**IE613: Online Machine Learning** **Jan-Apr 2018**

# Lecture 6: Standard Optimal Algorithm for Online Learning

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We will start this lecture by learning about Standard Optimal Algorithm, an online algorithm whose mistake bound is bounded by the Ldim of the hypothesis class. Next, we will go back to the unrealizable case and show that weighted majority algorithm can be used to achieve sublinear regret even when $|\mathcal{H}| = \infty$, where $\mathcal{H}$ is a hypothesis class. For proving this, we will construct a surrogate hypothesis class which is finite but includes all those hypothesis from $\mathcal{H}$ which gives accurate prediction. We will also show that the expected regret of weighted majority algorithm on this surrogate class is sublinear. Consequently, it can be seen that weighted majority algorithm gives sublinear regret irrespective of the size of $\mathcal{H}$, as long as $\text{Ldim}(\mathcal{H}) < \infty$.

## 6.1 Standard Optimal Algorithm (SOA)

SOA (see algorithm 1) is similar to the weighted majority algorithm, but instead of using majority voting, it uses Ldim for the prediction.

**Algorithm 1** Standard Optimal Algorithm (SOA)

1: **Input** A hypothesis class $\mathcal{H}$
2: **Initialize** $V_1 = \mathcal{H}$
3: **for** $t = 1, 2, ...$ **do**
4: receive $\mathbf{x_t}$
5: for $r \in \{0, 1\}$ let $V_t^{(r)} = \{h \in V_t : h(x_t) = r\}$
6: predict $p_t = \text{argmax}_{r\in\{0,1\}} \text{Ldim}\left(V_t^{(r)}\right)$
(in case of a tie predict $p_t = 1$)
7: receive true label $y_t$
8: update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$
9: **end for**

In the last lecture, we showed that mistake bound of any algorithm is greater than or equal to $\text{Ldim}(\mathcal{H})$ i.e., $M_A(\mathcal{H}) \geq \text{Ldim}(\mathcal{H})$. The following lemma shows that mistake bound of SOA is less than or equal to $\text{Ldim}(\mathcal{H})$. Therefore, there exist no other algorithm whose mistake bound is strictly lesser than that of SOA.

**Lemma 6.1** *Mistake bound of SOA* $M_{SOA}(\mathcal{H}) \leq$ *Ldim*$(\mathcal{H})$

**Proof:** Showing that $\text{Ldim}(V_{t+1}) \leq \text{Ldim}(V_t) - 1$ whenever SOA makes a wrong prediction is sufficient to prove the above claim (since the Ldim reduces by 1 everytime a mistake is made, the mistake bound cant not be more than $\text{Ldim}(\mathcal{H})$). Assume that the contrary is true i.e., $\text{Ldim}(V_{t+1}) = \text{Ldim}(V_t)$ even when SOA makes a wrong prediction. In that case, by the definition of $p_t$, we can see that $V_t^{(0)} = V_t^{(1)}$. If this was true, then one can construct a shattered tree of depth $\text{Ldim}(V_t) + 1$ for the class $V_t$, which is a contradiction. Hence, $\text{Ldim}(V_{t+1}) \leq \text{Ldim}(V_t) - 1$, and the lemma follows. ■

The above lemma gives a mistake bound for SOA in realizable case. Next, we consider the unrealizable case. In the previous lectures, we learned that weighted majority algorithm yields sublinear regret even under unrealizable setting. However, this result was proved with the assumption that $|\mathcal{H}| < \infty$. In the next section, the same result is proved with this assumption relaxed.

## 6.2 Unrealizable Case and $|\mathcal{H}| = \infty$

In this session, we analyse the expected regret of weighted majority algorithm when $\mathcal{H}$ is unrealizable and $|\mathcal{H}| = \infty$. We assume that $\text{Ldim}(\mathcal{H}) < \infty$. We start the analysis by constructing a surrogate hypothesis class (expressed as $\widetilde{\mathcal{H}}$) which is finite but includes those hypothesis from $\mathcal{H}$ which gives accurate prediction.

### 6.2.1 Construction of $\widetilde{\mathcal{H}}$

For each of the indices $(i_1, i_2, ..., i_L)$ an hypothesis (referred as $\text{Expert}(i_1, i_2, ..., i_L)$) where $1 \leq i_1 < i_2 ... < i_l \leq T, L \leq \text{Ldim}(\mathcal{H})$ is built as shown in algorithm 2. The finite dimensional surrogate hypothesis class, $\widetilde{\mathcal{H}}$, for $\mathcal{H}$ is defined as,

$$\widetilde{\mathcal{H}} = \{\text{Expert}(i_1, i_2, ..., i_L) : 1 \leq i_1 < i_2 ... < i_l \leq T, L \leq \text{Ldim}\mathcal{H}\}$$

**Algorithm 2** $\text{Expert}(i_1, i_2, ..., i_L)$

**Input** A hypothesis class $\mathcal{H}$; Indices $(i_1, i_2, ..., i_L)$
**Initialize** $V_1 = \mathcal{H}$
**for** $t = 1, 2, ...T$ **do**
  receive $\mathbf{x_t}$
  for $r \in \{0, 1\}$ let $V_t^{(r)} = \{h \in V_t : h(x_t) = r\}$
  define $\widetilde{y}_t = \text{argmax}_{r \in \{0,1\}} \text{Ldim}\left(V_t^{(r)}\right)$
    (in case of a tie predict $\widetilde{y}_t = 0$)
  **if** $t \in \{i_1, i_2, ..., i_L\}$ **then**
    predict $\hat{y}_t = 1 - \widetilde{y}_t$
  **else**
    predict $\hat{y}_t = \widetilde{y}_t$
  **end if**
  update $V_{t+1} = V_t^{(\hat{y}_t)}$
**end for**

### 6.2.2 Expected regret of Weighted Majority Algorithm (WMA)

We will first prove that the expected regret of WMA on $\widetilde{\mathcal{H}}$ is sublinear. Using this we will subsequently argue that the expected regret of WMA on $\mathcal{H}$ is also sublinear.

The number of hypothesis in $\widetilde{\mathcal{H}}$ is,

$$|\widetilde{\mathcal{H}}| = \widetilde{d} = \sum_{L=0}^{\text{Ldim}(\mathcal{H})} \binom{T}{L}$$

If $T \geq \mathrm{Ldim}(\mathcal{H}) + 2$, then $\widetilde{d} \leq (eT/\mathrm{Ldim}(\mathcal{H}))^{\mathrm{Ldim}(\mathcal{H})}$. We know that expected regret of WMA on $\widetilde{\mathcal{H}}$ is less than or equal to $\sqrt{2\log(|\widetilde{\mathcal{H}}|)T}$, and substituting the value of $|\widetilde{\mathcal{H}}|$ gives the expected regret as,

$$\sqrt{2\mathrm{Ldim}(\mathcal{H})\log\left(eT/\mathrm{Ldim}(\mathcal{H})\right)T}$$

Since $\log\left(eT/\mathrm{Ldim}(\mathcal{H})\right)$ is not as dominant as $\mathrm{Ldim}(\mathcal{H})$ and $T$, the expected regret is $\sqrt{2\mathrm{Ldim}(\mathcal{H})T}$.

Clearly the above regret is sublinear. Next, using the following lemma, we argue that the expected regret of WMA on $\mathcal{H}$ is less than that on $\widetilde{\mathcal{H}}$.

**Lemma 6.2** *Let $\mathcal{H}$ be any hypothesis class with Ldim($\mathcal{H}$) $< \infty$. Let $x_1, x_2, ..., x_T$ be any sequence of instances. For any $h \in \mathcal{H}$, there exists $L \leq$ Ldim($\mathcal{H}$) and indices $1 \leq i_1 < i_2 < ... < i_L \leq T$ such that when running Expert($i_1, i_2, ..., i_L$) on the sequence $x_1, x_2, ..., x_T$, the expert predicts $h(x_t)$ on each online round $t = 1, 2, ..., T$.*

**Proof:** Consider a hypothesis $h$ and a sequence $x_1, x_2, ..., x_T$. Let $(i_1, i_2, ..., i_L)$, $1 \leq i_1 < i_2 < ... < i_L \leq T$, be the rounds where SOA made wrong prediction. Since Expert$(i_1, i_2, ..., i_L)$ makes the same prediction as SOA in rounds $\{1, 2, ...T\} - \{i_1, i_2, ..., i_L\}$ and the opposite of the prediction made by SOA in rounds $\{i_1, i_2, ..., i_L\}$, the expert predicts $h(x_t)$ on each online round $t = 1, 2, ..., T$. ■

The following corollary can be obtained from the above lemma.

**Corollary 6.3** *Let $(x_1, y_1), (x_2, y_2), ..., (x_T, y_T)$ be a sequence of examples and let $\mathcal{H}$ be a hypothesis class with Ldim($\mathcal{H}$) $< \infty$. There exists $L \leq$ Ldim($\mathcal{H}$) and indices $1 \leq i_1 < i_2 < ... < i_L \leq T$ such that Expert($i_1, i_2, ..., i_L$) makes at most as many mistakes as the best $h \in \mathcal{H}$ does, namely, $\min_{h \in \mathcal{H}} \sum_{t=1}^{T} |h(x_t) - y_t|$ mistakes on the sequence of examples.*

From the above result, it is clear that expected regret of WMA on $\mathcal{H}$ is less than that on $\widetilde{\mathcal{H}}$. Therefore, expected regret of WMA on $\mathcal{H}$ is sublinear since the expected regret of WMA on $\widetilde{\mathcal{H}}$ is itself sublinear.