Linear Algebra and Differential Equations(Math 54): Lecture 5

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Recap

Last time, we learned perhaps the most important matrix operation: matrix multiplication.

Remember that we can write matrix multiplication as multiple matrix vector multiplications.

Also, it is important to remember the conditions under which $Ax = b$ always has solutions, and those $Ax = 0$ has only zero solution.

Identity matrix

The identity matrix I_n is an $n \times n$ matrix such that:

$$I_n(i,j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In other words, it is a matrix with ones on the diagonal and zeros elsewhere. And this is a matrix that commutes with all other matrices.If both are defined:

$$I_n A = A I_n = A$$

Elementary matrices

An elementary matrix is a matrix obtained by performing one elementary transformation on the identity matrix.

And an elementary row operation on a matrix A is equivalent to multiplying A by the corresponding elementary matrix on the left.

Matrix inversion

A matrix A is said to be invertible if there is a matrix B such that both AB and BA are equal to the identity matrix.

In this case, we say that B is the inverse of A, and if exists, the inverse is unique. $B = A^{-1}$

Is there any requirement on the shape of A?

Matrix inversion

If A is an invertible matrix, then solving the linear system can be done by finding the inverse of A.

$$Ax = b \Leftrightarrow b = A^{-1}x$$

In particular, the homogenous system Ax = 0 will have the unique solution zero.

Comparison

A has r rows and c columns

Ax = 0 implies x = 0	Ax = b has a solution for any b
There is a pivot in every column	There is a pivot in every row
Columns are linearly independent	Rows are linearly independent
Rows span $\mathbb{R}^c$	Columns span $\mathbb{R}^r$
Matrix function is injective	Matrix function is surjective
(can only happen if c<= r)	(can only happen if c >= r)

Matrix inversion

If A is invertible, then it is square and all the conditions on the previous slide are equivalent.

Conversely, if A is square, being invertible is equivalent to any one of the conditions on the previous slide.

Calculate the inverse

For an invertible matrix, there is a pivot in every row and every column. So row reducing the matrix to reduced echelon form will give you the identity matrix.

This leads to an algorithm to compute the inverse.

Calculate the inverse

Since elementary row operations are just multiplying by elementary matrices, reducing A to identity is equivalent to finding elementary matrices $E_1, E_2, \ldots, E_k$

$$E_k E_{k-1} \ldots E_1 A = I$$

$$E_k E_{k-1} \ldots E_1 = A^{-1}$$

Calculate the inverse

If we put A and I to side by side:

$$E_k E_{k-1} \dots E_1 [A|I] = [A^{-1}|I]$$

Since the elementary matrices are just row reductions, the actual algorithm is to put A and I side by side and then row reduce it.

Calculate the inverse

1. Form the matrix $[A|I]$
2. Row reduce it
3. If the form is $[I|A^{-1}]$
4. Otherwise A is not invertible

Example

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Another way to calculate

Finding the inverse of A is the same as finding a matrix B such that AB = I.

Denote the i th column of I as ei, then it is solving n linear systems:

$$Ab_i = e_i$$

$$B = [b_1|b_2|\dots|b_n]$$

Question: which one is a better algorithm?

2 by 2 matrix

Try to find the inverse for an arbitrary 2 by 2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$