MATH 1B Fall 2015
Section 8 Name: ______________________________

Power series is a series of the following form: $\sum_{n=1}^{\infty} a_n(x-a)^n$.

It is quite obvious to see that power series have the power of representing functions. i.e Given an value of x in the domain(interval of convergence), you can find one and only one value of the power series.

The two things you should notice are that:

- The term by term derivative has equal or smaller interval of convergence (might not be convergent at end points);
- The term by term integration has equal or larger interval of convergence (might not be divergent at end points).

From the power series expansion of a function, you can derive or integrate both sides to get sums of various series. Those questions can be pretty tricky.

Warm Up Questions (Find radius and interval of convergence):

1. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n(\ln n^2)}$

Comment: Just use ratio test to find the radius is 1 but the interval is $(-1, 1)$.

2. $\sum_{n=1}^{\infty} \frac{b}{2^n}(x-a)^n, \quad b > 0$

Comment: Radius is 2, the interval is $(a-2, a+2)$, and notice that b is not affecting anything.

Intermediate questions:) (Still radius and interval of convergence)

3. Let $f_n(x) = \frac{\sin(nx)}{n^2}$. Show that $\sum_{n=1}^{\infty} f_n(x)$ is defined for all x but $\sum_{n=1}^{\infty} f_n'(x)$ is divergent when $x = 2n\pi$.

Comment: $\sum f_n(x)$ is defined by the comparison test to $\frac{1}{n^2}$ but $\sum f_n'(x)$ is divergent because when you plug in $2n\pi$ you will get the harmonic series. This question is used to show that the interval of convergence of the derivative could be smaller than that of the original series.

4. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

Find the interval of convergence for f, f', f''.

Comment: The radius is 1, obtainable by ratio test, but the interval of convergence for f is $[-1, 1]$, for f' is $[-1, 1)$, for f'' is $(-1, 1)$.

5.

(a) Starting from the geometric series, $\sum_{n=0}^{\infty} x^n$, find the sum of the series :

$$\sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1.$$

(b) Find the sum of the following series:
$i) \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1, \qquad ii) \sum_{n=2}^{\infty} \frac{n^2}{2^n}$

Comment: Just take derivative of $\frac{1}{1-x}$ for part (a).
Part(b) is obtained by deriving $\sum x^n$ twice then multiplied by x, and the last one could be obtained by summing $\sum \frac{n^2}{x^n}$, which could be done by deriving and multiplying by x twice.

Challenges :D

6. Use the power series representation of $\tan^{-1} x$ to show:

$$\pi = 2\sqrt{3} \sum_{n=n}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Comment: Multiply $1/\sqrt{3}$ on both sides and the right hand side is just the power series expansion of $\arctan x$ at $1/\sqrt{3}$.

7. (This is pretty complicated)

(a) By completing squares, show

$$\int_0^{1/2} \frac{dx}{x^x - x + 1} = \frac{\pi}{3\sqrt{3}}.$$

(b) Use $x^3 + 1 = (x+1)(x^2 - x + 1)$ and the power series expansion of $x^3 + 1$ to show that:

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

Comment: This is too hard for most of you, so if you have done this or interested, just talk to me during office hours or ask me after section.