

Finite Projective Geometry based Fast, Conflict-free Parallel Matrix Computations

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Abstract

Matrix computations, especially iterative PDE solving (and the sparse matrix vector multiplication subproblem within) using conjugate gradient algorithm, and LU/Cholesky decomposition for solving system of linear equations, form the kernel of many applications, such as circuit simulators, computational fluid dynamics or structural analysis etc. The problem of designing approaches for parallelizing these computations, to get good speedups as much as possible as per Amdahl’s law, has been continuously researched upon. In this paper, we discuss approaches based on the use of finite projective geometry graphs for these two problems. For the problem of conjugate gradient algorithm, the approach looks at an alternative data distribution based on projective-geometry concepts. It is *proved* that this data distribution is an **optimal** data distribution for scheduling the main problem of dense matrix-vector multiplication. For the problem of parallel LU/Cholesky decomposition of general matrices, the approach is motivated by the recently published scheme for interconnects of distributed systems, perfect difference networks. We find that projective-geometry based graphs indeed offer an exciting way of parallelizing these computations, and in fact many others. Moreover, their applications ranges from architectural ones (interconnect choice) to algorithmic ones (data distributions).

Keywords: Distributed Computing; Parallel Algorithms; Parallel Processing

1 Introduction

Computations related to large matrices arise in many applications, from circuit simulators to well-known page ranking algorithm employed by Google. An omnipresent type of matrix

computation is linear system direct solver, which uses LU or Cholesky factorization as its *computational core*. **LU decomposition** of a matrix \mathbf{A} is its decomposition into a lower triangular matrix and an upper triangular matrix, i.e. $\mathbf{A} = \mathbf{L} \cdot \mathbf{U}$. **Cholesky decomposition** decomposes a *symmetric positive definite matrix* \mathbf{A} as $\mathbf{A} = \mathbf{L} \cdot \mathbf{L}^T$. Cholesky algorithm is numerically robust, and is often applicable due to the abundance of symmetric positive definite linear systems in scientific computing domain. In particular, Cholesky algorithm does not require pivoting, whereas LU decomposition typically does require pivoting. As the matrix sizes involved in applications such as circuit simulators can be in order of millions or even higher, expectedly, there has been plenty of research on parallelizing such algorithms [4]. When the matrix \mathbf{A} in the linear system $\mathbf{Ax} = \mathbf{b}$ is large but sparse, iterative methods are preferred over above methods. For *symmetric positive definite* matrices, the method of **conjugate gradients** [14] is often used. A large variety of scientific and engineering applications solve system of partial differential equations (PDEs) as well, using iterative solvers based on this method. The *computational core* within this algorithm is a multiplication of a sparse matrix with a dense vector(SpMV). Hence, as per Amdahl's law, best possible speedups can only be achieved when efforts are directed to parallelize these computational cores, SpMV and LU/Cholesky decomposition.

As such, parallelization of both matrix computations leads to interconnecting memory blocks and processing units, in the usual sense. Also, parallelization of the SpMV kernel entails problems such as load balancing, and the relatively low ratio of computation to communication. These are typical problems encountered during parallelization of *any* computational problem anyway [6]. To address these problems, a novel interconnection pattern was proposed by Karmarkar based on *finite projective geometries* [5]. The processors and memories are associated with elements of these geometries and the interconnections are based on their incidence relations. The *computations assigned* to a processor, and *corresponding data distribution*, also depend on the geometry and incidence relations. Because the geometry is symmetric in nature, *the computational load on each processor is balanced*. In fact, the load on memory blocks and interconnect is also balanced. The automorphisms governing these geometries are used to develop perfect-access patterns and perfect-access sequences, which ensure that all the processors and memories are simultaneously involved in *communication of data without any conflicts*. Algorithms to solve various problems on this architecture can be developed using these properties.

For CG Algorithm, a lot of research work is actively being carried out on the SpMV kernel implementations. Typically one uses preconditioned CG (PCG) as the workhorse algorithm for solving a linear system defined by large sparse symmetric positive definite matrix. The projective geometry based interconnection pattern proposed by Karmarkar aims at improving communication efficiency by *superior utilization of communication bandwidth* using perfect-access patterns and perfect-access sequences. In the solution to PCG, projective geometry *further improves* this efficiency, but in an *independent dimension*. We use the finite projective geometries introduced in [5] as a basis to define a novel data distribution, the (finite) projective data distribution. The projective data distribution *reduces the communication*

load of the algorithm, and this distribution is provably communication optimal. The software prototype developed by us and also described in the first half of this paper compares the performance using conventional data distribution and the projective data distribution. The optimal projective data distribution is patent pending [11].

A naive PCG (and SpMV) implementation faces many potential performance bottlenecks. These bottlenecks dominate the performance of the algorithm, and hence their elimination is a pre-requisite for the effect of the data distribution on the performance to manifest itself. Since these bottlenecks and their solutions are independent of the main focus – influence of Projective Geometry principles – of this paper, these solutions are only outlined in this paper.

A lot of research on parallelization of LU/Cholesky decomposition has been for dense matrices. Inter-processor data communication for such matrices has been quite a challenging issue. Here, we present our results for communication-efficient parallelization of this problem. The schemes are based on *more practical assumptions* than used in scheduling ideas suggested in [5], and are motivated by recent works[8, 9]. The choice of underlying graph is a 4-dimensional projective space, justified later. The scheduling schemes are based on indirect incidences based on subsumption relations between projective subspaces. They are evaluated based on the amount of communication and computation required in each of them. A comparison is also drawn with conventional parallel architectures such as mesh. We report these schemes in the latter half of this paper. As such, we have found even more applications of projective geometry based graphs in other areas, *most notably in error correction coding* and digital system design, that have been reported separately [1], [12], [2], [13].

2 Projective Spaces based Interconnect Topologies

We first provide an overview of the fundamental concepts of projective spaces, that have been used *throughout* our work. Projective spaces and their lattices are built using vector subspaces of the bijectively corresponding vector space, one dimension high, and their subsumption relations. Vector spaces being extension fields, Galois fields are used to practically construct projective spaces [7]. Consider a finite field $\mathbb{F} = \text{GF}(s)$ with s elements, where $s = p^k$: $k = +\text{ve integer}$.

An example *Finite Field* can be generated as follows. For each value of s in $\text{GF}(s)$, one needs to first find a *primitive polynomial* for the field. Such polynomials are well-tabulated in various literature. For example, for the (smallest) projective geometry, $\text{GF}(2^3)$ is used for generation. One primitive polynomial for this Finite Field is $(x^3 + x + 1)$. Powers of the root of this polynomial, x , are then successively taken, $(2^3 - 1)$ times, modulo this polynomial, modulo 2. This means, x^3 is substituted with $(x + 1)$, wherever required, since over base field $\text{GF}(2)$, $-1 = 1$. A *sequence* of such evaluations lead to generation of the sequence of $(s - 1)$ Finite field elements, **other than 0**. Thus, the sequence of 2^3 elements for $\text{GF}(2^3)$ is **0(by default), $\alpha^0 = 1, \alpha^1 = \alpha, \alpha^2 = \alpha^2, \alpha^3 = \alpha + 1, \alpha^4 = \alpha^2 + \alpha, \alpha^5 = \alpha^2 + \alpha + 1, \alpha^6 = \alpha^2 + 1$** .

A projective space of dimension d is denoted by $\mathbb{P}(d, \mathbb{F})$ and consists of one-dimensional subspaces of the $(d + 1)$ -dimensional vector space over \mathbb{F} . Zero-dimensional subspaces of the projective space are called *points*. The total number of points in $\mathbb{P}(d, \mathbb{F})$ are $P(d) = \frac{s^{d+1}-1}{s-1}$. Let us denote the collection of all the l -dimensional projective subspaces by Ω_l . To count the number of elements in each of these sets, we define the function [5]

$$\phi(n, l, s) = \frac{(s^{n+1} - 1)(s^n - 1) \cdots (s^{n-l+1} - 1)}{(s - 1)(s^2 - 1) \cdots (s^{l+1} - 1)} \quad (1)$$

The number of m -dimensional subspaces of $\mathbb{P}(d, \mathbb{F})$ is $\phi(d, m, s)$. Hence, the number of l -dimensional subspaces contained in an m -dimensional subspace (where $0 \leq 1 < m \leq d$) is $\phi(m, l, s)$, while the number of m -dimensional subspaces containing a particular l -dimensional subspace is $\phi(d-l-1, m-l-1, s)$. For more details on projective space construction, refer [7].

For example, to generate *Projective Geometry* corresponding to above Galois Field example ($\mathbb{GF}(2^3)$), the 2-d projective plane, we treat each of the above *non-zero* element as points of the geometry. Further, we pick various subfields(vector subspaces) of $\mathbb{GF}(2^3)$, and label them as various lines. Thus, the 7 lines of the projective plane are $\{1, \alpha, \alpha^3 = 1 + \alpha\}$, $\{1, \alpha^2, \alpha^6 = 1 + \alpha^2\}$, $\{\alpha, \alpha^2, \alpha^4 = \alpha^2 + \alpha\}$, $\{1, \alpha^4 = \alpha^2 + \alpha, \alpha^5 = \alpha^2 + \alpha + 1\}$, $\{\alpha, \alpha^5 = \alpha^2 + \alpha + 1, \alpha^6 = \alpha^2 + 1\}$, $\{\alpha^2, \alpha^3 = \alpha + 1, \alpha^5 = \alpha^2 + \alpha + 1\}$ and $\{\alpha^3 = 1 + \alpha, \alpha^4 = \alpha + \alpha^2, \alpha^6 = 1 + \alpha^2\}$.

The corresponding geometry can be seen as figures 1.

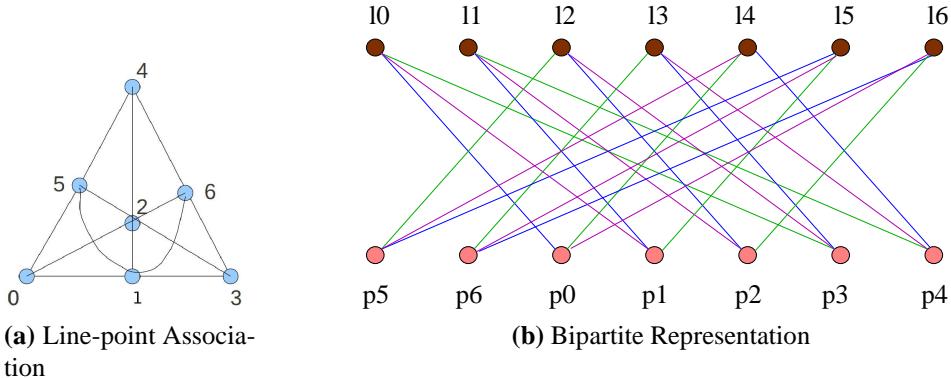


Figure 1: 2-dimensional Projective Geometry and Its Bipartite Representation

For Projective spaces, Karmarkar had in past evolved an architecture for parallel computing. There are problems, such as LU/Cholesky decomposition, which have been found to be amenable for parallel computation using processing units and memories, connected using subgraph of an instance of projective geometry. For such problems, once the *appropriate* geometry has been identified, a pair of dimensions d_m and d_p is chosen. The processing units are then bijectively associated with subspaces of dimension d_p , while memories are associated with subspaces of dimension d_m . A connection between a processing unit and memory is established if the corresponding subspaces have a non-trivial intersection [5].

In this architecture, accesses to memories happen in a structured fashion. By exploiting symmetry of the geometry, it is possible to *pair up all processing units and memories*, such that they communicate in a conflict-free manner. Each such set of pairs forms a *perfect-access pattern*. A collection of all such patterns together forms a *perfect-access sequence*, which ensures that every processing unit gets to communicate with every memory it is directly connected to, in an overall computation cycle. Once the problem is broken down to parallelizable atomic computations, and corresponding memory blocks for storing data, these computations can then be assigned to processor *connected to* these relevant memories, which depends on the problem and the underlying geometry. Load balancing is ensured by symmetry of the geometry. Thus, data required for computation is brought in parallelly, computations on each processor are carried out parallelly, leading to efficient and conflict-free use of resources.

2.1 Projective Spaces and Perfect Difference Networks

Recently, there has been discovery of a new class of parallel interconnect networks, perfect difference networks [8], [9]. The diameter of these graphs being 2, any node is *reachable* from any other node in one or two hops. For various schemes of routing, it is argued that this lesser diameter does help in improving the worst-case communication latency. Borrowing from this basic idea of reducing latency, the very thought of using a row of nodes in bipartite network graph as switches or buses to have graph diameter as 2 is worth looking at. In projective geometry, not every two lines meet on a plane. But when they meet, their distance is 2. Hence a bus can be used to communicate between them, something that we use later in an LU decomposition scheme. A projective plane being a PDN, the diameter-2 access has been *implicitly* used in the SpMV solving as well.

Next, we report our **concrete** investigations in the application of **projective geometry** for a prototypical computing core, viz. preconditioned conjugate gradient(**PCG**) method for solving large sparse linear systems.

3 Advantages of Projective Distribution in PCG Computation

The preconditioned conjugate gradient method has SpMV as its main computing subroutine (about 90% of the computing time is spent in SpMV). PCG being an iterative solver, SpMV is invoked repeatedly during one invocation of the solver. Hence, the solver is designed to prefer larger data movements during the setup, rather than during the iterations.

In a typical SpMV implementation within **PCG** algorithm, matrix and vector blocks are distributed to various processors. Matrix blocks are not modified during the iterations of PCG, and moreover are larger than vector blocks. Hence a parallel implementation of the algorithm is structured so as to move vector- but not matrix- blocks among the processors

during the iterations. Each processor can perform certain block - vector multiplications without needing additional data, whereas some multiplications can be carried out only after receiving an appropriate vector block. The *amount of communication* required for these vector blocks is an *important factor* influencing the performance of parallel SpMV.

The data distribution that exploits the conflict-free communications of a PG interconnect topology (dimension 2) identifies memories as points and computing nodes (processors) as lines in the projective space. A memory module and a processor are connected if the point corresponding to the memory module lies on the line corresponding to the processor. The matrix is distributed so that the block $A_{i,j}$ is allocated to the unique line passing through points i and j . Input sub-vector x_i is stored on processor i and output sub-vector y_j on processor j . With such a data distribution, the communication of input and output sub-vectors can be seen to be conflict free by virtue of presence of perfect access patterns and sequences. In our experiment, we use the same data distribution, with the difference that we *do not* focus on the conflict-free nature of communication¹, instead focusing on the communication load dictated by the data distribution.

In the *classical* row-wise distribution, every processor needs $n - 1$ vector blocks from $n - 1$ processors for it to complete one multiplication, and hence the communication complexity of this distribution is $O(n)$ per SpMV. In projective distribution, every processor needs to communicate $2p \simeq 2\sqrt{n}$ vector blocks to complete the multiplication. Hence, the communication complexity of one SpMV under the projective distribution is $O(\sqrt{n})$.

It is therefore expected that the performance of projective distribution should have a superior scaling behavior as compared to row-wise distribution. In next few sections, we provide details of our *thorough* investigation of this advantage.

Though we have used *preconditioned* CG, the performance of the solver is dominated by SpMV, and hence the focus of the discussion will be SpMV rather than the preconditioner.

4 Design Approach for Novel PCG Solver

4.1 Preliminaries

Within the context of a parallel implementation, the matrix A is viewed as a *blocked* matrix, and the vectors are viewed as blocked vectors. Notationally, we represent scalar elements of A as $a_{i,j}$, and the blocks of the matrix as $A_{k,l}$. This notational convenience of using lowercase letters for scalars and uppercase letters for matrix blocks can not be carried over to the vectors, since the convention is to use lowercase letters for full vector itself. Since the blocking of the various vectors also becomes relevant within the discussion in this report, we do need a notation for vector blocks. We introduce the notation X_k to denote a block of the vector x . The set of blocks $A_{i,*}$ will be called the i^{th} row-block of matrix A , and similarly the set of blocks $A_{*,j}$ will be called the j^{th} column-block of matrix A . A subset of a row-block

¹Hence we do not demand a PG-connected network

will be called a row sub-block and similarly a subset of a column-block will be called a column sub-block.

The parallel SpMV algorithm distributes the matrix tiles and input-/output- vector blocks to the processes. If a tile $A_{i,j}$ (or vector block X_l) is allocated to process k , we say process k **owns** the tile (or the vector block), and is the **owner process** for the tile (or block). The algorithm exploits the associativity in the computation $Y_i = \sum_j A_{i,j} \cdot X_j$, to parallelly compute multiplications of individual tiles $A_{i,j}$ with vector-blocks X_j . The results of such multiplications, which are partial-sums for Y_i , can then be added up to obtain the complete Y_i by its owner process, which defines to be the same process that owns X_i .

Our parallel implementation uses the **MPI** library for communication between various processes. We therefore use standard MPI terminology, **process** to denote an MPI process usually running on a separate computational processor, **rank** to denote the uniquely identifying integer rank associated with each process, **rank 0** or **root** to denote a rank that is often used in special ways. Since the PCG study and benchmarks were primarily focused on the data distribution rather than the interconnect topology, we shall mostly use the term **process** instead of the term “processor”.

4.2 Overview of PCG Algorithm

We reproduce an overview of the reference definition of a PCG, as defined in [14], for the sake of easy reference in Algorithm 1.

To summarize, the algorithm performs an initial SpMV to identify the initial and hence target residues. Subsequently, each iteration performs various vector operations and one SpMV operation. Since SpMV performance dominates the performance of the solver, our discussion will be mostly about SpMV i.e. steps 2, 8 and 12 of the algorithm.

Our experiment explores various decisions at both the algorithm and implementation level. To be able to easily compare and isolate the effect of these decisions, each of these decisions appears as either compile-time or most often a run-time decision. In such cases, the choice of one or the other option is indicated at the run-time, and the performance measurements are recorded for the active set of parameters. In the remaining part of this section, we will introduce these decisions. While our experimental performance results reflect the influence of many of these decisions, in this paper, we analyze the influence of only the data distribution.

4.3 Data Distribution

Conventional matrix data distributions are the 1-D or 2-D block distributions([6]). These use either 1 or 2 dimensions² of the matrix to identify the distribution. Thus, in block 1-D distribution, n row- or column-blocks are allocated to each of the n processes, and in block 2-D distribution, a square block (i, j) , $1 \leq i, j \leq n$ is allocated to each of the n^2 processes.

²in a Cartesian sense

Algorithm 1 PCG Algorithm

Input: Matrix A , vector b , initial vector x_0 , Pre-conditioner M ,
Max. number of iterations i_{max} , Error tolerance $\epsilon < 1$
Output: Vector x satisfying $Ax = b$ in less than i_{max} iterations with residual dropping
by a factor of ϵ

Algorithm

```
1    $i \leftarrow 0; x \leftarrow x_0;$ 
2    $r \leftarrow b - Ax;$            // SpMV
3    $d \leftarrow M^{-1}r;$ 
4    $\delta_{new} \leftarrow r^T d;$       // dot product
5    $\delta_0 \leftarrow \delta_{new};$ 
6    $\delta_{limit} \leftarrow \epsilon^2 \delta_0;$     // limiting residual
7   while ( $i < i_{max}$  and  $\delta_{new} > \delta_{limit}$ )
8        $q \leftarrow Ad;$            // SpMV
9        $\alpha \leftarrow \delta_{new}/(d^T q);$ 
10       $x \leftarrow x + \alpha d;$     // saxpy
11      if ( $i \equiv 0 \text{ mod } 50$ )
12           $r \leftarrow b - Ax;$     // SpMV
13      else
14           $r \leftarrow r - \alpha q;$   // saxpy
15      endif
16       $s \leftarrow M^{-1}r;$ 
17       $\delta_{old} \leftarrow \delta_{new};$ 
18       $\delta_{new} \leftarrow r^T s;$       // dot product
19       $\beta \leftarrow \delta_{new}/\delta_{old};$ 
20       $d \leftarrow s + \beta d;$       // saxpy
21       $i \leftarrow i + 1;$ 
22 endwhile
```

Conventional 1-D and 2-D block distributions can be considered as “Cartesian” distributions, since these are based on “Cartesian”-like coordinates.

The projective distribution uses $n \times n$ blocks, but a different relationship between the indices i and j to allocate matrix block (i, j) . We choose row-wise block 1-D distribution³ for our performance comparison, since its compute load distribution is very close to the projective distribution. To highlight the similarities as well as differences between these two distributions, we introduce a new classification of data distributions.

In this classification of distributions, the blocking (*or tiling*) of the matrix remains the same, viz, $n \times n$ square blocks, and each processor gets n tiles only. However, we *relax the constraint* of choosing the tiles *along a Cartesian dimension* only, allowing the tiles to be chosen more generally. Due to this relaxation, we call this family of distributions **Weak Cartesian** distributions. Both the data distributions we consider – the row-wise and projective – satisfy the following: (Note: the solver is run on n processes)

Definition 1 (Weak Cartesian Conditions). 1. *The $r \times r$ square matrix A is viewed as a blocked matrix, with $n \times n$ square blocks, each with $\lfloor \frac{r}{n} \rfloor$ rows and columns. The right and bottom boundary blocks $A_{*,n}$ and $A_{n,*}$ account for all the extra rows and columns in the case when r is not an integral multiple of n .*

2. *The diagonal block $A_{i,i}$ is allocated to process i .*
3. *Each process is allocated $n - 1$ other matrix blocks, apart from its designated diagonal block.*
4. *Each block is allocated to a unique process.*

Note that this classification does not require the additional $n - 1$ blocks to be from the same row or the column. These blocks can be located anywhere in the matrix, subject to the uniqueness condition. The uniqueness condition also implies that two diagonal blocks can not be allocated to one process.

We term data distributions which use n^2 square blocks for distribution among n processes, satisfying the weak cartesian conditions above as an n -process “**Weak Cartesian** distribution”.

During the parallel algorithm, a process will perform the tile multiplication $A_{i,j} \cdot X_j$ only for the tiles $A_{i,j}$ it owns. Since the indices i and j are not constrained in any way in a weak cartesian distribution, a process k may own tiles $A_{i,j}$ where $i \neq k$ and/or $j \neq k$. When $j \neq k$ (i.e. tiles from other columns), k will need to *receive* the corresponding input vector block (X_j in this case), before the beginning of computation of partial sums (*remember that the matrix tiles are not moved during multiplication*). On the other hand, when $i \neq k$ (i.e. tiles from other rows), $A_{i,j} \cdot X_j$ will be only a partial sum for Y_i . Such a partial sum will have to be *sent* back to the process that owns Y_i for completing the computation.

³Abbreviated as row-wise distribution hereafter

Hence, an SpMV based on a weak cartesian distribution will result in each process doing some communication to receive the input blocks it needs and send its input blocks to other processes at the beginning of the multiplication. After the individual multiplications have been computed, each process will perform some communication to send partial sums it has produced for others, and receive partial sums it needs. Different distributions may require one or both the types of communications.

4.3.1 Row-wise Distribution

The row-wise distribution allocates the matrix blocks $A_{i,*}$ to the process i . By our convention, process i starts an iteration with its own vector block X_i . Subsequently, it needs all the other vector blocks at some point or other within each iteration to complete the computation of Y_i . The low diameter interconnect topology based on projective spaces, described earlier, can be used with even the row-wise distribution, but that may not maximize the communication throughput. Hence, we alternatively use the projective space structure to define a novel data distribution, described in the next paragraph. Note the row-wise distribution requires communication for the input vector blocks only. Since each process has all the tiles in one row, each process ends up with complete output vector blocks.

4.3.2 Projective Distribution - A Family of Distributions

For PCG computation, memory modules as points and processes as lines in a **two-dimensional** projective space \mathcal{P}^2 , are used to define an interconnect scheme between $n = p^2 + p + 1$ processes and n memories for a prime power p . Lines and points are duals of each other in the projective plane. Each line has $p + 1$ points, and each point is on $p + 1$ lines. See section 2 for an example derivation for such geometry.

The perfect access patterns and sequences used in communication are defined using a mapping $c : N \rightarrow N^{p+1}$, where N is the set $\{0, 1, \dots, n-1\}$. The value $c(k)$ corresponds to line k , and is the set of points on this line.

Lemma 1. *In a two-dimensional projective space \mathcal{P}^2 , for any $i, j (i \neq j)$,*

- *There is a unique line $k = L_{i,j}$ such that $c(k)$ contains both the points i and j (Unique line through any two distinct points)*
- *There is a unique point $k' = P_{i,j}$ such that the lines $c(i)$ and $c(j)$ both contain k' (Unique point common to any two distinct lines)*

For a given p and hence n , this mapping defines a data distribution – the projective data distribution referenced in earlier sections. More generally, by varying p , we get a family of 2-D projective distributions, all of which follow the same data distribution procedure. We now define the data assignment for a given p (and hence n).

```

procedure PROJECTIVE DATA ASSIGNMENT( $A$ )
  1. Allocate the diagonal matrix block  $A_{i,i}$  to process  $i$ .
  2. To allocate blocks on process  $k$ , consider the line  $c(k)$  corresponding to the process.
  3. For every distinct pair of points  $i, j$  on line  $c(k)$ , allocate matrix block  $A_{i,j}$ 
    (and  $A_{j,i}$ ) to process  $k$ .
end procedure

```

4.3.3 Characteristics of Projective Distribution

We will now study some formal characteristics of the projective data distribution. Note that as yet, we have not used the fact that the matrix is sparse in any way, and hence, the results would hold true for dense matrices as well.

Lemma 2. *The distribution of matrix blocks by procedure Projective Data Assignment results in a symmetric partition of the matrix blocks.*

Proof. 1. Each diagonal block $A_{i,i}$ is allocated to process i .

2. Each non-diagonal block $A_{i,j}$ ($i \neq j$) is allocated to a unique process $k = L_{i,j}$, where $c(k)$ is the unique line through points i and j (by Lemma 1). Thus, all the blocks are covered, resulting in a *partition*.

3. Further $A_{j,i}$ is also allocated on the same process k , yielding a *symmetric* partition. □

Lemma 3. *The partition of matrix blocks by procedure Projective Data Assignment results in each process being allocated $O(p)$ row (column-) sub-blocks.*

Proof. Since the partition is symmetric, we prove the result for row sub-blocks and the corresponding result for column sub-blocks will follow from symmetry.

1. Consider the non-diagonal blocks allocated to process k , and a point i incident on line k : Each point $l (\neq i)$ on the line $c(k)$ will cause a matrix block $A_{i,l}$ to be allocated to process k . The set of such blocks $\{A_{i,l} : l \in c(k), l \neq i\}$ is a row sub-block of A .
2. Thus, each point on the line k results in one row sub-block of matrix A . Hence, the $p+1$ points on line k will result in $p+1$ row sub-blocks.
3. If point k is not incident on line k , the diagonal block $A_{k,k}$ which is always allocated to process k , will introduce an additional row sub-block.

□

Remark 1. : If a process k is allocated a sub-block $A_{i,j}$, it will need the vector block X_j to perform its allocated computations, and produce partial block Y_i .

Remark 2. : From Lemma 3, and the remark above, it is obvious, that each process will require $O(p)$ input blocks, and will produce partial sums for $O(p)$ output blocks.

To satisfy this input/output requirement, $O(p)$ units of communication need to be received, and $O(p)$ to be sent. Adding them up, we get the following **most important result** of our work.

Lemma 4. Projective Distribution has a communication complexity of $O(p) = O(\sqrt{n})$ per process.

The complexity of $O(p)$ is exact for dense matrix-vector multiplication, but only an upper bound for SpMV, since some vector-blocks may not need to be communicated based on the matrix structure in SpMV.

Remark 3. : Due to above distribution, process k contains matrix blocks from row-blocks i and j , and also from column-blocks i and j , where points $i, j \in c(k)$.

Remark 4. : Due to matrix blocks $A_{i,j}$ and $A_{j,i}$, process k will need vector blocks X_i and X_j from their owner processes, and will provide X_k to other processes. Similarly, it will produce partial results Y_i and Y_j ; and some other processes will contribute to Y_k .

Remark 5. A row-wise distribution needs to receive $n - 1$ vector blocks and produce only one vector block as result, to complete the block multiplication. Since it computes the result vector block completely, there is no communication for partial results. On the other hand, a column-wise distribution will need only one input vector block to complete the computation, but will need to send $n - 1$ partial results. In both the cases, the total communication complexity is $n - 1 = O(n)$ blocks per process.

As the number of row sub-blocks increases, the communication complexity for partial results will increase, while the communication complexity for input vector blocks will reduce. On the other hand, if the number of column sub-blocks increases, the reverse situation will manifest. The total communication complexity, which is the sum of number of row sub-blocks and number of column sub-blocks, will be at a minimum when both these numbers are roughly equal, which is approximately \sqrt{n} .

Lemma 5. In an n -process weak cartesian distribution, if a processor has r row sub-blocks and c column sub-blocks, then $r \times c \geq n$.

Proof. 1. Consider process k and the minimal rectangular sub-matrix of A , which has

- all the blocks allocated to k ,

- no entirely empty rows and no entirely empty columns.
2. This rectangular submatrix has r rows and c columns, and hence $r \times c$ blocks of the complete matrix. Note some of these blocks may not be allocated to process k .
 3. Since all the n blocks allocated to process k are in the submatrix, $r \times c \geq n$.

□

The construction of this submatrix is illustrated in Table 1. Rows 1, 4 and 9 are completely empty, and so are columns 1, 5, 8, 9. These entirely empty rows and columns are removed to yield the minimal submatrix described above. Notice in this example, that $n = 9$, $r = 6$ and $c = 5$ and indeed $5 \times 6 = 30 > 9$.

Note that in our discussion of Projective distribution, we have not used “sparsity” of the matrix. We state and prove the next two results too, for “general” matrices (potentially dense). We thus speak of “MV” or Matrix Vector multiplication, instead of particularly SpMV.

Lemma 6. *Matrix-Vector multiplication on any weak cartesian distribution has a per-process communication complexity bounded below by $O(\sqrt{n})$, where n is the number of processes.*

Proof. 1. Consider process i , which has n tiles allocated to it.

2. Let r be the number of row sub-blocks and c be the number of column sub-blocks on process i . Then the communication complexity at process i will be $r + c$ (c communications to receive input sub-vectors and r communications to send output sub-vector partial sums).
3. Since $r \times c \geq n$, $c \geq \frac{n}{r}$
4. Hence, the total communication complexity is bounded below by $r + \frac{n}{r}$, as r varies.

	1	2	3	4	5	6	7	8	9
1									
2			x			x			
3				x			x		
4									
5				x	x				
6						x			
7	x								
8					x				
9									

	2	3	4	6	7
2		x		x	
3			x		x
5			x	x	
6					x
7	x				
8				x	

Table 1: Sample allocation – 9×9 matrix, 6×5 minimal submatrix

5. $r + \frac{n}{r}$ will be minimum ($= 2\sqrt{n}$) when $r = \sqrt{n}$.

□

Again, based on matrix structure in SpMV, some vector-block communication will be elided, the result may not hold for *all* sparse matrices

Since matrix-vector multiplication on projective distribution has communication complexity of $O(\sqrt{n})$, which is the lower bound for matrix-vector multiplication on weak cartesian distributions, we have the next theorem.

Theorem 1. *For matrix-vector multiplication, projective distribution has the minimum communication complexity among the weak cartesian distributions.*

Based on the reasons mentioned earlier, this result too may not hold in its entirety for sparse matrices. Characterizing the sparse matrices with respect to this result is work in progress. Observe that a process k needs vector block X_j only if the process is allocated a block $A_{i,j}$ for some i at some point of time. Further, such a block can be allocated to process k if and only if the point j is on line k . Since every line has only $p+1$ points, obviously, a process k will need only $p+1$ (or p if point k is also on line k) vector blocks.

By similar logic, a process k will produce partial sums for only $p+1$ vector blocks. These characteristics of the projective distribution have been explained in the following example.

4.3.4 Illustration of Projective Distribution

We illustrate this distribution with a connection scheme for $p=3$ in table 2. It is easy to verify that the incidence properties (one line through any two points and vice versa) mentioned above hold for connection scheme. Table 3 shows the matrix blocks allocated to each process. Table 4 shows distribution in a matrix form; $(i, j)^{th}$ entry indicates the process to which the block $A_{i,j}$ is allocated. To make the pattern visible, blocks allocated to process 0 have been shown in a box (e.g. $\boxed{0}$), and those allocated to process 4 been marked with a bullet (e.g. $\bullet 4$) in Table 4. From these tables, it is easy to verify:

1. Incidence properties i.e. existence of one line through any two points, and a point on any two lines. This can be seen by observing in Table 2 that
 - (a) Given any two rows, there is one common entry between the two. e.g. Rows 2 and 6 have the entry “8” in common.
 - (b) Given any two entries (numbers), there is one row containing both the entries. e.g. Entries 1 and 5 are found in row “12”.
2. Number of vector blocks required (input) and produced (output) are $p+1$. Note the computations performed by process 0 as an example :
 - (a) Partial sum $\mathbf{y}_0 \leftarrow A_{0,0} \cdot \mathbf{x}_0$,

Table 2: PG Connections

Line	Points				
0	6	7	9	2	
1	7	8	10	3	
2	8	9	11	4	
3	9	10	12	5	
4	10	11	0	6	
5	11	12	1	7	
6	12	0	2	8	
7	0	1	3	9	
8	1	2	4	10	
9	2	3	5	11	
10	3	4	6	12	
11	4	5	7	0	
12	5	6	8	1	

Table 3: Projective Data Distribution - Blocks for each Process

Proc.	Allocated Blocks												
0	0,0	6,7	6,9	6,2	7,6	7,9	7,2	9,6	9,7	9,2	2,6	2,7	2,9
1	1,1	7,8	7,10	7,3	8,7	8,10	8,3	10,7	10,8	10,3	3,7	3,8	3,10
2	2,2	8,9	8,11	8,4	9,8	9,11	9,4	11,8	11,9	11,4	4,8	4,9	4,11
3	3,3	9,10	9,12	9,5	10,9	10,12	10,5	12,9	12,10	12,5	5,9	5,10	5,12
4	4,4	10,11	10,0	10,6	11,10	11,0	11,6	0,10	0,11	0,6	6,10	6,11	6,0
5	5,5	11,12	11,1	11,7	12,11	12,1	12,7	1,11	1,12	1,7	7,11	7,12	7,1
6	6,6	12,0	12,2	12,8	0,12	0,2	0,8	2,12	2,0	2,8	8,12	8,0	8,2
7	7,7	0,1	0,3	0,9	1,0	1,3	1,9	3,0	3,1	3,9	9,0	9,1	9,3
8	8,8	1,2	1,4	1,10	2,1	2,4	2,10	4,1	4,2	4,10	10,1	10,2	10,4
9	9,9	2,3	2,5	2,11	3,2	3,5	3,11	5,2	5,3	5,11	11,2	11,3	11,5
10	10,10	3,4	3,6	3,12	4,3	4,6	4,12	6,3	6,4	6,12	12,3	12,4	12,6
11	11,11	4,5	4,7	4,0	5,4	5,7	5,0	7,4	7,5	7,0	0,4	0,5	0,7
12	12,12	5,6	5,8	5,1	6,5	6,8	6,1	8,5	8,6	8,1	1,5	1,6	1,8

- (b) Partial sum $\mathbf{y}_2 \leftarrow A_{2,6} \cdot \mathbf{x}_6 + A_{2,7} \cdot \mathbf{x}_7 + A_{2,9} \cdot \mathbf{x}_9$,
- (c) Partial sum $\mathbf{y}_6 \leftarrow A_{6,2} \cdot \mathbf{x}_2 + A_{6,7} \cdot \mathbf{x}_7 + A_{6,9} \cdot \mathbf{x}_9$,
- (d) Partial sum $\mathbf{y}_7 \leftarrow A_{7,2} \cdot \mathbf{x}_2 + A_{7,6} \cdot \mathbf{x}_6 + A_{7,9} \cdot \mathbf{x}_9$,
- (e) Partial sum $\mathbf{y}_9 \leftarrow A_{9,2} \cdot \mathbf{x}_2 + A_{9,6} \cdot \mathbf{x}_6 + A_{9,7} \cdot \mathbf{x}_7$.

Thus, process 0 can perform all its computations using only 4 blocks \mathbf{x}_2 , \mathbf{x}_6 , \mathbf{x}_7 and \mathbf{x}_9 , and produces only 4 blocks \mathbf{y}_2 , \mathbf{y}_6 , \mathbf{y}_7 and \mathbf{y}_9 . Since the example under consideration has $p=3$, the number of blocks, 4, can be easily seen to be $p+1$.

These characteristics are not particular for our choice of the prime number, and a similar example for **any** prime (rather, prime power) can be easily constructed.

Table 4: Projective Distribution Example - Process owning each Block

Row	Column												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	7	6	7	11	11	• 4	11	6	7	• 4	• 4	6
1	7	1	8	7	8	12	12	5	12	7	8	5	5
2	6	8	2	9	8	9	0	0	6	0	8	9	6
3	7	7	9	3	10	9	10	1	1	7	1	9	10
4	11	8	8	10	• 4	11	10	11	2	2	8	2	10
5	11	12	9	9	11	5	12	11	12	3	3	9	3
6	• 4	12	0	10	10	12	6	0	12	0	• 4	• 4	10
7	11	5	0	1	11	11	0	7	1	0	1	5	5
8	6	12	6	1	2	12	12	1	8	2	1	2	6
9	7	7	0	7	2	3	0	0	2	9	3	2	3
10	• 4	8	8	1	8	3	• 4	1	1	3	10	• 4	3
11	• 4	5	9	9	2	9	• 4	5	2	2	• 4	11	5
12	6	5	6	10	10	3	10	5	6	3	3	5	12

4.3.5 Parallel Algorithm

Since SpMV is the strongest computational component within a preconditioned CG (PCG) solver, the parallel PCG algorithm has as one of its core component the parallel SpMV algorithm. Note that the matrix blocks need to be distributed to the individual processes only at the beginning of the solver. Subsequent iterations of the solver do not need the matrix blocks to be re-transmitted. The vector x , which keeps on changing for each iteration of the solver, needs to be sent to all the processes for each iteration.

procedure CG USING ROW-WISE DISTRIBUTION

Initial Step: Let each process i initiate the block computation $A_{i,i}X_i$, as a partial sum for Y_i .

while $i < n$ **do**

Let each process participate in sending and receiving different blocks
of the vector with other processes.

if block X_k is received **then**

Schedule the computation(s) using X_k viz corresponding to local blocks $A_{*,k}$,
thus providing other partial sums.

end if

$i \leftarrow i + 1$

end while

end procedure

procedure CG USING A PROJECTIVE DISTRIBUTION

Initial Step: Let each process i initiate the block computation $A_{i,i}X_i$, as a partial sum for Y_i .

while $i < (p + 1)$ **do**

Let each process participate in sending and receiving different blocks
of the vector with other processes.

if block X_k is received **then**

Schedule the computation(s) using X_k viz corresponding to local blocks $A_{*,k}$,
thus providing other partial sums.

end if

$i \leftarrow i + 1$

end while

while $i < (p + 1)$ **do**

Send the non-local partial sums produced on i are sent to their respective owner
processes.

At process i , add up the received non-local partial sums, produced by other
processes for i .

$i \leftarrow i + 1$

end while

end procedure

It is obvious from the algorithm definitions that the communication complexity for the row-wise distribution is $\mathbf{O}(n)$, while for the projective distributions it is $\mathbf{O}(\sqrt{n})$.

4.4 Vector Communication

Every SpMV execution requires a communication of vector blocks for the block computation to complete. The communication of vector blocks can be done in a variety of ways. An implementation can choose to broadcast the entire vector at the beginning of SpMV. However, this alternative suffers from two drawbacks:

1. The assumption that each process needs all the vector blocks may not be valid. As we have seen earlier, this assumption does not hold for projective distributions.
2. Even otherwise, all the processes wait for entire vector to be available, before carrying out any block-level multiplication. As the number of processes increases, the size of a vector block required for one block multiplication reduces significantly. Hence, broadcasting the vector introduces a large overhead, which can be avoided. Since the multiplication of one matrix block needs only one vector-block and not the entire

vector, the parallelism can be increased by scheduling vector communications at a block level.

Our implementations therefore do not use broadcast alternatives. Instead, each of the vector blocks required is explicitly communicated. The overhead of waiting for data to be available is relatively low.

4.5 Result Vector Ownership

Apart from the distribution of matrix A and vector x , the distribution of the resultant vector y also influences the communication in PCG. In the row-wise distribution, since every process owns complete rows of the matrix, at the end of SpMV, each process ends up with complete elements (i.e. *not* partial sums) of the resultant vector y . Consequently, the dot-products and saxpy's in the PCG algorithm can be carried out in parallel on all the processes, using the vector blocks for all the relevant vectors. This approach still requires the processes to reduce the partial dot-products, but the communication required in this case is of a single scalar after every dot product instead of the full vector. As a result, after SpMV, the vector blocks computed by the processes do not need to be collected on a single process. This approach can thus save significant amount of unnecessary communication.

Since the projective distributions allocate blocks from different rows and different columns, the vector blocks resulting are only partial sums. Communication steps are therefore necessary for these partial sums to be summed up together. However, even including this additional overhead, the communication complexity of the projective distribution is **far superior** to the row-wise distribution.

Hence our implementations consistently use the approach of not storing the entire vector on a single process, but keeping the resultant vector blocks on their owner processes.

Note that this optimization becomes relevant only when optimizing SpMV in context of the PCG algorithm, and may not be relevant at all when SpMV is considered stand-alone.

4.6 Packing Matrix, Input/Output Vector blocks

As explained above, the processes communicate vector blocks with other processes. When a process q sends the vector block X_q to another process s , some of the communication may be unnecessary. Since the matrix is sparse, it is quite likely that the processes s does not use all the elements of X_q when multiplying the blocks. If q knows which columns of X_q are required by s (or by every other process in general), each such communication can be made more lightweight by sending only the required vector elements instead of the entire vector block. The information about which matrix block (and hence its owning process) needs which elements depends upon matrix structure and hence can be computed during problem setup. In the simplest form of this approach, the sending process packs the vector block when sending and the receiving process unpacks the vector on receipt.

In the case of projective distribution, the same concept applies to the result vector block (partial sum). The rows that are completely zero will not contribute to the result vector block, and hence need not be communicated when reducing the partial sums at the end of SpMV.

This optimization turns out to be more powerful, when it is extended further to the matrix blocks, particularly the non-diagonal blocks. When storing a non-diagonal matrix block $A_{i,j}$, the all-zero rows **within this block** can be eliminated and the row numbers within this block renumbered. Similarly, the zero-columns can be eliminated and column numbers renumbered. When a block is completely packed this way, it turns out that the vector-unpacking step becomes redundant. Thus, the sending process still packs the vector block based on the structure at the receiving process, but the receiving process can use this packed vector block without unpacking it, since the column numbers in the packed matrix block and column numbers in the packed vector block are identical.

A projective distribution using packed matrix blocks will produce result vector blocks which are themselves packed. Hence, the process sending the partial sums incurs no additional packing/unpacking overhead. The receiving process unpacks the received partial sum based on the structure information.

5 Method and Experimental Results for PCG Performance Evaluation

The parallel CG (with Jacobi preconditioner) implementation was done in C++ for both row-wise and projective distributions. Both these implementations were carried out by the same researcher, using the same common code base for common operations. This ensured that the optimization style in both the codes was the same, and also that the benefit of common optimizations was uniformly available to both the distributions uniformly.

The experiment was carried out on the EKA cluster at Computational Research Laboratories. The individual nodes are 8-core Intel Clovertown (Xeon X5365 @ 3GHz), with 16 GB RAM and 4x4MB shared L2 cache, running HP XC. The performance was measured using different number of processes to solve a problem. The number of processes chosen were suitable for the projective distribution (viz. of the form $p^2 + p + 1$ where p is a prime power). Thus, the timings have been measured on 7, 13, 21, 31, 57, 73, ... blades. During the runs, 4 threads were used on each core. The sparse matrices in the dataset have been chosen from the University of Florida matrix market [3].

The time required for various steps has been measured in terms of cycles as returned by “rdtsc” instruction, which are then converted to time in seconds based on initial calibration. Operations such as reading the matrix, preprocessing the matrix, distributing it are part of the algorithm setup, and hence are not included in the timings. Some raw simulation details can be found in Table 5.

The plots show the number of processes on X-axis and the time in seconds on the Y-axis.

Table 5: Matrix Simulation Details

Matrix	No. Rows	No. Non-zeroes	Time (sec)	
			7 Processes	91 Processes
msdoor	415,863	20,240,935	9.0	3.0
F1	343,791	26,853,113	27.0	9.0
crankseg_2	63,838	14,148,858	1.8	0.55
Benelechi1	345,874	13,050,496	1.0	0.4
audikw_1	943,695	77,651,847	30.0	6.0
af_shell4	504,855	19,188,875	2.4	0.8
af_0_k101	503,625	17,550,675	1.4	0.5

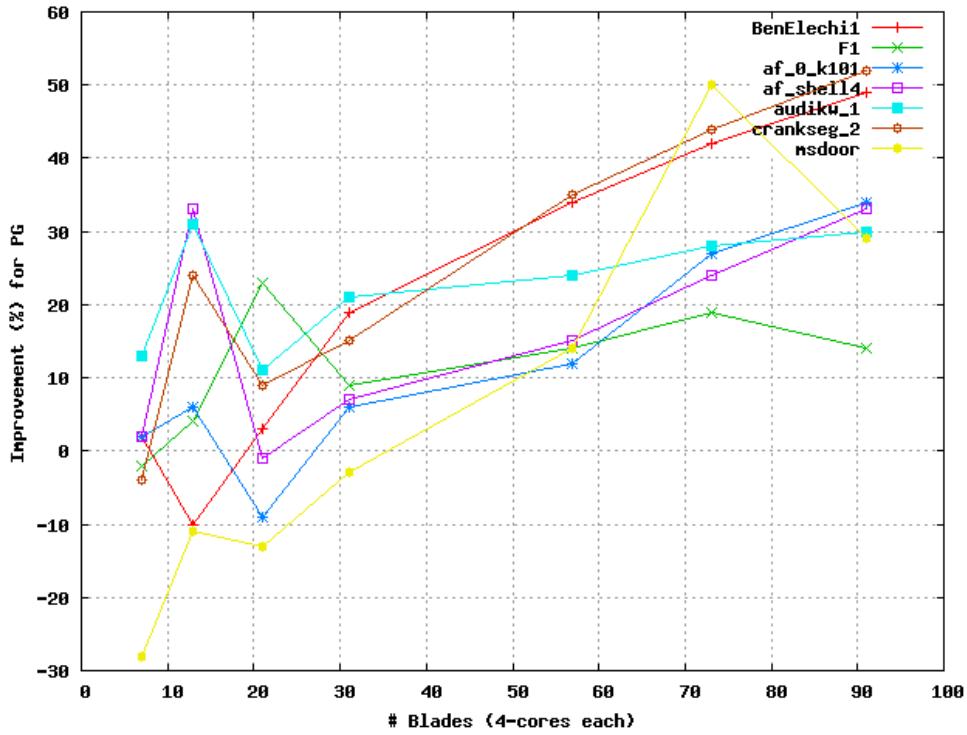


Figure 2: Percentage Improvement in projective Distribution w.r.t. row-wise

Figure 2 plots the performance improvement when using projective geometry distribution against when using the row-wise distribution. For CG algorithm's computation, performance improvement rate is higher when the number of processes is lower. As the number of processes increase, the rate of improvement drops in both the row-wise and projective geometry distributions. In row-wise distribution, the performance flattens around 20-30 processes, and actually starts degrading beyond 50 processes. In projective geometry distribution, the performance starts degrading only in a few graphs, and that too beyond 70 processes. In most

Table 6: Percentage Improvement while using projective geometry based distribution

Matrix	Percentage Improvement using projective						
	7	13	21	31	57	73	91
BenElechi1	2	-10	3	19	34	42	49
F1	-2	4	23	9	14	19	14
af-0-k1-1	2	6	-9	6	12	27	34
af-shell4	2	33	-1	7	15	24	33
audikw-1	13	31	11	21	24	28	30
crankseg-2	-4	24	9	15	35	44	52
msdoor	-28	-11	-13	-3	14	50	29

of the cases, the flattening goes much beyond 50 processes.

In the next part of the paper, we report our investigations in the application of **projective geometry** for another prototypical computing core, viz. **LU/Cholesky Decomposition**.

6 Overview of LU/Cholesky Decomposition

We use the popular trailing matrix update algorithm for decomposition [6]. The matrices involved in circuit simulation are generally extremely big in size, hence only a (matrix-)block level parallelization can be cost-effective. It also improves the probability of even distribution of computational load. To generalize the algorithm to block level, consider the partitioning of the $N \times N$ matrix A into square $B = \frac{N}{b}$ blocks of size $b \times b$. One can similarly partition L and U matrices into blocks $L_{i,j}$ and $U_{i,j}$ respectively, where $i, j \in 0, 1, \dots, B - 1$. Clearly, $L_{i,j}$ and $U_{j,i}$, where $j > i$ will be $\mathbf{0}$ matrices. Assuming well-conditioned matrices, the generalized algorithm without pivoting is then as follows.

In this algorithm, the decomposition is *in-place*, hence the $L_{i,j}$, $U_{i,j}$ matrices can be obtained from decomposed $A_{i,j}$ itself. In each iteration of main loop(line (1)), the operations to be carried out, and which we have optimized, are:

- **Row/Column Update** In the i^{th} iteration, the blocks of i^{th} row/column are updated(lines (4), (7)). These updates can be characterized by a triplet $\{(i, j, i) \text{ or } (i, i, k): i < j, k \leq B - 1\}$ of indices.
- **Trailing Matrix Update** In the i^{th} iteration, blocks $\{A_{j,k}: i < j, k \leq B - 1\}$ are updated(line (11)). This is the most **dominant** operation of this entire computation. These updates can be characterized by the triplet (i, j, k) (or (i, j, j) when $j=k$) of indices.

While parallelizing this computation, one can assign processing units which work on different values of indices j and k , every i^{th} iteration. This way, processing units will do independent computations simultaneously, every iteration. In such a case, these processing units will

Algorithm 2 Block LU Decomposition Algorithm

```
1: for  $i \leftarrow 0$  to  $B - 1$  do
2:    $A_{i,i} \leftarrow \text{blockLU}(A_{i,i})$ 

3:   for  $j \leftarrow i + 1$  to  $B - 1$  do
4:      $L_{j,i} \leftarrow A_{j,i}U_{i,i}^{-1}$ 
5:   end for

6:   for  $k \leftarrow i + 1$  to  $B - 1$  do
7:      $U_{i,k} \leftarrow L_{i,i}^{-1}A_{i,k}$ 
8:   end for

9:   for  $j \leftarrow i + 1$  to  $B - 1$  do
10:    for  $k \leftarrow i + 1$  to  $B - 1$  do
11:       $A_{j,k} \leftarrow A_{j,k} - L_{j,i}U_{i,k}$ 
12:    end for
13:  end for
14: end for
```

need blocks $L_{j,i}$ and $U_{i,k}$, and these blocks have one index in common with the block to be updated. This property is exploited in adapting this algorithm for Karmarkar's architecture, which we discuss in next few sections.

The *Cholesky decomposition* is another direct matrix decomposition method, $\mathbf{A} = \mathbf{L} \cdot \mathbf{L}^T$, where \mathbf{L} is a lower triangular matrix. The computation of \mathbf{L} follows nearly identical pattern of the 3 main steps as of LU decomposition per iteration. The major advantage of Cholesky computation is that *pivoting* is **not** required. Hence our schemes are expected to give clearer, better results, when adapted to Cholesky decomposition. However to save space, we only refer to LU decomposition in the forthcoming sections.

7 Projective Space Details for LU/Cholesky Decomposition

The topology for scheduling trailing matrix update is governed by choice of appropriate subspaces Ω_i of some projective space and their subsumption relations. This relation guides the specification of interconnection network for the multiprocesssing system. Our choice for topology is a *modified form* of the interconnect proposal in [5]. We map the block row and column indices of non-decomposed matrix A to **points** of some projective geometry. The distributed (main) memory blocks as well as processors are mapped to the **lines**, while either the computation or the communication gets mapped to the **planes** of the same geometry. Incidence relationships of lines onto planes is used to design the connections within the system.

```
procedure ASSIGNMENT PROCEDURE( $A$ )
```

1. Store block $A_{i,j}$ in the memory representing the line joining points associated with indices i and j .
2. Assign each computation characterized by a value of indices' triplet (i,j,k) to the processor-memory pair associated with a line mapped to plane corresponding to the triplet.

```
end procedure
```

Lemma 7. A 4-dimensional projective space, $\mathbb{P}(4, GF(2))$, has same number of lines and planes.

By above easy-to-prove lemma, our choice of 31-point 4-d projective geometry, generated over binary field, leads to a symmetric scheme involving same number of computations as well as processors. Some of the combinatorial numbers related to $\mathbb{P}(4, GF(2))$ can be found by evaluating function $\phi(\cdot, \cdot, \cdot)$ (c.f. section 2):

-
- There are 31 points, 155 lines ($\phi(4, 1, 2)$) and 155 planes ($\phi(4, 2, 2)$) in the geometry.
 - Each line has 3 points on it while a plane has 7 points.
 - Each plane has 7 lines, and exactly 3 lines belonging to a plane pass through any point.
 - A line is incident on 7 different planes.
 - A point is present on 15 different lines ($\phi(3, 0, 2)$) and 35 different planes ($\phi(3, 1, 2)$).
-

7.1 Projective Space Automorphisms

Projective subspaces are known to be sets of points. We use two automorphisms on these points, namely **Frobenius** and **Shift** automorphisms, to derive schedules.

The function corresponding to **Frobenius** automorphism is $\Phi(x) = x^p$, where $x \in \mathbb{F}$ and p is the characteristic of \mathbb{F} . Application of this automorphism in $\mathbb{P}(4, GF(2))$ corresponds to doubling of index of each point modulo 31 and taking its remainder modulo 31. Repeated application leads to 5 different automorphisms using the Frobenius map.

Similarly, for **Shift** automorphism, a ‘shift’ function can be defined on points as $L_x : (0, x^i) \rightarrow (0, x^{i+1})$, $\forall i \in 0, 1, \dots, 30$, where x is the generator polynomial. Clearly, application of L_x corresponds to incrementing the index of a point by 1, modulo 31. As earlier, repeated application of L_x leads to 31 different automorphisms. The most important advantage of working with these two automorphisms is that in a 4-d projective space $\mathbb{P}(4, GF(2))$, starting with a particular line or plane, it is **possible** to enumerate **all other** lines and planes using these two. These concepts will be later applied in establishing the interconnection network.

7.2 Perfect Matching Patterns

Using the automorphisms described above, we develop a sequence of patterns depicting perfect matchings in the bipartite graph made of lines and planes in $\mathbb{P}(4, GF(2))$. We denote a k-d projective subspace by a k-tuple of points. Since each plane subsumes 7 different lines, there can be a *sequence* of 7 patterns based on 7 different matchings. Define the first pattern as $S_1 : \Omega_2 \rightarrow \Omega_1$, such that

$$S_1(p) = L_x^a(\Phi^b(0, 1, 18)), \text{ if } p = L_x^a(\Phi^b(0, 1, 2, 5, 11, 18, 19)) \quad (2)$$

Thus the starting point is matching of plane $(0, 1, 2, 5, 11, 18, 19)$ to the line $(0, 1, 18)$ lying on it. Every other plane, obtained by applying b ($0 \leq b \leq 4$) Frobenius followed by a ($0 \leq a \leq 30$) shift automorphisms to this plane, is matched to the line obtained by applying the same sequence of automorphisms to line $(0, 1, 18)$. Varying a and b fully (31×5 cases), we obtain complete mapping for each of the 155 planes in form of a *perfect matching*. Similarly, we can create 6 other such perfect matching patterns S_2, \dots, S_7 by mapping the first plane to the other 6 lines lying on it, one at a time.

The inverses of each of these matching patterns also form *different* perfect matching patterns from the set of lines to the set of planes. These will be denoted by S_i^{-1} .

8 LU/Cholesky Decomposition Algorithm Mapping Scheme - I

The first decomposition scheme uses the concept of perfect matchings to design *direct* inter-connect network. This scheme is a concrete realization of very brief decomposition outline suggested by Karmarkar [5]. In this scheme, we use all the 155 processors and memory blocks. For easy implementation, each processor is connected to its own exclusive memory block (which can be main memory), and this pair is associated with a line of $\mathbb{P}(4, GF(2))$. In addition, the processor is also associated with the plane mapped to the line through the perfect matching S_1 . The plane signifies the computation which is scheduled for this processor. Each such processor is *directly* connected to 12 other processor-memory pairs to form its interconnection subnetwork - 6 pairs mapped to its plane through perfect matchings S_2 to S_7 , and 6 more through inverse perfect matchings S_2^{-1} to S_7^{-1} .

The data of matrix A is distributed among different memory blocks in manner similar to one used in ScaLAPACK [4]. It is first partitioned into $B \times B$ blocks of size $b \times b$. Each block index is then mapped to a point in geometry, obtained by taking its residue modulo 31. Block $A_{p,q}$ is then placed in the memory block given by points $p(\text{mod } 31)$ and $q(\text{mod } 31)$, using function $M : \Omega_0 \times \Omega_0 \rightarrow \Omega_1$.

$$\begin{aligned} M(i, j) &= \text{line joining points } i \& j \ \forall i, j \in 0, \dots, 30 \text{ and } i \neq j \\ M(i, i) &= L_x^i(\Phi^a(0, 1, 18)), a \in 0, 1, 2, 3, 4 (5 \text{ copies of } i^{\text{th}} \text{ diagonal block}) \end{aligned}$$

For mapping computations, the triplet of block indices triplet is first mapped to a triplet of points, in same way as above. It is then mapped to a plane using the map $C_1 : (\Omega_0 \times \Omega_0 \times \Omega_0) \rightarrow \Omega_2$.

$$\begin{aligned} C_1(i, j, i) \text{ or } C_1(i, i, k) &= S_1^{-1}(\text{line joining } i \text{ and } j \text{ or } k) \\ C_1(i, j, k) &= \text{plane through non-collinear points } i, j \text{ and } k \\ C_1(i, j, j) &= \text{plane passing through } i \text{ and the 5 lines } M(i, j) \end{aligned}$$

Our focus has been on a scheme with **improved** performance, in which we take an approach **different** than Karmarkar's, that we present in next section. Further, due to lack of space, we omit the details of the computation, other than re-used portions such as matrix-block mapping, as well as of the provably efficient communication schedule for this scheme. Only brief remarks follow. The details may be found in [10].

Over all iterations, this scheduling scheme makes use of all processing units in $\mathbb{P}(4, GF(2))$. Therefore, perfect matching patterns were applied here to develop communication strategies. This makes the design scalable – the use of a bigger geometry and remapping of the problem is easy in this case.

However, this design under-utilizes the parallelism available: either 15 or 35 out of 155 processors are working every cycle, though they are load-balanced among themselves. In later iterations, lesser number of trailing matrix computations need to be parallelized, and correspondingly lesser number of operational processors. Also, in later iterations, because of duplication of diagonal elements, certain blocks get communicated to extra number of processors. With these considerations, we propose and evaluate a **novel scheme**, which improves upon the resource usage as well as the wiring density.

9 LU/Cholesky Decomposition Algorithm Mapping Scheme - II

In previous scheme, each node had 12 direct connections, leading to high wire density. A better scheme inspired by perfect difference networks(PDN) is presented. Having node degree as 7, it saves upon interconnection cost and complexity.

Using the same geometry $\mathbb{P}(4, GF(2))$, each processor is once again paired up with one memory block, and each such pair is associated with a line of the geometry. These pairs are then interconnected to each other via **buses**; each bus mapped to a plane. Thus, we have 155 buses in all and each bus is connected to 7 processors corresponding to the lines that are incident on its representative plane.

9.1 Motivating the Scheme

To motivate the logic behind using multiple buses, we illustrate the update on $(j, k)^{\text{th}}$ block during 0^{th} phase. We would like to do all these updates in parallel. The charge of block $A_{j,k}$

is with the processor associated with the line through points j and k . Thus, each processor is typically in charge of 6 blocks. Recall from section 6 that the update of block $A_{j,k}$ during 0^{th} phase requires $L_{j,0}, U_{0,k}$ (normalized blocks of 0^{th} column and row respectively). These blocks are with the processors whose line contains point 0. Thus, processor of line through $[j, k]$ must have connection with those processors, which map to lines through $[j, 0]$ and $[0, k]$, so that $L_{j,0}, U_{0,k}$ can be communicated. If points 0, j , k are *collinear*, then this communication is local, i.e. to the processor's local memory. Else, there exists a unique plane through points 0, j , k . This plane would contain the above three lines (totally 7 lines per plane). Hence the requirement is to have interconnections between 7 processors associated with 7 lines of a plane, which can be naturally be implemented using a BUS. The processors with point 0 in their line, must broadcast and the other processors (without point 0) must "listen". Each listening processor, listens to the **unique** BUS (corresponding to the plane containing the line of the processor and point 0). For each plane, at a given time, at most one processor may broadcast. After fixed number of communication steps, each of the 155 processors will be able to "compute" its trailing matrix update concurrently.

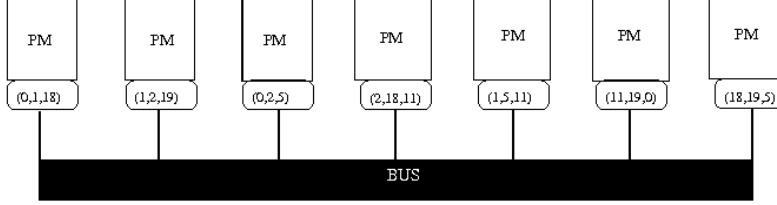
9.2 Data Distribution

The *distribution of data* among memory blocks is identical to the one in previous scheme. Every block $A_{i,j}$ with distinct i, j gets stored in the memory module of processor/line (i, j) . Each diagonal block $A_{i,i}$ is stored in 5 memory modules. This duplication helps in fast communication, as discussed later. The *distribution of computational load* (characterized by triplets described earlier) is done using the following modified function C_2 .

$$\begin{aligned} C_2 : \{C_2(i, j, k) &= \text{line through points } j \& k, j \neq k\} \wedge \\ \{C_2(i, j, j) &= \text{lines to which } A_{j,j} \text{ is allocated}\} \wedge \\ \{C_2(i, j, i) \text{ or } C_2(i, i, k) &= \text{line joining } (i \text{ and } j) \text{ or } (i \text{ and } k)\} \end{aligned}$$

9.3 Illustration of Mapping

We provide a running example of i^{th} iteration to illustrate the mapping. For illustration, the complete 0^{th} iteration is depicted in figure 3. The first step, block LU decomposition and its inverse, is simultaneously computed on 5 different processors per block, like in 1^{st} scheme. For usage in row and column updates, data from these 5 processors needs to be transferred to all processors, which will perform these updates in i^{th} iteration. A point(represented by i) lies on 15 lines in $\mathbb{P}(4, GF(2))$, and hence 15 processors perform these updates per iteration. Hence blocks $L_{i,i}^{-1}$ and $U_{i,i}^{-1}$ are transmitted on 5 buses, mapped to 5 processors having these blocks. Buses are chosen such that the 15 processors can receive them in conflict-free way. In $\mathbb{P}(4, GF(2))$, each line is contained in 7 planes, and hence each processor can potentially communicate with 7 buses. At each cycle, one processor on the bus transmits on the bus,



STEPS						
1	BlockLU(A(0,0))	BlockLU(A(0,0))				
	InverseL(L(0,0))	InverseL(L(0,0))				
	InverseU(U(0,0))	InverseU(U(0,0))				
2	L(0,0), U(0,0) sent to (13,14,0) (15,0,24)	L(0,0), U(0,0) sent to (26,28,0) (30,0,17)			Receives L(0,0), U(0,0) from (0,8,20)	
3	Computes L(1,0), L(18,0) U(0,1), U(0,18)	Computes L(2,0), L(5,0) U(0,2), U(0,5)			Computes L(11,0), L(19,0) U(0,11), U(0,19)	
4	Broadcasts L(1,0), U(0,1) on (0,2,4,10,22,5,7)	Broadcasts L(2,0), L(5,0) U(0,2), U(0,5) on (29,0,2,8,20,3,5)	L(18,0), U(0,18)	L(1,0), U(0,1)	Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (11,19,27,20,6,0,8)	L(18,0), U(0,18)
5	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (30,0,14,10,17,18)	Broadcasts L(2,0), L(5,0) U(0,2), U(0,5) on (29,0,2,8,20,3,5)	NO TRANSFER OF DATA ON THIS BUS		Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (3,11,19,12,29,23,0)	
6	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (0,16,1,18,21,9,25)	L(2,0), U(0,2)	L(2,0), U(0,2)	L(5,0), U(0,5)	Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (11,15,19,0,24,21,25)	L(5,0), U(0,5)
7	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (27,29,0,6,18,1,3)	Broadcasts L(2,0), L(5,0) U(0,2), U(0,5) on (23,27,0,12,5,2,6)	NO TRANSFER OF DATA ON THIS BUS		Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (10,26,11,23,0,19,4,0)	
8	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (23,0,8,1,18,12,20)	Broadcasts L(2,0), L(5,0) U(0,2), U(0,5) on (15,0,16,2,5,24,9)	NO TRANSFER OF DATA ON THIS BUS		Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (9,11,13,19,0,14,16,1)	
9	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (18,22,26,7,0,28,1)	L(19,0), U(0,19)	Broadcasts L(2,0), L(5,0) U(0,2), U(0,5) on (26,28,30,5,17,0,2)	L(11,0), U(0,11)	L(11,0), U(0,11)	L(19,0), U(0,19)
10	Broadcasts L(1,0), L(18,0) U(0,1), U(0,18) on (13,14,15,18,24,0,1)	NO TRANSFER OF DATA ON THIS BUS			Broadcasts L(11,0), L(19,0) U(0,11), U(0,19) on (22,30,7,0,17,11,19,1)	
11	Computes A'(1,18), A'(18,1)	Computes A'(1,1) A'(1,2), A'(2,1) A'(1,19), A'(19,1)	Computes A'(2,5)	Computes A'(2,2) A'(2,18), A'(18,2)	Computes A'(1,1) A'(1,2), A'(2,1) A'(1,11), A'(11,1)	Computes A'(11,19) A'(18,19), A'(19,18)
	A'(2,19), A'(19,2)	A'(5,2)	A'(2,11), A'(11,2)	A'(11,11), A'(11,1)	A'(19,11) A'(18,5), A'(5,18)	A'(19,5), A'(5,19)

Figure 3: Scheme II: Execution of 0th iteration on bus (0, 1, 2, 5, 11, 18, 19)

while remaining may or may not read the data from the bus. By carefully choosing the 5 processors in 1st step(starting with a line and taking its Frobenius automorphisms), and selecting 5 suitable buses, *matrix data $L_{i,i}^{-1}$ and $U_{i,i}^{-1}$ can be distributed in just one cycle without any conflict.* Duplicating 1st step on 5 processors thus saves many communication cycles. As an example, in 0th iteration, the 5 processors storing $A_{0,0}$ perform LU decomposition followed by computation of their inverses. These inverses are then transferred to other processors associated with 0-containing lines through 5 buses. Processor (0, 1, 18) transmits on bus (13, 14, 15, 18, 24, 0, 1), (0, 2, 5) on (26, 28, 30, 5, 17, 0, 2), (0, 4, 10) on (21, 25, 29, 10, 3, 0, 4), (0, 8, 20) on (11, 19, 27, 20, 6, 0, 8), and (0, 16, 9) on (22, 7, 23, 9, 12, 0, 16). The 5 processor nodes are linked via *Frobenius automorphisms*. For the i^{th} iteration, each of the above lines and planes are shifted i times, using *Shift automorphisms*. The *row/column updates* for 0th iteration are scheduled now on each of the 15 processors with blocks $A_{0,j}$ and $A_{j,0}$, $j \in 1, \dots, (B - 1)$, where B is number of blocks.

The *trailing matrix update* for the $(j,k)^{th}$ block is performed by processor corresponding to line (j,k) . This step requires $L_{j,i}$ and $U_{i,k}$, calculated by some other processors during row/column update. Hence these blocks need to be moved in to processor (j,k) . This communication is done in 7 steps. Each processor broadcasts these blocks on each of the 7 buses connected to it. In q^{th} step, the processor having these blocks broadcasts it on the bus mapped to it through perfect matching pattern S_q^{-1} , $q \in 1, \dots, 7$, mentioned earlier. The perfect matching pattern *ensures* that in each cycle, a bus is controlled by 1 processor only, and that a processor receives data from only 1 bus in one cycle. By the end of these steps, data has been broadcast once on every bus connected to a particular processor. E.g., in i^{th} iteration, a processor represented by line $(x, y, x + y)$ will need $L_{m,i}$ and $U_{i,m}$, if one of its indices is same as m . In such a case, the line corresponding to this processor and the line corresponding to processor containing $L_{m,i}$ or $U_{i,m}$ share a common point, and hence a bus represented by some plane. So at some stage within the 7 cycles, this processor will get the required data on the shared bus. Thus, at the end of these 7 steps, each processor will have the entire information needed by it to calculate trailing matrix updates. All the 155 processors now simultaneously compute the updates for the trailing matrix blocks that they possess, thus completing the i^{th} iteration.

9.4 Coherency and Synchronization Issues

Coherence is a primary design issue in multiprocessing scenario. Since we need only one level of memory hierarchy, coherence issue can (only) arise in context of 5 copies of diagonal blocks in 5 processors, during each iteration. These data copies are to be used in later cycles of the iteration, where reading of different, stale data can be a potential issue. However, these diagonal blocks as well as $L_{i,i}$, $U_{i,i}$ are all identical in each iteration, and hence there is no incoherence of data.

Data synchronization conflicts arise when two or more processors try to access some data object at the same time. Given that at a particular moment, all active processors are doing

identical computation, the only possible conflict is a write-write conflict for a particular matrix block in a given cycle, during trailing update. However, only one processor works with a particular block in a cycle *in a particular iteration*, and hence synchronization conflicts also don't arise in this scheme. Further, none of the processors involved in sending data has to receive any data in same cycle, so there is no I/O conflict.

9.5 Design Analysis

In this scheme, in most iterations, the parallelism provided by 155 processing elements is used up, bettering the degree of parallelism exploited by the earlier scheme-I. Hence this design is scalable. Also, the distribution of computation is almost balanced. The source of imbalance is at later stages, when the size of matrix to be updated “trails off” leading to a cut in required amount of parallelism. The resource usage is high, which leads to better time performance. The communication is also lesser than previous scheme, due to only L and U blocks being transferred during each iteration. Broadcasting per processor on each bus takes one cycle, hence there is no scope of changing bus data transfer mode to e.g. split transactions, which can boost the throughput further.

Having a possible target of distributed embedded systems, the buses are expected to be off-chip/backplane buses. Traditionally, shared buses for off-chip purposes have been implemented using tri-state buses that drive bidirectional lines. The advantage of tri-state bidirectional buses is that they take up fewer wires and have a smaller area footprint. This is important since we use many such buses, and hence their resource requirements need to be minimum. One issue is that due to large number of buses involved, there could be more power consumption. However, in system design, power consumption in general trades off with throughput, and similarly we gain throughput here while consuming some more power. Alternate interconnection patterns such as design of novel switches to do simultaneous communication can be considered to alleviate this problem.

10 Experimental Results for LU Decomposition

C++ programs for a mesh-based scheme(for comparison) and the 2 PG-based schemes were developed for both correctness and performance evaluation. By simulations on a uniprocessing system, the three schemes were indeed found to be working correctly. Also, the performance figures calculated analytically for these three schemes have been tested to be correct by instrumenting the corresponding programs. The sources of the programs are available with the authors on request. The tabulated experimental data is presented in table 7. For analysis, the active period of each processor is classified into three categories. A processor either spends time doing $O(b^3)$ computations, or $O(b^2)$ computations, or $O(b^2)$ communication. The $O(b^3)$ computations comprise of block LU decompositions, matrix inversions and matrix multiplications. The $O(b^2)$ computations comprise of matrix subtraction done during

trailing update. The table also contains the *average* number of cycles in which each processor is active for the 3 different categories. *Average processor utilization* in each category can then be defined as the average number of cycles taken by processor in that category, divided by total number of cycles in which all processors finish that category's job in parallel. The normalized computation times are reported too.

Table 7: Cycle Counts for Various Schemes

Scheme/ Size	Total Cycles			Time required			Average Cycles		
	Comp $O(b^3)$	Comm $O(b^2)$	Sub $O(b^2)$	Comp $O(b^3)$	Comm $O(b^2)$	Sub $O(b^2)$	Comp $O(b^3)$	Comm $O(b^2)$	Sub $O(b^2)$
Mesh:62×62	69	221	11	1189.56	61.35	3.05	4	7	3
Mesh:31×31	196	583	56	422.38	40.46	3.89	34	41	30
PG-1:24×24	856	2241	651	856.00	93.23	27.08	82	171	73
PG-1:12×12	5023	12635	4427	627.88	131.40	46.04	669	1234	633
PG-1:8×8	15291	37981	14118	565.77	175.47	65.23	2319	4050	2238
PG-2:24×24	393	846	188	393.00	35.19	7.82	82	44	73
PG-2:12×12	1693	2994	1097	211.63	31.14	11.41	669	214	633
PG-2:8×8	4458	6444	3285	164.95	29.77	15.18	2319	510	2238

An important observation was that across all schemes, as block size decreases, the performance improves almost linearly due to more fine-grained distribution of computational load. Another observation was that our schemes have much better processor utilization than mesh scheme for all different block sizes. Between the two schemes, for each given block size, the 2nd scheme needs much lesser amount of average communication cycles while having same average computation cycles, and hence improves upon the 1st scheme.

In terms of implementation complexity, our architecture is easy to implement for medium-sized matrices with cheap uniprocessors having one level of cache, or just main memory. Each processor will need to store approximately $\frac{n^2}{155}$ matrix elements, which for medium-sized matrices, can fit in its (main) memory or even L1 caches. Like cluster computing, we can use off-the-shelf cheap processors to interconnect and set up the configuration required for these computation schemes. However, unlike cluster computing, these schemes use special interconnects described earlier, not LAN. Given the low complexity of individual processors, a customized board with multiple lightweight microprocessor IP cores can be designed to significantly reduce the form factor of such system.

The main aim of these simulations was to validate the correctness of the communication and computation schedules based on projective geometry. It may be remarked that, as indicated in [9], the projective geometry like architectures can provide good implementations of important communication primitives in distributed high performance computing.

11 Conclusion and Future Work

For CG algorithm, although the performance difference trend between the two distributions is slightly unclear till about 20-30 processes, the percentage improvements seen beyond this cluster size show that the projective distribution seems to be performing better than row-wise distribution. In future, we have plans to (a) exploit the symmetry exposed in the projective distribution, (b) apply the projective distribution in multi/many core configurations as well as GPGPU configurations, (c) extend these ideas to more general sparse matrices, as well as (d) study other characteristics (e.g. load-balancing behaviour etc.) of projective distribution. For LU/Cholesky decomposition, we have introduced two new schemes for processor interconnection based on projective geometry graphs, which work efficiently. Results show that in terms of processor utilization and total time required, these schemes do better than the conventional mesh-based scheme. One direction of research is to make these schemes handle large-sized matrices. Possible interesting implementations include using distributed shared memory schemes over 1-level memory hierarchy to accommodate storage of bigger blocks. This work suggests suitability of using projective-geometry based graphs for application-specific system design, in contrast to generic design setup in [8]. As a matter of fact, variations of topologies derived from PG-based graphs that are succinct for other potential parallel computations have also yielded promising results, especially in the area of *error correction coding* [1] and digital systems design [2].

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