IE613: Online Machine Learning **Jan-Apr 2018**

Lecture 19: Hoeffdings and Bernsteins inequality

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19.1 Recap:

Sum of independent random variables

$$Pr\left\{|\frac{1}{n}\sum_{i=1}^{n}X_i - E[X_i]| \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

where X_i is an independent random variable with variance σ^2.

Now,

$$Pr\left\{|\frac{1}{n}\sum_{i=1}^{n}X_i - E[X_i]| \geq \epsilon\right\} \approx e^{\frac{-n\epsilon^2}{2\sigma^2}}$$

19.2 Hoeffding's inequality:

Let $X_1, X_2, ...X_n$ are independent bounded r.v's s.t. $X_i \in [a_i, b_i] \forall i$ w.p. 1. Then for any $t > 0$, we have

$$Pr\left\{S_n - E[S_n] \geq t\right\} \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

$$Pr\left\{S_n - E[S_n] \leq -t\right\} \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Combining the above two equations,

$$Pr\left\{|S_n - E[S_n]| \geq t\right\} \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Now, replacing t by $n\epsilon$,

$$Pr\left\{\frac{|S_n - E[S_n]|}{n} \geq \epsilon\right\} \leq 2\exp\left(\frac{-2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right) \tag{19.1}$$

These type of bounds are called concentration bounds (since the sum concentrates around the mean).
Proof:

$$\begin{aligned}Pr\left\{S_n - E[S_n] \geq t\right\} &= Pr\left\{e^{\beta(S_n - E[S_n])} \geq e^{\beta t}\right\}\\ &\leq \frac{E[e^{\beta(S_n - E[S_n]}]}{e^{\beta t}}\end{aligned}$$

Since, X_i's are independent random variables.

$$= e^{-\beta t} \prod_{i=1}^{n} E[e^{\beta(X_i - E[X_i]}]$$

Lemma: Let X be a r.v. with E[X]=0, $a < X < b$. Then for $\beta > 0$, $E[e^{\beta x}] \leq e^{\frac{\beta^2(b-a)^2}{8}}$
By convexity of the exponential function,

$$\begin{aligned}
e^{\beta x} &\leq \frac{x-a}{b-a} e^{\beta b} - \frac{b-x}{b-a} e^{\beta b} \\
e^{\beta x} &\leq \frac{b e^{\beta a)}}{\beta - a} - \frac{a e^{\beta b)}}{b-a} \\
&= \left(1 + \frac{a}{b-a} - \frac{a}{b-a} e^{\beta(b-a)}\right) e^{a\beta} \\
&= e^{\log\left(1 + \frac{a}{b-a} - \frac{a}{b-a} e^{\beta(b-a)}\right) + a\beta} \\
&= e^{\phi(u)}
\end{aligned}$$

where $u = \beta(b\text{-}a)$, $p = \frac{-a}{b-a}$, $\phi(u) = -pu + \log(1 - p + pe^u)$

Using Taylor's series second order approximation,
$\phi(u) = \phi(0) + u\phi'(0) + \frac{u^2}{2}\phi"(\theta)$ for some $\theta \in [0, u]$

$\phi(u) = -pu + \log(1 - p + pe^u)$
$\phi'(u) = -p + \frac{pe^u}{1-p+pe^u}$

Therefore,
$\phi(0) = 0$
$\phi'(0) = 0$

$\phi"(u) = \frac{(1-p+pe^u)(pe^u)-(pe^u)^2}{(1-p+pe^u)^2} = \frac{p(1-p)e^u}{(1-p+pe^u)^2} \leq \frac{1}{4}$

Therefore,
$E[e^{\beta x}] \leq e^{\frac{\mu^2}{2} * \frac{1}{4}} = e^{\frac{\beta^2(b-a)^2}{8}}$

Continuing with the proof,

$$\begin{aligned}
Pr\left\{S_n - E[S_n] \geq t\right\} &\leq e^{-\beta t} \prod_{i=1}^{n} E[e^{\beta(X_i - E[X_i])}] \\
&\leq e^{-\beta t} \prod_{i=1}^{n} e^{\frac{\beta^2(b_i-a_i)^2}{8}} \\
&= e^{-\beta t} + \beta^2 \sum_{i=1}^{n} \frac{(b_i - a_i)^2}{8} \qquad \forall \beta > 0
\end{aligned}$$

$$Setting \qquad \beta = \frac{4t}{\sum_{i=1}^{n}(b_i - a_i)^2} \leq e^{\frac{-2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}}$$

Replacing t by $n\epsilon$,

$$Pr\left\{\frac{S_n - E[S_n]}{n} \geq \epsilon\right\} \leq e^{\frac{-2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i-a_i)^2}}$$

$b_i = 1, a_i = 0 \qquad \forall i$
Therefore;

$$Pr\left\{\frac{S_n - E[S_n]}{n} \geq \epsilon\right\} \leq e^{-2n\epsilon^2}$$

The question now is : is it possible to improve Hoeffding's inequality so that it captures the knowledge of variance?
The answer is yes (Bernstein's inequality).
In the previous lectures, we have proved:
Chebyshev's inequality:-

$$Pr\left\{\frac{S_n - E[S_n]}{n} \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

and by Class Limit theorem,

$$Pr\left\{\frac{S_n - E[S_n]}{n} \geq \epsilon\right\} \leq e^{\frac{-n\epsilon^2}{2\sigma^2}}$$

19.3 Bernstein's inequality:

Theorem:$X_1, X_2, ...X_n$ are independent real value r.v.s with zero mean and $|X_i| < C \forall i$ w.p. 1. Let $\sigma^2 = \sum_{i=1}^{n} \frac{Var(X_i)}{n}$
Then, $\forall t > 0$,

$$Pr\left\{\frac{\sum_{i=1}^{n} X_i}{n} > \epsilon\right\} \leq \exp\left(\frac{-n\epsilon^2}{2\sigma^2 + \frac{2c\epsilon}{3}}\right)$$

$$\begin{aligned} Pr\left\{\sum_{i=1}^{n} X_i > t\right\} &= Pr\left\{e^{\beta\sum_{i=1}^{n} X_i} > e^{\beta t}\right\} \\ &= \prod_{i=1}^{n} E[e^{\beta X_i}]e^{-\beta t} \end{aligned}$$

Now,

$$
\begin{aligned}
F_i &= \sum_{r=2}^{\infty} \frac{\beta^{r-2} E[X_i^r]}{r!\sigma_i^2} \\
E[e^{\beta X_i}] &= 1 + \beta^2 \sigma_i^2 F_i \leq e^{\beta^2 \sigma_i^2 F_i} \qquad (Since, 1 + x \leq e^x)
\end{aligned}
$$

$$
\begin{aligned}
E[X_i^r] &= E[X_i^2 X_i^{r-2}] \leq E[X_i^2] C^{r-2} = \sigma_i^2 C^{r-2} \\
F_i &= \sum_{r=2}^{\infty} \frac{\beta^{r-2} E[X_i^r]}{r!\sigma_i^2} \\
&\leq \sum_{r=2}^{\infty} \frac{\beta^{r-2} \sigma_i^2 C^{r-2}}{r!\sigma_i^2} \\
&= \frac{1}{(\beta C)^2} \sum_{r=2}^{\infty} \frac{(\beta c)^r}{r!} \\
&= \frac{1}{(\beta C)^2} (e^{\beta C - 1 - \beta C})
\end{aligned}
$$

Now,

$$
\begin{aligned}
Pr\left\{\sum_{i=1}^{n} X_i > t\right\} &\leq \prod_{i=1}^{n} E[e^{\beta X_i}] e^{-\beta t} \\
&\leq e^{\frac{\sum_{i=1}^{n} \sigma_i^2 (e^{\beta C - 1 - \beta C})}{C^2} - \beta t} \\
&= e^{\frac{n\sigma^2 (e^{\beta C - 1 - \beta C})}{C^2} - \beta t} \qquad (Since, \left[\sigma^2 = \sum_{i=1}^{n} \frac{Var(X_i)}{n} = \sum_{i=1}^{n} \frac{\sigma_i^2}{n}\right])
\end{aligned}
$$

Now,$\beta = \frac{1}{C} \log(1 + \frac{tC}{n\sigma^2})$

$$
Pr\left\{\frac{1}{n} \sum_{i=1}^{n} X_i > \epsilon\right\} \leq \exp\left(\frac{-n\epsilon^2}{C^2} h\left(\frac{Ct}{n\epsilon^2}\right)\right)
$$

$h(u) = (1 + u)\log(1 + u) - u \qquad \forall u > 0$

$h(u) \geq \frac{u^2}{2 + \frac{2u}{3}}$

$$
Therefore, Pr\left\{\frac{1}{n} \sum_{i=1}^{n} X_i > \epsilon\right\} \leq \exp\left(\frac{-n\epsilon^2}{2\sigma^2 + \frac{2C\epsilon}{3}}\right)
$$

So, these results help to give probability of your estimate being away from the true value by some quantity.

19.4 Stochastic Multi-armed Bandits:

Known Parameters: number of arms (K) and number of rounds T(>K)
Unknown Parameters: K probability distributions, $\nu_1, \nu_2 ..... \nu_K$ on [0,1]

For each round t=1,2,....
1.) the learner/forecaster chooses $I_t \in \{1, 2...K\}$.
2.) given I_t, the environment draws reward. $X_{I_{t,t}} \sim \nu_{I_t}$.