Linear Algebra and Differential Equations(Math 54): Lecture 4

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Recap

Last time we learned linear combination and linear span as well as how solving a linear system is related to finding a vector in the span of the columns of a matrix.

In addition, we talked about matrix vector multiplication, which is a foundation of later matrix algebra, even matrix algorithms. Today we are going to dig deeper into matrix operations.

Matrix as function

As from yesterday, if A is a r by c matrix, and x is a column vector of length c, then Ax is a column vector of length r.

In this way, the matrix A has defined a function, also called A

$$
\begin{aligned}
A : \mathbb{R}^c &\to \mathbb{R}^r \\
x &\to Ax
\end{aligned}
$$

Thus, solving Ax = b is equivalent to finding the pre-image of b of the function A.

Linearity of matrix function

One important property of functions defined by matrices is that they are linear:

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$
$$A(c\mathbf{x}) = cA\mathbf{x}$$

Now an interesting question: is every linear function from $\mathbb{R}^r$ to $\mathbb{R}^c$ a matrix function?

Linear systems again

There are several ways we can interpret linear systems:

1. Intersection of lines and planes;
2. Writing one vector as a linear combination of others
3. Find the pre-image of a vector under a matrix function

Now lets see some qualitative properties of the solution sets.

Homogenous and inhomogeneous systems

If Ax1 = b, and Ax2 = b, then:

$$A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{b}\text{-}\mathbf{b} = 0$$

This means that the difference between two solutions to the inhomogeneous system Ax = b is a solution to the homogenous system Ax = 0.

Another way to say this is that given a solution x0 to the inhomogeneous system Ax = b, every solution to this system can be written as x0 + z where z is a solution to Ax = 0. (you can prove it)

Example

Find all solutions to the equation $x + y = 1$, and all the solutions to $x + y = 0$, draw the solutions, how do they compare?

Homogenous system and linearity

If Ax1 = 0 and Ax2 = 0, for any scalers c1, c2

$$A(c_1\mathbf{x}_2 + c_2\mathbf{x}_2) = c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 = 0.$$

Any linear combination of solutions to a homogenous system is still a solution, which means the set of solutions is a span. To get this span, the first step is row reduction

Homogenous system and linearity

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Has the following augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Homogenous system and linearity

We solve the system with back substitution and introduce free variables for columns without pivots:

$$\{(s, -t, -t, 0, t), \forall s, t \in \mathbb{R}\}$$

Another way to write the set of solutions will be:

$$s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \text{linear span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \}$$

Inhomogeneous systems

The solution set to equation Ax = b, if nonempty, is a translation of the solution set to the corresponding homogenous system Ax = 0.

In fact if Ax0 = 0, then for any x such that Ax = 0, A(x0 + x) = b

Symbolically, $x_0 + \{\text{solution to } Ax = 0\} = \{\text{solution to } Ax = b\}$

Can you prove this?

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Linear dependence and independence

Definition: A collection of vectors v1, v2, …, vn is said to be linearly dependent if one of the vectors can be written as a linear combination of others. If not, the collection is called linearly independent.

Equivalently, the collection of vectors are linearly dependent if there exist scalers a1, a2, …, an not all zeros such that:

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

Any collection including the zero vector is linearly dependent.

Row reduction and linear independence

The rows of a reduced echelon matrix are linearly independent if and only if there is no zero row.

So to check if a set of vectors are linearly independent, put them as rows in a matrix, row reduce the matrix, then check for a zero row.

Is this true for a matrix in echelon form?

When does Ax=b always have solutions?

In terms of rows: when the rows are linearly independent.

Recall that this means there is no nonzero linear combination of the rows of A that is zero.

In fact, if there is such a linear combination,then we can find a b such that a row of the following form after row reducing the augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & 1 \end{array}\right]$$

When does $Ax=b$ always have solutions?

In terms of columns: when the columns of A span the whole space.

This is because solving $Ax = b$ is equivalent to expressing b as a linear combination of the columns of A.

Is this equivalent to the previous slide? Why or why not?

When does Ax=b always have solutions?

In terms of the matrix function:

The function defined by : $A \in \mathbb{R}^{m \times n}$

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

is surjective. In other words, every vector in $\mathbb{R}^m$ is an image under the function A.

Zero solution

The zero vector is always a solution to a homogenous system.

That is, $Ax = 0$ is solved by $x = 0$.

When does Ax=0 have only zero solution?

In terms of columns: when the columns of A are linearly independent. Why?

A solution to Ax=0 is a way to write 0 as a linear combination of columns of A.

When does $Ax=0$ have only zero solution?

Since zero is a solution, it is the only solution if and only if $Ax = 0$ has a unique solution.

Recall from lecture 2, this means that there is a pivot location in every column in the reduced echelon form of the augmented matrix.

Question: if this is true, what is the span of the rows of A?

When does $Ax=0$ have only zero solution?

In terms of the matrix function: when the function A is injective. That is, only zero is mapped to zero.

Matrix algebra

Last time we learned vector arithmetic and matrix vector multiplication. This time we will proceed in matrix arithmetics.

Matrix addition is entry-wise addition, hence two matrices can be added only if they are of the same size.

Scaler multiplication of matrix is also entry-wise multiplication.

Matrix transpose

Matrix entries: the entry on the i th row j th column of matrix A is denoted as A_{ij} or $A(i,j)$

The transpose of matrix A is a new matrix, denoted as A^T such that:

$$A^T(i,j) = A(j,i)$$

In other words, transposition of a matrix is flipping the rows and the columns.

Matrix multiplication

Given two matrices A, B, they can be multiplied if number of columns of A equals number of rows of B.

The formula is given by a sum of matrix-vector multiplications:

$$AB(i,j) = A(i,1)B(1,j) + A(i,2)B(2,j) + \ldots + A(i,n)B(n,j)$$

where n is the number of columns of A, or the number of rows of B.

The result AB has the same number of rows a A and the same number of columns as B.

Example

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = ?$$

Matrix multiplication

Another way to see matrix multiplication is as multiple matrix vector multiplications.

Write the matrix B as one row of columns:

$$B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array}\right]$$

Then: $$AB = \left[\begin{array}{c|c|c|c} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_n \end{array}\right]$$

That is, the i th column of AB is obtained by multiplying A and the i th column of B.

Matrix multiplication

In general, matrix multiplication is not commutative.

In fact, one of AB or BA may not even make sense!
i.e size mismatch.

Question: can you find the set of matrices that commute with all other matrices in terms of multiplication.