A Comparison of the Binomial Asset Pricing Model and the Black-Scholes Formula in European Call Option Pricing

Vinay Vignesh Sidharth*

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Abstract

The objective of the study is to compare two widely used European options pricing algorithms for call options: the Binomial Asset Pricing Model (BAPM) and the Black-Scholes formula and identify the number of steps it takes for the binomial model to converge with the result of the Black-Scholes formula. The underlying assumptions of the models are listed followed by their numerical results under identical market conditions. We conclude the study by evaluating both approaches and offering recommendations for further studies on the topics.

Keywords — Derivatives,Options Pricing, Binomial Asset pricing Model, Black-Scholes Model, Covergence

1 Introduction

We start the study by introducing Derivatives, which are financial instruments whose value is derived from an underlying asset(stocks, commodities, or indices).One of the most common Financial Derivatives are Options Contract, the parties involved in the trade are the buyer and the seller of the options contract.

Options contract's gives the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (strike price) on or before a specified expiration date. There are two main types of options:

Call Option: Grants the right(but not the obligation) to buy the underlying asset.

Put Option: Grants the right(but not the obligation) to sell the underlying asset.

Options are also be classified based on the time of exercise i.e i) European and ii) American options. American options can be exercised at any time during the holding period whereas European options can be exercised only during the Expiry.Valuing options, plays a fundamental role in modern finance, among the various models developed to price options, the Binomial Asset Pricing Model (BAPM) and the Black-Scholes formula are two of the most widely recognized. The Binomial model, introduced by Cox, Ross, and Rubinstein (1979), utilizes a discrete-time framework, while the Black-Scholes model, developed by Black and Scholes (1973) and Merton (1973), provides a closed-form solution based on continuous-time assumptions.

*MSc Financial Mathematics, Queen Mary University of London. Email: vinayvsidharth@gmail.com

Options are traded both on exchanges and over-the-counter markets, they are primarily used for hedging, speculation, and portfolio diversification. Option Buyers pay a premium (the price which forms the central point of the study) to acquire these contracts, while sellers/writers assume the obligation to fulfill the transaction if exercised. Here arises an interesting question, How do we determine the fair price/value of the option contract?

This paper compares the two models which determine the price of the contract . The objective is to highlight the strengths and limitations of each approach , and provide insights into their respective suitability for different financial contexts.

2 The Binomial Asset Pricing Model

The Binomial Asset Pricing Model is a foundational model which approximates the evolution of asset prices by assuming discrete-time (i.e , at each time step, the price can either move up or down by fixed percentages). The model relies on backward induction(i.e by working from the option's expiration to the present) to compute the option price .

2.1 Mathematical Formulation

Let S_0 denote the current price of the underlying asset, K the strike price, T the time to maturity, r the risk-free rate, and σ the volatility of the asset. The time to maturity is divided into n intervals, with the length of each interval being $\Delta t = \frac{T}{n}$.

The up and down factors are given by:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$$

The risk-neutral probability p is computed as:

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

The option payoff at the expiration date is given by:

$$V(i, n) = \max(0, S_0 u^i d^{n-i} - K)$$

where i represents the number of up moves at time step n. The backward induction process is used to calculate the option price at each node of the binomial tree.

3 The Black-Scholes Formula

The Black-Scholes formula provides a closed-form solution for pricing European call options under the assumption of continuous trading, constant volatility, and a risk-free interest rate. The formula for a European call option is:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Terms: S_0 (spot price), K (strike price), r (risk-free rate), T (time to maturity), σ (volatility), $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. The formula is derived based on the assumption that the underlying asset follows a geometric Brownian motion.

4 Comparison of the Models

4.1 Computational Complexity

The Black-Scholes formula offers a closed-form solution, which makes it computationally efficient for European options. In contrast, the Binomial model requires multiple calculations to account for each possible path of the asset price, which can become computationally expensive as the number of steps increases. However, the Binomial model can be made arbitrarily accurate by increasing the number of steps n, whereas the Black-Scholes model is limited by its assumptions.

4.2 Practical Considerations

The Binomial model is more versatile, allowing for the pricing of American options (which can be exercised before expiration), options with time-varying volatility, and options with dividends. The Black-Scholes formula, while efficient, is limited to European options and may not perform well under conditions where volatility is stochastic or where dividends are significant.

4.3 Assumptions

Assumption	**Binomial Model**	**Black-Scholes Model**
Time Framework	Discrete time periods (e.g., steps or intervals).	Continuous time (trading occurs without interruptions).
Price Movements	Two possible price changes per period (up/down).	Continuous price changes (geometric Brownian motion).
Dividends	No dividends (unless explicitly modeled).	No dividends (unless extended).
Market Frictions	No transaction costs, taxes, or bid-ask spreads.	No transaction costs, taxes, or bid-ask spreads.
Underlying Distribution	Discrete price paths (binomial tree).	Log-normal distribution of asset prices.
Investor Risk Preference	Risk-neutral valuation (pricing under $\mathbb{Q}$-measure).	Risk-neutral valuation (pricing under $\mathbb{Q}$-measure).
Completeness	Complete markets (all risks hedgeable).	Complete markets (continuous trading ensures hedging).

Table 1: Comparison of Binomial Asset Pricing Model and Black-Scholes Model

5 Calculation

5.1 Data Collection

We collect stock price data [symbol : AAPL] for the period February 2024 to January 2025, covering 251 trading days. The dataset includes daily closing prices, sourced from NASDAQ. The data is downloaded in Excel format, which ensures ease of manipulation and further analysis.

The dataset consists of:

- **Date**: The trading date.
- **Open Price**: The price at the market opening.
- **High Price**: The highest price reached during the trading day.
- **Low Price**: The lowest price recorded.
- **Close/Last Price**: The final price at market close.
- **Volume**: The number of shares traded.

5.2 Pre-processing

Once the data is collected, we conduct the following steps to prepare it for analysis:

5.2.1 Compute Daily Returns

The daily return (R_t) is calculated using the closing price of consecutive days. The formula for simple returns is:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

where:

- P_t is the closing price on day t,
- P_{t-1} is the closing price on day $t-1$.

Alternatively, we can express it as:

$$R_t = \left(\frac{P_t}{P_{t-1}}\right) - 1 \tag{2}$$

This measures the percentage change in stock price from one day to the next.

5.2.2 Logarithmic Returns

To stabilize variance and make the data more suitable for modeling, we take the natural logarithm (log returns) instead of simple returns. The log return formula is:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{3}$$

Logarithmic returns are preferred over simple returns as:

- They are time additive, meaning the sum of log returns over multiple periods gives the total return.
- They normalize data, helping with statistical modeling.
- They often follow a more normal distribution, making them useful for modeling risk and volatility.

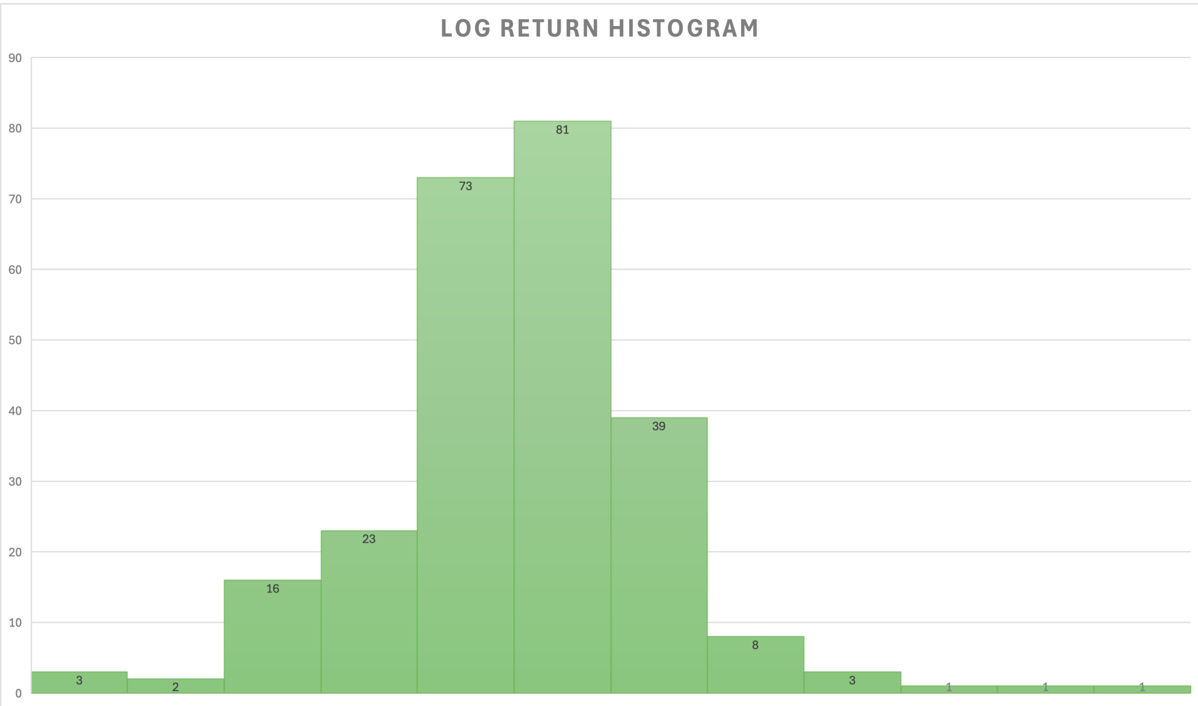

Figure 1: Log Normal Distribution of the Closing price

5.2.3 Daily and Annualized Volatility

After computing the log returns, we calculate the daily volatility (σ_{daily}) as the standard deviation of the log returns:

$$\sigma_{\text{daily}} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2} \tag{4}$$

where:

- r_t is the log return at time t,
- $\bar{r}$ is the mean of the log returns,
- N is the total number of trading days.

For our dataset, the daily volatility is found to be:

$$\sigma_{\text{daily}} = 1.43\% \tag{5}$$

To compute the annualized volatility (σ_{annual}), we scale the daily volatility by the square root of the number of trading days in a year ($T = 252$):

$$\sigma_{\text{annual}} = \sigma_{\text{daily}} \times \sqrt{T} \tag{6}$$

Substituting the values:

$$\sigma_{\text{annual}} = 1.43\% \times \sqrt{252} = 23.13\% \tag{7}$$

5.3 Option Pricing Algorithms

5.3.1 Black-Scholes Model

The Black-Scholes model calculates the price of a European call option as follows:

Algorithm 1 Black-Scholes Call Option Pricing

1: **Input:** S, K, T, r, σ
2: Compute $d_1 = \frac{\ln(S/K)+(r+0.5\sigma^2)T}{\sigma\sqrt{T}}$
3: Compute $d_2 = d_1 - \sigma\sqrt{T}$
4: Compute $N(d_1)$ and $N(d_2)$ using the cumulative standard normal distribution
5: Compute call option price: $C = SN(d_1) - Ke^{-rT}N(d_2)$
6: **Output:** C

5.3.2 Binomial Model

The Binomial Model computes the option price using discrete time steps:

Algorithm 2 Binomial Option Pricing

Input: S, K, T, r, σ, n
Compute $\Delta t = T/n$
Compute $u = e^{\sigma\sqrt{\Delta t}}$, $d = 1/u$
Compute risk-neutral probability: $p = \frac{e^{r\Delta t}-d}{u-d}$
for $i = 0$ to n **do**
    Compute terminal payoff: $C_i = \max(0, Su^i d^{n-i} - K)$
end for
for $j = n-1$ down to 0 **do**
    for $i = 0$ to j **do**
        Compute option value: $C_i = e^{-r\Delta t}(pC_{i+1} + (1-p)C_i)$
    end for
end for
Output: C_0

5.4 European Call Option Price Calculation

With the following parameters and using the algorithm above we estimate the options price :

$$S_0 = 227.63, \quad K = 227.63, \quad T = 1, \quad r = 0.045, \quad \sigma = 0.2303, \quad n = 100$$

The option prices computed using the Binomial model and the Black-Scholes formula are as follows:

$$\text{Binomial Option Price} = 25.7514$$

$$\text{Black-Scholes Option Price} = 25.8033$$

Steps (n)	**Binomial Price ($)**
2	23.4393
10	25.2911
40	25.6739
100	25.7514
200	25.7773
500	25.7929
3000	25.8016
8000	25.8027
100000	25.8028
200000	25.8033

Table 2: Comparison of Binomial Option Pricing with different steps

The results demonstrate that both models provide a similar very close approximation of the result, and when we increase the number of steps to 200,000 the results converge.

6 Conclusion

The Binomial Asset Pricing Model and the Black-Scholes formula are both valuable tools for pricing European call options. The Black-Scholes model is computationally efficient and widely used in practice, particularly for simple options in stable markets. However, the Binomial model is more flexible and can accommodate a wider range of option types and market conditions, making it a preferable choice in more complex scenarios.So in this particular problem setup we conclude that it takes 200,000 steps for the binomial model to converge with the result of the Black-Scholes model.

Future research could explore the application of both models to American options and investigate the impact of incorporating stochastic volatility in both frameworks.

7 References

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