Brief Introduction to Bond Math

Pre-requisite: Wall St. Prep course

Suggested Reference: Fixed Income Mathematics, Fabozzi 4th Edition

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Bond Math – Session 2, Chapter 5

Yield (Internal Rate of Return)

Goals of This Unit

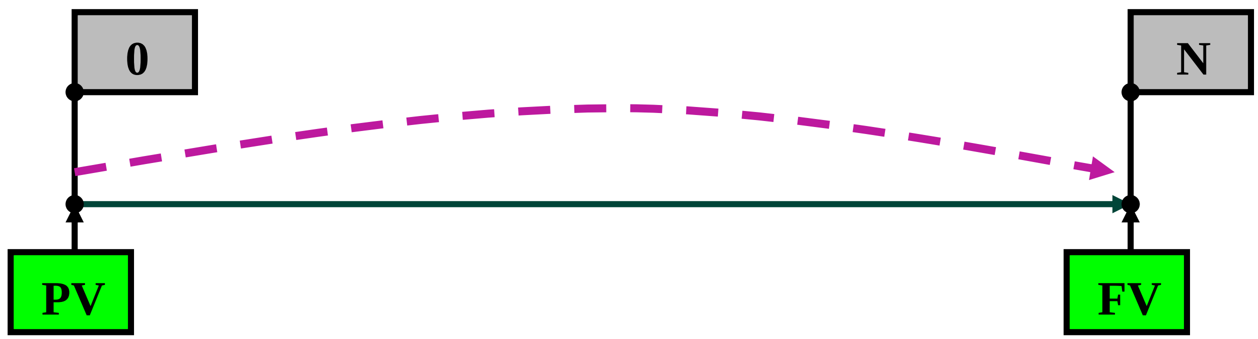
- Yield
  - Intuitive understanding
  - Calculation
  - Annualizing

- Price of Bond
  - Meaning
  - Calculation
  - Relationships with other factors (yield, coupon, time)
  - Accrued interest, daycounts, clean/dirty price

Time Value of Money Review - I

- Future Value

$$FV = PV(1+i)^N$$

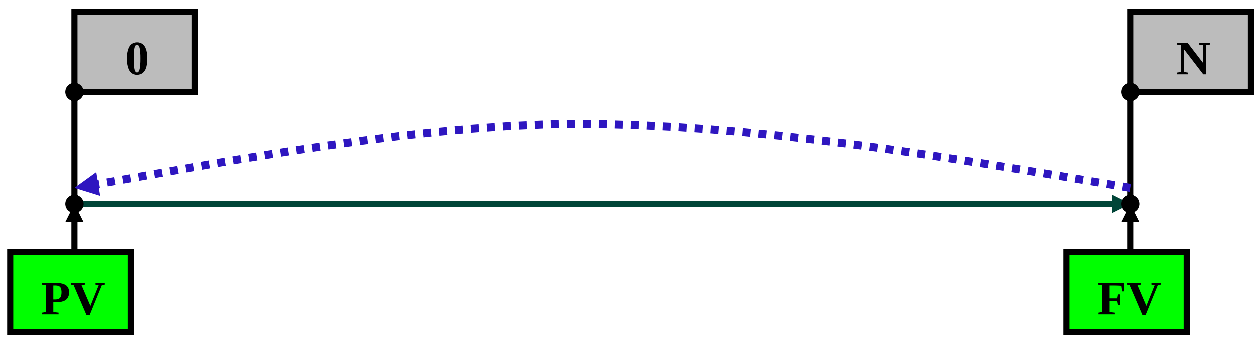


- We are **compounding**
- i is the **interest rate** to get us from PV to FV
- Solve for FV knowing PV, i, N

Time Value of Money Review - II

- Future Value

$$PV = FV\left[\frac{1}{(1+i)^N}\right]$$



- We are **discounting**
- i is the **discount rate** to get us from FV to PV
- Solve for PV knowing FV, i, N

Concept of Yield

- What if we knew PV, FV, and N – but had to solve for i?

$$\underbrace{PV}_{\text{Price of Bond}} = \underbrace{FV}_{\text{Bond Cashflow}}\left[\frac{1}{(1+i)^N}\right]$$

- i is the **discount rate** to get us from FV to PV
- i is the **discount rate** that will make the present value of the cashflow equal to the price of the bond
- i is the **yield** of the bond
  - The return that would make the cashflow FV at time N worth price PV

Calculating Yield - I

- Now let's look at a bond with multiple cashflows

$$PV = \frac{CF_1}{(1+y)^1} + \frac{CF_2}{(1+y)^2} + \frac{CF_3}{(1+y)^3} + \ldots + \frac{CF_N}{(1+y)^N}$$

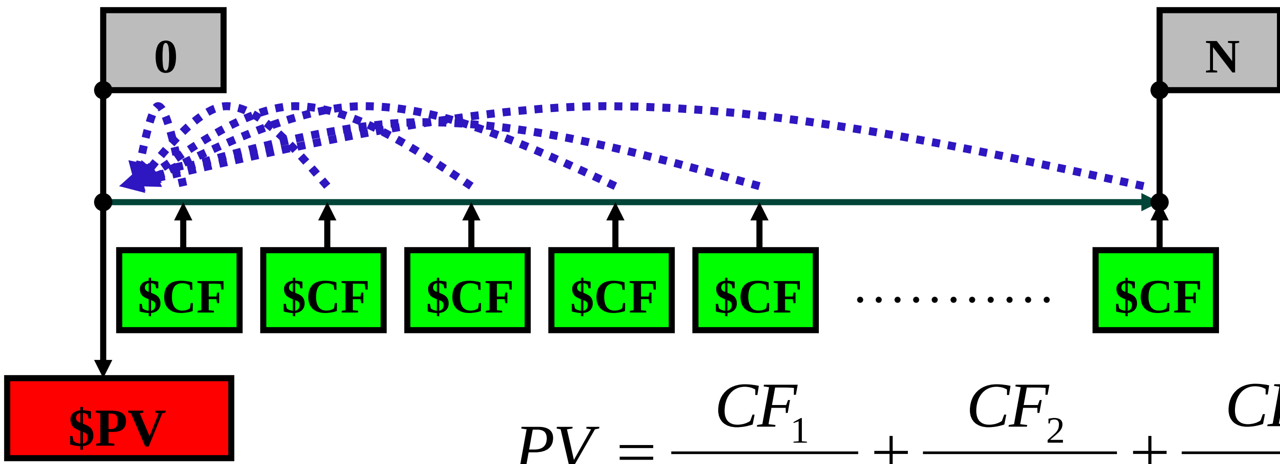
.....

$$PV = \sum_{t=1}^{N} \frac{CF_t}{(1+y)^t}$$

Know this formula!

- CF, PV, N are known
- We are solving for y
  - Iterative solution (Fabozzi p. 54)
  - Select a y
    - If resulting PV > price, pick a higher y
    - If resulting PV < price, pick a lower y

Calculating Yield - II



$$PV = \frac{CF_1}{(1+y)^1} + \frac{CF_2}{(1+y)^2} + \frac{CF_3}{(1+y)^3} + \ldots + \frac{CF_N}{(1+y)^N}$$

- Discounting every CF at the same y
- Solve for y so that sum of all discounting cashflows is PV . . . the price
- ***Yield*** is the single interest rate which equates the price of a security to the sum of the present values of its cash flows.
- What is the yield when p = $100?

Yield Names and Limitations

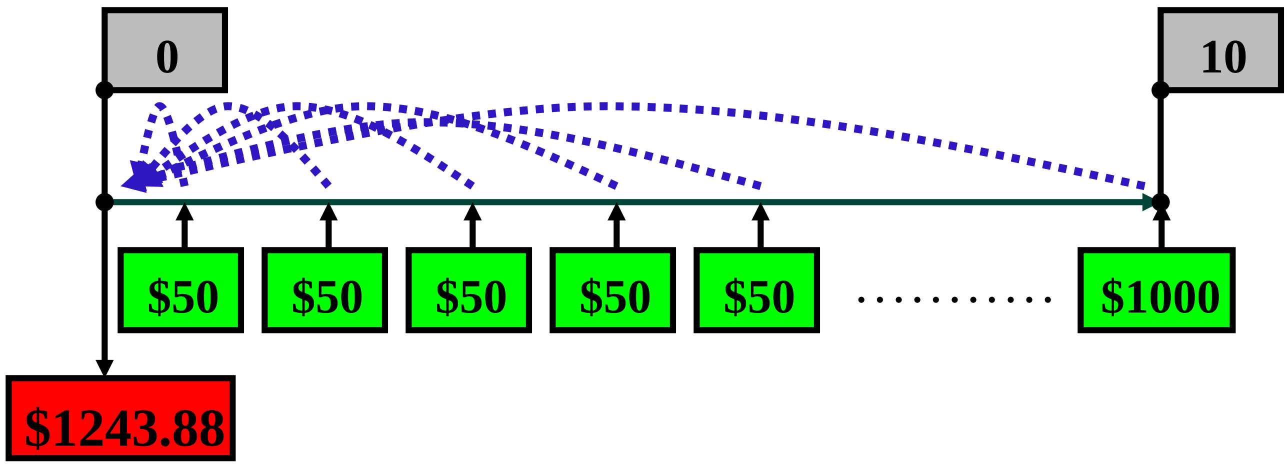
- Names
  - Yield to Maturity (YTM)
  - Internal Rate of Return
  - The Price
    - – Yield is as much a quote as price

- Limitations
  - Compounding this way assumes reinvesting each CF at same interest rate
  - So reverse – discounting this way – assumes discounting each CF at same discount factor
    - – Should you discount a cashflow 6 months from now at same rate as a cashflow 60 months from now?
  - Assumes holding bond to maturity

Yield Calculation Example - I

- From p. 56
- $1000 face **semi-annual** bond maturing in 5 years
- Coupon rate is 10%
- Priced at $1243.88

Yield Calculation Example - II



$$PV = \frac{\$50}{(1+y)^1} + \frac{\$50}{(1+y)^2} + \frac{\$50}{(1+y)^3} + \ldots + \frac{\$50}{(1+y)^{10}} + \frac{\$1000}{(1+y)^{10}}$$

- Coupon is 10% - why are CFs $50?
- Maturity is 5 years – why is n=10?
- Anyway, iteratively solving for y leads to 2.225%
- **Would we quote this bond as yielding 2.225% ? ? ?**

Annualizing Yields - I

- No, we would not quote it as yielding 2.225%
  - That is the **periodic yield**

- Bond is semi-annual
  - Just as we converted CFs from $100 to $50 by dividing by 2
  - Just as we converted N from 5 to 10 by multiplying by 2
  - We **annualize** the 2.225% periodic yield by multiplying by 2
  - The annualized semi-annual periodic yield is 4.45%
  - The **semi-annual bond-equivalent yield (BEY)**

Annualizing Yields - II

- Multiplying by 2 is not exactly correct
  - If you look at how interest compounding affects things (see p.60)
- Calculating it correctly gives you the **effective annual yield (EAY)**
  - See handout

$$(1+r_{f1})^{f1} = (1+r_{f2})^{f2}$$

- By convention
  - We divide/multiply by 2 to get from/to annual to semi-annual periodic
  - We usually quote semi-annual yield, not effective annual yield
  - Use handout formula when starting/going to frequencies 4,12, etc.
  - Must "pass through" the semi-annual periodic along the way

Examples

- Solve the YTM iteratively
- Use IRR function in Excel
- Use Bloomberg

Bond Math – Session 2, Chapter 6

The Price of a Bond

Quoting a Bond Price - I

- XYZ needs to borrow US$500MM

  XYZ 5 ¼ 12/14/2026

  Issue Size: US$500MM

- Sold in lots of US$1MM **face value** or **par** (usually a minimum size or round lots)
- Quoted as follows:

Quoted Price	Real Dollar Cost	Term	Meaning
$100	100% of $1m = $1m	Par	Bond is priced to market – coupon matches comparable current market yield
$102	102% of $1m = $1.02m	Premium	Bond is priced above market – coupon is higher than comparable market yields
$98	98% of $1m = $980,000	Discount	Bond priced below market – coupon is lower than comparable market yields

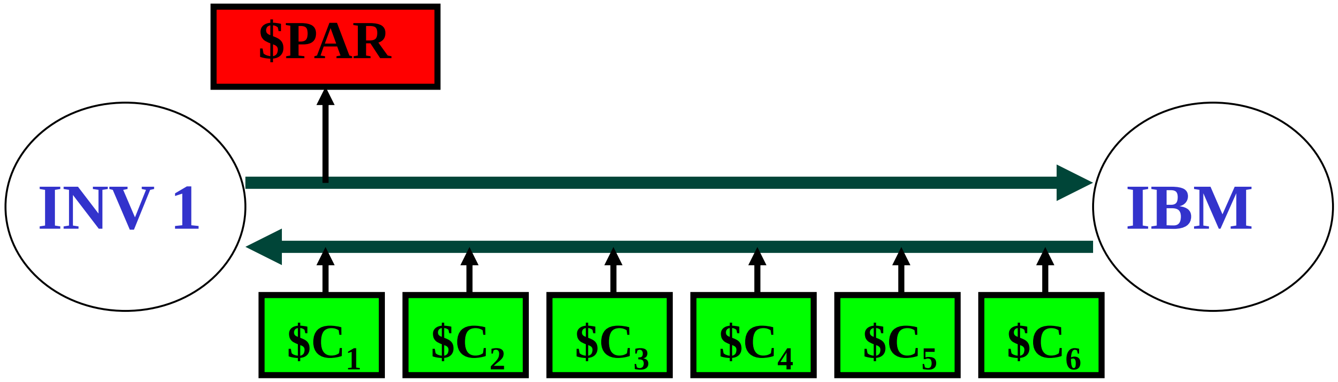
Quoting a Bond Price - II

- Quoting conventions by instrument:

Instrument	Sample Quote	Dollar Amount ($1000 face)	Rule
Corporate Bond	98.2564	$982.564	Priced to 4 decimal places
US TSY Notes, Bonds	100:02	$1,000.625 (100 + 2/32)	Left of ":" is percentage, right is in 32nds.
US TSY Bills	4.73%	$952.70 (100-4.73)	Percent discount from par
Mortgage-backed Securities	98-24+ 98-242	$987.6563 $987.5781	Same as TSY Bonds. The + is a 64th. A 3rd digit would be a 256th.

- Dates
  - ***Trade Date*** is the day the transaction occurs
  - ***Dated Date*** is the day the bonds begin accruing interest
  - ***Settlement Date*** is the day the money changes hands (T+1 or T+3)

Meaning of Price - I

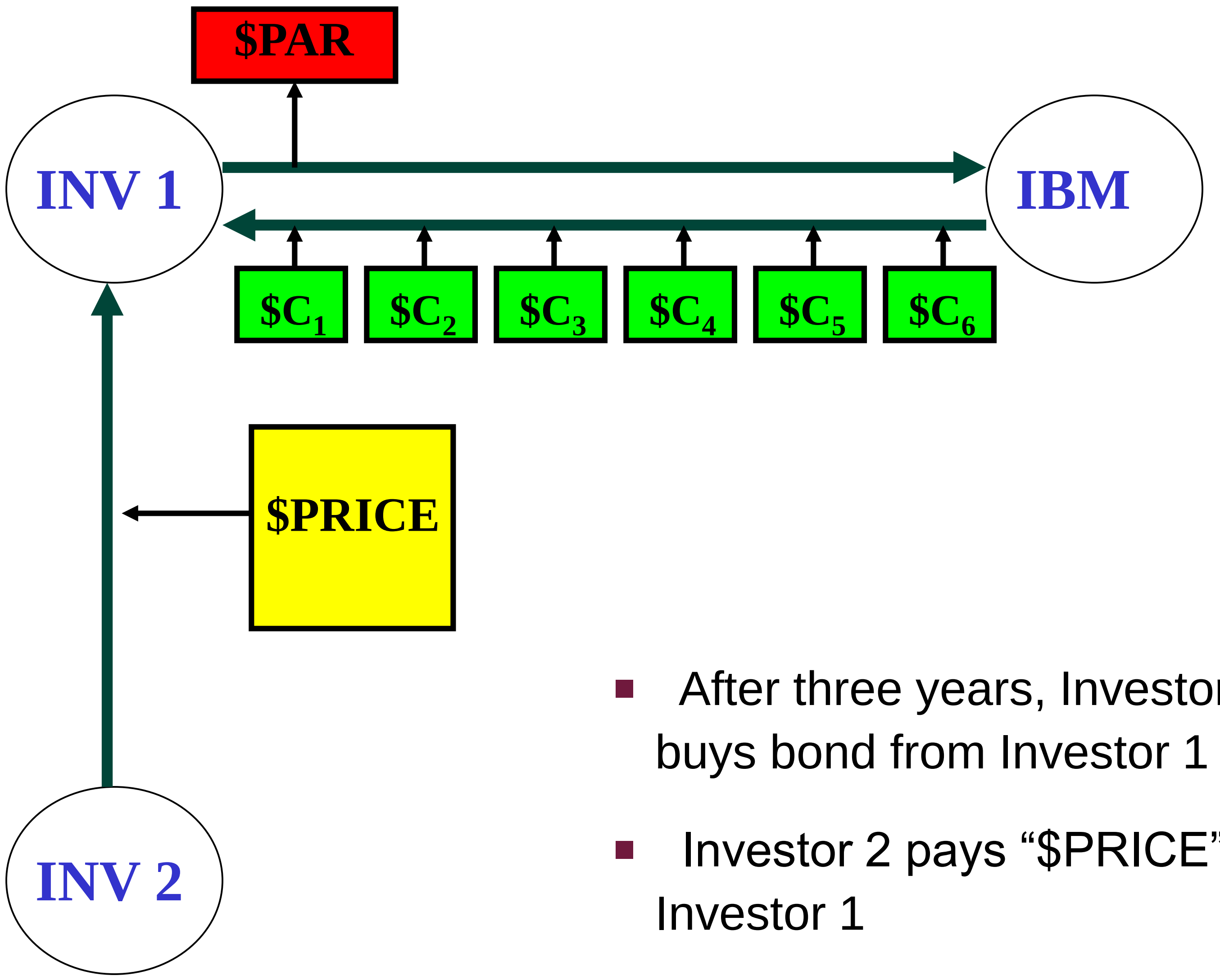


- IBM issues a new semi-annual bond, maturing in 10 years

- Investor 1 buys US$1MM face at par from IBM
  - Really buys from underwriter, simplified here

- Investor 1 owns the bond for 3 years and receives 6 coupon payments

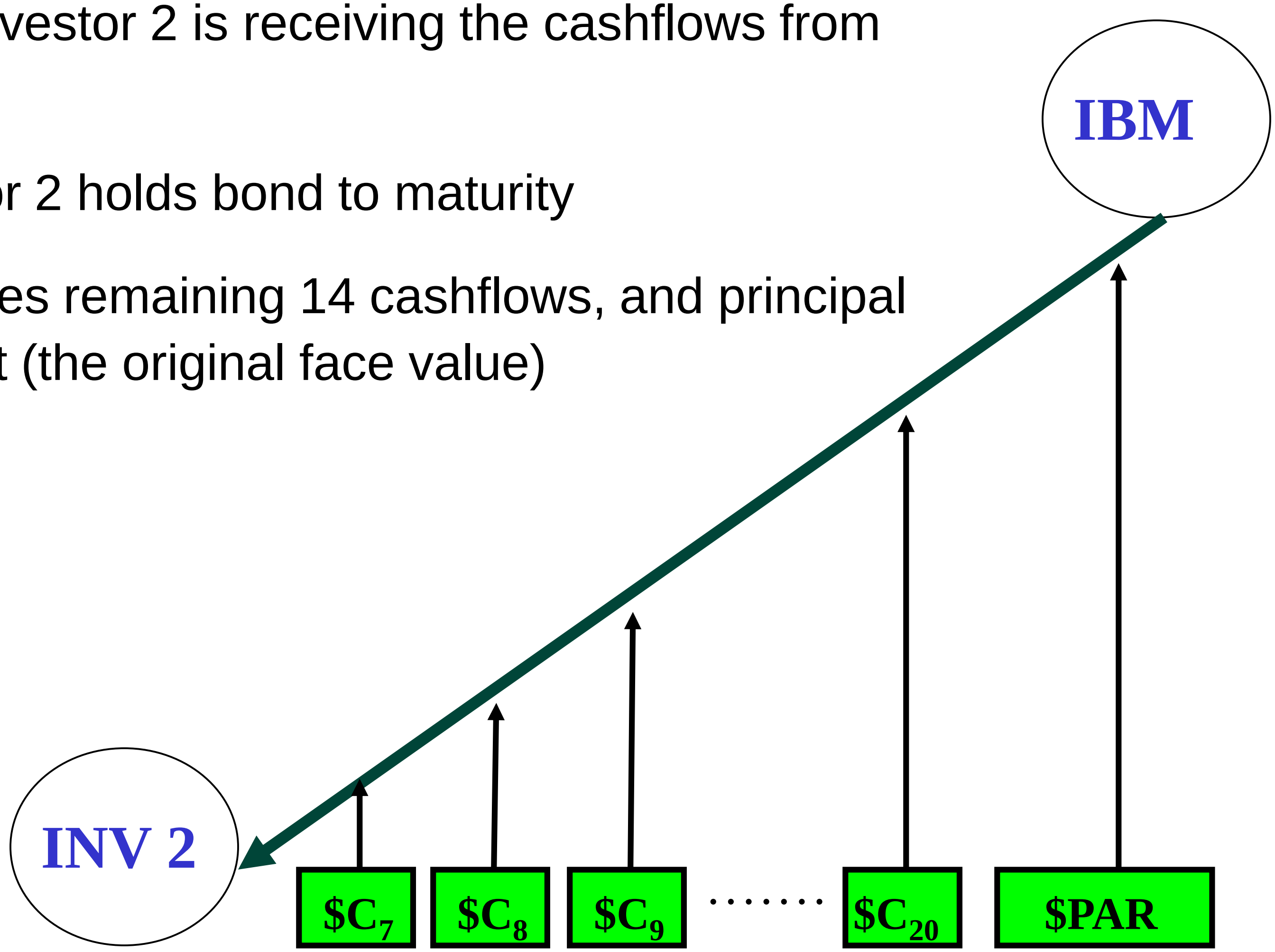
Meaning of Price - II



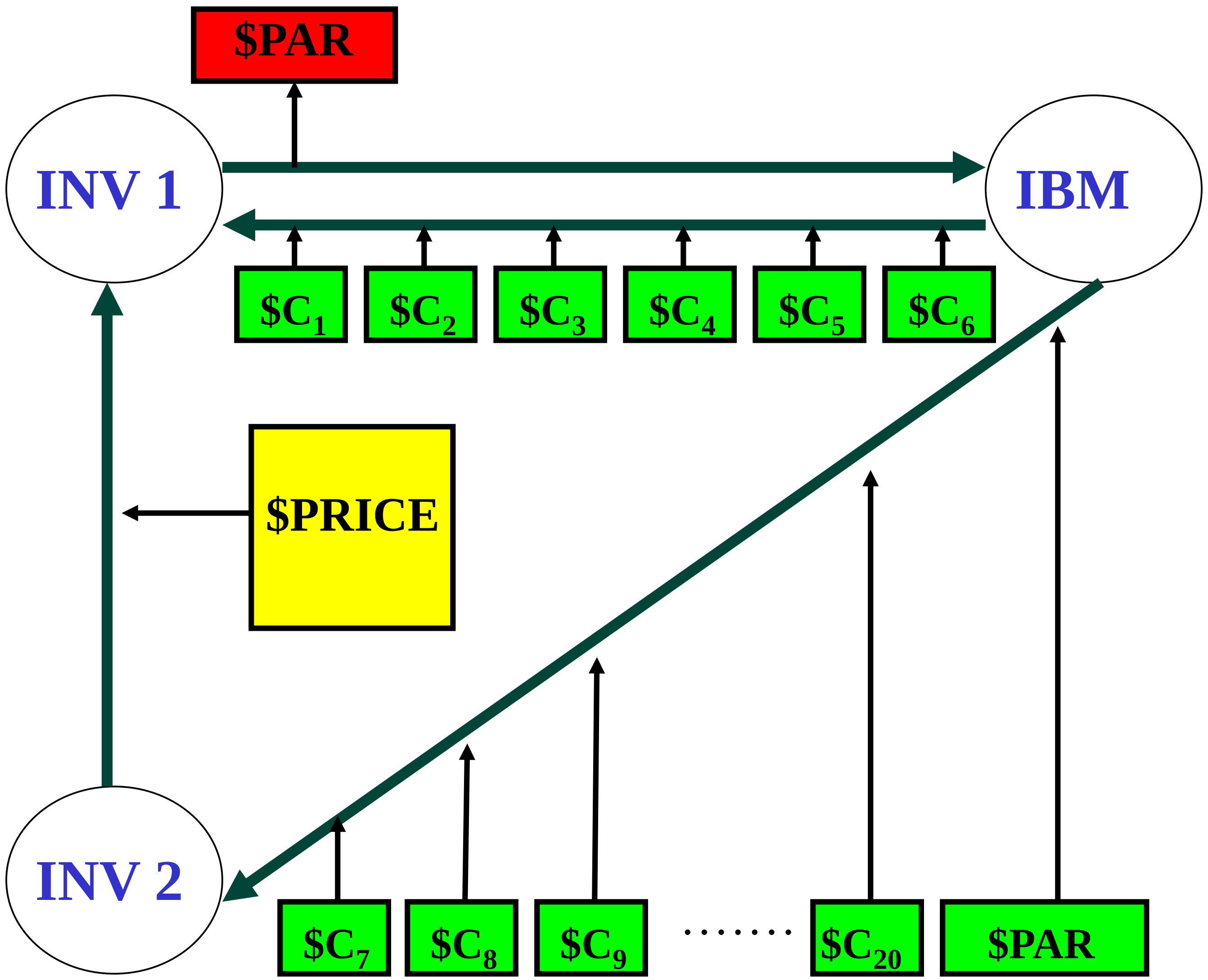
- After three years, Investor 2 buys bond from Investor 1
- Investor 2 pays "$PRICE" to Investor 1

Meaning of Price - III

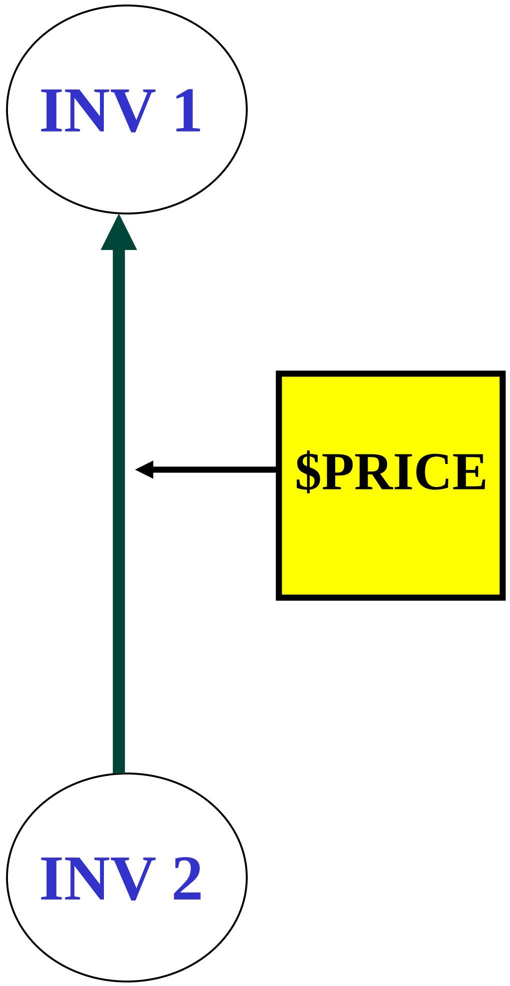
- Now Investor 2 is receiving the cashflows from IBM
- Investor 2 holds bond to maturity
- Receives remaining 14 cashflows, and principal payment (the original face value)



Meaning of Price - IV



Meaning of Price - V

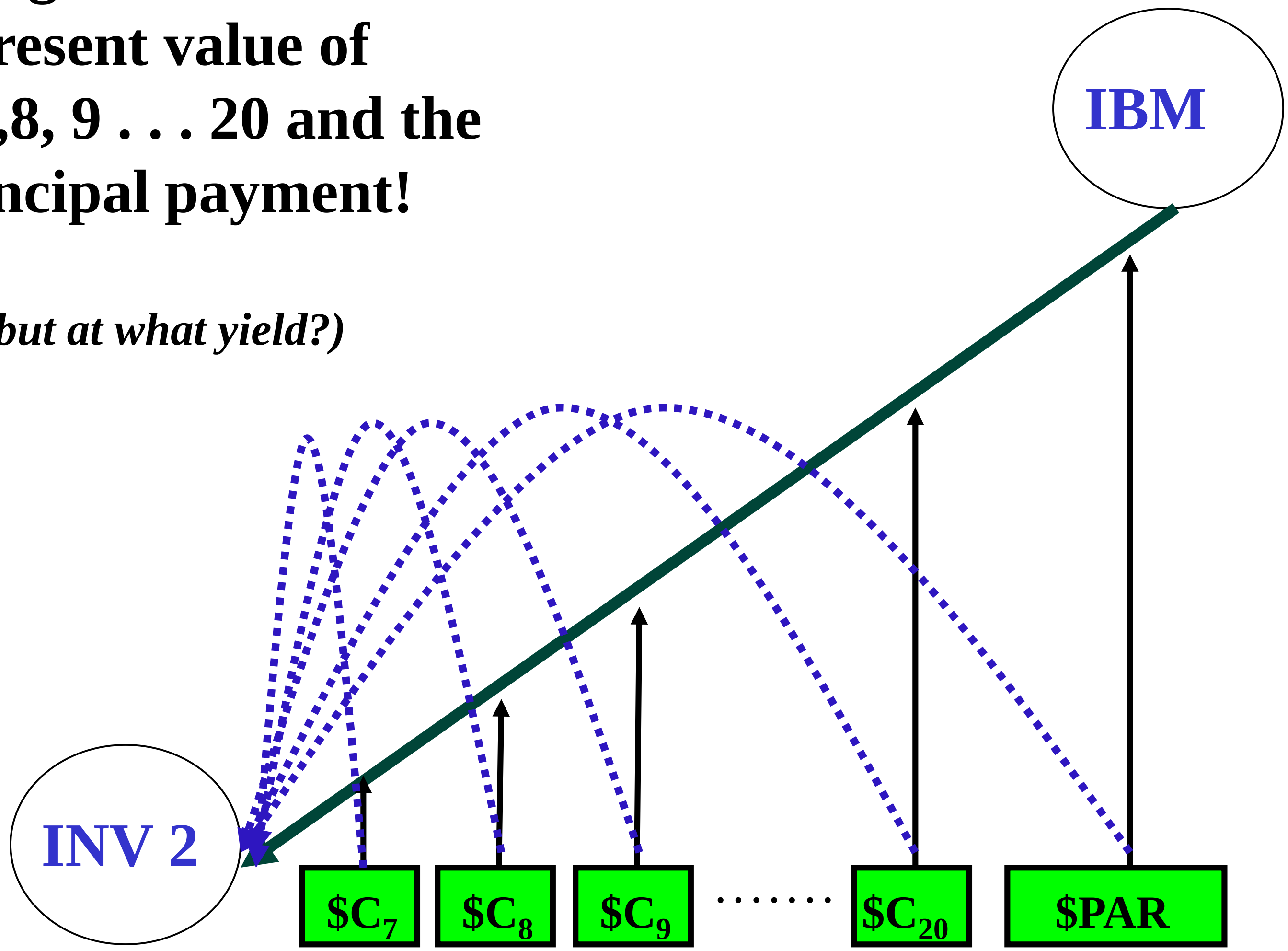


- Investor 2 paid Investor 1 for the right to receive
  - Cashflows 7, 8, 9, . . .20
  - The principal payment (par, face, loan amount, etc.)
- What should that price have been?

Meaning of Price - VI

You guessed it: the present value of CFs 7,8, 9 . . . 20 and the principal payment!

(but at what yield?)



Meaning of Price - VII

- “Price” is our valuation of the bond, not the market’s

- “Rich / cheap” analysis
  - Is your calculated price higher / lower than the market price?
  - Higher?
    - Buy the bond
  - Lower?
    - Sell the bond
  - Wait and see whether you were right

- Why price calculation may differ?
  - Different ideas of what yield to use
  - Different ideas of what cashflows to use (?)

Calculation - I

- The price of a bond is the present value of all future cashflows (coupons, principal payment), discounted at some market/expected yield

- The "heart" of bond math

- We can use either of the formulas we've learned

$$PV = \underbrace{\sum_{t=1}^{N} \frac{CF_t}{(1+y)^t}}_{\text{PV of Coupons}} + \underbrace{\frac{P}{(1+y)^N}}_{\text{PV of Principal}}$$

$$PV = \underbrace{c\left[\frac{1-\left[\frac{1}{(1+y)^N}\right]}{y}\right]}_{\text{PV of Coupons}} + \underbrace{\frac{P}{(1+y)^N}}_{\text{PV of Principal}}$$

Know these formulas!

Calculation - II

- Which yield to use?
  - The yield “on comparable bonds in the market . . . option-free bonds of the same credit quality and the same maturity.” (Fabozzi, p.66)
  - Is using the same yield for each cashflow always appropriate?

- Yield is quoted as annual interest rate
  - Must alter depending on compounding before plugging into PV equation
  - Must alter period numbers the same way

- Which cashflows to use?
  - What if we don’t know the cashflows ahead of time?
    - MBS, callables, derivatives, etc.
    - Build a team of highly skilled individuls (researchers/quants/developers etc)

Calculation - III

Par $1000

9% Coupon

20 years to maturity

Yields are at **12%**

- PV of coupons

$$PV = \left(\frac{0.09*1000}{2}\right)\left[\frac{1-\left[\dfrac{1}{\left(1+\left(\dfrac{0.12}{2}\right)\right)^{20*2}}\right]}{\left(\dfrac{0.12}{2}\right)}\right]$$

- PV of Principal

$$PV = \frac{1000}{\left(1+\left(\dfrac{0.12}{2}\right)\right)^{20*2}}$$

- Sum is $774.30
- Priced at a **discount**

Calculation - IV

Par $1000

9% Coupon

20 years to maturity

Yields are at **7%**

- PV of coupons

$$PV = \left(\frac{0.09 * 1000}{2}\right)\left[\frac{1 - \left[\frac{1}{\left(1 + \left(\frac{0.07}{2}\right)\right)^{20*2}}\right]}{\left(\frac{0.07}{2}\right)}\right]$$

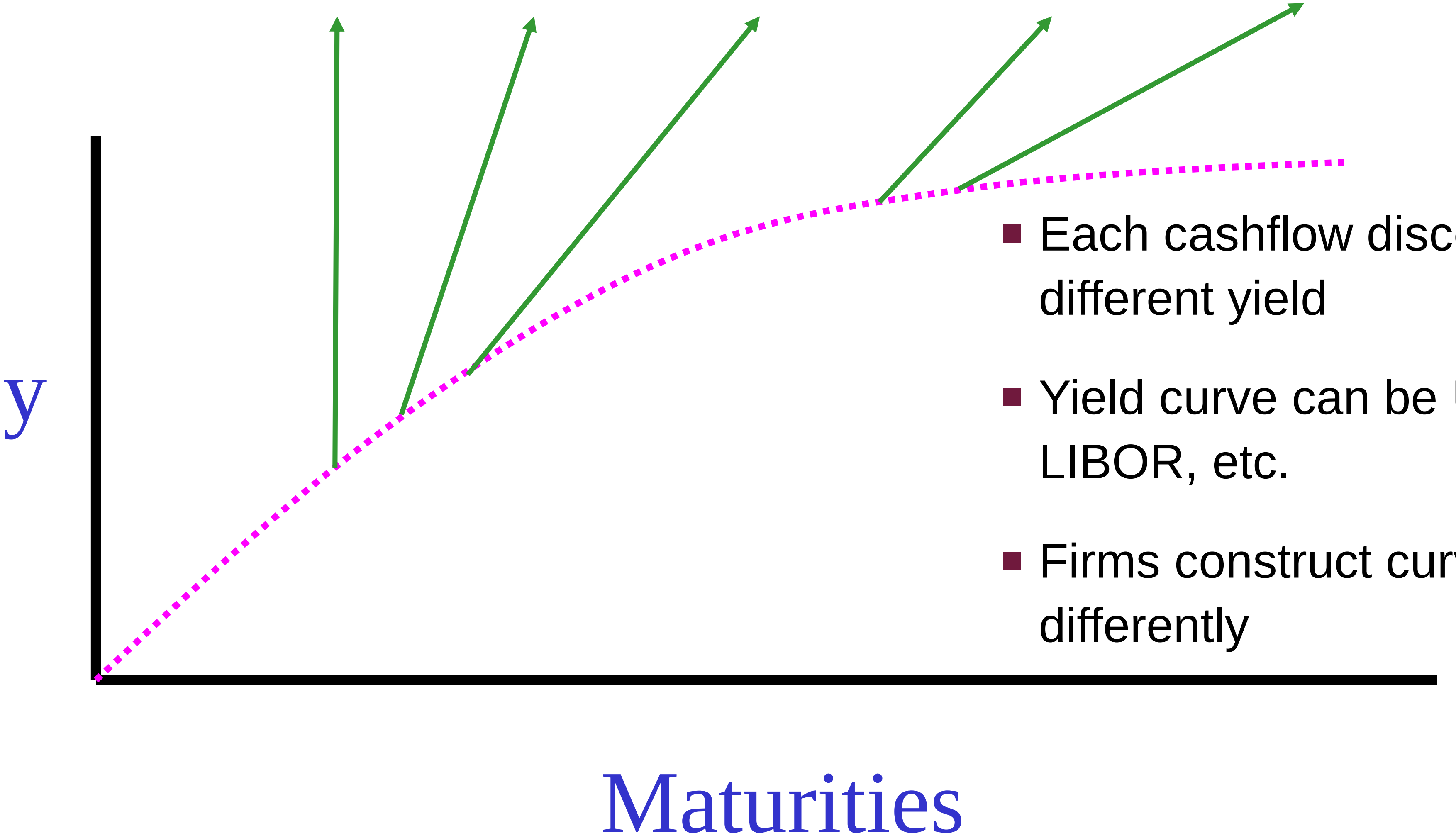
- PV of Principal

$$PV = \frac{1000}{\left(1 + \left(\frac{0.07}{2}\right)\right)^{20*2}}$$

- Sum is $1213.55
- Priced at a **premium**

Pricing with a Curve

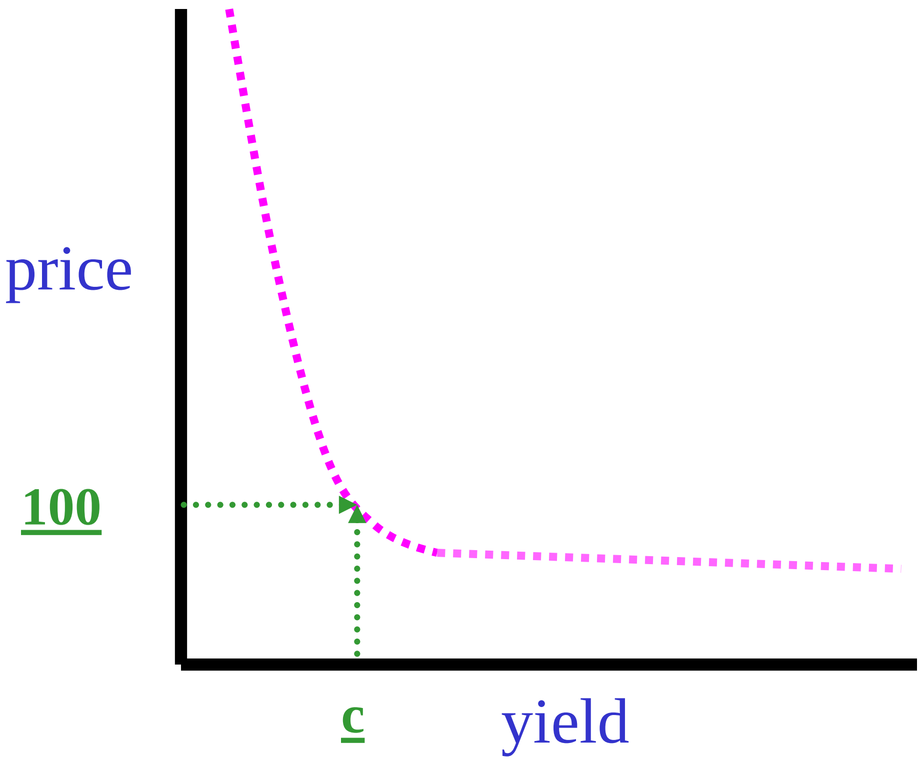
$$PV = \frac{\$50}{(1+y)^1} + \frac{\$50}{(1+y)^2} + \frac{\$50}{(1+y)^3} + \ldots + \frac{\$50}{(1+y)^{10}} + \frac{\$1000}{(1+y)^{10}}$$



- Each cashflow discounted at different yield
- Yield curve can be US TSYs, LIBOR, etc.
- Firms construct curves differently

Price Relationships - I

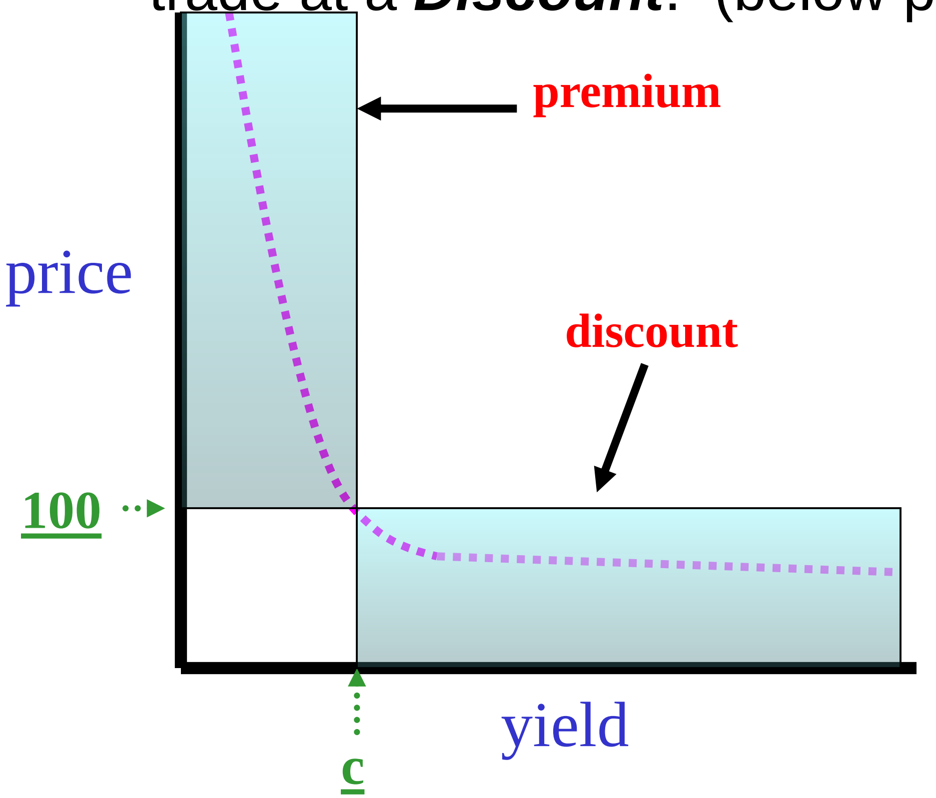
- Price / Yield relationship is “convex”
  - Bowed
  - Upward slope steeper
- Remember: “Price Up, Yield Down”…. “Price Down, Yield Up”. Why?



$$PV = \sum_{t=1}^{N} \frac{CF_t}{(1+y)^t} + \frac{P}{(1+y)^N}$$

Price Relationships - II

- Bonds with coupons at current market levels trade at par.
- Bonds with coupons that are higher than prevailing market levels, trade at a ***Premium***. (above par)
- Bonds with coupons that are lower than prevailing market levels trade at a ***Discount***. (below par)

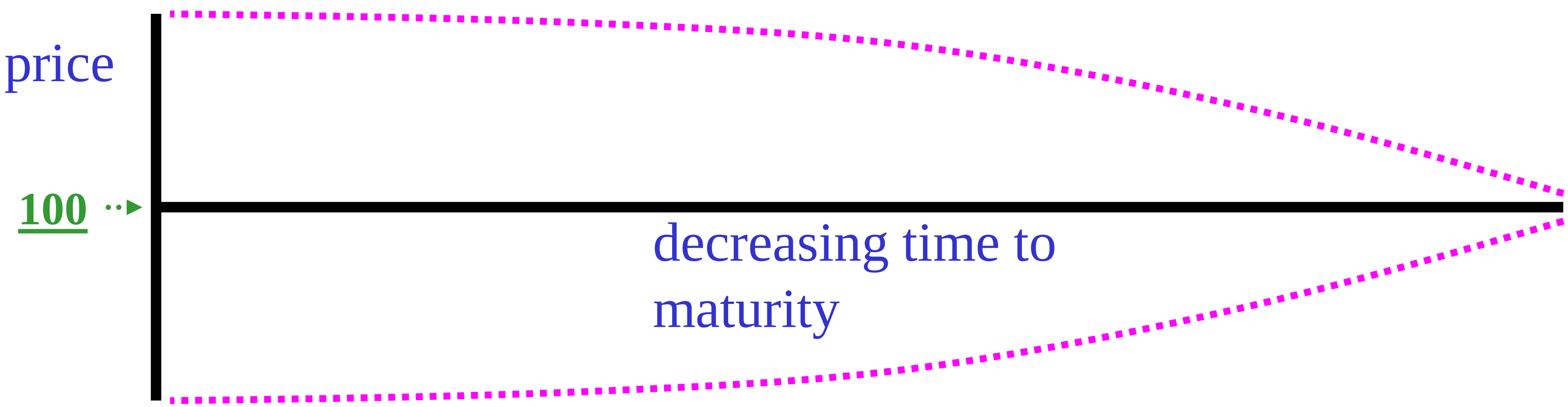


$$PV = \sum_{t=1}^{N} \frac{CF_t}{(1+y)^t} + \frac{P}{(1+y)^N}$$

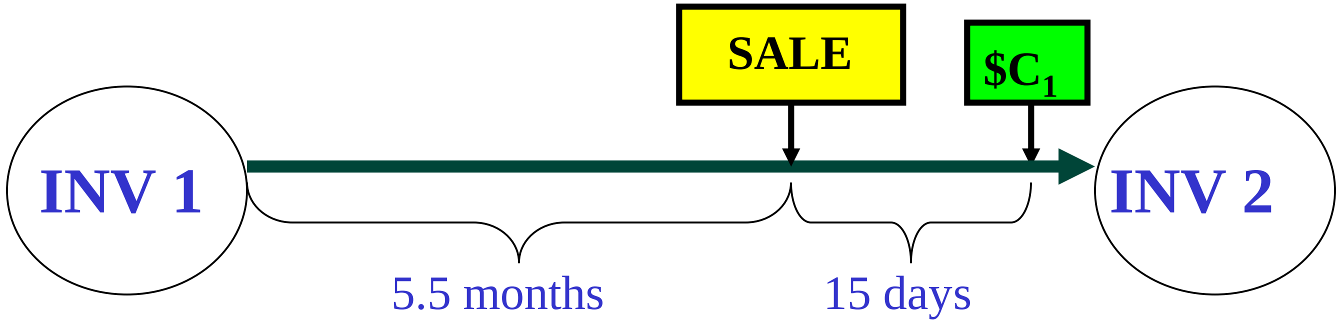
Price Relationships - III

- As time to maturity decreases
  - Premium priced bond gets cheaper
  - Discount bond gets more expensive
  - “Pull to par”

$$PV = \frac{\$CF}{(1+y)^1} + \frac{\$CF}{(1+y)^2} + \frac{\$CF}{(1+y)^3} + \ldots + \frac{\$CF}{(1+y)^{10}} + \frac{\$P}{(1+y)^{10}}$$



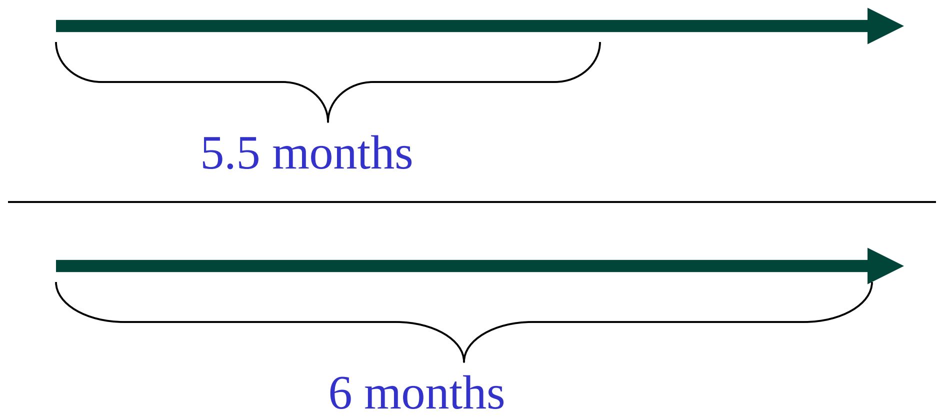
Accrued Interest - I



- Investor 1 sells to Investor 2 only 15 days before coupon paid
- Investor 2 gets the entire coupon
- Does Investor 1 lose the 5.5 months of “work” towards the coupon?
- No, Investor 2 must pay to Investor 1 the **accrued interest**
- The 5.5 months worth of interest that Investor 1 earned

Accrued Interest - II

$$AI = C\left[\frac{days_{pc \to settle}}{days_{inPeriod}}\right]$$



- Multiplying coupon by fractional portion of period elapsed before sale
  - Include *only one of the two bracketing dates*
  - If this is the first coupon, use the dated date instead of previous coupon date

- Different bonds use different conventions to count days

Daycounts

Days in Month / Days in Year

$$\left[\frac{days_{pc \to settle}}{days_{inPeriod}}\right]$$

Instrument	Daycount
US TSY Bills	Actual/360
US TSY Notes, Bonds, Strips	Actual/Actual
Agencies, Corporates, Munis	30/360

- Try the Bloomberg “DCX” Screen
- *Handbook of Global Fixed Income Calculations*, Dragomir Krgin (2002)

Clean Price and Dirty Price - I

- The full purchase price of a bond includes accrued interest – the ***dirty price*** or ***street price***

- Price not counting accrued interest is ***clean price*** or ***flat price***

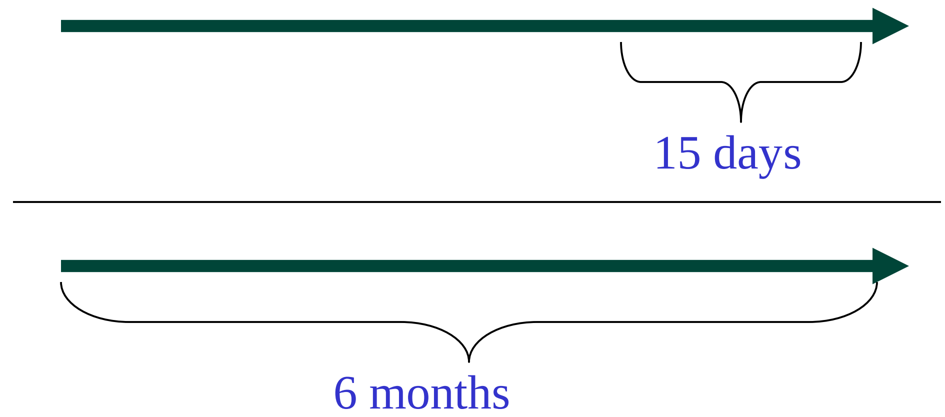
- Traders usually quote clean price

Dirty Price = Clean Price + Accrued Interest

$$\sum_{t=1}^{N} \frac{CF_t}{(1+y)^t} + \frac{P}{(1+y)^N} \qquad CF_1 \left[\frac{days_{pc \to settle}}{days_{inPeriod}} \right]$$

Clean Price and Dirty Price - II

$$w = \left[\frac{days_{settle \to nc}}{days_{inPeriod}} \right]$$



- Calculate dirty price directly by adjusting all periods by “w” offset
- Discounting over a shortened first coupon period
- Includes the portion of the first coupon that really belongs to the seller

$$PV = \frac{\$CF}{(1+y)^{w}} + \frac{\$CF}{(1+y)^{1+w}} + \frac{\$CF}{(1+y)^{2+w}} + \ldots + \frac{\$CF}{(1+y)^{n-1+w}} + \frac{\$P}{(1+y)^{n-1+w}}$$