Linear Algebra and Differential Equations(Math 54): Lecture 2

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Recap

Last time we introduced linear equations, linear systems and transforming them by “elementary transformations”. This time we are going to study the formal ways of solving linear systems, namely the Gaussian Elimination.

Augmented Matrix

For simplicity, we can abbreviate the system. An augmented matrix is a m by (n+1) matrix where m = number of equations, n = number of variables.

$$x + 2y + 3z = 6$$
$$x - y + z = 1$$
$$2x + 3y + 4z = 9$$

$$\Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -1 & 1 & 1 \\ 2 & 3 & 4 & 9 \end{array}\right]$$

Augmented Matrix

Through a series of "elementary transformations" we can reduce it to:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Gaussian Elimination

In each step, we reduce the number of non zero entries in the system, by one of the three "elementary transformations". This procedure is called "Gaussian Elimination".

Echelon Form

It was first used to describe a form of line up in air force.

Echelon Form

A rectangular matrix is in echelon form if every nonzero entry of every row (except for the first row) is strictly to the right of a nonzero entry of the previous row. In other words, the matrix is trapezoidal.

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Echelon Form

A matrix is in reduced echelon form if it is in echelon form and the leading nonzero entry of every row is 1 and all other entries in that column are zeros.

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot position

Definition: A pivot location in the matrix A is a location corresponding to a leading 1 in the reduced echelon form of A.

Can you spot pivot locations in the following matrix?

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Row Reduction Algorithm

1. Choose a pivot location in the leftmost column;
2. Bring it to the first row by switching rows;
3. Divide the whole row by the value of the pivot location;
4. Add multiples of this row to other rows to zero out all the nonzero entries in the column;
5. Repeat the process for other columns until the matrix in in reduced echelon form.

Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The result is unique

Although we make choices of pivot positions, the resulting reduced echelon form will always be the same.(It is unique!)

However if we don't bother reducing it all the way but stop at echelon form, the result is not unique but the shape is.

The shape is determined by the number of pivot positions!

The result is unique

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. Is it in echelon form?
2. Is it in reduced echelon form?
3. Where are the pivot positions?

Solving the linear systems

First, check for rows like this:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

What does it mean if you find a row like this?

Solving the linear systems

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

First notice that the first column is full of zeros. This means that x1 is a free variable. We will set x1 = s.

Then notice that there are only four equations in five variables, which means there is at least another free variable. We will set x5 = t.

Solving the linear systems

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then we will use back substitution to express other variables in terms of x1 and x5.

x4 = 1, x3 = 1 - x5, x2 = 3 - x5

$$\{(s, 3-t, 1-t, 1, t), \forall s, t\}$$

Number of solutions: revisit

With a system with augmented matrix in reduced echelon form.

There is no solution if and only if there is a pivot position in the last column.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Number of solutions: revisit

With a system with augmented matrix in reduced echelon form.

There is a unique solution if and only if there is no pivot in the last column and there is a pivot in every other column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Number of solutions: revisit

With a system with augmented matrix in reduced echelon form.

There are infinitely many solutions if and only if there is no pivot in the last column and there is at least another column with no pivot.

$$\left[\begin{array}{ccccc|c} 1* & 0 & 0 & * & * & \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$