Math 1B. Solutions to Sample Final Exam

Some Formulas

1. $\sin A\cos B = \dfrac{1}{2}\big[\sin(A-B)+\sin(A+B)\big]$

2. $\sin A\sin B = \dfrac{1}{2}\big[\cos(A-B)-\cos(A+B)\big]$

3. $\cos A\cos B = \dfrac{1}{2}\big[\cos(A-B)+\cos(A+B)\big]$

4. $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ $\qquad R = 1$

5. $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ $\qquad R = 1$

6. $(1+x)^k = \displaystyle\sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$ $\qquad R = 1$

1. (18 points) Write out the form of the partial fraction decomposition of the function

$$\frac{3x-7}{(x^2+x+19)^2(x^2+2x+1)(3x^2+3x)}\ .$$

Do not determine the numerical values of the coefficients.

The first step is to factor the denominator (more fully). The last factor $3x^2+3x$ factors as $3x(x+1)$, and the next to last factor x^2+2x+1 factors as $(x+1)^2$. We need to consolidate the factors $x+1$, so the denominator is

$$\begin{aligned}(x^2+x+19)^2(x^2+2x+1)(3x^2+3x) &= (x^2+x+19)^2(x+1)^2\cdot 3x(x+1)\\ &= 3x(x^2+x+19)^2(x+1)^3\ .\end{aligned}$$

The factor x^2+x+19 doesn't factor further, because its discriminant $1-4\cdot 19$ is negative.

Given the above fully factored form, the form of the partial fraction decomposition is

$$\frac{A}{x} + \frac{Bx+C}{x^2+x+19} + \frac{Dx+E}{(x^2+x+19)^2} + \frac{F}{x+1} + \frac{G}{(x+1)^2} + \frac{H}{(x+1)^3}\ .$$

2. (18 points) Find the following integral. Fully simplify your answer.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

This is done using a trigonometric substitution $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$. By drawing a right triangle with hypotenuse of length x, other sides of lengths 3 and $\sqrt{x^2-9}$, and angle θ opposite the side of length $\sqrt{x^2-9}$, we see that $\sqrt{x^2-9} = 3\tan\theta$, and so the integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta \\
&= 3\int \tan^2\theta\,d\theta \\
&= 3\int (\sec^2\theta - 1)\,d\theta \\
&= 3\tan\theta - 3\theta + C \\
&= \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C\,.
\end{aligned}$$

3. (22 points) Evaluate the integral

$$\int_{-1}^{\infty} \frac{dx}{x^2}$$

or show that it is divergent.

The integrand is discontinuous at 0, so it must be evaluated using three limits:

$$\int_{-1}^{\infty} \frac{dx}{x^2} = \lim_{t\to 0^-}\int_{-1}^{t}\frac{dx}{x^2} + \lim_{t\to 0^+}\int_{t}^{1}\frac{dx}{x^2} + \lim_{t\to\infty}\int_{1}^{t}\frac{dx}{x^2}\,.$$

The first of these is

$$\lim_{t\to 0^-}\int_{-1}^{t}\frac{dx}{x^2} = \lim_{t\to 0^-}\left[-\frac{1}{x}\right]_{-1}^{t} = \lim_{t\to 0^-}\left(-\frac{1}{t} - 1\right) = \infty\,.$$

Since this limit diverges, the improper integral diverges.

4. (24 points) Find the exact value of

$$\sum_{n=3}^{\infty} \ln\left(\frac{\arctan n}{\arctan(n+1)}\right)$$

or show that it is divergent.

Using properties of natural logarithm, we see that the terms of the sum can be written as

$$\ln\left(\frac{\arctan n}{\arctan(n+1)}\right) = \ln(\arctan n) - \ln(\arctan(n+1)) \ .$$

This is in telescoping form, so we approach the sum by the method of Example 7 on pages 707–708. After canceling terms, the n^{th} partial sum simplifies to

$$\begin{aligned} s_n &= (\ln(\arctan 3) - \ln(\arctan 4)) + (\ln(\arctan 4) - \ln(\arctan 5)) \\ &\qquad + \cdots + (\ln(\arctan(n-1)) - \ln(\arctan n)) + (\ln(\arctan n) - \ln(\arctan(n+1))) \\ &= \ln(\arctan 3) - \ln(\arctan(n+1)) \ . \end{aligned}$$

Therefore the sum is

$$\begin{aligned} \sum_{n=3}^{\infty} \ln\left(\frac{\arctan n}{\arctan(n+1)}\right) &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} (\ln(\arctan 3) - \ln(\arctan(n+1))) \\ &= \ln(\arctan 3) - \ln\left(\frac{\pi}{2}\right) \ . \end{aligned}$$

Note that it would *not* be correct to write

$$\begin{aligned} \sum_{n=3}^{\infty} \ln\left(\frac{\arctan n}{\arctan(n+1)}\right) &= \sum_{n=3}^{\infty} \Big(\ln(\arctan n) - \ln(\arctan(n+1))\Big) \\ &= \sum_{n=3}^{\infty} \ln(\arctan n) - \sum_{n=4}^{\infty} \ln(\arctan n) \\ &= \ln(\arctan 3) \ , \end{aligned}$$

because the sums in the second line are *divergent.*

5. (18 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^{n/2}}$$

Use the Root Test:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\arctan n)^{n/2}}\right|} &= \lim_{n\to\infty} \frac{1}{\sqrt{\arctan n}} \\ &= \frac{1}{\sqrt{\pi/2}} \\ &= \sqrt{\frac{2}{\pi}} < 1 \ . \end{aligned}$$

Therefore the sum converges absolutely.

6. (20 points) Suppose that the radius of convergence of the power series $\sum a_n x^n$ is 7. What is the radius of convergence of the power series $\sum a_n x^{3n}$?

By the definition of radius of convergence, the series $\sum a_n x^n$ converges when $|x| < 7$ and diverges when $|x| > 7$.

Replacing x with x^3 everywhere, we have that the series

$$\sum a_n (x^3)^n = \sum a_n x^{3n}$$

converges when $|x^3| < 7$ and diverges when $|x^3| > 7$. Therefore it converges when $|x| < \sqrt[3]{7}$ and diverges when $|x| > \sqrt[3]{7}$.

Again using the definition of radius of convergence, this says that the radius of convergence of the series $\sum a_n x^{3n}$ is $\sqrt[3]{7}$.

7. (18 points) Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \sin x$ centered at the number $a = \pi/6$. Recall that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

We have

$$\begin{aligned} f(x) &= \sin x & f\left(\tfrac{\pi}{6}\right) &= \tfrac{1}{2} \\ f'(x) &= \cos x & f'\left(\tfrac{\pi}{6}\right) &= \tfrac{\sqrt{3}}{2} \\ f''(x) &= -\sin x & f''\left(\tfrac{\pi}{6}\right) &= -\tfrac{1}{2} \\ f'''(x) &= -\cos x & f'''\left(\tfrac{\pi}{6}\right) &= -\tfrac{\sqrt{3}}{2} \end{aligned}$$

Therefore the Taylor polynomial is

$$T_3\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3 .$$

8. (24 points) In general, $(fg)'$ is not equal to $f'g'$. But, let $f(x) = e^{3x}$ and find all functions $g(x)$ such that $(fg)' = f'g'$.

Since $g'(x) = 3e^{3x}$, the equation becomes

$$f'e^{3x} + 3fe^{3x} = 3f'e^{3x} .$$

Dividing by e^{3x} gives $f' + 3f = 3f'$, so we have a linear differential equation $3f = 2f'$, or

$$f' - \frac{3}{2}f = 0 .$$

Now multiply by the integrating factor $e^{-3x/2}$ and solve the differential equation:

$$\begin{aligned} e^{-3x/2}f' - \frac{3}{2}e^{-3x/2}f &= 0 ; \\ \left(e^{-3x/2}f\right)' &= 0 ; \\ e^{-3x/2}f &= C ; \\ f &= Ce^{3x/2} . \end{aligned}$$

This can also be solved as a separable equation

$$\begin{aligned} f' &= \frac{3}{2} f \ ; \\ \frac{df}{f} &= \frac{3}{2}\, dx \ ; \\ \int \frac{df}{f} &= \frac{3}{2} \int dx \ ; \\ \ln|f| &= \frac{3x}{2} + C_0 \ ; \\ f &= \pm e^{C_0} e^{3x/2} \ ; \\ &= C e^{3x/2} \ . \end{aligned}$$

(Note that the second step assumed that f was not zero, so C does not have to be nonzero at the end because $f = 0$ also satisfies the differential equation.)

9. (20 points) Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

$$y'' - 2y' + 10y = xe^x + e^x \cos 3x$$

The auxiliary polynomial of the complementary equation is $r^2 - 2r + 10$, which has roots

$$r = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i \ ,$$

so

$$y_c = e^x (c_1 \cos 3x + c_2 \sin 3x) \ .$$

Therefore the trial solution is

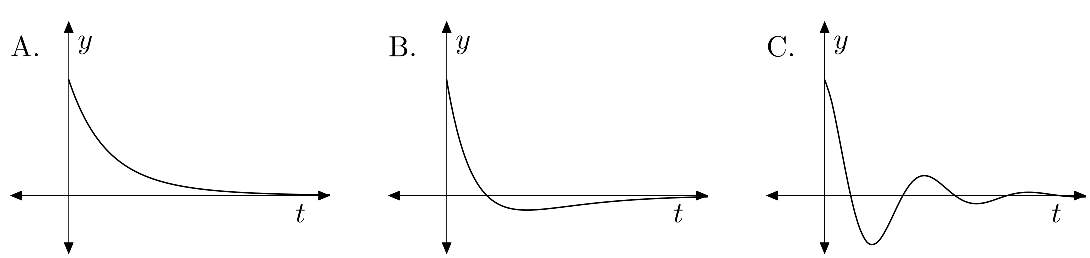
$$y_p = (Ax + B)e^x + Cxe^x \cos 3x + Dxe^x \sin 3x \ .$$

10. (18 points) The world's first (and possibly last) mountain unicycle has a suspension system that obeys the differential equation

$$my'' + 20y' + 3y = 0 \ .$$

For each of the following graphs, indicate which values of the mass m may exhibit such behavior. Use interval notation, and be careful about endpoints.

Note that a given value of m may correspond to more than one of the graphs (with different initial conditions). Assume that $m > 0$.



Graphs A and B are associated with critical damping and overdamping, so $c^2 - 4mk \ge 0$ (see Figure 4 on page 1158). Since we are given that $c = 20$ and $k = 3$, this means $20^2 - 12m \ge 0$, so $12m \le 400$, or $m \le 100/3$. In interval notation, the answer for both A and B is: $(0, 100/3]$.

Figure C (above) is associated with underdamping, so $c^2 - 4mk < 0$. Substituting $c = 20$ and $k = 3$ gives $m > 100/3$. In interval notation: $(100/3, \infty)$.

11. (25 points) Let

$$y = \sum_{n=0}^{\infty} c_n x^n$$

be a solution of the initial-value problem

$$y'' - x^2 y' + 6xy = 0\ , \qquad y(0) = 2\ , \qquad y'(0) = 3\ .$$

Write down all of the equations (recursion relations, etc.) that you would use to solve for the coefficients c_n. Do not solve these equations for the c_n.

Taking the termwise derivatives of the series for y gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}\ .$$

Substituting these and the series for y into the differential equation and simplifying gives

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} + 6x \sum_{n=0}^{\infty} c_n x^n = 0\ ;$$
$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n+1} + 6 \sum_{n=0}^{\infty} c_n x^{n+1} = 0\ .$$

We want to make the sums all involve x^n, so replace n with $n+2$ in the first sum, and with $n-1$ in the other two sums. This turns the equation into

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n + 6 \sum_{n=1}^{\infty} c_{n-1} x^n = 0\ .$$

It would be easiest if the sums all began at the same value of n, but they don't. However, when $n = 1$ the term in the middle series is zero, so we can change that sum to begin at $n = 1$ without affecting the equation. Also, split off the $n = 0$ term of the first sum. This gives

$$2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} (n-1) c_{n-1} x^n + 6 \sum_{n=1}^{\infty} c_{n-1} x^n = 0\ .$$

Now we can combine the sums:

$$2c_2 + \sum_{n=1}^{\infty} \Big((n+2)(n+1)c_{n+2}x^n - (n-1)c_{n-1}x^n + 6c_{n-1}x^n \Big) = 0 \ ;$$

$$2c_2 + \sum_{n=1}^{\infty} \Big((n+2)(n+1)c_{n+2} - ((n-1)-6)c_{n-1} \Big) x^n = 0 \ .$$

The only way a power series can be zero (as a function; i.e., for all x in its domain) is for all of its coefficients to be zero. This gives the equations

$$2c_2 = 0$$

(from the constant term) and

$$(n+2)(n+1)c_{n+2} - (n-7)c_{n-1} = 0 \qquad (n \geq 1)$$

(from the coefficient of x^n). The latter equation can be written as a recurrence:

$$c_{n+2} = \frac{(n-7)c_{n-1}}{(n+2)(n+1)} \ .$$

Now consider the initial conditions. We have $y = c_0 + c_1 x + c_2 x^2 + \dots$, so

$$2 = y(0) = c_0 \qquad \text{and} \qquad 3 = y'(0) = c_1 \ .$$

Therefore our equations are

$$\begin{aligned} c_0 &= 2 \ ; \\ c_1 &= 3 \ ; \\ 2c_2 &= 0 \ ; \qquad \text{and} \\ c_{n+2} &= \frac{(n-7)c_{n-1}}{(n+2)(n+1)} \ , \qquad n \geq 1 \ . \end{aligned}$$