Linear Algebra and Differential Equations(Math 54): Lecture 3

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Recap

Last time we looked at linear systems and their corresponding augmented matrices, as well as reducing them to reduced echelon form, which is unique. Then we can read off the solution from the reduced echelon form.

In addition, the number of pivot locations in the reduced echelon form also tells us the number of solutions to the system, zero, one or infinity.

Vectors

Vectors are just matrices. A column vector is a matrix with only one column.

Examples: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\mathbb{R}^n$ represents the set of all column vectors with n rows.

Vector arithmetic: Addition

Vectors can be added together if they have the same “length”.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$$

Vector arithmetic: Scaler multiplication

$$a * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$$

The constant a is called a scaler, because it scales entries in the vector.

Vector arithmetic

Vector addition and scaler multiplication satisfy the usual commutative, associative and distributive properties. Because these are satisfied by each entry of the vectors.

$$v + w = w + v$$

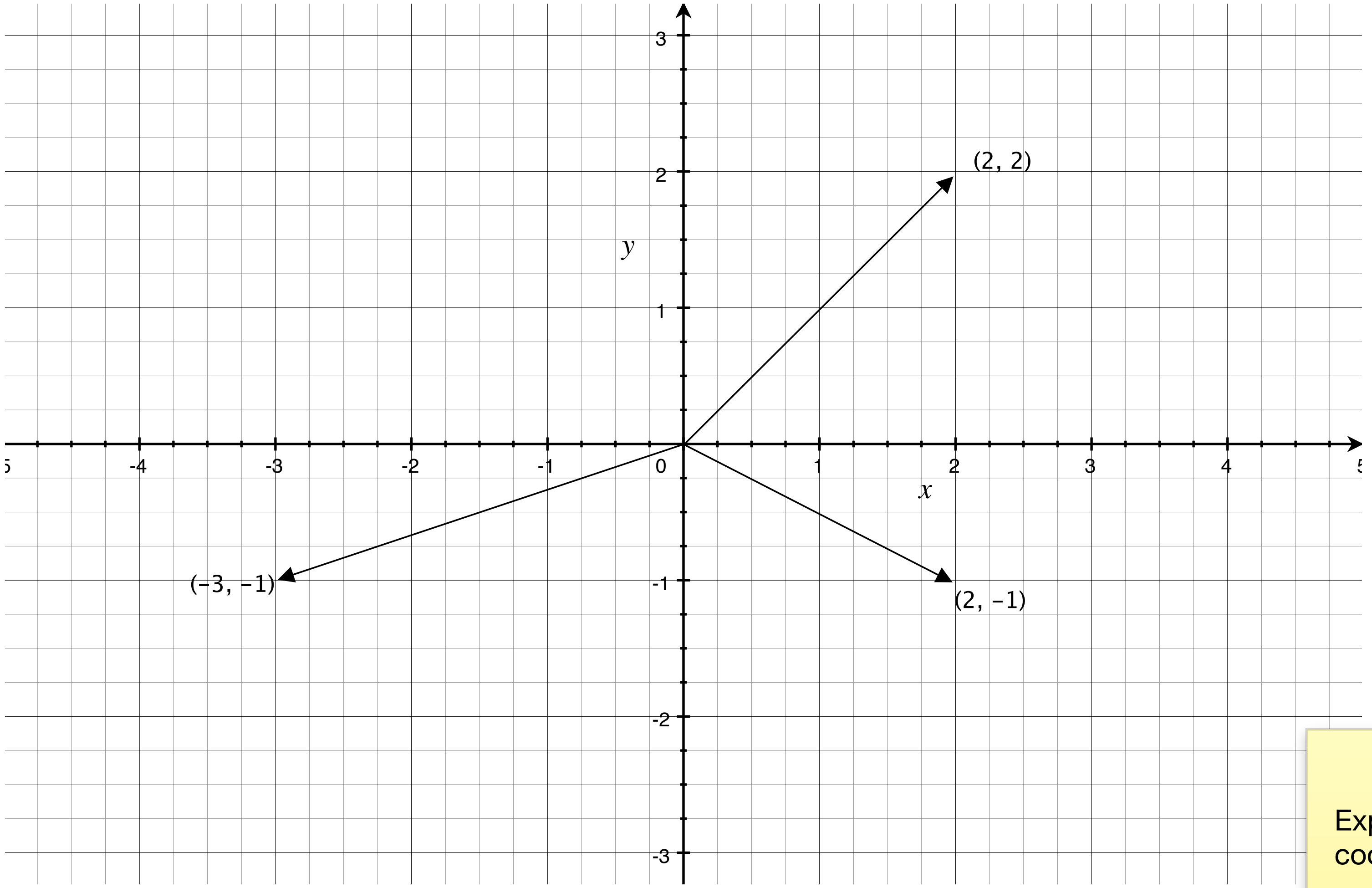
$$u + (v + w) = (u + v) + w$$

$$c(v + w) = cv + cw$$

$$(c + d)w = cw + dw$$

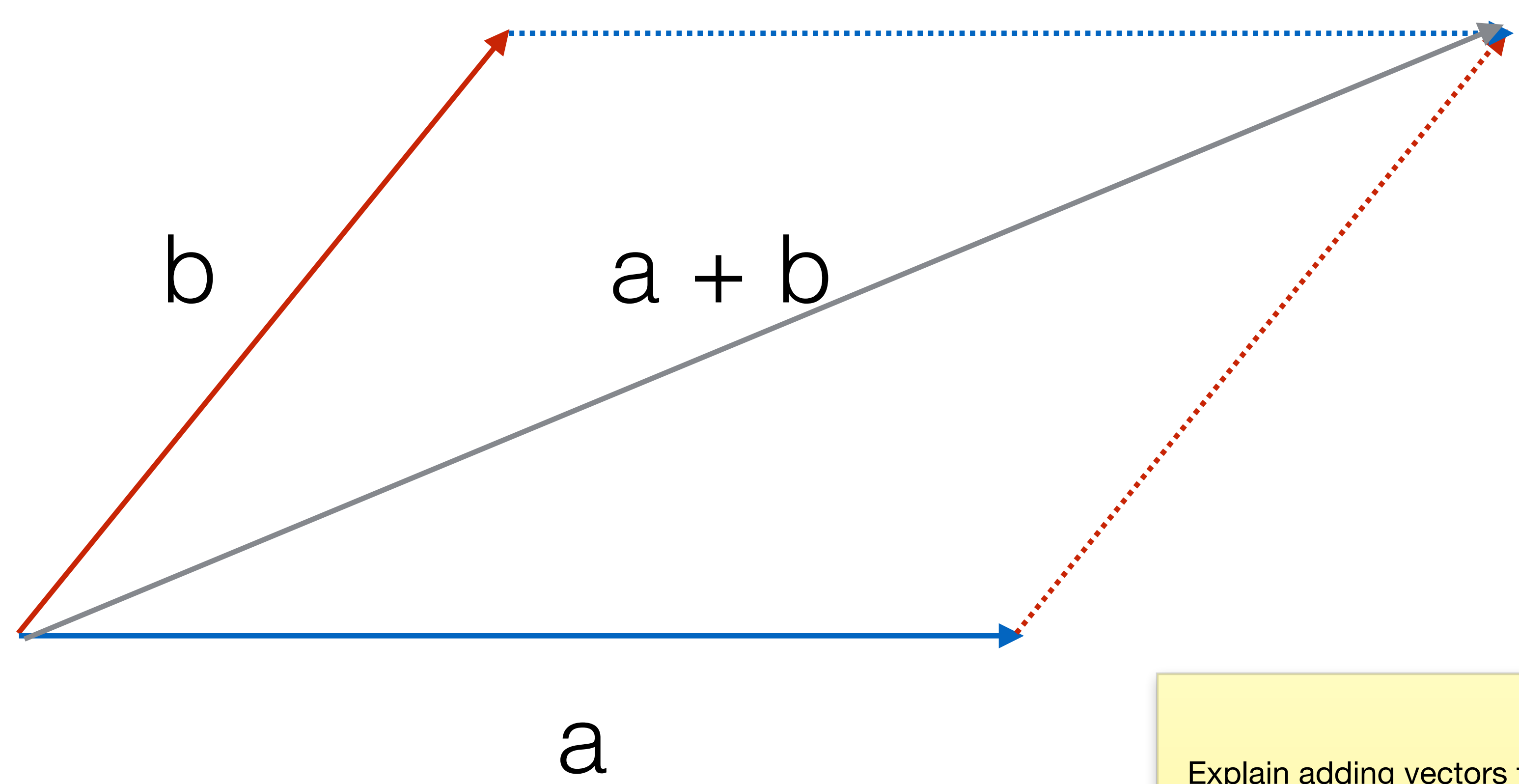
Geometric meanings

You can view vectors as arrows in coordinate systems.



Explain how to express vectors in 2d coordinates.

Geometric meanings



Explain adding vectors follow the parallelogram principal.

Do an example with coordinates on the board.

Geometric meanings

As we have seen, adding two vectors change the "direction" of the vector, however, scaler multiplication of vectors only change their "lengths".

Do an example of "scaling" on the board.

Linear combinations and span

Definition: Given a collection of vectors v1, v2, … vn in $\mathbb{R}^n$ and scalers, a1, a2, …, an in $\mathbb{R}$ an expression of the following form is said to be a linear combination of vectors v1, v2, …, vn:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

The set of all such expressions is called the span of v1, v2, … vn.

Example of span

What is the span of the following vectors:

ex1: $$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

ex2: $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

ex3: $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Linear span and linear equations

is $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ in the span of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

i.e are there x, y such that: $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

i.e are there x, y such that: $\begin{bmatrix} x + y \\ x + 3y \\ 2x + 7y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Linear span and linear equations

The linear system

$$\begin{aligned} 5x + 4y + 3z &= 2 \\ x + y + z &= 6 \\ x - y + z &= -3 \end{aligned}$$

is equivalent to the vector equation:

$$x \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

Linear span and linear equations

Solving the system

$$\begin{aligned} 5x + 4y + 3z &= 2 \\ x + y + z &= 6 \\ x - y + z &= -3 \end{aligned}$$

is the same as finding all the ways to express

$\begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

with x, y and z as the coefficients.

Matrix vector multiplication

As we see, a linear system can be written in an augmented matrix form, or a vector equation, but there is yet another way to write it.

$$\begin{bmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

Matrix vector multiplication

A column vector with c entries can be multiplied on the left by a matrix with c columns. If the matrix has r rows, then the resulting vector will have r entries as well.

matrix with r rows and c cols   vector with c rows

= vector with r rows

Matrix vector multiplication

The result is actually a linear combination of the columns of the matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} \\ a_{21} & a_{22} & \dots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1c}x_c \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2c}x_c \\ \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rc}x_c \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Matrix vector multiplication

The first linear algebra problem I mentioned in the first lecture

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} \\ a_{21} & a_{22} & \dots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rc} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_c \end{bmatrix}$$

Then it is the Ax = b we saw before!

Matrix vector multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} \\ a_{21} & a_{22} & \dots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{r1} \end{bmatrix} + \dots + x_c \begin{bmatrix} a_{1c} \\ a_{2c} \\ \vdots \\ a_{rc} \end{bmatrix}$$

Since matrix-vector multiplication is is a linear combination of columns, solving the linear system is equivalent to determining whether a vector is in the span of the columns of a matrix!