Homework 4

Information Theory for Complex Systems

Februari - 2023

There are 9 possible probabilities, which through symmetries can be reduced down 3.

	A	B	C
A	$P(AA) = p_1$	$P(AB) = p_2$	$P(AC) = p_2$
B	$P(BA) = p_2$	$P(BB) = p_3$	$P(BC) = p_3$
C	$P(CA) = p_2$	$P(CB) = p_3$	$P(CC) = p_3$

Since energy is only determined by interactions in pairs we can conclude that finding a maximum of ΔS_2 should be enough to determine p_1, p_2 and p_3.

Attempt a Lagrange optimization

$$\begin{aligned} L(p_1, p_2, p_3, \beta, \mu) = \Delta S_2 + \beta(u - J(4p_2 - p_1)) \\ + \mu(1 - p_1 - 4p_2 - 4p_3) \end{aligned} \tag{1}$$

where we know everything except for an expression for the block entropy difference. To construct an expression for ΔS_2 we need to know the probabilities of the spin states themselves

$$P(A) = ?, P(B) = ?, P(C) = ?. \tag{2}$$

Now we can calculate the block entropy difference

$$\begin{aligned} \Delta S_2 \quad &= \textstyle\sum_{z_{n-1}} p(z_{n-1}) \sum_{z_n} p(z_n|z_{n-1}) \log \frac{1}{p(z_n|z_{n-1})} = \\ &= P(A) \left(P(AA) \log \frac{1}{P(AA)} + P(AB) \log \frac{1}{P(AB)} + P(AC) \log \frac{1}{P(AC)} \right) + \\ &+ P(B) \left(P(BA) \log \frac{1}{P(BA)} + P(BB) \log \frac{1}{P(BB)} + P(BC) \log \frac{1}{P(BC)} \right) + \\ &+ P(C) \left(P(CA) \log \frac{1}{P(CA)} + P(CB) \log \frac{1}{P(CB)} + P(CC) \log \frac{1}{P(CC)} \right) = \\ &= -P(A) \left(p_1 \log p_1 + p_2 \log p_2 + p_2 \log p_2 \right) \\ &\quad -P(B) \left(p_2 \log p_2 + p_3 \log p_3 + p_3 \log p_3 \right) \\ &\quad -P(C) \left(p_2 \log p_2 + p_3 \log p_3 + p_3 \log p_3 \right) = \\ &= -(ap_1 \log p_1 + [2a + P(B) + P(C)] p_2 \log p_2 \\ &\quad +[2P(B) + 2P(C)] p_3 \log p_3) \end{aligned} \tag{3}$$

In this expression the only probabilities missing are $P(B)$ and $P(C)$. Since B and C appear in equal amounts of pairs with the same probabilities, the amounts of B and C must also be the same in the limit towards infinity. Therefore they split the proportion remaining after the A's,

$$P(B) = P(C) = \frac{1 - a}{2}. \tag{4}$$

Now we can simplifiy the block entropy further.

$$\Delta S_2 = -\left(ap_1 \log p_1 + [a+1]p_2 \log p_2 + 2[1-a]p_3 \log p_3\right) \tag{5}$$

U is the expectation value of the internal energy.

$$S[P] = \sum_i p_i \log \frac{1}{p_i} \tag{6}$$

$$\sum_i p_i h(i) = U \tag{7}$$

$$\sum_i p_i f_k(i) = n_k, (k = 1, \ldots, M) \tag{8}$$

$$\sum_i p_i = 1 \tag{9}$$