IE613: Online Machine Learning **Jan-Apr 2017**

Lecture 9: Expected Regret of EXP3

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In last lecture, we discussed about the pseudo regret bounds for EXP3 algorithm in adversarial bandit Setting. In this class, we will prove the theorem which gives the upperbound on pseudo regret. let we will rewrite the theorem:

Theorem 9.1 *If EXP3 is run with $\eta_t = \eta = \sqrt{\frac{2lnK}{TK}}$ then $\bar{R}_T \leq \sqrt{2KTlnK}$. Moreover, if EXP3 is run with $\eta_t = \sqrt{\frac{lnK}{tK}}$, then $\bar{R}_T \leq 2\sqrt{KTlnK}$.*

The following equality can be easily verified:

$$\mathbb{E}_{I_t\sim P_t}[\tilde{l}_{i,t}] = l_{i,t}; \quad \mathbb{E}_{i\sim P_t}[\tilde{l}_{i,t}] = \sum_{i=1}^{K} \tilde{l}_{i,t}P_{i,t} = \sum_{i=1}^{K} \frac{l_{i,t}}{P_{i,t}}\mathbf{1}_{\{I_t=i\}}P_{i,t} = l_{I_t,t} \qquad \mathbb{E}_{i\sim P_t}[l^2_{i,t}] = \frac{l^2_{I_t,t}}{P_{I_t,t}} \tag{9.1}$$

Proof: For any non increasing sequence $(\eta_t)_{t\in\mathbb{N}}$
Claim: $\bar{R}_T \leq \frac{K}{2}\sum_{t=1}^{T}\eta_t + \frac{lnK}{\eta_t}$
Step:1

$$\bar{R}_T = \sum_{t=1}^{T} l_{I_t,t} - \sum_{t=1}^{T} l_{j,t} = \sum_{t=1}^{T}\mathbb{E}_{i\sim P_t}[\tilde{l}_{i,t}] - \sum_{t=1}^{T}\mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}] \tag{9.2}$$

First term of (9.2) can be written as follows:

$$\mathbb{E}_{i\sim P_t}[\tilde{l}_{i,t}] = \frac{1}{\eta_t}ln\mathbb{E}_{i\sim P_t}[exp(-\eta_t(\tilde{l}_{i,t} - \mathbb{E}_{k\sim P_t}[\tilde{l}_{k,t}])) - \frac{1}{\eta_t}ln\mathbb{E}_{i\sim P_t}[exp(-\eta_t\tilde{l}_{i,t})] \tag{9.3}$$

Step:2
Now we will find the bound on first term of equation (9.3):

$$ln\mathbb{E}_{i\sim P_t}[exp(-\eta_t(\tilde{l}_{i,t} - \mathbb{E}_{k\sim P_t}[\tilde{l}_{k,t}])) = ln\mathbb{E}_{i\sim P_t}exp(-\eta_t\tilde{l}_{i,t}) + \eta_t\mathbb{E}_{k\sim P_t}[\tilde{l}_{k,t}]$$

using inequality $lnx \leq x-1, x \geq 0$, will rewrite above expression as follow:

$$\leq \mathbb{E}_{i\sim P_t}exp(-\eta_t\tilde{l}_{i,t}) - 1 + \eta_t\mathbb{E}_{k\sim P_t}[\tilde{l}_{k,t}]$$

Now , here we will use inequality $exp(-x) - 1 + x \leq \frac{x^2}{2}, \forall x \geq 0$,

$$= \mathbb{E}_{k\sim P_t}\{exp(-\eta_t\tilde{l}_{k,t}) - 1 + \eta_t\tilde{l}_{k,t}\}$$

$$\leq \mathbb{E}_{k\sim P_t}(\frac{\eta_t^2 \tilde{l}_{k,t}^2}{2})$$

Since, $0 < l_{I_t,t}^2 \leq 1$;

$$= \frac{\eta_t^2}{2}\frac{l_{I_t,t}^2}{P_{I_t,t}} \leq \frac{\eta_t^2}{2P_{I_t,t}} \tag{9.4}$$

Step:3

Now, here we would try to find the bound on second term of equation (9.4):

$-\frac{1}{\eta_t} ln\mathbb{E}_{k\sim P_t} exp(-\eta_t \tilde{l}_{k,t}) = -\frac{1}{\eta_t} ln \sum_{k=1}^{K} P_{k,t} exp(-\eta_t \tilde{l}_{k,t})$

$$= -\frac{1}{\eta_t} ln \sum_{k=1}^{K} \frac{exp(-\eta_t \tilde{l}_{k,t})}{\sum_i exp(-\eta_t L_{i,t})} \tag{9.5}$$

Let us define, $\phi_t(\eta) = \frac{1}{\eta} ln \frac{1}{K} \sum_{i=1}^{K} exp(-\eta \tilde{L}_{i,t}); \phi_0(\eta) = 0, \tilde{L}_{i,0} = 0$, then equation (9.5) is

$$= \phi_{t-1}(\eta) - \phi(\eta_t) \tag{9.6}$$

In next step, we will take expectation after adding the all terms.

Step:4

$\sum_{t=1}^{T} l_{I_t,t} - \sum_{t=1}^{T} l_{j,t} = \sum_{t=1}^{T} \mathbb{E}_{i\sim P_t}[\tilde{l}_{i,t}] - \sum_{t=1}^{T} \mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}]$

$$= \sum_{i=1}^{T} (\frac{1}{\eta_t} ln\mathbb{E}_{i\sim P_t}[exp(-\eta_t(\tilde{l}_{i,t} - \mathbb{E}_{k\sim P_t}[\tilde{l_{k,t}}])) - \frac{1}{\eta_t} ln\mathbb{E}_{i\sim P_t}[exp(-\eta_t \tilde{l}_{i,t})]) - \sum_{t=1}^{T} \mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}]$$

$$\leq \sum_{t=1}^{T} \frac{\eta_t}{2P_{I_t,t}} + \sum_{t=1}^{T} [\phi_{t-1}(\eta_t) - \phi_t(\eta_t)] - \sum_{t=1}^{T} \mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}] \tag{9.7}$$

After solving second term of last expression, we get

$$\sum_{t=1}^{T} [\phi_{t-1}(\eta_t) - \phi_t(\eta_t)] = \sum_{t=1}^{T-1} [\phi_t(\eta_{t+1}) - \phi_t(\eta_t)] - \phi_T(\eta_T) \leq -\phi_T(\eta_T) \tag{9.8}$$

Since $\phi_0(\eta_1) = 0$, $\phi_t(\eta)$ is increasing in η.So $\phi_t(\eta_{t+1}) - \phi_t(\eta_t) \leq 0$.

After replacing the middle terms of the equation (9.6) by equation (9.7), we get

$$\leq \sum_{t=1}^{T} \frac{\eta_t}{2P_{I_t,t}} - \phi_T(\eta_T) - \sum_{t=1}^{T} \mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}] \tag{9.9}$$

Taking expectation both side, we get

$$\bar{R}_T \leq \sum_{i=1}^{T} \frac{\eta_t K}{2} - \mathbb{E}[\phi_T(\eta_T)] - \sum_{t=1}^{T} \mathbb{E}_{I_t\sim P_t}[\tilde{l}_{j,t}]$$

Where expectation of first term(9.9), i.e, $\mathbb{E}_{I_t\sim P_t} \frac{1}{P_{I_t,t}} = K$.

To prove our claim we need to calculate, $\mathbb{E}[\phi_T(\eta_T)]$
Let us calculate first

$$-\phi_T(T) = \frac{lnK}{\eta_T} - \frac{1}{\eta_T} ln[\sum_{i=1}^{K} exp(-\eta_T \tilde{L}_{i,t})]$$

$$\leq \frac{lnK}{\eta_T} - \frac{1}{\eta_T} ln(exp(-\eta_T \tilde{L}_{j,t}))$$

$$= \frac{lnK}{\eta_T} + \sum_{t=1}^{t} \tilde{l}_{j,t}$$

Now,

$$-\mathbb{E}[\phi_T(\eta_T)] = \frac{lnK}{\eta_T} + \sum_{t=1}^{T} \mathbb{E}_{I_t \sim P_t}[\tilde{l}_{j,t}] \tag{9.10}$$

Form equations (9.9) and (9.10), we get

$$\bar{R}_T \leq \sum_{t=1}^{T} \frac{\eta_t K}{2} + \frac{lnK}{\eta_T} + \sum_{t=1}^{T} \mathbb{E}_{I_t \sim P_t}[\tilde{l}_{j,t}] - \sum_{t=1}^{T} \mathbb{E}_{I_t \sim P_t}[\tilde{l}_{j,t}]$$

which proves our claim.
Now,

$$\bar{R}_T \leq \sum_{t=1}^{T} \frac{\eta_t K}{2} + \frac{lnK}{\eta_T}$$

If $\eta_t = \sqrt{\frac{lnK}{tK}}$, we get

$$= \frac{K}{2} \sum_{t=1}^{T} \sqrt{\frac{lnK}{tK}} + \frac{lnK\sqrt{TK}}{\sqrt{lnK}}$$

$$= \frac{\sqrt{KlnK}}{2} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} + \frac{\sqrt{TK}lnK}{\sqrt{lnK}}$$

$$= \frac{\sqrt{KlnK}}{2} 2\sqrt{T} + \sqrt{TKlnK}$$

$$\bar{R}_T \leq 2\sqrt{TKlnK}$$

Now, if $\eta_t = \eta = \sqrt{\frac{2lnK}{TK}}$, we get

$$\bar{R}_T \leq \frac{KT\eta}{2} + \frac{lnK}{\eta}$$

$$= \sqrt{\frac{KTlnK}{2}} + \sqrt{\frac{TKlnK}{2}} = \sqrt{2KTlnK}$$

This prove the theorem, but till now, there are some thing which we need to prove to complete our proof.

We used that, $\phi_t(\eta_{t+1}) - \phi_t(\eta_t) \leq 0$, Since $\eta_{t+1} \leq \eta_t$,

Let, $P^{\eta}_{i,t} = \frac{exp(-\eta \tilde{L}_{i,t})}{\sum_{k=1}^{K} exp(-\eta \tilde{L}_{k,t})}$, then

$$\phi_t^{'}(\eta) = -\frac{1}{\eta^2} ln(\frac{1}{K}\sum_{i=1}^{K} exp(-\eta \tilde{L}_{i,t})) - \frac{1}{\eta}\frac{\sum_{i=1} Kexp(-\eta \tilde{L}_{i,t})}{\sum_{i=1}^{K} exp(-\eta \tilde{L}_{i,t})}$$

$$\frac{1}{\eta^2}\frac{1}{\sum_{i=1}^{K} exp(-\eta \tilde{L}_{i,t})}\sum_{i=1}^{K} exp(-\eta \tilde{L}_{i,t})(-\eta \tilde{L}_{i,t} - ln(\frac{1}{K}\sum_{i=1}^{K} exp(-\eta_{i,t})))$$

$$\Rightarrow \phi_t^{'}(\eta) = \frac{1}{\eta^2}\sum_{i=1}^{K} P^{\eta}_{i,t} ln(KP^{\eta}_{i,t})$$

$$= \frac{1}{\eta^2} KL(P^{\eta}_t, P_1) \geq 0,$$

where P_1 is uniform distribution over {1,2,...,K}.