Pagal K

3/01/020

Probability, Statistics and Randowm Processes

Quiz	(3)	:	30
Assignment	(1)	:	10
Mid Sems	(2)	:	30
End Sem	(1)	:	30

- Text Books:
  - ⇒ Introduction to probability and statistics for engineers & scientists, Ross, S.N., Academic Press
  - ⇒ A first course in probability, Ross, S.N., Prantice Hall

- Reference Books:
  - ⇒ An Indroduction to probability and statics, Rohtagi, V.K and Saleh

6 Jan 020

Experiment

Random Experiment : An experiment where -

- All possible outcomes are known in advance
- Any performance of the experiment results an outcome, i.e., known in advance
- Experiment can be repeated under identical condition

Sample Space

All possible outcomes of a random experiment

Eg., for a coin, $\Omega = \{H, T\}$

for a dice, $\Omega = \{1,2,3,4,5,6\}$

Event

Subset of the sample space.

Eg., $A = \{2,4,6\}$; $C = \{1\}$

$B = \{1,5,6\}$; $D = \phi$

$E = \{1,5,3\}$

$A \cup E = \{1,2,3,4,5,6\}$

Mutually Exclusive Events / Disjoint Events

Two events A & B in Ω are mutually exclusive if $A \cap B = \phi$

Exhaustive set of events

A set of events $A_1, A_2 \dots A_n$ are said to be exhaustive ~~ex~~ set of events if

$$\bigcup_{i=1}^{n} A_i = \Omega$$

Date
Page

Certain Event

Event that always occurs. E.g. Ω as $\Omega \subseteq \Omega$

Impossible Event

Event that never occurs

Translating words into sets

	Set
• Sample space	- Ω
• ω is a possible outcome	- $\omega \in \Omega$
• s is an impossible outcome	- $\omega \notin \Omega$
• A is an event	- $A \subseteq \Omega$
• A is occured	- $\omega_{actual} \in A$
• A or B	- $A \cup B$
• A and B	- $A \cap B$
• A or B but not both	- $(A \cap B^c) \cup (B \cap A^c)$
• At least one of the $A_1, A_2 \ldots A_n$ occurs	- $\bigcup_{i=1}^{n} A_i$
• All A_i occurs	- $\bigcap_{i=1}^{n} A_i$
• A implies B	- $A \subseteq B$

Equally likely Outcomes

A n sets of outcomes in said to be equally likely if each of then has the equal chance to occur

(Naive)
Classical definition of Probability -

Let Ω be the sample space of a random experiment that consists $n\,(<\infty)$ + equally likely set of outcomes. Further let A be an event that has m number of outcomes, then probability of A :

$$P(A) = \frac{m}{n} = \frac{\text{no. of favourable outcomes for } A}{\text{Total no. of outcomes}}$$

E.g., $\Omega = \{H, T\}$ $\qquad$ $\Omega = \{HH, HT, TH, TT\}$

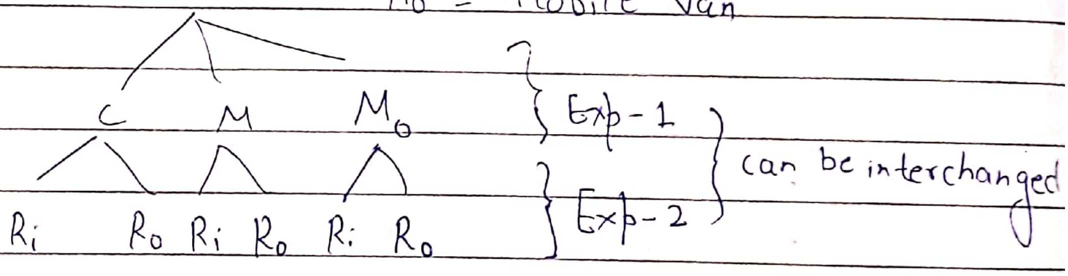
$P(H) = \frac{1}{2}$ $\qquad$ A : At least 1 H

$P(A) = \frac{3}{4}$

7 Jan 020 E.g. $R_i \equiv$ Rice $\qquad$ $C \equiv$ Canteen

$R_o \equiv$ Roti $\qquad$ $M \equiv$ Mess

$M_o \equiv$ Mobile Van



Multiplication Rule :

Suppose that 'r' experiments are performed as follows :

~~For first experiment there are n_1 options, for second n_2. Similarly for r^{th} experiment n_r options~~

For first experiment there are n_1 options and for each outcome of ~~the~~ first experiment there are n_2 possible outcomes in the second experiment and for each option of the first two experiments there are n_3 number of outcomes in the third experiment and so on.

Then ~~Th~~ there are total $n_1 n_2 n_3 \ldots n_r$ possible outcomes.

$n! = n * n-1 * n-2 * \cdots * 3 * 2 * 1$

$0! = 1$

Sampling Table:

How many ways are there so that k objects can be chosen from a set of n objects?

with replacement	n^k ①	${}^{n-1+k}C_k$ ②
without replacement	${}^nP_k$ ③	${}^nC_k$ ④
	order matters	order doesn't matter

① $n * n * n \cdots n$ (k times)

$= n^k$

④ $\dfrac{n * (n-1) * (n-2) * \cdots * (n-k+1)}{k!} = {}^nC_k = \binom{n}{k}$

③ $n * (n-1) * (n-2) * \cdots * (n-k+1) = {}^nP_k$

② ${}^{n-1+k}C_k$

Q) Birthday paradox -

$P(\text{No match}) = \dfrac{365 * (365-1) * \cdots * (365-k+1)}{(365)^k}$

$P(\text{at least 1 match}) = 1 - P(\text{No match})$

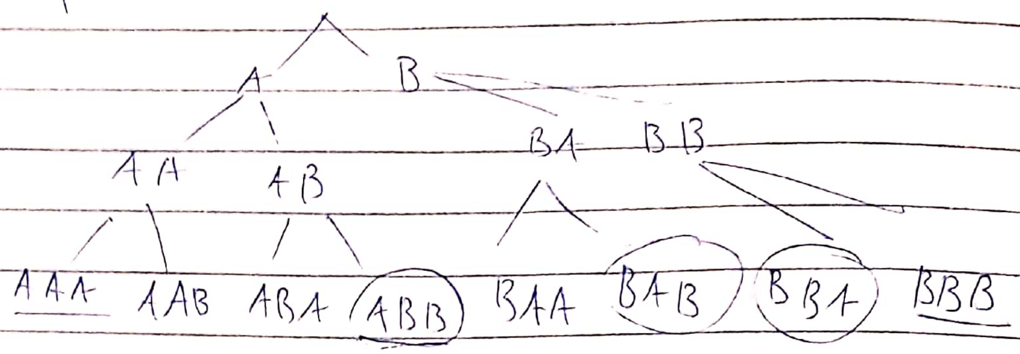
If $k = 23$ 50%

50 97%

57 99.7%

8 Jan 020

With replacement order doesn't matter -



$2^3 = 8$ (order matters)

Order doesn't matter = 4

(AAA) (ABB) (ABA) (BBB)

$n = 2 \quad k = 3 \qquad {}^{n+k-1}C_k = \binom{2+3-1}{3} = \binom{4}{3} = 4$

Case $k = 0$

$${}^{n+k-1}C_0 = 1$$

Case $k = 1$

$${}^{n+1-1}C_1 = n$$

Case $n = 2$

$${}^{n+k-1}C_k = k+1$$

$\boxed{A}$	$\boxed{B}$
0	k
1	$k-1$
2	$k-2$
⋮	⋮
k	0

$n = 2$

$k+1$

Case $n = 4$ $\quad k = 7$

$$\begin{array}{cccc} A & B & C & D \\ \cdot \;| & \cdot\cdot \;| & \cdot\cdot \;| & \cdot\cdot \\ \cdot\cdot\cdot \;| & \cdot\cdot\cdot \;| & \;| & \cdot\cdot \end{array}$$

$\rightarrow$ 10 positions

changing separators

$\Rightarrow$ Positions $= n + k - 1$

$${}^{n+k-1}C_{n-1}$$

$$= {}^{n+k-1}C_{k}$$

Bohr's Einstein

Identify indistinguisable and distinguisable objects.

(1) $\binom{n}{k} = \binom{n}{n-k}$

(2) $n \binom{n-1}{k-1} = k \binom{n}{k}$.

(3) Vandermond's identity –

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

(4) Binomial Theorem –

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)(x+y) = x^2 + xy + yx + y^2$$

$$= \sum_{k=0}^{2} \binom{2}{k} x^k y^{2-k}$$

f 1 – 1 onto

bijection surjective

9 Jan 2020

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

infinite but countable

↳ we can't use classical definition of probability ↲

$\Omega = [0, \infty) \subseteq \mathbb{R} \rightarrow$ infinite & uncountable

Axiomatic def^n of probability –

Let Ω be the sample space of a random experiment. Then,

(i) $0 \le P(E) \le 1$, for any event E

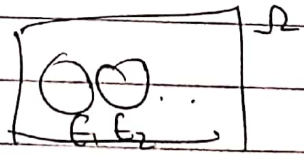
(ii) $P(\Omega) = 1$ & $P(\phi) = 0$

(iii) Let $E_1, E_2, \dots$ be a countable set of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \leftarrow \text{Countable add}^n$$

$P(E_1 \cup E_2 \cup E_3 \dots)$

$= P(E_1) + P(E_2) + \dots$

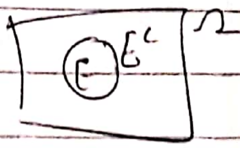


Result: ① $P(E^c) = 1 - P(E)$, for an event E.

$P(E \cup E^c) = P(\Omega)$

$P(E) + P(E^c) = 1$

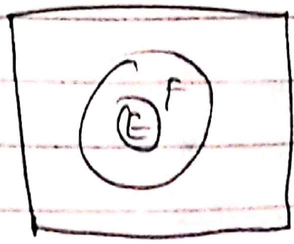
$P(E^c) = 1 - P(E)$



② $E \subseteq F$, then $P(E) \le P(F)$

$F - E = F \cap E^c$

$F = E \cup (F \cap E^c)$



$=$

$$A - B = A \cap B^c$$

Date
Page

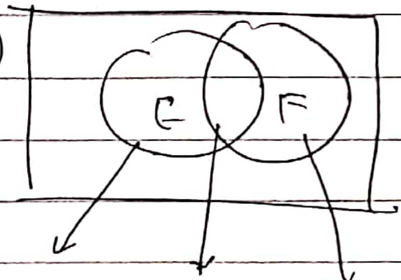
$$P(F) = P(E) + P(F \cap E^c)$$

$$P(F) - P(E) = P(F \cap E^c) \geq 0$$

$$P(F) \geq P(E)$$

③ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$E \cup F = (E \cap F^c) \cup (E \cap F) \cup (F \cap E^c)$$



$$P(E \cup F) = P(E \cap F^c) + P(E \cap F) + P(F \cap E^c)$$

$$E = (E \cap F^c) \cup (E \cap F)$$

$$P(E \cup F) = \underline{P(E \cap F^c) + P(E \cap F)} + \underline{P(F \cap E^c) + P(E \cap F)} - P(E \cap F)$$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

④ $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(G \cap E) + P(E \cap F \cap G)$

⑤ <u>Inclusion Exclusion formula</u> -

$$P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\Rightarrow \sum_{i=1}^{n} P(E_i) - \sum_{1 \leq i < j \leq n} P(E_i \cap E_j) + \sum P(E_i \cap E_j \cap E_k) - \cdots (-1)^{n+1} P(E_1 \cap E_2 \cap E_3 \cdots \cap E_n)$$

Veeee Vee

Pagal

10 Jan 020

Q) N students N hats
a) No one gets his/her hat
b) out of N, k gets their hat

→ (a) Compliment - At least one of them gets his/her own hat

E_i = i^{th} person getting his/her own hat.

$E_1 \cup E_2 \cup \dots E_N$ At least one of them get hat.

$P(\bigcup_{i=1}^{N} E_i) = \cancel{P(E_1) + P(E_2) + \dots + P(E_n)}$

$P(E_1) = \frac{(N-1)!}{N!} = \frac{1}{N}$

$P(E_2) = \frac{1}{N}$

$P(E_i) = \frac{1}{N} \quad (i = 1, 2, 3 \dots N)$

$P(E_1 \cap E_2) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}$

$P(E_i \cap E_j) = \frac{1}{N(N-1)} \quad ; 1 \le i < j \le N$

$P(E_1 \cap E_2 \cap \dots E_k) = \frac{(N-k)!}{N!}$

$P(E_1 \cap E_2 \cap \dots E_n) = \frac{(N-N)!}{N!} = \frac{1}{N!}$

$P(\bigcup_{i=1}^{N} E_i) = N \times \frac{1}{N} - {}^NC_2 \frac{1}{N(N-1)} + {}^NC_3 \frac{(N-3)!}{N!}$

$+ \dots (-1)^{k+1} \frac{(N-k)!}{N!} {}^NC_k$

$$+ (-1)^{N+1} \frac{1}{N!}$$

$$\Rightarrow 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + (-1)^{N+1} \frac{1}{N!}$$

$$P(E_1^c \cap E_2^c \cap \dots E_N^c) = 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} + \dots + (-1)^{N+1} \frac{1}{N!}\right)$$

$$= 1 - e^{-1}$$

Conditional Probability –

Let A and B be two events such that $P(E) > 0$

Then $P(A|B) = \begin{cases} \frac{P(A \cap B)}{P(B)} & ; \ P(B) > 0 \\ \text{not defined}, & \text{when } P(B) = 0 \end{cases}$

Eg., A : 6 (dice) $P(A) = \frac{1}{6}$

B → even no.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{\cancel{36} \times \frac{1}{2}}$$

- $P(B|B) = 1 = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)}$

- $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

- $P(\phi|B) = 0 = \frac{P(\phi \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Ve...

13 Jan 020

E.g., $A = \{2,3\}$ $\Omega = \{1,2,3,4,5,6\}$

$B = \{2,4,5,6\}$

$P(A) = \frac{2}{6}$ $P(B) = \frac{4}{6}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4}$$

$P(\Omega/B) = 1$ $P(B|B) = 1$

Result: If $P(B) > 0$ if $\{A_1, A_2 \ldots\}$ are mutually exclusive set of events.

Then

$$P\left(\bigcup_{i=1}^{\infty} A_i | B\right) = \sum_{i=1}^{\infty} P(A_i|B)$$ (Axiom - 3)

$$P\left(\bigcup_{i=1}^{\infty} A_i | B\right) = \frac{P\left(\bigcup_{i=1}^{\infty} A_i | B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} A_i \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{i=1}^{\infty} F_i\right)}{P(B)}, \quad F_i = A_i \cap B$$

$F_i \cap F_j = \phi$

Now, $F_i \cap F_j$

$(A_i \cap B) \cap (A_j \cap B)$

$\Rightarrow (A_i \cap A_j) \cap B$

$\Rightarrow \phi \cap B$ as A_i, A_j are disjoint

$= 0$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} F_i\right) = \Sigma P(F_i)$$

Scanned by CamScanner

$$\therefore \frac{P\left(\bigcup_{i=1}^{\infty} F_i\right)}{P(B)} = \frac{\sum_{i=1}^{\infty} P(F_i)}{P(B)}$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)}$$

$$= \sum_{i=1}^{\infty} P(A_i \mid B)$$

A_1, A_2 $- \; P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)}$

$$P(A_1 \cap A_2) = P(A_1 \mid A_2) * P(A_2)$$

$$- \; P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)}$$

$$P(A_1 \cap A_2) = P(A_2 \mid A_1)\, P(A_1)$$

$A_1 A_2 A_3$ $- \; P(\underbrace{A_1 \cap A_2}_{E} \cap A_3) = P(A_3 \mid A_1 A_2)\, P(A_2 \mid A_1)\, P(A_1)$

$$P(A_3 \cap E) = P(A_3 \mid E)\, P(E)$$

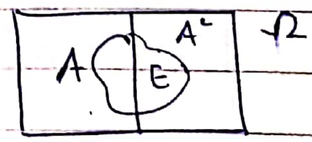
$$= P(A_3 \mid A_1 \cap A_2)\, P(A_1 \cap A_2)$$

$$= P(A_3 \mid A_1 \cap A_2)\, P(A_2 \mid A_1)\, P(A_1)$$

☆ ☆ $P(A_1 A_2 \ldots A_n) = P(A_n \mid A_1 A_2 \ldots A_{n-1})\, P(A_{n-1} \mid A_1 \ldots A_{n-2}) \ldots P(A_2 \mid A_1)\, P(A_1)$

Total Law of Probability -

$$E = (E \cap A) \cup (E \cap A^c)$$



$$\left.\begin{array}{l} A \cap A^c = \phi \\ A \cup A^c = \Omega \end{array}\right\} \text{partition}$$

$$P(E) = P(E \cap A) + P(E \cap A^c)$$

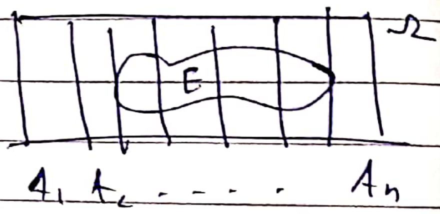
$$P(E) = P(E|A)P(A) + P(E|A^c)P(A^c)$$

Result: Let $A_1, A_2, \ldots$ be a partion on the sample space then for an event E

$$P(E) = P(E|A_1) + P(E|A_2)P(A_2) + \ldots$$

$$= \sum_{i=1}^{\infty} P(E|A_i)P(A_i)$$

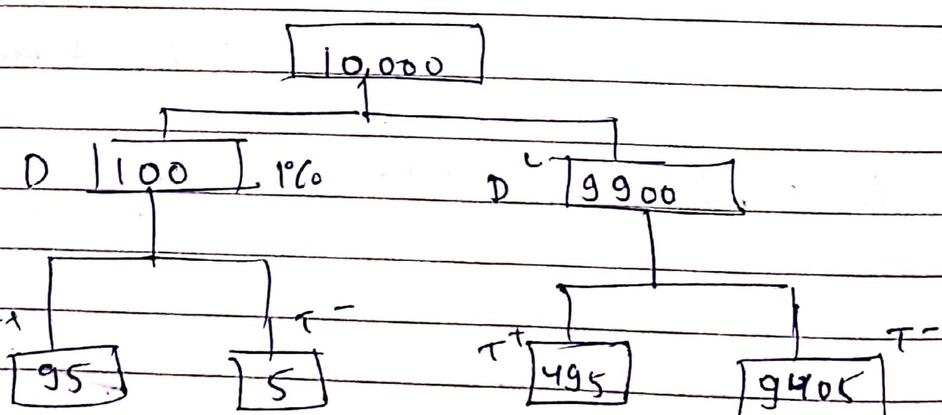
$$E = (E \cap A_1) \cup (E \cap A_2) \ldots$$



$$A_i \cap A_j = \phi$$
$$\cup A_i = \Omega$$

Baye's Theorem —

$$P(A_n / E) = \frac{P(A_n \cap E)}{P(E)} = \frac{P(E|A_n)P(A_1)}{\Sigma P(E/A_i)\,P(A_i)}$$



$$P(D|T^+) = \frac{9500}{95+495} = \approx 0.16$$

$$P(D) = 0.01 \qquad P(T|D) = 0.95$$
$$P(D^c) = 0.99 \qquad P(T^+|D^c) = 0.05$$

$$P(D|T) = \frac{P(D \cap T)}{P(T)}$$
$$= \frac{P(T|D)\,P(D)}{P(T)}$$
$$= \frac{P(T|D)\,P(D)}{P(T \cap D) + P(T \cap D^c)}$$
$$= \frac{P(T|D)\,P(D)}{P(T|D)\,P(D) + P(T|D^c)\,P(D^c)}$$

14 Jan 020

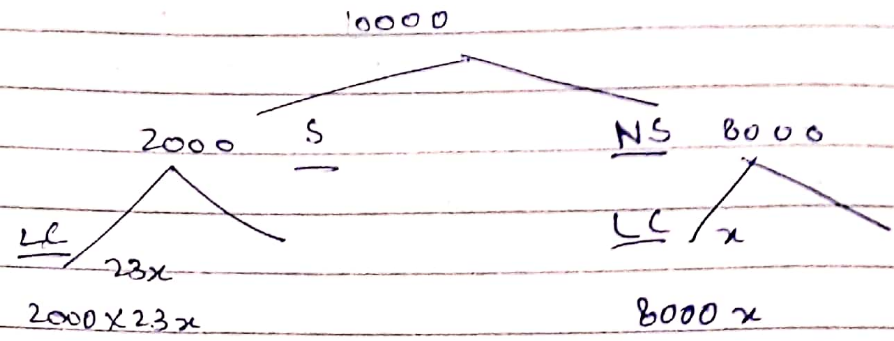
Q) 20% — Smokers 23x more prob of having cancer.

$$P(S|LC) = \frac{P(S \cap LC)}{P(LC)} = \frac{P(LC|S) \times P(S)}{P(LC)}$$
$$= \cancel{P(NS \cap LC)}$$
$$= \frac{P(LC/S)P(S)}{P(LC|S)P(S) + P(LC|NS)P(NS)}$$

$$P(LC/S) = 23\ P(LC/NS)$$
$$P(S) = 0.2 \qquad P(NS) = 0.8$$

$$= \frac{23\ P(LC/NS) \times 0.2}{23 \times P(LC/NS)\,0.2 + P(LC|NS)\,0.8}$$
$$= \frac{4.6}{4.6+0.8} = \frac{4.6}{5.4} = \frac{23}{27}$$

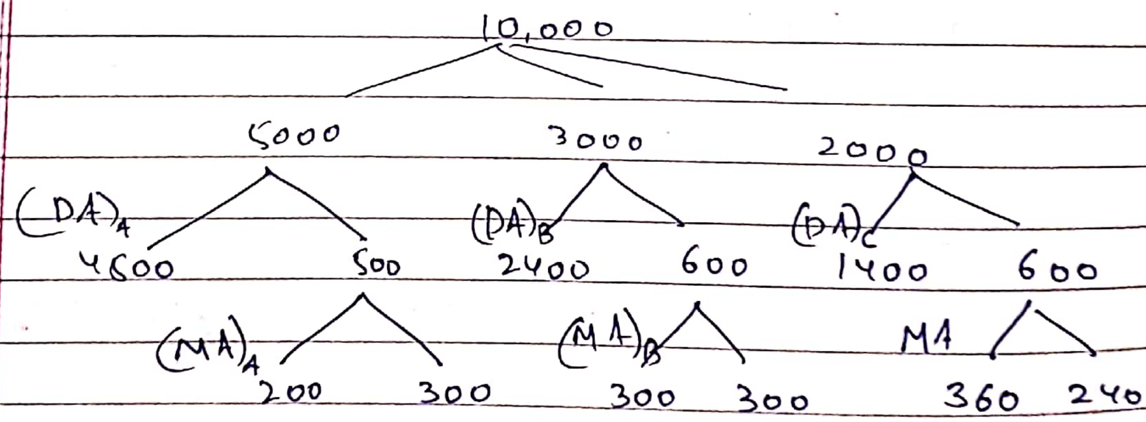
Vec



$$\Rightarrow \frac{\cancel{2000} \times 23x}{\cancel{2000} \times 23x + \cancel{8000}x} = \frac{46}{54} = \frac{23}{27}$$

Q)

	A	B	C
	50%	30%	20%
Fail	10%	20%	30%
After process	40%	50%	60%



$\frac{17 \times 5}{2^6}$

$17 \overline{)86(} 5.0$
85
100

(i) $P(D_A) = \frac{4500 + 2400 + 1400}{10000} = \frac{45 + 38}{100} = 0.83$

(ii) $P(MA) = \frac{200 + 300 + 360}{10000} = 0.086$

(iii) $P(MA/F) = \frac{P(MA)}{P(F)} = \frac{860}{1700} = 0.506$

$43 \overline{)100} (0.2$
86
140

Pagal

(iv) $P(A/MA) = \frac{\cancel{204}}{\cancel{800}} \quad \frac{10}{43} \cdot 0.23$

(v) $P(MA/B^c) = \frac{856}{\cancel{7000}} \cdot \cancel{0.1} \; 0.06$

15 Jan 020

Independance:

Two events A and B ~~are~~ in a sample space are said to be independent if -

$$P(A \cap B) = P(A)\, P(B)$$

If $P(A) > 0$, then A & B are independent iff

$$\frac{P(A \cap B)}{P(A)} = P(B)$$

$$P(B/A) = P(B)$$

Similarly, Suppose that $P(B) > 0$. Then A and B are independent iff

$$P(A/B) = P(A)$$

$A \cap B = \phi$
(different !)

Eg.,
$\Omega = \{1, 2, 3, 4, 5, 6\}$
$A := \{2, 3, 4\}$ $\quad B = \{4, 5\}$
$A \cap B = \{4\} \neq \phi \Rightarrow$ A & B are not disjoint

$P(A \cap B) = \frac{1}{6}$

$P(A) = \frac{3}{6}$ $\quad P(B) = \frac{2}{6}$

$P(A) \times P(B) = \frac{3}{6} \times \frac{2}{6} = \frac{1}{6}$

Eg:, $A = \{2, 3, 4\}$ $\qquad B = \{1, 5, 6\}$

$A \cap B = \phi$

$P(A \cap B) = P(\phi) = 0$

$P(A) = \frac{3}{6} \qquad P(B) = \frac{3}{6}$

$P(A \cap B) \neq P(A)\,P(B)$

Result: If A & B are disjoint and either $P(A) = 0$ or $P(B) = 0$, then A and B are independent

$A \cap B = \phi \qquad P(A \cap B) = 0 \Rightarrow P(A)\,P(B) = 0$

$\Downarrow$

$P(A) = 0$ or $P(B) = 0$

☆ Theorem: The following statement are equivalent -

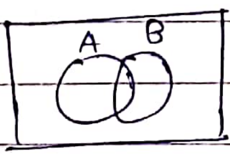
(i) A & B are independent

(ii) A^c & B " "

(iii) A & B^c " "

(iv) A^c & B^c " "

Proof: (i) $\Rightarrow$ (ii)



$P(A \cap B) = P(A)\,P(B)$ (Given)

Show: $P(A^c \cap B) = P(A^c)\,P(B)$

$\Rightarrow B = (A \cap B) \cup (A^c \cup B)$

$P(B) = P(A \cap B) + P(A^c \cup B)$

$P(A^c \cup B) = P(B) - P(A)\,P(B)$

$= P(B)\,P(A^c)$

Scanned by CamScanner

Independance of 3 events:

Three events A, B and C are said to be independent if the following condn's hold -

Independent
$$\left\{\begin{array}{l} P(A \cap B) = P(A)\,P(B) \\ P(B \cap C) = P(B)\,P(C) \\ P(C \cap A) = P(C)\,P(A) \\ P(A \cap B \cap C) = P(A)\,P(B)\,\&\,P(C) \end{array}\right.$$

(first three) } pair wise independence

- Pair-wise independence doesn't imply total independence

E.g., 2 coins $\Omega = \{HH, HT, TH, TT\}$

A: First H $P(A) = 1/2$

B: Second H $P(B) = 1/2$

C: Both same $P(C) = 1/2$

$A \cap B \cap C = \{HH\} \Rightarrow P(A \cap B \cap C) = \frac{1}{4}$

$P(A \cap B) = \{HH\}$ $P(A)P(B) = \frac{1}{4} = P(A \cap B)$

$P(A \cap C) = \{HH\}$ $P(A \cap C) = P(A)P(C)$

$P(B \cap C) = \{HH\}$ $P(B \cap C) = P(B)P(C)$

$P(A \cap B \cap C) \neq P(A)P(B)P(C)$

E.g., ~~P(A)~~ $P(A \cap B \cap C) = P(A)P(B)P(C)$, then we can't show that A, B, C are pair wise independent

$A = \phi$ $B = \{1, 2\}$ $C = \{2, 4\}$

$A \cap B \cap C = \phi$ $P(A \cap B \cap C) = 0$ $P(A)P(B)P(C) = 0$

$P(B\cap C) \neq P(B)\,P(C)$

$\frac{1}{6} \neq \frac{2}{6} \times \frac{2}{6}$

1296
671
625

Tutorial sheet-1

1) A $\equiv$ 2 W 2 B $\xrightarrow{1}$

B $\equiv$ 3 W 2 B

~~P(W)~~ T_w : Transferred ball was white.

W : White ball

$$P(T_w / W) = \frac{P(T_w \cap W)}{P(W)}$$

$$= \frac{P(W/T_w)\,P(T_w)}{P(W|T_w)\,P(T_w) + P(W|T_w^c)\,P(T_w^c)}$$

$$= \frac{\frac{4}{6} \times \frac{2}{4}}{\frac{4}{6}\times\frac{2}{4} + \frac{3}{6}\times\frac{2}{4}} = \frac{4}{7}$$

7) A : More than 4 throws to get a six

B : 6 doesn't turn up at the 1^{st} throw.

$P(A|B)$

216×6
1296

$P(A^c) = P(I_6) + P(II_6) + P(III_6) + P(IV_6)$

$= \frac{1}{6} + \frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

216
180
150
125
671

$= \frac{216 + 180 + 150 + 125}{216 \times 6} = \frac{671}{216\times 6} = \frac{671}{1296}$

Scanned by CamScanner

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{671}{1296 \times \frac{5}{6}}$$

$$P(A) = \frac{625}{1296}$$

11) $P(I) = 0.4$ $\quad$ $P(III) = 0.2$

$P(II) = 0.3$ $\quad$ $P(IV) = 0.1$

$$P(\overline{A}) = 0.6 \times 0.7 \times 0.8 \times 0.1$$
$$= 0.0384$$

$$1 - P(\overline{A}) = 0.9616$$

17 Jan 026

A set of events $\{A_1, A_2, A_3 \ldots A_n\}$ are said to be independent if

$$P(A_i \cap A_j) = P(A_i)\,P(A_j) \quad \forall\ i \neq j;\ i,j = 1,2,\ldots n$$
$$P(A_i \cap A_j \cap A_k) = P(A_i)\,P(A_j)\,P(A_k) \quad \forall\ i \neq j \neq k;\ i,j,k = 1,2\ldots n$$
$$\vdots$$
$$P(A_i \cap A_j \cap \ldots \cap A_n) = P(A_i)\,P(A_j)\ldots P(A_n)$$

Conditional Independence:

Two events A & B are said to be conditional independent given C if

$$P(A\cap B|C) = P(A|C)\,P(B|C)$$

– Q. Independence $\Rightarrow$ Cond^{n}al Independence ?

$\Rightarrow$ Not

Scanned by CamScanner

Pagal

E.g., $M \equiv$ Parents
$F \equiv$ Friends

$$P(M \cap F) = P(M)\ P(F)$$

$R \leftarrow$ The phone is ringing

$$P(M|R) + P(F|R) = 1$$

$\Rightarrow$ $0 < P(M|R) < 1$
$0 < P(F|R) < 1$

$P(A \cup B|C) = P(A|C)\ P(B|C)$
↳ for mutually exclusive events

$$P(M \mid R \cap F^c) = 1 \neq P(M|R)$$

– Q. Cond^{n}al Independence $\Rightarrow$ Independence ?
$\Rightarrow$ No $\Leftarrow$

E.g., A: Biased coin $P(H|A) = 3/4$
B: Unbiased coin $P(H|B) = 1/2$

$$P(A) = P(B) = 1/2$$

H_1 : Head in the first trial
H_2 : Head in the II trial

$$P(H_1H_2|A) = \frac{3}{4} \times \frac{3}{4} = P(H_1|A)\ P(H_2|A)$$

$$P(H_1H_2|B) = \frac{1}{2} \times \frac{1}{2} = P(H_1|B)\ P(H_2|B)$$

To show: $P(H_1H_2) \neq P(H_1)\ P(H_2)$
meaning non-independence

Scanned by CamScanner

$$P(H_1H_2) = P(H_1H_2|A)\,P(A) + P(H_1H_2|B)\,P(B)$$
$$= \frac{3}{4}\times\frac{3}{4}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$
$$= \frac{9}{32} + \frac{1}{8}$$
$$= \frac{13}{32}$$

$$P(H_1) = P(H_1|A)\,P(A) + P(H_1|B)\,P(B)$$
$$= \frac{3}{4}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}$$
$$= \frac{5}{8}$$

$$P(H_2) = P(H_2|A)\,P(A) + P(H_2|B)\,P(B)$$
$$= \frac{5}{8}$$

$$P(H_1H_2) \neq P(H_1)\,P(H_2)$$
$$\frac{13}{32} \neq \frac{25}{64}$$

\# $P(A\cap B|F) = P(A|F)\,P(B|F)$

$\Rightarrow P(A\cap B|F^c) = P(A|F^c)\,P(B|F^c)$?

$\Rightarrow$ No $\Leftarrow$

Q.

	Heart	Bandaid
Success	70	10
Fail	20	0

	Heart	Bandaid
S	2	81
F	8	9

Nikhil (N) $\quad\quad$ (N^c) Nikita

A : Success

$P(A|N,H) = \frac{70}{90} = 0.77$

$P(A^c|N^c,H) = \frac{2}{10} = 0.2$

$P(A|N, H^c) = 1$

$P(A|N^c, H^c) = \frac{81}{90} = 0.9$

$P(A|N) = \frac{80}{100} = 0.8$

$P(A|N^c) = \frac{83}{100} = 0.83$

$P(A) = \Sigma P(A|E_i) P(E_i)$

$P(A|C) = \Sigma P(A|E_i, C) P(E_i|C)$

$P(A|N) = P(A|N, H) P(H|N) + P(A|N, H^c) P(H^c|N)$

$P(A|N^c) = P(A|N^c, H) P(H|N^c) + P(A|N^c, H^c) P(H^c|N^c)$

~~P(A|H)~~ $P(A|N) < P(A|N^c)$

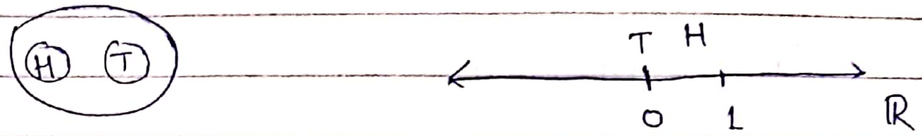
Scanned by CamScanner

20 Jan 020

Random Variable –

A random variable X is a function from sample space to the real line $\{X : \Omega \to \mathbb{R}\}$

E.g.,



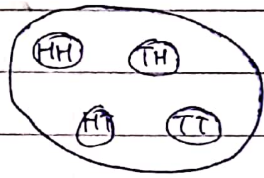
$x(\{H\}) = 1$ $\qquad$ $x = 1 \Leftrightarrow \{H\}$

$x(\{T\}) = 0$

$\{\omega \in \Omega \mid x(\omega) = 1\}$ $\qquad$ $\omega \to H$

$\therefore \quad P(X=1) = P\{\omega \in \Omega \mid X(\omega) = 1\}$

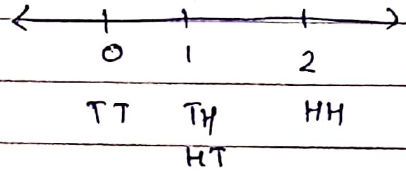
E.g., $\Omega = \{HH, HT, TH, TT\}$



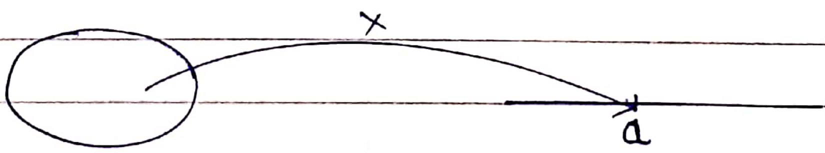
$HH \Leftrightarrow 2$

$TT \Leftrightarrow 0$

$\{HT, TH\} \Leftrightarrow 1$



Degenerate Random Variable –



$P(X=a) = 1$

Bernoulli (p) random variable –

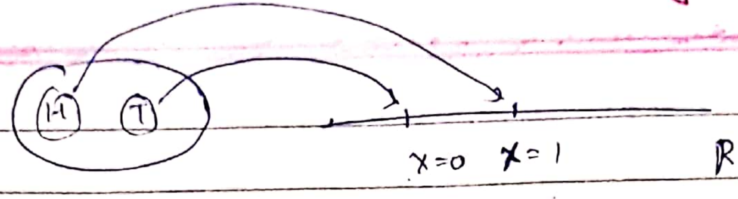
A random variable is said to be Bernoulli random distribution with parameter p, denoted as $X \sim \text{Bernoulli}(p)$, if X takes only two values namely 0 & 1 such that

$$P(X=1) = p$$

$$\& \ P(X=0) = 1-p \quad ; \quad 0 \le p \le 1$$

Scanned by CamScanner

Eg.,



Binomial Distribution

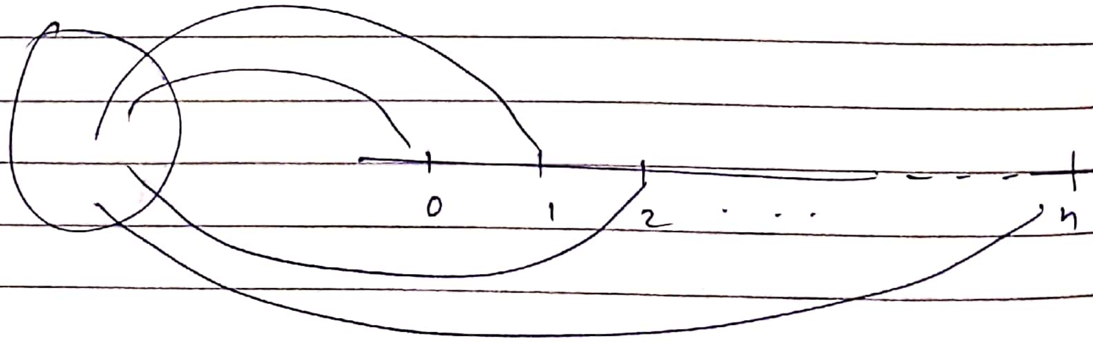
(n, p) : two parameters

Bernoulli Trials – A sequence of n trials such that
(i) each trial gives two outcomes, success & failure
(ii) The probability of success & failure remain unchanged in every trial
(iii) The trials are independent

A random variable X is said to have binomial distribution with parameter n & p, denoted as $X \sim \text{Binomial}(n,p)$, if

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad ; p \equiv \text{prob. of success}$$

$$k = 0, 1, 2 \dots n.$$



X counts the no. of successes in sequence of n Bernoulli~~s~~ trials

Date
Page

Eg., $n = 10$, $k = 4$

10 tosses 4 times Head

H T H H T H T T T T

$$P(X=k) = \binom{10}{4} \left(\frac{1}{2}\right)^4 \left(1-\frac{1}{2}\right)^{10-4}$$

$$\binom{n}{k} \quad p^k \quad (1-p)^{n-k}$$

$X = Z_1 + Z_2 + Z_3 + \cdots + Z_n$, when $Z_i \overset{iid}{\sim}$ Bernoulli(p)

21 Jan 020

$X \sim Bin(x, p)$ if

PMF $\equiv$ $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$; $k = 0, 1, 2 \ldots n$

X : the no. of successes in n-independent Bernoulli trials

X : $X_1 + X_2 + \cdots X_n$; $X_i \overset{iid}{\sim}$ Bernoulli(p)

iid $\equiv$ independent & identically distributed

$X=1$ $\quad X=2$

$P(X_1 = 1, X_2 = 1)$

$= P(X_1 = 1) P(X_2 = 1)$

$$X_i = \begin{cases} 1 ; & i^{th} \text{ trial is success} \\ 0 ; & i^{th} \text{ trial is failure} \end{cases}$$

☆ Result : $X \sim Bin(n, p)$. Then $n - X \sim Bin(n, 1-p)$

Scanned by CamScanner

$P(Y=k) = P(n-X=k) = \binom{n}{k}(1-p)^k p^{n-k}$ | $Y = n - X$

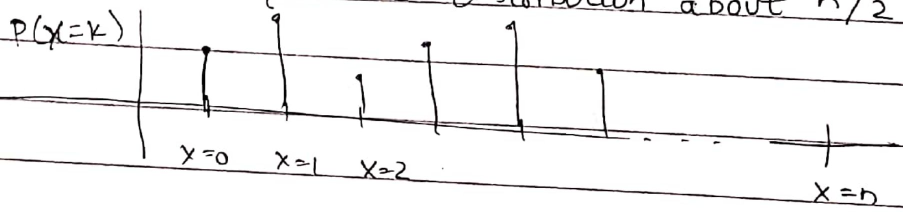
E.g., T T T H H T T

$k=5 \quad n=7$

$$P(Y=k) = \binom{n}{k}(1-p)^k p^{n-k}$$

$$P(X=n-k) = \binom{n}{n-k} p^{n-k}(1-p)^k$$

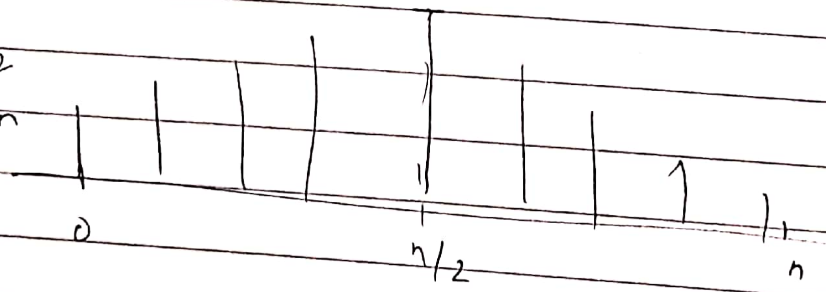
☆ Result: $X \sim Bin(n,p)$ when n is even & $p = 1/2$. Then X has symmetric distribution about $n/2$



$$\left.\begin{aligned} P(X=k) &= \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} \\ P(Y=k) &= \binom{n}{n-k}\left(\frac{1}{2}\right)^{n-k}\left(\frac{1}{2}\right)^k \end{aligned}\right\} \text{Symmetric}$$

$p = 1/2$

n = even



X is said to be discrete if the range of X is finite or countable infinite

Pagal

Probability Mass Function - (PMF)

Let X be a random variable and the suppourt of X is $\{a_1, a_2, \dots a_n\}$. Suppose that

$$P(X = a_i) = p_i$$

Then $\{p_1, p_2, \dots p_i\}$ is called probability mass function of X if

(i) $p_i > 0$ (ii) $\sum p_i = 1$

Cummulative Distributive Function (CDF) -

Let x be a random variable

$$0 \leq F_X(x) = P(X \leq x) \leq 1$$

(i) $x < y \Rightarrow F_X(x) \leq F_X(y)$

(Non-decreasing) $P_X(X \leq x) \leq P(X \leq y)$

$$F_X(y) = P(X \leq y) = P(X \leq x) + \underbrace{P(x < X \leq y)}_{\geq 0}$$

$$F_X(y) = F_X(x) + P(x < X \leq y)$$

$$\therefore F_X(y) > F_X(x)$$

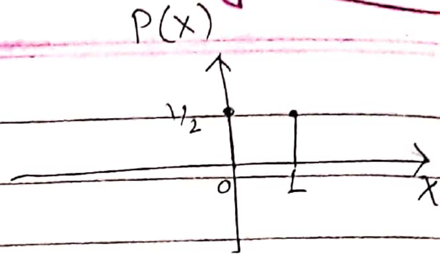
(ii) $F_X(x) = P(X \leq x)$ is atleast a right continuous func^n

(iii) $\lim_{x \to \infty} F_X(x) = 1$

$\lim_{x \to -\infty} F_X(x) = 0$

Pagal Pagal

Date ____
Page ____

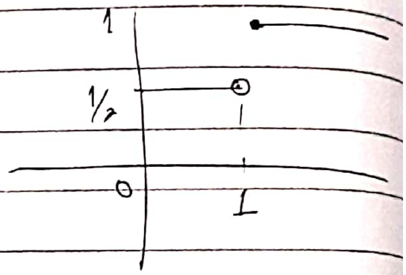
$X \sim Ber(1/2)$ 

$P(X=0) = 1/2$

$P(X=1) = 1/2$

$F_X(x) = P(X \le x)$

$$= \begin{cases} 0 & ; \; x < 0 \\ 1/2 & ; \; 0 \le x < 1 \\ \cancel{1/2} \; 1 & ; \; x \ge 1 \end{cases}$$



22 Jan 020

(i) $P(a < X \le b) = F_X(b) - F_X(a)$

$F_X(b) = P(X \le b)$

$= P(X \le a \cup a < X \le b)$

$= P(X \le a) + P(a < X \le b)$

$F_X(b) = F_X(a) + P(a < X \le b)$

$$P(a < X \le b) = F_X(b) - F_X(a)$$

(ii) $P(X=a) = F_X(a) - F_X(a-0)$

$X \le a = X < a \cup X = a$

$P(X \le a) = P(X < a \cup X = a)$

$F_X(a) = F_X(a-0) + P(X=a)$

E.g.,

PMF

$X=x$	1	2	3	4	5	6	7
$P(X=x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

$9k + 10k^2 = 1$

$10k^2 + 9k - 1 = 0$

$10k^2 + 10k - k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

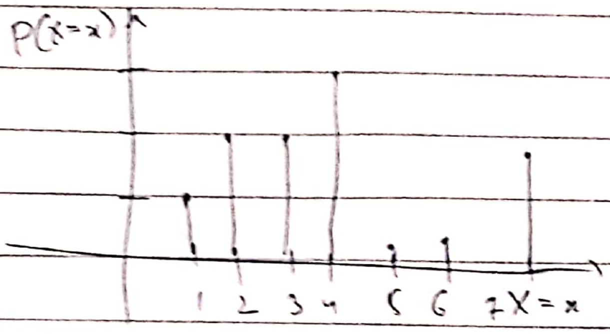
$k = 1/10$

√ee √ee Pagal

22 June 2069 Saturday

Date ____ Page ____

1	2	3	4	5	6	7
0.1	0.2	0.2	0.3	0.01	0.02	0.17



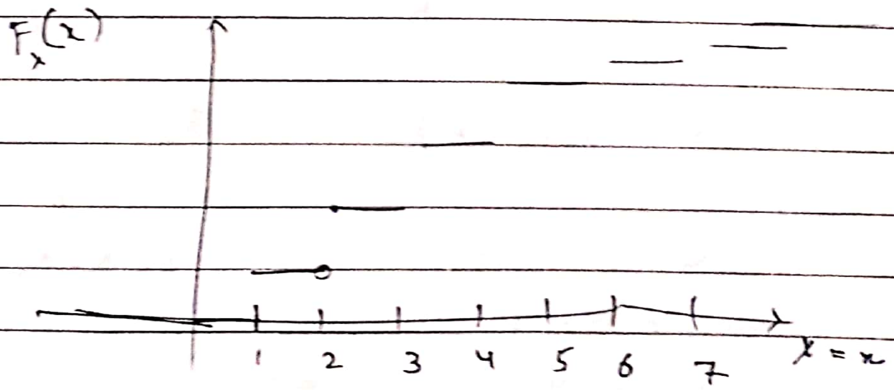
$X < 5.6 = 0.01$

$P(3 \le X < 5.6 / X > 3) =$

$$\frac{P(X<5.6 \cap X>3)}{P(X>3)}$$

$$= \frac{P(3 < x < 5.6)}{1 - P(X \le 3)} = \frac{0.31}{0.50} = 0.62$$

$$F_X(x) = \begin{cases} 0 & ; \quad X < 1 \\ 0.1 & ; \quad 1 \le X < 2 \\ 0.3 & ; \quad 2 \le X < 3 \\ 0.5 & ; \quad 3 \le X < 4 \\ 0.8 & ; \quad 4 \le X < 5 \\ 0.81 & ; \quad 5 \le X < 6 \\ 0.83 & ; \quad 6 \le X < 7 \\ 1 & ; \quad 7 \le X \end{cases}$$



Scanned by CamScanner

Expectation / Mean / Average -

2 2 3 3 3 5 5

$$\text{Avg} = \frac{2+2+3+3+3+5+5}{7}$$

$$= 2 \cdot \frac{2}{7} + 3 \cdot \frac{3}{7} + 5 \cdot \frac{2}{7}$$

$$= 2\, P(X=2) + 3\, P(X=3) + 5\, P(X=5)$$

$$= \Sigma\, x \cdot P(X=x)$$

★ The expectation of a random variable X is defined as
$$E(X) = \Sigma\, x \cdot P(X=x)$$

Eg.₁ $X \sim \text{Bern}(p)$ $\qquad P(X=1) = p$

$\qquad P(X=0) = 1-p$

$$E(X) = 0 \cdot P(X=0) + 1\, P(X=1)$$
$$= 0 \cdot (1-p) + 1 \cdot p$$
$$= p$$

E.g.₂ $X \sim \text{Bino}(n,p)$

$E(X)$

$= E(X_1 + \dots X_n)$

$= E(X_1) + E(X_2) + \dots + E(X_n)$

$= p + p + p + \dots p = np$

Result!

$X_1, X_2, X_3 \dots X_n$

$E(X_1 + X_2 + \dots X_n)$

$= E(X_1) + E(X_2) \dots + E(X_n)$

as $X = X_1 + X_2 + \dots X_n$; $X_i \overset{iid}{\sim} \text{Bern}(p)$

Payal

E.g., $n = 10$ $p = 1/2$

$E(X) = 5 = 10 \times \frac{1}{2}$

23 Jan 020

E.g., Tut

(18) ~~At least~~ No card on i^{th} number

$$1 - \frac{n-1}{n} \times \frac{n-2}{n-1} \times \frac{n-3}{n-2} \times \cdots \times \frac{1}{2} \times 1$$

$$= 1 - \frac{(n-1)!}{n!}$$

E_i = i^{th} flip has number i

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{N} P(E_i) + (-1)^1 P(E_i \cap E_j) + \cdots$$

$$(-1)^{n-1} P(E_1 \cap E_2 \cdots \cap E_n)$$

1 5   5 1 $\frac{5}{36}$ $\frac{5}{36}$ 6 1   1 6

2 4   4 2   2 5   5 2

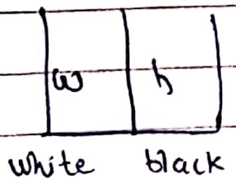
3 3   3 4   4 3

27 Jan 020

Hypergeometric Distribution -

A random variable X is said to have hypergeometric distribution, denotes as $X \sim$ Hypgeo (w, b, n), if the PMF of X is given by -

$$P(X = k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}} \quad ; k = 0, 1, 2 \ldots w$$



$w + b = N$ Probability of out of n balls drawn k are white ?

Success prob. of the white ball with replacement $= \frac{w}{w+b} = p$

$$P(X=k) = \binom{n}{k}\left(\frac{w}{w+b}\right)^k\left(1-\frac{w}{w+b}\right)^{n-k}$$

without replacement - w w b b w w w b

$$P(X=k) = \binom{n}{k}\frac{w}{w+b}\cdot\frac{w-1}{w+b-1}\cdot\frac{b}{w+b-2}\cdot\frac{b-1}{w+b-3}\cdots$$

$$= \cancel{\binom{n}{k}\frac{w\,(w+1-k)\,(b+1-n+k)}{(w+b)(w+b-1)\cdots(w+b+1-k)}}$$

$$= {}^{n}C_{k}\,\frac{w(w-1)\cdots(w-\overline{k-1})\;b(b-1)\cdots(b-\overline{n-k-1})}{(w+b)(w+b-1)\cdots(w+b-\overline{n-1})}$$

$$= \frac{n!}{k!\,(n-k)!}\;\frac{w(w-1)\cdots(w-\overline{k-1})\;b(b-1)\cdots(b-\overline{n-k-1})}{(w+b)(w+b-1)\cdots(w+b-\overline{n-1})}$$

$$= \frac{n!}{k!\,(n-k)!}\;\frac{w!}{(w-k)!}\;\frac{(w+b-n)!}{(w+b)!}\;\frac{b!}{(b-\overline{n-k})!}$$

$$= \frac{w!}{(w-k)!\,k!}\;\frac{b!}{(n-k)!\,(b-\overline{n-k})!}\;\frac{(w+b-n)!\,n!}{(w+b)!}$$

$$= \frac{{}^{w}C_{k}\;{}^{b}C_{n-k}}{{}^{w+b}C_{n}}$$

$$\sum_{k=0}^{n}\frac{{}^{w}C_{k}\;{}^{b}C_{n-k}}{{}^{w+b}C_{n}} = 1$$

$j^{r!}$ $\log a^{t}$ $w+b \to \infty$ $w \approx w-1$

Result: Let $w+b = N \to \infty$ s.t. $\frac{w}{w+b}$ = constant
Then PMF of Hypergeometric distribution converges to ~~to~~ the PMF of the binomial dist.

$X = X_1 + X_2 + X_3 + \cdots + X_n$; $X_i \overset{iid}{\sim} Ber(p)$
$X \sim Bin(n,p)$

$Y \sim Hygeo(w,b,n)$

$Y = Y_1 + Y_2 + \cdots + Y_n$; $Y_i \overset{identical}{\sim} Ber(p)$

$$\left.\begin{aligned} E(Y) &= \sum_{k=0}^{w} k\, P(X=k) \\ &= \sum_{k=0}^{w} k \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}} = n \cdot \frac{w}{w+b} \end{aligned}\right\} \star$$

Since $Y = Y_1 + \cdots + Y_n$

$E(Y) = E(Y_1) + E(Y_2) + \cdots + E(Y_n)$

$= \frac{w}{w+b} + \cdots \cdot \frac{w}{w+b}$ $\quad Y_i \sim Ber\left(\frac{w}{w+b}\right)$

$= n \cdot \frac{w}{w+b}$

Poission Distribution -

$X \sim Poission(\lambda)$

$$P(X=k) = \frac{e^{-\lambda}\lambda^{k}}{k!} \quad ; k = 0, 1, 2 \ldots$$

Result: $X \sim Bin(n,p)$, s.t. $n \to \infty$ and $p \to 0$
s.t. $n \times p = \lambda$ (const)
Then the PMF of binomial converges to PMF of poission

$$\Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \; ; \quad k = 0, 1, \ldots, n$$

$$p = \frac{\lambda}{n}$$

$$= \frac{n!}{k!\, n-k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \times \frac{n!}{(n-k)! \left(1 - \frac{\lambda}{n}\right)^k \times n^k}$$

$$= \frac{\lambda^k}{k!} (1 - \lambda/n)^n \times \frac{n(n-1) \times \ldots (n-(k-1))}{n^k \left(1 - \frac{\lambda}{n}\right)^k}$$

$\downarrow$

1

Now, $n \to \infty$

$$\left(1 - \frac{\lambda}{n}\right)^n \longrightarrow e^{-\lambda}$$

$$\left(1 - \frac{\lambda}{n}\right)^k \longrightarrow 1$$

$$n! \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) \to 1$$

$$\Rightarrow \frac{\lambda^k e^{-\lambda}}{k!} = P(Y=k), \quad Y \sim Poi(\lambda)$$

$\star$ $Y \sim$ Poisson (λ)

$$E(Y) = \lambda \quad \sum_{k=0}^{\infty} k\, P(X=k) = \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda^k}{k!}$$

Pagal

Date
Page

Geometric Distribution -

$X \sim Geo(p)$

X : counts the no. of trials to get the first success

$P(X=k) = (1-p)^{k-1} p$

$k = 1, 2, \ldots \infty$

T T T T T H

$(1-p)^5 p$

① Check if it is valid

② $E(x) = \frac{1}{p}$