Linear Algebra and Differential Equations(Math 54): Lecture 7

Ruochen Liang

June 28, 2016

Recap

Last time we talked about determinant and one way to compute it: the minor expansion. Also, we talked about how to compute the inverse of a matrix using determinant.

Invertibility

Recall that if a square matrix, the following conditions are equivalent:

1. The matrix is invertible
2. The rows are linearly independent
3. The columns are linearly independent
4. The function defined is injective
5. The function defined is surjective

2 by 2 invertible matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $a \neq 0$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$

Then the matrix is invertible if and only if the rows are linearly independent, in other words, if and only if

$$ad - bc \neq 0$$

If $a = 0$

then by inspection the matrix is invertible if and only if b, c are not both zero.

Example

$$\begin{bmatrix} 3 & 5 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 7 & 3 & 2 & 1 \\ 5 & 2 & 9 & 3 & 2 & 4 \\ 1 & 6 & 3 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 6 & 1 & 5 & 2 \\ 1 & 2 & 6 & 3 & 3 & 6 \end{bmatrix}$$

Properties of determinant

1. determinant of triangular matrix is the product of diagonal entries;
2. Scaling a row/column scales the determinant by the same constant;
3. Linearity in rows(we will see an example)
4. Any matrix with repeated row/column has zero determinant.(need to prove)
5. Switching two rows flips the sign of determinant.
6. Adding a multiple of one row to another does not change the determinant.

An example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a + a' & b + b' & c + c' \\ d & e & f \\ g & h & i \end{bmatrix}$$

Elementary matrices

Can we compute the determinants of elementary matrices?

$$det(E_m) = 1$$
$$det(E_\lambda) = \lambda$$
$$det(E_{sw}) = -1$$

Elementary matrices

For elementary matrix E:

$$det(EM) = det(E) * det(M)$$

This follows from the previous side and that multiplying elementary matrices on the left is equivalent to doing elementary row operations.

Determinants and invertibility

Theorem: A matrix is invertible if and only if it has a nonzero determinant.

Proof: If it is invertible, then it has a pivot position in every column, so it can be reduced to identity matrix, which means that the determinant is nonzero.

On the other hand, if the matrix has a nonzero determinant, it's reduced echelon form has a pivot location in every columns, meaning that the matrix is invertible.

Multiplication of determinants

Theorem: $det(AB) = det(A) * det(B)$

If either A or B is not invertible, then both sides are zero. Otherwise, both A and B can be reduced to identity matrix.

Then you can decompose both A and B into products of elementary matrices and then use the multiplicity of elementary matrices.