Linear Algebra and Differential Equations(Math 54): Lecture 6

Ruochen Liang

June 27, 2016

Recap

In the last week, we studied basics about linear systems and different ways to interpret them. Also, we learned some matrix operations such as matrix vector multiplication and matrix matrix multiplication.

In the end of Friday's lecture we talked about invertible matrix, in the 2 by 2 example, we see that it is invertible as long as the "determinant" is nonzero.

Determinant

We studied 2 by 2 determinant, below is the example for a 3 by 3 case.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\Delta = a * (ei - hf) - b * (di - fg) + c * (dh - eg)$$

Determinant

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The calculation is recursive

Example

Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix}$$

Determinants

In the previous slides, A_{ij}

represents the matrix obtained by deleting the i th row and j th columns of matrix A, it is called a minor.

Determinant can be calculated through minor expansion.

$$det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} det(A_{ik})$$

$$det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ki} det(A_{ki})$$

Determinants

I will use the absolute sign to represent determinants from now on:

$$det(A) = |A|$$

Example

Now compute the determinant of the previous matrix by minor expansion along the second row.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix}$$

Example

Compute the determinant by minor expansion(your choice) of the following matrix:

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Another formula for inverse

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} |A_{11}| & -|A_{21}| & |A_{31}| \\ -|A_{21}| & |A_{22}| & -|A_{32}| \\ |A_{13}| & -|A_{23}| & |A_{33}| \end{bmatrix}$$

First check the off-diagonal entries of A*Adj(A)

$$a_{21}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$$

Expand it out by definition we have:

$$a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Another formula for inverse

This is the minor expansion for the following determinant.

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The matrix has repeated rows, so the determinant is zero.

This is true for every off-diagonal entry.

Another formula for inverse

Actually, one can check that:

$$A * Adj(A) = det(A)I$$

$$A^{-1} = Adj(A)/det(A)$$

$$Adj(A)(i,j) = (-1)^{i+j} det(A_{ji})$$

So the i th row j th entry is the determinant of the matrix by removing the j th row and the i th col of A times a sign constant.

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determine Adj(A) and verify the formula on the previous slide.