Classical Game Theory
A Playful Introduction

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Chapter 1

Decision Making

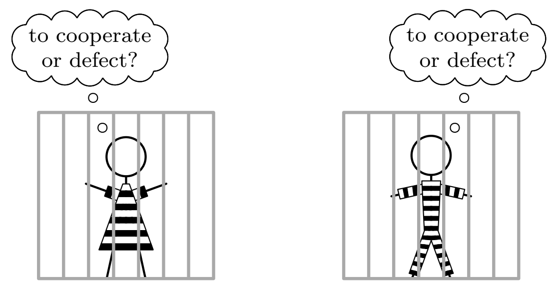


Figure 1.1: The Prisoner's Dilemma

Classical game theory could also be called "decision theory" since our central focus will be modelling real-world situations where our players have single decisions to make. Indeed, going forward we will use the term "game" in a very broad sense: A game is considered to be any situation involving multiple players, where the actions of one player may effect the outcome for another. Unlike the setting of combinatorial game theory where the only possibilities are win/lose/draw, here we will be considering far more general strategic situations where an outcome may be simultaneously good for both players or bad for both players or some combination of the two.

These general games are far too rich to hope for some kind of universal theorem that explains optimal play in every situation, but the tools of game theory can nonetheless model and analyze nearly any scenario in which the actions of one player affect the results for another. Precise mathematical representations can illuminate the essence of a strategic situation and, in some cases, strongly indicate best play. Other cases defy such conclusions, yet the formalism still helps to clarify the situation at hand. One great benefit of studying game theory is developing the ability to abstract key strategic ingredients from a wide variety of interactions.

To represent outcomes that are good for both players or bad for both players requires a richer payoff structure. To allow for this, we now associate each outcome with a pair of numbers (a, b) where a indicates the payoff to Rose and b indicates the payoff to Colin. We assume that Rose and Colin wish to maximize their payoff (with no concern for the payoff of the other player).

Definition 1.1. A *matrix game* is a game played between Rose and Colin using a fixed matrix A, known to both players, where each entry of A consists of an ordered pair (a, b) where a indicates the payoff to Rose and b the payoff to Colin. To play, Rose secretly chooses a row, Colin secretly chooses a column and then they reveal their answers. If this row and column selects the entry (a, b) from the matrix A, then Rose gets a *payoff* of a and Colin gets a *payoff* of b.

Let's get things started with a classic.

Game 1.2 (The Prisoner's Dilemma)**.** Rose and Colin have been caught robbing a bank, but the police don't have all the necessary evidence to charge them with the maximum penalty. The police isolate the players and offer each the option to give evidence to convict the other and, in return, receive less jail time. So each player can either cooperate with the other player (C) and stay silent, or may defect (D) by turning over evidence. In the matrix of payoffs below $-n$ indicates a prison sentence of n years.

		Colin	
		C	D
Rose	C	$-1, -1$	$-10, 0$
	D	$0, -10$	$-5, -5$

From Rose's perspective, if Colin cooperates, she gets a shorter prison term by defecting, and if Colin defects, she also gets a shorter prison term by defecting. In other words, no matter what Colin does, Rose does better by defecting. A similar argument shows that Colin should likewise choose defect. Still, if both players defect, each will get a -5 payoff, which is *worse for both players* than the -1 payoffs they could achieve by cooperating. This presents a significant, fundamental, and pervasive dilemma: rational individual choice leads to inferior outcomes.

1.1 Utility

In the Prisoner's Dilemma, we assumed that each player wished to minimize their own prison sentence with no regard for the other player. This might be a reasonable assumption in some situations, but could also be entirely wrong in others. What if Rose and Colin are in love? They might then value years apart in prison very differently than if they are enemies. The next section develops the vital notion of utility, which captures such meaningful variations.

Money vs. Utility

Imagine a scenario where one player, say Rose, considers buying health insurance. Assume for simplicity that the cost of this insurance is \$1000 per year. With 99% probability, Rose will stay healthy and not need the insurance. With 1% probability, she will have a significant health problem. In that case, if she's insured she will pay nothing extra, but if she is uninsured Rose will have to pay \$80,000 on average. The matrix below represents this 1-player game.[1]

[1] This type of game is known as a *game against the universe*.

		Chance	
		$^{99}/_{100}$ healthy	$^{1}/_{100}$ problem
Rose	buy ins.	$-\$1,000$	$-\$1,000$
	dont buy	$\$0$	$-\$80,000$

If Rose buys insurance her expected payoff will be $-\$1000$, but if she does not buy insurance, it will be $-\$800$. So not purchasing insurance gives Rose a better expected monetary return than purchasing insurance. Though this may seem surprising at first, the perspective of an insurance provider may clarify why such a payoff structure is likely: To be a viable business, the insurance company must set prices so, on average, income exceeds expenditure for each client.

This calculation suggests that Rose should *not* buy insurance. But how does Rose truly value the different possible outcomes in this scenario? The matrix shows Rose's payoff of $-\$1000$ when she buys insurance regardless of whether or not she has a significant health problem. It's unlikely this number reflects how she really values these two very different outcomes. Certainly Rose would much prefer to stay healthy than to have a serious health issue (even with insurance). And if she does fall ill, Rose would be especially relieved to have insurance. Ideally, we could associate with each outcome a number that accurately represents how Rose values the different events. Such numbers, called *utilities*, could further clarify Rose's decision to buy or not to buy insurance. We postpone for now the question of how to determine utilities and focus here on how these values impact the decision. Suppose the following matrix gives Rose's utility for each of the four outcomes.

		Chance	
		$^{99}/_{100}$ healthy	$^{1}/_{100}$ problem
Rose	buy ins.	$-1,000$	$-51,000$
	dont buy	0	$-300,000$

When Rose stays healthy, her utility is 0 when she does not buy insurance and -1000 when she does buy insurance, just as before. If Rose becomes seriously ill, then her utility will be $-51,000$ if she is insured and $-300,000$ if she is not insured. In this latter unfortunate situation, medical bills could financially ruin Rose.

What are Rose's expected payoffs in this new matrix of utilities? If she buys insurance, her expected utility is $(^{99}/_{100})(-1000) + (^{1}/_{100})(-51000) = -1500$. On the other hand, if Rose doesn't buy insurance, her expected utility is $(^{99}/_{100})(0) + (^{1}/_{100})(-300,000) = -3000$. So, in the end, using Rose's utility values alter the game. Purchasing insurance gives Rose a higher expected utility, despite a lower expected monetary payoff.

Even in circumstances where only money is at stake, it is quite possible for rational players to take actions that do not maximize expected monetary payoff. Let's consider a more pleasant situation for Rose, who will now play a one-player game starting with a hidden fair coin flip. Rose has the option of either receiving a sure $\$1,000,000$ or gambling on the result of the coin toss. If Rose gambles and the coin toss is heads, then she gets $\$2,200,000$, but if the coin came up tails, she gets nothing. The following 1-player game represents the situation.

		Chance	
		$^1/_2$ heads	$^1/_2$ tails
Rose	sure	$\$1,000,000$	$\$1,000,000$
	gamble	$\$2,200,000$	$\$0$

Now compute Rose's expected payoffs for each option. If Rose chooses the sure money, then she gets $\$1,000,000$. If instead she gambles, her expected payoff is $(^1/_2)(2,200,000) + (^1/_2)(0) = \$1,100,000$, which is $\$100,000$ higher.

The expected dollar reward is greater when Rose gambles, yet many individuals in her position strongly prefer the sure money. A key issues here the decreasing marginal utility of wealth. In short, a person will value their first 1 million dollars more than their second 1 million dollars. Another factor is Rose's risk tolerance for gambling with such large amounts of money. Rose might value the security of sure money more than the chance of a higher expected reward. Alternately, Rose might be neutral or even favorable towards risk. The game below accounts for all of these factors and assigns Rose utilities for the various outcomes.

		Chance	
		$^1/_2$ heads	$^1/_2$ tails
Rose	sure	$1,000,000$	$1,000,000$
	gamble	$1,600,000$	0

If this is accurate, then Rose's expected utility for gambling is $800,000$, less than the $1,000,000$ she gets by taking the sure payoff. Once again, we see that Rose will choose differently to maximize her expected utility instead of her expected monetary payoff.

In both examples here, Rose confronted a choice between two options. To get a more meaningful picture of these decision problems, we associated each of Rose's possible outcomes with her utility—a number representing her value for each outcome relative to the others. Introducing utility radically transformed the nature of the game and generated a different choice from Rose in both cases. All of this reasoning hinges on the assumption that Rose can meaningfully assign utilities to the various outcomes... but is this really possible?

Von Neumann and Morgensterns' Lottery

Among the many important contributions to game theory due to Von Neumann and Morgenstern is a robust theory of utility. They proved that under certain natural assumptions, rational players can associate outcomes with utilities in such a way that the rational action in any decision problem is to make the choice that maximizes expected utility. Moreover, they explained how to determine these utilities using the idea of a lottery.

To introduce their idea, suppose Rose has two possible outcomes, X and Z, and that she values X with utility 0 and Z with utility 10. Given a lottery resulting in outcome X with probability $\frac{1}{2}$ and outcome Z with probability $\frac{1}{2}$, then Rose's expected utility from this lottery would be $\frac{1}{2}0 + \frac{1}{2}10 = 5$. More generally, in a lottery where Rose gets X with probability p and Z with probability $1-p$, her expected utility is $(p)0+(1-p)10 = 10-10p$.

Now introduce a third outcome Y and suppose that Rose ranks the three outcomes $X < Y < Z$, so she prefers Z to Y and she prefers Y to X. Von Neumann and Morgenstern suggested that Rose's utility for Y could be determined by comparing Y to lotteries involving outcome X with probability p and Z with probability $1-p$ for various values of p. If $p = 1$, then the lottery gives Rose an outcome of X every time — she would certainly prefer Y to that lottery. At the other extreme, when $p = 0$, the lottery gives Rose an outcome of Z every time, so she prefers this lottery to Y. For some number $0 < p < 1$ Rose views the lottery as equivalent to Y. From above, Rose's expected utility in this lottery is $10 - 10p$, so this is the utility assigned to Y. More generally, if there are many outcomes $Y, Y', Y'', \ldots$ that Rose prefers over X but less than Z (i.e. $X < Y, Y', Y'', \ldots, < Z$), then this procedure assigns to each one a utility.

In this process, X need not have utility 0 nor Z utility 10. The same operations work with any utility x for X and any utility z for Z as long as $x < z$. The lottery says how to determine the utilities of all of the outcomes given utilities for X and Z, but how will we find the utilities associated with the least and most favorable outcomes? The answer is that the utility x for the least favorable outcome X and the utility z for the most favorable outcome Z may be arbitrarily chosen, subject to the constraint $x < z$. In other words, utility is not an absolute measure, but rather a kind of comparative scale similar to temperature metrics. The Celsius scale, for example, declares the freezing point of water to be 0 and the boiling point for water to be 100 and then extends linearly. The Fahrenheit scale sets the freezing point of water at 32 and the boiling point 212. We could likewise build another perfectly reasonable temperature scale by choosing x to be the freezing point and z to be the boiling point. Endpoints on a utility scale work similarly.

Von Neumann and Morgenstern's lottery provides a method to assign utilities that measure how a player compares any collection of outcomes. We can use these utilities instead of basing our analysis on years in prison or dollar reward. This presents a much more nuanced view of players' motivations and facilitates more careful analysis of rational play. In the Prisoner's Dilemma, for example, having players assign a utility to each of the four possible outcomes would create a 2×2 matrix game more meaningfully representing the essence of the situation. Forthcoming chapters feature strategic situations and represent outcomes as payoffs for the players. In these games, we will always assume that the given payoffs accurately reflect the players' utility for a corresponding outcome.

1.2 Matrix Games

This section develops the basic theory of matrix games, a generally applicable model for representing decision problems. We first introduce a handful of dilemmas that help demonstrate the variety of matrix games beyond the Prisoner's Dilemma. Concepts of dominance and equilibria will then help us better understand the play of these games in some cases.

More Dilemmas

To get an idea what types of decision problems can be represented with matrix games, we will introduce a number of meaningful examples. There is no complete theory advising players how to play an arbitrary matrix game, so we will not be presenting a "solution" to most of these games here or ever. Our goal is to simply to demonstrate the power of the model.

Game 1.3 (Coordination Game)**.** Rose and Colin are test subjects in a psychology experiment. They have been separated, and each player gets to guess either X or Y. Both players get \$1 if their guesses match, and nothing if they do not.

		Colin X	Colin Y
Rose	X	1, 1	0, 0
Rose	Y	0, 0	1, 1

In a game such as this one, *communication* between the players would result in an advantageous outcome. If they players knew the game and were permitted to communicate prior to play, it would be easy for them to agree to make the same choice.

This situation is in contrast with the Prisoner's Dilemma. In that game defect always yields a better payoff than cooperate—a reality unaffected by the existence of communication. Indeed, were advance communication part of the Prisoner's Dilemma, each player would have incentive to try to convince the other one to cooperate, but then to defect in the actual game.

Game 1.4 (Dating Dilemma)**.** Suppose Rose and Colin have started dating, and they are on the phone deciding what to do this evening. Rose wants to go to the ball game while Colin prefers to go to the film. Unfortunately, Colin's phone battery dies mid-conversation, so further communication is impossible! Each player must individually decide to go to the Ball game (B) or to the Film (F). The matrix of payoffs is below.

		Colin B	Colin F
Rose	B	2, 1	0, 0
Rose	F	0, 0	1, 2

They players prefer to spend the evening together, so payoffs where the players are in separate places are worst possible for both players. The tricky part of this dilemma is that Rose would prefer to end up with Colin at the ballgame, whereas Colin would rather be with Rose at the Film. Coordination could help here, but there is still a conflict between the players.

In some situations it may be possible for one of the players to seize the initiative and move first (breaking the usual simultaneity of matrix games). Suppose that Rose committed to going to the ball game and Colin knew of this decision. The his best move is to attend the ball game, too, giving Rose her favorite outcome. This feature appears in numerous games and, particularly in economics, is called first mover advantage.

Finally, note that although this situation is artificial, scenarios with this type of payoff structure appear commonly in interpersonal dynamics. In a circumstance the same players might repeatedly encounter this sort of game, the players might arrange to coordinate, but alternate between the two activities. This game will reappear later in our consideration of cooperation.

Game 1.5 (Volunteering Dilemma)**.** Rose and Colin have been a mostly happy couple for some time now, but they disagree about who will do the dishes. Each has the option of either volunteering (V) to do the dishes or staying silent (S). Here is the matrix of payoffs.

		Colin	
		S	V
Rose	S	$-10, -10$	$0, -2$
	V	$-2, 0$	$-1, -1$

If neither player volunteers to do the dishes, the payoff is quite bad for both (perhaps they cannot eat dinner). Each player would most like to stay silent and have the other volunteer. Both, though, certainly want to avoid the bad SS outcome. This dynamic is so common that the aboriginal people of Tierra del Fuego have a word for it. *Mamihlapinatapai* (noun): The situation when two people are staring at one another, each hoping the other will volunteer to do something that both want done, but neither wants to do.

Game 1.6 (Stag Hunt)**.** Rose and Colin remain a little annoyed with one another over the dishes, each feels a bit uncertain of the trust between them. Now they are headed off to the woods on a hunting trip. Each player has two strategies — work together and hunt for a stag (S), or go for a rabbit alone (R). Here is the matrix of payoffs.

		Colin	
		S	R
Rose	S	$3, 3$	$0, 2$
	R	$2, 0$	$1, 1$

Obviously, both players do best here if they cooperate and hunt the Stag. Really the only sticky point is that if one player suspects the other may go for a rabbit, then that player has incentive to choose R, too. Communication is likely to help here, as long as the players trust each other enough to cooperate.

Strategies and Dominance

Next we will introduce a natural concept of dominance that can be used to identify inferior strategies. in some instances, repeatedly applying this concept will allow us to determine rational play. In fact, dominance already appeared in the Prisoner's Dilemma—it was the key tool guiding our investigation.

For a matrix game A define a *pure strategy* for Rose to be a choice of row and a *pure stratey* for Colin to be a choice of column. If there are two rows, i and i', with the property that no matter what column Colin chooses, Rose's payoff from i is greater than or equal to her payoff from i', then row i *dominates* row i'. Also as before, if, no matter what column Colin chooses, Rose's payoff will be strictly greater when choosing row i than when choosing row i', then row i *strictly dominates* row i'. We define domination and strict domination analogously for Colin's column strategies.

A rational player maximizing payoff would never play a strictly dominated strategy. In the Prisoner's Dilemma, for instance, the strategy (D) of defecting strictly dominates the strategy (C) of cooperating. Eliminating the dominated strategies leaves (D) as the only rational move. Let's see another example where iteratively deleting dominated strategies is the key to determining rational play.

Game 1.7 (Competing Pubs)**.** Consider two pubs in a tourist town competing for business. On a given night, 200 locals and 200 tourists head to these pubs. For simplicity, assume that each person will order exactly one drink. Each pub prices drinks at either \$5, \$6, or \$7. The tourists are not discerning, and $1/2$ of them will go to each pub. The locals, on the other hand, know the drink prices, so if one pub is cheaper than the other, that pub will get all of the locals. If prices are equal at both pubs, then the locals will also split $1/2$ and $1/2$. Assuming each pub wants to maximize revenue results in the following matrix game.

		Pub 2 \$5	Pub 2 \$6	Pub 2 \$7
	\$5	$1000, 1000$	$1500, 600$	$1500, 700$
Pub 1	\$6	$600, 1500$	$1200, 1200$	$1800, 700$
	\$7	$700, 1500$	$700, 1800$	$1400, 1400$

We see that the \$5 strategy strictly dominates \$7 for both players. Eliminating this row and column leaves a 2×2 matrix where strategy \$5 dominates \$6 for both players. After this dominated row and column are eliminated, the rational solution remains: both pubs charge \$5 and get a 1000 payoff. Note that both pubs could get a payoff of \$1400 if they both set prices at \$7. So, just as in the Prisoner's Dilemma, rational play by the individual pubs results in an inferior outcome.

The iterated removal of strictly dominated strategies led to the above conclusion. Since it is clearly irrational to play a strictly dominated strategy, this approach is a powerful tool in analyzing rational play. What about eliminating strategies which are dominated (but not strictly dominated)? While this type of reduction does help find certain equilibria (as we'll see later), it can lead to some surprising results.

Best Responses and Pure Equilibria

As usual, in a game-theoretic situation, we need to consider the actions of the other player when making a decision. To facilitate our discussion, we introduce some terminology. Fix a matrix game A and assume that Colin chooses the pure strategy of column j. A *best pure response* to column j is a choice of row for Rose that maximizes her payoff under the assumption that Colin plays column j. Similarly if i is a row, then a *best pure response* to row i is a choice of column for Colin that maximizes his payoff under the assumption that Rose plays row i.

In the special case of 2×2 matrices, a graphical technique called a *movement diagram* helps to visualize best responses. We construct this diagram as follows:

- For each column, draw an arrow from the outcome Rose likes least to the one she likes best (in case of a tie, use a doubleheaded arrow).

- For each row, draw an arrow from the outcome Colin likes least to the one he likes best (in case of a tie, use a doubleheaded arrow).

The following figure shows a simple example.

$$
\left[\begin{tikzcd}
(1,2) & (3,1) \arrow[l] \\
(0,1) \arrow[u] \arrow[r] & (2,2) \arrow[u]
\end{tikzcd}\right]
$$

Figure 1.2: A movement diagram

In the Figure 1.2 movement diagram, the upper left cell has both vertical and horizontal arrows pointing toward it. This means that row 1 is a best response to column 1, and that column 1 is a best response to row 1. This implies a certain stability for this pair of strategies. Namely, if each player thinks the other is going to play 1, then each should play 1. Take note that this stability does *not* in any way mean that these strategies are somehow optimal — only that each is a best response to the other. This significant type of stability merits a formal definition: A *pure Nash equilibrium* is a pair of pure strategies, one for Rose and one for Colin, each of which is a best responses to the other.

For identifying pure Nash equilibria and visualizing dynamics of play, movement diagrams can be quite helpful. The following figure depicts these diagrams for our four dilemmas.

Prisoner's Dilemma

$$
\left[\begin{tikzcd}
(-1,-1) \arrow[r] \arrow[d] & (-10,0) \arrow[d] \\
(0,-10) \arrow[r] & (-5,-5)
\end{tikzcd}\right]
$$

Dating Dilemma

$$
\left[\begin{tikzcd}
(2,1) & (0,0) \arrow[l] \arrow[d] \\
(0,0) \arrow[u] \arrow[r] & (1,2)
\end{tikzcd}\right]
$$

Volunteering Dilemma

$$
\left[\begin{tikzcd}
(-10,-10) \arrow[r] \arrow[d] & (2,-2) \\
(-2,2) & (-1,-1) \arrow[l] \arrow[u]
\end{tikzcd}\right]
$$

Stag Hunt

$$
\left[\begin{tikzcd}
(3,3) & (0,2) \arrow[l] \arrow[d] \\
(2,0) \arrow[u] \arrow[r] & (1,1)
\end{tikzcd}\right]
$$

Figure 1.3: Movement diagrams for our dilemmas

1.3 Game Trees

Game trees are useful when studying combinatorial games, and they are useful again in the setting of classical game theory. Indeed, game trees, provide natural models of more general multi-stage strategic interactions. Here, we will add three new features — general payoffs, chance nodes, and information sets — to enhance the power of game trees to model sequential interactions.

General Payoffs

When studying W-L-D game trees, the only possible outcomes were $+-$, $-+$, or 00. We could view these outcomes as $(1,-1)$, $(-1,1)$ and $(0,0)$ in our present setting. More generally, we will now allow outcomes in our game trees to be any possible ordered pair (a,b) indicating a payoff of a Rose and b to Colin. See Figure 1.4 for an example.

A *strategy* for a player is as before — a set of decisions indicating which move to make at each node where that player has a decision to make. As with W-L-D game trees, we can

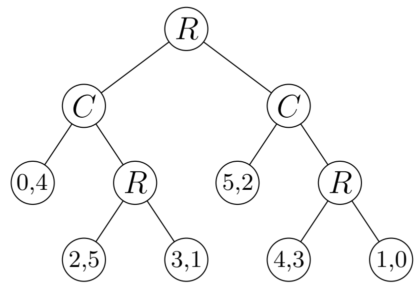


Figure 1.4: A game tree with general payoffs

work backward up a general game tree to try to determine how rational players will play. This process appears in Figure 1.5, where each nonterminal node is marked with the payoffs the players will receive if they play rationally from that point down.

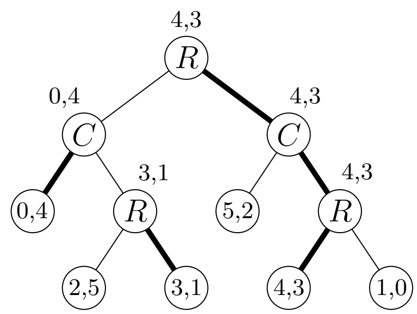


Figure 1.5: Rational play

This process works well unless there are payoffs (a, b) and (a', b') that agree in exactly one coordinate and can break down the working-backward procedure.

Randomness

Next we will enhance our game trees by incorporating randomness of the type appearing when players roll dice, toss a coin, or call on another source of randomness to make a decision. To enable game trees to model such situations, we introduce a new type of nonterminal node called a *chance* node (indicated in our figures by Ch). Suppose a game reaches a chance node N and the possible next nodes from N are $N_1, N_2, \ldots, N_k$. Then the game will randomly move from N to one of the nodes $N_1, \ldots, N_k$ according to given probabilities. More precisely, each N_i comes with a probability p_i (used to label the edge from N to N_i). If node N is reached, the game will then move to node N_i with probability p_i. Naturally, the numbers $p_1, \ldots, p_k$ will all be nonnegative and will sum to 1. Just as the game tree is known to both players, these probabilities are also common information.

Game 1.8 (Coin Toss)**.** In this game, Rose calls either “heads” or “tails” and then a fair coin is tossed. If Rose correctly predicts the coin toss, the outcome is $(1, -1)$ and otherwise it is $(-1, 1)$. Figure 1.6 depicts a game tree for this game. Here the node marked R is one where Rose has a choice, the nodes marked Ch are chance nodes.

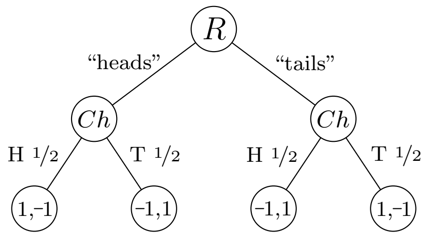


Figure 1.6: A game tree for Coin Toss

Incomplete Information

Finally, we would like to use game trees in scenarios in which the players do not have full information about the state of the game. Consider a Blackjack position, in which each player starts with one card face up and another face down. A blackjack position in a game between Rose and Colin might appear as follows to an outside spectator.

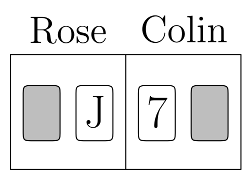


Figure 1.7: Spectator's view

Rose may peek at her face down card, so she knows both of her cards and one of Colin's. Colin knows both of his cards, but only one of Rose's. When we refer to the position of a game, we will always assume that a position includes *all* information, even that which may be hidden from the players. For instance, the position corresponding to the spectator's view from Figure 1.7 might be the one depicted in Figure 1.8.

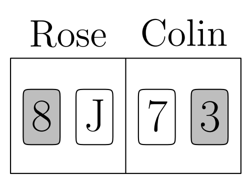


Figure 1.8: The position

Now consider the situation from Rose's perspective. What does she know about the position? Since Rose can see both face up cards and can also peek to see her downward facing card, she can distinguish between many positions. But since she can't see Colin's face down card, there are also many positions between which she cannot distinguish. The figure below shows a number of possible Blackjack positions. Positions indistinguishable to Rose are connected with a dotted line.

In general, the position of a game always includes all relevant information about the state of the game. Each player may or may not be able to distinguish between different positions of the game. A set consisting of all of the positions that appear the same to Rose is called an

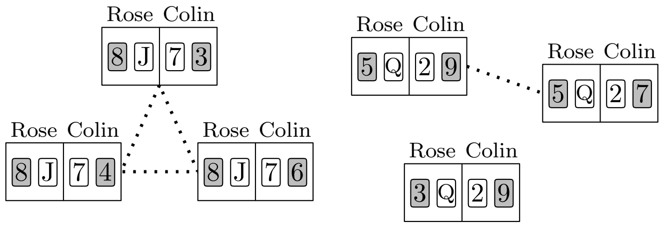


Figure 1.9: Some positions from Rose's perspective

information set for Rose.[2] In the Blackjack example, one information set for Rose consists of all positions in which she has a face down 8 and a face up J and Colin has a face up 7. Information sets for Colin are defined analogously.

Now we will add dotted lines to a game tree to represent an information set. If two nodes in a game tree are labelled for the same player, who cannot distinguish between those two positions in the game, indicate this by a dotted line connecting these two nodes of the tree. Note that in this case, the available moves from these two nodes should look the same to the player choosing between them. Next we will see how to use information sets to model a game with coins which plays a little like poker or blackjack.

Game 1.9 (Coin Poker)**.** At the start of play, Rose and Colin each put one chip in the pot as ante and each player tosses a coin. Rose sees the result of her toss, but not Colin's, and vice versa. It is then Rose's turn to play, and she may either fold, ending the game and giving Colin the pot, or bet and place 2 more chips in the pot. If Rose bets, then it is Colin's turn to play and he may either fold giving Rose the pot, or he may call and place 2 chips in the pot. In this latter case, both coin tosses are revealed. If both players have the same coin toss, the pot is split between them. Otherwise, the player who tossed heads wins the entire pot.

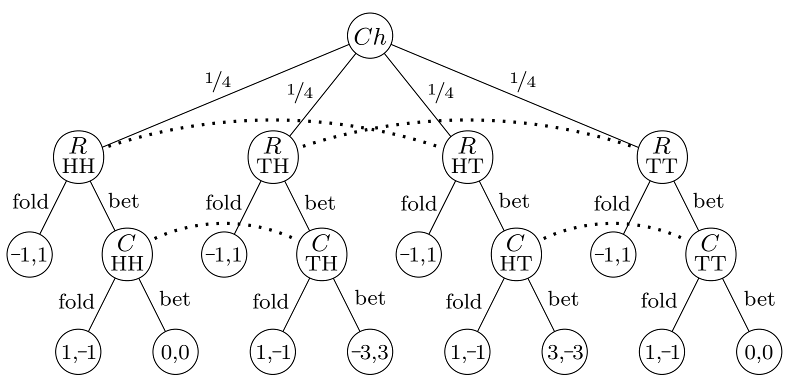


Figure 1.10: A game tree for Coin Poker

In this example, Rose has two information sets. The first consists of the two nodes where she flipped heads and the second contains the two nodes where she flipped tails, so call these

[2]This standard terminology is admittedly somewhat confusing since an information set indicates a lack of information.

information sets H$*$ and T$*$. Colin similarly has two information sets, one where he flipped heads and one where he flipped tails, so call these $*$H and $*$T.

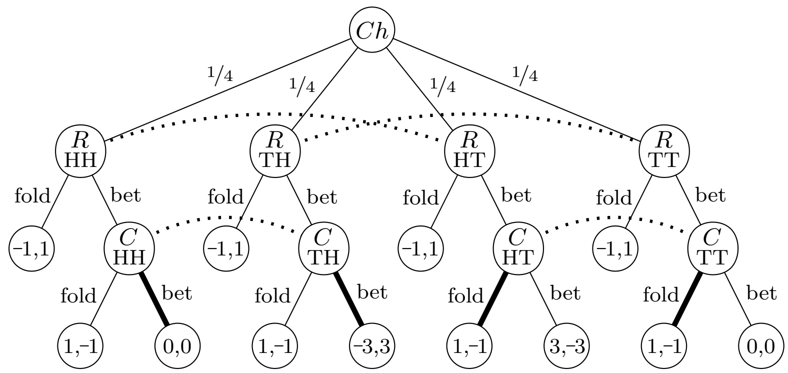


Figure 1.11: Colin's strategy H

What about strategy in this game tree? If Rose has T and Colin has T, Rose would probably like to bet, but if Rose has T and Colin has H, she would probably prefer to fold. Rose unfortunately cannot distinguish between these two positions. Rose's strategy must determine her move using only the information she has. In other words, a strategy for Rose must make the same choice in both of the above circumstances. Expanding our earlier definition, we now define a *strategy* for a player in a game tree to be a choice of action at each information set where that player has a decision. In our example, Rose has information sets H$*$ and T$*$ corresponding to the result of her coin toss. In each information set, she can choose to either bet or fold, so Rose has a total of four strategies. She can Always bet (A), Never bet (N), bet only with Heads (H) or bet only with tails (T). Although Colin's position in this game is quite different, he also has the choice to bet or fold in each of his information sets, so we will also use A, N, H, and T to indicate his strategies, too.

Adding randomness and incomplete information markedly expands game trees' capacity to model a wide variety of situations with sequential decisions. Unfortunately, these same improvements mean the recursive working-backward procedure in general no longer applies to determine rational play.

1.4 Trees vs. Matrices

To tap the modelling power of game trees, we need a new method to study them in general. In fact, every game tree can be represented by a matrix game, and conversely any matrix game can be represented by a game tree. So these two different types of games are really just two different ways to visualize the same game. The matrix representation is generally known as the normal or strategic form (since it crystallizes the strategies), while the tree representation is called the extensive form (since it is based on the extended sequence of possible decisions).

Trees to Matrices

Since it's possible to represent any game tree as a matrix game, any theorem proven in the real of matrix games also applies to game trees. This is especially helpful since matrix games are nice to work with mathematically. Let's first convert the Coin-Poker game tree into a matrix.

As we saw, in the game Coin-Poker, Rose and Colin each have four strategies, denoted A, N, H, and T. Suppose that Rose adopts the strategy of always betting (A) and Colin adopts the strategy of betting only with heads (H). Then the outcome depends only on the chance node and we will have one of the following possibilities.

probability	coins (RC)	payoffs
$1/4$	TT	$(1, -1)$
$1/4$	TH	$(-3, 3)$
$1/4$	HT	$(1, -1)$
$1/4$	HH	$(0, 0)$

In this case, the expected payoff for Rose is given by $1/4(1) + 1/4(-3) + 1/4(1) + 1/4(0) = -1/4$ and the expected payoff for Colin is $1/4(-1) + 1/4(3) + 1/4(-1) + 1/4(0) = 1/4$. More compactly, we can compute the expected payoffs for both players simultaneously as

$$1/4(1, -1) + 1/4(-3, 3) + 1/4(1, -1) + 1/4(0, 0) = (-1/4, 1/4)$$

.

Similarly, we can compute for any possible choice of strategies for Rose and Colin all of the expected payoffs, which always have the form $(a, -a)$. We can now take a rather sophisticated *strategic* view of this game. Instead of having Rose and Colin sequentially play the game, imagine they each simply choose a strategy ahead of time. We know the expected payoff for every possible strategy pair, so we can represent the game tree as a matrix game. The Coin-Poker game tree results in the following zero-sum matrix game.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cccc} A & T & H & N \end{array} \\ \text{Rose} \begin{array}{c} A \\ T \\ H \\ N \end{array} & \begin{bmatrix} (0,0) & (5/4, -5/4) & (-1/4, 1/4) & (1,-1) \\ (-5/4, 5/4) & (-1/4, 1/4) & (-1, 1) & (0,0) \\ (1/4, -1/4) & (1/2, -1/2) & (-1/4, 1/4) & (0,0) \\ (-1,1) & (-1,1) & (-1,1) & (-1,1) \end{bmatrix} \end{array}$$

A quick analysis of this matrix reveals strategy A strictly dominates T for Rose. Without the row corresponding to T, Colin's strategy H dominates all of his other strategies. Assuming Colin plays H, Rose does best to play either H or A and the payoffs will be $(-1/4, 1/4)$. Looking back at the original game we see that Colin's strategy of H will guarantee him a payoff of $1/4$ while Rose's strategies A and H each will guarantee Rose a payoff of $-1/4$.

This process of turning a game tree into a matrix generalizes naturally.

Procedure 1.10 (Tree-to-Matrix)**.** Consider an arbitrary game tree with strategies $S_1, S_2, \ldots, S_m$ for Rose and $T_1, T_2, \ldots, T_n$ for Colin. For each choice of a strategy S_i for Rose and a strategy

T_j for Colin, we know how every non-chance node in the game tree will act. Therefore, each possible outcome will occur with some probability, and we can compute the expected payoffs for this pair of strategies S_i and T_j. Define a matrix A by setting the (i, j) entry to be the pair of numbers (a, b) where a and b are the expected payoffs for Rose and Colin playing strategy S_i opposite T_j. The matrix A is called the *strategic form* of the game.

Let's do one more example of this tree-to-matrix process with a game that is not zero sum.

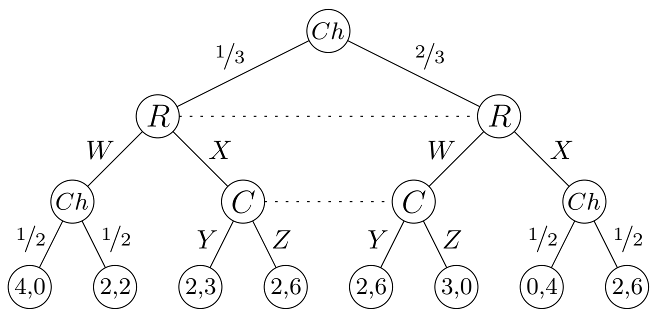


Figure 1.12: A game tree

Example 1.11. In Figure 1.12, Rose and Colin each have just a single information set, and in each of these information sets each player has just two options. Say Rose has strategies W and X and Colin has strategies Y and Z. Let's compute the expected payoffs when Rose plays W and Colin plays Y. With probability $1/3$, the initial chance node chooses left. Since Rose chooses W a second chance node gives expected payoffs of $1/2(4, 0) + 1/2(2, 2) = (3, 1)$. With probability $2/3$, the initial chance node chooses right. If Rose plays W and Colin Y, then the game ends at a terminal node with payoffs $(2, 6)$. Altogether, we find expected payoffs when Rose plays W and Colin plays Y are

$$1/3\Big(1/2(4,0) + 1/2(2,2)\Big) + 2/3(2,6) = (7/3, 13/3).$$

We may likewise compute the expected payoffs for all possible choices of strategies. Combining these gives us the following strategic form matrix.

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \quad Y \quad & \quad Z \quad \end{array} \\ \text{Rose} & \begin{array}{c} W \\ X \end{array} & \begin{bmatrix} (7/3, 13/3) & (3, 1/3) \\ (4/3, 13/3) & (4/3, 16/3) \end{bmatrix} \end{array}$$

In this matrix game, Rose's strategy of W strictly dominates X. Eliminating this dominated row leaves the matrix $\big[(7/3, 13/3), (3, 1/3)\big]$ in which Colin has a dominant strategy of Y. So, assuming our players are rational, Rose will play W and Colin will play Y.

This convenient matrix form suppresses all of the details concerning who can move where and when. The matrix focuses on comparing how a strategy for Rose matches up with a strategy for Colin.

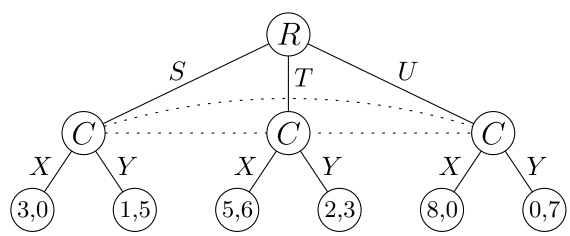
Matrix-to-Tree

The matrix form is generally most convenient for dealing with games. However, it is also the case that every matrix game can be turned into a game tree. To introduce our matrix-to-tree process, we will again consider an example.

Example 1.12. Consider the following 3×2 matrix game

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Colin}} \\ & & \multicolumn{1}{c}{X} & \multicolumn{1}{c}{Y} \\ \cline{3-4} & S & 3,0 & 1,5 \\ \cline{3-4} \text{Rose} & T & 5,6 & 2,3 \\ \cline{3-4} & U & 8,0 & 0,7 \\ \cline{3-4} \end{array}$$

We usually imagine Rose and Colin revealing their choices simultaneously in a matrix game, whereas in a game tree their decisions are necessarily sequential. Nevertheless, it is straightforward to model a matrix game with a tree. We will have Rose make her decision and move first and have Colin make his decision and move second. Since Colin is not permitted to have knowledge of Rose's choice, we put all of his decision nodes in one large information set. For our example game, the corresponding game tree follows.



In this game tree, Rose makes the first choice of S, T, or U, and then Colin chooses either X or Y, with no knowledge of Rose's choice. This is just the same as the play of the original matrix game. The general process is a straightforward extension of this example.

Procedure 1.13 (Matrix-to-Tree)**.** Let M be a matrix game with m rows and n columns. Construct a game tree where the root node offers Rose the choices $1, 2, \ldots, m$ each corresponding to a row. Every node directly below the root node will be a decision node for Colin with the choices $1, 2, \ldots, n$ each corresponding to a column, and we place all of these nodes in the same information set. Below each of these decision nodes for Colin is a terminal node, and the terminal node is given payoffs from the (i, j) entry of M if Rose selected row i and Colin selected row j. This tree is called the *extensive form* of M.

We have now demonstrated that the expressive power of matrix games and game trees are theoretically equivalent. Both of these models capture all features one might hope for in modelling most decision problems between two players. Going forward we will focus primarily on matrix games, since these are structurally simpler and more convenient to work with mathematically.

Chapter 2

Nash Equilibrium and Applications

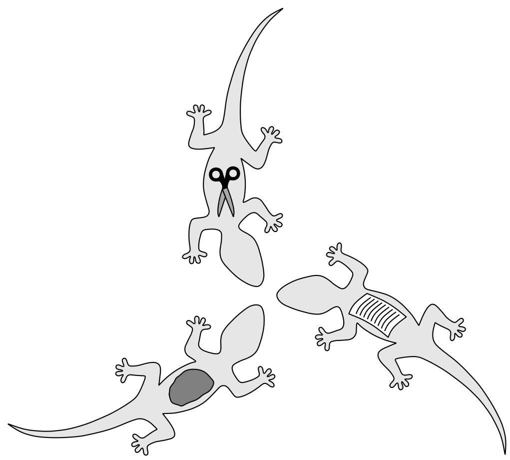


Figure 2.1: Lizards playing Rock-Paper-Scissors

Matrix games can represent a wide variety of decision problems between two players, but how should rational players play in this general framework? This chapter develops Nash's Equilibrium Theorem, one of the great triumphs of classical game theory. The proof indicates how rational players in an environment of repeated play will move towards a certain type of equilibrium point. In the many places where a matrix game is played repeatedly, Nash's theorem has great predictive power in addition to its mathematical beauty. This chapter introduces Nash's Theorem and explores applications in evolutionary biology and business. Chapter 4 presents a proof of Nash's Equilibrium Theorem.

To get things started, try playing the following game a few times in succession.

Game 2.1 (A Number Game)**.** This is a game between player 1 and player 2. Player 1 secretly selects an integer $0 \le n_1 \le 100$ and player 2 secretly selects an integer $0 \le n_2 \le 100$. Then the players' choices are revealed, and player 1 gets a payoff of $(100 - n_1 - n_2)n_1$ while player 2 gets a payoff of $(100 - n_1 - n_2)n_2$.

See Section 2.4 for a discussion of rational play in this game.

2.1 Mixed Strategies

Consider the following 2×2 matrix game

$$\begin{bmatrix} (1,2) & \longleftarrow & (3,1) \\ \downarrow & & \uparrow \\ (2,1) & \longrightarrow & (2,2) \end{bmatrix}$$

If Rose expects Colin will select the first column, then she should play the second row to get her best payoff. If Rose is playing the second row, then Colin would do best to choose the second column, but If Colin is playing the second column, then Rose would do best to play the first row. However, in this case, Colin does best by playing the first column. As suggested by the movement diagram, we are going in circles here! Each player wants to predict what action the other will take, but would prefer their own actions to be unpredictable. A natural way to create unpredictability is to introduce a source of randomness. For instance, Colin might flip a coin and choose column 1 if it comes up heads and column 2 if it is tails. Then Colin will randomly play each column with probability $1/2$. Rose might similarly flip two coins and choose row 1 if she gets heads both times and otherwise choose row 2. Then she will play row 1 with probability $1/4$ and row 2 with probability $3/4$.

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} 1/2 & \quad 1/2 \end{array} \\ \text{Rose} & \begin{array}{c} 1/4 \\ 3/4 \end{array} & \begin{bmatrix} (1,2) & (3,1) \\ (2,1) & (2,2) \end{bmatrix} \end{array}$$

The idea of using this kind of randomness in constructing a strategy is essential in going forward, so we will take time to formalize it.

Mixed Strategies

For an arbitrary $m \times n$ matrix game A, define a *mixed strategy* for Rose to be a row vector $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \dots & p_m \end{bmatrix}$ with the property that $p_i \geq 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} p_i = 1$. When Rose *plays* $\mathbf{p}$, she randomly chooses to play row i with probability p_i. Analogously for Colin, a *mixed strategy* is a column vector $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$ for which $q_j \geq 0$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} q_j = 1$. When Colin *plays* $\mathbf{q}$ he randomly chooses column j with probability q_j. (Unless stated otherwise, we assume that Rose and Colin's probabilistic choices are independent.) Above, for example, Rose played the mixed strategy $\begin{bmatrix} 1/4 & 3/4 \end{bmatrix}$ and Colin played the mixed strategy $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Consider the mixed strategy $\mathbf{p} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ for Rose. In this strategy, Rose plays the first row with probability 1 and all other rows with probability 0. This operates exactly like Rose's pure strategy choice of row 1, so, in this case, we will also call $\mathbf{p}$ a pure strategy. More generally, we say that a mixed strategy for Rose or Colin with one entry equal to 1 and all others 0 is a *pure* strategy. Pure strategies are then just special cases of mixed ones.

We will be interested in computing expected payoffs for our players in the circumstance when each is using a mixed strategy. To do so, it will be helpful to introduce two more matrices. For a matrix game A, each entry is an ordered pair of numbers, say (x, y). If we take just the first coordinate, x, of each such entry, we have an ordinary matrix (i.e. a matrix where each entry is a single real number) that tells us the payoffs for Rose. We call this *Rose's payoff matrix*. Similarly, the second coordinates of each entry from our game$A$ form a matrix C, called *Colin's payoff matrix*.

Example 2.2.

$$\underset{\text{Matrix Game}}{A = \begin{bmatrix} (1,4) & (2,0) & (3,3) \\ (3,0) & (1,2) & (2,1) \end{bmatrix}} \qquad \underset{\text{Rose's Payoff Matrix}}{R = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}} \qquad \underset{\text{Colin's Payoff Matrix}}{C = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}}$$

To compute the players' expected payoffs, assume that Rose plays $\mathbf{p}$ and Colin plays $\mathbf{q}$. To compute Rose's expected payoff, consider her payoff matrix R. Rose gets a payoff corresponding to the (i, j) entry of this matrix with probability $p_i q_j$. It follows that Rose's expected payoff is given by $\mathbf{p}R\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i r_{i,j} c_j$. Colin's payoffs are given by his payoff matrix C, so Colin's expected payoff in this case is $\mathbf{p}C\mathbf{q}$.

Example 2.2 Continued. Assume Rose plays $\mathbf{p} = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix}$ and Colin plays $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

Then

$$\begin{aligned} \text{Rose's expected payoff} &= \mathbf{p}R\mathbf{q} = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 2, \\ \text{Colin's expected payoff} &= \mathbf{p}C\mathbf{q} = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 4 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 4/3. \end{aligned}$$

Best Response

How can we determine best responses (both pure and mixed) for our players? First, we will introduce a new term. Suppose that $\mathbf{p}$ is a mixed strategy for Rose, and the i^{th} entry of $\mathbf{p}$ is 0. In this case, Rose will never select row i when she plays $\mathbf{p}$. On the other hand, if the i^{th} entry of $\mathbf{p}$ is positive, then she will select row i some of the time. We say that $\mathbf{p}$ *calls on* row i if $p_i > 0$. Similarly, we say that Colin's mixed strategy $\mathbf{q}$ *calls on* column j if $\mathbf{q}_j > 0$. Equipped with this new term, let's think about best responses in a particular example.

Example 2.3. Consider the following 3×2 matrix game A together with Rose's payoff matrix R.

$$A = \begin{bmatrix} (3,2) & (-1,1) \\ (3,1) & (0,3) \\ (1,4) & (2,6) \end{bmatrix} \qquad R = \begin{bmatrix} 3 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

We will suppose throughout this example that Colin will always play the mixed strategy $\mathbf{q} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Then vector $R\mathbf{q}$ below indicates Rose's expected payoff for each possible pure strategy she selects.

$$R\mathbf{q} = \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix}$$

When Rose plays a pure strategy of the first row, her expected payoff is 1. If she plays either the second or the third row, then her expected payoff is $3/2$. Thus, the pure strategies that maximize Rose's expected payoff are rows 2 and 3.

Now suppose that Rose is going to play the mixed strategy $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. What mixed strategies will give her the highest possible expected payoffs? Could she possibly do better with a mixed strategy than with a pure one? Rose's expected payoff when she plays $\mathbf{p}$ is

$$\mathbf{p}R\mathbf{q} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix} = p_1 + (3/2)p_2 + (3/2)p_3.$$

To maximize Rose's expected payoff, we maximize the quantity on the right hand side of the previous equation over all possible mixed strategies for Rose. In order for $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$ to be a mixed strategy, we must have $p_1, p_2, p_3 \geq 0$ and $p_1 + p_2 + p_3 = 1$. It follows that

$$p_1 + (3/2)p_2 + (3/2)p_3 \leq 3/2\,(p_1 + p_2 + p_3) = 3/2.$$

So, we see that Rose cannot get an expected payoff which is greater than $3/2$. We already know that Rose's pure strategies of playing row 2 and row 3 achieve this highest possible expected payoff of $3/2$. Thus, Rose cannot get a higher expected payoff using a mixed strategy than she could with a pure strategy. Are there any mixed strategies different from the pure strategies of row two and row three that achieve this best possible expected payoff? To see the answer, return to the inequality from above: $p_1 + (3/2)p_2 + (3/2)p_3 \leq (3/2)p_1 + (3/2)p_2 + (3/2)p_3$. Note that when p_1 is not zero (i.e. $p_1 > 0$) the right hand side is strictly larger. This tells us that Rose's expected payoff will be $< 3/2$. On the other hand, when $p_1 = 0$ the left and right hand side will be equal, so Rose's expected payoff will be exactly $3/2$. We conclude that the mixed strategies that give Rose the highest possible expected payoff are precisely those of the form $\mathbf{p} = \begin{bmatrix} 0 & p_2 & p_3 \end{bmatrix}$ (note that this also includes the pure strategies of row two and three as special cases). Therefore, Rose's optimal mixed strategies are precisely those that only call on rows that are best pure responses.

With this guiding example, we now extend the notion of best responses to include the use of mixed strategies. Consider a matrix game A and suppose that Colin is going to play a mixed strategy $\mathbf{q}$. We say that a pure strategy, row i, is a *best response* to $\mathbf{q}$ if this row gives Rose the maximum expected payoff over all of her pure strategies. Similarly, we say that a mixed strategy $\mathbf{p}$ for Rose is a *best response* if it gives her the maximum expected payoff over all mixed strategies. We similarly define best pure responses and best responses for Colin. The following proposition says how to find all best responses to a mixed strategy for either Colin or Rose.

Proposition 2.4. *Consider a matrix game with payoff matrices R, C.*

Rose's best responses to Colin's mixed strategy $\mathbf{q}$	
Pure	*Every row* i *for which the* i^{th} *entry of* $R\mathbf{q}$ *is max.*
Mixed	*Every* $\mathbf{p}$ *which only calls on best pure responses to* $\mathbf{q}$

Colin's best responses to Rose's mixed strategy $\mathbf{p}$	
Pure	*Every column* j *for which the* j^{th} *entry of* $\mathbf{p}C$ *is max.*
Mixed	*Every* $\mathbf{q}$ *which only calls on best pure responses to* $\mathbf{p}$

Proof. We prove the result only for Rose's best responses, since Colin's case follows by a similar argument. When Colin plays $\mathbf{q}$ and Rose plays row i, her expected payoff will be $(R\mathbf{q})_i$ (i.e. the i^{th} entry of the vector $R\mathbf{q}$). So, Rose's best pure responses are precisely those rows i for which the i^{th} entry of $R\mathbf{q}$ is a maximum entry.

Next, define M to be the maximum entry in the vector $R\mathbf{q}$. If Rose plays the mixed strategy $\mathbf{p} = \begin{bmatrix} p_1 & \dots & p_m \end{bmatrix}$ and Colin plays $\mathbf{q}$, then her expected payoff will be

$$\mathbf{p}R\mathbf{q} = \begin{bmatrix} p_1 & \dots & p_m \end{bmatrix} (R\mathbf{q}) = \sum_{i=1}^{m} p_i (R\mathbf{q})_i \leq \sum_{i=1}^{m} p_i M = M.$$

We see that Rose cannot do better than an expected payoff of M with a mixed strategy. If her mixed strategy $\mathbf{p}$ calls on a row i that is not a best pure response to $\mathbf{q}$, then the i^{th} entry of $R\mathbf{q}$ is strictly smaller than M so $p_i(R\mathbf{q})_i < p_i M$. The above equation then shows that $\mathbf{p}$ will give Rose an expected payoff which is $< M$. On the other hand, if every row called on by $\mathbf{p}$ is a best pure response to $\mathbf{q}$, then $p_i(R\mathbf{q})_i = p_i M$ will hold for every $1 \leq i \leq m$ and Rose will get her best possible expected payoff of M. □

2.2 Nash Equilibrium

The single most important theorem in classical game theory pairs best responses to produce what's called a Nash Equilibrium. For a matrix game A a pair of mixed strategies $\mathbf{p}$ for Rose and $\mathbf{q}$ for Colin form a *Nash Equilibrium* if $\mathbf{p}$ is a best response to $\mathbf{q}$ and $\mathbf{q}$ is a best response to $\mathbf{p}$.

Theorem 2.5 (Nash)**.** *Every matrix game has a Nash Equilibrium.*

In Chapter 4, we prove this important theorem. For now, let's explore an example.

Example 2.6. Consider the following 5×5 matrix game A together with the payoff matrices R for Rose and C for Colin.

$$A = \begin{bmatrix} (1,4) & (-1,1) & (3,7) & (2,-3) & (0,2) \\ (5,3) & (1,-1) & (1,0) & (3,1) & (2,4) \\ (0,-1) & (2,3) & (-1,2) & (1,0) & (4,1) \\ (1,3) & (3,7) & (2,-4) & (-2,6) & (6,0) \\ (3,2) & (0,-1) & (1,0) & (4,3) & (-2,5) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 5 & 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & 1 & 4 \\ 1 & 3 & 2 & -2 & 6 \\ 3 & 0 & 1 & 4 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 1 & 7 & -3 & 2 \\ 3 & -1 & 0 & 1 & 4 \\ -1 & 3 & 2 & 0 & 1 \\ 3 & 7 & -4 & 6 & 0 \\ 2 & -1 & 0 & 3 & 5 \end{bmatrix}$$

Consider the strategies $\mathbf{p} = \begin{bmatrix} 0 & 9/17 & 3/17 & 5/17 & 0 \end{bmatrix}$ for Rose and $\mathbf{q} = \begin{bmatrix} 2/31 \\ 0 \\ 0 \\ 12/31 \\ 17/31 \end{bmatrix}$ for Colin. If Colin plays $\mathbf{q}$, then $R\mathbf{q} = \begin{bmatrix} 26/31 \\ 80/31 \\ 80/31 \\ 80/31 \\ 20/31 \end{bmatrix}$. This gives Rose's expected payoff for each possible row she might select. We see that rows two, three, and four are Rose's best pure responses to $\mathbf{q}$ — they give her an expected payoff of $80/31$. More generally, her best responses to $\mathbf{q}$ will be all mixed strategies which only call on rows two, three, and four (i.e. mixed strategies of the form $\begin{bmatrix} 0 & p_2 & p_3 & p_4 & 0 \end{bmatrix}$). In particular, her strategy $\mathbf{p}$ is indeed a best response to $\mathbf{q}$.

On the other side, assume that Rose is going to play $\mathbf{p}$. Then the vector $\mathbf{p}C = \begin{bmatrix} 39/17 & 35/17 & -14/17 & 39/17 & 39/17 \end{bmatrix}$ indicates Colin's expected payoff for each possible column he might choose. We see that columns one, four, and five are Colin's best pure responses to $\mathbf{p}$ and these give him an expected payoff of $39/17$. More generally, his best responses to $\mathbf{p}$ will be all mixed strategies which only call on columns one, four, and five (i.e. all mixed strategies of the form $\begin{bmatrix} q_1 \\ 0 \\ 0 \\ q_4 \\ q_5 \end{bmatrix}$). Thus, Colin's strategy $\mathbf{q}$ is a best response to $\mathbf{p}$.

Since Rose's strategy $\mathbf{p}$ is a best response to Colin's strategy $\mathbf{q}$ and $\mathbf{q}$ is also a best response to $\mathbf{p}$, this pair of strategies forms a Nash Equilibrium. This equilibrium does *not* indicate exactly how the players should necessarily play the matrix game A. In fact, this particular matrix game has four more Nash Equilibria! Though a Nash Equilibrium does not prescribe how to play the game, it does provide a kind of stability. If Rose and Colin are going to play this game repeatedly, and they have settled into the mixed strategies of Rose playing $\mathbf{p}$ and Colin playing $\mathbf{q}$, then neither player has any incentive to change. In this way, a Nash Equilibrium is a stable point.

Computing Equilibria in 2×2 Games

The complicated problem of computing Nash Equilibria in matrix games is well beyond the scope of this book. We nonetheless *do* have the tools to find Nash Equilibria in 2×2 games. A key concept here is the following.

Definition 2.7. Let A be a 2×2 matrix game with payoff matrices R for Rose and C for Colin. We say that a mixed strategy $\mathbf{p}$ for Rose *equates Colin's results* if both entries of $\mathbf{p}C$ are equal (so Colin will get the same payoff no matter what he does). Similarly, we say that a mixed strategy $\mathbf{q}$ for Colin *equates Rose's results* if both entries of $R\mathbf{q}$ are equal (so Rose will get the same payoff no matter what she does).

Lemma 2.8. *For every 2×2 matrix game, we have:*

1. *If Rose has no dominant row, then Colin has a mixed strategy equating her results.*

2. *if Colin has no dominant column, then Rose has a mixed strategy equating his results.*

Proof. We will give the argument for (1), part (2) follows from a similar argument. Let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the payoff matrix for Rose in our 2×2 matrix game. We will assume that Rose does not have a dominant strategy and we will show that there is a mixed strategy $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$ for Colin that equates Rose's results. In order for Colin's mixed strategy to equate Rose's results, the equation $aq + b(1-q) = cq + d(1-q)$ must hold. This simplifies to $(a - c + d - b)q = (d - b)$, so we must have $q = (d-b)/((a-c)+(d-b))$. Since Rose has no dominant strategy, it is not possible for $a \geq c$ and $b \geq d$ and also not possible for $a \leq c$ and $b \leq d$. It follows that either $d > b$ and $a > c$ or $d < b$ and $a < c$. In either case, the number q will satisfy $0 \leq q \leq 1$, so $\mathbf{q}$ is a mixed strategy which equates Rose's results, as desired. $\square$

Theorem 2.9. *In every 2×2 matrix game, one of the following holds.*

1. *Iterated removal of dominated strategies reduces the matrix to 1×1. This row and column form a pure Nash Equilibrium.*

2. *Rose and Colin both have mixed strategies that equate the other player's results and these form a Nash Equilibrium.*

Proof. Suppose first that one player has a dominant strategy. Without (significant) loss of generality, assume that Rose has a dominant strategy of row i. Choose column j to be a best pure response for Colin when Rose plays row i. Now the pure strategies of row i and column j form a pure strategy Nash Equilibrium.

Next suppose that neither player has a dominant strategy. In this case the previous lemma implies that Rose has a mixed strategy $\mathbf{p}$ that equates Colin's results, and similarly Colin has a mixed strategy $\mathbf{q}$ that equates Rose's results. Now $\mathbf{p}$ and $\mathbf{q}$ are best responses to each other, so they form a Nash Equilibrium. $\square$

Example 2.10. Find a Nash Equilibrium in the following 2×2 matrix game.

$$A = \begin{bmatrix} (1,-2) & (0,2) \\ (-1,3) & (3,-1) \end{bmatrix}$$

The payoff matrices are $R = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$ for Rose and $C = \begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$ for Colin. Neither player has a dominant strategy, so by the above theorem, each player has a mixed strategy that equates the other's results and these form a Nash Equilibrium. First let us find a strategy $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ for Rose that equates Colin's results. Equating Colin's first and second column payoffs gives us

$$p(-2) + (1-p)3 = p(2) + (1-p)(-1).$$

Thus $8p = 4$ and $p = 1/2$, so $\mathbf{p} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$. Now we will look for a strategy $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$ for Colin that equates Rose's results. Equating Rose's first and second row payoffs yields

$$q(1) + (1-q)0 = q(-1) + (1-q)3.$$

This gives us $5q = 3$ so $q = 3/5$ and $\mathbf{q} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$. Since the strategies $\mathbf{p}$ and $\mathbf{q}$ both equate the other player's results, they are best responses to one another. Therefore $\mathbf{p} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$ form a Nash Equilibrium.

2.3 Evolutionary Biology

This section adopts the perspective of modern evolutionary biology to view evolution as a game. Charles Darwin first articulated evolution as a competition in which success corresponds to abundant offspring in future generations. If a genetic mutation develops that gives rise to a variation beneficial to survival, this new variant will outcompete the original, and eventually replace it (perhaps over the course of many generations). Consider this so-called "survival of the fittest" as nature adopting a dominant strategy in the evolutionary competition. However, this simple notion of dominance cannot fully describe the rich interactions evident in nature. In many cases, mutation leads to a variant that neither dominates nor is dominated by the original. This new variant might have advantages in some situations and disadvantages in others. A game-theoretic approach to the competition of evolution facilitates a more nuanced view of these dynamics. It also equips us to predict genetic balances in populations.

Hawk vs. Dove

We begin this section with a famous game in evolutionary biology, Hawk vs. Dove that first introduced game theory to the study of evolutionary biology in a landmark paper of John Maynard Smith and George Price.

Example 2.11 (Hawk vs. Dove)**.** Imagine members of a certain species engage in pairwise competitions for a scarce resource. Suppose for simplicity that the species has two genetic variants, call them *hawk* and *dove*. The hawks of the species are very aggressive and will always fight for the resource. In contrast, the doves are passive and will wait around to see if the opponent gives up and goes away. Many variations of this game appear in nature. For instance, male dung beetles, Onthophagus acuminatus, come in two varieties; some have a large pair of front horns, while others have very small (or possibly even nonexistent) horns.

In other species, this type of distinction might be less physically obvious and instead might express as a predisposition toward aggressive or passive behavior.

Next assign some utilities to these outcomes to enable calculations. Assume that the resource is worth 10. If two hawks engage, they will fight, and the winner earns the resource for $+10$, and the loser scores -20 for suffering defeat. On average, the expected payoff when two hawks engage is -5 for each. When a hawk and a dove compete, the hawk takes the resource for 10 and the dove gets 0. When two doves compete, they waste time posturing, which costs each -1. One of the doves will eventually give up and the other will get the $+10$ resource, so the expected payoff for each dove will be 4. We can express this in a familiar matrix game as follows:

	Hawk	Dove
Hawk	$-5, -5$	$10, 0$
Dove	$0, 10$	$4, 4$

Suppose that the population is presently split so the probability that an individual is a hawk is p. Now consider the situation from the perspective of an individual I who might be a hawk or a dove. When I competes, the probability that I will face a hawk is p and the probability I will encounter a dove is $1 - p$. From the perspective of I, this gives the following one-player game.

		Chance	
		p Hawk	$1 - p$ Dove
I	Hawk	-5	10
	Dove	0	4

Now we can compute I's expected payoff if I is a hawk or if I is a dove. If I is a hawk, then I's expected payoff will be $p(-5) + (1-p)10 = 10 - 15p$. On the other hand, if I is a dove, the expected payoff is $p(0) + (1-p)4 = 4 - 4p$. These two payoffs will be equal when $10 - 15p = 4 - 4p$, so $p = 6/11$. When $p > 6/11$ the payoff is higher if I is a dove, while if $p < 6/11$ the payoff is higher if I is a hawk. The following figure illustrates this situation with a number line.

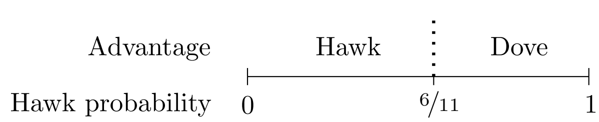


Figure 2.2: Advantage in Hawk-Dove

Imagine for a moment that the species is presently divided such that the probability an individual is a hawk is larger than 0 but smaller than $6/11$. Since our computations show that the hawks have an advantage (i.e. a hawk will receive a higher expected payoff than will a dove in our pairwise competitions), we can expect that the average hawk will be fitter and healthier than the average dove. The average hawk will therefore have more healthy offspring than the average dove. Consequently, over time the division of the species will shift so that hawks comprise a higher percentage of the population. That is, if $0 < p < 6/11$ initially, we expect to see p increase over time. Conversely, if the species is presently divided so that the probability an individual is a hawk is greater than $6/11$ but smaller than 1, then doves will have the advantage. In this case, the doves will be fitter and have more offspring. That is, if we start with $6/11 < p < 1$, then we expect over time to see p decrease.

We have now argued that the population distribution should eventually settle so that the probability an individual is a hawk is $6/11$. Notice the key to the calculation was finding a strategy for the column player that equated the row player's results. Thanks to the symmetry between the row and column players in our original matrix, these calculations give a Nash Equilibrium consisting of the strategies $\begin{bmatrix} 6/11 & 5/11 \end{bmatrix}$ and $\begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix}$.

Symmetry and Nash Equilibria

The Hawk-Dove game exhibits a type of symmetry between the row and column player common to this type of evolutionary game. Namely, both the row and column players have the same strategies and corresponding payoffs. Define a square matrix game A to be *symmetric* if the payoff matrices R, C for Rose and Colin satisfy $R^\top = C$. In other words, a matrix game A is symmetric if Rose's payoff in entry (i, j) is always the same as Colin's payoff for the entry (j, i). So, both the Hawk-Dove game and the Prisoner's Dilemma are symmetric. For a symmetric matrix game A we say that a column vector $\mathbf{d}$ is a *symmetric Nash Equilibrium* if the strategies $\mathbf{d}^\top$ for Rose and $\mathbf{d}$ for Colin are a Nash Equilibrium. A slight modification of the proof of Nash Equilibrium yields the following result.

Theorem 2.12. *Every symmetric matrix game has a symmetric Nash Equilibrium.*

Whenever pure strategies correspond to genetic variants in an evolutionary game, we should always expect the population distribution to form a Nash Equilibrium $\mathbf{d}$. If the population distribution did not to give an equilibrium, there would be a pure strategy with a better expected payoff than $\mathbf{d}$. The corresponding genetic variant would then be more successful than the average, and over time the percentage of individuals of that type would increase. So, our population distribution should form a symmetric Nash equilibrium, and the above theorem conveniently guarantees that such an equilibrium must always exist. In the Hawk-Dove game, for example, $\begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix}$ is a symmetric Nash Equilibrium.

Example 2.13 (Common Side-Blotched Lizard)**.** The males of this type of lizard have either orange, blue, or yellow coloration on their throats, and each of the three types has a different mating strategy. The orange-throated males are strongest, and their strategy is to control large amounts of territory with many rocks since the female lizards enjoy sunning themselves on these rocks. The blue-throated males are of medium strength and they generally control a smaller amount of territory. In competition, the stronger orange lizards beat blue ones. The yellow-throated males are the smallest and weakest, but they have a very similar color to that of the female. Instead of strategically controlling territory, the camouflaged yellow-throated male sneaks into the territory of another lizard and mates with females there. Since the orange lizard controls so much territory, yellow can take advantage and beat orange. However, the blue-throated male can defend his smaller territory against yellow. Altogether, then, orange lizards beat blue, blue lizards beat yellow, and yellow lizards beat orange. The following matrix game models the situation (the simple here payoffs adequately approximate the true situation for our purposes).

	Orange	Blue	Yellow
Orange	$0, 0$	$1, -1$	$-1, 1$
Blue	$-1, 1$	$0, 0$	$1, -1$
Yellow	$1, -1$	$-1, 1$	$0, 0$

This zero-sum game is identical to Rock-Paper-Scissors! These lizards are essentially playing a kind of evolutionary version of this familiar game. Thanks to the symmetry between the three strategies here, it is natural to suspect that $\mathbf{d} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ will be a symmetric Nash Equilibrium. To check this, let R be the payoff matrix for the row player and consider the vector $R\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Every pure strategy is a best response to $\mathbf{d}$, so Proposition 2.4 implies that $\mathbf{d}^\top$ is a best response to $\mathbf{d}$. So, $\mathbf{d}$ is a symmetric Nash Equilibrium.

Evolutionary Stability

We have argued that populations should settle into a symmetric Nash Equilibrium based on evolutionary considerations. From an evolutionary standpoint, however, not all equilibria are the same. Some are less stable than others, and these less stable equilibria tend not to be selected. The Nash Equilibrium in the Hawk-Dove game exhibits strong stability and the population would tend to settle on a distribution with a $6/11$ probability of hawk. Next we will consider a game with a different type of equilibrium.

Example 2.14. Recall the coordination game from Chapter 7.

	X	Y
X	$1, 1$	$0, 0$
Y	$0, 0$	$1, 1$

Let's consider this as an evolutionary game. So, our species will have two variants, X and Y, and in our pairwise competitions, Xs will fare better when opposite other Xs and similarly Ys will do better opposite other Ys. There are two pure Nash Equilibria here consisting of X, X and Y, Y and these correspond to one of the two variants dying off and the other taking over. Yet, there is also another symmetric Nash Equilibrium in this game given by the strategy $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ that results in a population distribution evenly split between the X and Y variants.

As in the Hawk Dove game, imagine that the probability an individual is an X is p. Does variant X or Y have the advantage? The perspective of a single individual yields the following one-player game.

		Chance	
		p X	$1-p$ Y
I	X	1	0
	Y	0	1

Here individual X gets an expected payoff of p, while Y gets an expected payoff of $1-p$ in this game. Hence Xs will do better when $p > 1/2$, and Ys will do better when $p < 1/2$, and both will do equally well when $p = 1/2$. We can again illustrate this advantage on a number line as in the Hawk-Dove game.

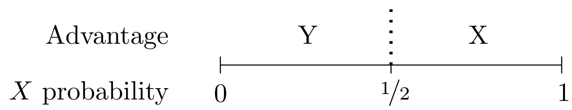


Figure 2.3: Advantage in the Coordination game

Suppose that the present population distribution has a higher percentage of Xs than Ys. In this case, the X individuals will be more successful than the Ys on average. As a result, we expect that the percentage of type X will increase even more. This will result over time in a population with a higher and higher percentage of Xs. Eventually the population will be 100% type X. If, however, the population were distributed with a higher percentage of Ys, then the Ys would be more successful than the Xs. Then the population would shift to have an even smaller percentage of Xs and, in time, the population would be 100% type Y. Although $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is indeed a symmetric Nash Equilibrium, it is not very stable in the sense that any small fluctuation is likely to lead over time to one of the pure Nash Equilibria consisting of all X or all Y.

So the Nash Equilibrium in the Hawk-Dove game is stable in a way the $\begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$ equilibrium in the Coordination game is not. How can we formalize this notion of stability in general? Maynard Smith famously answered this question with the following definition of a stable equilibrium of the type that should be favored in an evolutionary setting. In particular, an equilibrium of this type could not be invaded by an alternative strategy.

Definition 2.15. Let A be a symmetric matrix game with payoff matrix R for the row player. A strategy $\mathbf{d}$ is an *evolutionarily stable strategy* if it satisfies the following:

1. $\mathbf{d}$ is a symmetric Nash Equilibrium,

2. For every pure strategy $\mathbf{p}$ that is a best response to $\mathbf{d}$, we have $\mathbf{d}^\top R\mathbf{p} > \mathbf{p}^\top R\mathbf{p}$.

In order for a symmetric Nash Equilibrium $\mathbf{d}$ to be an evolutionarily stable strategy, any pure strategy alternative that is a best response to $\mathbf{d}$ must have the property that $\mathbf{d}$ is a better response to this alternative than it is to itself. Why are these strategies stable? Imagine the population consists of variants $X_1, \ldots, X_n$ and that the present distribution is given by the evolutionarily stable strategy $\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. Now suppose that, due to random fluctuations, the X_i population increases slightly. If X_i is not a best response to $\mathbf{d}$, then in pairwise competitions, these X_i individuals will do worse than the rest of the population on average, and this will cause the X_i population to decrease, returning toward distribution $\mathbf{d}$. Now suppose that X_i is a best response to $\mathbf{d}$ and let $\mathbf{p}$ be the pure strategy of playing X_i (i.e. so $\mathbf{p}$ is a column vector with a 1 in the i^{th} position and 0 elsewhere). Since X_i is a best response to $\mathbf{d}$ it would not be surprising to see a small increase in the percentage of individuals of type X_i. However, if the original distribution $\mathbf{d}$ satisfies $\mathbf{d}^\top R\mathbf{p} > \mathbf{p}^\top R\mathbf{p}$, then the average player gets a better payoff competing against X_i than X_i gets competing against itself. In this new population distribution with a slightly higher percentage of variant X_i, X_i players are once again at a disadvantage, and this will cause their numbers to decline, returning the population to distribution $\mathbf{d}$.

Let's verify this definition in some familiar games. We identified symmetric Nash Equilibria of $\mathbf{d} = \begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix}$ in Hawk-Dove, $\mathbf{d} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ in our Lizard Game, and $\mathbf{d} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ in the coordination game. In a symmetric Nash Equilibrium, $\mathbf{d}^\top$ must be a best response to $\mathbf{d}$, so by Proposition 2.4, any pure strategy that is called on by $\mathbf{d}^\top$ must be a best pure response to $\mathbf{d}$. So, in all three of our games, every pure strategy is a best response to $\mathbf{d}$. A little calculation shows that, in the given games, every pure strategy $\mathbf{p}$ satisfies:

$$\begin{array}{rl} \text{Hawk-Dove} & \mathbf{d}^\top R\mathbf{p} > \mathbf{p}^\top R\mathbf{p} \\ \text{Lizard} & \mathbf{d}^\top R\mathbf{p} = \mathbf{p}^\top R\mathbf{p} \\ \text{Coordination} & \mathbf{d}^\top R\mathbf{p} < \mathbf{p}^\top R\mathbf{p} \end{array}$$

The first fact implies that $\mathbf{d} = \begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix}$ is an evolutionarily stable strategy in Hawk-Dove. This corresponds to the strong stability property uncovered in our earlier discussion. In the Coordination game, the symmetric Nash Equilibrium $\mathbf{d} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is not an evolutionarily stable strategy. This is an unstable equilibrium. Finally, in the lizard game, our symmetric Nash Equilibrium is not evolutionarily stable, but it is right on the borderline. Accordingly, the male population in this species of lizard does not maintain a precise distribution between orange, blue, and yellow, instead it fluctuates a little over time around the equilibrium.

2.4 Cournot Duopoly

Game-theoretic situations appear frequently in the world of economics, where actions of one party often impact the outcome for another party. This section presents a famous model concerning two companies in competition (a duopoly).

Setup

The demand curve communicates one particularly important relationship in economic theory, that between the price of a certain good and the amount the public will purchase at a given price. For convenience, we will set our timescale at 1 year, so if (p, q) is a point on the demand curve, then at price p, the public will demand q units of the good over the course of 1 year. Barring exceptional circumstances, the higher the price the lower the amount the public will demand. Thus, the demand curve generally has a negative slope as depicted as in Figure 2.4.[1]

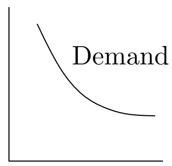


Figure 2.4: The demand curve

Going forward, we assume that the demand curve is given by the following simple linear equation,

$$P = a - bQ.$$

Here a and b are constants, P denotes the price and Q denotes the quantity of goods demanded.

Although real world demand curves are generally not linear, linear functions frequently can approximate important parts of the demand curve. Moreover, since the analysis here is robust under changes to our assumption on the demand curve, the conclusions we reach apply quite broadly. The model here is carefully chosen to be rich enough to reveal the essential features of interest, but simple enough to allow for easy computations.

Another simplifying assumption relates to production costs. Namely, we assume that producing each unit has a cost of c. So, the total cost for a company producing Q goods is given by the equation

$$\text{Cost} = cQ.$$

Monopoly

Start by considering a monopoly — just one company producing goods. This company makes only one decision; the quantity of goods to produce. Given this quantity, the demand curve sets the price. So, if the company decides to produce Q units, the price will be$P = a - bQ$, which gives the following:

$$\begin{aligned}\text{Cost} &= cQ \\ \text{Revenue} &= PQ = (a - bQ)Q.\end{aligned}$$

[1] Although it may initially seem backwards, the standard convention is to write Quantity on the horizontal axis and Price on the vertical. The motivation for this will be discovered later in this section.

Profit is equal to revenue minus cost, so

$$U = (a - bQ)Q - cQ = -bQ^2 + (a - c)Q.$$

The company naturally wants to choose Q to maximize profit. Profit is given by a quadratic function of Q whose graph would be a downward-opening parabola. Accordingly, the value of Q that will maximize profit will be the unique point where the derivative is equal to zero. So, to find the quantity which maximizes profit, set $\frac{dU}{dQ} = 0$ and solve for Q.

$$0 = \tfrac{dU}{dQ} = -2bQ + a - c$$

To maximize profit, the company should produce exactly $Q^* = \frac{a-c}{2b}$ units. With this level of production, yearly profit will be

$$\begin{aligned} U^* &= -b(Q^*)^2 + (a - c)Q^* \\ &= -\tfrac{(a-c)^2}{4b} + \tfrac{(a-c)^2}{2b} \\ &= \tfrac{(a-c)^2}{4b}. \end{aligned}$$

A monopoly poses no game-theoretic question. This is just an optimization problem! Let's add a second company, and see how the dynamics change.

Duopoly

Now maintain the same setup and introduce a second company. Assume that these two companies compete in the same marketplace and produce equivalent (i.e. interchangeable) goods, so the public will not see any significant distinction between the two brands. Call these companies Company 1 and Company 2.

As before, each company has just a single decision to make: how many goods to produce. Assume that Company 1 produces Q_1 units while Company 2 Produces Q_2 units. In this case, the total number of goods produced will be $Q_1 + Q_2$. According to the above demand curve, the price of the goods can be computed by

$$P = a - b(Q_1 + Q_2).$$

Assume that each company is interested in maximizing their profit and pause for a moment to recognize all the ingredients for a game. Company 1 and Company 2 both choose a number of goods to produce, Q_1 and Q_2, respectively. Based on these choices, each company will make a certain profit, we view as their payoff. Notice that the actions of one company affects the price and therefore profit of the other company. We have a game here! If maximum production were capped at, say, N, then we could express this as a matrix game. Each player must choose a number between 0 and N and then payoffs correspond to pairs of choices.

To analyze rational competition in this situation, we need profit functions (the payoffs in our game) for both companies. The revenue for each company is given by the price times the quantity sold, so

$$\text{Revenue for Company } i \ = PQ_i = (a - bQ_1 - bQ_2)Q_i.$$

Assume that each company has a production cost of c per unit. So, if Company i produces Q_i units, then

$$\text{Cost for Company } i \ = cQ_i$$

What about profit? The profit for each company is given by revenue minus costs. So, the profit for Company i, denoted U_i, is given by

$$U_1 = (a - bQ_1 - bQ_2)Q_1 - cQ_1 = -bQ_1^2 + (a - c - bQ_2)Q_1$$
$$U_2 = (a - bQ_1 - bQ_2)Q_2 - cQ_2 = -bQ_2^2 + (a - c - bQ_1)Q_2$$

First consider this situation from the perspective of Company 1. Suppose Company 1 knows (based on previous actions, or industry information, etc.) that Company 2 will be producing Q_2 units. How many units should they produce? In game-theoretic terminology, what is best response for Company 1 when Company 2 plays Q_2? If we treat Q_2 as fixed, then the profit function U_1 is a function of the single variable Q_1. Since U_1 is a quadratic function of Q_1 and its graph is a downward-opening parabola, the maximum profit will be achieved at the unique point where the derivative $\frac{dU_1}{dQ_1}$ is equal to zero. This gives us the equation

$$0 = \tfrac{dU_1}{dQ_1} = -2bQ_1 + a - bQ_2 - c.$$

So, the quantity Company 1 should produce to maximize profit is

$$Q_1^* = \tfrac{a - bQ_2 - c}{2b}.$$

In game-theoretic terminology, when Company 2 produces Q_2, the best response for Company 1 is to produce Q_1^* units determined by the above equation. A similar analysis concludes that when Company 1 produces Q_1 units, the best response for Company 2 is to produce Q_2^* units where

$$Q_2^* = \tfrac{a - bQ_1 - c}{2b}.$$

Suppose these two companies have a long history of competing in this marketplace. They would each know a lot about the other company's actions, and we would expect these rational players to use strategies that are best responses to one another. In other words, we would expect these two strategies to form a Nash Equilibrium. Suppose exactly this: the production quantities Q_1^* and Q_2^* form a Nash Equilibrium.

If Q_1^* is a best response to Q_2^*, then from above, we have

$$Q_1^* = \tfrac{a - bQ_2^* - c}{2b}.$$

Similarly, if Q_2^* is a best response to Q_1^*, then we have

$$Q_2^* = \tfrac{a - bQ_1^* - c}{2b}.$$

Given these two equations in two unknowns, we can solve for the unknowns. Multiply the first equation through by $4b$ and $4bQ_1^* = 2a - 2bQ_2^* - 2c$. Multiply the second by $2b$, so $2bQ_2^* = a - bQ_1^* - c$. Now substitute to find $4bQ_1^* = 2a - (a - bQ_1^* - c) - 2c$. Thus $Q_1^* = \frac{a-c}{3b}$ and, similarly, $Q_2^* = \frac{a-c}{3b}$. So, in other words, we have just computed that the only Nash Equilibrium in this game is given by the strategy pair $Q_1^* = Q_2^* = \frac{a-c}{3b}$.

Since this is the *only* Nash Equilibrium in the game, it is the anticipated behavior from two well-informed, rational companies. How does this compare with the monopoly situation? With just one company, $\frac{a-c}{2b}$ was the optimal number of goods to produce. In the duopoly case, Nash Equilibrium has each company producing $\frac{a-c}{3b}$, which means the total number of goods produced will be $\frac{2(a-c)}{3b}$. So, the total number of goods produced is greater for the duopoly. As a result, the price will be lower when there are two companies competing than when there is just one.

The profit for Company 1 at the Nash Equilibrium is

$$U_1^* = (a-c)Q_1^* - b(Q_1^* + Q_2^*)Q_1^* = \frac{(a-c)^2}{3b} - \frac{2(a-c)^2}{9b} = \frac{(a-c)^2}{9b}.$$

Likewise, Company 2 will have profits of $U_2^* = \frac{(a-c)^2}{9b}$. So, the combined profits of the two companies will be $\frac{2(a-c)^2}{9b}$, which is less than the monopoly profit of $\frac{(a-c)^2}{4b}$.

But if the two companies worked together, they could operate just like a monopoly. Each company, for instance, could produce half the optimal monopoly quantity, i.e. $\frac{a-c}{4b}$, sell at the monopoly price, and earn half the monopoly profit of $\frac{(a-c)^2}{8b}$. This is more profit than at the duopoly Nash Equilibrium! Here is the striking reality: If both companies compete rationally, each profits strictly less than if they work together and limit production.

This dynamic parallels the Prisoner's Dilemma. Rational individual competition results in an outcome less desirable than what could be achieved through cooperation. This strategic dynamic appears frequently in business competition. In fact, anti-trust laws exist to prevent companies from signing agreements to cooperate in this manner. The above example demonstrates why such agreements improve profits for companies, and how preventing this action benefits consumers with lower prices.

In closing, note that the game at the start of this chapter is equivalent to the duopoly problem in the case when $a = 100$, $b = 1$ and $c = 0$. Specifically, in the game each player chooses a quantity Q_i and the first player hopes to maximize profit of $(100 - Q_1 - Q_2)Q_1$ while the second hopes to maximize profit of $(100 - Q_1 - Q_2)Q_2$. These values are artificial, but analysis reveals the central feature. The best total can be achieved when each player chooses 25, but the only Nash Equilibrium occurs when each player chooses $33^1/_3$.

Chapter 3

Zero-Sum Matrix Games

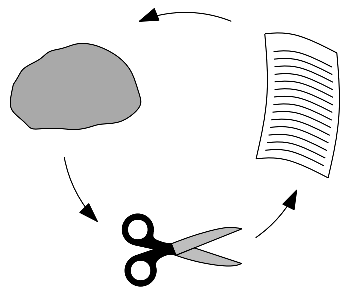


Figure 3.1: Rock-Paper-Scissors

We have been studying a very general class of games for which an outcome could yield any possible payoffs to the two players. In this chapter we will specialize to games called *zero-sum* games for which the payoffs always have the form $(a, -a)$, so the sum of the payoffs to the two players is zero. In such circumstances, whatever is gained by one player is lost by another, so the two players are in perfect competition. Zero-sum games arise in sports, politics, competitive games, and in other real-world situations as well.

For the purposes of a zero-sum game we do not need to specify both the payoff to Rose and to Colin, since either one implies the other. We will adopt the convention of taking Rose's perspective and indicating a zero-sum matrix game by giving Rose's payoff matrix. So in this setup, the *outcome* is just a single number indicating what Rose wins and what Colin loses. Accordingly, Rose wants the outcome to be as large as possible while Colin wants it to be as small as possible. For instance, the following matrix models Rock-Paper-Scissors.

Game 3.1 (Rock-Paper-Scissors)**.**

		Colin		
		Rock	Paper	Scissors
	Rock	0	-1	1
Rose	Paper	1	0	-1
	Scissors	-1	1	0

More generally we have the following formal definition.

Definition 3.2. A *Zero-Sum Matrix Game* is a game played between Rose and Colin according to the following rules. There is a fixed matrix A that is known to both players. Rose secretly chooses one of the rows and Colin secretly chooses one of the columns (neither player knows the other's choice). Then, both players reveal their choices. If Rose chose row i and Colin chose column j, then the (i, j) entry of matrix A is the *payoff* of the game. This payoff indicates how much Colin must pay Rose (of course, if the payoff is negative, then Rose will have to pay Colin).

Two Finger Morra is another example. It's very similar to Rock-Paper-Scissors, but much lesser known.

Game 3.3 (Two Finger Morra)**.** This game is played between two players similarly to Rock-Paper-Scissors. On the count of three, each player *plays* either one or two fingers with their right hand, and simultaneously *guesses* either one or two fingers with their left hand. Each player wants the number of left-hand guess fingers to match the number played by the opponent's right.

If either both players guess correctly or both guess incorrectly, then the game is a tie and nothing is exchanged. If just one player guesses correctly, that player wins from the other an amount equal to the total number of fingers played by both players.

Denote the strategy of playing i and guessing j by pigj, and the following zero-sum matrix game rerpresents Two Finger Morra.

	p1g1	p1g2	p2g1	p2g2
p1g1	0	2	−3	0
p1g2	−2	0	0	3
p2g1	3	0	0	−4
p2g2	0	−3	4	0

3.1 Von Neumann Solutions

So far in our investigation of matrix games we have introduced many important ideas, most notably the concept of a Nash Equilibrium. However, there is no general theory indicating how rational players should play in an arbitrary game. In contrast to this, a famous theorem of Von Neumann gives a robust prescription for optimal play in zero-sum matrix games. Von Neumann's Theorem can be derived as a consequence of Nash's Equilibrium Theorem, but instead we will develop it from basic principles. This approach aligns with the historical development, and has the advantage of introducing some important geometric principles that are useful for reasoning about games.

Consider a zero-sum matrix game A and suppose that Rose has decided to play the mixed strategy $\mathbf{p}$. Now the row vector $\mathbf{p}A$ contains the expected payoff for each possible column which Colin might choose. Rose wants the expected payoff to be large regardless of Colin's choice. Define the *guarantee* of $\mathbf{p}$ to be the minimum entry of the vector $\mathbf{p}A$. So, if the guarantee of $\mathbf{p}$ is 2, then the expected payoff will be at least 2 when Rose plays $\mathbf{p}$ no matter which column Colin chooses.[1] Colin wants the expected payoffs to be small, so he would like

[1]Note that if $\mathbf{p}$ has a guarantee of v, it is not necessarily true that when Rose plays $\mathbf{p}$, the payoff will always be at least v. What is true is that the *expected* payoff is at least v.

to find a mixed strategy $\mathbf{q}$ so that all entries of $A\mathbf{q}$ are small. Accordingly, we define the *guarantee* of $\mathbf{q}$ to be the maximum entry of $A\mathbf{q}$. So, if the guarantee of $\mathbf{q}$ is 3, then when Colin plays $\mathbf{q}$, the expected payoff will be at most 3 no matter which row Rose plays.

Example 3.4. Given the zero-sum matrix game $A = \begin{bmatrix} 6 & 2 & -3 \\ 3 & 3 & 0 \\ 0 & -3 & 2 \end{bmatrix}$, find the guarantees of Rose's mixed strategy $\mathbf{p} = \begin{bmatrix} 1/6 & 1/2 & 1/3 \end{bmatrix}$ and Colin's mixed strategy $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

For Rose, $\mathbf{p}A = \begin{bmatrix} 1/6 & 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} 6 & 2 & -3 \\ 3 & 3 & 0 \\ 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 5/2 & 5/6 & 1/6 \end{bmatrix}$, so $\mathbf{p}$ has a guarantee of $1/6$.

For Colin, $\begin{bmatrix} 6 & 2 & -3 \\ 3 & 3 & 0 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2 \\ -1/3 \end{bmatrix}$, so $\mathbf{q}$ has a guarantee of 2.

To work with guarantees, it will be useful to have an inequality to compare two vectors. For two vectors of the same dimension, say $\mathbf{u} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$, we define $\mathbf{u} \leq \mathbf{v}$ if $u_i \leq v_i$ for every $1 \leq i \leq n$. In other words, $\mathbf{u} \leq \mathbf{v}$ when each coordinate of $\mathbf{u}$ is less than or equal to the corresponding coordinate in $\mathbf{v}$. Below we see two vectors which can be compared with $\leq$ and two vectors which cannot be compared with $\leq$.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \end{bmatrix} \not\leq \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \end{bmatrix} \not\leq \begin{bmatrix} 1 & 0 \end{bmatrix}$$

This vector inequality provides a way to express the guarantees of mixed strategies for Rose and Colin. Namely, a mixed strategy $\mathbf{p}$ for Rose has a guarantee of at least v if and only if $\mathbf{p}A \geq \begin{bmatrix} v & \dots & v \end{bmatrix}$. Similarly, $\mathbf{q}$ has a guarantee of at most v if and only if $A\mathbf{q} \leq \begin{bmatrix} v \\ \vdots \\ v \end{bmatrix}$.[2]

Suppose Rose is playing the mixed strategy $\mathbf{p}$ in the zero-sum matrix game A. If $\mathbf{p}$ has a guarantee of r, then all entries of $\mathbf{p}A$ are at least r. So whichever pure strategy Colin chooses, the expected payoff will always be at least r. The next result shows that the same will hold true for any mixed strategy of Colin. In short, if Rose plays a mixed strategy with a guarantee of r, the expected payoff will always be at least r, no matter what Colin does.

Lemma 3.5. *Let $\mathbf{p}$ and $\mathbf{q}$ be mixed strategies for Rose and Colin in the zero-sum matrix game A.*

1. *If $\mathbf{p}$ has a guarantee of r, then $\mathbf{p}A\mathbf{q} \geq r$.*

[2]Note that this inequality for vectors appeared implicitly with domination. In a zero-sum matrix game A, Rose's strategy of row i dominates row i' if row i is $\geq$ row i'. For Colin, column j dominates column j' if column j is $\leq$ column j'.

2. If $\mathbf{q}$ *has a guarantee of* c*, then* $\mathbf{p}A\mathbf{q} \leq c$*.*

3. If $\mathbf{p}$ *and* $\mathbf{q}$ *have guarantees of* r *and* c*, then* $r \leq c$*.*

Proof. Let $\mathbf{p} = \begin{bmatrix} p_1 & \dots & p_m \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$. Now (1) and (2) follow from the inequalities

$$(\mathbf{p}A)\mathbf{q} \geq \begin{bmatrix} r & \dots & r \end{bmatrix} \mathbf{q} = r(q_1 + \ldots + q_n) = r$$
$$\mathbf{p}(A\mathbf{q}) \leq \mathbf{p} \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix} = c(p_1 + \ldots + p_m) = c$$

and the last part comes from combining the previous two. □

Rose and Colin both want to do their best in a zero-sum matrix game. Rose wants a strategy with the highest possible guarantee, while Colin wants one with he lowest possible guarantee. The above lemma says that Rose's guarantee will always be less than or equal to Colin's guarantee. But how close can these guarantees be? Von Neumann's famous Minimax Theorem answers this question. It states that, in fact, Rose's maximum guarantee is always equal to Colin's minimum guarantee. This special number is called the value of the game.

Theorem 3.6 (Von Neumann's Minimax Theorem)**.** *Every zero-sum matrix game* A *has a unique number* v*, called the* value *of* A*, which is the maximum guarantee of a mixed strategy for Rose and the minimum guarantee of a mixed strategy for Colin.*

This theorem concludes the satisfying fact that every zero-sum matrix game A has a special value v, which may be viewed as the expected payoff under rational play. Rose has a mixed strategy $\mathbf{p}$ to guarantee the expected payoff is at least v and Colin has a mixed strategy $\mathbf{q}$ to guarantee the expected payoff is at most v. This pair of strategies and resulting payoff v provide strong information about the matrix game A. Accordingly, we will call the number v together with two such strategies $\mathbf{p}$ and $\mathbf{q}$ a *Von Neumann Solution* to the game A. Note that while the value v is unique (every game has just one value) there may be more than one mixed strategy for Rose or Colin with a guarantee of v. So, in general, a zero-sum matrix game may have many Von Neumann solutions.

Why is this theorem sometimes called the minimax theorem? Colin wants to choose a mixed strategy which has the smallest guarantee. In other words, he is looking for a mixed strategy $\mathbf{q}$ which **mini**mizes the **max**imum entry of $A\mathbf{q}$. This is the "minimax" by which the theorem is frequently known. On the other side, Rose wants a mixed strategy with the highest possible guarantee. So she wants a mixed strategy $\mathbf{p}$ which **maxi**mizes the **min**imum value of $\mathbf{p}A$ - which may be called a "maximin". Von Neumann's Theorem tells us that Colin's minimax will always be equal to Rose's maximin for every zero-sum matrix game A.

Example 3.7. Consider the matrix game

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 & -1 \\ -2 & 3 & 2 & 1 & 2 \\ 0 & -1 & 4 & 0 & 3 \\ -1 & 5 & 1 & 1 & 2 \\ 3 & 0 & 4 & -1 & 4 \end{bmatrix}$$

together with the mixed strategies

$$\mathbf{p} = \begin{bmatrix} 3/8 & 0 & 0 & 7/24 & 1/3 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 11/24 \\ 1/4 \\ 0 \\ 7/24 \\ 0 \end{bmatrix}.$$

A quick calculation reveals that $\mathbf{p}A = \begin{bmatrix} 13/12 & 13/12 & 13/8 & 13/12 & 37/24 \end{bmatrix}$, so $\mathbf{p}$ has a guarantee of $13/12$. Since $A\mathbf{q} = \begin{bmatrix} 13/12 \\ 1/8 \\ -1/4 \\ 13/12 \\ 13/12 \end{bmatrix}$, then $\mathbf{q}$ also has a guarantee of $13/12$, so this is the value of the game. It follows that $13/12$ together with the mixed strategies $\mathbf{p}$ and $\mathbf{q}$ form a Von Neumann Solution for A.

The general problem of finding Von Neumann solutions is quite involved — it is a question central to the field of Linear Programming.[3] However, for small matrix games, we will see how to compute these solutions.

3.2 2×2 Games

For a general 2×2 matrix game we have already developed a method for finding Nash Equilibria, and this simplifies to give us a way of computing Von Neumann solutions. As a reminder, the key to our method was finding strategies that equate the opponent's results. Below we restate the definition of this concept in the special case of zero-sum matrix games.

Definition 3.8. For a zero-sum matrix game A, we say that a mixed strategy $\mathbf{p}$ for Rose *equates Colin's results* if all entries of $\mathbf{p}A$ are equal. Similarly, we say that a mixed strategy $\mathbf{q}$ for Colin *equates Rose's results* if all entries of $A\mathbf{q}$ are equal.

Next we show how to compute Von Neumann solutions in 2×2 games.

Proposition 3.9. *For every 2×2 zero-sum matrix game A, one of the following holds.*

[3]In fact, Von Neumann's Theorem and its generalization to LP duality helped to provide the theoretical underpinning for linear programming, a subject central to mathematical optimization and its extensive industrial applications.

1. *Iterated removal of dominated strategies reduces the matrix to a 1×1 matrix $\begin{bmatrix} v \end{bmatrix}$. The number v and the associated pure row and column strategies form a Von Neumann Solution.*

2. *Rose and Colin have mixed strategies $\mathbf{p}$ and $\mathbf{q}$ equating the opponents results. Then $\mathbf{p}A = \begin{bmatrix} v & v \end{bmatrix}$ and $A\mathbf{q} = \begin{bmatrix} v \\ v \end{bmatrix}$. The number v with $\mathbf{p}$ and $\mathbf{q}$ form a Von Neumann Solution.*

Proof. If a player has a dominanted strategy, then removing it reduces the matrix to the form $\begin{bmatrix} a \\ b \end{bmatrix}$ or $\begin{bmatrix} c & d \end{bmatrix}$. Either way, one of the two numbers in the matrix will be less than or equal to the other. Removing the corresponding dominated strategy leaves a 1×1 matrix. The entry of the original matrix selected by this process will always be the largest in its column and the smallest in its row. It follows that this value together with these pure strategies form a Von Neumann Solution.

Now assume neither player has a dominant strategy. In this case, Lemma 2.8 tells us that there exists a mixed strategy $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ for Rose that equates Colin's results, and a mixed strategy $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$ for Colin that equates Rose's results. So there must exist real numbers u and w so that $\mathbf{p}A = \begin{bmatrix} u & u \end{bmatrix}$ and $A\mathbf{q} = \begin{bmatrix} w \\ w \end{bmatrix}$. Now

$$w = pw + (1-p)w = \mathbf{p} \begin{bmatrix} w \\ w \end{bmatrix} = \mathbf{p}A\mathbf{q} = \begin{bmatrix} u & u \end{bmatrix} \mathbf{q} = qu + (1-q)u = u.$$

So $u = w$ is the value of the matrix, and (2) holds. □

The world of sports provides a familiar arena for sophisticated competition in zero-sum games. Sporting events generally feature two individuals or teams in direct opposition and such situations can always be modelled with zero-sum games. Expert players and teams practice intensely, and this focused repetition often results in highly strategic play. While sportscasters commonly deliver blanket statements like "always play to your strengths," reality is often more subtle than this. Ideally, rational players will make the easiest way to beat them as difficult as possible. A sophisticated player knows the opponent's strengths and weaknesses and assumes the opponent has reciprocal knowledge. If a player is easy to beat in some way, a strategic opponent will focus on and exploit this. As such, players want the easiest way to beat them to be as difficult as possible. In cases where each side has just two options we can use the methods developed above to see how our players should act.

Example 3.10 (Tennis)**.** Rose and Colin are playing a game of tennis. Rose has come to the net and played a somewhat shallow ball to Colin, who now has the advantage. We assume that Colin will either play a return to Rose's forehand (F) or backhand side (B) (ignoring the possibility of, say, a lob) and Rose must guess by moving to her forehand (F) or backhand side (B). If Rose guesses wrong, she will not get her racket on the ball. Rose, of course, can still win the point if Colin hits it into the net or out of bounds. The following matrix shows the probability Rose will win the point depending upon the F/B decisions of the two players in this particular scenario. Be aware that this matrix only applies to this very specific situation and not to any other part of the game.

$$
\begin{array}{ccc}
 & & \text{Colin} \\
 & & \begin{array}{cc} \text{F} & \text{B} \end{array} \\
\text{Rose} & \begin{array}{c} \text{F} \\ \text{B} \end{array} & \begin{array}{|c|c|} \hline 80\% & 10\% \\ \hline 20\% & 50\% \\ \hline \end{array}
\end{array}
$$

Suppose now that Rose works very hard and improves her backhand during the off season. When she plays Colin in the following year and encounters this particular scenario again, the probabilities have changed to the following.

$$
\begin{array}{ccc}
 & & \text{Colin} \\
 & & \begin{array}{cc} \text{F} & \text{B} \end{array} \\
\text{Rose} & \begin{array}{c} \text{F} \\ \text{B} \end{array} & \begin{array}{|c|c|} \hline 80\% & 10\% \\ \hline 20\% & 60\% \\ \hline \end{array}
\end{array}
$$

Assume that Rose and Colin know each other's game well, so Colin knows Rose's backhand is now better. These are sophisticated players! How will Colin's strategies compare in these two games? Will Rose play more or less to her improved backhand side?

Let us pause to consider strategy in this game. Momentarily suppose that when Rose plays according to her current strategy, it is easier for Colin to win by playing to her backhand than by playing to her forehand. Then Colin will generally play to Rose's backhand, and on average Rose will lose more points than she should. Rose could do better by going to her backhand more often, thus making it more difficult to beat her that way. Indeed, Rose should adopt a strategy of guessing F or B so she will be equally difficult to beat no matter what choice Colin makes. This is exactly the idea of playing to equate the opponent's results, and it is precisely what we see in upper echelons of sporting competition.

Let's look at some numbers in the tennis example. The matrices at the start of the chapter have percentages in the boxes to indicate on average who wins what percentage of points. Say the outcome is listed as 80%. This means the expected number of points for Rose will be $.8(1)+.2(-1)$ since she will win one point with probability .8 and lose one point with probability .2. For each entry in the first matrix, this gives us the following familiar type of zero-sum matrix game.

$$A = \begin{bmatrix} 3/5 & -4/5 \\ -3/5 & 0 \end{bmatrix}$$

This matrix has no dominated strategy. To solve it we will find a strategy for Rose to equate Colin's results. Since

$$\mathbf{p}A = \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ -3/5 & 0 \end{bmatrix} = \begin{bmatrix} 6/5p - 3/5 & -4/5p \end{bmatrix},$$

Colin's results will be the same when $6/5p - 3/5 = -4/5p$, or when $p = 3/10$. Therefore, Rose should guess forehand 30% of the time and backhand 70% of the time. Assuming Rose does this, the matrix product $\begin{bmatrix} 3/10 & 7/10 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ -3/5 & 0 \end{bmatrix} = \begin{bmatrix} -6/25 & -6/25 \end{bmatrix}$ tells us (by Proposition

3.9) that this game has value $-6/25$. If each player plays rationally, this game will have an expected advantage of $6/25$ of a point to Colin. What happens when Rose improves her backhand? Our sophisticated players are now playing the new game

$$A' = \begin{bmatrix} 3/5 & -4/5 \\ -3/5 & 1/5 \end{bmatrix}.$$

This game also has no dominated strategy, so to solve it we will find a strategy for Rose that equates Colin's results. The new strategy $\mathbf{p'} = \begin{bmatrix} p' & 1-p' \end{bmatrix}$ for Rose gives us

$$\mathbf{p}'A' = \begin{bmatrix} p' & 1-p' \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ -3/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 6/5p' - 3/5 & -p' + 1/5 \end{bmatrix}.$$

Equating Colin's results yields $6/5p' - 3/5 = -p' + 1/5$, so $p' = 4/11$ which is approximately 36.4%. This means Rose should now go to her forehand about 36.4% of the time and to her backhand about 63.6% of the time. Although her backhand volley has improved, Rose should actually go for her backhand shot less as a result! To find the new value of the game we compute $\mathbf{p}'A' = \begin{bmatrix} -9/55 & -9/55 \end{bmatrix}$ giving a value of $-9/55$. Overall, Rose's backhand improvement lead to a better result for her in this game, which is now only worth an expected advantage of $9/55$ of a point for Colin. The fact that Rose plays her better backhand less often may not be so surprising after all. When Rose's backhand improves, Colin will probably want to play a smaller percentage of shots there. With more shots are going toward Rose's forehand, she'll probably want to move that way to take advantage.

Professional athletes appear keenly aware of the strategic nature of the games they play, and in-depth statistical analysis generally shows that they compete in an essentially optimal manner. The key principle highlighted here is always to play such that the easiest way to beat you is as difficult as possible.

3.3 Two Dimensional Games

This section introduces a graphical technique for finding Von Neumann Solutions to $2 \times n$ and $m \times 2$ zero-sum matrix games. Let's start with an example.

Example 3.11. Consider the zero-sum matrix game

$$A = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Suppose that Rose is going to play the mixed strategy $\begin{bmatrix} p & 1-p \end{bmatrix}$. By treating p as a variable, we can investigate expected payoffs in a variety of situations. If Colin plays the first column, the expected payoff is given by the linear function $C_1(p) = p(4) + (1-p)(-1) = 5p - 1$. A similar calculation shows that when Colin plays the second column or the third column, the expected payoffs are given by $C_2(p) = -5p + 3$ or $C_3(p) = -2p + 1$. Figure 3.2 shows graphs of these linear functions using the horizontal axis for the variable p (which satisfies $0 \leq p \leq 1$) and the vertical axis for the expected payoff.

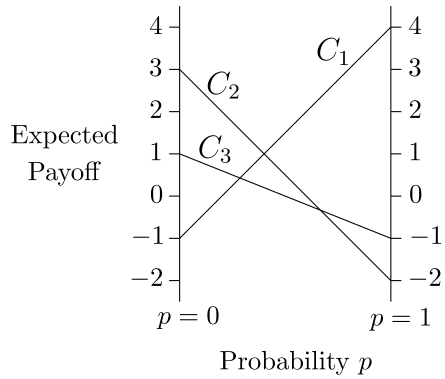


Figure 3.2: Graphing the payoffs

Combining linear functions C_1, C_2, C_3 gives the equation

$$\begin{bmatrix} p & 1-p \end{bmatrix} A = \begin{bmatrix} C_1(p) & C_2(p) & C_3(p) \end{bmatrix}.$$

For every $0 \leq p \leq 1$, the guarantee of $\begin{bmatrix} p & 1-p \end{bmatrix}$ is the minimum entry of the vector $\begin{bmatrix} p & 1-p \end{bmatrix} A$. Treating p as a variable gives us a *guarantee function* for Rose, denoted G_R, and defined by

$$G_R(p) = \min\{C_1(p), C_2(p), C_3(p)\}.$$

For every $0 \leq p \leq 1$ we have $G_R(p)$ as the guarantee for Rose's mixed strategy $\begin{bmatrix} p & 1-p \end{bmatrix}$. Since G_R is defined by the minimum of the functions C_1, C_2, C_3, it is straightforward to graph it (see Figure 3.3).

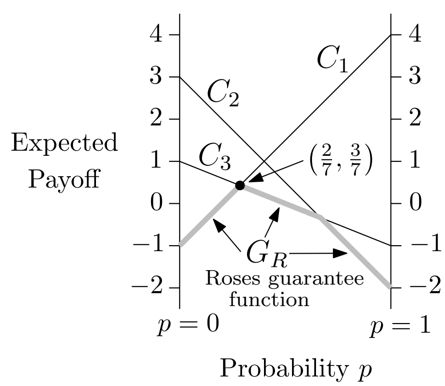


Figure 3.3: Rose's guarantee function

Rose wants to find a mixed strategy with the highest possible guarantee, so Rose wants to maximize $G_R(p)$ over all $0 \leq p \leq 1$. The graph in Figure 3.3 shows that this maximum will be achieved at the point where the lines C_1 and C_3 intersect. If $C_1(p) = C_3(p)$ then $5p - 1 = -2p + 1$ so $7p = 2$ and $p = 2/7$. At $p = 2/7$ we have $C_1(2/7) = C_3(2/7) = 3/7$. Thus the highest point on the graph of G_R has coordinates $(2/7, 3/7)$. In terms of the game, this tells us that the Rose can achieve a maximum guarantee of $3/7$ by the mixed strategy $\begin{bmatrix} 2/7 & 5/7 \end{bmatrix}$ (i.e. the mixed strategy $\begin{bmatrix} p & 1-p \end{bmatrix}$ for $p = 2/7$).

A full Von Neumann Solution also includes a mixed strategy for Colin with a guarantee of $3/7$. When Rose plays $\begin{bmatrix} 2/7 & 5/7 \end{bmatrix}$, the expected payoff is $3/7$ when Colin plays either the first or third column. The payoff is higher if Colin plays the second column. Thus a mixed strategy for Colin with a guarantee of $3/7$ will not use the second column. That is, Colin's strategy should have the form $\mathbf{q} = \begin{bmatrix} q \\ 0 \\ 1-q \end{bmatrix}$. Following our analysis of 2×2 games, we now seek a mixed strategy of this form that equates Rose's results. In particular,

$$4q - 1(1-q) = (-1)q + (1-q)$$

implies $7q = 2$, so $q = 2/7$ and $\mathbf{q} = \begin{bmatrix} 2/7 \\ 0 \\ 5/7 \end{bmatrix}$. Now calculate

$$A\mathbf{q} = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2/7 \\ 0 \\ 5/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 3/7 \end{bmatrix}.$$

Thus Colin's strategy $\mathbf{q}$ has a guarantee of $3/7$. We conclude that the value $3/7$ together with the strategies $\mathbf{p} = \begin{bmatrix} 2/7 & 5/7 \end{bmatrix}$ for Rose and $\mathbf{q} = \begin{bmatrix} 2/7 \\ 0 \\ 5/7 \end{bmatrix}$ for Colin form a Von Neumann Solution for A.

Now let's extend these ideas to more general matrices. Consider a $2 \times n$ zero-sum matrix game given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}.$$

As before, assume that Rose is playing the mixed strategy $\begin{bmatrix} p & 1-p \end{bmatrix}$ and view p as a variable with $0 \leq p \leq 1$. If Colin plays according to the pure strategy of choosing column j, then the expected payoff is given by $C_j(p) = pa_{1j} + (1-p)a_{2j}$. As in the example above, the expected payoff when Colin plays column j is an easily-graphed linear function of p. The functions $C_1, \dots, C_n$ give the convenient equation

$$\begin{bmatrix} p & 1-p \end{bmatrix} A = \begin{bmatrix} C_1(p) & C_2(p) & \dots & C_n(p) \end{bmatrix}.$$

Based on this, define Rose's *guarantee function* G_R by the rule

$$G_R(p) = \min\{C_1(p), C_2(p), \dots, C_n(p)\}.$$

Now $G_R(p)$ is the guarantee of Rose's mixed strategy $\begin{bmatrix} p & 1-p \end{bmatrix}$. As before, the graph of G_R is the minimum of the lines $C_1, \dots, C_n$.

Determine a highest point (p^*, v^*) on the graph of G_R (this may be found by computing intersection points of some of the lines $C_1, \dots, C_n$). By construction, Rose's mixed strategy $\begin{bmatrix} p^* & 1-p^* \end{bmatrix}$ has a guarantee of v^*. We will show that Colin also has a mixed strategy with a guarantee of v^*, thus giving a Von Neumann solution. If there is a column of the matrix with both entries less than or equal to v^*, then this column is a pure strategy with a guarantee of v^*, so we are done. Otherwise, the point (p^*, v^*) will satisfy $0 < p^* < 1$ and will lie at the intersection of two lines C_i and C_j, one with positive slope and the other with negative slope. Consider the 2×2 matrix A' obtained by deleting all but columns i and j from the matrix A. Since $C_i(p^*) = C_j(p^*) = v^*$, in this smaller matrix we have

$$\begin{bmatrix} p^* & 1-p^* \end{bmatrix} A' = \begin{bmatrix} v^* & v^* \end{bmatrix}.$$

So, when playing A', Rose's mixed strategy $\begin{bmatrix} p^* & 1-p^* \end{bmatrix}$ equates Colin's results. By our assumptions, one of C_i and C_j has positive slope while the other has negative slope. It follows from this that Rose does not have a dominant strategy in the matrix A'. Therefore, by Lemma 2.8 we may choose a mixed strategy $\begin{bmatrix} q^* \\ 1-q^* \end{bmatrix}$ for Colin that equates Rose's results in A'. Now Proposition 3.9 implies that $A' \begin{bmatrix} q^* \\ 1-q^* \end{bmatrix} = \begin{bmatrix} v^* \\ v^* \end{bmatrix}$. So v^* together with the mixed strategies $\begin{bmatrix} p^* & 1-p^* \end{bmatrix}$ and $\begin{bmatrix} q^* \\ 1-q^* \end{bmatrix}$ form a Von Neumann solution in the matrix A'. Now we will use this to obtain a Von Neumann solution to A. Define a mixed strategy $\mathbf{q}$ for Colin in the original matrix game A by having Colin play columns i and j with the same probabilities as in our solution to A', and having Colin play all other columns with probability 0 (i.e. we form $\mathbf{q}$ from $\begin{bmatrix} q^* \\ 1-q^* \end{bmatrix}$ by adding a 0 entry for each column we deleted in going from A to A'). Then $A\mathbf{q} = A' \begin{bmatrix} q^* \\ 1-q^* \end{bmatrix} = \begin{bmatrix} v^* \\ v^* \end{bmatrix}$, so we have found a Von Neumann solution to the original game A consisting of the value v^* and the strategies $\begin{bmatrix} p^* & 1-p^* \end{bmatrix}$ and $\mathbf{q}$. We summarize this process next.

Procedure 3.12 (Solving $2 \times n$ Dimensional Games)**.**

1. Let Rose play $\begin{bmatrix} p & 1-p \end{bmatrix}$ and graph the expected payoffs for each of Colin's pure strategies $C_1, \ldots, C_n$ (each a linear function of p).
2. Rose's guarantee function is $G_R(p) = \min\{C_1(p), \ldots, C_n(p)\}$
3. Identify a highest point (p^*, v^*) on the graph of G_R.
4. The value of the game is v^*.
5. Rose's strategy $\begin{bmatrix} p^* & 1-p^* \end{bmatrix}$ has a guarantee of v^*.
6. If Colin has a pure strategy with a guarantee of v^*, this gives a solution. Otherwise, the point (p^*, v^*) lies at the intersection of two lines C_i and C_j, one with positive slope and the other with negative slope. Obtain a strategy for Colin with a guarantee of v^* by playing these two columns i and j to equate Rose's results.

A similar process works for an $m \times 2$ zero-sum matrix game. Suppose Colin plays the mixed strategy $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ and view q as a variable. If Rose plays the pure strategy of row i, then her expected payoff will be $R_i(q)$ for a linear function R_i. Now Colin's guarantee function is given by the *maximum* of $R_1, \ldots, R_m$ (note that we previously defined Rose's guarantee function as a *minimum*). To find the solution we will determine the *lowest* point on the graph of G_C (whereas before we found the *highest* point on the graph of G_R). This produces the following procedure.

Procedure 3.13 (Solving $m \times 2$ Dimensional Games)**.**

1. Let Colin play $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ and graph the expected payoffs for each of Rose's pure strategies $R_1, \ldots, R_m$ (each a linear function of q).
2. Colin's guarantee function is $G_C(q) = \max\{R_1(q), \ldots, R_m(q)\}$.
3. Identify a lowest point (q^*, v^*) on the graph of G_C.
4. The value of the game is v^*.
5. Colin's strategy $\begin{bmatrix} q^* \\ 1-q^* \end{bmatrix}$ has a guarantee of v^*.
6. If Rose has a pure strategy with a guarantee of v^* this gives a solution. Otherwise, the point (q^*, v^*) lies at the intersection of two lines R_i and R_j, one with positive slope and the other with negative slope. Obtain a strategy for Rose with a guarantee of v^* by playing rows i and j to equate Colin's results.

3.4 Proof of the Minimax Theorem

This section presents a proof of Von Neumann's Theorem. From the many proofs for this significant result, this one was chosen for its emphasis on the essential geometric nature of the problem.

Hyperplanes

Our proof involves the geometry of n-dimensional Euclidean space $\mathbb{R}^n$. We will denote points in $\mathbb{R}^n$ using column vectors. So if $\mathbf{x} \in \mathbb{R}^n$ is the point with coordinate i equal to x_i we write $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. If $\mathbf{y} \in \mathbb{R}^n$ is given by $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, then the *dot product* of $\mathbf{x}$ and $\mathbf{y}$ is defined to be

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

Fix a particular nonzero point $\mathbf{y}$ and consider dot products of other points with $\mathbf{y}$. For every real number d, define

$$H_d = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = d\}.$$

The set H_0 consists of all $\mathbf{x} \in \mathbb{R}^n$ which have dot product with $\mathbf{y}$ equal to 0 and this forms a subspace of $\mathbb{R}^n$ of dimension $n-1$. Every other set H_d may be obtained by translating H_0 by a scalar multiple of $\mathbf{y}$. We call any set of the form H_d a *hyperplane* with *normal* $\mathbf{y}$.

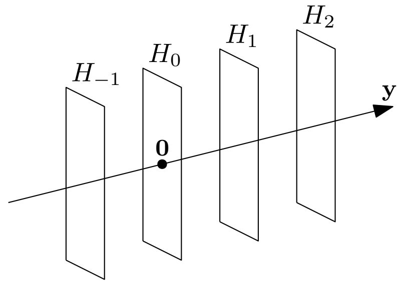


Figure 3.4: Hyperplanes

Figure 3.4 shows the points which have dot products -1, 0, 1, and 2 with $\mathbf{y}$. Observe that each of the hyperplanes H_d divides $\mathbb{R}^n$ into those points which have dot product $\geq d$ with $\mathbf{y}$ and those which have dot product $\leq d$ with $\mathbf{y}$.

Convexity

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the midpoint between $\mathbf{x}$ and $\mathbf{y}$ is given by the equation $\frac{1}{2}\mathbf{x}+\frac{1}{2}\mathbf{y}$. This is apparent since each coordinate of the midpoint is the average of the corresponding coordinates of $\mathbf{x}$ and $\mathbf{y}$. More generally, the line segment between $\mathbf{x}$ and $\mathbf{y}$, denoted by $\overline{\mathbf{xy}}$, is given by $\overline{\mathbf{xy}} = \{t\mathbf{x} + (1-t)\mathbf{y} \mid 0 \leq t \leq 1\}$.

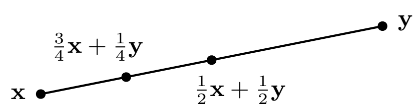


Figure 3.5: A line segment

Recall that a *linear combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is an expression of the form

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots c_k\mathbf{x}_k,$$

where $c_1, \ldots, c_k$ are any real numbers. We call this a *convex combination* if in addition $c_1, \ldots, c_k \geq 0$ and $\sum_{i=1}^k c_i = 1$. Define the *convex hull* of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ to be the set of all points which may be written as convex combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$. Note that the convex hull of $\mathbf{x}, \mathbf{y}$ is precisely $\overline{\mathbf{xy}}$. Other convex combinations appear in Figure 3.6.

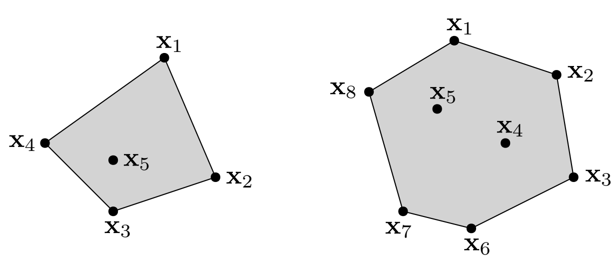


Figure 3.6: Two convex hulls in $\mathbb{R}^2$

For an arbitrary set $S \subseteq \mathbb{R}^n$, we say that S is *convex* if for every $\mathbf{x}, \mathbf{y} \in S$ the entire line segment $\overline{\mathbf{xy}}$ is contained in S. A necessary property of convex hulls is that these sets are indeed convex.

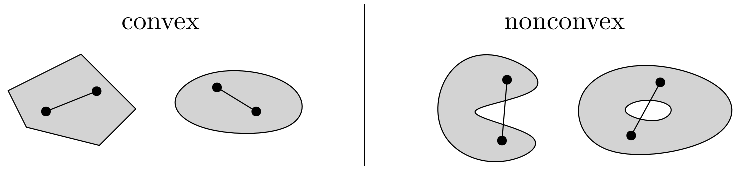


Figure 3.7: Convex and nonconvex sets

Proposition 3.14. *The convex hull of* $\mathbf{x}_1, \ldots, \mathbf{x}_k$ *is a convex set.*

Proof. Let $\mathbf{y}$ and $\mathbf{z}$ be in the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ and consider a point on the line segment $\overline{\mathbf{yz}}$ given by $t\mathbf{y} + (1-t)\mathbf{z}$ (so $0 \le t \le 1$). To complete the proof, it suffices to show that this point is also in the convex hull. By assumption, $\mathbf{y}$ and $\mathbf{z}$ are both convex combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, so we may choose $c_1, \ldots, c_k \ge 0$ summing to 1 and $d_1, \ldots, d_k \ge 0$ summing to 1, so that

$$\begin{aligned} \mathbf{y} &= c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k \text{ and} \\ \mathbf{z} &= d_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k. \end{aligned}$$

Now, the point of interest is

$$t\mathbf{y} + (1-t)\mathbf{z} = (tc_1 + (1-t)d_1)\mathbf{x}_1 + \ldots + (tc_k + (1-t)d_k)\mathbf{x}_k.$$

All of the coefficients of the $\mathbf{x}_i$ on the right in the above equation are nonnegative, and $\left(t\sum_{i=1}^k c_i\right) + \left((1-t)\sum_{i=1}^k d_i\right) = t + (1-t) = 1$ so this point lies in the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, as desired. □

In fact, the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is the unique minimal convex set which contains these points.

The Proof

We will use $\mathbf{0}$ to denote the origin and $\mathbf{1}$ to denote the point with all coordinates equal to 1. We will also call on the transpose to move between column vectors and row vectors. For example, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. For convenience, we restate the Minimax Theorem before giving the proof.

Theorem (Von Neumann)**.** *For every zero-sum matrix game, the maximum guarantee of a mixed strategy for Rose is equal to the minimum guarantee of a mixed strategy for Colin.*

Proof. Fix a zero-sum matrix game $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix}$, and let r be the largest number[4] so that Rose has a mixed strategy with a guarantee of r. It follows from Lemma 3.5 that Colin does not have a mixed strategy with a guarantee strictly smaller than r. So, to complete the

[4] That such a number r exists follows from the continuity of the function $\mathbf{p} \to \min(\mathbf{p}A)$ and the fact that the set of all mixed strategies $\mathbf{p}$ is compact.

proof, it suffices to show that Colin has a mixed strategy with a guarantee of at most r. To prove this by contradiction, now suppose that Colin does not have a mixed strategy with a guarantee of at most r.

Modify the matrix A by subtracting r from each entry. Observe that this decreases every payoff by r. After this change, the highest guarantee Rose can achieve is 0, and Colin does not have a mixed strategy with a guarantee which is ≤ 0. We will see that this leads us to a contradiction.

Define the following set

$$\begin{aligned} S &= \{A\mathbf{q} \mid \mathbf{q} \text{ is a mixed strategy for Colin}\} \\ &= \{q_1\mathbf{a}_1 + \ldots + q_n\mathbf{a_n} \mid q_1, \ldots, q_n \geq 0 \text{ and } \textstyle\sum_{i=1}^n q_i = 1\} \end{aligned}$$

Note that S is precisely the convex hull of the columns of A, so in particular, S is convex. Now define another set as follows.

$$S^+ = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} \geq \mathbf{x} \text{ for some } \mathbf{x} \in S\}.$$

If the point $\mathbf{0}$ was contained in S^+, we would have $\mathbf{0} \geq \mathbf{x}$ for some $\mathbf{x} \in S$. However then there would be a mixed strategy $\mathbf{q}$ for Colin with $A\mathbf{q} = \mathbf{x} \leq \mathbf{0}$, which contradicts the assumption that Colin does not have a mixed strategy with a guarantee of ≤ 0. Therefore $\mathbf{0}$ is not contained in S^+. Let $\mathbf{y}$ be a closest point[5] in S^+ to $\mathbf{0}$ and define the number $d = \mathbf{y} \cdot \mathbf{y} > 0$ and the hyperplane

$$H_d = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = d\}.$$

Next, we proceed with a sequence of three claims.

Claim 1. *S^+ is convex.*

Let $\mathbf{u}^+$ and $\mathbf{w}^+$ be points in S^+, let $0 \leq t \leq 1$ and consider the point $t\mathbf{u}^+ + (1-t)\mathbf{w}^+$. To prove the claim, it suffices to show that this point is in S^+. Since $\mathbf{u}^+, \mathbf{w}^+ \in S^+$, we may choose points $\mathbf{u}, \mathbf{w} \in S$ with $\mathbf{u} \leq \mathbf{u}^+$ and $\mathbf{w} \leq \mathbf{w}^+$. It follows form the convexity of S that $t\mathbf{u} + (1-t)\mathbf{w}$ is in S. Now $t\mathbf{u}^+ + (1-t)\mathbf{w}^+ \geq t\mathbf{u} + (1-t)\mathbf{w}$ so $t\mathbf{u}^+ + (1-t)\mathbf{w}^+$ is in S^+, as desired.

Claim 2. *$\mathbf{y} \cdot \mathbf{z} \geq d$ for every $\mathbf{z} \in S^+$.*

If there were a point $\mathbf{z} \in S^+$ for which $\mathbf{z} \cdot \mathbf{y} < d$, then $\mathbf{z}$ would lie on the same side of the hyperplane H_d as $\mathbf{0}$, as shown on the left in the figure below. Then, by the convexity of S^+, the line segment between $\mathbf{z}$ and $\mathbf{y}$ would also be contained in S^+. However, then $\mathbf{y}$ would not be a closest point in S^+ to $\mathbf{0}$. Thus, the claim holds, and the situation is as on the right in the figure.

Claim 3. *$\mathbf{y} \geq 0$.*

[5] The fact that there is a closest point follows from the fact that S^+ is closed.

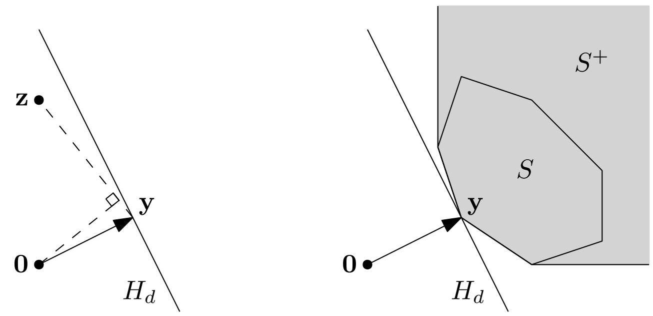


Figure 3.8: The hyperplane H_d

Let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and let $\mathbf{y}' = \mathbf{y} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. The definition of S^+ implies that $\mathbf{y}' \in S^+$ so Claim 2 gives us $d \leq \mathbf{y} \cdot \mathbf{y}' = \mathbf{y} \cdot \mathbf{y} + y_1 = d + y_1$. It follows that $y_1 \geq 0$. A similar argument for the other coordinates gives us $\mathbf{y} \geq 0$.

With this last claim in place, we can now complete the proof. Define $s = \mathbf{y} \cdot \mathbf{1}$ and note that $s > 0$. Then define $\mathbf{p} = \frac{1}{s}\mathbf{y}$. The vector $\mathbf{p}^\top$ is a mixed strategy for Rose since $\mathbf{p} \geq 0$ and the sum of its entries is $\mathbf{p} \cdot \mathbf{1} = \frac{1}{s}(\mathbf{y} \cdot \mathbf{1}) = 1$. For every point $\mathbf{z} \in S^+$, Claim 2 implies $\mathbf{p} \cdot \mathbf{z} = \frac{1}{s}\mathbf{y} \cdot \mathbf{z} \geq \frac{d}{s}$. So, in particular, every column $\mathbf{a}_i$ of A must satisfy $\mathbf{p} \cdot \mathbf{a}_i \geq \frac{d}{s}$. However, this means that $\mathbf{p}^\top$ is a mixed strategy for Rose with a guarantee of at least $\frac{d}{s}$. This contradicts our initial assumption and completes the proof. $\square$

Chapter 4

Nash's Equilibrium Theorem

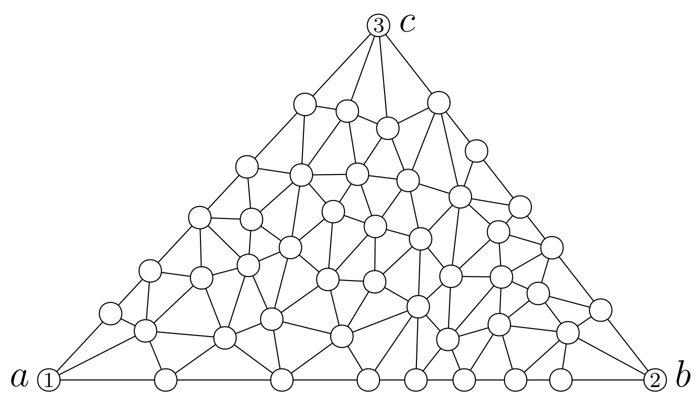


Figure 4.1: Triangle Solitaire

This chapter presents a proof of Nash's Equilibrium Theorem. The intricate proof of this deep result even involves the above game of triangle solitaire! For accessibility, the argument here focuses on the special case of 2×2 games. Each result cleanly generalizes to higher dimensions.[1]

We now know how to calculate a Nash Equilibrium in a 2×2 matrix game, and it's easy to hope that higher dimensional cases could be handled similarly, perhaps with more complicated algebraic expressions. This is simply not the case. Analogously, the familiar quadratic formula finds roots of a second degree polynomial and there are similar formulas for finding roots of polynomials of degrees three and four. However, there is no such formula for a fifth degree polynomial... the situation is too complex to admit simple algebraic solutions. It is nevertheless still possible to reason and prove things about the roots of these polynomials.[2]. Likewise, there are explicit formulae for finding Nash Equilibria in small cases, but such conveniences do not exist for higher dimensions. Hence, we take a less direct approach to

[1] These details appear explicitly in and appendix to the full book.

[2] For instance, there is an easy proof that a polynomial of degree 5 must have at least one root. This follows from the fact that a polynomial $f(x)$ of degree 5 either has $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ or has $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$. In either case, we can deduce that f has both positive and negative values, so by the Intermediate Value Theorem it must have a root.

proving the existence of a Nash Equilibrium. The following diagram outlines the major pieces of the argument.

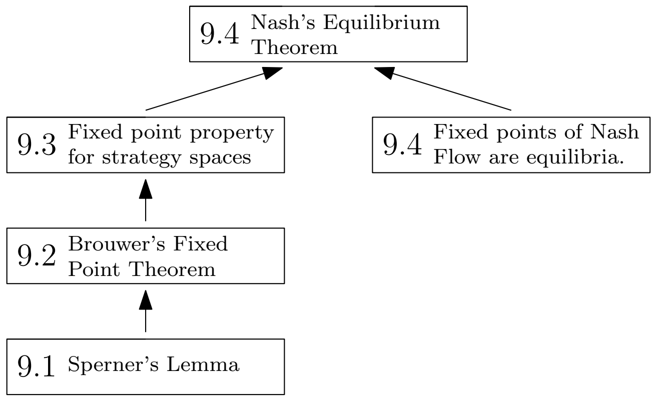


Figure 4.2: Steps in proving Nash's Theorem

We begin with the following one-player game.

Game 4.1 (Triangle Solitaire)**.** This game uses the triangle in Figure 4.1.[3] To play, write a 1, 2, or 3 in each of the empty circles according to the rule that every circle on the side between the corners labelled i and j must get either an i or a j. So for instance, every circle on the bottom of the big triangle must be filled with either a 1 or a 2, but those in the interior can be labelled 1, 2, or 3. The goal in this game is to minimize the number of little triangles with three differently-labelled corners. How well can you do?

4.1 Sperner's Lemma

This section builds to a proof a beautiful general result due to Emanuel Sperner. Despite its significance, it is traditionally called a Lemma. We begin with a one-dimensional version that involves a subdivided line segment. The two-dimensional version calls on the triangle from the start of the chapter.

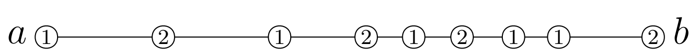


Figure 4.3: Sperner's Lemma in 1D

Lemma 4.2 (Sperner 1D)**.** *Let $\overline{ab}$ be a line segment that is subdivided into edges by adding some new vertices. Assume that each vertex is labelled according to the following rules:*

- *a is labelled 1 and b is labelled 2.*
- *Every other vertex is labelled 1 or 2.*

Then there exist an odd number of edges with endpoints of both numbers.

[3]A larger version of this game board can be found at the end of the book.

Proof. Imagine starting at a and walking along the segment to b. Starting at a vertex labelled 1 and walking to one of label 2 means switching numbers an odd number of times. Thus, there are an odd number of edges with ends of different labels. □

This one-dimensional result figures into the following proof that the game of triangle solitaire will always have at least one small triangle with all three labels.

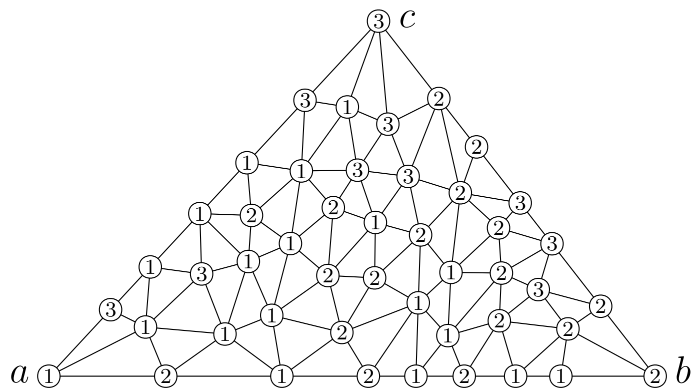


Figure 4.4: Sperner in 2D

Lemma 4.3 (Sperner 2D)**.** *Let $\triangle abc$ be subdivided into small triangles by adding new vertices and edges. Assume that each vertex is given a label according to the following rules:*

- *the labels on a, b, c are (respectively) 1,2,3.*
- *Every vertex on $\overline{ab}$ is labelled 1 or 2.*
- *Every vertex on $\overline{bc}$ is labelled 2 or 3.*
- *Every vertex on $\overline{ac}$ is labelled 1 or 3.*
- *Every vertex inside $\triangle abc$ is labelled 1, 2, or 3.*

Then there are an odd number of small triangles with vertices of all three labels.

Proof. Imagine Figure 4.3 as a floorplan for a house, so each little triangle is a room with three walls. Then add a door along every wall (edge) that has one end labelled 1 and the other labelled 2. Now consider the possibilities. A room with at least one door must have one vertex of label 1 and one of label 2. If the third vertex has label 3, then the room has just one door. Otherwise, the third vertex is labelled either 1 or 2, and, in either case, the room will have exactly two doors. No room has more than two doors, so a person walking through the house must walk only forward or backward along a pathway. Figure 4.5 highlights the pathways in this labelling.

Some rooms have no doors and some pathways form cycles — these are irrelevant. Focus on the pathways that have a beginning and an end. The first and last door on any such pathway must either be a door to the outside (along $\overline{ab}$) or a door into a room with just one door. Since each such pathway has two ends, the total number of doors to the outside plus

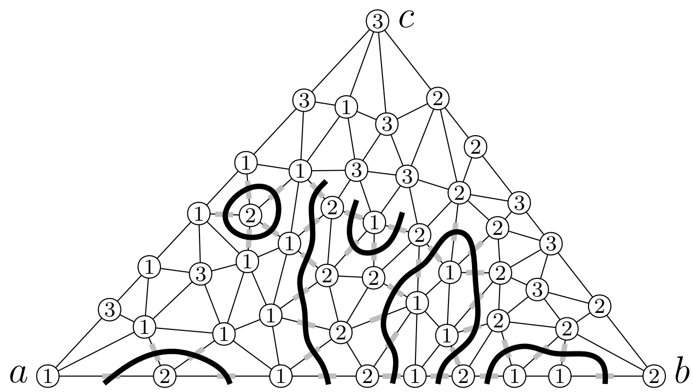


Figure 4.5: Pathways

the number of rooms with exactly one door is even. The number of doors to the outside must be odd by Sperner 1D. Therefore the number of rooms with exactly one door must also be odd. In particular, there is at least one room with just one door. This is a small triangle with vertices of all three labels! □

Higher Dimensions

To extend Sperner's Lemma to higher dimensions requires sets in $\mathbb{R}^n$ that behave like line segments in $\mathbb{R}^1$ and triangles in $\mathbb{R}^2$. The notions of hyperplane and convex hull we saw with Von Neumann's Theorem in Chapter 3 recur here. One way to describe a triangle is as the convex hull of 3 points in $\mathbb{R}^2$ which do not lie on a common line. This idea generalizes to the following definition of an n-dimensional simplex.

Definition 4.4. An *n-simplex* is the convex hull of $n+1$ points in $\mathbb{R}^n$ that do not lie on a common hyperplane.

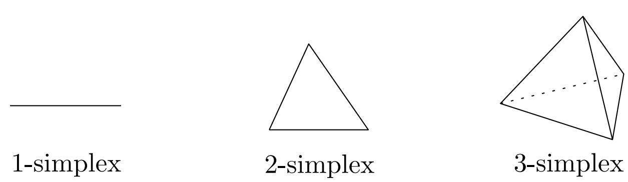


Figure 4.6: Small dimensional simplices

Observe that the 1D Sperner Lemma involved a 1-simplex that was subdivided and had vertices labelled 1 and 2. The 2D Sperner Lemma involved a 2-simplex that was subdivided with vertices labelled 1, 2, and 3. More generally, the n-dimensional Sperner's Lemma uses an n-simplex that has vertices labelled $1, 2, \ldots, n+1$.

Lemma 4.5 (Sperner)**.** *Consider a n-simplex given as the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ that has been subdivided into small simplexes. Suppose that each vertex is given a label from $1, 2, \ldots, n+1$ satisfying the following rule:*

- *If a vertex has label i, then it does not lie in the convex hull of the points $\mathbf{x}_1, \ldots \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_{n+1}$.*

Then the number of small simplexes with all $n+1$ labels is odd.

The proof of this n-dimensional result is a generalization of the proof for the 2D version.

4.2 Brouwer's Fixed Point Theorem

Some subsets of $\mathbb{R}^n$ exhibit a special property called the fixed point property. Perhaps surprisingly, Sperner's Lemma features prominently in the proof of a topological result called Brouwer's Fixed Point Theorem concerning this characteristic.

Fixed Points

This section focuses on subsets $X \subseteq \mathbb{R}^n$ and functions of the form $f : X \to X$. Define a *fixed point* of such a function f to be a point $x \in X$ for which $f(x) = x$ (i.e. it is a point that is fixed by the function).

Examples 4.6.

1. The function $f_1 : [0,1] \to [0,1]$ given by $f_1(x) = 1 - x$ has $1/2$ as a fixed point since $f_1(1/2) = 1/2$.
2. The function $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f_2(x,y) = (y+1, x-1)$ has $(1,0)$ as a fixed point since $f_2(1,0) = (1,0)$.
3. The function $f_3 : \mathbb{R} \to \mathbb{R}$ given by $f_3(x) = e^x$ has no fixed point, since there is no solution to the equation $x = e^x$.

Moving forward, it will be useful to know if *all* continuous functions on a particular set have a fixed point.

Definition 4.7. We say that a set $X \subseteq \mathbb{R}^n$ has the *fixed point property* if every continuous function $f : X \to X$ has a fixed point.

Note that this definition only concerns continuous functions. To prove that a set X does *not* have the fixed point property, simply find one continuous function from X to itself that has no fixed points.

Examples 4.8.

1. Consider a circle $C \subseteq \mathbb{R}^2$. Now, choose an angle $0 < \theta < 2\pi$ and define a function $f : C \to C$ by the rule that f rotates each point around the circle by an angle of θ. Since this is a continuous function with no fixed point, we conclude that C does not have the fixed point property.
2. Consider the set consisting of the entire real number line $\mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ is a continuous function with no fixed point. Therefore $\mathbb{R}$ does not have the fixed point property.

It is considerably more difficult to prove that a set X does have the fixed point property, since this means showing that *every* continuous function $f : X \to X$ has a fixed point. In the case when X is an interval on the real line, we have good tools to solve this problem.

Theorem 4.9 (Brouwer 1D)**.** *The closed interval* $[0, 1]$ *has the fixed point property.*

Proof. We must show that every continuous function from the interval $[0, 1]$ to itself has a fixed point. Let $f : [0, 1] \to [0, 1]$ be such a function and define the new function $g : [0, 1] \to \mathbb{R}$ by the rule $g(x) = x - f(x)$. Now g is also a continuous function and $g(0) = -f(0) \leq 0$ while $g(1) = 1 - f(1) \geq 0$. It now follows from the Intermediate Value Theorem that there exists a point c in $[0, 1]$ so that $g(c) = 0$. This point c satisfies $0 = g(c) = c - f(c)$ so it must then be a fixed point of the function f. Since this holds for every continuous function $f : [0, 1] \to [0, 1]$, we have established that $[0, 1]$ has the fixed point property, as desired. □

Next, we will prove that a triangle has the fixed point property.

Triangles

Define Δ_2 to be the solid planar triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. More formally,

$$\Delta_2 = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x + y \leq 1\}.$$

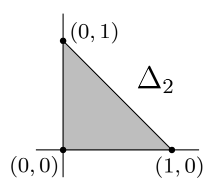


Figure 4.7: The triangle Δ_2

To prove that Δ_2 has the fixed point property, we need to consider continuous functions $f : \Delta_2 \to \Delta_2$. While it's common to visualize functions $f : \mathbb{R} \to \mathbb{R}$ using a graph, this technique is not so helpful for picturing a function $f : \Delta_2 \to \Delta_2$. Instead, draw Δ_2 and draw a collection of arrows that indicate the output of particular points under f (an arrow indicates that the initial point (x, y) is mapped to the terminal point $f(x, y)$). Since only continuous functions concern us, these arrows give a good idea how f acts on nearby points, too.

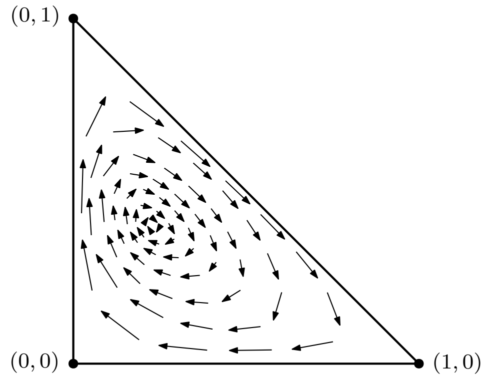


Figure 4.8: A continuous function $f : \Delta_2 \to \Delta_2$

Let (x, y) be a point in the triangle Δ_2 and suppose that $f(x, y) = (x', y')$. Define $(x', y') - (x, y) = (x' - x, y' - y)$ to be the *direction of* (x, y). Observe that in Figure 4.8, each arrow indicates the direction of its initial point.

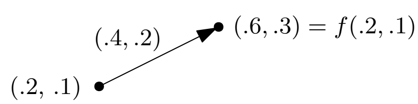


Figure 4.9: The direction of $(.2, .1)$ is $(.4, .2)$

Now for an unusual move. Instead of considering all possible directions for a point, divide the directions into three groups: West, Southeast, and Northeast. This divides the points in the triangle into three sets that give a very rough indication of where the points go when we apply the function f. Formally, assign each point in Δ_2 a label 1 (for West), 2, (for Southeast), and 3 (for Northeast) according to the following rule: If $(x', y') = f(x, y)$ then

$$(x, y) \text{ has label} \begin{cases} 1 & \text{if } x' < x \\ 2 & \text{if } x' \geq x \text{ and } y' < y \\ 3 & \text{if } x' \geq x \text{ and } y' \geq y \end{cases}$$

Figure 4.10 helps visualize these directions. In words, when the direction associated with a point has an angle of θ, the label will be a 1 if $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$, a 2 if $\frac{3\pi}{2} < \theta < 2\pi$, and a 3 if $0 \leq \theta \leq \frac{\pi}{2}$.

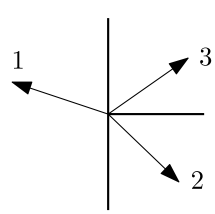


Figure 4.10: Assigning labels to directions

Consider a point (x, y) that is not a fixed point, and suppose it has a direction with angle $\frac{\pi}{4}$. By the above labeling scheme, this point will have label 3. Because f is continuous, every point near (x, y) will have a similar direction. So all points sufficiently close to (x, y) also have label 3. Next suppose that (x, y) has direction with angle $\frac{\pi}{2}$ and is thus assigned label 3. This point would be on the boundary between points labelled 1 and labelled 3, so all points sufficiently close would be labelled 1 or 3. Similarly if (x, y) has direction with angle $-\frac{\pi}{2}$ all points sufficiently close would be labelled 1 or 2, and if the direction has angle 0, all points sufficently close would be labelled 2 or 3. The only way for (x, y) to have points of all three labels arbitrarily close to it is for (x, y) to be a fixed point.

The following proof utilizes exactly this feature of the labeling. For any continuous function f, we ignore everything except the associated labeling of the points. We show that there exists a point (x, y) which has points of all three labels arbitrarily close to it, and from this we deduce that f has (x, y) as a fixed point.

Theorem 4.10 (Brouwer 2D)**.** *The triangle Δ_2 has the fixed point property.*

Proof. Let $f : \Delta_2 \to \Delta_2$ be continuous, and label the points of Δ_2 in accordance with the above description.

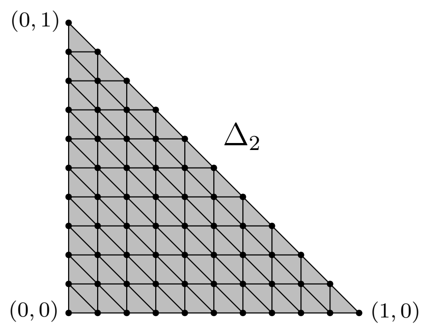


Figure 4.11: A subdivision of Δ_2

Claim. *For every $\ell \geq 0$ either there is a fixed point, or a point in Δ_2 with distance $\leq \frac{1}{2^\ell}$ to points of all three labels.*

Subdivide the triangle Δ_2 into small triangles using a fine mesh (as in Figure 4.11) so that the center of each small triangle is distance at most $\frac{1}{2^\ell}$ from each of its three vertices. Now focus on the vertices of this subdivision. If one of them is a fixed point, then there is nothing left to prove, so assume no vertex is a fixed point. Consider the $1, 2, 3$ labeling of these vertices and observe that this labeling satisfies the assumptions of Sperner 2D (with $a = (1, 0)$, $b = (0, 1)$ and $c = (0, 0)$). The point $(0, 0)$, for instance, must get a label of 3, the point $(1, 0)$ must get a label of 1, and the points on the line segment between these two all have the form $(x, 0)$ so will be labeled 1 or 3. It follows that there is a small triangle with vertices of all three labels. By construction, the center of that triangle is distance $\leq \frac{1}{2^\ell}$ from points of all three labels.

By applying the above claim for $\ell = 1, 2, 3, \ldots$ we either find a fixed point (thus completing the proof) or we generate a sequence of points in the triangle $(x_1, y_1), (x_2, y_2), \ldots$ so that (x_i, y_i) is distance $< 1/2^i$ from points of all three labels. It follows[4] that there is a particular

[4]Since Δ_2 is compact, this sequence has a convergent subsequence.

point (x^*, y^*) in the triangle that has points of all three labels arbitrarily close to it. So (x^*, y^*) is a fixed point. □

Higher Dimensions

To generalize this to higher dimensions we introduce some special n-simplexes defined as

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \ldots + x_n \leq 1\}.$$

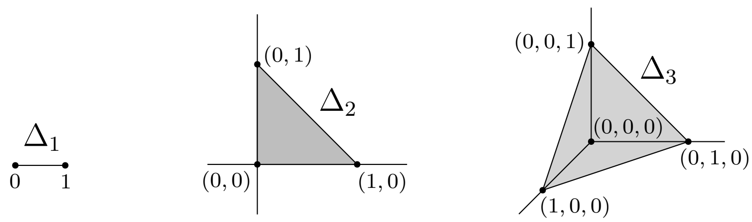


Figure 4.12: More simplicies

So Δ_1 is the line segment $[0, 1]$ and Theorem 4.9 showed it has the fixed point property. Theorem 4.10 proved that Δ_2 also has the fixed point property. The Brouwer Fixed Point Theorem asserts that this holds true in general.

Theorem 4.11 (Brouwer)**.** *The simplex Δ_n has the fixed point property for every $n \geq 1$.*

The proof of this more general result is a straightforward generalization of the proof of Theorem 4.10 the calls upon the general Sperner Lemma.

4.3 Strategy Spaces

So far in this chapter, we have no hint whatsoever of a matrix game! To rectify this, we will use certain subsets of $\mathbb{R}^n$ to describe all possible pairs of mixed strategies for Rose and Colin and then apply The Brouwer Fixed Point Theorem to prove that these subsets have the fixed point property. This constitutes the main step in the anticipated proof of Nash's Equilibrium Theorem.

2×2 matrices

Suppose that Rose and Colin are playing a 2×2 matrix game A. Recall that a mixed strategy for Rose is a vector of the form $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ where $0 \leq p \leq 1$. Although there are two entries in this vector, the single real number p entirely determines $\mathbf{p}$. Likewise, a mixed strategy for Colin is a vector $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$, but just the single number q is enough to completely describe this strategy. So we may identify Rose's mixed strategies with numbers $0 \leq p \leq 1$ and Colin's mixed strategies with numbers $0 \leq q \leq 1$.

How can we simultaneously describe a mixed strategy for Rose and one for Colin? In the case of a 2×2 matrix game as above, a pair of real numbers (p, q) with $0 \leq p \leq 1$ and

$0 \le q \le 1$ is enough. This set of ordered pairs make up what is formally called a strategy space.

Definition 4.12. The *strategy space* of a 2×2 matrix game is

$$S_{2,2} = \{(p,q) \in \mathbb{R}^2 \mid 0 \le p \le 1 \text{ and } 0 \le q \le 1\}.$$

The strategy space $S_{2,2}$ is the familiar set of points that make up a unit square. Now associate each point (p,q) in the unit square $S_{2,2}$ with the pair of mixed strategies $\begin{bmatrix} p & 1-p \end{bmatrix}$ for Rose and $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ for Colin. This valuable approach equips us with nice geometric interpretation of all possible pairs of strategies for Rose and Colin.

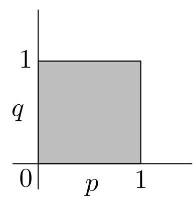


Figure 4.13: The strategy space of a 2×2 game

Strategy spaces will reappear with Nash Flow in the following section. For now, recognize that the strategy space $S_{2,2}$ is a square that encodes pairs of mixed strategies for the players. The desired result here is that the strategy space of every matrix game has the fixed point property. We will need another concept from the world of topology to achieve this.

Topological Equivalence

Thanks to The Brouwer Fixed Point Theorem, we know that the triangle Δ_2 has the fixed point property. To deduce from this the fact that square $S_{2,2}$ also has the fixed point property will involve showing that Δ_2 and $S_{2,2}$ have a certain kind of equivalence — topological equivalence — and that any topologically equivalent sets either both have the fixed point property or neither does.

Central to this key notion of equivalence is the definition of a bijection. A function $f : X \to Y$ is a bijection if every $y \in Y$ is the image of exactly one point $x \in X$. So, a bijection gives a correspondence that pairs up the points between X and Y. Assuming f is a bijection, define an inverse function $f^{-1} : Y \to X$ by the rule that $f^{-1}(y) = x$ where x is the unique point in X for which $f(x) = y$. Bijections are precisely those functions that have inverses.

Definition 4.13. We say that two sets $X, Y \subseteq \mathbb{R}^n$ are *topologically equivalent* if there is a bijection $g : X \to Y$ with the property that both g and g^{-1} are continuous.

Example 4.14. Consider the intervals $[0,1]$ and $[0,2]$ in $\mathbb{R}$. The function $g : [0,1] \to [0,2]$ given by $g(x) = 2x$ is a bijection between $[0,1]$ and $[0,2]$. Since both g and g^{-1} are continuous, then $[0,1]$ and $[0,2]$ are topologically equivalent.

Notably, the following proposition shows that whenever one of two topologically equivalent sets has the fixed point property, the other does, too. For example, Theorem 4.9 proved that the interval $[0, 1]$ has the fixed point property, so the topological equivalence from the previous example means that $[0, 2]$ also has the fixed point property.

Proposition 4.15. *If X and Y are topologically equivalent and X has the fixed point property, then Y has the fixed point property.*

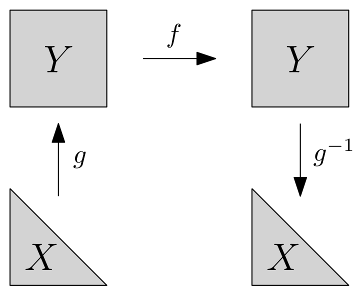


Figure 4.14: Inheriting the fixed point property

Proof. Assume that X and Y are topologically equivalent and also that X has the fixed point property. To prove that Y has the fixed point property, let $f : Y \to Y$ be an arbitrary continuous function. Since X and Y are topologically equivalent, there exists a continuous bijection $g : X \to Y$ so that g^{-1} is also continuous. Now combine the functions g, f, and g^{-1} as in Figure 4.14. More precisely, construct a new function from X to itself given by the rule

$$x \to g^{-1}(f(g(x))).$$

Since this function is continuous and X has the fixed point property, it must have a fixed point. So, we may choose a point $x \in X$ for which $x = g^{-1}(f(g(x))$. Next apply the function g to both sides of this equation to get

$$g(x) = f(g(x)).$$

Now set $y = g(x)$ and observe that $y \in Y$ and $f(y) = y$. Thus f has a fixed point and, since f was an arbitrary continuous function, we conclude that Y has the fixed point property. $\square$

With these tools in hand, we are now ready to prove that strategy space $S_{2,2}$ has the fixed point property.

Corollary 4.16. *The strategy space $S_{2,2}$ has the fixed point property.*

Proof. With Theorem 4.10 and the above proposition, we can prove that $S_{2,2}$ has the fixed point property by showing that $S_{2,2}$ is topologically equivalent with Δ_2. This follows from the continuous functions $g : \Delta_2 \to S_{2,2}$ and $g^{-1} : S_{2,2} \to \Delta_2$ defined by:

$$\begin{aligned} g(x, y) &= \left(x + \tfrac{x+y-|x-y|}{2}, y + \tfrac{x+y-|x-y|}{2}\right) \\ g^{-1}(x, y) &= \left(x - \tfrac{x+y-|x-y|}{4} +, y - \tfrac{x+y-|x-y|}{4}\right). \square \end{aligned}$$

Note that there is nothing particularly special about the triangle and square here. It is tangential to our investigations here, but, in fact, every solid polygon and every circle plus its interior are topologically equivalent to one another, and to many more shapes as well.[5]

Higher Dimensions

In a 2×2 matrix game, a mixed strategy for Rose has the form $\begin{bmatrix} p & 1-p \end{bmatrix}$ so it can be described with a single real number p. More generally, in an $m \times n$ matrix game, Rose's mixed strategies will have the form $\mathbf{p} = \begin{bmatrix} p_1 & \dots & p_m \end{bmatrix}$. Here again there is some redundancy. Since $p_1 + \ldots + p_m = 1$, the last coordinate of Rose's mixed strategy can be deduced from the earlier ones: $p_m = 1 - (p_1 + \ldots + p_{m-1})$. Therefore, each possible mixed strategy for Rose can be associated with a point $(p_1, \ldots, p_{m-1})$ where $p_1, \ldots, p_{m-1} \geq 0$ and $p_1 + \ldots + p_{m-1} \leq 1$. We will adopt this convenient description of Rose's mixed strategies. Correspondingly, a mixed strategy for Colin $\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ can be associated with a point $(q_1, \ldots, q_{n-1})$ for which $q_1, \ldots, q_{n-1} \geq 0$ and $q_1 + \ldots + q_{n-1} \leq 1$.

With this interpretation, define the *strategy space* of a general $m \times n$ matrix game A to be

$$\begin{aligned} S_{m,n} = \{(p_1, \ldots, p_{m-1}, & q_1, \ldots, q_{n-1}) \in \mathbb{R}^{m+n-2} \mid \\ & p_1, \ldots p_{m-1}, q_1, \ldots, q_{n-1} \geq 0, \\ & \textstyle\sum_{i=1}^{m-1} p_i \leq 1, \text{ and } \sum_{j=1}^{n-1} q_j \leq 1\}. \end{aligned}$$

Just like in the 2×2 case, each point in the strategy space corresponds to a pair of strategies, one for Rose and one for Colin. In the same way we used the 2D version of The Brouwer Fixed Point Theorem to prove that $S_{2,2}$ has the fixed point property, the general version of Brouwer's Theorem can be used to prove that $S_{m,n}$ has the fixed point property.

Lemma 4.17. *For every pair of positive integers m, n, the strategy space $S_{m,n}$ has the fixed point property.*

4.4 Nash Flow and the Proof

This section finally concludes our proof of Nash's Equilibrium Theorem. Most of the hard work is already done. We have proved that the strategy space has the fixed point property. What remains is to introduce a continuous function called Nash Flow on the strategy space and use the fixed point property to locate a Nash Equilibrium.

Nash Flow

Imagine that Rose and Colin are playing a game of ping-pong. Rose notices that she is somewhat more successful today when she hits to Colin's backhand rather than his forehand.

[5] More generally, any two closed convex sets in $\mathbb{R}^n$ are topologically equivalent.

It would be silly for Rose to respond by hitting every single ball to Colin's backhand — he would quickly realize and exploit her strategy. It would make more sense for Rose instead to adjust and play a slightly higher percentage of her shots to his backhand.

This situation has a very natural and important game-theoretic analogue. Suppose that Rose and Colin are playing a 2×2 matrix game, with Rose using the strategy $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ and Colin using the strategy $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$. If Rose observes that she does better playing the second row than playing the first against strategy $\mathbf{q}$, she might decide to modify her strategy $\mathbf{p}$ to play row 2 more often. As in the ping-pong game above, a subtle adaptation makes more sense an abrupt change to playing the second row 100% of the time. This adjustment to improve is the idea behind Nash Flow.

For a 2×2 matrix game A, Nash Flow is a function, denoted f_A, that maps a point (p, q) in the strategy space to another point in the strategy space $f_A(p, q) = (p', q')$. Formally, then, $f_A : S_{2,2} \to S_{2,2}$. We think of this as updating Rose and Colins' strategies $\begin{bmatrix} p & 1-p \end{bmatrix}$ and $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ to new strategies $\begin{bmatrix} p' & 1-p' \end{bmatrix}$ and $\begin{bmatrix} q' \\ 1-q' \end{bmatrix}$. Each player modifies an initial strategy to do better against the other player.

Example 4.18. Figure 4.15 depicts the Nash Flow function f_A for the matrix $A = \begin{bmatrix} (1,2) & (2,3) \\ (0,3) & (3,1) \end{bmatrix}$.[6]

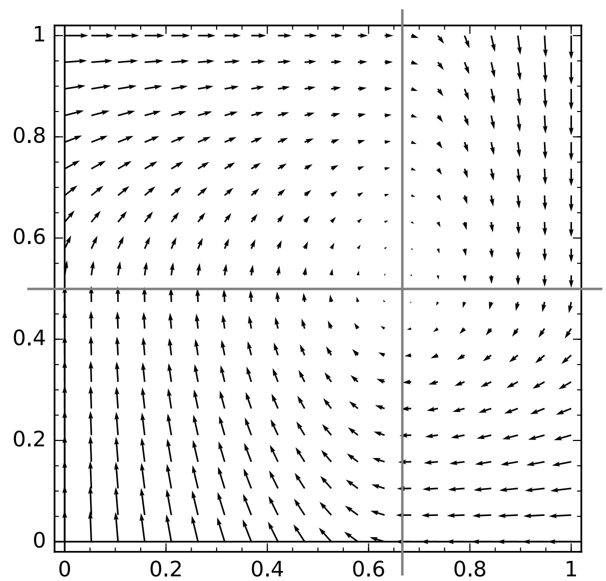


Figure 4.15: Nash Flow for a 2×2 matrix

In this figure, arrows point to the right in the region where $q > 1/2$. This corresponds to Rose increasing the probability that she plays the first row. Rose indeed gets a better payoff if she plays the first row instead of the second row when $q > 1/2$. On the other hand, in the region where $q < 1/2$, the vectors are all directed to the left, indicating Rose's adjustment decreasing the probability that she chooses the first row. When $q < 1/2$, Rose in fact gets a

[6]This plot was created using the computing program Sage.

better payoff playing the second row instead of the first. Near the point $(2/3, 1/2)$, the arrows shink to length zero. This suggests that Rose has little incentive to modify her strategy in the region. Indeed $(2/3, 1/2)$ is the unique fixed point of the Nash Flow f_A, corresponding to the fact that $p = 2/3$ and $q = 1/2$ form the unique Nash Equilibrium of A, so neither player has any incentive to change from this point.

Definition 4.19 (Nash Flow)**.** The Nash Flow for a 2×2 matrix game A is a continuous function $f_A : S_{2,2} \to S_{2,2}$. As usual, let R and C denote Rose and Colin's payoff matrices and let (p, q) be a point in the strategy space. The precise definition of the function will describe how to compute the new point $f_A(p, q) = (p', q')$.

Point (p, q) in the strategy space is associated with the strategies $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ for Rose and $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$ for Colin. Rose's (expected) payoffs for playing row 1 or 2 against Colin's strategy $\mathbf{q}$ are given by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = R\mathbf{q}.$$

So a_1 is Rose's payoff if she plays row 1 against Colin's $\mathbf{q}$ strategy, and a_2 is her payoff if she plays row 2. Similarly, if Rose plays $\mathbf{p}$, Colin's payoffs for playing either column 1 or column 2 are

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \mathbf{p}C.$$

To articulate the players' updated strategies, first define the *plus function* as follows:

$$(x)^+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Now, at last, we can define Nash Flow for the point (p, q). Let $f_A(p, q) = (p', q')$ where

$$p' = \frac{p + (a_1 - a_2)^+}{1 + |a_1 - a_2|} \qquad q' = \frac{q + (b_1 - b_2)^+}{1 + |b_1 - b_2|}.$$

Let's investigate the situation. The behavior of p' and q' are very similar, so this analysis focuses on p'. First note that the denominator in the expression for p' is positive and the numerator nonnegative, so $p' \geq 0$. Since this denominator is always at least as large as the numerator, $p' \leq 1$. Therefore $0 \leq p' \leq 1$, so p' does correspond to a strategy for Rose.

If $a_1 = a_2$, then $p' = p$ and Rose will not alter her strategy. She gets the same payoff playing row 1 or row 2 so she has no incentive to change. Next suppose Rose does better playing row 2 than row 1, so $a_2 > a_1$. In this case the formula says $p' = \frac{p}{1+t}$ where $t = |a_1 - a_2| > 0$. If $p = 0$, then Rose is already playing the pure strategy of row 2 and $p' = 0$, so she does not change. On the other hand, if $p > 0$ then $p' < p$ so Rose's new strategy has her playing the first row with lower probability and the second with higher probability. In the $a_2 < a_1$ case, Rose does better playing the first row than the second. Here, the formula simplifies to $p' = \frac{p+t}{1+t}$ where $t = a_1 - a_2 > 0$. If $p = 1$, then Rose is already playing the pure strategy of choosing row 1 and $p' = p$ so she does not change. Otherwise, $p < 1$ and $p' > p$, so Rose's new strategy has her play the first row with lower probability and the second with higher, just as desired.

In sum, this modification gives Rose a sensible response. If row 1 and row 2 give her equal payoffs, she does not change strategy. If row 1 gives Rose a better payoff than row 2, then she modifies her strategy to pay row 1 more frequently (if possible). Similarly, if row 2 gives a better payoff than row 1, then Rose alters her strategy to play row 2 more frequently (if possible). Rose's new strategy p' will be exactly the same as her original p if and only if p is a best response to q. A similar analysis for Colin results in the following key property.

Lemma 4.20. *For every 2×2 matrix A, the fixed points (p, q) of Nash flow f_A are precisely those points for which $\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$ form a Nash Equilibrium of A.*

Nash flow can be very helpful for getting a sense of the strategy space for a particular game. Let's reconsider this function for another one of the dilemmas from Chapter 7.

Example 4.21. Figure 4.16 depicts the Nash Flow associated with the Dating Dilemma game given by the matrix $\begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix}$. This game has two pure Nash Equilibria corresponding to the case when the players arrive at the same place. These outcomes correspond to the fixed points $(0,0)$ and $(1,1)$ in the Nash Flow. There is an additoinal Nash Equilibrium when Rose plays $\begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$ and Colin plays $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

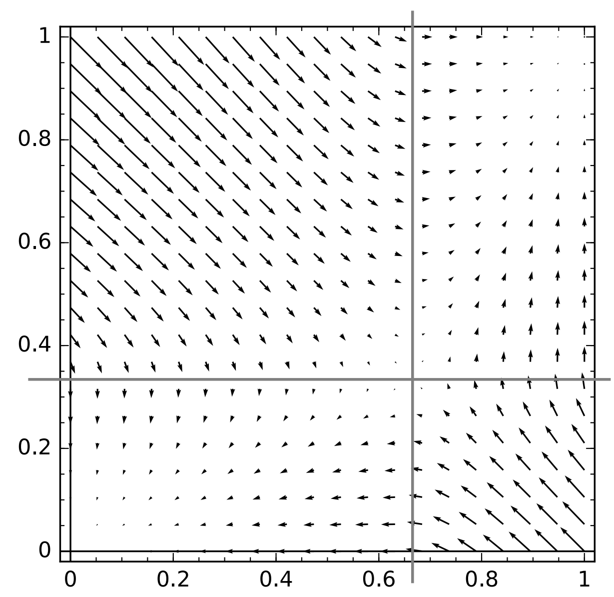


Figure 4.16: Nash Flow for the Dating Dilemma

In closing, we note the considerable value of Nash Flow for studying stability properties of Nash Equilibria (as in the discussion of evolution in Chapter 2). It's evident in the Dating Dilemma that the two pure strategy Nash Equilibria are very stable — all points in the strategy space near either of these points is directed toward it.

The Proof

We are finally ready to prove Nash's Equilibrium Theorem. This proof is, in fact, a fairly straightforward consequence of Lemma 4.16 and Lemma 4.20.

Theorem. *Every 2×2 matrix game A has a Nash Equilibrium.*

Proof. The existence of a Nash Equilibrium for the 2×2 game A follows from:

1. Nash Flow is a continuous function $f_A : S_{2,2} \to S_{2,2}$ with the property that every fixed point corresponds to a Nash Equilibrium of A (Lemma 4.20).

2. The strategy space $S_{2,2}$ has the fixed point property (Corollary 4.16), and thus f_A has a fixed point. □

Higher Dimensions

The full proof of Nash's Equilibrium Theorem in arbitrary dimensions follows from precisely the same reasoning as for 2×2 games. Just as we defined for every 2×2 matrix game A a Nash Flow $f_A : S_{2,2} \to S_{2,2}$, it is possible to define a (continuous) Nash Flow function for an arbitrary $m \times n$ matrix game. The key is the following generalization of Lemma 4.20.

Lemma 4.22. *For every $m \times n$ matrix A, the fixed points of Nash flow $f_A : S_{m,n} \to S_{m,n}$ are precisely those points which correspond to a Nash Equilibrium of A.*

Once this is in place, the general proof of the Nash's Equilibrium Theorem follows from the same reasoning as in the 2×2 case.

Theorem (Nash)**.** *Every matrix game has a Nash Equilibrium.*

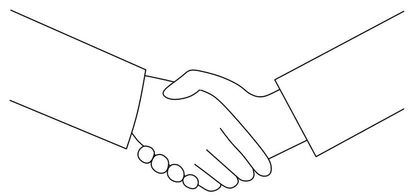
Proof. Let A be an $m \times n$ matrix game with strategy space $S_{m,n}$. The proof that A has a Nash Equilibrium follows from the two properties:

1. Nash Flow is a continuous function $f_A : S_{m,n} \to S_{m,n}$ with the property that every fixed point corresponds to a Nash Equilibrium of A. (Lemma 4.22).

2. The strategy space $S_{m,n}$ has the fixed point property (Lemma 4.17), so f_A has a fixed point. □

Chapter 5

Cooperation



So far in our analysis of games, we haven't seen any cooperation, even though many games model decision problems where experience suggests that cooperation may be beneficial. Could game theory be useful in determining how and when rational players cooperate? This chapter expands our horizons beyond the one-time play of a matrix game to explore this question. First, a change in perspective: Instead of *playing* a matrix game, imagine that two players *negotiate* about their actions prior to play. This setting gives rise to the concept of a negotiation set and establishes context for a beautiful theorem about selecting a fair outcome. In the end, the chapter returns to game play embellished by a new type of game with many stages. The well-known Folk Theorem indicates a mechanism by which cooperation may appear in such an environment.

Imagine a decision problem for Rose and Colin. Assume that each player has two choices. In combination, these will select one of four possible outcomes (denoted W, X, Y, Z).

	Colin	
Rose	W	X
	Y	Z

As usual, each player assigns a utility to indicate a value on each of these four outcomes relative to the others. Assume the following 2×2 matrix game results.

	Colin	
Rose	$4, 2$	$1, 4$
	$0, 1$	$3, 0$

If probabilistic actions are involved, outcome X may occur with probability $\frac{1}{3}$ and outcome Z with probability $\frac{2}{3}$. In this case, Rose and Colin's expected payoffs are given by

$\frac{1}{3}(1,4)+\frac{2}{3}(3,0)=(\frac{7}{3},\frac{4}{3})$. It is natural to view this possibility of having X occur with probability $\frac{1}{3}$ and Z occur with probability $\frac{2}{3}$ as a type of "outcome" in its own right and to denote it by $\frac{1}{3}X+\frac{2}{3}Z$. We will use the term *mixed outcome* to describe such situations.

How should Rose and Colin play this game without any communication? What if Rose and Colin sit down and negotiate in advance? Could they reach a more favorable mixed outcome through cooperation?

5.1 The Negotiation Set

Perhaps surprisingly, a geometric framework can assist Rose and Colin's negotiations in a matrix game. To set a baseline for the negotiations, we first investigate how Rose and Colin might expect to fare in a game with no communication.

Security Levels

Consider the game from the introduction from Rose's perspective. If she plays a mixed strategy $\mathbf{p}=\begin{bmatrix}p & 1-p\end{bmatrix}$, then we can compute her expected payoff for each possible choice Colin makes. Rose's payoff matrix is $R=\begin{bmatrix}4 & 1\\ 0 & 3\end{bmatrix}$, so when she plays mixed strategy $\mathbf{p}=\begin{bmatrix}p & 1-p\end{bmatrix}$, the vector of interest is given by

$$\mathbf{p}R=\begin{bmatrix}p & 1-p\end{bmatrix}\begin{bmatrix}4 & 1\\ 0 & 3\end{bmatrix}=\begin{bmatrix}4p & 3-2p\end{bmatrix}.$$

Thus, if Rose plays the mixed strategy $\mathbf{p}$, her expected payoff is $4p$ when Colin chooses the first column and $3-2p$ when he chooses the second. A conservative plan for Rose would be to choose a mixed strategy so that the smallest entry of $\begin{bmatrix}4p & 3-2p\end{bmatrix}$ is as large as possible. In words, this strategy gives Rose the highest guaranteed expected payoff. For this particular game, this can be achieved when $4p=3-2p$, so $p=\frac{1}{2}$. This would guarantee Rose an expected payoff of 2 no matter what Colin does. This may well look familiar — right now Rose is just playing as she would in the zero-sum matrix game R. She is operating only on the basis of R and completely ignoring Colin's payoffs. This may not be particularly strategic play for Rose, but it does at least offer her a baseline guarantee.

Definition 5.1. For an arbitrary matrix game, Rose's *security level* is the maximum expected payoff she can guarantee herself. Similarly, Colin's *security level* is the maximum expected payoff he can guarantee himself.

The above example generalizes to yield the following theorem that says how to calculate Rose's security level for an arbitrary matrix game.

Theorem 5.2. *In a matrix game with Rose's payoff matrix R, Rose's security level is the Von Neumann value of R.*

Proof. Let v together with the strategies $\mathbf{p}$ for Rose and $\mathbf{q}$ for Colin be a Von Neumann solution to the zero-sum matrix game R. If Rose plays $\mathbf{p}$, then $\mathbf{p}R\geq\begin{bmatrix}v & \dots & v\end{bmatrix}$ so Rose will be guaranteed an (expected) payoff at least v no matter what Colin does. On the other hand, if Colin plays $\mathbf{q}$ then $R\mathbf{q} \leq \begin{bmatrix} v \\ \vdots \\ v \end{bmatrix}$, so Rose will have an (expected) payoff at most v. Thus, v is Rose's security level. □

Computing Colin's security value is similar, but with a slight twist (or rather a flip). To find Colin's security level, consider his payoff matrix C. His security level will be the highest possible guarantee he can achieve using a mixed strategy $\mathbf{q}$. In other words, we seek $\mathbf{q}$ so that the minimum entry of $C\mathbf{q}$ is as large as possible. This is nearly the same as the usual zero-sum game format, except that Colin, the column player, is now looking to maximize (instead of minimize) the payoff. To adjust for this, transpose the matrix to interchange the roles of the players.

Theorem 5.3. *In a matrix game with Colin's payoff matrix C, Colin's security level is the Von Neumann value of $C^\top$.*

Proof. This is a straightforward exercise left to the reader. □

In the 2×2 example game above, Colin's payoff matrix is given by $C = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$. So $C^\top = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$. In the zero-sum matrix game $C^\top$, the second column dominates the first. Removing the dominated column leaves a smaller matrix in which the first row dominates the second. Thus $C^\top$ has a Von Neumann value of 1 and Colin has a security level of 1 in this game.

At first glance, it might seem sensible for Rose (or Colin) to play in such a way to guarantee herself (himself) the security value. But this may be a bad idea! Matrix game A might contain one strictly dominated column — a poor choice for Colin. Assuming Colin plays rationally, he never chooses this dominated column, so there is no reason for Rose to worry about him playing it. Accounting for this may permit Rose a better guarantee. In general, rational play of matrix games against rational players usually involves careful consideration of the payoffs for both players.

The Payoff Polygon

We have now determined what Rose and Colin can guarantee themselves independently. What if they communicate? Imagine that, instead of playing the game immediately, Rose and Colin first negotiate about what might be a fair or reasonable outcome. For clarity, suppose they are bound to act in accordance with the agreement resulting from their discussion. Consequently, concepts developed here apply whenever there exists some social, legal, or moral mechanism in place to prevent default on the agreement.

Let's return to the introductory game with the four outcomes W, X, Y, Z and the corresponding payoffs as shown below.

		Colin		
Rose	W	$(4,2)$	X	$(1,4)$
	Y	$(0,1)$	Z	$(3,0)$

The payoff associated with each outcome is an ordered pair (x, y), so we can visualize these payoffs as points in the plane, as shown in Figure 5.1. In this setting, Rose prefers the selected point to be as far to the right as possible, while Colin wants it to be as high up as possible.

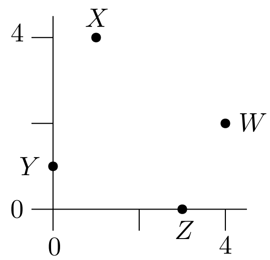


Figure 5.1: Payoffs

Consider the mixed outcome $\frac{1}{2}W + \frac{1}{2}Z$. To achieve this, it would require Rose to choose row 1 and Colin to choose column 1 $\frac{1}{2}$ of the time, and for the other $\frac{1}{2}$ of the time for Rose to choose row 2 and Colin to choose column 2. So this particular mixed outcome is one that could only be achieved by players who are cooperating in a kind of negotiated setting. If the players do coordinate to achieve the mixed outcome $\frac{1}{2}W + \frac{1}{2}Z$, then their expected payoffs will be $\frac{1}{2}(4,2) + \frac{1}{2}(3,0) = (\frac{7}{2}, 1)$. Geometrically, this is the midpoint between W and Z in Figure 5.1. More generally, the players may agree to select outcomes W, X, Y, Z with probabilities p_W, p_X, p_Y, p_Z whenever p_W, p_X, p_Y, p_Z are nonnegative numbers which sum to 1. The corresponding expected payoffs will then be given by $p_W(4,2) + p_X(1,4) + p_Y(0,1) + p_Z(3,0)$. Geometrically, this is a convex combination of the four points associated with W, X, Y, Z. In general, then, the set of all possible expected payoffs of mixed outcomes is precisely the convex hull of the points $(4,2)$, $(1,4)$, $(0,1)$, and $(3,0)$. This is the payoff polygon for the matrix game in Figure 5.2.

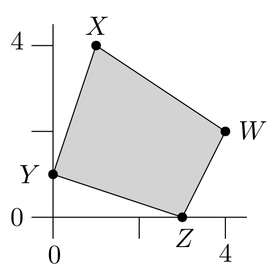


Figure 5.2: The payoff polygon

Next, generalize this to an arbitrary matrix game.

Definition 5.4. The *payoff polygon* of a matrix game A is the convex hull of the ordered pairs appearing as entries of A.

The payoff polygon provides a useful geometric viewpoint for comparing various mixed outcomes. To get a sense of these polygons, let's revisit the dilemmas from Chapter 1 and consider the associated payoff polygons.

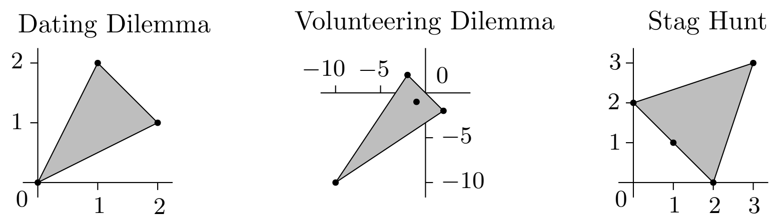


Figure 5.3: Payoff polygons for some Dilemmas

In terms of modelling real world situations, the mixed outcomes here may or may not make sense. In the Dating Dilemma, for example, if Rose and Colin will go on many dates, it is certainly reasonable for them to go to a ball game (B) half the time and to a film (F) half the time. This gives the mixed outcome $\frac{1}{2}B + \frac{1}{2}F$, which would correspond to the point $\frac{1}{2}(2,1) + \frac{1}{2}(1,2) = (\frac{3}{2}, \frac{3}{2})$ in the payoff polygon. There are other circumstances where the game in question will not be repeated. Perhaps Rose and Colin are planning a wedding. Rose wants to get married in France, while Colin wants to get married in Hawaii. This decision game could generate a similar payoff polygon, but the situation is different. Likely, Rose and Colin will not repeat this same game. They still may flip a fair coin to make the decision and this is a perfectly reasonable process if they both agree. However, it is certainly possible that Rose or Colin might be unwilling to let a coin flip decide this important question. In that case, they have a dilemma indeed!

The Negotiation Set

The payoff polygon depicts all possible payoffs associated with the mixed outcomes. It is a little inconvenient to have to consider all of these points for the purposes of negotiation. Is there a sensible way to focus on just some reasonable points in the payoff polygon?

At the start of this section, we considered playing a matrix game and we defined the security levels for Rose and Colin. Recall that the players' security levels are the highest possible expected payoff they can guarantee themselves. If our players instead negotiate, it would thus be irrational for either player to agree to a mixed outcome that has an expected payoff less than the security level.

Optimality offers another consideration. It would be irrational for Rose and Colin to agree on a mixed outcome with expected payoff (x, y) if there were another point (x', y') in the payoff polygon with $(x, y) < (x', y')$ (i.e. $x < x'$ and $y < y'$). Why would they agree to (x, y) if another point is strictly better for both players? More generally, if there is a point (x'', y'') with $(x, y) \leq (x'', y'')$ (so each player does at least as well at this new point) and $(x, y) \neq (x'', y')$ (so at least one player does strictly better) then the players should favor the point (x'', y'') over (x, y). Based simply on optimality, then, negotiations should ignore points in the payoff polygon (x, y) if it is possible to increase one or the other coordinate (or both) and stay in the polygon. Taking all of this into account generates the following definition.

Definition 5.5. The *negotiation set* is the set of all points (x, y) in the payoff polygon satisfying the following properties:

1. x is at least Rose's security level and y is at least Colin's security level.

2. There is no point (x', y') in the payoff polygon with $(x, y) \leq (x', y')$ and $(x, y) \neq (x', y')$.

Von Neumann and Morgenstern introduced the negotiation set to focus attention on the reasonable negotiated outcomes. This well-accepted notion serves as a helpful guide in negotiations. For the earlier 2×2 example, we already computed Rose's security level to be 2 and Colin's as 1, so the negotiation set is as follows.

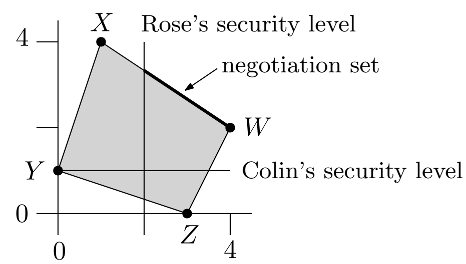


Figure 5.4: The negotiation set

Now let us return to dilemmas from Chapter 7 to consider the associated payoff polygons and negotiation sets as shown in Figure 5.5.

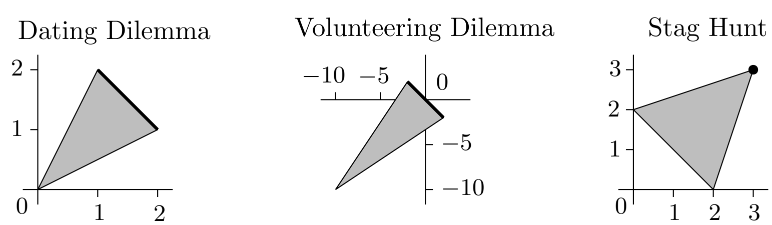


Figure 5.5: Negotiation sets for some dilemmas

In the Stag Hunt, the negotiation set consists just of a single point. Von Neumann and Morgenstern would argue in this case that this point is the only reasonable outcome for the players. In other games such as the Dating Dilemma and the Volunteering Dilemma, the negotiation set is larger, and serves just to focus attention on the sensible outcomes for a negotiation.

5.2 Nash Arbitration

A negotiation set, as developed above, contains all reasonable possibilities for a negotiated outcome. Our goal in this section is to prove a beautiful theorem, which selects a particular point from the negotiation set that is arguably a "fair" outcome. First, we abstract further from the setting of games to an even more general framework for negotiations.

In the previous section, we considered a matrix game A and associated it with a payoff polygon, security levels, and a negotiation set. Now define the *status quo* point associated with A to be the point (x_0, y_0) at the intersection of the lines corresponding to Rose and Colin's security levels. As suggested by the terminology, view the status quo point as a starting point in the negotiation. Note that the status quo point indicates the security levels for both Rose and Colin. Given a payoff polygon and its status quo point, we can quickly determine the associated security levels, and thus the negotiation set.

General Negotiations

In fact, these concepts of payoff polygon, status quo point, and negotiation set apply even in places where there is no game! Indeed, many actual negotiations that are not based directly on games can be modelled using these ideas, and this can be a useful construct for sharpening our understanding of such situations. Imagine, for example, a dispute at a company between labor and management. Labor would like to see salaries increase by \$500 per year, while Management would prefer they decrease by \$500 per year. Labor would like to increase pension payments by \$100 per month, while Management would prefer to decrease them by \$100 per month. Finally, Labor would like to increase the afternoon break time by 20 minutes, while management would like to decrease it by 20 minutes. (It is not necessary for each such possibility to be added or subtracted by the same amount).

Assume that labor and management have linear utility scales for each of these three possibilities, where the value of the present arrangement is zero and the utilities associated with each possible increase or decrease are shown in Table 5.2. Labor, for instance, would value an increase in salary, denoted S^+, at 4, whereas management would value this outcome at -6. On the other hand, the corresponding decrease in salary, denoted S^-, is valued at -4 by labor and at 6 by management.

Payoffs (labor, management)

	Salary (S)	Pension (P)	Break time (B)
Increase (+)	$4, -6$	$8, -3$	$2, -1$
Decrease (−)	$-4, 6$	$-8, 3$	$-2, 1$

More generally, imagine all of these three quantities moving either up or down, and assume the utility that labor and management associate with an outcome is given by the sum of the three quantities. For example, an outcome of $S^+P^-B^+$ gives labor a payoff of $4 - 8 + 2 = -2$ and management a payoff of $-6 + 3 - 1 = -4$.

Just as with matrix games, we can plot the utilities for labor and management associated with these outcomes. The convex hull of these points form the payoff polygon, and we will regard the origin as the status quo point, since the utility for both labor and management of the present situation is 0. Using this status quo point, we can then determine the negotiation set. All of this appears in Figure 5.6.

Consider the point corresponding to $\frac{3}{8}S^-P^+B^+ + \frac{5}{8}S^-P^+B^-$ (i.e. the point on the line segment between $S^-P^+B^+$ and $S^-P^+B^-$ that is $\frac{3}{8}$ of the way from $S^-P^+B^+$ to $S^-P^+B^-$). Using the interpretation from the previous section, we could think of this point as corresponding to an outcome where $\frac{3}{8}$ of the time we choose $S^-P^+B^-$ and the other $\frac{5}{8}$ of the

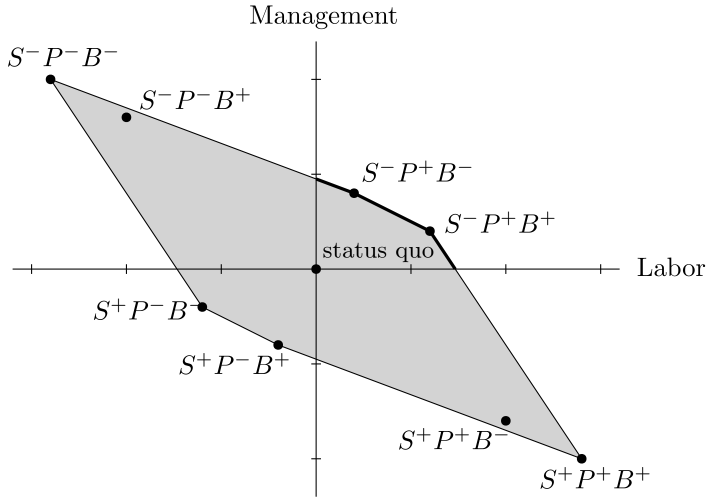


Figure 5.6: The payoff polygon for our labor-management dispute

time we choose $S^-P^+B^+$. This would mean decreasing salaries by \$500 per year, increasing pensions by \$100 per month, and $\frac{3}{8}$ of the time decreasing the break by 20 minutes, and the other $\frac{5}{8}$ of the time increasing the break by 20 minutes.

However, break time is an inherently continuous quantity. So instead of choosing this strange sounding combination, we could just increase the break time by 5 minutes (yielding the same average since $\frac{3}{8}(-20) + \frac{5}{8}(20) = 5$). Indeed, all salary, pension, and break time quantities are inherently continuous. As a result, we can associate each point in our polygon with a particular change in salary, pension, and break time. The payoff polygon thus produces a detailed encoding of the utilities the two parties associate with any possible adjustment to salaries, pensions, and break time within the given bounds.

Arbitration

Von Neumann and Morgenstern focused on the negotiation set as a set of reasonable outcomes. Could there be a way of choosing a particular point in this negotiation set that is somehow "fair"? This brings us to the concept of an arbitration scheme.

Formally speaking, we'd like to find a rule that, given a convex polygon P and a status quo point (x_0, y_0), will select a *solution point* (x, y) from the polygon. Such a rule we define to be an *arbitration scheme*. Note that we will always assume that the polygon contains at least one point (x, y) with $x \geq x_0$ and $y \geq y_0$ — otherwise there would be no negotiation set.

Figure 5.7: An arbitration scheme

Next we introduce a particular arbitration scheme.

Definition 5.6 (Nash Arbitration)**.** If there is a point (x, y) in the polygon with $x > x_0$ and $y > y_0$, then choose such a point (x, y) for which $(x-x_0)(y-y_0)$ is maximum. Otherwise, the negotiation set has size 1 and the solution point (x, y) is the unique point in the negotiation set.

Before considering any questions of fairness, return to the labor-management negotiation. What point in the payoff polygon does Nash Arbitration select? The optimization required to find the solution to Nash Arbitration is fairly straightforward to compute. In the above example, the status quo point is $(0, 0)$, so the function to maximize is xy (more generally it is $(x - x_0)(y - y_0)$ for status quo point (x_0, y_0)). The maximum will be one of the points in the negotiation set, so it will lie on one of three line segments. To find the maximum over all three, consider each one individually, and take the best. Start with the middle line segment. This segment is given by the equation $y = -\frac{1}{2}x + 5$ with the bounds $2 \le x \le 6$. So substituting for y, we are looking to maximize the function $f(x) = x(-\frac{1}{2}x+5)$ for $2 \le x \le 6$. The maximum will either be an endpoint (i.e. $x = 2$ or $x = 6$) or an interior point $2 < x < 6$ for which $f'(x) = 0$. Checking these points reveals that the maximum is attained when $x = 5$, and in fact this turns out to be the solution point to Nash Arbitration. This corresponds to decreasing salaries by \$500 per year, increasing pensions by \$100 per month, and increasing break time by 10 minutes per day.

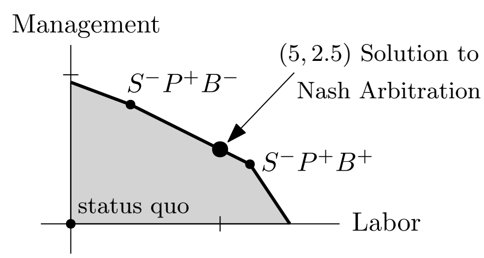


Figure 5.8: Nash Arbitration for our labor-management dispute

Nash Arbitration is a procedure that will always select a point from the payoff polygon, so it is indeed a valid arbitration scheme (this is the only requirement to be an arbitration scheme). But is the solution point determined by this scheme really fair? How can we even make sense of this question from a rigorous mathematical perspective? Nash provided a particularly attractive answer to these questions. His answer involves the following four axioms, always satisfied by his arbitration scheme.

Definition 5.7 (Nash's Axioms)**.**

1. Rationality. The solution point is in the negotiation set.

2. Linear Invariance. If either Rose or Colin's utilities are transformed by a linear function with positive slope, the solution point is transformed by applying this same function to the player's payoff.

3. Symmetry. If the payoff polygon is symmetric about the line of slope 1 through (x_0, y_0), then the solution point (x, y) is on this line.

4. Independence of Irrelevant Alternatives. Suppose (x, y) is the solution point when the arbitration scheme is applied to polygon P and status quo point (x_0, y_0). Let Q be another polygon completely contained in P also containing (x, y). In this case, (x, y) is also the solution point when the arbitration scheme is applied to Q with status quo point (x_0, y_0).

Von Neumann and Morgenstern suggested the first axiom here when they introduced the concept of negotiation set, and it is quite sensible and well accepted. The second axiom is also quite reasonable. For instance, if one player doubled his utilities, we would expect the new solution point to double in this coordinate. Similarly, if one player were to add a fixed value of t to each of her utilities, this should shift the solution point by adding t to the appropriate coordinate. The third axiom also exhibits a natural fairness property. In a game like the Dating Dilemma or the Volunteering Dilemma where the positions of the players are symmetric, neither player should be favoured, so the solution point should lie on the line of slope 1 through (x_0, y_0). The last axiom also seems quite sensible. After all, if (x, y) is the best solution for the polygon P and status quo point (x_0, y_0) and we modify the situation by shrinking the polygon in such a way that we still keep the point (x, y), we should expect the solution point for this new polygon (still using the status quo point (x_0, y_0)) to be the same.

If we do accept these axioms as fair properties, then we should focus our attention on arbitration schemes which satisfy them. Shockingly, Nash proved that his arbitration scheme is the only one that can satisfy all four of these axioms! So, if you accept these axioms as fair, you must also accept his arbitration scheme as fair!

Theorem 5.8 (Nash)**.** *The only arbitration scheme satisfying all four of Nash's axioms is Nash arbitration.*

Proof. Denote Nash arbitration by $\mathcal{N}$ and let $\mathcal{A}$ be another arbitration scheme which satisfies all of Nash's axioms. To prove the theorem, it suffices to show that, for an arbitrary polygon P with status quo point (x_0, y_0), the solution point selected by $\mathcal{N}$ and by $\mathcal{A}$ are the same.

Begin by modifying the polygon P. Subtract x_0 from each of Rose's payoffs and y_0 from each of Colin's to shift the status quo point to $(0, 0)$. If $\mathcal{N}$ and $\mathcal{A}$ select the same solution point for this new polygon, then axiom 2 implies that they also select the same solution point from the original.

If there is no point (x, y) in the payoff polygon with $x > 0$ and $y > 0$, the negotiation set has size one, and the first axiom implies that this point is selected by both $\mathcal{N}$ and $\mathcal{A}$. Assume, then, that there is a point in P with $x > 0$ and $y > 0$. Let (x^*, y^*) be the point in P with $x^* > 0$ and $y^* > 0$ that maximizes x^*y^*. Now modify P by dividing all x-coordinates by x^* and all y-coordinates by y^*. Again, by the second axiom, it suffices to show that for this new polygon, the solution point selected by $\mathcal{N}$ and $\mathcal{A}$ is the same. However, now we know that for this new polygon $\mathcal{N}$ will select $(1, 1)$.

We claim that the entire polygon P now lies on or below the line given by the equation $x + y = 2$. Suppose (for a contradiction) that there is a point (x_1, y_1) in P for which $x_1 + y_1 > 2$. Now consider the line segment between $(1, 1)$ and (x_1, y_1), and zoom in on the region of the plane near the point $(1, 1)$ as shown in Figure 5.9.

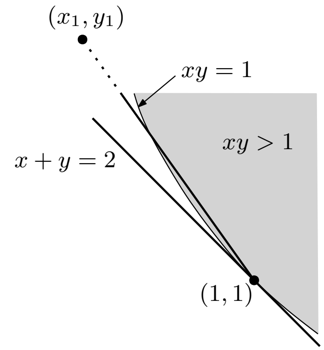


Figure 5.9: The line segment $(1, 1)$ to (x_1, y_1) enters the region $xy > 1$

Here we see a branch of the hyperbola given by the equation $xy = 1$ passing through $(1, 1)$. At this point, the line tangent to the hyperbola is the line with equation $x+y = 2$. All points that lie above this branch of our hyperbola satisfy $xy > 1$ and this region is shaded in the figure. Now, the line segment between $(1, 1)$ and (x_1, y_1) must enter the region $xy > 1$ and it follows that the line segment between $(1, 1)$ and (x_1, y_1) contains a point (x, y) with $xy > 1$. By the convexity of P, this entire line segment is in P, so P contains a point (x, y) with $xy > 1$. This is a contradiction, so polygon P must lie on or below the line given by $x + y = 2$ as claimed.

Now use the fact that polygon P lies on or below the line $x + y = 2$ to choose a certain triangle that contains P. That is, choose a triangle T so that one vertex of T lies on the line $x = y$, the opposite side lies on the line $x + y = 2$ in such a way that T is symmetric about the line $x = y$ and T contains P.

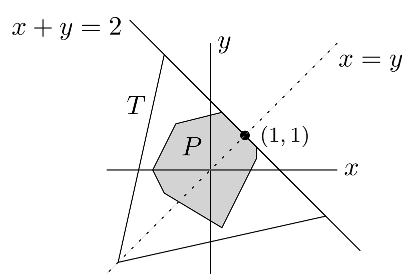


Figure 5.10: A triangle T which contains P

Axiom 3 implies that when arbitration scheme $\mathcal{A}$ is applied the triangle T with status quo point $(0, 0)$ it must select the point $(1, 1)$. Now axiom 4 implies that $\mathcal{A}$ must also select this point when applied to P with status quo point $(0, 0)$. This completes the proof since this is the same point selected by $\mathcal{N}$. $\square$

In light of Nash's Theorem, accepting his four axioms as fair means also accepting his arbitration scheme as fair. If, in the example above, labor and management agree that Nash's axioms are fair, they must also accept the Nash Arbitration output as a fair solution. However, not everyone agrees with Nash's axioms! More specifically, some theorists consider the fourth axiom to be somewhat questionable and, consequently, there is no universal agreement on what the fair outcome of this type of arbitration should be. Kalai-Smorodinsky

Arbitration, for example, uses a different axiom in place of Nash's fourth and it has been shown that there is a unique arbitration scheme satisfying these axioms.

5.3 Repeated Games and The Folk Theorem

So far in considering cooperation, we have relied on an external agent to force the players to honor the solution they agreed to in negotiations. Applying Nash Arbitration to the Prisoner's Dilemma, for example, yields a solution point that requires both players to cooperate. But if nothing binds the players actually to cooperate when they play, rationality dictates that they should defect. Is there any setting in which rational players might cooperate without an external force?

Reducing a decision problem to a matrix game removes the possibility of future interaction — a potentially significant factor. While a matrix game is concerned only immediate payoffs, real interactions are rarely final and this sometimes motivates individuals to make different choices. This section enriches the types of games to account for the possibility of repeated play. In this more complicated setting, cooperation may be rational even when it does not give the best immediate return.

Since the forthcoming games are extremely complex in terms of possible strategies, we will not make any attempt to give general solutions or even guidelines how to play. We will instead focus on simply demonstrating that there exist pairs of strategies that cooperate in equilibrium. The main theorem from this section, known as the Folk Theorem, pushes this phenomena to its extreme. This famous theorem says that, in some sense, any plausible payoffs can be achieved by a pair of strategies in equilibrium. The Folk Theorem applies broadly in the social sciences as it provides a way in which cooperation may emerge without any external agent.

Repeated Games

Concepts from this section apply in very broad settings, but we begin with the 2×2 matrix game PD that is a variant of the Prisoner's Dilemma.

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The matrix game PD.

While the numbers in this matrix game are different from those in the standard Prisoner's Dilemma, this is strategically just the same. Pure strategies for each player are still cooperate (C) and defect (D), and, as before, Defect is strictly dominant for both players. However (as in the usual Prisoner's Dilemma) when both players defect, the payoffs of $(1, 1)$ are worse than the $(2, 2)$ with cooperation.

We previously discussed the possibility of playing a matrix game over and over again (especially in the context of probabilistic strategies), and now we consider repeated play in a very different manner. Namely, we are going to introduce a new type of game that itself has many stages of play. Here is a first example.

Game 5.9 (Probabilistic Repeated Prisoner's Dilemma)**.** Begin by playing one round of the above game of Prisoner's Dilemma and recording the scores. Then each player flips a coin, and if both flip tails the game ends. Otherwise, play another round of the game (again recording the scores) and then toss coins to see if the game continues. Keep playing until one of the coin tosses results in a pair of tails. The goal (as always) is to maximize total payoff without considering the other player's payoff.

At first blush it might seem that this game should be essentially similar to the one-time matrix game PD, but it's not. This is a far more complicated realm. The key difference in this repeated game is that the choices of each player in later rounds are permitted to depend on what the other player did in earlier rounds. As a result, this repeated game becomes far more strategically complex than just a one-time play of PD. Unlike the one-time play of PD — with just a single pure Nash equilibrium (where both players defect) — we will prove that the probabilistic repeated version has a rich variety of pure Nash equilibria (many of which feature cooperation). With this example in hand, consider the general repeated matrix game central to this section.

Definition 5.10. For a matrix game A and a number $0 < \delta < 1$, define the game $\text{Repeat}(A, \delta)$ as follows. On each round, Rose and Colin play the matrix game A. Then they toss a biased coin to see if the game continues. With probability δ the coin comes up heads, and the game continues to another round. With probability $1 - \delta$ the coin comes up tails and the game ends. The payoff for each player is the sum of that player's payoffs over all of the rounds. Note that the particular game we introduced at the start of this section is equivalent to $\text{Repeat}(PD, \frac{3}{4})$.

There is another interpretation of the game $\text{Repeat}(A, \delta)$ that appears in economics. Instead of flipping a coin and possibly ending the game, imagine that the players are going to play the game A infinitely many times, but the payoff for each player will be determined by the payoff for the first round plus δ times the payoff on the second round, plus δ^2 times the payoff on the third round, and so on. To see that these two interpretations are essentially equivalent, consider a player who gets a payoff of 2 in each round. This new interpretation gives a total payoff of $2 + 2\delta + 2\delta^2 + \ldots$. In the original probabilistically repeated interpretation, our player will get 2 on the first round, with expected payoff for the second round of 2δ (i.e. the payoff of 2 times the probability the game continues to the second round), and the expected payoff for the third round will be $2\delta^2$, and so on. This new interpretation is known as "discounting the future" and is based on the simple economic property that one would rather receive a dollar today than receive a dollar tomorrow. In a business-related repeated-play scenario, then, the players should value a payoff of \$$n$ in an early round more than a payoff of \$$n$ in a later round.

Strategies and Equilibria in Repeated Games

What about the notion of strategy in a repeated matrix game? Generally, a (pure) strategy in a game (eg. a combinatorial game, game tree, or matrix game) is a plan that tells a player what to do each time there is a decision to make. Since a repeated matrix game has many rounds, a strategy must tell a player what to do on each one. This strategy may now also take into account what the other player has done on earlier rounds. This brings us to the following formalization.

Definition 5.11. A *pure strategy* in the game Repeat(A, δ) is a rule that indicates what choice to make on the k^{th} round, depending on what has happened in rounds $1, \ldots, k-1$.

All of the following are pure strategies in the game Repeat(PD, δ).

Strategy	Rule
Always Defect	Defect on every round.
Always Cooperate	Cooperate on every round.
Grim Trigger	Cooperate every round until the other player defects. Then defect on all future rounds.
Silly Prime	Cooperate unless the total number of times the other player has defected is a prime.

Of course, rational play dictates that each player should defect in a one-time game of Prisoner's Dilemma and a pair of defect strategies give the only possible Nash equilibrium. What happens in repeated games? With the below definition of Nash equilibrium extended to the setting of repeated games, we will look to find cooperative equilibria in this new setting.

If S and T are pure strategies in Repeat(A, δ), we say that S is a *best response* to T if S gives a highest possible payoff over all possible strategies played against T. We say that the pair S, T form a *pure Nash equilibrium* if S is a best response to T and T is a best response to S. In order to investigate equilibria and best responses, the below formula for the sum of a geometric series is frequently useful.

Proposition 5.12. $t + t\delta + t\delta^2 + \ldots = \frac{t}{1-\delta}$ *for every* $0 < \delta < 1$.

Proof. $(1-\delta)(t + t\delta + t\delta^2 + \ldots) = t + (t\delta - t\delta) + (t\delta^2 - t\delta^2) + \ldots = t$.[1] □

So, whenever a player receives a payoff of t on each round in Repeat(A, δ), her total payoff is $\frac{t}{1-\delta}$. With this, it is straightforward to find the payoffs when the strategy Always Cooperate and Always Defect plays against either itself or the other.

Strategies			Payoffs
Always Cooperate	vs.	Always Cooperate	$\left(\frac{2}{1-\delta}, \frac{2}{1-\delta}\right)$
Always Defect	vs.	Always Defect	$\left(\frac{1}{1-\delta}, \frac{1}{1-\delta}\right)$
Always Defect	vs.	Always Cooperate	$\left(\frac{3}{1-\delta}, 0\right)$

[1] This proof uses the fact that the sequence $t, t\delta, t\delta^2, \ldots$ is absolutely convergent.

This table demonstrates that Always Cooperate is not a best response either to itself or to Always Defect. So, Always Cooperate does not form a pure Nash Equilibrium with either itself or Always Defect. On the other hand, it is straightforward to verify that Always Defect is a best response to itself. Indeed, if one player were going to defect every round no matter what, then the other player would get the highest payoff by also defecting on every round. So, two strategies of Always Defect do form a pure Nash equilibrium. In this repeated game, the strategies Always Defect and Always Cooperate act much like the strategies defect and cooperate in the one-round game.

Grim Trigger

The strategy Grim Trigger effectively has two phases. In the first phase, Grim Trigger is cooperative and plays C each time. If the other player ever fails to cooperate, it turns into a retaliatory phase and plays D every time. Suppose that a player plays against Grim Trigger. What are the possible payoffs? If the player cooperates on each round, then Grim Trigger will stay in its cooperative phase and will also cooperate each round. Thus, the payoff will be $\frac{2}{1-\delta}$. Now suppose the player chooses a strategy that cooperates for the first k rounds and then defects on round $k+1$. The Grim Trigger strategy will then be cooperative for rounds $1, \ldots, k+1$ and then turn retaliatory and defect from round $k+2$ onward. In light of this, there is no point to cooperating on round $k+2$ or afterward. In other words, the best payoff against Grim Trigger is either that obtained by cooperating every round, or by cooperating for k rounds and then defecting afterward. The following calculation shows the payoff for this latter strategy.

$$\begin{aligned}
\text{Payoff} &= 2 + 2\delta + \ldots + 2\delta^{k-1} + 3\delta^k + \delta^{k+1} + \delta^{k+2} + \ldots \\
&= \left(2 + 2\delta + 2\delta^2 + \ldots\right) + \delta^k - \left(\delta^{k+1} + \delta^{k+2} + \ldots\right) \\
&= \tfrac{2}{1-\delta} + \delta^k - \tfrac{\delta^{k+1}}{1-\delta} \\
&= \tfrac{2}{1-\delta} + \delta^k \left(1 - \tfrac{\delta}{1-\delta}\right)
\end{aligned}$$

When $\delta > \frac{1}{2}$ the quantity $1 - \frac{\delta}{1-\delta}$ is negative and a player is best off cooperating every round. On the other hand, when $\delta < \frac{1}{2}$ this quantity is positive, and a player earns the best payoff when δ^k is as large as possible. Since $0 < \delta < 1$, this is achieved at $k = 0$. It follows that when $\delta < \frac{1}{2}$, always defecting results in the best payoff against Grim Trigger. In the boundary case when $\delta = \frac{1}{2}$, then $1 - \frac{\delta}{1-\delta} = 0$ and the above calculation shows that cooperating every time, or cooperating for k rounds and defecting thereafter (for every k) all yield the same (best possible) payoff.

What happens if two copies of Grim Trigger play against one another? By definition, each of these strategies will cooperate each round, so both players will get a payoff of $\frac{2}{1-\delta}$. In the case when $\delta \geq \frac{1}{2}$, cooperating every round gives the highest possible payoff vs. Grim Trigger as shown above. So, in this case, Grim Trigger is a best response to Grim Trigger. Finally, a Pure Nash equilibrium consisting of cooperating strategies!

Let us pause to recount the key features of Grim Trigger that brought about this equilibrium. In a way, Grim Trigger insists that the other player adopt the plan of cooperating on each round. Someone playing against Grim Trigger who deviates from this plan on round k

can get a higher payoff on round k, but on all future rounds, $k+1, k+2, \ldots$, Grim Trigger will punish the player entering its retaliatory phase, and then the player gets poor payoffs. So long as $\delta > \frac{1}{2}$, future earnings from cooperating with Grim Trigger exceed the one-time payoff from defecting, so a player is better off cooperating.

The Folk Theorem

The version of the Folk Theorem presented here is an elementary but instructive one. In fact, there is not just one Folk Theorem, but rather a body of related results known by this name. These theorems all assert (under various assumptions) that any "reasonable" payoffs can result from two strategies that form a pure Nash equilibrium in a repeated game. In particular, these theorems demonstrate a way in which cooperation may emerge in equilibrium strategies (without any appeal to an external agent) for a wide range of repeated games.

The proof below is essentially a sophisticated generalization of the key principle behind the Grim Trigger equilibrium above. This theorem requires one extra bit of terminology. We need a modified notion of the payoff in a repeated matrix game that facilitates meaningful comparison of payoffs in the games Repeat(A, δ) and Repeat(A, δ') when $\delta \neq \delta'$. Define the *normalized payoff* of Repeat(A, δ) to be the (usual) payoff multiplied by $1-\delta$. Note that a player who earns 2 in each round of the game Repeat(A, δ) will then get a normalized payoff of $(1-\delta)\frac{2}{1-\delta} = 2$, independent of the value of δ.

Theorem 5.13 (Folk)**.** *Let A be a matrix game with status quo point (x_0, y_0) and let (x, y) be a point in the payoff polygon with $x > x_0$ and $y > y_0$. Then for all δ sufficiently close to* 1*,* Repeat(A, δ) *has a pair strategies with normalized payoffs (x, y) that form a pure Nash equilibrium.*

Proof. We will construct a pair of coordinated probabilistic strategies that form a pure Nash equilibrium and have (x, y) as the expected payoffs for the players (it is possible to remove this randomization, but we omit this detail). The constructed strategies, $\mathcal{R}$ for Rose and $\mathcal{C}$ for Colin, will have two phases: a cooperative phase and a retaliatory phase. Just like Grim Trigger, $\mathcal{R}$ and $\mathcal{C}$ begin in a cooperative phase, and remain there until the other player deviates from a certain predetermined course. Once one player deviates the other will move to the retaliatory phase and play to punish in every future round.

Since (x, y) is a point in the payoff polygon, we can express (x, y) as a convex combination of entries of A. Therefore, there exist nonnegative numbers $p_1, \ldots, p_\ell$ that satisfy $p_1 + \ldots + p_\ell = 1$ together with a selection of rows $r_1, \ldots, r_\ell$ and columns $c_1, \ldots, c_\ell$ that satisfy the following equation (here $a_{i,j}$ denotes the (i, j) entry of A).

$$(x, y) = \sum_{i=1}^{\ell} p_i a_{r_i, c_i}.$$

Since the numbers $p_1, \ldots, p_\ell$ are nonnegative and $p_1 + \ldots + p_\ell = 1$, these are probabilities we can use to define the cooperative phase of the strategies. On each round of play while $\mathcal{R}$ and $\mathcal{C}$ are in the cooperative phase, the players choose a random index $1 \leq i \leq \ell$ according to the rule that i is selected with probability p_i. If the index i has been selected, then the predetermined course will be for Rose to play row r_i and for Colin to play the column c_i. Assuming the players follows this course, the above equation tells us that the expected payoff on each round will be exactly (x, y). It follows that the expected normalized payoffs for the play of Repeat(A, δ) will also be (x, y).

Both the strategies $\mathcal{R}$ for Rose and $\mathcal{C}$ for Colin will stay in the cooperative phase so long as the other player follows the above predetermined course. However, if Colin were to deviate from this course, Rose's strategy $\mathcal{R}$ would go into retaliatory phase. In this phase Rose plays according to a mixed strategy $\mathbf{p}$, chosen so that Colin's expected payoff on each round will be at most his security value y_0. Similarly, if Rose were to deviate from the course, Colin's strategy $\mathcal{C}$ would go into retaliatory phase in which Colin plays according to a mixed strategy $\mathbf{q}$ to limit Rose's expected payoff on each round to her security value x_0.

At this point, we have constructed a strategy $\mathcal{R}$ for Rose and a strategy $\mathcal{C}$ for Colin. When both players adopt these strategies, both will stay in the cooperative phase and the resulting normalized payoffs will be (x, y). It remains to prove that these strategies form a pure Nash equilibrium when δ is sufficiently close to 1. So, to complete the proof, we need to show that for δ sufficiently close to 1, $\mathcal{C}$ is a best response to $\mathcal{R}$ and vice versa.

Assume that Colin plays according to $\mathcal{C}$ and that Rose plays according to $\mathcal{R}$ up until round k, when she deviates. We will prove that when δ is high enough, Rose gets a worse payoff than if she continued to follow the strategy $\mathcal{R}$. On round k, Rose will surprise Colin and she may get a very high payoff, but she cannot get a higher payoff than M, which we define to be the maximum value in Rose's payoff matrix. For round $k + 1$ and beyond, C will be in the retaliatory phase, so on each such round Rose will get an expected payoff of at most x_0. We can compute Rose's expected payoffs for rounds k and beyond as follows

$$\delta^{k-1}M + \delta^k x_0 + \delta^{k+1}x_0 + \ldots = \delta^{k-1}\left(M + \frac{\delta x_0}{1-\delta}\right).$$

On the other hand, if Rose were to continue cooperating, her expected payoffs from rounds k and higher would be

$$\delta^{k-1}x + \delta^k x + \ldots = \delta^{k-1}\left(\frac{x}{1-\delta}\right).$$

Continuing to follow $\mathcal{R}$ gives Rose a better payoff whenever $\frac{x-\delta x_0}{1-\delta} > M$. Since $x > x_0$, the limit as δ approaches 1 from below of $\frac{x-\delta x_0}{1-\delta}$ is ∞. So, if $0 < \delta < 1$ is sufficiently close to 1, Rose's payoff will be better when she continues to follow $\mathcal{R}$ than when she deviates from it. In other words, for all sufficiently high values of δ, the strategy $\mathcal{R}$ is a best response to $\mathcal{C}$. A similar argument for $\mathcal{C}$ shows that these strategies form a pure Nash equilibrium, as desired. □

Axelrod's Olympiad

The Folk Theorem demonstrates the existence of cooperative equilibria in a wide variety of repeated games. However, the strategies, like the Grim Trigger, constructed to prove this theorem would be rather unreasonable in most circumstances. For instance, if one abandoned any friendship as soon as a friend commits even a small annoyance, very soon no friends would remain – a sad circumstance, indeed! Furthermore, the mere existence of cooperating strategies in equilibrium doesn't really indicate that cooperation is a good idea—only that it may be rational as a response in certain settings. Is there a way to argue that cooperation really is advantageous in an appropriate setting?

In response to this question, Robert Axelrod devised a tournament where players compete in games of repeated Prisoner's Dilemma. In his competition, the competitors are strategies for playing repeated Prisoner's Dilemma, denoted $\mathcal{S}_1, \ldots, \mathcal{S}_k$. These strategies are viewed as genetic variants of a species that exhibits different strategies in pairwise interactions. The first generation is evenly divided among the different variants $\mathcal{S}_1, \ldots, \mathcal{S}_k$. Now the expected payoff is computed for each strategy $\mathcal{S}_i$ competing against a randomly chosen opponent. Strategies that did well in the first round make up a higher percentage of the population in the second round and strategies that did worse make up a smaller percentage (as one would expect in a true evolutionary competition). This competition continues for many stages until the population distribution has stabilized. The strategy making up the highest percentage of the final population wins.

Axelrod invited a number of social scientists to submit computer programs designed to play against one another in a repeated game of Prisoner's Dilemma. Each program chose whether to cooperate or defect each round based on what happened during the previous rounds. So, in other words, each program executed a strategy $\mathcal{S}_i$ for this repeated Prisoner's Dilemma game. Axelrod had these programs compete in an olympiad structured as the evolutionary competition described above. The champion was the following delightfully simple submission by Anatol Rapoport.

Strategy 5.14 (Tit-for-Tat)**.** Cooperate on the first round. On each future round, do what the other player did on the last round.

After publicizing his findings, Axelrod ran his olympiad again. The second competition featured some strategies specifically designed to beat Tit-for-Tat, but Tit-for-Tat won yet again! The strategies coded to beat Tit-for-Tat tended to do poorly when squared off against one another. Tit-for-Tat has many nice properties, but its key feature is that it encourages cooperation. It will retaliate if the other player defects, so as not to be a sucker. However, after retaliating, it will be forgiving in the sense that it will cooperate again.

Tit-for-Tat should not be viewed as *the* best strategy for repeated play of Prisoner's Dilemma. Even in Axelrod's evolutionary competitions, many other strategies performed nearly as well. Nonetheless, Axelrod's Olympiads and numerous subsequent related competitions all reach one very robust conclusion: In evolutionary play of repeated Prisoner's Dilemma (and similar games), the strategies that emerge successful are highly cooperative. Although decision making is considerably more nuanced than any of our models, Axelrod's competition still does capture some of its essence, and already here we can see the great value of cooperation.

Chapter 6

N-Player Games

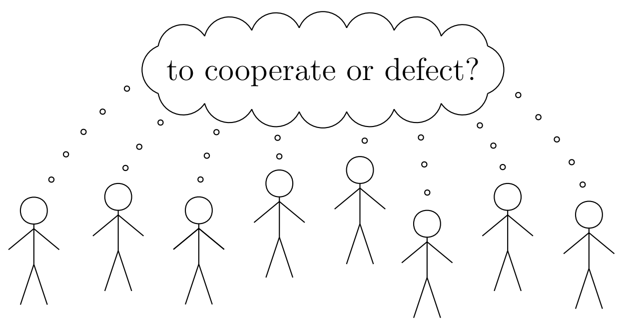


Figure 6.1: Tragedy of the Commons

The previous six chapters introduced the classical theory of two-player games, a powerful framework for representing decision problems between two parties. Rich theoretical ideas like Nash's Equilibrium Theorem and Nash Arbitration help identify rational play and even suggest fair negotiated outcomes. But what if there are more than two players? Welcome to the world of n-player games, involving circumstances that feature many individuals making decisions, all of which may affect the outcome for everyone. The tools of classical n-player game theory expressively model the dynamics of a collection of rational players. Many ideas from the context of 2-player games — notably, Nash's Equilibrium Theorem — still apply in this more general setting, but we will also encounter some new complexities. Let's begin with another classic.

Game 6.1 (Tragedy of the Commons)**.** This is a game played between 100 farmers who share a field for grazing cattle. Each farmer can decide to either put one or two cows in the field. Each farmer gets +1 for each cow she has on the field. However, the field only sustainably supports 100 cows, and each cow beyond that depletes field's nutrients. This costs everyone. Every farmer gets $-1/50$ for each extra cow.

Say that a farmer who puts one cow in the field is cooperating (C) and a farmer who puts two cows in the field is defecting (D). To analyze this, consider the perspective of farmer i, and suppose that k of the other farmers have chosen to defect. The 2×1 matrix below depicts the payoff for each possible choice of farmer i.

		k other D's
farmer i	C	$1 - k(1/50)$
	D	$2 - (k+1)(1/50)$

Farmer i achieves maximum payoff by defecting and putting two cows in the field. This holds true for every possible value of k, so farmer i gets the highest payoff by defecting no matter what the other farmers do. Thus (D) is the rational play for every farmer. Still, if everyone defects, every farmer will get a payoff of $2 - 100/50 = 0$, obviously worse than the payoff of 1 for every farmer obtained when everyone cooperates. As in the Prisoner's Dilemma, we are confronted here with another situation in which rational individual play leads to an inferior outcome.

The heart of this dilemma is the fact that the penalty for a farmer putting an extra cow in the field is a cost shared among all the farmers, while the only the owner of the extra cow earns the benefit. Although this is clearly a toy example, the dynamic present here appears in a variety of more meaningful situations:

- People have the option of cooperating by disposing trash responsibly or defecting by littering. Each person who litters gets a bonus for getting rid of trash. However, for every player who litters, there is a small penalty to everyone for the accumulated trash.

- Fishing companies can cooperate with sustainable fishing or defect by fishing in an unsustainable way. Unsustainable fishing yields a higher profit for a company. However, the fish population declines with each company that fishes unsustainably, thus penalizing every company.

- Nations can cooperate by limiting CO_2 emissions, or defect by continuing as is. A country that defects will have greater productivity. However, each country that defects contributes to the global accumulation of CO_2, thus causing a penalty to everyone.

In such situations, we may prefer that all player cooperate, but rationality dictates they should all defect. To arrange for cooperation, requires that we *change the game* by adding incentives or penalties to alter the payoffs associated with the different outcomes. How might constructs such as social norms and laws be used to alter the payoffs in each of the three instances above?

6.1 Matrix Games

Our investigations of two-player games centered on matrix games. Higher dimensional matrix games likewise figure vitally in the study of n-player games. Just as in the study of two-player matrix games, n-player matrix games can be used to model a wide variety of decision problems.

Basic Definitions

We begin with 3-player matrix games. Rose and Colin are still playing, as before, and now a new player, called Larry, will join them. Represent this game not by a 2-dimensional matrix with just rows and columns, but instead by a 3-dimensional matrix with rows, columns, and *layers*. As usual, Rose will choose a row of the matrix and Colin will choose a column; now Larry will choose a layer. These three choices select a matrix entry of the form (x, y, z), indicating payoffs of x to Rose, y to Colin, and z to Larry.

Example 6.2. The following is a 3-player matrix game between Rose, Colin, and Larry. So, for instance, if Rose chooses A, Colin chooses C, and Larry chooses F the payoffs will be $(1, 4, -1)$.

Larry E

		Colin	
		C	D
Rose	A	2,−1,3	1,1,−2
	B	0, 3, 2	2, 0, 1

Larry F

		Colin	
		C	D
Rose	A	1,4,−1	3, 1, 1
	B	−1,1,0	1, 1, 1

Figure 6.2: A 3-player matrix game

Although it becomes challenging to draw matrix games between large numbers of players, these are nonetheless helpful constructs. To play a matrix game with players $1, 2, \ldots, n$ requires an n-dimensional matrix, say with dimensions $d_1 \times d_2 \ldots \times d_n$. Each entry of the matrix will have the form $(x_1, x_2, \ldots, x_n)$ indicating a payoff of x_i to player i in the event the players' choices select this entry.

For instance, a matrix can model the 100-player Tragedy of the Commons. Each of the 100 players has just two options (cooperate or defect), so the matrix game has dimensions $\underbrace{2 \times 2 \times \ldots \times 2}_{100}$. Every cell of this matrix corresponds to a choice of cooperate or defect for each of the players, and cell entries are of the form $(x_1, \ldots, x_{100})$, indicating a payoff of x_i to player i.

Dilemmas

Such n-player matrix games provide an insightful mathematical model for situations involving many players. We have already articulated one dilemma, The Tragedy of the Commons, as an n-player version of The Prisoner's Dilemma. Other two-player dilemmas from Chapter 1 similarly scale. Below are two games that give each of 100 players two choices, and can be represented by a $\underbrace{2 \times 2 \times \ldots \times 2}_{100}$ matrix. These two examples give only a taste of the rich variety of phenomena this framework can model.

Game 6.3 (General Volunteering Dilemma)**.** This is a game played between 100 roommates. The bathroom needs to be cleaned, and everyone must either cooperate by offering to clean it (C) or defect (D) by not offering. If everyone defects, then the bathroom stays dirty and resulting in a penalty of -10 to each player. If at least one person cooperates by volunteering to clean the bathroom, then the players who defect get a payoff of 2 for the clean bathroom. If k players cooperate, then they split the work and each gets a payoff of $2 - {}^{20}/_{k}$.

		Other Roommates	
		$k \geq 1$ play C	all play D
Roommate i	C	$2 - {}^{20}\!/_{k+1}$	-18
	D	2	-10

We note that this General Volunteering Dilemma is a simple generalization of the two-player Volunteering Dilemma 1.5. As there, each player here wants to Defect while at least one other player cooperates — a familiar strategic dynamic.

Game 6.4 (Investing Dilemma)**.** This is a game played among 100 investors. Each player has \$1 and can choose to either invest (I) or hold (H) this money. If 90% of the players choose to invest, then each investor earns \$1. Otherwise, each investor loses \$1. The matrix below indicates the outcome for player i depending on the other players.

		Other Investors	
		≤ 88 play I	≥ 89 play I
Investor i	I	-1	1
	H	0	0

This game is essentially a many player version of Stag Hunt. Each player benefits if everyone invests. Yet there may be some uncertainty about whether the other players can all be trusted to choose I. Like Stag Hunt, this game has two pure Nash Equilibria, one where every player plays I and the other where each chooses H. Everyone prefers the first equilibria, but potential mistrust makes for a tricky dynamic.

Dominance

In both zero-sum and general 2-player matrix games, the notion of dominance enhanced our investigation. There is an analogous concept for n-player matrix games that will prove similarly useful. Suppose player i has two pure strategies s and s' with the property that no matter what every other player does, her payoff will always be at least as good when she selects s as when she selects s'. In this case, we say that strategy s *dominates* strategy s'. Analogously, if no matter what every other player does, player i does strictly better when playing s than when playing s' we say that s *strictly dominates* s'.

Example 6.5. In the matrix game from Figure 6.3, Larry has a dominant strategy of Layer E. Eliminating his dominated strategy, leaves a (2×2) game in which Rose has a dominant strategy of Row B. It follows (assuming the players are rational) that Rose should play B, Colin should play C, and Larry should play E.

In the Tragedy of the Commons, each individual farmer achieved a higher payoff from defecting rather than cooperating no matter what the other players did. This corresponds to the property that in the associated matrix game the strategy of defect strictly dominates that of cooperate for every player. Next we will introduce an amusing n-player game to analyze using dominance.

Larry E

		Colin	
		C	D
Rose	A	1,−1,3	2,−1,1
	B	2, 0, 3	3,−1,2

Larry F

		Colin	
		C	D
Rose	A	3,−1,2	−1,2,0
	B	1,−1,0	2, 2, 1

Figure 6.3: Dominance in a 3-player matrix game

Game 6.6 (The $2/3$ of the Average Game)**.** This is an n-player game in which each player secretly chooses an integer between 1 and 100 and then the choices are revealed. The person or persons who selected the number(s) closest to $2/3$ of the average number evenly divide a pot of money.

What are rational players likely to do in this game? Since each player is calling out a number between 1 and 100, the highest possible average is 100. Therefore, $2/3$ of the average will always be less than 67. It follows that the strategy of playing 67 dominates all higher numbers. If we eliminate strategies 68 and above for all of the players, then no player will guess a number larger than 67, so the average will necessarily be at most 67. But then $2/3$ of the average will be less than 45, so the strategy of playing 45 dominates all of the higher strategies. This trend continues and iterated removal of dominated strategies will further reduce the game.

This analysis suggests that a collection of perfectly rational players who share common knowledge that the other players are rational will all choose 1. Note that the method of iteratively removing strictly dominated strategies relies heavily on the rationality of the other players and the extent to which this is known. When the $2/3$ of the Average game is actually played, it generally takes some time before the players converge to choosing small numbers.

6.2 Coalitions

One new consideration in n-player games is that of coalitions. For instance, in a 3-player game, two of the players might decide to team up against the third. This section explores n-player games with an eye toward such coalitions and it culminates in an entirely new way of thinking about n-player games.

For simplicity and concreteness, assume for the remainder of this chapter that all payoffs are actual dollar amounts and that each player receives one unit of utility from each dollar. The important side effect of this is that players in a coalition may decide to distribute their winnings differently than the actual result from play (i.e. they may arrange side payments). For instance, if Rose and Colin form a coalition against Larry in a 3-player game and the payoffs were $(5, 1, -2)$, Rose and Colin might agree to split the $5 + 1 = 6$ dollars so they each get 3. Transferable utility is the technical term for this significant assumption. Additionally, assume that the players can communicate and cooperate with one another. These assumptions lead to a mathematically rich and interesting realm of game theory (albeit one that is not universally applicable).

Coalitions in Matrix Games

Before introducing any general ideas, let's begin by considering coalitions in a particular case.

Example 6.7. Consider the following 3-player matrix game.

Larry E

		Colin C	Colin D
Rose	A	2,−1,3	1,1,−2
	B	0, 3, 2	2, 0, 1

Larry F

		Colin C	Colin D
Rose	A	1,4,−1	3, 1, 1
	B	−1,1,0	1, 1, 1

Suppose first that Rose and Colin decide to form a coalition against Larry. Rose's pure strategies are A and B while Colin has the pure strategies C and D. Acting as a coalition, they can decide what Rose will do and what Colin will do. This gives the Rose-Colin coalition the four pure strategies: AC, AD, BC, and BD. This Rose-Colin coalition can also play a mixed strategy. For instance, it could play AC with probability $1/2$ and BD with probability $1/2$. Note that this strategy requires Rose and Colin to coordinate their actions, so they would indeed need to be working as a team.

Assuming this coalition has formed, the game reduces to the following 2-player matrix game between Rose-Colin and Larry. Here the payoff to Rose-Colin indicates the sum of the payoffs in the original game for Rose and Colin. So, for instance, in the original game Rose Playing A and Colin playing D and Larry playing F gives a payoff of 3 to Rose and 1 to Colin. In the new game, then, when Rose-Colin play AD and Larry plays F, the payoff to Rose-Colin is $3 + 1 = 4$.

		Larry E	Larry F
Rose-Colin	AC	$1, 3$	$5, -1$
	AD	$2, -2$	$4, 1$
	BC	$3, 2$	$0, 0$
	BD	$2, 1$	$2, 1$

What is the maximum total payoff the Rose-Colin coalition can guarantee? This is precisely the Rose-Colin security value in the above 2-player game. It follows from Theorem 5.2 that this is the Von-Neumann value of the zero-sum matrix given by the Rose-Colin payoffs as shown in Figure 6.4. To compute this, we can invoke Procedure 3.13. In this case, Larry has a guarantee function G_L and the value of the game is given by the lowest point on the graph of this function. As shown in the figure, this value is $12/5$, so this is the maximum total expected payoff the Rose-Colin coalition can guarantee.

What about Larry? What is the maximum value can he guarantee himself working alone? The worst case scenario for Larry is when Rose and Colin form a coalition and play to minimize his payoff. This case gives the zero-sum matrix on the left in Figure 6.5. So, the maximum value Larry can guarantee working on his own is the value of this zero-sum matrix, which is $1/7$.

$$\begin{array}{cc} & \text{Larry} \\ \text{Rose-Colin} & \begin{array}{c|cc} & E & F \\ \hline AC & 1 & 5 \\ AD & 2 & 4 \\ BC & 3 & 0 \\ BD & 2 & 2 \end{array} \end{array}$$

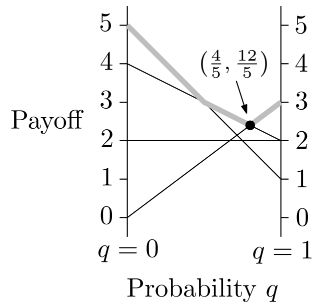


Figure 6.4: Rose-Colin vs. Larry

$$\begin{array}{cc} & \text{Rose-Colin} \\ \text{Larry} & \begin{array}{c|cccc} & AC & AD & BC & BD \\ \hline E & 3 & -2 & 2 & 1 \\ F & -1 & 1 & 0 & 1 \end{array} \end{array}$$

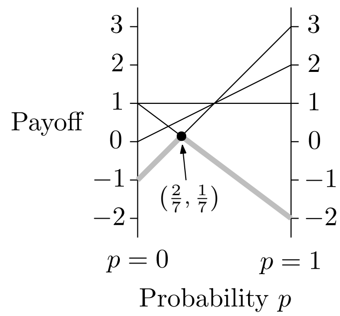


Figure 6.5: Rose-Colin playing to limit Larry

Coalitional Form

The above example suggests an important general concept, namely the total payoff a coalition can guarantee itself (independent of the actions of the other players). More formally, for any matrix game M introduce a function, denoted f_M, called the *coalitional form* of M. The coalitional form f_M is defined by the rule that for every subset S of players, $f_M(S)$ denotes the maximum total expected payoff the players in the set S can guarantee themselves when working as a coalition. Let's return to our 3-player game and determine its coalitional form.

Example 6.7 Continued. Let's call the matrix from this example M and determine its coalitional form, f_M. The coalitional form is defined for every subset S of players, even the empty set. However, the empty set doesn't ever win any money so $f_M(\emptyset) = 0$. Another extreme is the case when all of our players work together. In this case, the three players can decide to play any combination of the three strategies. They could, for instance, decide that Rose will play B, Colin will play C, and Larry will play E getting payoffs of $(0, 3, 2)$. In fact, $0+3+2 = 5$ is the highest sum the three can achieve, so $f_M(\{\text{Rose}, \text{Colin}, \text{Larry}\}) = 5$. Since it is a bit cumbersome to write $f(\{\ldots\})$ when applying f to a nonempty set, we will simplify the notation. Going forward, we will drop the inner set braces (eg. write $f_M(\text{Rose}, \text{Colin}, \text{Larry}) = 5$). Previous considerations of the Rose-Colin coalition determined that, working together, the best guarantee for Rose and Colin is $^{12}/_{5}$, so $f_M(\text{Rose, Colin}) = {}^{12}/_{5}$. We've likewise seen that Larry's best guarantee working alone is $^{1}/_{7}$, so $f_M(\text{Larry}) = {}^{1}/_{7}$.

A similar analysis for the other coalitions produce the full coalitional form for this game.

$$\begin{array}{ll} f_M(\emptyset) = 0 & f_M(\text{Rose, Colin, Larry}) = 5 \\ f_M(\text{Rose}) = 1 & f_M(\text{Colin, Larry}) = 13/5 \\ f_M(\text{Colin}) = 3/5 & f_M(\text{Rose, Larry}) = 17/7 \\ f_M(\text{Larry}) = 1/7 & f_M(\text{Rose, Colin}) = 12/5 \end{array}$$

Next we look beyond this particular example to consider the general case. Suppose that M is an n-player matrix game, and we are interested in determining the maximum expected payoff that a certain subset S of players can guarantee itself (i.e. we want to determine $f_M(S)$). Assume further that S is not empty and is also not equal to the set of all players. Define T to be the set of all players not in S and note that T is also not empty and not equal to the set of all players. The worst case for coalition S is if all of the players in T decided to work together in a coalition against coalition S. This now resembles just a 2-player game where the S-coalition plays against the T-coalition. The pure strategies for the S-coalition correspond to any combination of pure strategies for the players in S. The pure strategies for the T-coalition consist of any combination of pure strategies for the players in T. The maximum amount that S can guarantee itself is precisely the S security value in this 2-player game.

Procedure 6.8 (Finding the Coalitional Form)**.** Consider an n-player matrix game M for players $\{1, \ldots, n\}$. The coalitional form of M, denoted f_M, is given by the following rule.

1. $f_M(\emptyset) = 0$
2. $f_M(1, \ldots, n)$ is equal to the maximum sum of payoffs of the players over all cells of the matrix.
3. When $S \subseteq \{1, \ldots, n\}$ has at least one, but not all, of the players, let T be the set of players not in S and define a zero-sum 2-player matrix game M_S as follows. Each row of M_S corresponds to a pure strategy for S (i.e. a choice of pure strategy for each player in S) and each column of M_S corresponds to a pure strategy for T (i.e. a choice of pure strategy for each player in T). The entry in position i, j of M_S is the total payoff that S achieves when playing strategy i opposite strategy j for T. Now $f_M(S)$ is the value of M_S.

Note that the above procedure generalizes the above example. To determine the maximum guarantee for Rose-Colin, we computed the value of the zero-sum matrix game where Rose-Colin plays against Larry and each entry is the total payoff to Rose-Colin. Similarly, to compute Larry's maximum guarantee, we computed the value of the zero-sum matrix game where Larry plays against Rose-Colin and each entry is Larry's payoff.

Coalitional Games

The coalitional form f_M associated with a matrix game M gives us a very different way of thinking about the game. This representation ignores the strategic considerations and instead only tells us about the strength of the coalitions. We now distance ourselves from even having a matrix and view a coalitional form as a game in and of itself.

Definition 6.9. We define a *coalitional game* to consist of a set of players $\{1, 2, \ldots, n\}$ and a function f that assigns each subset $S \subseteq \{1, 2, \ldots, n\}$ a real number $f(S)$ indicating the total payoff that S can obtain as a coalition. Assume that f always satisfies the following

1. $f(\emptyset) = 0$
2. $f(S \cup T) \geq f(S) + f(T)$ whenever S and T are disjoint.

The first condition just enforces the rule that the empty set cannot guarantee a positive payoff. The second condition is natural since whenever S and T are disjoint sets of players, the total that these sets of players can obtain when working together should always be at least the sum of what S and T can obtain individually. Let's begin our exploration of coalitional games with a simple but instructive example.

Game 6.10 (Divide the Dollar)**.** This is a game played between three players $1, 2, 3$. They have one dollar to divide among the three players, and if any two out of the three agree on a division, then this is the outcome. We represent this coalitional game with a function called d (for Divide and Dollar)

$$\begin{aligned} 0 &= d(\emptyset) = d(1) = d(2) = d(3) \\ 1 &= d(1,2) = d(1,3) = d(2,3) = d(1,2,3) \end{aligned}$$

Imagine for a moment how the players might negotiate a split of this dollar. Player 1 might suggest a division of $(1/3, 1/3, 1/3)$ since this gives each player the same amount. However, Player 2 might suggest to Player 3 a two-way split, so each would get $1/2$ by forming a coalition and agreeing upon the division $(0, 1/2, 1/2)$. But then Player 1 could offer Player 3 an even better split, suggesting $(1/3, 0, 2/3)$. Player 2 might respond by offering to form a coalition with Player 1 and divide the dollar as $(1/2, 1/2, 0)$. As you can probably see, this could go on forever!

Just as there is no all-encompassing theory to explain how two rational players should play a general matrix game, there does not exist a complete theory to resolve an arbitrary coalitional game. In particular, we will not be giving some sort of "answer" to the Divide the Dollar game. Many useful concepts have nonetheless been developed to study and help understand coalitional games. The remainder of this chapter introduces a few of these ideas.

Consider a coalitional game f for the players $1, \ldots, n$. As in Chapter 5, imagine that the players are going to negotiate about the play of this game. Since the payoffs are transferrable dollars, it will be possible for the players to exchange money after the game is played. If the players are going to agree to divide up the winnings so that player i gets x_i dollars, what properties should we expect of the payoff vector $(x_1, \ldots, x_n)$?

Certainly every individual deserves a payoff at least as large as each individual's own guarantee in the game. So we should restrict our attention to those payoffs $(x_1, \ldots, x_n)$ which satisfy $x_i \geq f(i)$ for every player i. There is also a global optimality condition for the entire group. Namely, the total payoff should be equal to $f(1, \ldots, n)$ (i.e. the maximum that the players can achieve working together). If the payoffs $(x_1, \ldots, x_n)$ were to add up to less, then it would be possible for the players all to form a coalition and (possibly after exchanging money) every player could get a higher payoff. This motivates the following important definition.

Definition 6.11. If f is a coalitional game for the players $1, \ldots, n$, then a vector of payoffs $(x_1, \ldots, x_n)$ with x_i going to player i is an *imputation* if it satisfies the following properties:

1. Individual Rationality. $x_i \geq f(i)$ for every $1 \leq i \leq n$

2. Group Rationality. $x_1 + x_2 + \ldots x_n = f(1, 2, \ldots, n)$

Von Neumann and Morgenstern introduced the concept of an imputation as a reasonable vector of payoffs in a coalitional game. This well-accepted notion will be our focus going forward in our investigation of these games. Let's examine the set of imputations for Divide the Dollar.

The set of imputations for Divide the Dollar consists of all triples (x_1, x_2, x_3) with $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 1$. This set is a triangle in $\mathbb{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ as depicted in Figure 6.6.

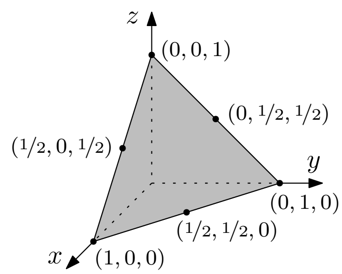


Figure 6.6: A triangle in $\mathbb{R}^3$

To analyze a negotiation between two players, Chapter 10 introduced the concept of a negotiation set. In the special case when the negotiation set consisted of a single point, the problem of selecting a reasonable and fair negotiated outcome was easy — just pick that point. In the setting of n-player coalitional games, certain games are similarly uninteresting from the standpoint of negotiations since they have just a single imputation. Next we introduce a term for these games.

Definition 6.12. A coalitional game f for the players $1, \ldots, n$ is called *inessential* if

$$f(1, 2, \ldots, n) = f(1) + f(2) + \ldots + f(n),$$

and otherwise we call f *essential*

Lemma 6.13. *A coalitional game f for the players $1, \ldots, n$ is inessential if and only if it has exactly one imputation (in this case the unique imputation must be $(f(1), \ldots, f(n))$).*

Proof. If f is inessential, then $(f(1), f(2), \ldots, f(n))$ is an imputation. Furthermore, it must be the only one since any other imputation must have coordinate i at least $f(i)$ and must sum to $f(1, \ldots, n) = f(1) + \ldots + f(n)$. Next suppose f is essential and define $m = f(1, \ldots, n) - (f(1) + \ldots + f(n))$, noting that $m > 0$. For every player j, we can form an imputation $(x_1, \ldots, x_n)$ by the rule:

$$f(i) = \begin{cases} f(i) & \text{if } i \neq j \\ f(j) + m & \text{if } i = j \end{cases}$$

This gives us $n \geq 2$ imputations, thus completing the proof. $\square$

At this point in our study of coalitional games, we have focused on the set of imputations and have defined inessential games that have only a single imputation. Ideally, we would like a way to select a single imputation, or perhaps a small set of imputations, that would be sensible negotiated outcomes for essential games. We use the term "solution concept" for a rule that selects one or more imputations and now introduce two solution concepts based on the following notion of dominance.

Domination

The initial Divide the Dollar analysis ran in circles as players formed coalitions and then broke them to form different coalitions over and over and over. Some terminology here will explain precisely what was going on there and, in fact, this revisits a many-player variant of the now-familiar notion of dominance.

Consider a coalitional game f for the players $1, \ldots, n$ and suppose we want an imputation to split the total payoff. Assume further that a player has proposed the imputation $(x_1, \ldots x_n)$. Suppose also that there exists a subset of players S and another imputation $(y_1, \ldots, y_n)$ with the following properties:

- $y_i > x_i$ for all $i \in S$.
- $f(S) \geq \sum_{i \in S} y_i$.

In this case, the players in S could form a coalition in the game and then split the total payoff so that each player in S gets a higher payoff than in the imputation $(x_1, \ldots, x_n)$. Consequently, these players have a legitimate objection to the proposed division $(x_1, \ldots, x_n)$. In this case, we say that the imputation $(y_1, \ldots, y_n)$ *S-dominates* $(x_1, \ldots, x_n)$.

With this notion of dominance, we can better understand the repeated cycle of one suggested Divide the Dollar proposal being rejected in favor of another. Player 1 first suggested the imputation $(1/3, 1/3, 1/3)$, but this was $\{2,3\}$-dominated by the imputation $(0, 1/2, 1/2)$, which was $\{1,3\}$-dominated by $(1/3, 0, 2/3)$. This, in turn was $\{1,2\}$-dominated by $(1/2, 1/2, 0)$. Domination, then, is what caused movement from one proposal to the next. This suggests some value in finding imputations that are not dominated by any others.

Definition 6.14. The *core* of a coalitional game f for the players $1, \ldots, n$ is the set of all imputations $(x_1, \ldots, x_n)$ with the property that $\sum_{i \in S} x_i \geq f(S)$ for every set S of players. In other words, an imputation $(x_1, \ldots, x_n)$ is in the core if every set of players receives a total payoff at least as much as the players in S could guarantee themselves by working as a coalition.

In fact, the core of f is precisely the set of all imputations that are not dominated by any other imputation. Since domination is such a fundamental principle, the core is a natural and important set of imputations. There do exist coalitional games with an empty core, so in these cases the core isn't particularly useful. In fact, it's straightforward to verify that the core of Divide-the-Dollar has nothing in it.

Next, another solution concept for coalitional games based on domination.

Definition 6.15. A *stable set* is a set J of imputations with the following properties:

1. No imputation in J dominates another imputation in J.

2. Every imputation not in J is dominated by one in J.

There is nothing in the definition of a stable set that guarantees a stable set be somehow optimal, but these sets certainly do have some nice features. Since every imputation is dominated by one in the stable set, it is reasonable to argue that we may restrict our attention to a stable set without significant loss. Also, since no imputation in a stable set dominates another, we avoid running in circles by restricting our attention to a stable set. Let's return to Divide-the-Dollar to find a stable set.

We claim that $J = \{(1/2, 1/2, 0),\ (1/2, 0, 1/2),\ (0, 1/2, 1/2)\}$ is a stable set in Divide-the-Dollar. It's easy to check that none of the imputations in J dominates another, so the first condition is satisfied. For the second condition, consider an arbitrary imputation (x_1, x_2, x_3) and suppose (without loss) that $x_1 \leq x_2 \leq x_3$. If $x_1 \leq x_2 < 1/2$ then this imputation is $\{1, 2\}$-dominated by $(1/2, 1/2, 0)$. Otherwise we must have $(x_1, x_2, x_3) = (0, 1/2, 1/2)$, but then our imputation is already in J.

Cores and stable sets are two solution concepts based on the fundamental notion of domination. Unfortunately, some games, like Divide-the-Dollar, have no core. There are other coalitional games with no stable sets. So, on the downside, these solution concepts are somewhat limited in their applicability, but on the significant upside they are valuable when they do apply. The next section turns to questions of fairness and introduces another solution concept called Shapley Value — something that always exists and is unique!

6.3 Shapley Value

Nash Arbitration appeared in Chapter 10 as a rule that can be used to select an arguably fair point in a payoff polygon. Here, we introduce a roughly analogous idea for coalitional games. The new setting is an n-player coalitional game f. We would like to determine an imputation $(x_1, \ldots, x_n)$ that, in some sense, fairly represents the value player i contributes to the coalitions.

More precisely, define a *valuation scheme* to be a rule that takes as input a coalitional game f and outputs an imputation $(x_1, \ldots, x_n)$ for this game. A valuation scheme may be regarded as a very special type of function of the following form.

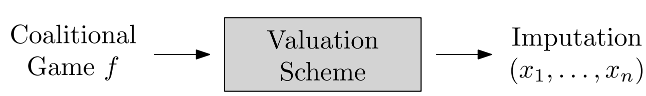


Figure 6.7: A valuation scheme

Introducing Shapley Value

Central to this section is the following valuation scheme due to Lloyd Shapley.

Definition 6.16 (Shapley Value)**.** Given an n-player coalitional game, we will give a rule to determine an imputation $(x_1, \ldots, x_n)$. To compute the x_i values, consider all $n!$ ways to order the n players. For each such ordering, imagine growing a larger and larger coalition by starting with the empty set and adding one player at a time according to this order, until all n players are part of the coalition. Determine how much value player i adds to this growing coalition as follows: If S is the set of players before i in the ordering, then player i adds a value of $f(S \cup \{i\}) - f(S)$ to the growing coalition. The payoff x_i assigned to player i will be the average value this player adds over all possible orderings.

Example 6.17. Consider a coalitional game with players $\{1, 2, 3\}$ given by the following function f

$$\begin{array}{llll} & f(1) = 2 & f(1,2) = 8 & \\ f(\emptyset) = 0 & f(2) = 3 & f(1,3) = 7 & f(1,2,3) = 10 \\ & f(3) = 1 & f(2,3) = 6 & \end{array}$$

To compute the Shapley Value, consider all $3! = 6$ orderings of the players, and for each determine the value that each player adds to the growing coalition. For instance, for the ordering 123 the first player adds $f(1) = 2$ to the growing coalition. The second player contributes $f(1,2) - f(1) = 8 - 2 = 6$, and the third contributes $f(1,2,3) - f(1,2) = 10 - 8 = 2$. The table below indicates the value each player adds for each possible ordering.

	Value added by		
Ordering	Player 1	Player 2	Player 3
123	2	6	2
132	2	3	5
213	5	3	2
231	4	3	3
312	6	3	1
321	4	5	1
Total	23	23	14
Average	$^{23}/_{6}$	$^{23}/_{6}$	$^{14}/_{6}$

The Shapley Value is given by the averages in the above table so it is $(^{23}/_{6}, ^{23}/_{6}, ^{7}/_{3})$.

A quick check reveals that $(^{23}/_{6}, ^{23}/_{6}, ^{7}/_{3})$ is indeed an imputation for this game f since the entries sum to $10 = f(1,2,3)$ and each player i gets a payoff of at least $f(i)$. Next, a proof that this holds in general.

Proposition 6.18. *Shapley Value is a valuation scheme.*

Proof. Suppose that Shapley Value applied to f gives $(x_1, \ldots, x_n)$. We need to prove that $(x_1, \ldots, x_n)$ is an imputation, so we must show that it satisfies both the Individual Rationality and Group Rationality conditions.

For Individual Rationality, let i be an arbitrary player and show $x_i \geq f(i)$. To do so, consider an arbitrary ordering of the players, and think about how much player i adds to the growing coalition by joining it. If S is the set of players before i then this is given by $f(S \cup \{i\}) - f(S)$. The definition of a coalitional game implies that $f(S \cup \{i\}) - f(S) \geq f(i)$. Thus, for every possible ordering, player i contributes at least $f(i)$ to the value of the growing coalition. Since x_i is an average of these values, it follows that $x_i \geq f(i)$ as desired.

To prove Group Rationality, we need to show that $(x_1, \ldots, x_n)$ satisfies $x_1 + \ldots + x_n = f(1, \ldots, n)$. To do this, first consider the ordering of the players $1, 2, \ldots, n$ and think about how much this ordering will add to the total payoffs for the players. In this case, the sum of the values added by all of the players to the coalition is

$$\begin{array}{rl} & f(1) \\ & f(1,2) - f(1) \\ & f(1,2,3) - f(1,2) \\ & \vdots \\ + & f(1,\ldots,n) - f(1,\ldots,n-1) \\ \hline & f(1,\ldots,n) \end{array}$$

So, for this ordering, the total value added by the players is $f(1, \ldots, n)$. A similar argument shows that the total value added for every ordering will be $f(1, \ldots, n)$. Therefore $x_1 + \ldots + x_n$, which is the sum of the average values added by the players, will equal $f(1, \ldots, n)$ as desired. □

Shapley's Axioms

We have now defined Shapley Value and proven that it is indeed a valuation scheme. That said, finding a valuation scheme isn't really anything special. Another (somewhat silly) valuation scheme is this: For every n-player coalitional game f give every player $i \neq n$ the value $x_i = f(i)$, that player's guarantee, and give everything else to player n. In other words, set $x_n = f(1 \ldots n) - f(1) - f(2) \ldots - f(n-1)$. This valuation scheme certainly doesn't seem like a particularly fair one, but how can we argue that Shapley Value is fair and this one isn't?

Just as Nash introduced fairness axioms for his arbitration scheme, Shapley likewise introduced certain axioms that articulate appealing properties of a valuation scheme. Again parallel to Nash, Shapley then proved that Shapley Value is the unique valuation scheme that satisfies all of these axioms. So, if you accept his axioms as fair, you must accept Shapley Value as fair! Here are the axioms.

Definition 6.19 (Shapley's Axioms)**.** These are axioms for an n-player valuation scheme $\mathcal{V}$. We will assume that applying $\mathcal{V}$ to the coalitional game f gives the imputation $(x_1, \ldots, x_n)$.

1. Irrelevance. If there is a player i who adds nothing to any coalition (i.e. $f(S) = f(S \cup \{i\})$ for every $S \subseteq \{1, \ldots, n\}$) then $x_i = 0$.

2. Symmetry. If there are two players i, j so that every set $S \subseteq \{1, \ldots, n\}$ with $i, j \notin S$ satisfies $f(S \cup \{i\}) = f(S \cup \{j\})$ (i.e. players i and j are symmetric), then $x_i = x_j$.

3. Invariance Under Sums. Suppose $\mathcal{V}$ assigns the coalitional games f' and f'' the imputations $(x'_1 \ldots x'_n)$ and $(x''_1 \ldots x''_n)$. If $f = f' + f''$ then $(x_1 \ldots x_n) = (x'_1 \ldots x'_n) + (x''_1 \ldots x''_n)$.

The first axiom seems clearly fair. After all, why would a player who never contributes anything deserve a positive payoff? The second is a natural fairness property similar to Nash's symmetry axiom: If two players are symmetric, they should get equal payoffs. The last axiom also looks reasonable. If player i gets a payoff of x_i for the coalitional game f and a payoff of x'_i for the coalitional game f', then $x_i + x'_i$ looks like a sensible payoff for $f + f'$. However, these axioms are very powerful in conjunction as evidenced by the following theorem of Shapley, the centerpiece of this section.

Theorem 6.20 (Shapley)**.** *The only n-player valuation scheme satisfying the above axioms is Shapley Value.*

To prove Shapley's Theorem, we need to do two things. First, we must prove that Shapley Value does indeed obey all three of Shapley's Axioms. Then we need to show that any other valuation scheme that obeys these axioms must be the same as Shapley Value. The lemma below accomplishes the first task.

Lemma 6.21. *Shapley Value obeys Shapley's Axioms.*

Proof. The fact that Shapley Value obeys the first axiom is an immediate consequences of its definition. The third axiom can also be deduced straight from the definition. For the second axiom, consider a coalitional game where players i and j satisfy the symmetry condition that $f(S \cup \{i\}) = f(S \cup \{j\})$ for every set S not including i or j. Now, for every possible ordering σ of the players, another ordering σ' comes from interchanging the positions of players i and j. It follows from the symmetry property that the value i adds to the growing coalition for σ will be equal to the value j adds to the growing coalition for σ' and vice-versa. It follows that Shapley Value will assigns i and j the same payoffs. $\square$

Proof of Shapley's Thoerem

Now we turn to the "hard" direction of the proof, showing that Shapley Value is the unique valuation scheme that obeys all three of Shapley's Axioms. The proof of this utilizes some special coalitional games. For a nonempty set $T \subseteq \{1, \ldots, n\}$, define the coalitional game h_T by the rule that for every subset of players $S \subseteq \{1, \ldots, n\}$

$$h_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

So, the function h_T assigns a coalition a value of 1 if this coalition contains every member of T, and otherwise assigns this coalition a value of 0. A straightforward check reveals that h_T is a coalitional game for every nonempty set T, and more generally, for any number $c \geq 0$ the game $c \cdot h_T$ is a coalitional game. The next lemma gives a key property of these games.

Lemma 6.22. *Let $\mathcal{V}$ be a valuation scheme that satisfies Shapley's axioms 1 and 2. Let $T \subseteq \{1, \ldots, n\}$ be nonempty and let $c \geq 0$. The imputation $(x_1, \ldots, x_n)$ that $\mathcal{V}$ assigns to $c \cdot h_T$ is given by the rule*

$$x_i = \begin{cases} \frac{c}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}$$

Proof. If $i \notin T$, then player i adds nothing to any coalition (i.e. $f(S) = f(S \cup \{i\})$ holds for all S). In this case, the first axiom implies that $x_i = 0$. If i and j are two players in T, then

i and j will obey the symmetry condition in axiom two, so $x_i = x_j$. Consequently, every player not in T gets 0 while every player in T gets the same value. To find this common value, note that, because we have a valuation scheme, $x_1 + \ldots + x_n = c \cdot h_T(1, \ldots, n) = c$. It follows that every $i \in T$ will satisfy $x_i = \frac{c}{|T|}$ as desired. $\square$

We now have all the necessary ingredients for the proof of Shapley's Theorem. A short detour back to Divide the Dollar will help illuminate the main idea in the forthcoming argument. Recall that Divide the Dollar is a coalitional game with players $\{1, 2, 3\}$ and function d given by the rule:

$$d(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 1 & \text{if } |S| \geq 2 \end{cases}$$

Suppose that we have a valuation scheme $\mathcal{V}$ that satisfies all of Shapley's Axioms. What happens when we apply this valuation to the above coalitional game d? The central idea is the following equation that expresses d in terms of the special functions h_T,

$$d = h_{\{1,2\}} + h_{\{1,3\}} + h_{\{2,3\}} - 2h_{\{1,2,3\}}.$$

This equation is straightforward to verify (for instance on the input $\{1, 2, 3\}$ the left hand side evaluates to 1 and the right hand side evaluates to $1 + 1 + 1 - 2 = 1$). Adding $2h_{\{1,2,3\}}$ to both sides of this equation gives us

$$d + 2h_{\{1,2,3\}} = h_{\{1,2\}} + h_{\{1,3\}} + h_{\{2,3\}}$$

(an equation without any negative terms). By assumption, valuation scheme $\mathcal{V}$ satisfies the first two axioms, so the previous lemma shows that $\mathcal{V}$ must assign $2h_{\{1,2,3\}}$ the imputation $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, and must assign $h_{\{1,2\}}$, $h_{\{1,3\}}$, and $h_{\{2,3\}}$ the imputations $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$, and $(0, \frac{1}{2}, \frac{1}{2})$, respectively. If $\mathcal{V}$ assigns the Divide-the-Dollar game d the imputation (x_1, x_2, x_3), then, since all terms in the previous equation are nonnegative, we may apply axiom (3) to it to obtain

$$(x_1, x_2, x_3) + (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}) = (\tfrac{1}{2}, \tfrac{1}{2}, 0) + (\tfrac{1}{2}, 0, \tfrac{1}{2}) + (0, \tfrac{1}{2}, \tfrac{1}{2}).$$

From this, we conclude that $(x_1, x_2, x_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We have thus demonstrated that any valuation scheme $\mathcal{V}$ that satisfies all three of Shapley's Axioms must assign the imputation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to Divide-the-Dollar, just as Shapley Value does. The proof of Shapley's Theorem is a generalization of this argument.

Proof of Theorem 6.20. We have already shown that Shapley Value obeys Shapley's axioms. To complete the proof, we must show that Shapley Value is the only scheme satisfying all three axioms. To do this, let $\mathcal{V}$ be a valuation scheme satisfying all three of the axioms and we will show that $\mathcal{V}$ and Shapley Value agree on every input. Let f be an arbitrary n-player coalitional game and, in particular, we will show that $\mathcal{V}$ and Shapley Value both assign the same imputation to f. The key to the proof is the following claim:

Claim. *The coalitional game f may be expressed as a linear combination of h_T games.*

To prove the claim, we proceed in steps. At each step, we construct a new function that maps subsets of $\{1, \ldots, n\}$ to numbers. For the first step, define for each player i the number $c_{\{i\}} = f(i)$. Now define a new function f_1 (using f) by the rule

$$f_1 = f - c_{\{1\}}h_{\{1\}} - c_{\{2\}}h_{\{2\}} - \ldots - c_{\{n\}}h_{\{n\}}.$$

The key property of f_1 is that it will assign the value 0 to any set of size ≤ 1 (this follows immediately from its definition). For the next step, define for any two distinct players i, j the value $c_{\{i,j\}} = f_1(i, j)$. Then define the function f_2 from f_1 by

$$f_2 = f_1 - c_{\{1,2\}}h_{\{1,2\}} - c_{\{1,3\}}h_{\{1,3\}} \ldots - c_{\{n-1,n\}}h_{\{n-1,n\}}.$$

Observe that f_2 will assign 0 to any set of size ≤ 2. Continuing in this manner, will eventually construct a function f_n that evaluates to 0 on every set of size $\leq n$, i.e. f_n is the zero function. Together, these equations yield an expression for the original function f as a linear combination of h_T games, thus completing the proof of the claim. In symbols, this is

$$f = \sum_{T \subseteq \{1,\ldots,n\}} c_T h_T. \tag{6.1}$$

The equation here obtained is not quite in the right form for us to apply the axioms. If there is a term on the right in Equation (6.1) with a negative coefficient, say $-10c_T$, we add $10c_T$ to both sides of the equation to get rid of this negative sign. Doing this for all such terms gives an equation of the form

$$f + \sum_{T \subseteq \{1,\ldots,n\}} a_T h_T = \sum_{T \subseteq \{1,\ldots,n\}} b_T h_T. \tag{6.2}$$

where a_T and b_T are nonnegative for every T. It follows from Lemma 6.22 that for every $T \subseteq \{1, \ldots, n\}$ both $\mathcal{V}$ and Shapley Value assign the same imputation to the game $a_T h_T$. Both similarly assign the same imputation to $b_T h_T$. Since both $\mathcal{V}$ and Shapley Value satisfy the third axiom (and every sum of coalitional games is a coalitional game) this means that they both assign the same imputation to the coalitional game $\sum_{T \subseteq \{1,\ldots,n\}} a_T h_T$ and similarly both assign the same imputation to the coalitional game $\sum_{T \subseteq \{1,\ldots,n\}} b_T h_T$. By applying the third axiom to Equation (6.2) we deduce that both $\mathcal{V}$ and Shapley Value assign the same imputation to f, as desired. □

This establishes Shapley's important theorem. Although this theorem is rightly held in high regard, it is nonetheless not the case that Shapley Value is considered to be *the* fair valuation scheme (just as Nash arbitration is not regarded as *the* fair arbitration scheme). There are many other meaningful valuation schemes, so Shapley Value should be viewed as important part of a bigger picture.

Using Shapley Value

Shapley Value is a valuation scheme with widespread application, even frequently in places where there is no obvious game. As an indication of this, consider a voting scenario in which the voters do not have equal power.

Example 6.23. In the United Nations Security Council, there are 5 permanent members and 10 temporary members. In order to pass a (substantive) resolution, all 5 of the permanent members must vote in favor and at least 4 of the temporary members must vote in favor.

Coalitional games can model voting situations like this. For a given subset S of voters, the key question is whether or not S, working as a coalition, has the strength to pass a resolution (i.e. if every member in S votes for the resolution, and everyone not in S votes against, does it pass?). For every set S of voters, define

$$f(S) = \begin{cases} 1 & \text{if the coalition } S \text{ can pass a resolution.} \\ 0 & \text{otherwise} \end{cases}$$

This function f is a special type of coalitional game that conveniently encodes this voting situation. Since this situation is both common and relevant, we give this type of game a general definition.

Definition 6.24. A coalitional game f for the players $1, \ldots, n$ is called a *voting game* if $f(1, \ldots, n) = 1$ and every $S \subseteq \{1, \ldots, n\}$ has $f(S) = 0$ or $f(S) = 1$.

Using the above process, we can construct a voting game to model any kind of voting scenario. Conversely, a voting game describes a certain type of voting situation. In short, voting games are a class of coalitional game which accurately model any kind of voting scenario.

What does the Shapley Value of a voting game mean? In fact, the Shapley value gives a good measure of the relative strengths of the different voters in the game. This type of game is not one where it makes sense for the players to split the payoffs, but the Shapley Value - here known as the Shapley-Shubik Index - nevertheless holds a significant meaning.

Example 6.23 Continued. To compute the Shapley-Shubik Index for the U.N. Security Council, treat this as a voting game and compute the Shapley value for each member. For an ordering of the players, the value of the growing coalition will stay at 0 until the point where a voter is added who gives the coalition enough strength to pass a resolution, and at that point the value jumps to 1. So there is just one voter, called the *swing voter*, who adds value to the growing coalition, and specifically a value of 1. Let T be a temporary member and consider how many times T will be the swing voter over all orderings. For this to happen, all of the permanent members must be before T in the ordering, and exactly 3 of the temporary members must be in before T. (So, T must be in the 9th position.) The total number of ways this happens is $\ell = \binom{9}{3}8!6! = 2438553600$.[1] The value assigned to T by the Shapley-Shubik Index will be this total number of times T is the swing voter, divided by the total number of orderings ($15! = 1307674368000$), which is approximately .00186. To compute the value for a permanent member, use the symmetry of the game to note that any two permanent members get the same value. Since the sum of the values for all members is 1, an easy computation shows that every permanent member will have a value of approximately .19627. Therefore, every permanent member has a little more than 100 times the voting power of each temporary member.

In closing, note that the Shapley-Shubik Index is also not the only metric that can be used to measure voting strength. There are different competing measures — such as the

[1] This is because there are $\binom{9}{3}$ ways to choose which temporary members will be before T, and once this choice has been made there will be $8!$ ways to order the members in front of T and $6!$ ways to order the members behind T.

Banzhaf Index — so the Shapley-Shubik Index certainly cannot be viewed as *the* solution. Such indices nonetheless provide reasonable quantifications of the weighty property of voting strength.

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