes [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}) \\
&- 2\mu_0 \mu_r\ J\ [\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]
\end{aligned}$$

For given symmetric rank 2 tensors $\mathbf{A}$ and $\mathbf{B}$, we know:

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}.$$

Moreover, any general tensor $\mathbf{A}$ can be uniquely decomposed into symmetric and skew-symmetric parts, $\mathbf{A} = \mathbf{A}^{sym} + \mathbf{A}^{skw}$ where,

$$\begin{aligned}
\mathbf{A}^{sym} &= \frac{1}{2}[\mathbf{A} + \mathbf{A}^T], \\
\mathbf{A}^{skw} &= \frac{1}{2}[\mathbf{A} - \mathbf{A}^T].
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathfrak{C} = &\ \lambda\ \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2[\mu - \lambda \ln J]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&- \frac{\mu_0 \mu_r}{2}\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}](\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}) - \mu_0 \mu_r\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&- 2\mu_0 \mu_r\ J\ (\mathbf{C}^{-1} \otimes [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{skw}) - 2\mu_0 \mu_r\ J\ [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]
\end{aligned}}$$

1.3 Magnetic induction vector

$$
\begin{aligned}
\mathbb{B} &= -\frac{\partial \Psi}{\partial \mathbb{H}} \\
&= \frac{\mu_0 \mu_r}{2} \frac{\partial [J \ \mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbb{H}} \\
&= \frac{\mu_0 \mu_r}{2} \frac{\partial [J \ C^{-1}_{IJ} \ H_I \ H_J]}{\partial H_K} \mathbf{E}_K \\
&= \frac{\mu_0 \mu_r}{2} \left[J \ C^{-1}_{IJ} \frac{\partial H_I}{\partial H_K} \ H_J + J \ C^{-1}_{IJ} \ H_I \frac{\partial H_J}{\partial H_K} \right] \mathbf{E}_K \\
&= \frac{\mu_0 \mu_r}{2} \left[J \ C^{-1}_{IJ} \ \delta_{IK} \ H_J + J \ C^{-1}_{IJ} \ H_I \ \delta_{JK} \right] \mathbf{E}_K \\
&= \frac{\mu_0 \mu_r}{2} \left[J \ C^{-1}_{JI} \ \delta_{IK} \ H_J + J \ C^{-1}_{IJ} \ \delta_{JK} \ H_I \right] \mathbf{E}_K \\
&= \frac{\mu_0 \mu_r}{2} \left[J \ C^{-1}_{JK} \ H_J + J \ C^{-1}_{IK} \ H_I \right] \mathbf{E}_K \\
&= \frac{\mu_0 \mu_r}{2} \left[J \ C^{-1}_{KJ} \ H_J + J \ C^{-1}_{KI} \ H_I \right] \mathbf{E}_K \\
\mathbb{B} &= \mu_0 \mu_r \ J \ [\mathbf{C}^{-1} \cdot \mathbb{H}]
\end{aligned}
$$

1.4 Magneto-elasticity tensor

$$
\begin{aligned}
\mathbf{D} &= \frac{\partial \mathbb{B}}{\partial \mathbb{H}} \\
&= \frac{\partial [\mu_0 \mu_r \ J \ (\mathbf{C}^{-1} \cdot \mathbb{H})]}{\partial \mathbb{H}} \\
&= \mu_0 \mu_r \ J \ \frac{\partial (C^{-1}_{IJ} \ H_J)}{\partial H_K} \ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0 \mu_r \ J \ C^{-1}_{IJ} \ \frac{\partial H_J}{\partial H_K} \ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0 \mu_r \ J \ C^{-1}_{IJ} \ \delta_{JK} \ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0 \mu_r \ J \ C^{-1}_{IK} \ \mathbf{E}_I \otimes \mathbf{E}_K \\
\mathbf{D} &= \mu_0 \mu_r \ J \ \mathbf{C}^{-1}
\end{aligned}
$$

1.5 Coupling tensor between magnetic and elastic fields

This quantity defines the sensitivity of 2^{nd} Piola-Kirchhoff stress $\mathbf{S}$ w.r.t. the applied magnetic field $\mathbb{H}$.

$$
\begin{aligned}
\mathbb{P} &= \frac{\partial \mathbf{S}}{\partial \mathbb{H}} = \frac{\partial}{\partial \mathbb{H}} \left(2 \frac{\partial \Psi}{\partial \mathbf{C}} \right) = 2 \ \frac{\partial^2 \Psi}{\partial \mathbf{C} \otimes \partial \mathbb{H}} \\
&= -\frac{\mu_0 \mu_r}{2} \ J \ \left[\mathbf{C}^{-1} \otimes \frac{\partial}{\partial \mathbb{H}} \left(\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H} \right) \right] + \mu_0 \mu_r \ J \ \frac{\partial [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}}{\partial \mathbb{H}}
\end{aligned}
$$

Side calculation 1:

$$
\begin{aligned}
\frac{\partial\left(\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}\right)}{\partial \mathbb{H}} &= \frac{\partial\left(C_{IJ}^{-1}\ H_I\ H_J\right)}{\partial H_K} \mathbf{E}_K \\
&= \left[C_{IJ}^{-1}\ \delta_{IK}\ H_J + C_{IJ}^{-1}\ H_I\ \delta_{JK}\right]\ \mathbf{E}_K \\
&= \left[C_{JI}^{-1}\ \delta_{IK}\ H_J + C_{IJ}^{-1}\ \delta_{JK}\ H_I\right]\ \mathbf{E}_K \\
&= \left[C_{JK}^{-1}\ H_J + C_{IK}^{-1}\ H_I\right]\ \mathbf{E}_K \\
&= \left[C_{KJ}^{-1}\ H_J + C_{KI}^{-1}\ H_I\right]\ \mathbf{E}_K \\
&= 2[\mathbf{C}^{-1} \cdot \mathbb{H}]
\end{aligned}
$$

Side calculation 2:

$$
\begin{aligned}
\frac{\partial[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}}{\partial \mathbb{H}} &= \frac{\partial\left[C_{IM}^{-1}\ H_M\ C_{JN}^{-1}\ H_N\right]}{\partial H_K}\ \mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \left[C_{IM}^{-1}\ \delta_{MK}\ C_{JN}^{-1}\ H_N + C_{IM}^{-1}\ H_M\ C_{JN}^{-1}\ \delta_{NK}\right]\ \mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \left[C_{IK}^{-1}\ C_{JN}^{-1}\ H_N + C_{IM}^{-1}\ H_M\ C_{JK}^{-1}\right]\ \mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P} &= -\mu_0\mu_r\ J\ \left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})\right] + \mu_0\mu_r\ J\ \left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1}\right] \\
&= \mu_0\mu_r\ J\ \left[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1}\right]
\end{aligned}
$$

References

[1] Peter Wriggers. *Nonlinear Finite Element Methods*. Springer-Verlag GmbH, Sept. 24, 2008. ISBN: 3540710000. URL: https://www.ebook.de/de/product/7511919/peter_wriggers_nonlinear_finite_element_methods.html.