Elementary Graph Algorithms

Lecture 11

Graphs

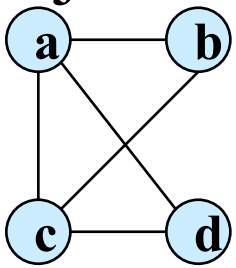
- *Graph* $G = (V, E)$
  - » V = set of vertices
  - » E = set of edges $\subseteq (V \times V)$
- Types of graphs
  - » Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - » Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - » Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - » Dense: $|E| \approx |V|^2$.
  - » Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

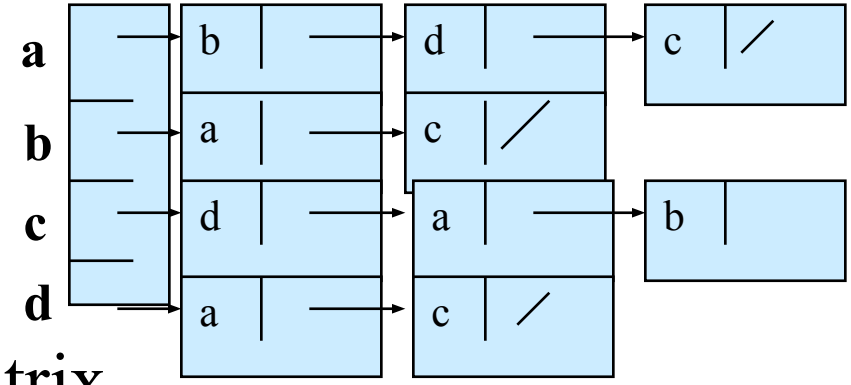
Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - » Symmetric if G is undirected.
  - » Not necessarily so if G is directed.
- If G is connected:
  - » There is a path between every pair of vertices.
  - » $|E| \geq |V| - 1$.
  - » Furthermore, if $|E| = |V| - 1$, then G is a tree.

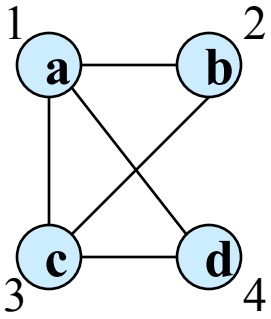
Representation of Graphs

- Two standard ways.
  - » Adjacency Lists.





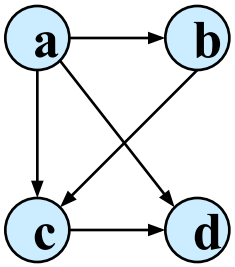
  - » Adjacency Matrix.

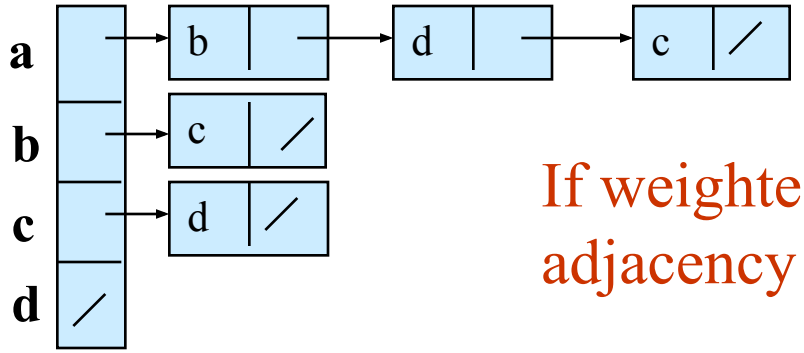


	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

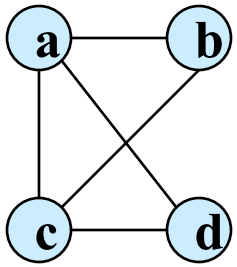
Adjacency Lists

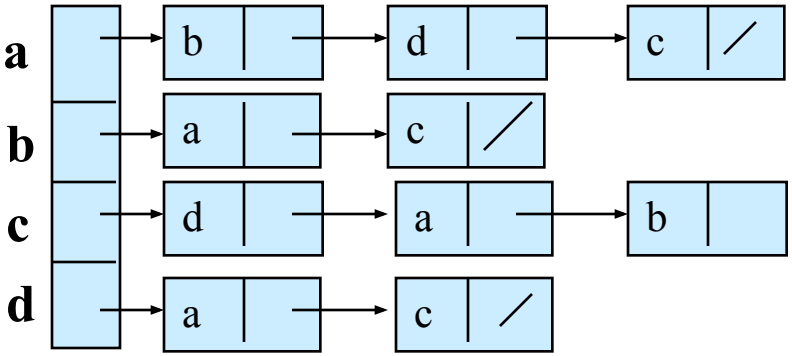
- Consists of an array Adj of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.





If weighted, store weights also in adjacency lists.





Storage Requirement

- For directed graphs:
  - » Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

No. of edges leaving v

  - » Total storage: $\Theta(V+E)$
- For undirected graphs:
  - » Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

No. of edges incident on v. Edge (u,v) is incident on vertices u and v.

  - » Total storage: $\Theta(V+E)$

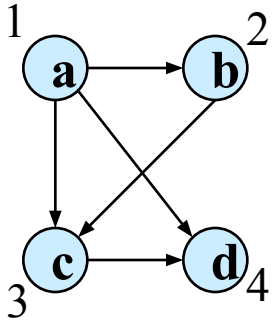
Pros and Cons: adj list

- Pros
  - » Space-efficient, when a graph is sparse.
  - » Can be modified to support many graph variants.
- Cons
  - » Determining if an edge $(u,v) \in G$ is not efficient.
    - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.
    - $\Theta(V)$ in the worst case.

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$



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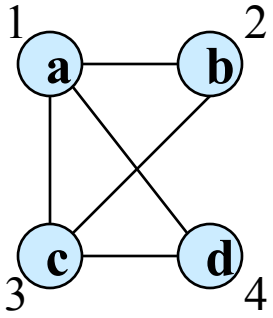


	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(V^2)$.
  - » Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Graph-searching Algorithms

- Searching a graph:
  - » Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - » Breadth-first Search (BFS).
  - » Depth-first Search (DFS).

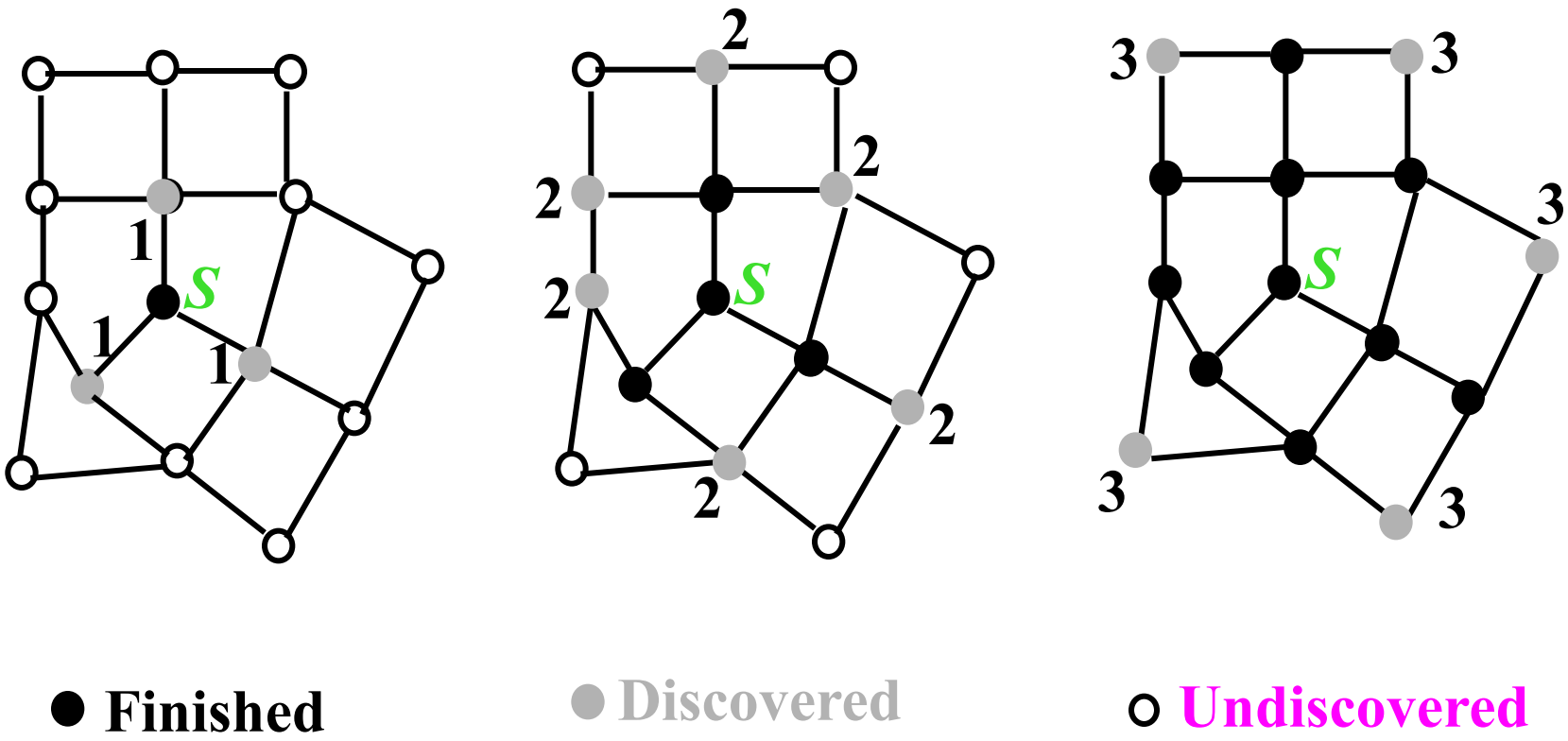
Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - » $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - » $\pi[v] = u$ such that (u, v) is last edge on shortest path s --> v.
    - u is v's predecessor.
  - » Builds breadth-first tree with root s that contains all reachable vertices.

Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - » A vertex is "discovered" the first time it is encountered during the search.
  - » A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - » White – Undiscovered.
  - » Gray – Discovered but not finished.
  - » Black – Finished.
    - Colors are required only to reason about the algorithm. Can be implemented without colors.

BFS for Shortest Paths



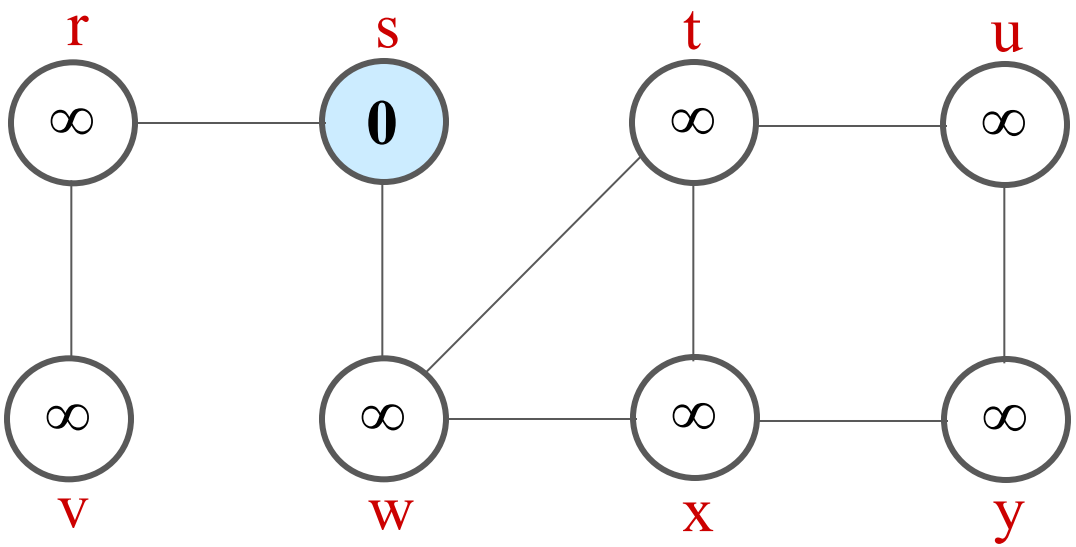
BFS(G,s)

```
1.  for each vertex u in V[G] − {s}
2.        do color[u] ← white
3.           d[u] ← ∝
4.           π[u] ← nil
5.  color[s] ← gray
6.  d[s] ← 0
7.  π[s] ← nil
8.  Q ← Φ
9.  enqueue(Q,s)
10. while Q ≠ Φ
11.       do u ← dequeue(Q)
12.          for each v in Adj[u]
13.             do if color[v] = white
14.                then color[v] ← gray
15.                     d[v] ← d[u] + 1
16.                     π[v] ← u
17.                     enqueue(Q,v)
18.          color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

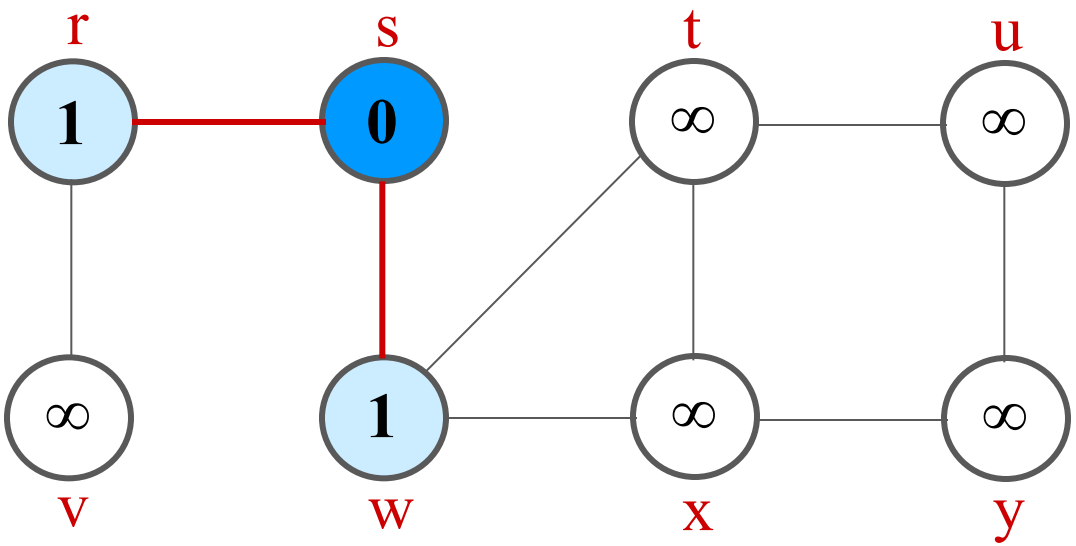
Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

Example (BFS)



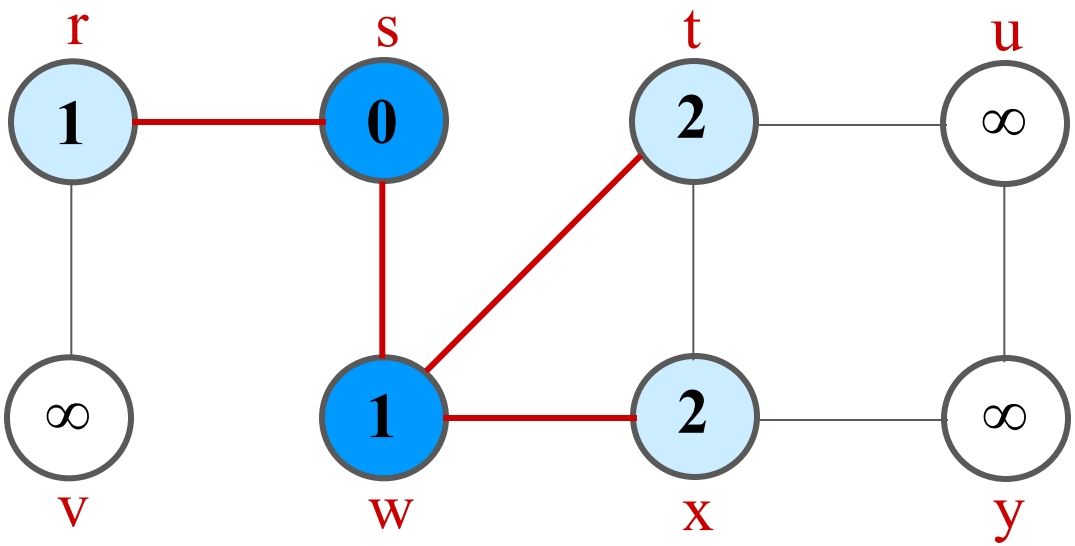
Q: s
0

Example (BFS)



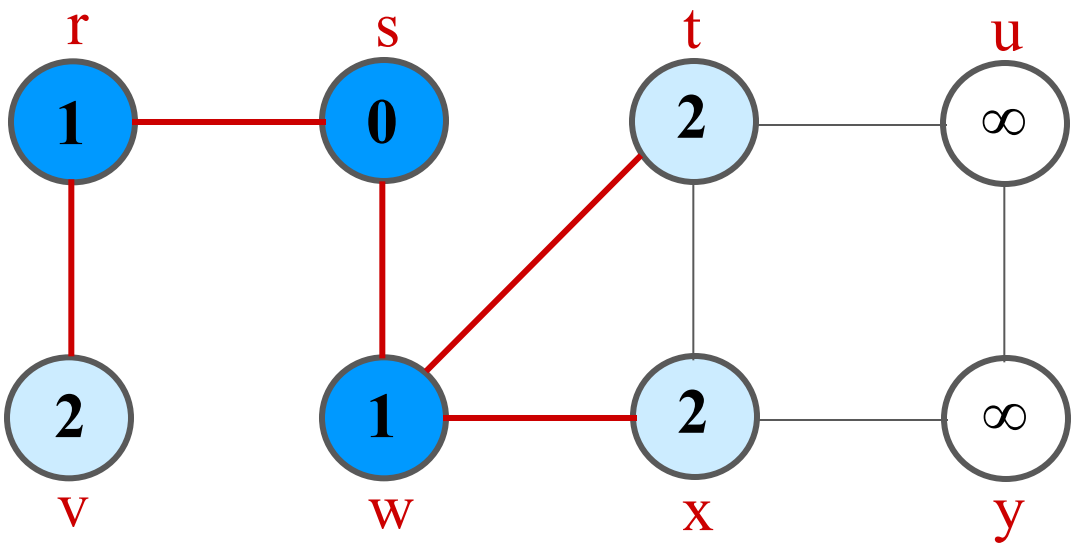
Q:	w	r
	1	1

Example (BFS)



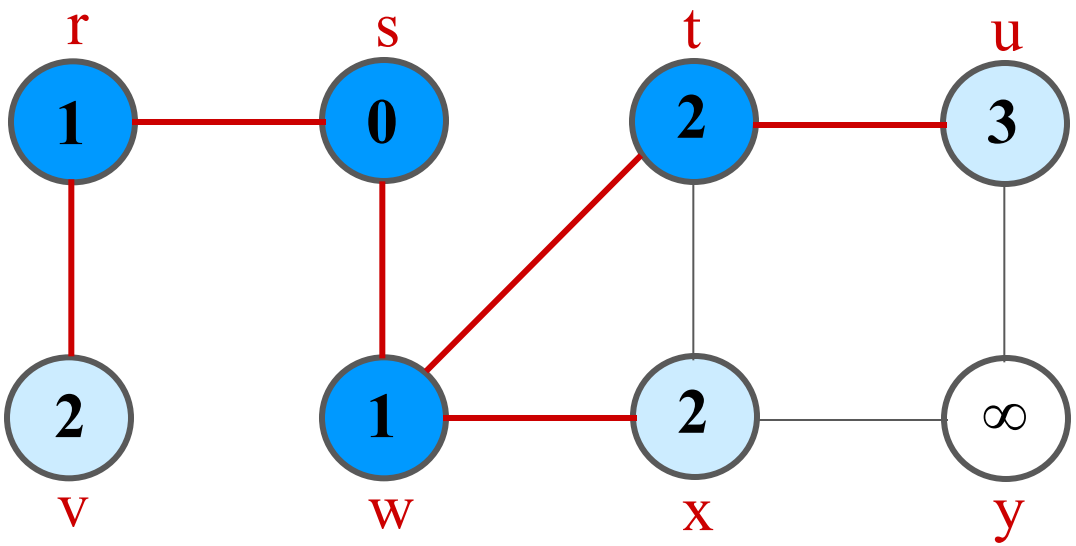
Q:	r	t	x
	1	2	2

Example (BFS)



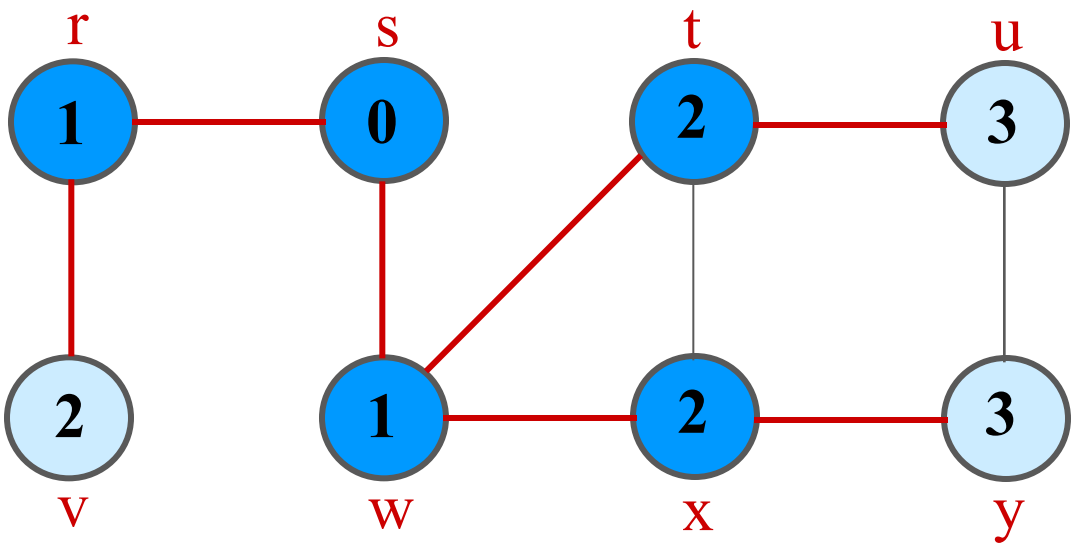
Q:	t	x	v
	2	2	2

Example (BFS)



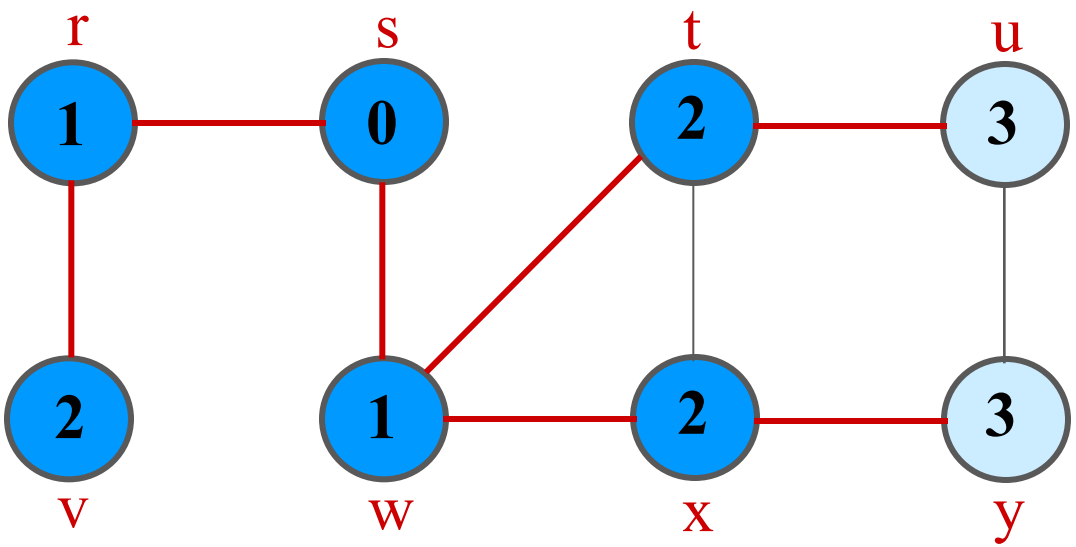
Q:	x	v	u
	2	2	3

Example (BFS)



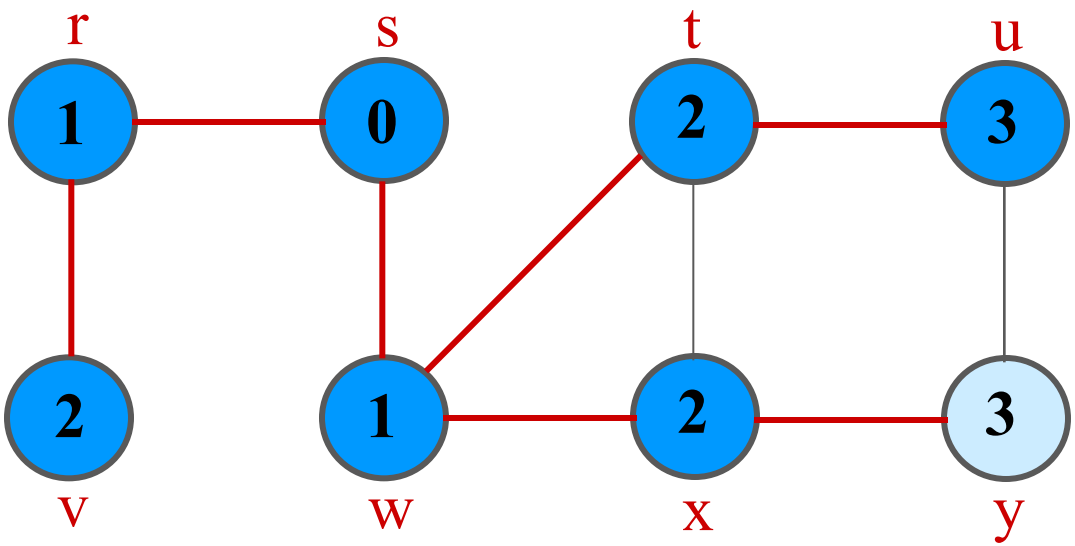
Q:	v	u	y
	2	3	3

Example (BFS)



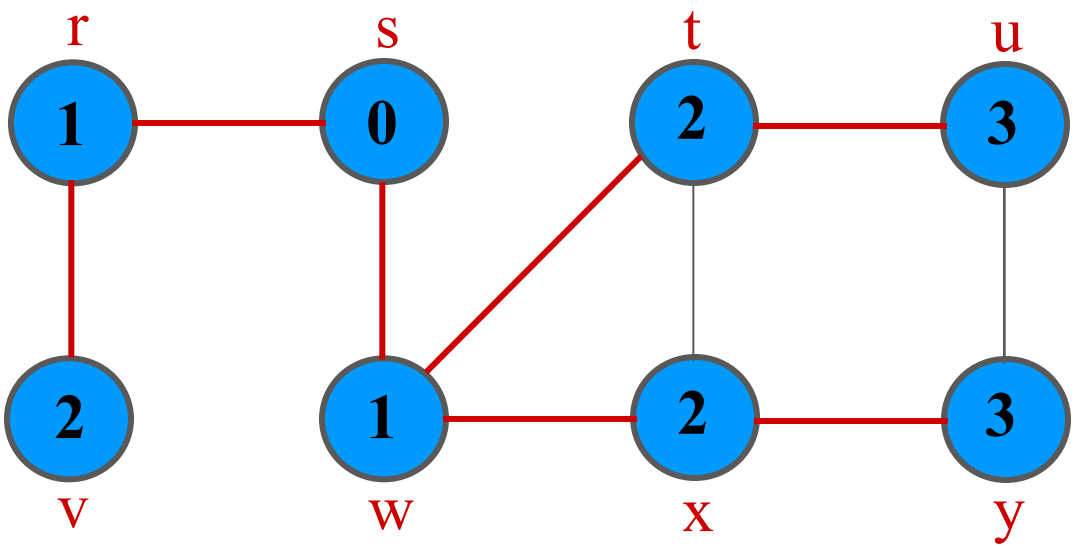
Q:	u	y
	3	3

Example (BFS)



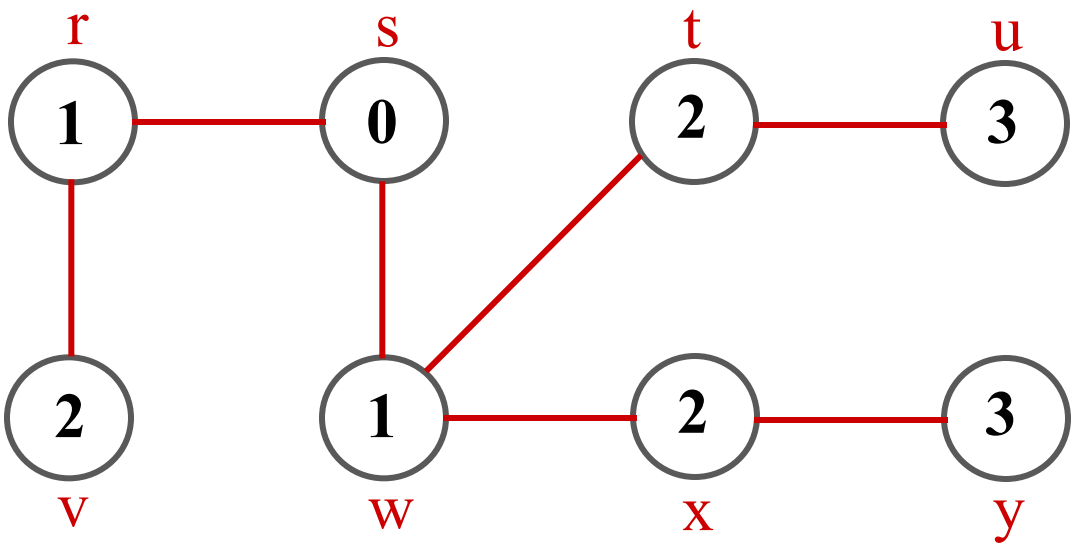
Q: y
3

Example (BFS)



Q: ∅

Example (BFS)



BF Tree

Analysis of BFS

- It takes $O(V)$ to initialize the distance and predecessor for each vertex
- Each vertex is visited at most one time, because only the first time that it is reached is its distance `null`, and so each vertex is enqueued at most one time.
- Since we examine the edges incident on a vertex only when we visit from it, each edge is examined at most twice, once for each of the vertices it's incident on.
- Thus, breadth-first search spends O(V+E) times in visiting vertices

Breadth-first Tree

- For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - » $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} \cup \{s\}$
  - » $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** if:
  - » V_π consists of the vertices reachable from s and
  - » for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**.
  $|E_\pi| = |V_\pi| - 1$.

Depth-first Search (DFS)

- Explore edges out of the most recently discovered vertex v.
- When all edges of v have been explored, backtrack to explore other edges leaving the vertex from which v was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - » 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - » $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Pseudo-code

DFS(G)

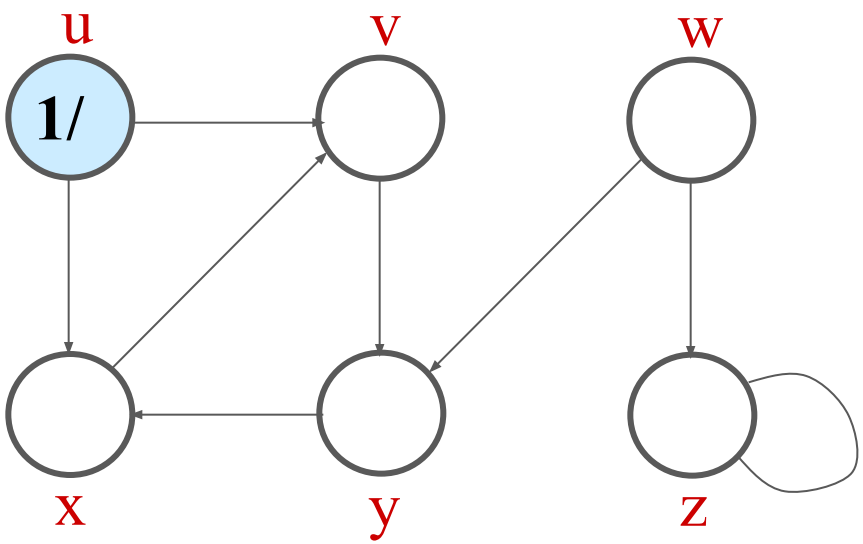
```
1. for each vertex u ∈ V[G]
2.     do color[u] ← white
3.        π[u] ← NIL
4. time ← 0
5. for each vertex u ∈ V[G]
6.     do if color[u] = white
7.        then DFS-Visit(u)
```

Uses a global timestamp ***time***.

DFS-Visit(u)

```
1.  color[u] ← GRAY   ▽ White vertex u
    has been discovered
2.  time ← time + 1
3.  d[u] ← time
4.  for each v ∈ Adj[u]
5.      do if color[v] = WHITE
6.          then π[v] ← u
7.               DFS-Visit(v)
8.  color[u] ← BLACK    ▽ Blacken u;
    it is finished.
9.  f[u] ← time ← time + 1
```

Example (DFS)



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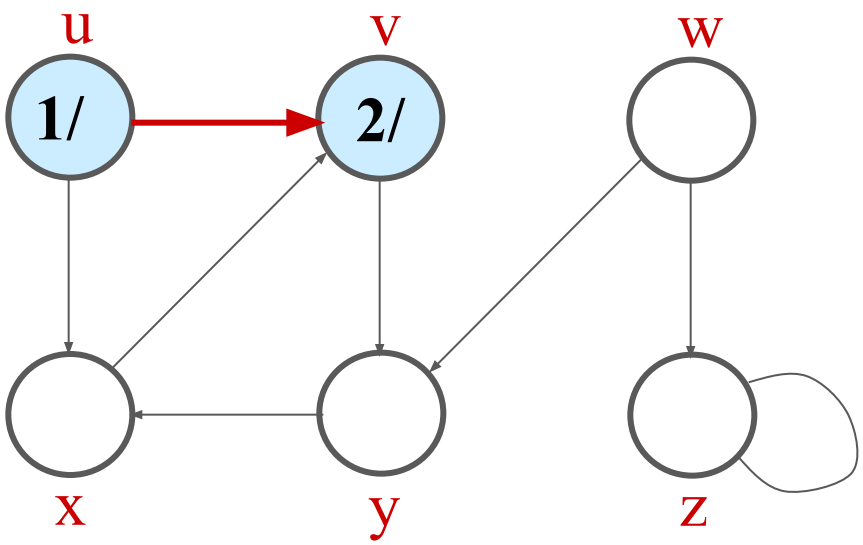

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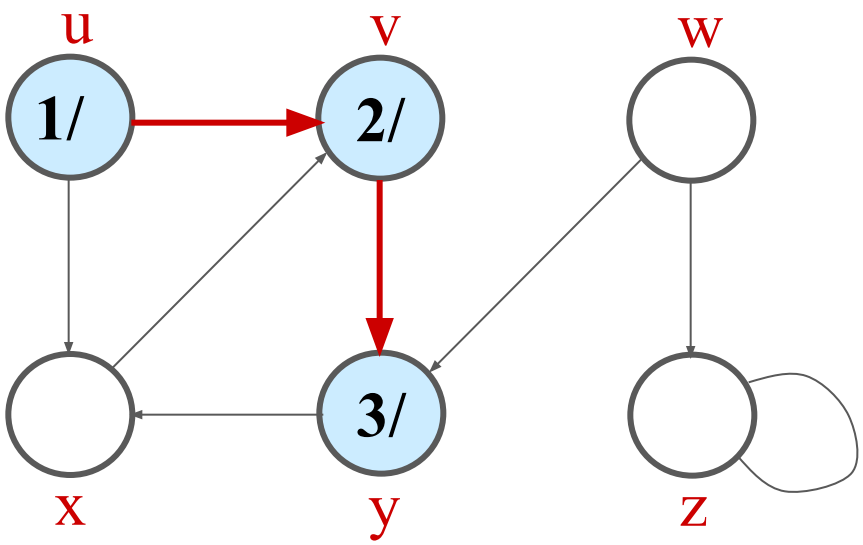

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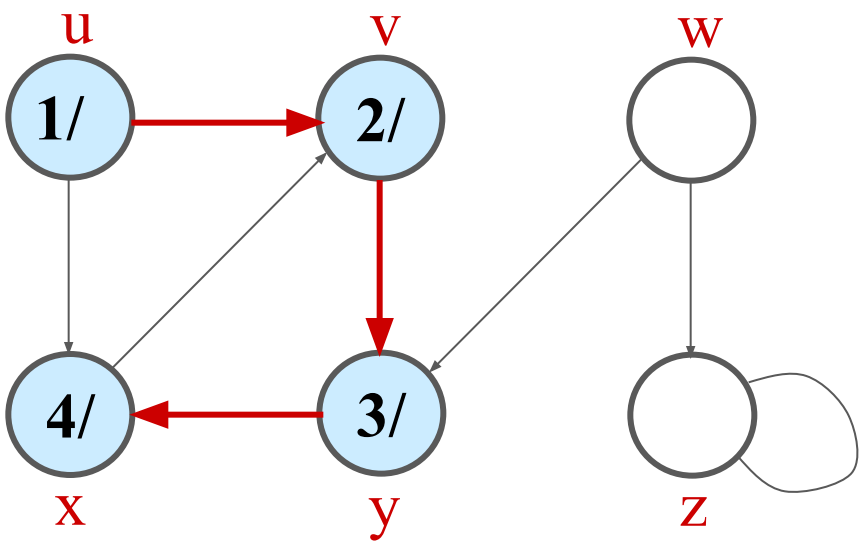

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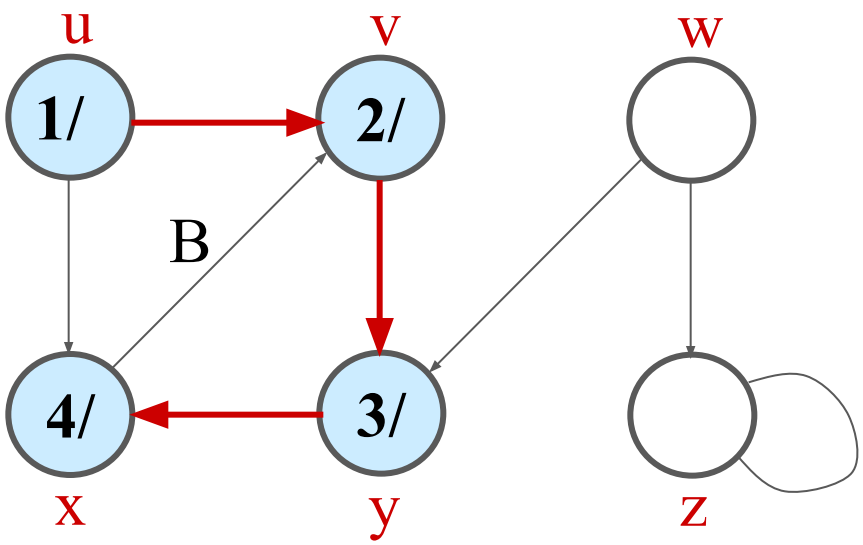

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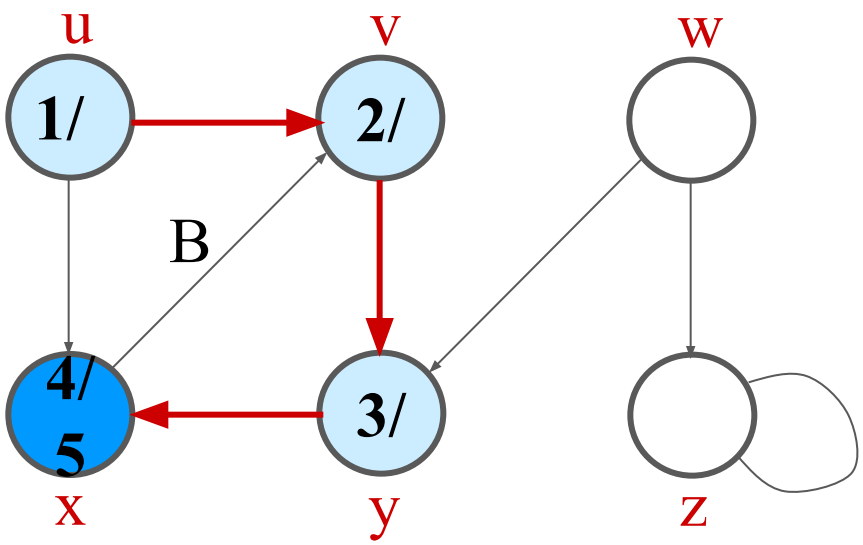

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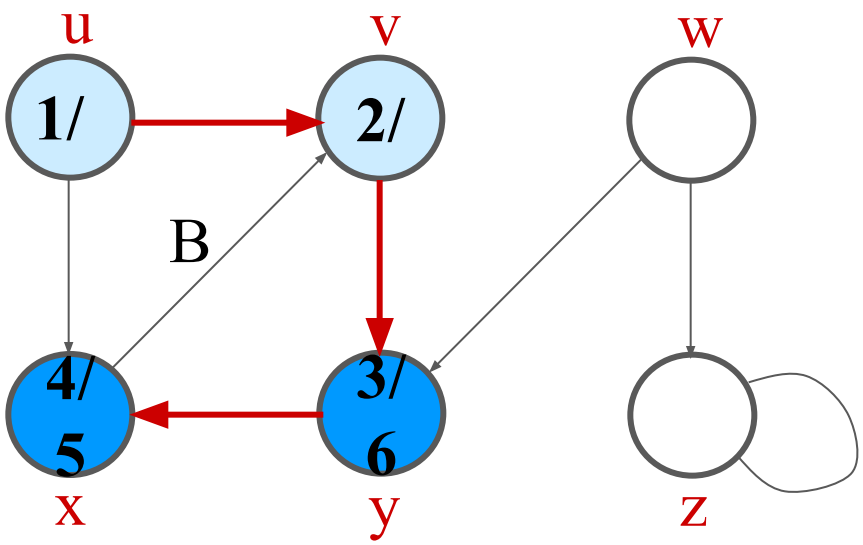

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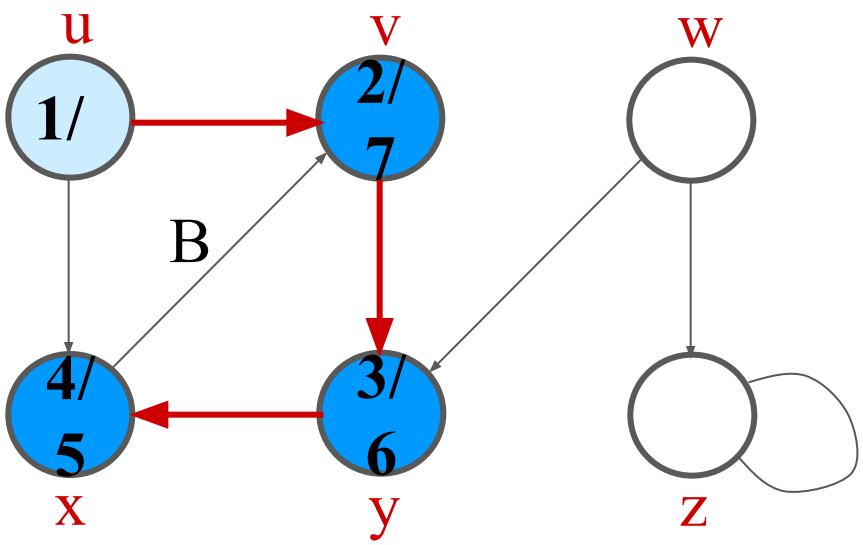

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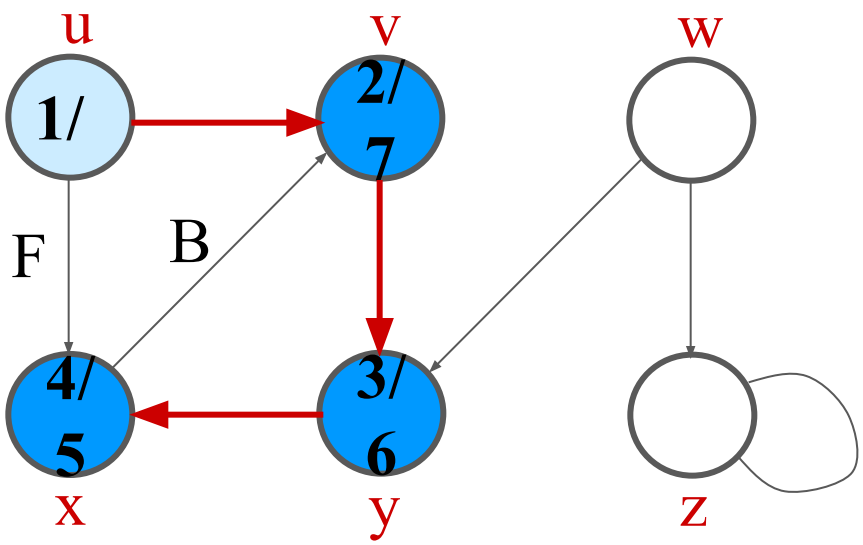

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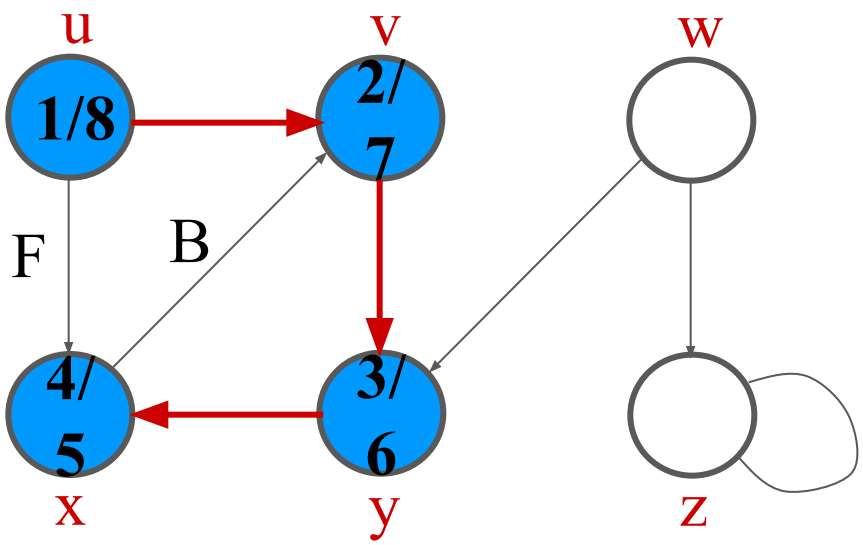

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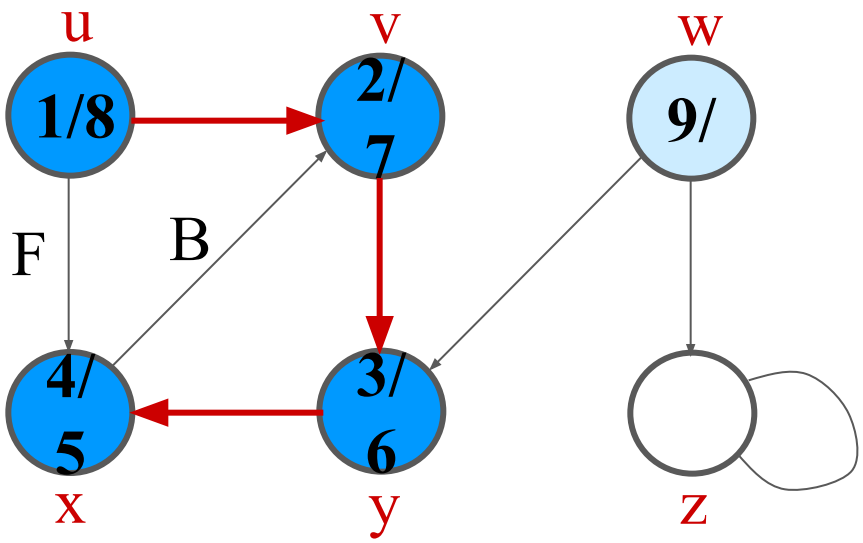

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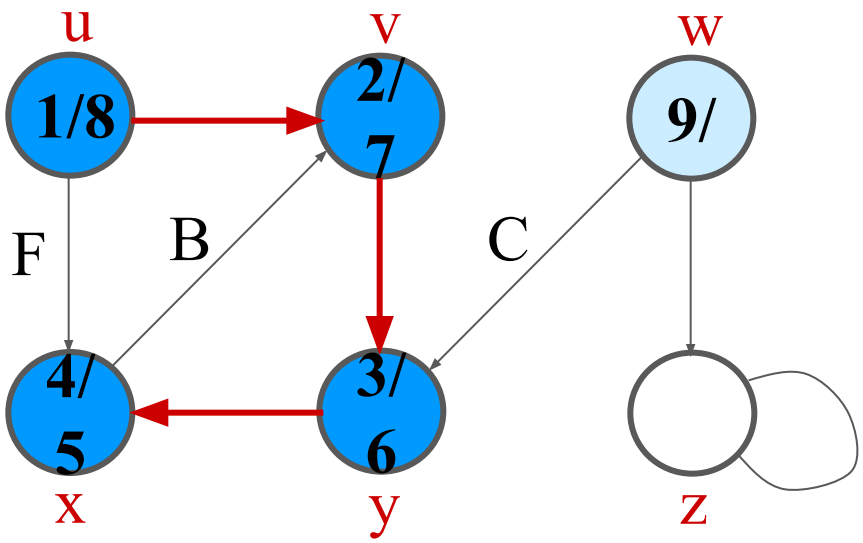

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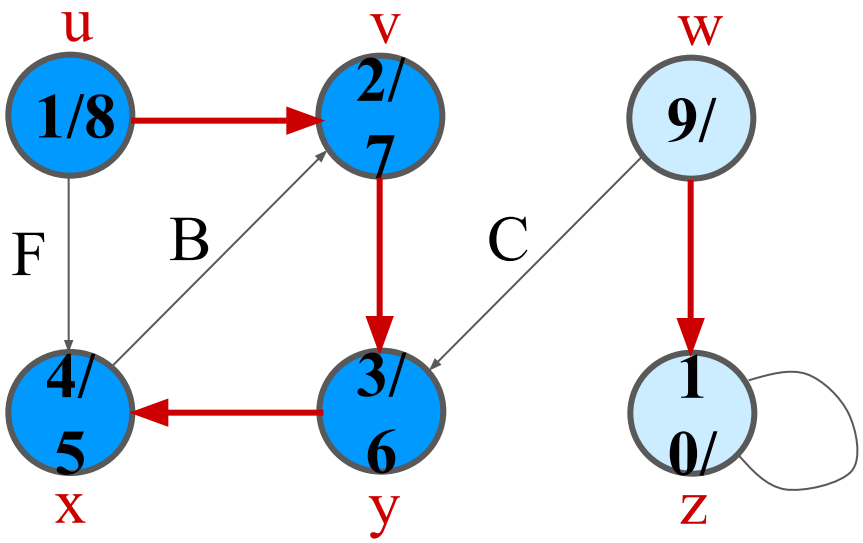

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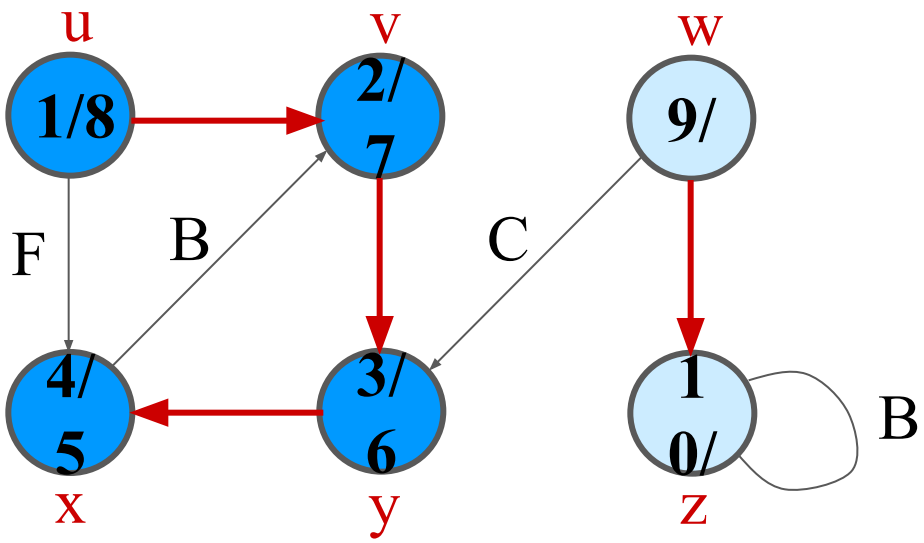

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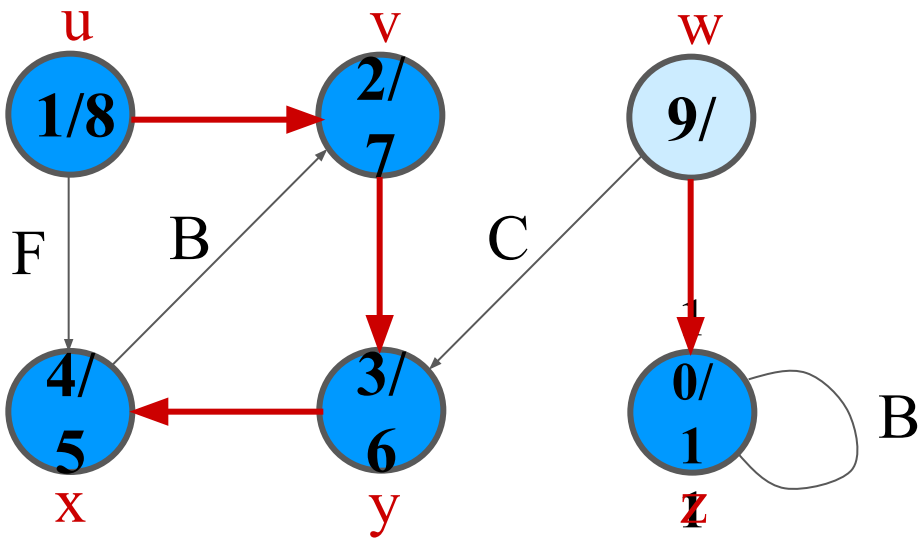

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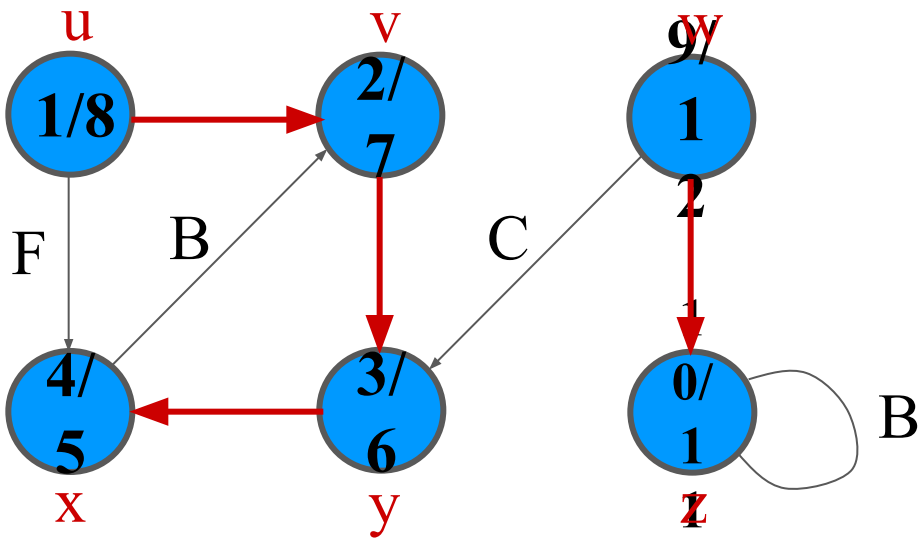


Pseudo-code

DFS(G)

```
1. for each vertex u ∈ V[G]
2.     do color[u] ← white
3.        π[u] ← NIL
4. time ← 0
5. for each vertex u ∈ V[G]
6.     do if color[u] = white
7.        then DFS-Visit(u)
```

Uses a global timestamp ***time***.

DFS-Visit(u)

```
1.  color[u] ← GRAY   ▽ White vertex u
    has been discovered
2.  time ← time + 1
3.  d[u] ← time
4.  for each v ∈ Adj[u]
5.      do if color[v] = WHITE
6.          then π[v] ← u
7.               DFS-Visit(v)
8.  color[u] ← BLACK    ▽ Blacken u;
    it is finished.
9.  f[u] ← time + 1
```

Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed $|\text{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\text{Adj}[v]| = \Theta(E)$
- Total running time of DFS is $\Theta(V+E)$.

Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}$.
  - » How does it differ from that of BFS?
  - » The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:

Forest: An acyclic graph G that may be disconnected.

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

DAG Theorem

A directed graph *G* is acyclic if and only if a DFS of *G* yields no back edges.

Proof: Suppose there is a back edge (*u*,*v*):
Then *v* is an ancestor of *u* in DFS forest. Thus, there is a path from *v* to *u* in G and (*u*,*v*) completes the cycle.
Suppose there is a cycle *c*: Let *v* be the first vertex in *c* to be discovered and *u* is the predecessor of *v* in *c*.

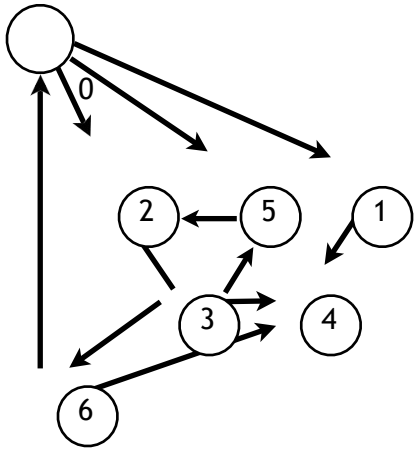
- Upon discovering *v* the whole cycle from *v* to *u* is white
- We visit all nodes reachable on this white path before DFS(*v*) returns, i.e., vertex *u* becomes a descendant of *v*
- Thus, (*u,v*) is a back edge

Thus, we can verify whether *G* is a DAG using DFS

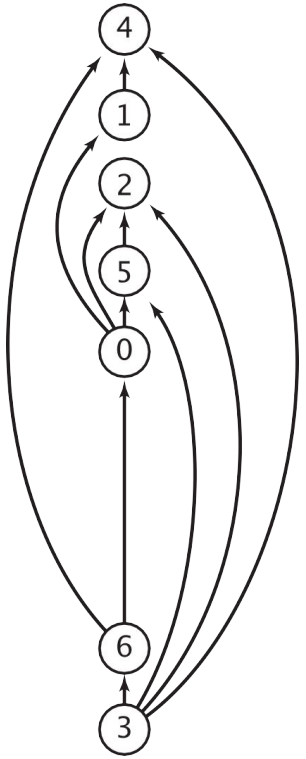
Precedence scheduling

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

Digraph model. vertex = task; edge = precedence constraint.



precedence constraint graph



feasible schedule

Topological Sorting

- Sorting of a directed acyclic graph (DAG)
- A topological sort of a DAG
  is a linear ordering of all its vertices
  such that for any edge (*u*,*v*) in the DAG,
  u appears before *v* in the ordering

Steps:

- Run depth-first search.
- Return vertices in reverse postorder

- Cycles make topological sort impossible.
- Select any node with no in-edges
  - print it
  - delete it
  - and delete all the edges leaving it
- Repeat

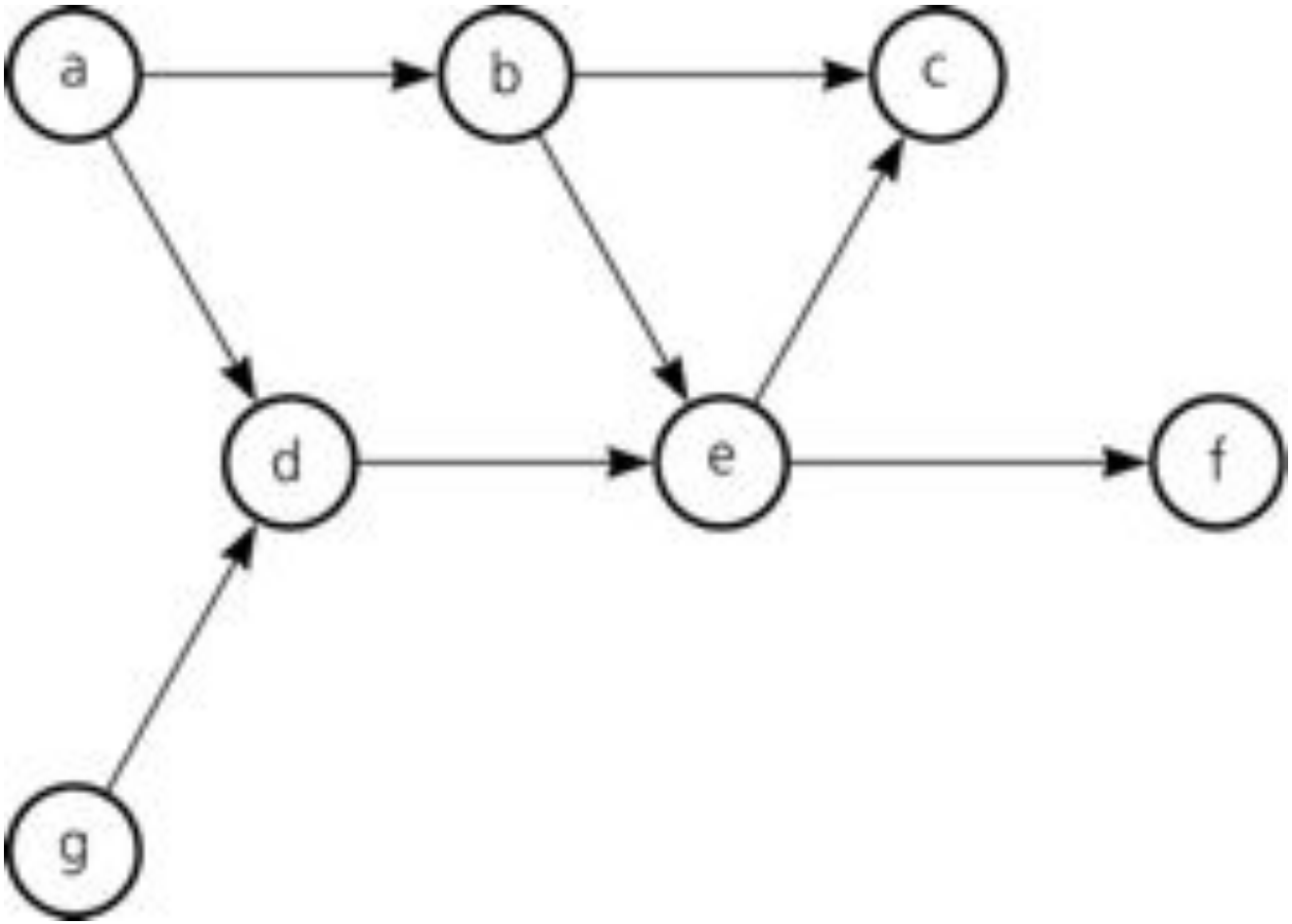
Topological Sorting

The following algorithm topologically sorts a

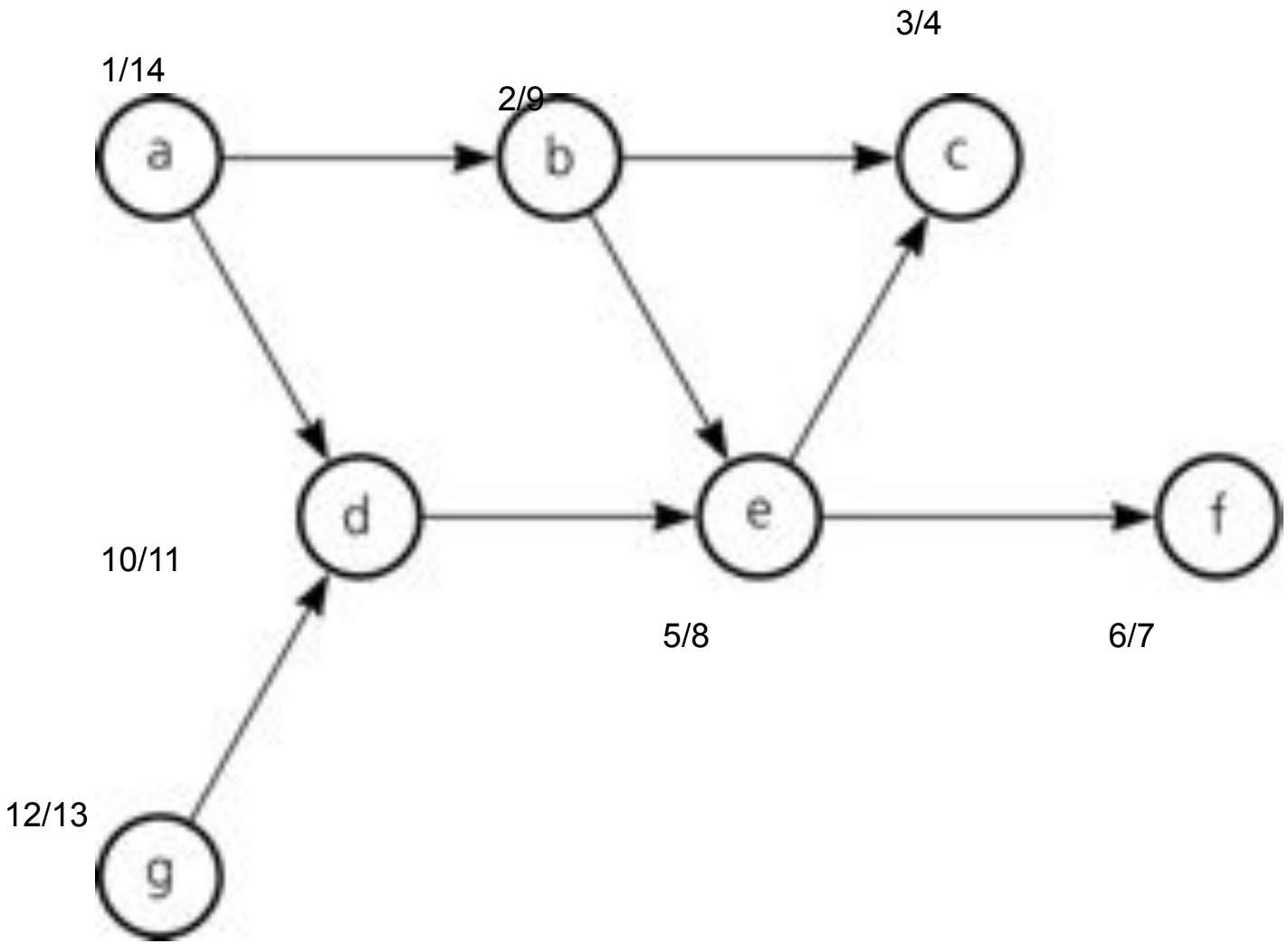
DAG The linked lists comprises a total ordering

```
TopologicalSort(G)
   Call DSF(G) to compute
      v.endtime for each vertex v
   As each vertex is finished,
   insert it at the beginning of a linked list  Return
   the linked list of vertices
```

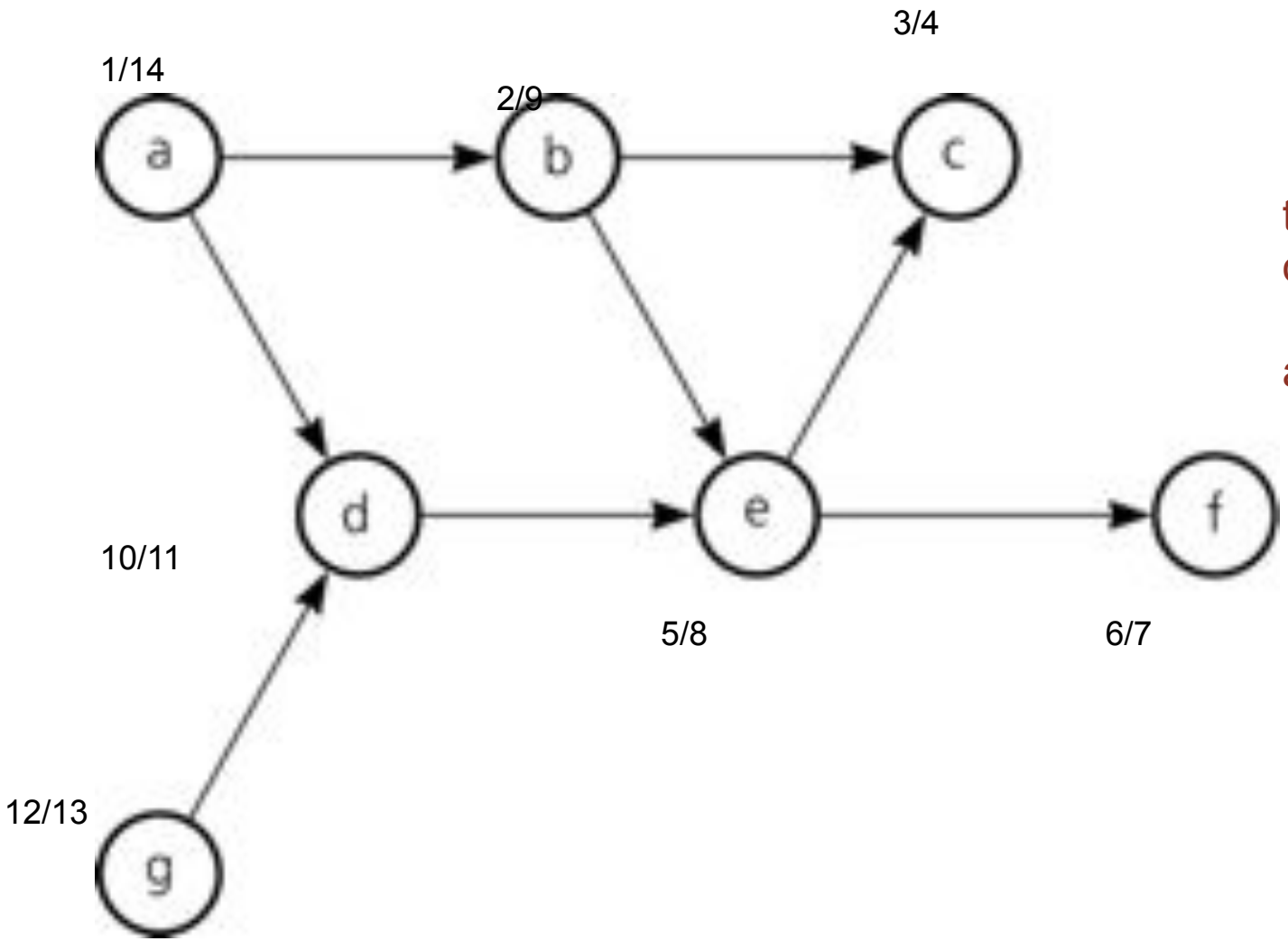
Perform Topological sorting :



Perform Topological sorting :



postorder:
c f e b d g a



topological order:

a g d b e f c

Topological sort in a DAG: correctness proof

Proposition. Reverse DFS postorder of a DAG is a topological order.

Pf. Consider any edge v→w. When dfs(v) is called:

Case 1:

dfs(w) has already been called and returned.

Thus, w was done before v.

Case 2:

dfs(w) has not yet been called.

dfs(w) will get called directly or indirectly by dfs(v) and will finish before dfs(v).

Thus, w will be done before v.

Case 3:

dfs(w) has already been called,

but has not yet returned.
Can't happen in a DAG: function call stack contains path from w to v, so v→w would complete a cycle.

Topological Sorting: Running Time

Running time

- depth-first search: $O(V+E)$ time
- insert each of the $|V|$ vertices to the front of the linked list: $O(1)$ per insertion

Thus the total running time is $O(V+E)$