Master thesis: documentation

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1 Constitutive relations for coupled magneto-elastic material

Neo-Hookean hyperelastic material model for coupled magneto-elastic material:

$$\boxed{\Psi = \Psi(J, \mathbf{C}, \mathbb{H}) = \frac{\mu}{2}[\mathbf{C} : \mathbf{I} - \mathbf{I} : \mathbf{I} - 2\ln J] + \frac{\lambda}{2}(\ln J)^2 - \frac{\mu_0 \mu_r}{2}[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]} \tag{1}$$

where J: Jacobian,
$\mathbf{C}$: Right Cauchy-Green deformation tensor,
$\mathbb{H}$: Applied magnetic vector field,
μ: Shear modulus,
λ: Lamé 1^{st} parameter,
μ_0: vacuum magnetic permeability,
μ_r: relative magnetic permeability of magneto-elastic material.

1.1 2^{nd} Piola-Kirchhoff stress

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial \Psi}{\partial \mathbf{C}} \\ &= 2\left[\frac{\mu}{2}\left\{\frac{\partial[\mathbf{C} : \mathbf{I}]}{\partial \mathbf{C}} - 2\frac{\partial \ln J}{\partial \mathbf{C}}\right\} + \frac{\lambda}{2}\frac{\partial(\ln J)^2}{\partial \mathbf{C}} - \frac{\mu_0\mu_r}{2}\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}}\right]\end{aligned}$$

Side calculation:

$$\begin{aligned}\frac{\partial[\mathbf{C} : \mathbf{I}]}{\partial \mathbf{C}} &= \mathbf{I} \\ \frac{\partial \ln J}{\partial \mathbf{C}} &= \frac{1}{J}\frac{\partial J}{\partial \mathbf{C}} \\ \frac{\partial(\ln J)^2}{\partial \mathbf{C}} &= 2\ \ln J\ \frac{\partial \ln J}{\partial \mathbf{C}} \\ \frac{\partial J}{\partial \mathbf{C}} &= \frac{1}{2}J\mathbf{C}^{-1} \quad \text{c.f. [see 1, page 46 Equation (3.124)]}\end{aligned}$$

$$\mathbf{S} = \mu\mathbf{I} - \mu\mathbf{C}^{-1} + \lambda \ln J \mathbf{C}^{-1} - \mu_0\mu_r\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}}$$

Side calculation: Note $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ is symmetric $\implies$ $\mathbf{C}^{-1}$ is also symmetric

$$\begin{aligned}\frac{\partial[J\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbf{C}} &= \frac{\partial[JC^{-1}_{IJ}H_I H_J]}{\partial C_{KL}}\ \mathbf{E}_K \otimes \mathbf{E}_L \\ &= C^{-1}_{IJ}\ H_I\ H_J\ \frac{\partial J}{\partial C_{KL}}\ \mathbf{E}_K \otimes \mathbf{E}_L + J\ H_I\ H_J\ \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}\ \mathbf{E}_K \otimes \mathbf{E}_L \\ &= C^{-1}_{IJ}\ H_I\ H_J\ \frac{J}{2}\ C^{-1}_{KL}\ \mathbf{E}_K \otimes \mathbf{E}_L + J\ H_I\ H_J\ \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}\ \mathbf{E}_K \otimes \mathbf{E}_L\end{aligned}$$

c.f. [see 1, page 519]:

$$\frac{\partial C^{-1}_{IJ}}{\partial C_{KL}} = -\frac{1}{2}[C^{-1}_{IK}\ C^{-1}_{LJ} + C^{-1}_{IL}\ C^{-1}_{KJ}]$$

$$
\begin{aligned}
\frac{\partial[J\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]}{\partial\mathbf{C}} &= \frac{J}{2}\ C^{-1}_{IJ}\ H_I\ H_J\ C^{-1}_{KL}\ \mathbf{E}_K\otimes\mathbf{E}_L + \frac{-J}{2}\ [C^{-1}_{IK}\ C^{-1}_{LJ}\ H_I\ H_J + C^{-1}_{IL}\ C^{-1}_{KJ}\ H_I\ H_J]\ \ \mathbf{E}_K\otimes\mathbf{E}_L \\
&= \frac{J}{2}\ C^{-1}_{IJ}\ H_I\ H_J\ C^{-1}_{KL}\ \mathbf{E}_K\otimes\mathbf{E}_L - \frac{J}{2}\ [C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{LI}\ H_I\ C^{-1}_{KJ}\ H_J]\ \ \mathbf{E}_K\otimes\mathbf{E}_L \\
&= \frac{J}{2}\ C^{-1}_{IJ}\ H_I\ H_J\ C^{-1}_{KL}\ \mathbf{E}_K\otimes\mathbf{E}_L - J[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}\ \mathbf{E}_K\otimes\mathbf{E}_L \\
&= \frac{J}{2}[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\mathbf{C}^{-1} - J[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}
\end{aligned}
$$

$$
\boxed{\mathbf{S} = \mu\mathbf{I} - [\mu-\lambda\ln J]\mathbf{C}^{-1} - \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\mathbf{C}^{-1} + \mu_0\mu_r\ J\ [(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}}
$$

1.2 4^{th} order material elasticity tensor

$$
\begin{aligned}
\mathfrak{C} &= 2\frac{\partial\mathbf{S}}{\partial\mathbf{C}} \\
&= 2\frac{\partial S_{KL}}{\partial C_{MN}}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N \\
= 2&\left\{-\mathbf{C}^{-1}\otimes\frac{\partial[\mu-\lambda\ln J]}{\partial\mathbf{C}} - [\mu-\lambda\ln J]\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}} - \frac{\mu_0\mu_r}{2}[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\ \mathbf{C}^{-1}\otimes\frac{\partial J}{\partial\mathbf{C}}\right\} \\
&+ 2\left\{-\frac{\mu_0\mu_r}{2}\ J\ \mathbf{C}^{-1}\otimes\frac{\partial[\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]}{\partial\mathbf{C}} - \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1}:\mathbb{H}\otimes\mathbb{H}]\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}}\right\} \\
&+ 2\left\{\mu_0\mu_r[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}\otimes\frac{\partial J}{\partial\mathbf{C}} + \mu_0\mu_r\ J\ \frac{\partial[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}}{\partial\mathbf{C}}\right\} \\
= 2&\left\{C^{-1}_{KL}\ \frac{\lambda}{J}\frac{\partial J}{\partial C_{MN}} - [\mu-\lambda\ln J]\left\{\frac{-1}{2}\ (C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML})\right\}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N \\
&+ 2\left\{-\frac{\mu_0\mu_r}{2}\ [C^{-1}_{IJ}\ H_I\ H_J]C^{-1}_{KL}\frac{1}{2}\ J\ C^{-1}_{MN} - \frac{\mu_0\mu_r}{2}J\ C^{-1}_{KL}\frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N \\
&+ 2\left\{-\frac{\mu_0\mu_r}{2}\ J\ [C^{-1}_{IJ}\ H_I\ H_J]\left\{\frac{-1}{2}(C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML})\right\}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N \\
&+ 2\left\{\mu_0\mu_r[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}\ \frac{1}{2}J\ C^{-1}_{MN} + \mu_0\mu_r\ J\ \frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}}\right\}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N \\
= \lambda&\ C^{-1}_{KL}\ C^{-1}_{MN} + [\mu-\lambda\ln J]\ (C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML}) - \frac{\mu_0\mu_r}{2}\ J\ [C^{-1}_{IJ}\ H_I\ H_J]C^{-1}_{KL}\ C^{-1}_{MN} \\
&- \mu_0\mu_r J\ C^{-1}_{KL}\ \frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}} + \frac{\mu_0\mu_r}{2}\ J\ [C^{-1}_{IJ}\ H_I\ H_J](C^{-1}_{KM}\ C^{-1}_{NL} + C^{-1}_{KN}\ C^{-1}_{ML}) \\
&+ \mu_0\mu_r\ J\ [C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}\ C^{-1}_{MN} + 2\mu_0\mu_r\ J\ \frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}}\ \mathbf{E}_K\otimes\mathbf{E}_L\otimes\mathbf{E}_M\otimes\mathbf{E}_N
\end{aligned}
$$

Side calculation 1:

$$
\begin{aligned}
\frac{\partial[C^{-1}_{IJ}\ H_I\ H_J]}{\partial C_{MN}} &= H_I\ H_J\ \frac{\partial C^{-1}_{IJ}}{\partial C_{MN}} \\
&= \frac{-1}{2}\ H_I\ H_J\ [C^{-1}_{IM}\ C^{-1}_{NJ} + C^{-1}_{IN}\ C^{-1}_{MJ}] \\
&= \frac{-1}{2}[C^{-1}_{MI}\ H_I\ C^{-1}_{NJ}\ H_J + C^{-1}_{NI}\ H_I\ C^{-1}_{MJ}\ H_J] \\
&= -[C^{-1}_{MI}\ H_I\ C^{-1}_{NJ}\ H_J]^{sym} \\
&= -[(\mathbf{C}^{-1}\cdot\mathbb{H})\otimes(\mathbf{C}^{-1}\cdot\mathbb{H})]^{sym}
\end{aligned}
$$

Side calculation 2:

$$
\begin{aligned}
\frac{\partial[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J]^{sym}}{\partial C_{MN}} &= \frac{1}{2}\frac{\partial\left[C^{-1}_{KI}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{LI}\ H_I\ C^{-1}_{KJ}\ H_J\right]}{\partial C_{MN}} \\
&= \frac{1}{2}\frac{\partial C^{-1}_{KI}}{\partial C_{MN}}\ H_I\ C^{-1}_{LJ}\ H_J + \frac{1}{2}C^{-1}_{KI}\ H_I\ \frac{\partial C^{-1}_{LJ}}{\partial C_{MN}}\ H_J \\
&\quad + \frac{1}{2}\frac{\partial C^{-1}_{LI}}{\partial C_{MN}}\ H_I\ C^{-1}_{KJ}\ H_J + \frac{1}{2}C^{-1}_{LI}\ H_I\ \frac{\partial C^{-1}_{KJ}}{\partial C_{MN}}\ H_J \\
&= \frac{1}{2}\frac{-1}{2}\ \left[C^{-1}_{KM}C^{-1}_{NI} + C^{-1}_{KN}C^{-1}_{MI}\right]\ H_I\ C^{-1}_{LJ}\ H_J \\
&\quad + \frac{1}{2}\frac{-1}{2}C^{-1}_{KI}\ H_I\ \left[C^{-1}_{LM}C^{-1}_{NJ} + C^{-1}_{LN}C^{-1}_{MJ}\right]\ H_J \\
&\quad + \frac{1}{2}\frac{-1}{2}\ \left[C^{-1}_{LM}C^{-1}_{NI} + C^{-1}_{LN}C^{-1}_{MI}\right]\ H_I\ C^{-1}_{KJ}\ H_J \\
&\quad + \frac{1}{2}\frac{-1}{2}C^{-1}_{LI}\ H_I\ \left[C^{-1}_{KM}C^{-1}_{NJ} + C^{-1}_{KN}C^{-1}_{MJ}\right]\ H_J \\
&= -\frac{1}{4}\ \left[C^{-1}_{KM}\ C^{-1}_{NI}\ H_I\ C^{-1}_{LJ}\ H_J + C^{-1}_{KN}\ C^{-1}_{MI}\ H_I\ C^{-1}_{LJ}\ H_J\right] \\
&\quad - \frac{1}{4}\ \left[C^{-1}_{KI}\ H_I\ C^{-1}_{LM}\ C^{-1}_{NJ}\ H_J + C^{-1}_{KI}\ H_I\ C^{-1}_{LN}\ C^{-1}_{MJ}\ H_J\right] \\
&\quad - \frac{1}{4}\ \left[C^{-1}_{LM}\ C^{-1}_{NI}\ H_I\ C^{-1}_{KJ}\ H_J + C^{-1}_{LN}\ C^{-1}_{MI}\ H_I\ C^{-1}_{KJ}\ H_J\right] \\
&\quad - \frac{1}{4}\ \left[C^{-1}_{LI}\ H_I\ C^{-1}_{KM}\ C^{-1}_{NJ}\ H_J + C^{-1}_{LI}\ H_I\ C^{-1}_{KN}\ C^{-1}_{MJ}\ H_J\right] \\
&= -\mathbb{X} - \mathbb{Y}
\end{aligned}
$$

The tensors

$$
\mathbb{X} := \frac{1}{4}\left[C^{-1}_{KI}H_IC^{-1}_{LM}C^{-1}_{NJ}H_J + C^{-1}_{KI}H_IC^{-1}_{LN}C^{-1}_{MJ}H_J + C^{-1}_{LI}H_IC^{-1}_{KM}C^{-1}_{NJ}H_J + C^{-1}_{LI}H_IC^{-1}_{KN}C^{-1}_{MJ}H_J\right]
$$
$$
\mathbb{Y} := \frac{1}{4}\left[C^{-1}_{KM}C^{-1}_{NI}H_IC^{-1}_{LJ}H_J + C^{-1}_{KN}C^{-1}_{MI}H_IC^{-1}_{LJ}H_J + C^{-1}_{LM}C^{-1}_{NI}H_IC^{-1}_{KJ}H_J + C^{-1}_{LN}C^{-1}_{MI}H_IC^{-1}_{KJ}H_J\right]
$$

are both symmetric rank-4 tensors such that for given symmetric rank-2 tensors $\mathbf{M}, \mathbf{N}, \mathbf{P}, \mathbf{Q}$, we have:

$$
\begin{aligned}
\mathbf{N} &= \mathbb{X} : \mathbf{M}, \\
\mathbf{Q} &= \mathbb{Y} : \mathbf{P}.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C} = &\lambda\ \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2[\mu - \lambda \ln J]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&- \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}](\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}) + \mu_0\mu_r\ J\ (\mathbf{C}^{-1} \otimes [(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}) \\
&- \mu_0\mu_r\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} + \mu_0\mu_r\ J\ ([(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym} \otimes \mathbf{C}^{-1}) \\
&- 2\mu_0\mu_r\ J\ (\mathbb{X} + \mathbb{Y})
\end{aligned}
$$

For given symmetric rank 2 tensors $\mathbf{A}$ and $\mathbf{B}$, we know: $\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}$.

$$
\boxed{
\begin{aligned}
\mathfrak{C} = &\lambda\ \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2[\mu - \lambda \ln J]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&- \frac{\mu_0\mu_r}{2}\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}](\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}) - \mu_0\mu_r\ J\ [\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]\ \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \\
&+ 2\mu_0\mu_r\ J\ (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym} \otimes \mathbf{C}^{-1}) - 2\mu_0\mu_r\ J\ (\mathbb{X} + \mathbb{Y})
\end{aligned}
}
$$

1.3 Magnetic induction vector

$$
\begin{aligned}
\mathbb{B} &= -\frac{\partial \Psi}{\partial \mathbb{H}} \\
&= \frac{\mu_0\mu_r}{2} \frac{\partial [J\ \mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}]}{\partial \mathbb{H}} \\
&= \frac{\mu_0\mu_r}{2} \frac{\partial [J\ C_{IJ}^{-1}\ H_I\ H_J]}{\partial H_K} \mathbf{E}_K \\
&= \frac{\mu_0\mu_r}{2} \left[J\ C_{IJ}^{-1} \frac{\partial H_I}{\partial H_K}\ H_J + J\ C_{IJ}^{-1}\ H_I \frac{\partial H_J}{\partial H_K} \right] \mathbf{E}_K \\
&= \frac{\mu_0\mu_r}{2} \left[J\ C_{IJ}^{-1}\ \delta_{IK}\ H_J + J\ C_{IJ}^{-1}\ H_I\ \delta_{JK} \right] \mathbf{E}_K \\
&= \frac{\mu_0\mu_r}{2} \left[J\ C_{JI}^{-1}\ \delta_{IK}\ H_J + J\ C_{IJ}^{-1}\ \delta_{JK}\ H_I \right] \mathbf{E}_K \\
&= \frac{\mu_0\mu_r}{2} \left[J\ C_{JK}^{-1}\ H_J + J\ C_{IK}^{-1}\ H_I \right] \mathbf{E}_K \\
&= \frac{\mu_0\mu_r}{2} \left[J\ C_{KJ}^{-1}\ H_J + J\ C_{KI}^{-1}\ H_I \right] \mathbf{E}_K
\end{aligned}
$$

$$
\boxed{\mathbb{B} = \mu_0\mu_r\ J\ [\mathbf{C}^{-1} \cdot \mathbb{H}]}
$$

1.4 Magnetic tensor

$$
\begin{aligned}
\mathbf{D} &= \frac{\partial \mathbb{B}}{\partial \mathbb{H}} \\
&= \frac{\partial [\mu_0\mu_r\ J\ (\mathbf{C}^{-1} \cdot \mathbb{H})]}{\partial \mathbb{H}} \\
&= \mu_0\mu_r\ J\ \frac{\partial (C_{IJ}^{-1}\ H_J)}{\partial H_K}\ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0\mu_r\ J\ C_{IJ}^{-1}\ \frac{\partial H_J}{\partial H_K}\ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0\mu_r\ J\ C_{IJ}^{-1}\ \delta_{JK}\ \mathbf{E}_I \otimes \mathbf{E}_K \\
&= \mu_0\mu_r\ J\ C_{IK}^{-1}\ \mathbf{E}_I \otimes \mathbf{E}_K
\end{aligned}
$$

$$\boxed{\mathbf{D} = \mu_0\mu_r \; J \; \mathbf{C}^{-1}}$$

1.5 Magneto-elastic coupling tensor

This quantity defines the sensitivity of 2^{nd} Piola-Kirchhoff stress $\mathbf{S}$ w.r.t. the applied magnetic field $\mathbb{H}$.

$$\begin{aligned}
\mathbb{P} = -\frac{\partial \mathbf{S}}{\partial \mathbb{H}} &= -\frac{\partial}{\partial \mathbb{H}}\left(2\frac{\partial \Psi}{\partial \mathbf{C}}\right) = -2\;\frac{\partial^2 \Psi}{\partial \mathbf{C} \otimes \partial \mathbb{H}} \\
&= \frac{\mu_0\mu_r}{2}\; J \;\left[\mathbf{C}^{-1} \otimes \frac{\partial}{\partial \mathbb{H}}\left(\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}\right)\right] - \mu_0\mu_r \; J \; \frac{\partial[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}}{\partial \mathbb{H}}
\end{aligned}$$

Side calculation 1:

$$\begin{aligned}
\frac{\partial\left(\mathbf{C}^{-1} : \mathbb{H} \otimes \mathbb{H}\right)}{\partial \mathbb{H}} &= \frac{\partial\left(C^{-1}_{IJ}\; H_I\; H_J\right)}{\partial H_K}\mathbf{E}_K \\
&= \left[C^{-1}_{IJ}\;\delta_{IK}\;H_J + C^{-1}_{IJ}\;H_I\;\delta_{JK}\right]\;\mathbf{E}_K \\
&= \left[C^{-1}_{JI}\;\delta_{IK}\;H_J + C^{-1}_{IJ}\;\delta_{JK}\;H_I\right]\;\mathbf{E}_K \\
&= \left[C^{-1}_{JK}\;H_J + C^{-1}_{IK}\;H_I\right]\;\mathbf{E}_K \\
&= \left[C^{-1}_{KJ}\;H_J + C^{-1}_{KI}\;H_I\right]\;\mathbf{E}_K \\
&= 2[\mathbf{C}^{-1} \cdot \mathbb{H}]
\end{aligned}$$

Side calculation 2:

$$\begin{aligned}
\frac{\partial[(\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})]^{sym}}{\partial \mathbb{H}} &= \frac{1}{2}\frac{\partial\left[C^{-1}_{IM}\;H_M\;C^{-1}_{JN}\;H_N + C^{-1}_{JM}\;H_M\;C^{-1}_{IN}\;H_N\right]}{\partial H_K}\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \frac{1}{2}\left[C^{-1}_{IM}\;\delta_{MK}\;C^{-1}_{JN}\;H_N + C^{-1}_{IM}\;H_M\;C^{-1}_{JN}\;\delta_{NK}\right]\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&\quad + \frac{1}{2}\left[C^{-1}_{JM}\;\delta_{MK}\;C^{-1}_{IN}\;H_N + C^{-1}_{JM}\;H_M\;C^{-1}_{IN}\;\delta_{NK}\right]\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \frac{1}{2}\left[C^{-1}_{IK}\;C^{-1}_{JN}\;H_N + C^{-1}_{IM}\;H_M\;C^{-1}_{JK}\right]\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&\quad + \frac{1}{2}\left[C^{-1}_{JK}\;C^{-1}_{IN}\;H_N + C^{-1}_{JM}\;H_M\;C^{-1}_{IK}\right]\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K
\end{aligned}$$

Observing the symmetry over the indices I and J.

$$\begin{aligned}
&= \left[C^{-1}_{IK}\;C^{-1}_{JN}\;H_N + C^{-1}_{IM}\;H_M\;C^{-1}_{JK}\right]\;\mathbf{E}_I \otimes \mathbf{E}_J \otimes \mathbf{E}_K \\
&= \mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1}
\end{aligned}$$

$$\boxed{\mathbb{P} = \mu_0\mu_r \; J \;\left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H})\right] - \mu_0\mu_r \; J \;\left[\mathbf{C}^{-1} \otimes (\mathbf{C}^{-1} \cdot \mathbb{H}) + (\mathbf{C}^{-1} \cdot \mathbb{H}) \otimes \mathbf{C}^{-1}\right]}$$

References

[1] Peter Wriggers. *Nonlinear Finite Element Methods*. Springer-Verlag GmbH, Sept. 24, 2008. ISBN: 3540710000. URL: https://www.ebook.de/de/product/7511919/peter_wriggers_nonlinear_finite_element_methods.html.