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Math 1B

Solutions to Review Problems

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1. (a). Substitute $u = \sin x$:

$$\int \sin^4 x \cos^3 x \, dx = \int u^4(1-u^2)\, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(b). Integrate by parts with

$$\begin{array}{rr} u = \arcsin x & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v = x \end{array}$$

The indefinite integral is

$$\begin{aligned} \int \arcsin x \, dx &= x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2}\int \frac{du}{\sqrt{u}} \\ &= x \arcsin x + \sqrt{u} + C = x \arcsin x + \sqrt{1-x^2} + C, \end{aligned}$$

so

$$\int_0^1 \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2}\,\Big|_0^1 = \frac{\pi}{2} - 1.$$

[**Note:** if we had computed the definite integral directly, it would have involved an improper integral after the integration by parts.]

(c). Integrate by parts with

$$\begin{array}{rr} u = \sin 2x & dv = e^x\,dx \\ du = 2\cos 2x\,dx & v = e^x \end{array}$$

Then

$$\int e^x \sin 2x \, dx = e^x \sin 2x - 2\int e^x \cos 2x \, dx.$$

Integrate by parts again with

$$\begin{array}{rr} u = \cos 2x & dv = e^x\,dx \\ du = -2\sin 2x\,dx & v = e^x\ : \end{array}$$

$$\begin{aligned} \int e^x \sin 2x \, dx &= e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x \, dx\right) \\ 5\int e^x \sin 2x \, dx &= e^x \sin 2x - 2e^x \cos 2x + C \\ \int e^x \sin 2x \, dx &= \frac{e^x}{5}\big(\sin 2x - 2\cos 2x\big) + C. \end{aligned}$$

(d). Let $u = 4 - x^2$; then $du = -2x\,dx$ and

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_4^0 (4-u)\sqrt{u}\,du = -\frac{1}{2}\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_4^0 = -\frac{32}{3} + \frac{32}{5} = \frac{64}{15}.$$

(e). Integrate by parts with

$$\begin{array}{cc} u = x & dv = \cos 2x\,dx \\ du = dx & v = \frac{1}{2}\sin 2x \end{array}$$

to get

$$\begin{aligned} \int x\sin^2 x\,dx &= \frac{1}{2}\int x(1-\cos 2x)\,dx = \frac{x^2}{4} - \frac{1}{2}\int x\cos 2x\,dx \\ &= \frac{x^2}{4} - \frac{1}{2}\left(\frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right) \\ &= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C. \end{aligned}$$

(f). Substitute $u = \tan^2\theta$:

$$\int_0^{\pi/4} \tan^2\theta\sec^4\theta\;d\theta = \int_0^1 u^2(u^2+1)\,du = \left(\frac{u^5}{5} + \frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}.$$

(g). Let $x = \tan\theta$; then $dx = \sec^2\theta$:

$$\begin{aligned} \int \frac{dx}{(1+x^2)^2} &= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\;d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)\;d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}\arctan x + \frac{1}{4}\sin(2\arctan x) + C \\ &= \frac{1}{2}\big(\arctan x + \sin(\arctan x)\cos(\arctan x)\big) + C \\ &= \frac{1}{2}\left(\arctan x + \frac{\tan(\arctan x)}{\sec^2(\arctan x)}\right) + C \\ &= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C. \end{aligned}$$

It's OK if you got as far as $\frac{1}{2}\arctan x + \frac{1}{4}\sin(2\arctan x) + C$.

(h). The denominator factors as $(x+1)(x^2-4x+7)$. So we look for partial fractions:

$$\begin{aligned}\frac{x^2+7x-6}{(x+1)(x^2-4x+7)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-4x+7};\\ x^2+7x-6 &= A(x^2-4x+7) + (Bx+C)(x+1).\end{aligned}$$

Successively setting x equal to -1, 0, and 1 gives the equations

$$\begin{aligned}-12 &= 12A,\\ -6 &= 7A + C, \text{ and}\\ 2 &= 4A + 2(B+C).\end{aligned}$$

Therefore we can read off: $A = -1$, $C = 1$, and $B = 2$. Thus

$$\begin{aligned}&\int \frac{x^2+7x-6}{(x+1)(x^2-4x+7)}\,dx\\ &\quad = -\int \frac{dx}{x+1} + \int \frac{2x+1}{x^2-4x+7}\,dx = -\ln|x+1| + \int \frac{2(x-2)+5}{(x-2)^2+3}\,dx\\ &\quad = -\ln|x+1| + \ln|x^2-4x+7| + \frac{5}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + C.\end{aligned}$$

(i). Let $u = 1 + \sqrt{x}$; then $x = (u-1)^2$; $dx = 2(u-1)\,du$ and the integral is

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = -\frac{2}{u} + \frac{1}{u^2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C.$$

2. (a). $\displaystyle\int_0^1 \ln x\,dx = -1$. See Example 8 on page 525.

(b). First use partial fractions to find the indefinite integral:

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{A}{x-1} + \frac{B}{x+1}\\ 1 &= A(x+1) + B(x-1).\end{aligned}$$

Letting $x = 1$ gives $A = 1/2$; $x = -1$ gives $B = -1/2$. Therefore

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

and therefore

$$\int_2^\infty \frac{dx}{x^2-1} = \lim_{t\to\infty} \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \Big|_2^t = \frac{1}{2}\lim_{t\to\infty} \ln\left|\frac{t+1}{t-1}\right| - \frac{1}{2}\ln\frac{1}{3} = -\frac{1}{2}\ln 3.$$

(c). $\displaystyle\int_1^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\frac{1}{3}\ln(x^3+2)\Big|_1^t = \infty$. Therefore the integral diverges. You can also see this more quickly by using the Limit Comparison Test for improper integrals to compare the integral with the divergent integral of $1/x$.

3. (a).

$$\begin{aligned}\int \tan^n x\,dx &= \int \tan^{n-2} x(\sec^2 x - 1)\,dx = \int \tan^{n-2} x\sec^2 x\,dx - \int \tan^{n-2} x\,dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx\ .\end{aligned}$$

(b).

$$\begin{aligned}\int \tan^6 x\,dx &= \frac{\tan^5 x}{5} - \int \tan^4 x\,dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int \tan^2 x\,dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - \int dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C.\end{aligned}$$

4.

$$\begin{aligned}\int \frac{\sin^4 x}{\cos^3 x}\,dx &= \int \frac{(1-\cos^2 x)^2}{\cos^3 x}\,dx \\ &= \int \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^3 x}\,dx \\ &= \int \sec^3 x\,dx - 2\int \sec x\,dx + \int \cos x\,dx \\ &= \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) - 2\ln|\sec x + \tan x| + \sin x + C \\ &= \frac{1}{2}\sec x\tan x - \frac{3}{2}\ln|\sec x + \tan x| + \sin x + C.\end{aligned}$$

Here the fourth step used Example 8 on pages 475–476 and Formula 1 on page 475.

5. $\displaystyle\int_0^4 \sqrt{1+\left(\tfrac{3}{2}\sqrt{x}\right)^2}\,dx = \int_0^4 \sqrt{1+\tfrac{9}{4}x}\,dx = \frac{4}{9}\cdot\frac{2}{3}\left(1+\tfrac{9}{4}x\right)^{3/2}\Big|_0^4 = \frac{8}{27}(10\sqrt{10}-1).$

6. (a). The area is

$$\int_0^1 2\pi y\,ds = 2\pi\int_0^1 \sqrt{x}\sqrt{1+\left(\tfrac{1}{2\sqrt{x}}\right)^2}\,dx = 2\pi\int_0^1 \sqrt{x+\tfrac{1}{4}}\,dx$$
$$= \frac{4\pi}{3}\left(x+\frac{1}{4}\right)^{3/2}\Big|_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$$

(b). The area is

$$\int_0^2 2\pi x\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^2 x\sqrt{1+9x^4}\,dx.$$

To evaluate this, substitute $u = x^2$, $du = 2x\,dx$ and use Formula 21 in the table of integrals to get

$$\pi\int_0^4 \sqrt{1+9u^2}\,du = 3\pi\int_0^4 \sqrt{1/9+u^2}\,du$$
$$= \frac{3\pi u}{2}\sqrt{1/9+u^2} + \frac{3\pi}{18}\ln|u+\sqrt{1/9+u^2}|\;\Big|_0^4$$
$$= 2\pi\sqrt{145} + \frac{\pi}{6}\ln|12+\sqrt{145}|.$$

(Formula 21 is obtained by a trigonometric substitution $u = \frac{1}{3}\tan\theta$.)

7. The curve can be written $y^2 = x^3(1-x)$, and it is evident that the right-hand side is nonnegative only if $0 \le x \le 1$. Therefore the region can be represented by $-\sqrt{x^3-x^4} \le y \le \sqrt{x^3-x^4}$, $0 \le x \le 1$.

It is symmetric about the x-axis, so by the Symmetry Principle, $\bar{y} = 0$—no computation is needed.

To compute $\bar{x}$, we first compute the area. This involves completing the square:

$x - x^2 = 1/4 - (x - 1/2)^2$ and substituting $u = x - 1/2$:

$$
\begin{aligned}
A &= 2\int_0^1 \sqrt{x^3 - x^4}\,dx \\
&= 2\int_0^1 x\sqrt{x - x^2}\,dx \\
&= 2\int_{-1/2}^{1/2} \left(u + \frac{1}{2}\right)\sqrt{\frac{1}{4} - u^2}\,du \\
&= 2\int_{-1/2}^{1/2} u\sqrt{\frac{1}{4} - u^2}\,du + \int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - u^2}\,du \\
&= 0 + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}\,.
\end{aligned}
$$

In the last step, the first integral is zero because the integrand is an odd function, and the second integral is the area of a semicircle of radius $1/2$.

Then, using the same steps as before, and also Formula 31 in the back of the book (or a trigonometric substitution $u = \frac{1}{2}\sin\theta$),

$$
\begin{aligned}
\overline{x} &= \frac{2}{A}\int_0^1 x\sqrt{x^3 - x^4}\,dx \\
&= \frac{16}{\pi}\int_0^1 x^2\sqrt{x - x^2}\,dx \\
&= \frac{16}{\pi}\int_{-1/2}^{1/2} \left(u + \frac{1}{2}\right)^2\sqrt{\frac{1}{4} - u^2}\,du \\
&= \frac{16}{\pi}\int_{-1/2}^{1/2} u^2\sqrt{\frac{1}{4} - u^2}\,du + \frac{16}{\pi}\int_{-1/2}^{1/2} u\sqrt{\frac{1}{4} - u^2}\,du + \frac{4}{\pi}\int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - u^2}\,du \\
&= \frac{16}{\pi}\left(\frac{u}{8}\left(2u^2 - \frac{1}{4}\right)\sqrt{\frac{1}{4} - u^2} + \frac{1}{16\cdot 8}\sin^{-1} 2u\right)\Bigg|_{-1/2}^{1/2} + 0 + \frac{4}{\pi}\cdot\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 \\
&= \frac{16}{\pi}\cdot\frac{\pi}{16\cdot 8} + \frac{1}{2} = \frac{5}{8}\,.
\end{aligned}
$$

Therefore the centroid is $(5/8, 0)$.

8. (a).

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{(n+1)^5 - (n-1)^5}{n^4} \\
&= \lim_{n\to\infty} \frac{(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1)}{n^4} \\
&= \lim_{n\to\infty} \frac{2(5n^4 + 10n^2 + 1)}{n^4} = 10.
\end{aligned}
$$

(b). As in the Ratio Test, we have

$$\frac{((n+1)!)^2}{(2(n+1))!} \Big/ \frac{(n!)^2}{(2n)!} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)} = \frac{1}{4}.$$

Therefore the corresponding *series* converges, so the terms of that series have to go to zero. This implies that the limit in the problem is zero.

(c). By l'Hôpital's rule,

$$\lim_{n\to\infty}(1+n)^{1/n} = \lim_{x\to\infty}(1+x)^{1/x} = \lim_{x\to\infty}\exp\left(\frac{\ln(1+x)}{x}\right) = \lim_{x\to\infty}\exp\left(\frac{\frac{1}{1+x}}{1}\right) = 1.$$

9. (a). Use partial fractions:

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3};$$
$$1 = A(n+3) + Bn.$$

When $n = 0$ this gives $A = 1/3$; when $n = -3$ this gives $B = -1/3$. So

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+3)} &= \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)\\ &= \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{3}\sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)\\ &= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}.\end{aligned}$$

(b). $$\sum_{n=1}^{\infty}\frac{4^{n+1}}{5^n} = \sum_{n=1}^{\infty}\frac{16}{5}\left(\frac{4}{5}\right)^{n-1} = \frac{16/5}{1-4/5} = 16\,.$$

10. (a). It is easy to see that the function $\frac{e^{1/\ln x}}{x(\ln x)^2}$ is positive, and it is decreasing because its derivative is $-\frac{(\ln x+1)^2}{x^2(\ln x)^4}e^{1/\ln x} < 0$. Therefore the Integral Test can be applied, and the series converges because

$$\int_2^{\infty}\frac{e^{1/\ln x}}{x(\ln x)^2}\,dx = \lim_{t\to\infty} -e^{1/\ln x}\Big|_2^t = e^{1/\ln 2} - 1 < \infty.$$

(b). Divergent by the Limit Comparison Test, comparing with $\sum \frac{1}{n}$.

(c). $e^{1/x} - 1 = \left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + \dots\right) - 1 = \frac{1}{x} + \dots$ so the series diverges by the Limit Comparison Test, comparing it with $\sum \frac{1}{n}$.

(d). Convergent by the Alternating Series Test.

(e). Since $\cos(n\pi) = (-1)^n$, this series is $\sum \frac{1}{\sqrt{n}}$, which diverges. (It is *not* an alternating series!)

(f). This integral converges absolutely (and is therefore convergent) since

$$\left|(-1)^n \frac{1+\cos n}{n^{3/2}}\right| = \frac{1+\cos n}{n^{3/2}} \le \frac{2}{n^{3/2}}$$

and since $\sum \frac{1}{n^{3/2}}$ converges.

(g). Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\arctan n)^n}} = \lim_{n\to\infty} \frac{1}{\arctan n} = \frac{2}{\pi} < 1,$$

so the series converges. (Or, use the Limit Comparison Test and compare it with $\sum \frac{1}{(\pi/2)^n}$. It's the same thing, really.)

(h). This series diverges, because its terms don't $\to 0$.

11. From the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ we have $\frac{1}{n} - \sin\frac{1}{n} = \frac{1}{3!}\left(\frac{1}{n}\right)^3 - \frac{1}{5!}\left(\frac{1}{n}\right)^5 + \dots$, which grows like $\left(\frac{1}{n}\right)^3$. So this suggests that we use the Limit Comparison Test to compare the series with $\left(\frac{1}{n^3}\right)^p$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left(\frac{1}{n} - \sin\frac{1}{n}\right)^p}{\left(\frac{1}{n^3}\right)^p} &= \lim_{n\to\infty} \left(n^3\left(\frac{1}{3!}\left(\frac{1}{n}\right)^3 - \frac{1}{5!}\left(\frac{1}{n}\right)^5 + \dots\right)\right)^p \\
&= \lim_{n\to\infty} \left(\frac{1}{3!} - \frac{1}{5!}\left(\frac{1}{n}\right)^2 + \dots\right)^p \\
&= \left(\frac{1}{3!}\right)^p
\end{aligned}$$

This is neither 0 nor ∞, so the series $\sum \left(\frac{1}{n} - \sin\frac{1}{n}\right)^p$ converges if and only if the series $\sum \frac{1}{n^{3p}}$ converges. This happens if and only if $3p > 1$; *i.e.*, $p > 1/3$.

12. (a). By the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(n+1)(x+10)^{n+1}}{((n+1)^2+1)4^{n+1}}\right| \Big/ \left|\frac{n(x+10)^n}{(n^2+1)4^n}\right| = \frac{|x+10|}{4},$$

so the radius of convergence is 4 (centered at $x=-10$). At $x=-14$ the series converges by the Alternating Series Test. At $x=-6$ the series diverges by the Limit Comparison Test, comparing with $\sum\frac{1}{n}$. So the interval of convergence is $[-14,-6)$.

(b). Apply the Ratio Test:

$$\lim_{n\to\infty}\frac{((n+1)!)^2}{(2n+2)!}|x|^{n+1} \Big/ \frac{(n!)^2}{(2n)!}|x|^n = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}|x| = \frac{|x|}{4}$$

so the radius of convergence is 4. At $x=\pm 4$, the coefficients are getting *larger* because the ratio is $\frac{4(n+1)^2}{(2n+1)(2n+2)} = \frac{2n+2}{2n+1} > 1$. So it diverges at those two points and the interval of convergence is therefore $(-4,4)$.

(c). Use the Root Test:

$$\lim_{n\to\infty} n^{-\ln n/n}|x| = \lim_{n\to\infty} e^{-(\ln n)^2/n}|x| = |x|,$$

so the radius of convergence is 1. At $x=1$ this series converges by the Limit Comparison Test with $\sum\frac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{1}{n^{\ln n}} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} n^{2-\ln n} = 0\,.$$

(This uses Exercise 40a on page 727.) The series at $x=-1$ also converges, because it converges absolutely. The interval of convergence is therefore $[-1,1]$.

13. (a). Use the geometric series and integrate termwise:

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{4n};$$

$$\int\frac{dx}{1+x^4} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{4n+1} + C.$$

(b). Use the binomial series and replace x with x^2:

$$\sqrt[3]{1+x} = (1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)x^2}{2!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3}{3!} + \ldots$$

$$= 1 + \frac{x}{3} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{2\cdot 5\cdots(3n-4)}{3^n(n!)}x^n$$

$$\sqrt[3]{1+x^2} = 1 + \frac{x^2}{3} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{2\cdot 5\cdots(3n-4)}{3^n(n!)}x^{2n}$$

14. Yes. First the easy part. $\lim_{t\to 0^+}\int_t^\pi \frac{\sin x}{x}\,dx$ exists because $\frac{\sin x}{x}$ extends to a continuous function at $x=0$.

Now we check the other part. Let $[\![x]\!]$ be the greatest integer function, which is defined on page 105. Then

$$\begin{aligned}\lim_{t\to\infty}\int_\pi^t \frac{\sin x}{x}\,dx &= \lim_{t\to\infty}\left(\sum_{n=2}^{[\![t/\pi]\!]}\int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x}\,dx + \int_{[\![t/\pi]\!]\pi}^{t}\frac{\sin x}{x}\,dx\right)\\ &= \lim_{t\to\infty}\left(\sum_{n=2}^{[\![t/\pi]\!]}\int_0^\pi(-1)^{n-1}\frac{\sin x}{x+(n-1)\pi}\,dx + \int_{[\![t/\pi]\!]\pi}^{t}\frac{\sin x}{x}\,dx\right).\end{aligned}$$

Since $\frac{\sin x}{x+(n-1)\pi}\le\frac{\sin x}{x+n\pi}$ for $0\le x\le\pi$, the first term has a finite limit as $t\to\infty$, by the Alternating Series Test. The second term is bounded in absolute value by $\int_0^\pi \frac{\sin x}{x+[\![t/\pi]\!]\pi}\,dx \le \frac{1}{[\![t/\pi]\!]}$, so it also converges (to 0) by the Squeeze Theorem. So the limit exists and therefore the integral converges.

Alternate solution: As before, since the integrand extends to a continuous function at $x=0$, the integral $\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx$ converges. So it is enough to check whether $\displaystyle\int_\pi^\infty \frac{\sin x}{x}\,dx$ converges.

By integration by parts with

$$\begin{array}{rl} u=\frac{1}{x} & dv=\sin x\,dx \\ du=-\frac{dx}{x^2} & v=-\cos x \end{array},$$

we have

$$\begin{aligned}\int_\pi^t\frac{\sin x}{x}\,dx &= -\frac{\cos x}{x}\bigg|_\pi^t - \int_\pi^t\frac{\cos x}{x^2}\,dx\\ &= -\frac{1}{\pi}-\frac{\cos t}{t}-\int_\pi^t\frac{\cos x}{x^2}\,dx\,.\end{aligned}$$

First of all, by the Squeeze Theorem for infinite limits,

$$\lim_{t\to\infty}\frac{\cos t}{t}=0\,;$$

see Exercise 57a on page 142, where the same result is proved for $\frac{\sin t}{t}$.

For the remaining integral, we have $0\le 1+\cos x\le 2$ for all x, so

$$0\le\frac{1+\cos x}{x^2}\le\frac{2}{x^2}$$

for all $x > 0$. Since $\int_\pi^\infty \frac{2\,dx}{x^2}$ converges, the Comparison Theorem for improper integrals on page 525 implies that $\int_\pi^\infty \frac{1+\cos x}{x^2}\,dx$ converges. Subtracting the convergent integral $\int_\pi^\infty dx/x^2$ then gives that $\int_\pi^\infty \frac{\cos x}{x^2}\,dx$ converges.

This implies that $\displaystyle\lim_{t\to\infty}\int_\pi^t \frac{\cos x}{x^2}\,dx$ converges, so

$$\begin{aligned}\int_\pi^\infty \frac{\sin x}{x}\,dx &= \lim_{t\to\infty}\int_\pi^t \frac{\sin x}{x}\,dx \\ &= \lim_{t\to\infty}\left(-\frac{1}{\pi} - \frac{\cos t}{t} - \int_\pi^t \frac{\cos x}{x^2}\,dx\right) \\ &= -\frac{1}{\pi} - \lim_{t\to\infty}\int_\pi^t \frac{\cos x}{x^2}\,dx\end{aligned}$$

also converges.

(This solution came from some student ideas on the problem that came up in office hours.)

15. The data are:

$$\begin{aligned} f(x) &= \ln\sin x & f\left(\tfrac{\pi}{2}\right) &= 0 \\ f'(x) &= \cot x & f'\left(\tfrac{\pi}{2}\right) &= 0 \\ f''(x) &= -\csc^2 x & f''\left(\tfrac{\pi}{2}\right) &= -1 \\ f'''(x) &= 2\csc^2 x\cot x & f'''\left(\tfrac{\pi}{2}\right) &= 0 \end{aligned}$$

Therefore $T_3(x) = -\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2$.

16. (a). First, if $f(x) = e^{x^3}$ then $f'(x) = 3x^2e^{x^3}$, $f''(x) = (6x+9x^4)e^{x^3}$, $f'''(x) = (6+54x^3+27x^6)e^{x^3}$, and $f^{(4)}(x) = (180x^2+324x^5+81x^8)e^{x^3}$. This has a maximum value of $M = 585e$ on $[0,1]$. We want to find n such that

$$\frac{M(b-a)^5}{180n^4} = \frac{585e}{180n^4} < 0.00001\ ,$$

so $n^4 > \frac{585e}{0.00180}$; $n > 30.66$. Since n must be even, n must be at least 32.

(b). We have $\displaystyle e^{x^3} = \sum_{n=0}^\infty \frac{x^{3n}}{n!}$, so

$$\int_0^x e^{t^3}\,dt = \sum_{n=0}^\infty \frac{x^{3n+1}}{(3n+1)(n!)}.$$

(c). By (b), we have $\displaystyle\int_0^1 e^{x^3}\,dx = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(n!)}$.

It's possible to do this part by using the remainder term in Taylor's formula. I wouldn't recommend it, though: it would require computing the 21^{st} derivative of e^{x^3} . This is not easy, so a more *ad hoc* approach will be necessary.

Let's start by looking at the first few terms of the series to see what we can find out:

n	0	1	2	3	4	5	6	7	8	9	10
term	1	$\frac{1}{4}$	$\frac{1}{14}$	$\frac{1}{60}$	$\frac{1}{312}$	$\frac{1}{1920}$	$\frac{1}{13{,}680}$	$\frac{1}{110{,}880}$	$\frac{1}{1{,}008{,}000}$	$\frac{1}{10{,}160{,}640}$	$\frac{1}{101{,}606{,}400}$

We need at least the terms through $n = 6$, and since $\frac{1}{110{,}880} + \frac{1}{1{,}008{,}000} > 0.00001$, we need the $n = 7$ term, too.

Next we'll show that the sum of the terms for $n \ge 8$ is < 0.00001 . Then we'll be done: we need eight terms, for $n = 0, \dots, 7$.

We notice (and in fact can prove) that after the $n = 8$ term, each successive term is smaller by a factor of at least 9 . So the term for $n \ge 8$ is bounded by $\frac{1}{1{,}008{,}000 \cdot 9^{n-8}}$. (This is shown by induction, as in the proof of the Ratio Test.) Using the formula for the sum of a geometric series, the sum of these terms is therefore seen to be less than $\frac{1}{1{,}008{,}000} \cdot \frac{1}{1-1/9} \approx 0.0000012 < 0.00001$. This is the desired bound, so eight terms are enough.

(d). We have two expressions for the coefficient of x^{99} in the power series expansion for e^{x^3} . One comes from part (b), and the other from the Taylor series. Setting them equal gives

$$\frac{1}{33!} = \frac{f^{(99)}(0)}{99!};$$
$$f^{(99)}(0) = \frac{99!}{33!}.$$

17. Taylor's polynomial (with remainder) for e^x about $x = 0$ is

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^z}{(n+1)!} x^{n+1},$$

where (in this case) $x = \sqrt{2}$ and $0 < z < \sqrt{2}$. The function e^z has its largest absolute value (for $0 \le z \le \sqrt{2}$) at $z = \sqrt{2}$, so we want to find n such that

$$\frac{e^{\sqrt{2}}}{(n+1)!} 2^{(n+1)/2} < 0.00001.$$

After squaring and rearranging, this becomes $((n+1)!)^2/2^{n+1} > 10^{10}e^{2\sqrt{2}}$. Since $e < 3$ and $\sqrt{2} < 1.5$, we have $e^{2\sqrt{2}} < 3^3 = 27$, so it will be good enough (maybe too good) if we find n such that

$$\frac{((n+1)!)^2}{2^{n+1}} > 27 \times 10^{10}.$$

If $n = 9$ then we get $12859560000 \approx 1 \times 10^{10}$, which is not big enough. If $n = 10$, though, we get $778003380000 \approx 78 \times 10^{10}$, which is plenty big. Therefore we need 11 terms ($n = 0$ through $n = 10$).

18. We have $e^{x+2} = \displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$. This is close, but the denominator is wrong. We can replace n with $n+3$ and split off the first three terms:

$$e^{x+2} = \sum_{n=-3}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!} = 1 + (x+2) + \frac{(x+2)^2}{2} + \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}.$$

Now divide by $(x+2)^3$ and solve:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{e^{x+2}}{(x+2)^3} - \frac{1}{2(x+2)} - \frac{1}{(x+2)^2} - \frac{1}{(x+2)^3} = \frac{e^{x+2} - x^2/2 - 3x - 5}{(x+2)^3}.$$

19. (This problem is similar to Example 14 on page 699. Also see Exercise 81 on page 701, which was one of the additional exercises for week 4.)

(a). By induction we will show that the sequence is increasing and bounded (from above) by 1. First we use induction to show that it is increasing; *i.e.,* to show that $a_n > a_{n-1} \geq 0$ for all $n \geq 2$. (i) The inequality is true for $n = 2$ since $a_2 = 1/4$ is greater than $a_1 = 0$. (ii) Assume that the inequality is true for n. Then, first of all, a_n and a_{n-1} are nonnegative, so we can square both sides of $a_n > a_{n-1}$ to get $a_n^2 > a_{n-1}^2$. Adding this to $a_n > a_{n-1}$ gives

$$a_n^2 + a_n > a_{n-1}^2 + a_{n-1};$$
$$a_{n+1} = \frac{a_n^2 + a_n + 1}{4} > \frac{a_{n-1}^2 + a_{n-1} + 1}{4} = a_n.$$

So the inequality is therefore true also for $n+1$. By induction the sequence is therefore ≥ 0 and increasing.

Next consider the condition $a_n < 1$. This is true for $n = 1$. If it is true for a_n, then adding $a_n < 1$ and $a_n^2 < 1$ gives

$$a_n^2 + a_n < 2; \qquad \frac{a_n^2 + a_n + 1}{4} < \frac{3}{4} < 1.$$

Therefore the sequence is monotone and bounded, so it has a limit. Call the limit L. As an added bonus we know that $0 < L \le 1$.

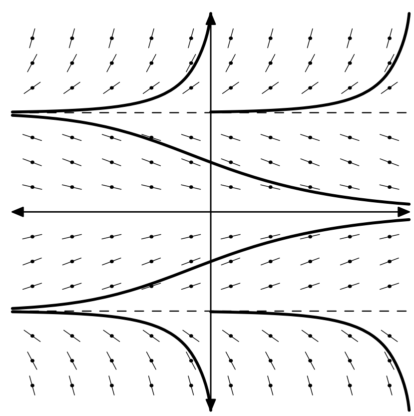
How do we know to choose the upper bound 1? Any number will do, as long as it satisfies $(x^2 + x + 1)/4 \le x$. This condition holds for $\frac{3-\sqrt{5}}{2} \le x \le \frac{3+\sqrt{5}}{2}$, and 1 is a convenient choice.

(b). Replacing n with $n+1$ in the limit gives an equation in L that we can use to narrow down the set of possible limits:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n^2 + a_n + 1}{4} = \frac{L^2 + L + 1}{4}.$$

Solving for L gives $L^2 - 3L + 1 = 0$; $L = \frac{3\pm\sqrt{5}}{2}$. Since only $\frac{3-\sqrt{5}}{2}$ lies in the range $0 < L \le 1$, we must have $L = \frac{3-\sqrt{5}}{2}$.

20. First, notice that the slope depends only on y, so all the little lines in a horizontal row will be the same.



Next, notice that the right-hand side is zero when y equals -1, 1, or 0; hence the dotted lines at $y = \pm 1$ (I couldn't draw a dotted line at $y = 0$).

Finally, note that the slope is positive and increasing as y gets larger starting at $y = 1$, and negative and decreasing as y becomes more negative starting at $y = -1$. Also, note the sign of the slope in the intervals $(0, 1)$ and $(-1, 0)$.

21. Use the formula $y_{n+1} = y_n + .25(x_n + \ln y_n)$ with $y_0 = 1$ and $x_n = .25n$ to obtain the following table:

n	x_n	y_n
0	0	1
1	.25	$1 + .25(0 + \ln 1) = 1$
2	.50	$1 + .25(.25 + \ln 1) = 1.0625$
3	.75	$1.0625 + .25(.50 + \ln 1.0625) = 1.203$
4	1.00	$1.203 + .25(.75 + \ln 1.203) = 1.436$

Therefore $y(1) \approx 1.436$.

22. Since $\frac{dy}{dx} = e^{x-y}$, we can separate variables to get $e^y\,dy = e^x\,dx$; integrating then gives $e^y = e^x + C$. Plugging in the initial condition $x = 0$, $y = 1$ gives $C = e - 1$, so $e^y = e^x + e - 1$; $y = \ln(e^x + e - 1)$.

23. (a). This equation is not separable, because when you write it as $y' = (x + 2y)/x$, you cannot separate the factor $x + 2y$ into a function of x times a function of y. It is linear, because you can write it in standard form as $y' - (2/x)y = 1$.

(b). This is separable, because it can be written as $y' = (y - 1)\tan x$. It is linear, because it can be written as $y' - (\tan x)y = -\tan x$.

(c). This is separable because it can be written as $y' = x \cdot (1/y)$. It is not linear, because when you solve for y' you get $y' = x/y$, and the y is on the bottom.

24. (a). This is a separable equation, so it can be written as

$$y\,dy = e^x\,dx.$$

Integrating gives $\frac{1}{2}y^2 = e^x + C$; $y = \pm\sqrt{2e^x + C}$ (where the second C is different from the first).

(b). The equation can be rewritten in the form of a linear equation: $y' - e^x y = e^{2x}$. Multiplying by the integrating factor $e^{-\int e^x\,dx} = e^{-e^x}$ gives:

$$\begin{aligned} e^{-e^x}y' - e^{-e^x}e^x y &= e^{-e^x}e^{2x} \\ \left(e^{-e^x}y\right)' &= e^{-e^x}e^{2x} \\ e^{-e^x}y &= \int e^{-e^x}e^{2x}\,dx\,. \end{aligned}$$

The integral can be solved by substituting $t = e^x$, $dt = e^x\,dx$ and then using integration by parts with

$$\begin{array}{cc} u = t & dv = e^{-t}\,dt \\ du = dt & v = -e^{-t} \end{array}.$$

We have:

$$\int e^{-e^x}e^{2x}\,dx = \int e^{-t}t\,dt = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C = -e^{-e^x}(e^x + 1) + C\,.$$

Therefore

$$\begin{aligned} e^{-e^x}y &= -e^{-e^x}(e^x + 1) + C\,; \\ y &= Ce^{e^x} - e^x - 1\,. \end{aligned}$$

(c). The characteristic polynomial is $r^2+6r+25$. The quadratic equation gives the roots as $(-6\pm\sqrt{36-100})/2=-3\pm 4i$. Therefore the general solution is

$$y=e^{-3x}(c_1\cos 4x+c_2\sin 4x)\ .$$

(d). The characteristic polynomial is $r^2-10r+25=(r-5)^2$. Therefore the general solution is $y=(c_1+c_2x)e^{5x}$.

(e). The characteristic polynomial is $r^2-2r-15=(r-5)(r+3)$. Therefore the general solution is $y=c_1e^{5x}+c_2e^{-3x}$.

(f). The characteristic polynomial is r^2+3r+5. The quadratic equation gives the roots as $(-3\pm\sqrt{9-20})/2$, so the general solution is

$$y=e^{-3x/2}(c_1\cos(x\sqrt{11}/2)+c_2\sin(x\sqrt{11}/2))\ .$$

25. By part (e) of the previous problem, the general solution is $y=c_1e^{5x}+c_2e^{-3x}$. Plugging in $y(0)=7$ and $y'(0)=3$ gives the equations

$$\begin{aligned} c_1+c_2&=7\\ \text{and}\qquad 5c_1-3c_2&=3. \end{aligned}$$

Multiplying the first equation by 3 and adding it to the second equation (this is what you need to do in order to get the c_2 terms to cancel) gives $8c_1=24$, so $c_1=3$. Therefore $c_2=4$ and the solution is $y=3e^{5x}+4e^{-3x}$.

26. The second order equation $y''+y=0$ has general solution $y=c_1\cos x+c_2\sin x$, so it is periodic of period 2π. Therefore $y(0)=0$, $y(2\pi)=1$ are boundary conditions that are never satisfied.

27. In all parts the general solution of the complementary equation is $y_c=c_1e^{2x}+c_2xe^{2x}$.

(a). Ax^3+Bx^2+Cx+D.

(b). $Ax^4e^{2x}+Bx^3e^{2x}+Cx^2e^{2x}+Dx+E$. Note that we multiplied the first three terms by x^2 to avoid collision with the terms in y_c, but we didn't need to do this for the last two terms.

(c). Using the trigonometric identity $\sin A\cos B = \frac{1}{2}\big[\sin(A-B)+\sin(A+B)\big]$, the differential equation can be written $y''-4y'+4y = \frac{1}{2}e^{2x}\big[-\sin((\pi-1)x)+\sin((\pi+1)x)\big]$, giving a trial solution of the form

$$Ae^{2x}\cos((\pi-1)x) + Be^{2x}\sin((\pi-1)x) + Ce^{2x}\cos((\pi+1)x) + De^{2x}\sin((\pi+1)x))\ .$$

One can apply trigonometric identities again to put it in the form

$$Ae^{2x}\cos x\cos\pi x + Be^{2x}\cos x\sin\pi x + Ce^{2x}\sin x\cos\pi x + De^{2x}\sin x\sin\pi x$$

(with different values of A, B, C, and D).

(d). Not possible; $e^{1/x}$ is not a product of functions of the required form.

28. For all parts of this problem, the general solution of the complementary equation is $y_c = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}$.

(a). We use the method of undetermined coefficients, with trial solution

$$y = Ax^3 + Bx^2 + Cx + D.$$

The derivatives are $y' = 3Ax^2 + 2Bx + C$ and $y'' = 6Ax + 2B$. Therefore we have

$$4y'' + y = 4(6Ax+2B) + Ax^3 + Bx^2 + Cx + D = Ax^3 + Bx^2 + (24A+C)x + (8B+D).$$

This is supposed to equal x^3, so $A = 1$, $B = 0$, $24A + C = 0$, and $8B + D = 0$. These last two equations give $C = -24$ and $D = 0$, so the particular solution is $y_p = x^3 - 24x$ and the general solution is

$$y = x^3 - 24x + c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}.$$

(b). This part requires the method of variation of parameters. We look for a solution of the form

$$y = u_1(x)\cos\frac{x}{2} + u_2(x)\sin\frac{x}{2}.$$

the equations we need to solve are

$$u_1'\cos\frac{x}{2} + u_2'\sin\frac{x}{2} = 0;$$
$$4\left(-u_1'\cdot\frac{1}{2}\sin\frac{x}{2} + u_2'\cdot\frac{1}{2}\cos\frac{x}{2}\right) = 8\sec^2\frac{x}{2}.$$

The second equation can be rewritten as $-u_1' \sin\frac{x}{2} + u_2' \cos\frac{x}{2} = 4\sec^2\frac{x}{2}$. Multiplying the first equation above by $\sin\frac{x}{2}$ and adding $\cos\frac{x}{2}$ times the (rewritten) second equation gives $u_2'(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}) = 4\sec\frac{x}{2}$ and therefore

$$u_2 = 4\int \sec\frac{x}{2}\,dx = 8\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C_2$$

(from Formula 14 in the back of the book). The first equation above then gives

$$u_1' = -u_2' \tan\frac{x}{2} = -4\sec\frac{x}{2}\tan\frac{x}{2}\ ,$$

so $u_1 = -8\sec\frac{x}{2} + C_1$ and the solution is

$$\begin{aligned} y &= \left(-8\sec\frac{x}{2} + C_1\right)\cos\frac{x}{2} + \left(8\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C_2\right)\sin\frac{x}{2} \\ &= -8 + C_1\cos\frac{x}{2} + \left(8\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C_2\right)\sin\frac{x}{2}. \end{aligned}$$

(c). By the Principle of Superposition, this answer is obtained by adding the results from the previous two parts:

$$y = x^3 - 24x - 8 + C_1\cos\frac{x}{2} + \left(8\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C_2\right)\sin\frac{x}{2}.$$

29. This circuit satisfies the differential equation

$$0.001Q'' + RQ' + 10^4 Q = E;$$

since the voltage is constant (after the switch is thrown), we can take derivatives of both sides to get

$$0.001I'' + RI' + 10^4 I = 0.$$

This has complementary equation $0.001r^2 + Rr + 10^4$. We are looking for solutions that oscillate at 440 Hz, so we need a pair of imaginary roots $\alpha + \beta i$ with $\beta/440 = 2\pi$. Therefore, from the quadratic equation, we have

$$\begin{aligned} \frac{\sqrt{4\cdot 0.001\cdot 10^4 - R^2}}{0.002} &= 880\pi; \\ \sqrt{40 - R^2} &= 1.76\pi; \\ R &= \sqrt{40 - (1.76\pi)^2} \\ &\approx 3.07\,\Omega. \end{aligned}$$

30. Substitute $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$, and the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ into the differential equation:

$$x \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n.$$

Replacing n with $n+1$ in the first and third series gives

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + 3\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n.$$

In the first two power series, we can start the sums at $n = 0$ because the added terms are all zero. Therefore, the sums combine to give

$$\sum_{n=0}^{\infty}\big(n(n+1)c_{n+1} - n(n-1)c_n - 2(n+1)c_{n+1} + 3c_n - 1\big)x^n = 0.$$

Thus, for all $n = 0, 1, 2, \ldots$ we have

$$(n^2 - n - 2)c_{n+1} - (n^2 - n - 3)c_n = 1. \qquad (*)$$

The initial conditions $y'(0) = -5$ and $y'''(0) = 6$ give us $c_1 = -5$ and $c_3 = 1$, respectively. Plugging $n = 0$, $n = 1$, $\ldots$ into (*) gives

$$\begin{aligned}
\text{Put } n = 0: &\quad -2c_1 + 3c_0 = 1\,; \\
n = 1: &\quad -2c_2 + 3c_1 = 1\,; \\
n = 2: &\quad 0c_3 + c_2 = 1\,; \\
n = 3: &\quad 4c_4 - 3c_3 = 1\,; \qquad c_4 = \frac{3c_3 + 1}{4}\,; \\
n = 4: &\quad 10c_5 - 9c_4 = 1\,; \qquad c_5 = \frac{9c_4 + 1}{10}\,; \\
n = 5: &\quad 18c_6 - 17c_5 = 1\,; \qquad c_6 = \frac{17c_5 + 1}{18}\,;
\end{aligned}$$

The equations for $n = 3$, 4, and 5 give recurrences. We can easily check that all equations for $n \geq 3$ give recurrences, since $n^2 - n - 2 \neq 0$ for all $n \geq 3$. Set aside the recurrences for now. This leaves just the initial conditions $c_1 = -5$ and $c_3 = 1$, as well as the above equations for $n = 0$, 1, and 2.

The equation for $n = 2$ gives $c_2 = 1$, and the equation for $n = 1$ gives $-2 + 3c_1 = 1$, so $c_1 = 1$. This contradicts the value of c_1 from the initial conditions, so the initial-value problem *has no solutions.*

If you had ignored the initial condition $y'(0) = -5$, then you would only have had $c_3 = 1$ from the initial conditions. The equation you get from (*) with $n = 2$ gives $c_2 = 1$; from the equation with $n = 1$ we then get $c_1 = 1$, and from the equation with $n = 0$ we get $c_0 = 1$.

Using the recurrences obtained from (*) with $n = 4$, 5, and 6, together with the initial condition $c_3 = 1$, gives, successively, $c_4 = 1$, $c_5 = 1$, and $c_6 = 1$, respectively. The pattern seems to be $c_n = 1$ for all $n \geq 3$, and in fact this satisfies (*) for all $n \geq 3$. Combining this with $c_0 = c_1 = c_2 = 1$ obtained earlier, we have

$$y = \sum_{n=0}^{\infty} 1 \cdot x^n = \frac{1}{1-x} .$$

We can check that this is in fact a solution to the differential equation.

31. Plug in $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ (and note that it's easier to keep the $n = 0$ term of y', which is zero anyway):

$$\begin{aligned} \sum_{n=0}^{\infty} n c_n x^{n-1} &= \frac{1}{2x} \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \frac{c_n}{2} x^{n-1} . \end{aligned}$$

Equating coefficients gives $n c_n = c_n/2$ for $n = 0, 1, 2, \ldots$; therefore $(n - 1/2) c_n = 0$, so $c_n = 0$ for all n. The only solution is $y = 0$.

Solving it the conventional way, as a separable equation, gives:

$$\begin{aligned} y' &= \frac{y}{2x} \\ \frac{dy}{y} &= \frac{dx}{2x} \\ \int \frac{dy}{y} &= \int \frac{dx}{2x} \\ \ln|y| &= \frac{1}{2} \ln|x| + C_0 \\ y &= \pm\sqrt{|x|} \cdot e^{C_0} = C\sqrt{|x|} . \end{aligned}$$

If $C \neq 0$ then $C\sqrt{|x|}$ is not a solution at $x = 0$ (it is not differentiable there). That is why the series method only found the solution with $C = 0$. This situation stems from the fact that the coefficient function $\frac{1}{2x}$ is not continuous at $x = 0$.