18-May

Mean of random variable

- Mean of random variable, generally called expected value
- It is the average outcome of random process that is repeated many times
- Technically, it's a weighted average of possible outcomes of random variable where each outcome is weighted by it's probability of occurence.

Ex
- Rolling a dice
- Random variable(X) { 1, 2, 3, 4, 5, 6 }
- Probability distribution of X =

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- Now if you roll a dice multiple times
- If you roll dice multiple times each time dice would have outcome let's say

Ist time = 3
II time - 6
III time - 3
IV " = 1
V = 4
⋮
Multiple times (1000)

$\Rightarrow$ Mean of it = $\frac{3+6+3+1+4\cdots\cdots}{1000}$ is mean of random variable

- Suppose these are the outcomes of rolling a dice 6 times ⇒ 5, 3, 4, 5, 3, 3

Mean $= \dfrac{5+3+4+5+3+3}{6} = \dfrac{23}{6}$

Giving new perspective to mean

$= \dfrac{2(5) + 3(3) + 1(4)}{6}$

$= \boxed{\frac{1}{3}(5) + \frac{1}{2}(3) + \frac{1}{6}(4)}$

⇒ If you notice, the above is : Sum of Probability of outcome × outcome is mean of random variable

⇒ 5, 3, 5, 4, 3, 3

⇒ Probability of 5 $= \frac{2}{6} = \frac{1}{3}$

⇒ " " 3 $= \frac{3}{6} = \frac{1}{2}$

⇒ " " 4 $= \frac{1}{6}$

~~Conclusion: Expected value is~~

- In the previous example, you rolled a dice

X	1	2	3	4	5	6
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- In expected value, you take each possible outcome and multiply with

with its probability

E.V / $\Rightarrow \left(\frac{1}{6} \times 1\right) + \left(\frac{1}{6} \times 2\right) + \left(\frac{1}{6} \times 3\right) + \left(\frac{1}{6} \times 4\right) + \left(\frac{1}{6} \times 5\right) + \left(\frac{1}{6} \times 6\right)$

Mean of random variable

$\Rightarrow \boxed{3.5}$

Interpretation

It means if you perform the rolling a dice multiple times then the average value / mean of random variable will be 3.5

- $E[X] = \sum_{i=1}^{n} x_i P(x_i)$
- where n unique values for X

Variance of random variable

Ex: Rolling dice

- X 1 2 3 4 5 6

$E(X) = 3.5$

By normal mean we find expected value similarly we will find variance of random variable using normal variance formula

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Normal variance formula

$$= \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \Rightarrow \text{Average.}$$

Variance of random variable

X	1	2	3	4	5	6

$$\Rightarrow \sum_{i=1}^{n} (X - E[X])^2$$

X is value that it can take

E[X] ⇒ mean of random variable.

⇒ Now you to find average

$$\Rightarrow 1 — \boxed{E[(X - E[X])^2]} \Rightarrow \text{Variance of random variable}$$

formula

You can also expand :-

$$\Rightarrow E[(X^2 + (E[X])^2 - 2XE[X])] \quad (\because E[X+Y] = E[X] + E[Y])$$

$$\Rightarrow E[X^2] + E[(E[X])^2] - E[2XE[X]] \quad (\because E[XY] = E[X]\,E[Y] \text{ for X \& Y being independent})$$

$$\Rightarrow E[X^2] + E[(E[X])^2] - E[2]\,E[X]\,E[E[X]] \quad (\because E[C] = C)$$

$$\Rightarrow E[X^2] + E[(E[X])^2] - 2E[X]\,E[X]$$

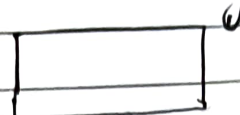
$$\Rightarrow E[X^2] + (E[X])^2 - 2(E[X])^2$$

$$\Rightarrow \boxed{E[X^2] - (E[X])^2} — 2 \text{ formula for variance}$$

E[X] → some constant

E[E[X]] = ∴ E[X]

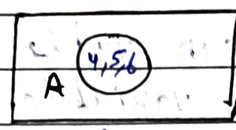
Ven diagrams in Probability

Rectangle $\Rightarrow$ 

- represents sample space $\Rightarrow P(U) = 1$

Event :

Ex $\rightarrow$ Experiment : Rolling a dice
$\rightarrow$ Event A $\rightarrow$ getting a num ≥ 4



$\rightarrow P(A)' = P(U) - P(A)$

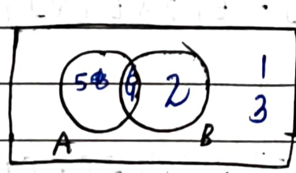
$P(A)' = 1 - \frac{1}{2}$

$= \frac{1}{2}$

2 events

A $\rightarrow$ num ≥ 4 $\Rightarrow \{4, 5, 6\}$
B $\rightarrow$ even num $\{2, 4, 6\}$

$\{1, 2, 3, 4, 5, 6\}$



$\Rightarrow P(A \cap B) = \{6, 4\} \Rightarrow \frac{2}{6} = \frac{1}{3}$

$\Rightarrow P(A) = \{5, 6, 4\} \Rightarrow \frac{3}{6} = \frac{1}{2}$

$\Rightarrow P(B) = \{2, 4, 6\} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow P(U) = 1$

$\Rightarrow P(A \cup B) = \{5, 6, 4, 2\} = \frac{4}{6} = \frac{2}{3}$

Contigency Table in Probability

- experiment → rolling a dice
- Event A → ≥ 4 → X
- Event B → even number → Y

- You will take 2 random variables, one for event A, and one for B (X and Y)

	even	not even
≥4	2 (4,6)	1 (5)
<4	1 (2)	2 (1,3)

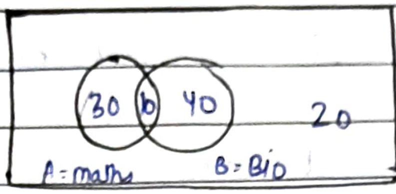
When Useful

Ex: • Class of 100 students
- 40 → only Bio
- 30 → only maths
- 10 → both

Contengincy table →

	math	not math
bio	10	40
not bio	30	20

We can easily convert into Venn diagram



- same representation

TYPES OF PROBABILITY

① Joint Probability

- Let's say you have 2 random variables X and Y.
- The joint probability of X and Y, denoted as $P(X = x, Y = y)$ is the probability that X takes the value x, and Y takes the value y at same time.

- In short, you can say probability of 2 events at same time

Ex :- Let's say we have 2 random variables. X and Y

- X represents P class $\{1, 2, 3\}$
- x can be any, 1, 2, 3 depending on passenger
- Y represents survival $\{0, 1\}$

contigency table

Pclass / Survived	1	2	3
0	80	97	372
1	136	87	119

- joint probability is probab of 2 events at same time.

- Ex: $P(X=1, Y=0) = \frac{80}{891} = 0.089$

  - This represents the probability of passenger in P class 1, and simultaneously he is dead (simply divide contengency by total) you'll get probabilities

Pclass / Survived	1	2	3
0	0.089	0.10	0.41
1	0.15	0.09	0.13

(Joint probability) distribution

concept of Joint Probability distribution

- When you find joint probability of one possible outcome is called joint probability. ~~distribution~~

- When you find joint probabity of every possible outcome is called joint probability distribution

(2) Marginal probability / Simple / unconditional

- Probability of an event occuring irrespective of outcome of some other event

Ex Same example, we have events A and B. A is random variable that represents Pclass, and B is random variable that represents survival

Marginal probabilities are separate probabilities

Ex: We have same contengency table

Pclass / Survived	1	2	3	
0	80	97	372	549
1	136	87	119	342
	216	184	491	891

Marginal prob

I. X represents Pclass

- $P(X=1) = \frac{80+136}{891} = \frac{216}{891}$

- It tells the probability of person being in Pclass 1 irrespective of he survived or not

- Same goes for $P(X=2)$, & $P(X=3)$

II
- Y represents survived {0, 1}
- $P(Y=0) = \frac{549}{891}$
- It represents prob of person not survived irrespective of in Pclass he was.
- Same goes for $P(Y=1)$

③ Conditional Probability

- measure of probability of an event occuring, given that another event already occured
- In this, both events are not happening simultaneously just like Joint prob.
- In this; let's say we have 2 events A and B, where B has already occured, due to this what is the effect on probability of A
- Represent = $P(A|B)$, B has already occured now what is the probability of A

Q] 3 unbiased coins are tossed. What is conditional probability that atleast two coins show heads, given that atleast one coin shows head

Ans Write sample space for these ques.

{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

- Event A → atleast 2 heads
- Event B → atleast 1 head

where B already has occured

- Since B is already occured, we will reduce the sample space acc to B.

- B says it requires atleast 1 head therefore remove those who don't have even single head (TTT)

- New sample space

= { HHH, HHT, HTH, THH, HTT, THT, TTH }

- Now A event is Prob of atleast 2 heads, search in new sample space

$$= \boxed{P(A|B) = \frac{4}{7}}$$

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Q2 Two fair six-sided dice are rolled. What is the conditional probability that sum of numbers rolled is 7 given that first die shows odd no.

A→

Ans. Write sample space:

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

- Event A → sum of numbers is 7
- " B → first die shows odd no
- Since B event already has occurred ie first die shows odd no.
- Therefore, remove all those in which first die is even

New sample space

$$= \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \end{matrix}\right\}$$

→ find prob of A in new sample space
→ A → sum of no = 7

$$\boxed{P(A|B) = \frac{3}{18} = \frac{1}{6}}$$

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- Direct formula for calculating conditional probability

$= P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\longrightarrow$ Joint prob / $\longrightarrow$ Marginal prob of B

Q

X

	Pclass	1	2	3
Y	Survived			
	0	80	97	372
	1	136	87	119

$\leftarrow$ given

- What is the probability of ~~X~~ Y=0 given X = 3

Sol • ~~P(X=~~ $\Rightarrow P(Y=0 \mid X=3)$

$\Rightarrow = \frac{P(Y \cap X)}{P(X)}$

$= \frac{P(Y=0 \cap X=3)}{P(X=3)}$

- ~~the~~ find joint probability

$\Rightarrow P(Y=0 \cap X=3) = \frac{372}{891} = 0.41$

$\Rightarrow P(X=3) = \frac{372+119}{891} = 0.54$

$P(Y|A) = 0.75$

That means, if you are travelling in

P class 3, then there is a 0.75 chance you will die

Refer to jupyter notebook to see / directly finding using python code

Different types of events:

① Independent events: where occurence of one event doesn't effect the other.

Ex: ① Flipping a coin & rolling a die
② Drawing a card, with replacement

② Conditional Probability in independent:

- $P(A|B) = P(A)$
- because for independent event :- $P(A \cap B) = P(A) \cdot P(B)$

② Dependent events: where occurence of one event affect the others

- Conditional Prob in dependent :-
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Ex: Drawing a card with replacement

(3) Mutually exclusive events

- events that can't occur at same time
- If one event occurs, the other cannot

Ex: Tossing a coin and getting both H and T is not possible, hence this is mutually exclusive

- In mutually exclusive events :-

$\Rightarrow P(A \cap B) = 0$

$\Rightarrow \therefore P(A|B) = 0 \qquad \left(\because P(A|B) = \frac{P(A \cap B)}{P(B)} = 0 \right)$

BAYES THEORM

- respectful :
-

$$P(A|B) = \underset{\text{prior}}{P(A)} \times \frac{P(B|A) \rightarrow \text{likelihood}}{P(B) \rightarrow \text{marginal / evidence}}$$

Mathematical Proof

$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$ —— 1

$\Rightarrow P(B|A) = \frac{P(B \cap A)}{P(B)} \qquad (\because P(A \cap B) = P(B \cap A))$

$\Rightarrow P(B \cap A) = P(A \cap B) = P(B|A) \, P(A)$ —— 2

put 2 in 1

$$\Rightarrow \boxed{P(A|B) = \frac{P(B|A)\ P(A)}{P(B)}}, \text{ Hence proved}$$

* Idea of how it is used in Machine learning

- Data ⇒

gender	survived
M	0
M	0
F	1
F	0
M	1

- You are required to tell, a new query → M will die or not ⇒ {0, 1}
- The question can be framed as find the Probability of this man dying or survive given that he is male
- that means we are required to find → P(0|M) for dying

  OR

  P(1|M) for survive
- you can find it using Bayes Theorm

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

$$P(0|M) = \frac{P(M|0)\ P(0)}{P(M)}$$ → These all values can be extracted from dataset

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$\Rightarrow P(O) \rightarrow$ Prob of dying

$\Rightarrow P(O) = \frac{3}{5}$ (marginal)

$\Rightarrow P(M) = \frac{3}{5}$ (marginal)

$\Rightarrow P(M|O) \rightarrow$ find prob of Male given, person is dead.

$\rightarrow P(M|O) = \frac{2}{3}$

put all the values in, $P(O|M)$

$$= \frac{P(M|O) \times P(O)}{P(M)}$$

$$= \frac{\frac{2}{3} \times \frac{3}{5}}{\frac{3}{5}} = \frac{\frac{\cancel{6}^{2}}{\cancel{15}_{3}}}{\frac{\cancel{3}}{\cancel{5}}} = \frac{2}{3}$$

$\boxed{P(O|M) = \frac{2}{3}}$ we Basically find probability of ~~dead~~ person will die, given he is male

Similarly we will find probability of person ~~being~~ will ~~...~~ survive given that he is men

Basically this value $\Rightarrow P(1|M)$?

$$P(1|M) = \frac{P(M|1)\,P(1)}{P(M)} \quad \text{— } ①$$

$\Rightarrow P(1) = \frac{2}{5}$ (marginal)

$\Rightarrow P(M) = \frac{3}{5}$ (marginal)

$\Rightarrow P(M|1) \Rightarrow$ Prob of men, given that he ~~is~~ ~~dead~~ survived

$\Rightarrow P(M|1) = \frac{1}{2}$

put these values in ①

$$= \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{3}{5}} = \frac{\frac{2}{10}}{\frac{3}{5}} = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow \boxed{P(1|M) = \frac{1}{3}}$$

Prob of dying, given he is man $\Rightarrow P(0|m) = \frac{2}{3}$

" " survived, " " " $\Rightarrow P(1|M) = \frac{1}{3}$

Since Prob of dying is more, he will die

This is how, we use Bayes theorm