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Master 2 — Internship Report

SHARP OBSERVABILITY ESTIMATES AND F-EQUIVALENCE FOR PARABOLIC EQUATIONS

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Contents

Summary

This report gathers the main outcomes of my Master 2 internship at CERMICS, the applied mathematics laboratory of École nationale des ponts et chaussées, carried out under the supervision of Amaury Hayat and Emmanuel Trélat. It is organized around two research articles that address complementary aspects of the observability and stabilization of parabolic partial differential equations. Their juxtaposition points toward the research that will continue in my coming PhD with Amaury Hayat and Emmanuel Trélat: on the one hand, trying to show observability properties on sub-Riemannian Laplacians. On the other, developing a general framework for F-equivalence that unifies stabilization for linear and nonlinear equations.

The first article, *Sharp weighted observability inequalities for Riemannian heat equations*, was entirely done during this internship with Amaury Hayat and Emmanuel Trélat. It originates from a question raised by Emmanuel Trélat and investigates the optimal small-time weights that can appear in integrated observability inequalities for the heat equation on compact Riemannian manifolds. Besides clarifying the geometric content of these inequalities, the results identify the intrinsic barrier on the time-weights and works for finite and infinite-time integrated observability. This allows us to partially answer positively to a conjecture of Ervedoza and Zuazua from 2011 about the geometric lower bound of the infinite-time integrated observability constant.

At present, the first article has not yet been released as a preprint. As explained in its conclusion, we anticipate further progress that will place the results within a more abstract framework capable of encompassing sub-Riemannian Laplacians. We therefore postpone public release to consolidate these expected improvements and present a unified treatment.

The second article, *F-equivalence for parabolic systems and applications to nonlinear PDE*, comes from my previous M1 internship with Amaury Hayat. Following a referee report in Spring 2025, we thoroughly revised and expanded the paper during the present internship. In particular, the entire nonlinear part was changed, which led to new applications and examples, including the Navier–Stokes system and a quasilinear heat equation. Because these revisions substantially reshaped the scope and structure of the article relative to the version included in my previous internship report, I chose to reproduce the complete, updated paper here rather than only the new sections. The revised version is now available on arXiv `https://arxiv.org/abs/2508.21605` and has been submitted to the *Journal de l'École Polytechnique*.

The first part develops sharp weighted observability inequalities for the Riemannian heat equation. The second part presents the theory of F-equivalence for parabolic systems and

its applications to nonlinear equations, together with stabilization results and illustrative examples. Taken together, these two parts synthesize the core contributions of my internship and outline several avenues for future work.

Chapter 1

Sharp weighted observability inequalities for Riemannian heat equations

1.1 Introduction

1.1.1 Context and aims

Let $(\mathcal{M}, g)$ be a smooth connected compact d-dimensional Riemannian manifold, and let $\omega \subset \mathcal{M}$ be a nonempty open subset such that $\overline{\omega} \neq \mathcal{M}$, which will be our observation set. We will denote by Δ_g the positive Laplacian, which is a self-adjoint operator on $L^2(\mathcal{M}) := L^2(\mathcal{M}, \mu)$, where μ is the Riemannian measure. In this article, if a constant depends on time T, it will be denoted by a subscript, otherwise it means that it is independent of T.

Using global Carleman estimates, it was shown, first for bounded open subsets of $\mathbb{R}^d$ in [FI96; Ima95], then for compact Riemannian manifolds $(\mathcal{M}, g)$ in [LR95], that the heat equation is observable in ω for any finite time $T > 0$. That is, there exists $C_T > 0$ such that

$$\forall f \in L^2(\mathcal{M}), \quad \|e^{-T\Delta_g} f\|^2_{L^2(\mathcal{M})} \leq C_T \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.1}$$

The previous inequality is called *observability inequality*, and it is well-known, see [FI96; Güi85; Mil10; Mil04], that the constant in (1.1) blows up exponentially as T goes to 0. Thus, we can take C_T of the form

$$\exists C, K > 0,\ \forall T > 0, \quad C_T = Ce^{K/T}, \tag{1.2}$$

and the constant K is called *the finite-time exponential observability cost*, for more details see [LL12; Mil06a; LL21].

Beyond its intrinsic analytic interest, the observability inequality (1.1) is the cornerstone of control for parabolic PDEs. In [DR77], it was first established that (1.1) is equivalent to exact (internal) controllability to zero in time $T > 0$ of

$$\begin{cases} \partial_t y + \Delta_g y = 1_\omega \, u, \\ y(0) = f, \end{cases} \tag{1.3}$$

That is, for every initial condition $f \in L^2(\mathcal{M})$, there exists $u \in L^2((0,T) \times \mathcal{M})$ such that $y(T) = 0$. This duality was then popularized in optimal control theory by Lions in [Lio88a; Lio88b], for more details, see also more modern expositions in [LT00; Zua06; Tré24]. It is also worth noting that in [Lio88b], Lions showed that the best constant C_T in (1.1) is equal to *null-controllability cost* for the heat equation, i.e it is the best constant in the following inequality

$$\|u\|^2_{L^2((0,T)\times\mathcal{M})} \leq C_T \|f\|^2_{L^2(\mathcal{M})}, \tag{1.4}$$

for all initial data f and control u such that the solution of (1.3) satisfies $y(T) = 0$.

Beyond inequality (1.1), three closely related formulations of observability for the heat semigroup emerged:

(i) *Finite-time integrated observability.* There exists $\widetilde{\gamma} > 0$ such that for every $T > 0$, there exists $\widetilde{C}_T > 0$ for which we have

$$\forall f \in L^2(\mathcal{M}), \quad \int_0^T e^{-\frac{\widetilde{\gamma}}{t}} \|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})} dt \leq \widetilde{C}_T \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.5}$$

This inequality was first obtained as a consequence of Carleman estimate methods for Euclidean bounded domains in [FZ00]. Then it was extended to compact Riemannian manifolds in [Mil06a].

(ii) *Infinite-time integrated observability.* There exists $\gamma > 0$ and $C_\infty > 0$ such that

$$\forall f \in L^2(\mathcal{M}), \quad \int_0^\infty e^{-\frac{\gamma}{t}} \|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})} dt \leq C_\infty \int_0^\infty \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.6}$$

This was first obtained in [Zua01] for smooth bounded domains of $\mathbb{R}^d$. It is the infinite-time version of (1.5).

(iii) *Sharp observability.* Fernández-Cara and Zuazua in [FZ00] noticed thanks to Laplace's method (see Lemma 4), that (1.5) implies the following inequality for all $T > 0$ (for some $K', C'_T > 0$)

$$\forall f \in L^2(\mathcal{M}), \quad \|e^{-K'\sqrt{\Delta_g}} f\|^2_{L^2(\mathcal{M})} \leq C'_T \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} \, dt. \tag{1.7}$$

We will call this the *sharp observability inequality.* Notice that for large $T > 0$, the previous inequality is sharper than (1.1).

A key result from Miller in [Mil06a] is that (i) holds if and only if inequality (1.1) holds for all $T > 0$, and the exponential observability costs are equal, meaning that the best constant $\widetilde{\gamma}$

in (1.5) is equal to the best constant K in (1.2), hence we will write

$$\widetilde{\gamma} = K. \tag{1.8}$$

Similarly, it was shown in the same article that (i) implies (ii), and that

$$\widetilde{\gamma} \geq \gamma. \tag{1.9}$$

Thus, constants $\widetilde{\gamma}$ and γ are important geometric quantities in control theory, and therefore, it would be interesting to understand their dependence on the geometry of ω. Although an explicit computation of these quantities is not possible, we have estimates on them. The first geometric lower bound is due to Zuazua in [Zua01], where he obtained the following lower bound for both finite and infinite time integrated observability constants

$$\widetilde{\gamma}, \gamma \geq \frac{\mathcal{D}(\omega)^2}{2}, \tag{1.10}$$

where $\mathcal{D}(\omega) = \sup\{\rho \in \mathbb{R}_{>0} \mid \exists x_0 \in \mathcal{M},\, B(x_0, \rho) \subset \mathcal{M} \setminus \omega\}$ is the largest radius of a ball included outside ω. Then, in [Mil04], Miller improved this result and obtained the following lower bound but only for the finite-time integrated observability constant

$$\widetilde{\gamma} \geq \frac{\mathcal{L}(\omega)^2}{2}, \tag{1.11}$$

where $\mathcal{L}(\omega) = \sup_{x \in \mathcal{M}} d(x, \omega)$ is the largest distance of a point in $\mathcal{M}$ to the observation set ω. Notice that we always have $\mathcal{L}(\omega) \geq \mathcal{D}(\omega)$, and that these two quantities are equal in some cases, for example if $\mathcal{M}$ is a sphere and ω is a spherical cap. It is also worth to notice that the lower bound (1.11) fails to be sharp in general but can be sharp in some cases, this is known since the work of Laurent and Léautaud in [LL21]. Typically if $\mathcal{M} = \mathbb{S}^2$, and ω is a spherical cap that satisfies the Geometric Control Condition (GCC), then by Theorem 1.1 of [EZ11], the optimal γ in (1.6) satisfies $\gamma \leq \mathcal{L}(\omega)^2/2$.

Hence a natural question is whether the same lower bound holds for the infinite-time integrated observability constant. This is an old open problem, which was first explicitly asked in [EZ11], and we state here a weaker conjecture.

Conjecture 1. *There exists $\alpha > 0$ such that, for any smooth connected compact Riemannian manifold $(\mathcal{M}, g)$, and any nonempty open subset $\omega \subset \mathcal{M}$ such that $\overline{\omega} \neq \mathcal{M}$, if there exists $\gamma > 0$ such that (1.6) holds for some $C_\infty > 0$, then*

$$\gamma \geq \alpha \mathcal{L}(\omega)^2. \tag{1.12}$$

Until now, this was still an open question, which is answered in the present article, see Corollary 1 and the comments that follow it for more details. For those more interested in the cost of null-controllability for the heat equation, we refer to [Sei88; SY96; Sei08; Lis12; Lis14; Lis15; Lis17; Ngu22; TT07; Mil10; Mil06b; Nak+20; EV18; TT11; DE17].

Now we describe our aim and strategy in this article. We are interested in weighted integrated observability of the form

$$\forall f \in L^2(\mathcal{M}), \qquad \int_0^T h(t)\, \|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})}\, dt \;\leq\; \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)}\, dt, \tag{1.13}$$

where $h \in C^0_b([0,T))$ (bounded and continuous functions space) is nonnegative and nondecreasing near 0, with $T \in (0,+\infty]$. In the Riemannian case, Carleman methods naturally produce the exponential weight $h(t) = A_T e^{-\widetilde{\gamma}/t}$ (or $h(t) = A_T e^{-\gamma/t}$). Our goal, is to determine the best possible weight h, in the precise sense of identifying the minimal admissible decay of $h(t)$ as $t \to 0$ for which (1.13) can still hold, and to relate this decay rate to the geometry of the observation set ω. This is done in Theorem 1, which is the main result of this article, and it will allow us to answer Conjecture 1 positively in Corollary 1. This also shows that the sharp observability inequality (1.7) is indeed the sharpest inequality we can obtain using an integrated observability approach, see Corollary 2.

1.1.2 Settings and results

We keep the same settings and notations as in Subsection 1.1.1. We start by defining the set of admissible weights for an integrated version of observability inequality for any $T \in (0,+\infty]$ as follows

$$\mathcal{O}([0,T]) = \{h \in C^0_b([0,T)) \mid h \geq 0 \text{ and } h \text{ locally nondecreasing at } 0\}, \tag{1.14}$$

where $C^0_b([0,T))$ is the space of continuous bounded functions on $[0,T)$. This leads us to the following definition.

Definition 1. *Let $T > 0$ and $h \in \mathcal{O}([0,T])$. We say that we have an integrated observability inequality with weight h, if we have*

$$\forall f \in L^2(\mathcal{M}), \quad \int_0^T h(t)\|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})} dt \leq \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.15}$$

We now state our main result, which says that if an integrated observability inequality holds with respect to a weight h, then h must decay at least as fast as $e^{-\kappa\mathcal{L}(\omega)^2/t}$, for every $\kappa < \frac{1}{4}$ as $t \to 0$. More precisely, we have the following theorem.

Theorem 1. *Let $T \in (0,\infty]$, and $h \in \mathcal{O}([0,T])$ such that we have an integrated observability inequality with weight h. Then, setting $\mathcal{L}(\omega) = \sup_{x\in\mathcal{M}} d(x,\omega)$, we will show that for all $\kappa < \frac{1}{4}$, there exists $A_T > 0$ such that*

$$\forall t \in [0,T), \quad h(t) \leq A_T e^{-\kappa\mathcal{L}(\omega)^2/t}. \tag{1.16}$$

The proof of this theorem is done in Subsection 1.2.1. Compared with the classical results of Zuazua [Zua01] and Miller [Mil04] mentioned in Subsection 1.1.1, Theorem 1 is more general, as we do not assume that the weight is of the form $h(t) = A_T e^{-\gamma/t}$. Hence the exponential barrier delivered by Carleman estimates is not an artefact of the method but an intrinsic geometric constraint.

Another important and immediate consequence of Theorem 1 is that it allows us to answer positively to Conjecture 1.

Corollary 1. *Let $\gamma > 0$ such that there exists $C_\infty > 0$, for which the following infinite time observability inequality holds*

$$\forall f \in L^2(\mathcal{M}), \quad \int_0^{+\infty} e^{-\frac{\gamma}{t}} \|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})} dt \leq C_\infty \int_0^{+\infty} \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.17}$$

Then $\gamma \geq \frac{1}{4}\mathcal{L}(\omega)^2$.

This shows that Conjecture 1 holds with $\alpha = 1/4$.

Remark 2. *We do not know if the constant $1/4$ is optimal in Corollary 1, but we believe it is not, and that the optimal constant should be $1/2$, which would be consistent with the lower bounds discussed in Subsection 1.1.1. Furthermore, using the main result of [EZ11, Theorem 1.1], we see that for $\mathcal{M} = \mathbb{S}^2$, and with a spherical cap ω satisfying GCC, inequality (1.17) holds for any $\gamma > \mathcal{L}(\omega)^2/2$. Hence the optimal constant α of Conjecture 1 is in $[1/4, 1/2]$.*

Now, we denote by $(\lambda_n)_{n\geq 0}$ the nondecreasing sequence of eigenvalues of Δ_g, and by $(\phi_n)_{n\geq 0}$ the associated orthonormal eigenfunctions basis of $L^2(\mathcal{M})$, then for every $h \in \mathcal{O}([0,T])$ we have

$$\int_0^T h(t)\|e^{-t\Delta_g} f\|^2_{L^2(\mathcal{M})} dt = \sum_{n=0}^{+\infty} |f_n|^2 \int_0^T h(t) e^{-2t\lambda_n} dt, \tag{1.18}$$

with $f_n = \langle \phi_n \,, f\rangle_{L^2(\mathcal{M})}$. This leads us to the following definition.

Definition 2. *Let $T > 0$ and $h \in \mathcal{O}([0,T])$ we define the following function*

$$\forall \lambda \geq 0, \quad H_T(\lambda) = \sqrt{\int_0^T h(t) e^{-2t\lambda} dt}, \tag{1.19}$$

which is bounded and continuous on $[0, +\infty)$.

Hence, if we have an integrated observability inequality with weight h, this allows us to rewrite (1.15) as

$$\forall f \in L^2(\mathcal{M}), \quad \|H_T(\Delta_g) f\|^2_{L^2(\mathcal{M})} \leq C_T \int_0^T \|e^{-t\Delta_g} f\|^2_{L^2(\omega)} dt. \tag{1.20}$$

Recall that the operator $H_T(\Delta_g)$ is defined as follows

$$\forall f \in L^2(\mathcal{M}), \quad H_T(\Delta_g) f = \sum_{n=0}^{+\infty} H_T(\lambda_n) f_n \phi_n. \tag{1.21}$$

This allows us to pass from an integrated observability inequality to a final time one. This is the way Fernández-Cara and Zuazua in [FZ00] obtained the sharp observability inequality (1.7) from (1.5) using Laplace's method (see Lemma 4), as only the high frequencies contribute to the final time observability. This leads us to the following corollary of Theorem 1.

Corollary 2. *Let $T > 0$ and $h \in \mathcal{O}([0,T])$, such that we have an integrated observability inequality with weight h. Then we will show that for all $\beta < 1/\sqrt{2}$, there exists $A'_T > 0$ such that*

$$\forall \lambda \geq 0, \quad H_T(\lambda) \leq A'_T e^{-\beta \mathcal{L}(\omega)\sqrt{\lambda}}. \tag{1.22}$$

This shows that the inequality (1.7) is indeed the sharpest inequality we can obtain using an integrated observability approach as described above.

1.2 Main proofs

1.2.1 Proof of Theorem 1

In this Subsection we prove Theorem 1. Let $T \in (0,+\infty]$ and $h \in \mathcal{O}([0,T])$ such that we have an integrated observability inequality with respect to h. We will show that for all $\kappa < \frac{1}{4}$, there exists $A_T > 0$ such that

$$\forall t \in [0,T), \quad h(t) \leq A_T e^{-\kappa \mathcal{L}(\omega)^2/t}. \tag{1.23}$$

Before starting the proof, we recall the Kannai transform, it is a tool which first appears in [CGT82], that allows to relate the heat semigroup to the wave semigroup as follows

$$\forall t > 0, \quad e^{-t\Delta_g} = \int_0^{+\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} \cos(s\sqrt{\Delta_g}) ds. \tag{1.24}$$

Hence if we denote by $p_t(x,y)$ the heat kernel on $\mathcal{M}$, and by $w_t(x,y)$ the Schwartz Kernel of $\cos(t\sqrt{\Delta_g})$, we have in the distributional sense on $\mathcal{M} \times \mathcal{M}$

$$\forall t \geq 0, \quad p_t(x,y) = \int_0^{+\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} w_s(x,y) ds. \tag{1.25}$$

This allows us to show the following lemma, which is one of the key to prove Theorem 1.

Lemma 1. *For all $\delta > 0$, there exists $C > 0$ such that*

$$\forall k \geq 0,\ \forall t > 0,\ \forall x, y \in \mathcal{M}, \quad d(x,y) \geq \delta \implies |\Delta_{g_x}^k p_t(x,y)| \leq C\sqrt{2}^{2(k+d)+1} \frac{(2(k+d+1))!}{\delta^{2k}}. \tag{1.26}$$

Proof. Applying $\Delta_{g_x}^k$ to (1.25), and using $\partial_s^2 w_s(x,y) = -\Delta_{g_x} w_s(x,y)$ on $(0,+\infty) \times \mathcal{M}$, we get

$$\forall t > 0, \quad \Delta_{g_x}^k p_t(x,y) = (-1)^k \int_0^{+\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{s^2}{4t}} \partial_s^{2k} w_s(x,y) ds. \tag{1.27}$$

Recall that we denote by $(\lambda_n)_{n\geq 0}$ the nondecreasing sequence of eigenvalues of Δ_g (with $\lambda_0 = 0$), and by $(\phi_n)_{n\geq 0}$ the associated orthonormal basis of $L^2(\mathcal{M})$, we have by Mercer theorem that

$$\forall t \in \mathbb{R}, \quad w_t(x,y) = \phi_0^2 + \sum_{n=1}^{+\infty} \cos(t\sqrt{\lambda_n}) \phi_n(x)\phi_n(y). \tag{1.28}$$

Recall that by a Sobolev inequality we have $H^s(\mathcal{M})$ is continuously embedded in $L^\infty(\mathcal{M})$, from this we deduce there exists $C>0$ such that

$$\forall n\geq 1,\quad \|\phi_n\|_{L^\infty}\leq C\lambda_n^{\frac{d}{4}}. \tag{1.29}$$

Then, we set

$$\forall t\in\mathbb{R},\ \forall x,y\in\mathcal{M},\quad h(t,x,y)=\frac{t^{2d+1}}{(2d+1)!}\phi_0^2+\sum_{n=1}^{+\infty}(-1)^n\frac{\sin(t\sqrt{\lambda_n})}{\lambda_n^{d+1/2}}\phi_n(x)\phi_n(y), \tag{1.30}$$

and Weyl law ensures us that there exists $C_d>0$ such that

$$\forall t\in\mathbb{R},\ \forall x,y\in\mathcal{M},\quad |h(t,x,y)-\frac{t^{2d+1}}{(2d+1)!}\phi_0^2|\leq\sum_{n=1}^{+\infty}\frac{C_d}{n^{1+1/d}}<+\infty. \tag{1.31}$$

Hence we have $h\in C^0(\mathbb{R}\times\mathcal{M}\times\mathcal{M})$ and, as it has a polynomial growth in t, we have $h(t,x,y)\in\mathcal{S}'(\mathbb{R}_t)$ and $\partial_t^{2d+1}h(t,x,y)=w_t(x,y)$. Now let $x,y\in\mathcal{M}$ such that $d(x,y)\geq\delta$, the finite speed of propagation of the wave equation gives us that, for all $|t|<\delta$ we have $w_t(x,y)=0$, and so $h(t,x,y)=P(t,x,y)$, where P is a polynomial of degree $2d$ in t and continuous in x,y, thus setting $f=h-P$ is continuous in t,x,y, and we have

$$\forall t\in\mathbb{R},\quad |t|\leq\delta\implies f(t,x,y)=0. \tag{1.32}$$

Now we can rewrite (1.27) as follows, again we fix $x,y\in\mathcal{M}$ such that $d(x,y)\geq\delta$, and we have the following equality

$$\forall t>0,\quad \Delta_{g_x}^k p_t(x,y)=(-1)^k\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{s^2}{4t}}\partial_s^{2(k+d)+1}f(s,x,y)ds. \tag{1.33}$$

As $f(t,x,y)\in\mathcal{S}'(\mathbb{R}_t)$, we can integrate by parts $2(k+d)+1$ times, and we get

$$\forall t>0,\quad \Delta_{g_x}^k p_t(x,y)=(-1)^{k+1}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}\frac{H_{2(k+d)+1}(\frac{s}{\sqrt{2t}})}{\sqrt{2t}^{2(k+d)+1}}e^{-\frac{s^2}{4t}}f(s,x,y)ds. \tag{1.34}$$

Where $H_{2(k+d)+1}$ is the $(2(k+d)+1)$-th probabilistic Hermite polynomial. Now let $x,y\in\mathcal{M}$, such that $d(x,y)\geq\delta$, thus using the fact that $f(t,x,y)=0$ for $|t|\leq d(x,y)$, we can rewrite the previous equality as

$$\forall t>0,\quad \Delta_{g_x}^k p_t(x,y)=(-1)^{k+1}\frac{1}{\sqrt{2\pi}}\int_{|s|>\delta}\frac{H_{2(k+d)+1}(\frac{s}{\sqrt{2t}})}{\sqrt{2t}^{2(k+d+1)}}e^{-\frac{s^2}{4t}}f(s,x,y)ds. \tag{1.35}$$

Thus we have the following estimate

$$\forall t>0,\quad |\Delta_{g_x}^k p_t(x,y)|\leq\frac{1}{\sqrt{2\pi}}\int_{|s|>\delta}|s^{2(d+1)}\frac{H_{2(k+d)+1}(\frac{s}{\sqrt{2t}})}{\sqrt{2t}^{2(k+d+1)}}e^{-\frac{s^2}{4t}}\frac{1}{s^{2(d+1)}}f(s,x,y)|ds. \tag{1.36}$$

Hence as $\frac{1}{s^{2(d+1)}}f(s,x,y)\in L^1(\mathbb{R}_s)$, to conclude we just have to find a uniform bound in $s>\delta$

and $t > 0$ of

$$|s^{2(d+1)} \frac{H_{2(k+d)+1}(\frac{s}{\sqrt{2t}})}{\sqrt{2t}^{2(k+d+1)}} e^{-\frac{s^2}{4t}}|. \tag{1.37}$$

We set $x = \frac{s}{\sqrt{2t}}$ and $N = 2(k+d)+1$, and thus we have

$$|s^{2(d+1)} \frac{H_{2(k+d)+1}(\frac{s}{\sqrt{2t}})}{\sqrt{2t}^{2(k+d+1)}} e^{-\frac{s^2}{4t}}| \leq \frac{1}{\delta^{2k}} |x^{N+1} H_N(x) e^{-\frac{x^2}{2}}|. \tag{1.38}$$

As the right-hand side of the previous equation decays super-exponentially, one can show it attains its maximum for $|x| \leq N+1$, and therefore we have

$$\sup_{x\in\mathbb{R}} |x^{N+1} H_N(x) e^{-\frac{x^2}{2}}| \leq (N+1) \sup_{x\in\mathbb{R}} |x^N H_N(x) e^{-\frac{x^2}{2}}|. \tag{1.39}$$

Hence we can now use the fact that the generating function of the probabilistic Hermite polynomials is

$$\forall z \in \mathbb{C}, \quad e^{xz-\frac{1}{2}z^2} = \sum_{k=0}^{+\infty} H_k(x) \frac{z^k}{k!}. \tag{1.40}$$

Now by Cauchy formula, we have

$$|H_N(x)| \leq \frac{N!}{R^N} \sup_{|z|=R} |e^{xz-\frac{1}{2}z^2}|, \tag{1.41}$$

then taking $R = \frac{x}{\sqrt{2}}$, we get $\sup_{|z|=R} |e^{xz-\frac{1}{2}z^2}| = e^{\frac{x^2}{2}}$, and therefore

$$\sup_{x\in\mathbb{R}} |x^{N+1} H_N(x) e^{-\frac{x^2}{2}}| \leq \sqrt{2}^N (N+1)!. \tag{1.42}$$

Which gives us the desired result.

□

Another lemma that we need in order to prove Theorem 1 is the following one.

Lemma 2. *There exists $\tau_d > 0$ depending only on d, such that for all $x \in \mathcal{M}$, we have*

$$\forall k \in \mathbb{N},\ \forall t \in (0,\tau_d), \quad \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} \geq \frac{\omega_d}{2(2\pi)^d} \frac{1}{(2t)^{2k+d/2}} \Gamma(2k+d/2). \tag{1.43}$$

Proof. Let $x \in \mathcal{M}$, $k \geq 0$ and $t \in (0,1)$, keeping the notations of Lemma 1, we have

$$\|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} = \sum_{n=0}^{+\infty} \lambda_n^{2k} e^{-2t\lambda_n} \phi_n(x)^2. \tag{1.44}$$

Now, we set $N_x(\lambda) = \sum_{\lambda_n \leq \lambda} \phi_n(x)^2$, and the Hörmander local Weyl's law from [Hör68] tells us that

$$N_x(\lambda) = \frac{\omega_d}{(2\pi)^d} \lambda^{d/2} + R(\lambda), \tag{1.45}$$

and $|R(\lambda)| \leq C\lambda^{\frac{d-1}{2}}$ with $C \geq 1$, which is independent of x. Then using Stieltjes integral, we can write the previous sum as follows

$$\sum_{n=0}^{+\infty} \lambda_n^{2k} e^{-2t\lambda_n} \phi_n(x)^2 = \int_0^{+\infty} \lambda^{2k} e^{-2t\lambda} dN_x(\lambda) \tag{1.46}$$

$$= \underbrace{\int_0^{+\infty} \lambda^{2k} e^{-2t\lambda} \frac{\omega_d}{(2\pi)^d} \frac{d}{2} \lambda^{d/2-1} d\lambda}_{M_k(t)} + \int_0^{+\infty} \lambda^{2k} e^{-2t\lambda} dR(\lambda). \tag{1.47}$$

For the first integral, using a change of variable, we recognize the Gamma function, and we get

$$M_k(t) = \frac{\omega_d}{(2\pi)^d} \frac{1}{(2t)^{2k+d/2}} \Gamma(2k + d/2). \tag{1.48}$$

For the second integral, notice that $f_t(\lambda) = \lambda^{2k} e^{-2t\lambda}$ attains its maximum at $\frac{k}{t}$, hence we split the integral into two parts, one for $\lambda < \frac{k}{t}$ and one for $\lambda > \frac{k}{t}$. For the first part, as $f_t(\lambda)$ is increasing on $(0, \frac{k}{t})$, we have

$$\left| \int_0^{k/t} \lambda^{2k} e^{-2t\lambda} dR(\lambda) \right| \leq 2 \sup_{\lambda \in (0, \frac{k}{t})} |R(\lambda)| \left(\frac{k}{t}\right)^{2k} e^{-2k} \leq 2C \left(\frac{k}{t}\right)^{2k + \frac{d-1}{2}} e^{-2k}. \tag{1.49}$$

From this and thanks to Stirling formula, we deduce that there exists $K_d > 0$, depending only on d, such that

$$\left| \int_0^{k/t} \lambda^{2k} e^{-2t\lambda} dR(\lambda) \right| \leq K_d \sqrt{t} M_k(t). \tag{1.50}$$

For the second part, we get by integration by parts

$$\int_{k/t}^{+\infty} f_t(\lambda) dR(\lambda) = -R(\frac{k}{t}) f_t(\frac{k}{t}) - \int_{k/t}^{+\infty} R(\lambda) f_t'(\lambda) d\lambda. \tag{1.51}$$

Our previous argument showed that $|R(\frac{k}{t}) f_t(\frac{k}{t})| \leq K_d \sqrt{t} M_k(t)$, and for the second term, as $f_t(\lambda)$ is decreasing on $(\frac{k}{t}, +\infty)$, we have[1]

$$\left| \int_{k/t}^{+\infty} R(\lambda) f_t'(\lambda) d\lambda \right| \leq -C \int_{k/t}^{+\infty} \lambda^{\frac{d-1}{2}} f_t'(\lambda) d\lambda \tag{1.52}$$

$$= C \left(\frac{k}{t}\right)^{\frac{d-1}{2}} f_t(\frac{k}{t}) + C \int_{k/t}^{+\infty} \frac{d-1}{2} \lambda^{\frac{d-3}{2}} f_t(\lambda) d\lambda. \tag{1.53}$$

Now it suffices to bound that last integral, using the Gamma function bound, we have

$$\left| \int_{k/t}^{+\infty} \frac{d-1}{2} \lambda^{\frac{d-3}{2}} f_t(\lambda) d\lambda \right| \leq \frac{d-1}{2} \frac{\Gamma(2k + \frac{d-1}{2})}{(2t)^{2k + \frac{d-1}{2}}} \leq \frac{d-1}{2} \sqrt{t} M_k(t). \tag{1.54}$$

From this, we deduce that there exists $K_d' > 0$, depending only on d, such that

$$\left| \int_0^{+\infty} \lambda^{2k} e^{-2t\lambda} dR(\lambda) \right| \leq K_d' \sqrt{t} M_k(t). \tag{1.55}$$

[1]Here we implicitly suppose that $d > 1$, because the case $d = 1$ is simpler as it suffices to bound $f_t(k/t)$.

Hence for $t < \frac{1}{4K_d'^2}$, we have

$$\|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} \geq \frac{1}{2} M_k(t), \tag{1.56}$$

which gives us the desired result. $\square$

Remark 3. *Notice that in the previous proof, we used a strong version of local Weyl's law, which gives a remainder of order $\lambda^{(d-1)/2}$. This was crucial to ensure that we can bound uniformly in k by $M_k(t)$ the last part of inequality (1.49). The Stirling formula tells us that it would have been impossible with a remainder of order $\lambda^{d/2-\varepsilon}$ for $\varepsilon \in (0, \frac{1}{2})$.*

Now to handle the case $T = +\infty$, we need the following simple lemma.

Lemma 3. *There exists $K > 0$, such that for all $k > 0$, we have*

$$\forall x \in \mathcal{M},\ \forall t > k^2, \quad \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} \leq K \lambda_1^{2k+\frac{d}{2}}. \tag{1.57}$$

Proof. Using the same notations as in the previous lemma, and thanks to (1.29) we have

$$\|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} = \sum_{n=1}^{+\infty} \lambda_n^{2k} e^{-2t\lambda_n} \phi_n(x)^2 \leq C^2 \sum_{n=1}^{+\infty} \lambda_n^{2k+\frac{d}{2}} e^{-2t\lambda_n}. \tag{1.58}$$

Now if we denote by m_1 the multiplicity of the first eigenvalue, we see that as $t \to +\infty$, the dominant term is $m_1 C^2 \lambda_1^{2k+\frac{d}{2}} e^{-2t\lambda_1}$. Hence, it suffices to show that the following sequence

$$S_k = \sum_{n=m_1+1}^{+\infty} \left(\frac{\lambda_n}{\lambda_1}\right)^{2k+\frac{d}{2}} e^{-2\lambda_1 k^2 (\frac{\lambda_n}{\lambda_1} - 1)}, \tag{1.59}$$

is uniformly bounded in k. The Weyl law tells us that there exists $C_d > 0$ such that $\lambda_n \sim C_d n^{2/d}$, and this allows us to conclude the uniform boundedness in k thanks to elementary calculus. $\square$

Now we prove Theorem 1, for this it suffices to treat the $T = +\infty$ case. Let $k \geq 1$, as $L^2(\mathcal{M})$ is dense in $\mathcal{D}'(\mathcal{M})$, we can take a sequence $(f_n)_{n\in\mathbb{N}}$ of $L^2(\mathcal{M})$ functions, such that $f_n \to \Delta_g^k \delta_x$ in $\mathcal{D}'(\mathcal{M})$. Then as $e^{-t\Delta_g}$ is a smoothing operator for $t > 0$, we can pass to the limit in the integrated observability and this gives us

$$\int_0^{+\infty} h(t) \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\mathcal{M})} dt \leq C_\infty \int_0^{+\infty} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} dt. \tag{1.60}$$

Hence, Lemma 2 gives us that there exists $K_d > 0$ such that

$$\int_0^{\tau_d} \frac{h(t)}{(2t)^{\frac{d}{2}+2k}} \Gamma(2k + d/2) K_d dt \leq C_\infty \int_0^{+\infty} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} dt. \tag{1.61}$$

Then we split the integral in two parts, one for $t < k^2$ and another for $t > k^2$. For the first part, setting $\delta = d(x,\omega)$ allows us to apply Lemma 1 to get the following estimate

$$\int_0^{k^2} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} dt \leq k^2|\omega| C_\infty C^2 4^{k+d+1/2} \frac{(2(k+d+1))!^2}{\delta^{4k}}. \tag{1.62}$$

Now dividing both sides of (1.62) by $(2k)!\Gamma(2k+d/2)$ and multiplying by a^{2k} for some $a>0$, we get thanks to the Stirling formula

$$\int_0^{k^2} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} \frac{a^{2k}}{(2k)!\Gamma(2k+d/2)} dt \leq \left(\frac{2a}{\delta^2}\right)^{2k} R_d(k), \tag{1.63}$$

where $R_d(k)$ grows at most polynomially in k. Hence for all $B>2$, there exists $C_B>0$ such that $2^{2k}R_d(k) \leq C_B B^{2k}$, then summing over $k\in\mathbb{N}$, we get

$$\sum_{k=1}^{+\infty} \int_0^{k^2} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} \frac{a^{2k}}{(2k)!\Gamma(2k+d/2)} dt \leq C_B \sum_{k=1}^{+\infty} \left(\frac{Ba}{\delta^2}\right)^{2k}. \tag{1.64}$$

Hence the right-hand side converges if $a < \frac{\delta^2}{B}$. Now for the second part, we use Lemma 3 to get the following estimate

$$\int_{k^2}^{+\infty} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} dt \leq K \lambda_1^{2k+\frac{d}{2}} e^{-2k^2\lambda_1}. \tag{1.65}$$

Thus dividing both sides by $(2k)!\Gamma(2k+d/2)$ and multiplying by a^{2k} for any $a < \frac{\delta^2}{B}$, and summing over $k\in\mathbb{N}$, we get

$$\sum_{k=1}^{+\infty} \int_{k^2}^{+\infty} \|\Delta_g^k p(t,x,\cdot)\|^2_{L^2(\omega)} \frac{a^{2k}}{(2k)!\Gamma(2k+d/2)} dt < \sum_{k=1}^{+\infty} K \frac{\lambda_1^{2k+\frac{d}{2}} a^{2k}}{(2k)!\Gamma(2k+d/2)} < +\infty. \tag{1.66}$$

From this, we finally deduce that

$$\int_0^{\tau_d} \frac{h(t)}{t^{\frac{d}{2}}} \cosh(\frac{a}{2t}) dt < +\infty, \tag{1.67}$$

which concludes the proof of Theorem 1 thanks to Lemma 5.

1.2.2 Proof of Corollary 2

Here we prove Corollary 2. Let $T\in(0,+\infty)$ and $h\in\mathcal{O}([0,T])$ such that we have an integrated observability inequality with weight h. By Theorem 1, for all $\kappa < \frac{1}{4}$, there exists $A_T>0$ such that

$$\forall t\in[0,T), \quad h(t) \leq A_T e^{-\kappa\mathcal{L}(\omega)^2/t}. \tag{1.68}$$

Thus, we deduce the following bound

$$\forall \lambda \geq 0, \quad H_T(\lambda)^2 \leq A_T \int_0^T e^{-\kappa\mathcal{L}(\omega)^2/t - 2\lambda t} dt. \tag{1.69}$$

Hence using Lemma 4, there exists $A'_T > 0$ such that we have

$$\forall \lambda > 0, \quad H_T(\lambda)^2 \leq A'_T \frac{1}{\lambda^{3/4}} e^{-2\sqrt{2\lambda\kappa\mathcal{L}(\omega)^2}}. \tag{1.70}$$

Now, as H_T is bounded on $[0, +\infty)$, it suffices to prove the bound for $\lambda \geq 1$, and hence taking square roots in the previous inequality concludes as $\sqrt{2\kappa} < 1/\sqrt{2}$.

1.3 Conclusions and perspectives

One thing we would like is to generalize our results to more abstract functional settings, that would allow us to treat more cases, especially we deeply think that our method could be adapted to the case of sub-Riemannian Laplacians.

In fact the only thing that we would need is to adapt the proof of Lemma 2 to that setting, as it was said in Remark 3, we used a strong version of local Weyl's law, which is not known in the sub-Riemannian setting, see [CHT21] to see the state of the art on Weyl Laws in sub-Riemannian geometry.

One other perspective would be to generalize our results but for L^p observability inequalities, especially the L^1 case, using the fact that the heat kernel is stochastically complete, i.e. (see [Gri09] for more details)

$$\forall t > 0,\ \forall x \in \mathcal{M}, \quad \int_{\mathcal{M}} p_t(x, y) dy = 1. \tag{1.71}$$

1.4 Appendix

In this Appendix we gather some technical lemmas that we used in the main proofs.

Lemma 4. *Let $T \in (0, +\infty)$ and $\lambda, \alpha > 0$, we have*

$$\int_0^T e^{-\lambda t - \frac{\alpha}{t}} dt \underset{\lambda \to +\infty}{\sim} \sqrt{\pi} \frac{\alpha^{\frac{1}{4}}}{\lambda^{\frac{3}{4}}} e^{-2\sqrt{\alpha\lambda}}. \tag{1.72}$$

Proof. By the change of variables $t \mapsto t/T$, we only need to treat the case $T = 1$. Set

$$I(\lambda) := \int_0^1 e^{-\lambda t - \alpha/t} dt, \qquad s := \sqrt{\alpha\lambda}. \tag{1.73}$$

With the change of variable $t = \sqrt{\alpha/\lambda}\, y$ we obtain

$$I(\lambda) = \sqrt{\frac{\alpha}{\lambda}} \int_0^{\sqrt{\lambda/\alpha}} \exp\big(-s\,(y + y^{-1})\big)\, dy. \tag{1.74}$$

The phase $\psi(y) := y + y^{-1}$ has a unique nondegenerate minimum at $y_0 = 1$, with $\psi(1) = 2$, $\psi'(1) = 0$, and $\psi''(1) = 2$. Fix $\eta \in (0, 1)$ and split the integral into a local part $|y - 1| \leq \eta$ and its complement. On the complement, $\psi(y) \geq 2 + c_\eta$ for some $c_\eta > 0$, hence that contribution is $O\big(e^{-(2+c_\eta)s}\big) = o\big(e^{-2s}\big)$.

On the local part, write $y = 1 + u$ and use the Taylor expansion

$$\psi(1+u) = 2 + u^2 + O(u^3) \quad (u \to 0). \tag{1.75}$$

Therefore,

$$\int_{|u|\le\eta} \exp\big(-s\,\psi(1+u)\big)\,du = e^{-2s}\int_{|u|\le\eta} e^{-su^2}\big(1+O(s|u|^3)\big)\,du. \tag{1.76}$$

With the change of variable $v = \sqrt{s}\,u$, we get

$$\int_{|u|\le\eta} e^{-su^2}\big(1+O(|u|^3)\big)\,du = \frac{1}{\sqrt{s}}\int_{|v|\le\eta\sqrt{s}} e^{-v^2}\Big(1+O\Big(\frac{|v|^3}{s^{1/2}}\Big)\Big)\,dv = \frac{\sqrt{\pi}}{\sqrt{s}}\,(1+o(1)), \tag{1.77}$$

as $s \to \infty$. Collecting the estimates yields

$$I(\lambda) = \sqrt{\frac{\alpha}{\lambda}}\,e^{-2s}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,(1+o(1)) = \sqrt{\pi}\,\frac{\alpha^{1/4}}{\lambda^{3/4}}\,e^{-2\sqrt{\alpha\lambda}}\,(1+o(1)), \quad \lambda \to \infty, \tag{1.78}$$

which proves the claimed asymptotic. □

Lemma 5. *Let $T \in (0,+\infty]$ and $h \in \mathcal{O}([0,T])$, such that there exists $\gamma > 0$ for which we have*

$$\int_0^T h(t)e^{\frac{\gamma}{t}}\,dt < +\infty. \tag{1.79}$$

Then for every $\varepsilon > 0$, we have $h(t) = O(e^{-\frac{(\gamma-\varepsilon)}{t}})$ near $t = 0$.

Proof. Set $I := \int_0^T h(t)\,e^{\gamma/t}\,dt < \infty$. Since $h \in \mathcal{O}([0,T])$ is locally nondecreasing at 0, there exists $t_0 > 0$ such that h is nondecreasing on $(0,t_0]$. Now, fix $\varepsilon > 0$ and choose $\delta \in (0,1)$ small enough such that

$$\frac{\gamma}{1+\delta} \;\ge\; \gamma - \frac{\varepsilon}{2}. \tag{1.80}$$

For any $t \in (0,t_0]$ with $(1+\delta)t < T$, by monotonicity we have $h(\tau) \ge h(t)$ for $\tau \in [t,(1+\delta)t]$. Hence

$$I \;\ge\; \int_t^{(1+\delta)t} h(\tau)\,e^{\gamma/\tau}\,d\tau \;\ge\; h(t)\int_t^{(1+\delta)t} e^{\gamma/\tau}\,d\tau. \tag{1.81}$$

Moreover, for $\tau \in [t,(1+\delta)t]$ we have $e^{\gamma/\tau} \ge e^{\gamma/((1+\delta)t)}$, thus

$$\int_t^{(1+\delta)t} e^{\gamma/\tau}\,d\tau \;\ge\; \delta t\,e^{\gamma/((1+\delta)t)}. \tag{1.82}$$

Combining the two estimates gives

$$h(t) \;\le\; \frac{I}{\delta t}\,e^{-\gamma/((1+\delta)t)}. \tag{1.83}$$

Finally, for the chosen δ and for $t > 0$ small enough we can absorb the factor $1/t$ into a subexponential term, namely

$$\frac{1}{t} \le e^{\varepsilon/(2t)}. \tag{1.84}$$

Therefore,

$$h(t) \;\le\; C\,\exp\Big(-\frac{\gamma}{(1+\delta)t}+\frac{\varepsilon}{2t}\Big) \;\le\; C\,\exp\Big(-\frac{\gamma-\varepsilon}{t}\Big), \tag{1.85}$$

for some constant $C > 0$ and all sufficiently small $t > 0$. This proves $h(t) = O\big(e^{-(\gamma-\varepsilon)/t}\big)$ as $t \to 0^+$. $\square$

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Chapter 2

F-equivalence for parabolic systems and applications to nonlinear PDE

2.1 Introduction

Consider the following nonlinear control problem

$$\partial_t u(t) = Au(t) + \mathcal{F}(u(t)) + Bw(t), \tag{2.1}$$

where A is an unbounded linear operator, $\mathcal{F}$ is a nonlinear perturbation that can also be unbounded, B is a given control operator, w is a control that can be chosen, and $u(t) \in H$, where H is a given Hilbert space. An interesting question in control theory is to know whether we can rapidly stabilize the system with the control w, that is whether the following holds.

Problem 1. *For any $\lambda > 0$, there exists an operator $K = K_\lambda \in \mathcal{L}(D(A); U)$ such that, by choosing $w(t) = K_\lambda u(t)$, the system* (2.1) *is (locally) exponentially stable with decay rate λ.*

"Locally" here only makes sense when $\mathcal{F} \neq 0$ and means locally around the equilibrium $u^* = 0$ and U refers to a given Hilbert space such that $BK \in \mathcal{L}(D(A); H)$.

This question has been extensively investigated in the last decades in different frameworks and under different assumptions on A, B and $\mathcal{F}$. Even when the system is linear (i.e. $\mathcal{F} = 0$), answering this question in all generality is challenging. The first works date back (at least) to Slemrod [Sle72] in 1972 in the case where $\mathcal{F} = 0$ and B is a bounded operator. Many results in this framework were obtained using tools from optimal control and Linear-Quadratic (LQ) theory by Lions, Barbu, Lasiecka, Triggiani and many others [Lio71; Bar18; LT00a; LT00b; Urq05; Ves13]. This question was considered in the semilinear framework, i.e. $\mathcal{F}(H) \subset H$ in [Tré17]. Other approaches successfully obtained results even when B is unbounded; one can cite, for instance, the observability approach of [TWX19; Liu+22; MWY23; KWY25], in particular [TWX19] shows that when (A, B) is exactly null controllable the system can be rapidly stabilized[1] and [Liu+22] gives a very nice characterization of stabilizability in terms

[1] See also [Zab20] for exponential stability without requesting rapid stabilization, i.e. arbitrary λ.

of observability. Other results, inspired by optimal control approaches and using Riccati or Hamilton-Jacobi-Bellman equations, were obtained on semilinear systems either on particular systems of interest (for instance, [BK14], see also [CK17]) or when A is parabolic [BW03; Bar18]. One can also cite the Gramian approach (see, for instance, [Urq05]) and in particular the recent result of [Ngu24b] where the author managed modify the Gramian approach to get a quantitative rapid stabilization and even a finite-time stabilization for a semilinear system (i.e. $\mathcal{F}(H) \subset H$) when A is skew-adjoint (see [Ngu24a] on the application to the 1D Schrödinger equation). Nevertheless, in these approaches, it often happens that the control feedback laws are not explicit, as they rely either on solving a minimization problem and an algebraic Riccati equation [Kom97] (or even a Hamilton-Jacobi-Bellman equation [Bel52], see also [KVW23] for a learning approach to alleviate this difficulty) or because they rely on the knowledge of the semigroup $e^{\mathcal{A}^* t}$.

Other methods have been introduced, specifically for the parabolic cases. For instance, the impressive Frequency Lyapunov which obtains a quantitative rapid stabilization and even a finite-time stabilization for the multidimensional heat equation [Xia24] (and later for the 2D Navier-Stokes equation [Xia23a]) by relying on Carleman estimates and a specific Lyapunov function. However, it requires the control to be distributed, that is K takes value in H. For this most challenging case, where B is unbounded and the control belongs to a finite-dimensional space (that is K takes value in $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$) it is worth highlighting, still in the parabolic framework, the work of [BT14] where the authors manage to deal both with a wide class of linear parabolic systems and, notably, with some nonlinear systems where the nonlinear perturbation $\mathcal{F}$ is not semilinear (i.e. $\mathcal{F}(H)$ does not belong to H) as long as the system is approximately controllable [BT11].

Another method was introduced to tackle the aforementioned limitations and deal with this most challenging case in a general framework: the *F-equivalence* (for *feedback equivalence*[2]). The principle is simple: instead of trying directly to find a feedback K, this method solves a different mathematical problem:

Problem 2. *Given an operator $\widetilde{A}$, find $(T, K) \in \mathcal{L}(H) \times \mathcal{L}(D(A), \mathbb{R}^k)$ such that T is an isomorphism from H into itself and maps (in H) the system*

$$\partial_t u(t) = \mathcal{A}u(t) + BKu(t) \tag{2.2}$$

to the system

$$\partial_t u(t) = \widetilde{\mathcal{A}}u(t). \tag{2.3}$$

Of course, $\widetilde{\mathcal{A}}$ generates an exponentially stable semigroup on H, the existence of such a pair implies the exponential stability of the original system in H. This approach is sometimes called *generalized backstepping* or *Fredholm backstepping* for, when T is a Volterra transform of the second kind, it coincides with the well investigated *backstepping* method for 1D systems

[2]This name was, in fact, first introduced by Brunovsky in [Bru70] for linear finite-dimensional system.

[BK02; KS08; Hu+15] (see also [Xia19] or [CEH23] and references therein). In the last ten years, F-equivalence approaches have been used to achieve rapid stabilization of many systems, first for particular systems [CL14; CL15; CHO17; CGM18; DG19; GLM21; Gag+22; Cor+22; LM23] and recently in increasingly general settings [Gag+24; HL24].

By definition, the F-equivalence problem is *a priori* asking for more than the rapid stabilization, which is only its consequence. However, it actually turns out that for many systems the sufficient conditions of existence of an F-equivalence are relatively permissive and for skew-adjoint systems they were shown to be even better than the usual known sufficient conditions for rapid stabilization [HL24, Section 3.1]. This can be explained since the F-equivalence allows to look at the problem directly as a stabilization problem rather than deriving a feedback from the resolution of the (optimal) control problem. This avoids usual admissibility conditions on B (see, for instance, [Cor07, Section 2.3] or [TW09]) which usually ensures that the system (2.1) is well-posed for a whole class of control, which is not needed for the stabilization problem (one only needs the system to be well-posed along $w = Ku$). By considering the equivalence with the simpler system (2.3), it also avoids regularity additional conditions on K as long as (2.1) is well-posed. This, together with the explicitness of the feedback constructed makes the F-equivalence interesting both as a problem and for its application to the rapid stabilization.

However, when it comes to parabolic systems, the existing F-equivalence conditions are likely too conservative compared to usual condition of rapid stabilization: they are stronger than asking for the exact null controllability of (A, B) of [TWX19], let alone the approximate controllability as in [BT14].

Another limit is that all the existing results of F-equivalence assume uniformly bounded multiplicities of the eigenvalues of A. For skew-adjoint systems this condition is necessary as soon as there is a finite number of controls (i.e. K takes value in $\mathbb{R}^k$). For parabolic system, however, this is likely too conservative as well. While, strictly speaking, this does not restrict this approach to 1D systems, it is still a strong limitation in practice when looking at systems that are multidimensional in space.

In this paper, we tackle these two limitations. We show the following (see Theorem 9 for a more detailed version):

Theorem 4. *Let A be a parabolic (unbounded) operator on a Hilbert space H with a Riesz basis of eigenvectors and $B \in (D(A^*)')^k$. For all $\lambda \in \mathbb{R}_{>0}$, there exists an explicitly computable $m(\lambda)$ such that either*

- *$k < m(\lambda)$ and there is no exponentially stable operator $\widetilde{A}$ such that there exists a solution to the F-equivalence problem 2.*
- *$k \geq m(\lambda)$, in this case if B satisfies the λ-approximate controllability condition (H_B), there exists an explicit $\widetilde{A}$, T and $K \in \mathcal{L}(H; \mathbb{R}^k)$ such that $\widetilde{A}$ is exponentially stable with decay rate λ and (T, K) are solutions of the F-equivalence problem 2.*

This implies in particular that the original system (2.1) with $w(t) = Ku$ and $\mathcal{F} = 0$ is well-posed (see Proposition 2).

A consequence of this theorem is the exponential stability of the nonlinear system (2.1) (see also Theorem 13):

Theorem 5. *Let A be a parabolic (unbounded) operator on a Hilbert space H with a Riesz basis of eigenvectors and let $B \in (D_{-s}(A))^k$ (with $s \in [0,1]$) satisfies the λ-approximate controllability condition (H_B). If $\mathcal{F}$ satisfies the following assumption: We set $\gamma = \min(1-s, 1/2)$, we have $\mathcal{F}$ is a map from $D_\gamma(A)$ to $D_{-1/2}(A)$, such that there exists $\eta, K > 0$ and $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ non-decreasing continuous at 0 with $\Phi(0) = 0$, which satisfies the following conditions: for all $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$, we have*

$$\|\mathcal{F}(u)\|_{D_{-1/2}(A)} \leq \Phi(\|u\|_H)\, \|u\|_{D_\gamma(A)}, \tag{2.4}$$

$$\begin{aligned} \|\mathcal{F}(u) - \mathcal{F}(v)\|_{D_{-1/2}(A)} \leq\, & K(\|u\|_{D_\gamma(A)} + \|v\|_{D_\gamma(A)})\, \|u-v\|_H \\ & + K\Phi(\|u\|_H + \|v\|_H)\, \|u-v\|_{D_\gamma(A)}. \end{aligned} \tag{2.5}$$

then there exists an explicit $K \in \mathcal{L}(H, \mathbb{R}^k)$ such that the system (2.1) *is exponentially stable with $w(t) = Ku$.*

As intended, the λ-approximate controllability assumption (H_B) is much less restrictive than the F-equivalence conditions in a general setting given by [HL24]. This either significantly improves or recovers the recent results of [CL15; GLM21; Gag+22; LM23; HL24]. As an illustration, in the case of the $1d$ heat equation and Burgers' equation on a torus studied in [Gag+22; HL24] with $B = (B_1, B_2)$ where $B_1 : \; x \mapsto \sum_{n\geq 1} b_n^1 \sin(nx)$ is odd and $B_2 \; x \mapsto \sum_{n \geq 0} b_n^2 \cos(nx)$ is even, the F-equivalence condition of [GLM21; HL24] amounts to

$$b_0^2 \neq 0 \text{ and } \exists \gamma \in [0, 1/2),\ \forall j \in \{1,2\},\ \forall n \geq 1,\ c \leq |b_n^j| \leq Cn^\gamma, \tag{2.6}$$

and our λ-approximate controllability assumption (H_B) amounts to the (much) less restrictive condition

$$\forall j \in \{1,2\},\ \forall n \leq \sqrt{\lambda},\ b_n^j \neq 0. \tag{2.7}$$

In particular, if one wants to stabilize at any rate λ, the previous conditions amount to the approximate controllability of (A, B) by the generalized Fattorini criterion (see [BT14] and Lemma 11). The example of the Kuramoto-Shivashinski system of [CL15] is discussed in Section 2.4.2. It is also worth noting in (2.4)–(2.5) that Φ is not necessarily Lipschitz which allows to derive examples as the one in Section 2.4.4.

In Theorem 5 there is also no requirement on the multiplicity of the eigenvalues of A, which makes it suitable, for instance, to multidimensional systems in space, in contrast to essentially all the previous F-equivalence approaches [CL14; CL15; GLM21; LM23; Gag+22; Gag+24; HL24].

While our main goal of this work is to improve the conditions for the F-equivalence problem for parabolic systems and extend them to multidimensional systems, one can note that the rapid stabilization result Theorem 13 can still be compared to previous results for parabolic systems that use different approaches. In particular, compared to [Ray19], the system is not necessarily linear and, to [BT14; Bad+20], the conditions on the nonlinearity $\mathcal{F}$ are different (in particular Φ does not need to be Lipschitz and, thanks to the F-equivalence, our conditions on the nonlinearity are expressed only with respect to A and do not depend *a priori* on the feedback, see Remark 14). Note that compared to [BT14], and similarly to [BT11], B can belong to $(D(A^*)')^k$ and does not have to belong to the smaller space $D((A^*)^{-s})^k$ for some $s < 1$.

In Section 2.4 we illustrate this result on several examples of applications. Among others, we study the rapid stabilization of a heat equation with potential on a Riemannian manifold, the (nonlinear) Kuramoto-Sivashinsky, the Navier-Stokes equations and a quasilinear heat equation.

Overall, the paper is organized as follows: in Section 2.2 we introduce our setting, notations and useful propositions. In Section 2.3 we state our main results, i.e. Theorems 9 and 13 that are shown in Section 2.5, and in Section 2.4 we give examples of applications on concrete systems.

While revising this manuscript we were made aware of another preprint [GKN25] treating the F-equivalence problem for parabolic systems in the particular case of linear systems with a similar method but relying on Cauchy matrices to derive the control. It would be an interesting question to know whether such an approach could also be applied to the general systems we consider in this paper.

2.2 Settings and notations

2.2.1 Functional setting

Let $(H, \langle \cdot\,, \cdot \rangle_H)$ be a Hilbert space and A be an unbounded operator on H satisfying the following conditions:

(A1) There exists a family of eigenvectors $(e_n)_{n\geq 1}$ of A that is a Riesz basis of H. Then for every $x \in H$, Ax has a meaning (not necessarily in H) and we define the domain of A as

$$D(A) = \{x \in H \mid Ax \in H\}.$$

(A2) The sequence $(\mathrm{Re}(\lambda_n))_{n\geq 1}$, where λ_n are the eigenvalues associated to $(e_n)_{n\in\mathbb{N}^*}$, is non-increasing and we have

$$\mathrm{Re}(\lambda_n) \underset{n\to+\infty}{\to} -\infty.$$

(A3) There exists $C > 0$ such that

$$\forall n \geq 1,\ |\,\mathrm{Re}(\lambda_n)| \geq C|\,\mathrm{Im}(\lambda_n)|.$$

Remark 6. *In proofs, we often replace the Riesz basis assumption in (A1) by a Hilbert basis. This does not change the result, since any Riesz basis is a Hilbert basis for an equivalent inner product.*

Definition 3. *Let A be an unbounded operator on H. We will say that A is diagonal parabolic if it satisfies (A1), (A2) and (A3).*

In particular, any parabolic self-adjoint operator is a diagonal parabolic operator.

Remark 7. *Hypothesis (A2) implies that the multiplicity of each eigenvalue of A is finite, but the supremum of all such multiplicities can be infinite. Hypothesis (A2) is here to ensure that A is the infinitesimal generator of an analytic semigroup.*

For A diagonal parabolic operator, hypothesis (A2) ensures that $\{\mathrm{Re}(\lambda_n) \mid n \geq 1\}$ has a maximum m_A. We define $c_A := \max(0, m_A)$, this constant will be useful for stating our main result. Besides, from (A1) there exists an inner product of H such that $(e_n)_{n\geq 1}$ is orthonormal and the associated norm is equivalent to the norm associated with $\langle \cdot\,, \cdot \rangle_H$ (see [HL24, Section 2]). Therefore in the following we will assume, without loss of generality, that e_n is an orthonormal basis of H. Also, for every $x \in H$ we define x_n to be $\langle x\,, e_n \rangle_H$.

Since A is closed, $D(A)$ is a Hilbert space endowed with the inner product

$$\forall y, z \in D(A),\ \langle y\,, z \rangle_{D(A)} := \langle y\,, z \rangle_H + \langle Ay\,, Az \rangle_H. \tag{2.8}$$

Notice that $(e_n/\sqrt{1 + |\lambda_n|^2})_{n\geq 1}$ forms an orthonormal basis of $D(A)$. We see that $(D(A), H, D(A)')$ is a Gelfand triple.

Finally, note that a parabolic diagonal operator A is normal (see Proposition 13), in particular we have $D(A) = D(A^*)$. This allows us to see it as an element of $\mathcal{L}(H, D(A)')$. This is formalized in Appendix 2.10.2.

Later, it will be useful to consider AM where $M \in \mathcal{L}(D(A)')$, and we wish that $AM \in \mathcal{L}(H, D(A)')$. However, this is not generally the case, so we need to work with another operator algebra. This is the reason for the following proposition.

Proposition 1. *We define the following algebra*

$$\mathcal{L}_H(D(A)') = \{M \in \mathcal{L}(D(A)') \mid M_{|H} \in \mathcal{L}(H)\}, \tag{2.9}$$

which, endowed with the norm,

$$||M||_{\mathcal{L}_H(D(A)')} := \sup_{\substack{||x||^2_{D(A)'} + ||y||^2_H = 1 \\ (x,y)\in D(A)'\times H}} \sqrt{||Mx||^2_{D(A)'} + ||My||^2_H}, \tag{2.10}$$

is a Banach algebra. Furthermore, we have the embedding of $\mathcal{L}_H(D(A)')$ in $\mathcal{L}(D(A)' \times H)$

$$\forall M \in \mathcal{L}_H(D(A)'),\ \forall (x, y) \in D(A)' \times H,\ \varphi(M)(x, y) := (Mx, My). \tag{2.11}$$

Proof. It is clear that $\mathcal{L}_H(D(A)')$ is an algebra and that φ is injective. Notice that the norm given above was designed such that

$$\forall M \in \mathcal{L}_H(D(A)'),\ ||\varphi(M)||_{\mathcal{L}(D(A)'\times H)} = ||M||_{\mathcal{L}_H(D(A)')}.$$

Hence, $\mathcal{L}_H(D(A)')$ is a Banach algebra. $\square$

We denote by $\mathcal{GL}_H(D(A)')$ the group of invertible elements in $\mathcal{L}_H(D(A)')$, then the above embedding gives us the following characterization:

$$M \in \mathcal{GL}_H(D(A)') \iff (M, M_{|H}) \in \mathcal{GL}(D(A)') \times \mathcal{GL}(H). \tag{2.12}$$

Notice that now, for every $M \in \mathcal{L}_H(D(A)')$, we have $AM \in \mathcal{L}(H, D(A)')$.

2.2.2 Generalized Sobolev spaces

To define the generalized Sobolev spaces, let us fix $\delta \geq 0$ such that $-A+\delta$ is invertible.[3] We denote here and in the following $-A+\delta$ for $-A+\delta I$. Then we can define

$$\forall s \in \mathbb{R}, \quad D_s(A) := D((-A+\delta)^s). \tag{2.13}$$

Endowed with the usual graph norm, and after metric completion if $s < 0$, these become Hilbert spaces. Let $s \in \mathbb{R}$, one can show that the following norm is equivalent

$$\forall x = \sum_{n\geq 1} x_n e_n \in D_s(A), \quad \|x\|^2_{D_s(A)} := \sum_{n\geq 1}(1+|\lambda_n|^2)^s |x_n|^2. \tag{2.14}$$

We can identify $D_{-s}(A)$ with $D_s(A)'$, and the triple $(D_{-s}(A), H, D_s(A))$ forms a Gelfand triple. Note that $D_1(A) = D(A)$ and $D_0(A) = H$.

We refer to this scale of spaces as generalized Sobolev spaces, as they share the same properties as classical Sobolev spaces. For instance, when A is a power of the Laplace operator on a closed manifold, these spaces coincide with the usual Sobolev spaces. For more details, see Subsection 2.4.1 and in particular (2.32).

2.2.3 Frequency decomposition

Let $\lambda > 0$. Here, we focus on defining a decomposition of our spaces into low and high frequencies. We define the low and high frequency spaces as

$$\begin{aligned} L_\lambda &= \operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) \geq -\lambda\}, \\ H_\lambda &= \overline{\operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) < -\lambda\}}^H, \\ D(A)'_\lambda &= \overline{\operatorname{span}\{e_n \mid \operatorname{Re}(\lambda_n) < -\lambda\}}^{D(A)'}. \end{aligned} \tag{2.15}$$

[3]Note that it is sectorial since A is a diagonal parabolic operator.

We have the orthogonal decomposition $H = L_\lambda \oplus H_\lambda$ and $D(A)' = L_\lambda \oplus D(A)'_\lambda$. We set $N(\lambda) := \dim L_\lambda < +\infty$ and define $m(\lambda)$ to be the greatest multiplicity of any eigenvalue in L_λ. We denote by P_L and P_H the orthogonal projections on L_λ and H_λ in H.

2.2.4 Control setting

In this paper, we seek to stabilize A using only a finite number of scalar controls, which means that our control system looks like

$$\partial_t u = Au + Bw(t),$$

with $w(t) \in \mathbb{C}^m$ and B a given control operator. Let E be a normed vector space. We will use the canonical isomorphism from E^m to $\mathcal{L}(\mathbb{C}^m, E)$ to identify an element $B \in \mathcal{L}(\mathbb{C}^m, E)$ with $(B_1, \dots, B_m) \in E^m$ in the following way

$$B : z \in \mathbb{C}^m \mapsto \sum_{j=1}^{m} z_j B_j \in E. \tag{2.16}$$

Let $B \in (D(A)')^m$ for some fixed $m \geq 1$, when the supremum of the multiplicities of A is infinite, the pair (A, B) is not approximately controllable, making it impossible to achieve stabilization at any desired rate. Instead, we set a target stabilization rate $\lambda > 0$ and seek to determine whether we can stabilize the system at this rate. Note that, for any given $\lambda > 0$, we have the following lemma

Lemma 6. *There exist $N_1(\lambda), \dots, N_{m(\lambda)}(\lambda) \in \mathbb{N}^*$ and a partition $(e_n^1)_{n\geq 1}, \dots, (e_n^{m(\lambda)})_{n\geq 1}$ of $(e_n)_{n\geq 1}$ such that, if we set $L_\lambda^j = span((e_n^j)_{1\leq n\leq N_j(\lambda)})$ and $\mathcal{H}^j = \overline{span((e_n^j)_{n\geq 1})}^H$ (thus $H = \bigoplus_{j=1}^{m(\lambda)} \mathcal{H}^j$) , then the multiplicities of eigenvalues are simple in L_λ^j and*

$$L_\lambda = \bigoplus_{k=1}^{m(\lambda)} L_\lambda^k, \quad H = \bigoplus_{j=1}^{m(\lambda)} \mathcal{H}^j.$$

Moreover, for each $j \in \{1, \dots, m(\lambda)\}$, A induces a diagonal parabolic operator on $\mathcal{H}^j$ such that

$$A = A_1 + \cdots + A_{m(\lambda)}, \quad D(A) = \bigoplus_{j=1}^{m(\lambda)} D(A_j), \quad D(A)' = \bigoplus_{j=1}^{m(\lambda)} D(A_j)'.$$

This is shown in Appendix 2.13.

In the following, we define P_L^j as the orthogonal projection onto L_λ^j in $\mathcal{H}^j$. We then set $P_H^j = Id_{\mathcal{H}^j} - P_L^j$. In order to achieve a stabilization at decay rate λ, we make the following assumption on our control operator:

(H_B) $m \geq m(\lambda)$, and for all $j \in \{1, \dots, m(\lambda)\}$, we have $B_j \in D(A_j)'$ and

$$\langle B_j , e_n^j \rangle_{D(A)'} \neq 0, \ \ \forall\, n \in \{1, \dots, N_j(\lambda)\},$$

Definition 4. *Let $B = (B_1, \dots, B_m) \in (D(A)')^m$. We say that B is F_λ-admissible if it satisfies (H_B).*

Let us briefly comment on this assumption (H_B). We first allow our control operators to be unbounded, which means they belong to a larger space than H. For well-posedness reasons, we know that $D(A)'$ is optimal, and here it is allowed, which makes it slightly less restrictive than the condition of [BT14] where $B \in D_{-s}(A)^m$ with $s < 1$ and similar to the condition of [Bad+20] (which consider in addition a non-autonomous setting) . Then the condition on the scalar product of the B_j in (H_B) is here to ensure that the low-frequency system is controllable.

Finally, we define the concept of target operator, which we will need to define the concept of F-equivalence.

Definition 5. *Let A be a diagonal parabolic operator and $\lambda > 0$. We say that an unbounded normal operator D on H is a λ-target if it is the infinitesimal generator of a differentiable semigroup on H with a growth rate of at most $-\lambda$. This means that*

$$\exists C > 0, \forall x \in H, \forall t \geq 0, \ ||e^{tD} x||_H \leq C e^{-\lambda t} ||x||_H.$$

Remark 8. *In Definition 5, the operator D could have a domain different from A, and this happens, for instance, in [Cor+22]. In the following, however, we will only consider λ-targets having the same domain as A.*

2.3 Main results

2.3.1 F-equivalence results

Let $\lambda > 0$, A be a diagonal parabolic operator in H, D be a λ-target with domain $D(A)$, and $B \in (D(A)')^m$. We can now define the concept of F-equivalence.

Definition 6 (F-equivalence)**.** *Let $(T, K) \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^m)$. We say that (T, K) is an F-equivalence of (A, B, D), or that it is an F-equivalence between (A, B) and D, if*

$$\begin{cases} T(A + BK) = DT \ \text{in } \mathcal{L}(H, D(A)'), \\ TB = B \ \text{in } D(A)'. \end{cases} \tag{2.17}$$

Furthermore, if $K \in \mathcal{L}(L_\lambda, \mathbb{C}^m)$, we say that (T, K) is a parabolic F-equivalence.

Before presenting our main result, we want to emphasize few points. As one can imagine, finding an F-equivalence is a challenging problem. The condition $TB = B$ in 6 is included for two reasons. First, to make the problem linear in (T, K), and secondly, in the hope of achieving uniqueness, i.e., that there exists one and only one F-equivalence of (A, B, D), which greatly aids in finding a solution. As Theorem 25 shows (see also [Cor15]), if (A, B) is finite-dimensional and with $D = A - \lambda$, then there exists one and only one F-equivalence of (A, B, D). Unfortunately, in our case, we will show in Section 2.6 that in general, there is no uniqueness to the F-equivalence problem. However, at the same time, we introduce a new formalism that we call *weak F-equivalence*, which allows us to regain uniqueness. More precisely, we show that the uniqueness is linked with the approximate controllability of (A, B). For more details, see Section 2.6. Our first main theorem is the following

Theorem 9 (Parabolic F-equivalence)**.** *Let A be a diagonal parabolic operator, and let $\lambda \in \mathbb{R}_{>0}$. Suppose $B \in (D(A)')^{m(\lambda)}$ is an F_λ-admissible control operator (see Definition 4). For $\mu \geq \lambda + c_A$, we define*

$$D = (A_L - \mu)P_L + A_H P_H. \tag{2.18}$$

Then, D is a λ-target and for almost every $\mu \geq \lambda + c_A$, there exists a parabolic F-equivalence (T, K) between (A, B) and D.

Remark 10. *Note that the choice of λ-target D does not depend on B and only depends on λ, μ and A.*

The proof of Theorem 9 is provided in Subsection 2.5.1. Let us comment on the above theorem. First, the definition of D is very natural if the goal is to obtain a λ-target, that is an operator with growth rate at most $-\lambda$. The hope behind is to find a feedback operator K acting only on the low-frequency space, as one can expect intuitively and as is classically used for parabolic systems see, for instance, [BT14; Tré17; Xia24]. Secondly, besides D being intuitive, it is novel to perform F-equivalence in a generic framework with a target operator different from[4] $A - \lambda$, as done in [Gag+22; Gag+24; HL24]. This is, of course, made possible by the parabolic nature of the system.

In fact, here, it would have been impossible to take $D = A - \lambda$, because if, for example, $B \in H$, then BK would be a compact operator on H. If there exists $T \in \mathcal{GL}(H)$ such that

$$T(A + BK) = (A - \lambda)T,$$

we should have $\sigma(A + BK) = \sigma(A - \lambda)$. However, we know that if BK is compact, A and $A + BK$ should asymptotically have the same spectrum (see, for instance, [EN99, Chapter IV, Sec. 1]), which would be absurd.

2.3.2 Rapid stabilization results

Here, we apply F-equivalence to the stabilization of parabolic systems. We start with the linear case, then we consider semilinear equations. As before, let $\lambda > 0$, A be a diagonal parabolic operator in H, D be a λ-target, and $B \in (D(A)')^m$. As expected, finding a solution to the F-equivalence problem also ensures rapid stabilization of the linear system:

Proposition 2. *Let $(T, K) \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^m)$ be a F-equivalence of (A, B, D). Then $A + BK$ is an unbounded operator on H with dense domain $T^{-1}(D(A))$ which generates a*

[4] Note that another target operator was also used in [Cor+22] in the particular case of the Saint-Venant system.

differentiable semigroup with a growth of at most $-\lambda$. In particular, the Cauchy problem

$$\begin{cases} \partial_t u = (A+BK)u, \quad \forall t > 0, \\ u(0) = u_0 \in H, \end{cases} \tag{2.19}$$

is well-posed in $C^0([0,+\infty);H) \cap C^\infty(0,+\infty;H)$ with $u(t) \in D(A+BK)$, $\forall t > 0$, and we have the following exponential stability estimate:

$$\exists C > 0, \forall u_0 \in H, \forall t \geq 0, \ ||u(t)||_H \leq Ce^{-\lambda t}||u_0||_H.^5 \tag{2.20}$$

Proof. See Appendix 2.14. □

The above proposition demonstrates the utility of the F-equivalence approach for the stabilization of linear systems. To show that (A,B) is exponentially stable at rate λ, one only needs to demonstrate the existence of an F-equivalence between (A,B) and some target operator D as described above. Additionally, notice that this F-equivalence approach ensures that the problem is well-posed.

Now, using F-equivalence we aim to stabilize the following type of nonlinear control system

$$\partial_t u = Au + Bw(t) + \mathcal{F}(u), \tag{2.21}$$

where $\mathcal{F}$ is a nonlinear map, possibly highly nonlinear and with regularity as low as that of the operator A (see Assumption 1).

More precisely, let (T,K) be an F-equivalence of (A,B,D) given by Theorem 9, we aim to show local well-posedness and exponential stability of the following system

$$\begin{cases} \partial_t u = (A+BK)u + \mathcal{F}(u), \\ u(0) = u_0 \in H. \end{cases} \tag{2.22}$$

As D is a diagonal parabolic operator that generates a exponentially stable semigroup, $A+BK$ is too, thanks to the F-equivalence property. For this kind of operator, nonlinear perturbation are well-known, see [Paz83; Tré17], essentially, when we see $\mathcal{F}$ as a small Lipschitz perturbation, optimal conditions[6] for well-posedness and stability are given in Appendix 2.12.

This would give us a result where the condition on the unboundedness on $\mathcal{F}$ are stated with respect to $D_\gamma(A+BK)$, which is what typically happens in the existing literature, see [BT14; BT11]. Here F-equivalence will allow us to give unboundedness conditions on $D_\gamma(A)$ directly, because thanks to Lemma 10, if $B \in D_{-s}(A)$ with $s \in [0,1]$, then $T \in \mathcal{GL}(D_{1-s}(A))$ and this will ensure by F-equivalence that $D_r(A+BK) = D_r(A)$ for all $r \in [-1, 1-s]$. Hence we introduce the following assumption on $\mathcal{F}$.

[5]Here $u(\cdot)$ is the solution with initial condition u_0.

[6]Optimal in the sense that, as $\mathcal{F}$ goes from $D_{1/2}(A+BK)$ to $D_{-1/2}(A+BK)$, hence it is as regular as $A+BK$, allowing quasi-linear perturbation.

Assumption 1. *Let $s \in [0,1]$ be such that $B \in (D_{-s}(A))^m$, we set $\gamma = \min(1-s, \frac{1}{2})$. Then $\mathcal{F}$ is a map from $D_\gamma(A)$ to $D_{-1/2}(A)$, such that there exists $\eta, K > 0$ and $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ non-decreasing continuous at 0 with $\Phi(0) = 0$, which satisfies the following conditions: for all $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$, we have*

$$\|\mathcal{F}(u)\|_{D_{-1/2}(A)} \leq \Phi(\|u\|_H)\, \|u\|_{D_\gamma(A)}, \tag{2.23}$$

$$\begin{aligned} \|\mathcal{F}(u) - \mathcal{F}(v)\|_{D_{-1/2}(A)} \leq{}& K(\|u\|_{D_\gamma(A)} + \|v\|_{D_\gamma(A)})\, \|u-v\|_H \\ &+ K\Phi(\|u\|_H + \|v\|_H)\, \|u-v\|_{D_\gamma(A)}. \end{aligned} \tag{2.24}$$

Remark 11 (Admissibility of B and domain of $\mathcal{F}$)**.** *Notice that if $B \in (D_{-1/2}(A))^m$ (for instance, if B is an admissible control operator, which is very common in control problems) then $\gamma = \frac{1}{2}$. In this situation, we allow $\mathcal{F}$ to be defined on $D_{1/2}(A)$ and to take values in $D_{-1/2}(A)$, meaning that $\mathcal{F}$ may be in some sense as irregular as the operator A itself. This is optimal in the sense that, for such generators, further lowering the regularity of $\mathcal{F}$ would in general destroy local well-posedness.*

Remark 12 (Sufficient conditions on $\mathcal{F}$)**.** *In practice we often work with $\Phi = Id_{\mathbb{R}}$ (but not always, see Subsection 2.4.4), then notice that a sufficient condition on $\mathcal{F}$ to ensure Assumption 1 is $\mathcal{F}(0) = 0$ and for any $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$*

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_{D_{-1/2}(A)} \leq C(\|u\|_H + \|v\|_H)\|u-v\|_{D_\gamma(A)}. \tag{2.25}$$

Even with $\Phi = Id_{\mathbb{R}}$, (2.23) *itself is not sufficient in general to ensure the previous inequality. To understand where* (2.24) *comes from, suppose that $\mathcal{F}(u) = B(u,u)$ where B is a bilinear map satisfying, for any $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$,*

$$\|B(u,v)\|_{D_{-1/2}(A)} \leq C\|u\|_H\|v\|_{D_\gamma(A)}. \tag{2.26}$$

Then, if B is symmetric (as in Subsection 2.4.2), (2.23) *is indeed sufficient to ensure* (2.25)*. Otherwise (as in Subsection 2.4.4 and 2.4.3) from* (2.26) *we deduce that for all $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$*

$$\|B(u,u) - B(v,v)\|_{D_{-1/2}(A)} \leq C(\|u\|_H\|u-v\|_{D_\gamma(A)} + \|v\|_{D_\gamma(A)}\|u-v\|_H). \tag{2.27}$$

Which ensures that Assumption 1 holds with $\Phi = Id_{\mathbb{R}}$ and $K = C$.

The above discussion leads to the following theorem.

Theorem 13. *Let A be a diagonal parabolic operator, $B \in (D_{-s}(A))^m$ with $s \in [0,1]$ and $\mathcal{F}$ be a map satisfying Assumption 1. Let $\lambda \in \mathbb{R}_{>0}$, suppose that B is an F_λ-admissible control operator, and let (T, K) be an F-equivalence given by Theorem 9. Then, there exists $\delta > 0$ such that for every $u_0 \in H$, if $\|u_0\|_H \leq \delta$, there exists a unique solution*

$u \in C_b^0([0,+\infty); H) \cap L^2((0,+\infty); D_\gamma(A))$ (with $\gamma = \min(1-s, \frac{1}{2})$) to the following system

$$\begin{cases} \partial_t u = (A + BK)u + \mathcal{F}(u), \\ u(0) = u_0. \end{cases} \tag{2.28}$$

In addition, the system is exponentially stable, more precisely setting $C = ||T||_{\mathcal{L}(H)}||T^{-1}||_{\mathcal{L}(H)}$, we have

$$\forall t \geq 0, \quad ||u(t)||_H \leq Ce^{-\lambda t}||u_0||_H. \tag{2.29}$$

Remark 14 (Comparison with [BT11; BT14])**.** *The main goal of this paper is to study the F-equivalence of parabolic systems and the rapid stabilization results is only a useful consequence. As mentioned earlier, stabilization results with nonlinear perturbations for parabolic systems is already known (see [BT11; BT14]). Notice that our assumptions on $\mathcal{F}$ are similar. However, ours are slightly more general thanks to the use of Φ, which allows us to have sharper hypotheses in some cases (see Section 2.4.4). Apart from this, one contribution of F-equivalence here, is that it allows us to express the unboundedness condition on $D_\gamma(A)$ and not on $D_\gamma(A + BK)$, which gives a simpler and more natural condition.*

Remark 15 (Expliciteness of T and K)**.** *Equation* (2.29) *shows that for numerical applications, or for quantitative finite time stabilization as in [Xia24], the knowledge of T is crucial in order to compute C. Note that in the proof of Theorem 9, we give a complete explicit expression of T, see Propositions 8 and 10.*

The proof of this result is done in Section 2.5.2, and note that, it is in fact a direct corollary of Theorem 9 and results on well-posedness and stability in Appendix 2.12. Moreover, as shown in the proof of Theorem 9, K is obtained by solving a finite-dimensional linear system of equations. Thus, the above theorem provides a very simple and explicit way to stabilize a whole class of nonlinear parabolic PDE.

2.4 Applications and examples

2.4.1 Heat equation on manifolds

Let $(\mathcal{M}, g)$ be a compact oriented and connected d-dimensional Riemannian manifold, and set $H = L^2(\mathcal{M})$, $A = \Delta_g$ and $(e_n)_{n\geq 1}$ an orthonormal basis of eigenvectors such that

$$0 = -\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_k \to +\infty. \tag{2.30}$$

The Weyl law tells us that $N(\lambda) \underset{\lambda\to+\infty}{\sim} \frac{\mathrm{Vol}(\mathcal{M})\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}}$, where we recall that $N(\lambda) = \dim(L_\lambda)$ (see (2.15)). Because the eigenvalues are non-decreasing, and from the definition of L_λ, we have $N(\lambda_n) = n$. Hence, we have

$$|\lambda_n| \underset{n\to+\infty}{\sim} \frac{4\pi^2}{(\omega_d \mathrm{Vol}(\mathcal{M}))^{\frac{2}{d}}} n^{\frac{2}{d}}. \tag{2.31}$$

Using the classical characterization of Sobolev spaces on manifolds as in [CPR01], we have

$$\forall s \in \mathbb{R}, \quad D_s(A) = H^{2s}(\mathcal{M}). \tag{2.32}$$

Notice that Δ_g is a diagonal parabolic operator on $L^2(\mathcal{M})$, so we have the following immediate corollary of our main result.

Corollary 3. *Let $(\mathcal{M}, g)$ be a compact oriented and connected d-dimensional Riemannian manifold. We set $H = L^2(\mathcal{M})$, $A = \Delta_g$, and fix $\lambda \in \mathbb{R}_{>0}$. Let $B \in (H^{-2}(\mathcal{M}))^{m(\lambda)}$ be an F_λ-admissible control operator. Then there exists $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R}^{m(\lambda)})$ such that the Cauchy problem*

$$\begin{cases} \partial_t u = \Delta_g u + BKu, & \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{2.33}$$

is well-posed,[7] *and we have the following stability estimate:*

$$\exists C > 0, \forall u_0 \in L^2(\mathcal{M}), \forall t \geq 0, \ ||u(t)||_{L^2} \leq Ce^{-\lambda t}||u_0||_{L^2}. \tag{2.34}$$

Before presenting concrete examples, notice that we have all the information we need about the asymptotic behavior of $N(\lambda)$ and λ_n. Interestingly, we see that the growth rate of the eigenvalues is entirely governed by the topology of the manifold, while the prefactor depends on its geometry, specifically its volume.

Despite all this, we still have no information on $m(\lambda)$. In fact, it is well-known that Δ_g has "generically" simple eigenvalues, which implies $m(\lambda) = 1$ for all $\lambda > 0$. For a more precise definition, see [Mic72; Uhl76].

We conclude this subsection with some examples. Let $\mathcal{M}$ be a compact oriented and connected Riemannian manifold of dimension $d \leq 3$. If we make no further assumptions on it, we have the following result.

Proposition 3. *Let $p \in \mathcal{M}$, and denote by δ_p the Dirac distribution at p. Let $\lambda > 0$, and define $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R})$ as*[8]

$$\forall f \in L^2(\mathcal{M}), \quad Kf = -\lambda \int_{\mathcal{M}} f \, d\mu_g. \tag{2.35}$$

Then, if λ is small enough (more precisely, $\lambda < -\lambda_2$), the Cauchy problem

$$\begin{cases} \partial_t u = \Delta_g u + \delta_p K u, & \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{2.36}$$

is well-posed, and there exists $C > 0$ such that

$$\forall u_0 \in L^2(\mathcal{M}), \quad ||u(t)||_{L^2} \leq Ce^{-\lambda t}||u_0||_{L^2}. \tag{2.37}$$

[7] In the sense of Proposition 2.

[8] Here, $d\mu_g$ is the measure induced by the Riemannian volume form.

Proof. Since $\mathcal{M}$ is connected, $\lambda_1 = 0$ is a simple eigenvalue, and hence $e_1 = \frac{1}{\sqrt{\mathrm{Vol}(\mathcal{M})}}$ and $\lambda_2 < 0$. Let $\lambda \in (0, -\lambda_2)$, then $m(\lambda) = 1$. By the Sobolev embedding theorem, $\delta_p \in H^{-\frac{d}{2}-\varepsilon}(\mathcal{M}) \subset H^{-2}(\mathcal{M})$. Since $(e_n)_{n\geq 1}$ forms a Hilbert basis of $L^2(\mathcal{M})$, it is straightforward to show that, in $H^{-\frac{d}{2}-\varepsilon}(\mathcal{M})$, we have

$$\delta_p = \sum_{n\geq 1} e_n(p) e_n. \tag{2.38}$$

Now we can apply Theorem 9. Since $L_\lambda = \mathrm{span}(e_1)$, we have $(\Delta_g)_L = 0$ and $(\delta_p)_L = \frac{1}{\sqrt{\mathrm{Vol}(\mathcal{M})}}$. Solving the one-dimensional F-equivalence is then trivial, and we obtain

$$K = -\lambda \sqrt{\mathrm{Vol}(\mathcal{M})} \langle \cdot, e_1 \rangle_{L^2}. \tag{2.39}$$

□

To stabilize the heat equation at any desired rate using only a Dirac control, we assume that $(\mathcal{M}, g)$ is such that Δ_g has only simple eigenvalues. As we discussed earlier, this situation typically arises when a manifold is chosen randomly. We now present the following stronger result.

Proposition 4. *Suppose that dim $\mathcal{M} \leq 3$ and that Δ_g has only simple eigenvalues, then for almost every $p \in \mathcal{M}$, the pair (Δ_g, δ_p) is approximately controllable. This implies that for every $\nu > 0$, there exists $K \in \mathcal{L}(L^2(\mathcal{M}), \mathbb{R})$ such that the Cauchy problem*

$$\begin{cases} \partial_t u = \Delta_g u + \delta_p K u, & \forall t > 0, \\ u(0) = u_0 \in L^2(\mathcal{M}), \end{cases} \tag{2.40}$$

is well-posed, and there exists a constant $C > 0$ such that

$$\forall u_0 \in L^2(\mathcal{M}), \quad ||u(t)||_{L^2} \leq C e^{-\nu t} ||u_0||_{L^2}. \tag{2.41}$$

Proof. The nodal set of e_n (i.e. $e_n^{-1}(\{0\})$) has measure zero for every $n \geq 1$, so for almost every $p \in \mathcal{M}$, we have

$$\forall n \geq 1, \quad e_n(p) \neq 0. \tag{2.42}$$

Recall that $\delta_p \in H^{-\frac{d}{2}-\varepsilon}(\mathcal{M})$ and that $\delta_p = \sum_{n\geq 1} e_n(p) e_n$ with $\langle \delta_p, e_n \rangle = e_n(p) \neq 0$. Using Lemma 11, we deduce that for almost every $p \in \mathcal{M}$, the pair (Δ_g, δ_p) is approximately controllable. Now let $\lambda > \nu$ as in Theorem 9. Since $m(\lambda) = 1$, the approximate controllability implies that δ_p is F_λ-admissible. Therefore, Theorem 9 together with Proposition 2 allows us to conclude.

□

Remark 16. *As always with parabolic F-equivalence, the feedback K can be easily constructed using the same method as in the example following Corollary 4.*

In the following subsections, we provide some concrete applications of Theorem 13 to some classical PDE controlled systems.

2.4.2 Kuramoto–Sivashinsky equation

Here, we focus on the Kuramoto–Sivashinsky equation on the one-dimensional torus, which is given by

$$\partial_t u + \Delta^2 u + \Delta u + \frac{1}{2}\partial_x(u^2) = 0. \tag{2.43}$$

This equation was introduced by Yoshiki Kuramoto and Gregory Sivashinsky to study flame front propagation, for more details see [Kur78; Siv80; Siv77]. To apply Theorem 13, we need to establish the appropriate setting. We work in $H = L^2(\mathbb{T})$, and define $A = -(\Delta^2 + \Delta)$, hence we have

$$\forall s \in \mathbb{R}, \quad D_s(A) = H^{4s}(\mathcal{M}). \tag{2.44}$$

The eigenbasis of A is $(e_n)_{n\in\mathbb{Z}}$, defined as follows

$$\forall n \in \mathbb{Z}, \forall \theta \in \mathbb{T}, \quad e_n(\theta) = \frac{1}{(2\pi)^{d/2}} e^{i\sum_{k=1}^{d} \frac{n_k}{l_k}\theta_k}. \tag{2.45}$$

In order to apply our result we can reindex by $\mathbb{N}_{>0}$ in the following way

$$\forall n \geq 1,\ \tilde{e}_n := \begin{cases} e_k \text{ if } n = 2k+1,\ k \geq 0, \\ e_{-k} \text{ if } n = 2k,\ k \geq 1. \end{cases} \tag{2.46}$$

With this notation we similarly define $\tilde{\lambda}_n$, we deduce from the previous section that

$$\tilde{\lambda}_n = -\left\lfloor \frac{n}{2} \right\rfloor^4 + \left\lfloor \frac{n}{2} \right\rfloor^2 \underset{n\to+\infty}{\sim} -\frac{n^4}{16}. \tag{2.47}$$

Hence A is a diagonal parabolic operator on H. For simplicity, we will continue to use the family $(e_n)_{n\in\mathbb{Z}}$ for the Sobolev norms.

We want to define $\mathcal{F}(u) = -\frac{1}{2}\partial_x(u^2)$ as a map from $L^2(\mathbb{T})$ to $H^{-2}(\mathbb{T})$, we will use the following lemma.

Lemma 7. *Let $u, v \in L^2(\mathbb{T})$. Then there exists a constant $C > 0$ such that*

$$||\partial_x(uv)||_{H^{-2}} \leq C||u||_{L^2}||v||_{L^2}. \tag{2.48}$$

Proof. Let $u, v \in C^\infty(\mathbb{T})$, we have

$$uv = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} u_k v_{n-k} \right) e_n. \tag{2.49}$$

We define $\langle n\rangle = \sqrt{1+|n|^2}$ for all $n \in \mathbb{Z}$. Then by definition of Sobolev norms we have

$$||\partial_x(uv)||_{H^{-2}}^2 = \sum_{n\in\mathbb{Z}} n^2 \left|\sum_{k\in\mathbb{Z}} u_k v_{n-k}\right|^2 \langle n\rangle^{-4}. \tag{2.50}$$

Now, applying Cauchy-Schwarz inequality gives

$$||\partial_x(uv)||_{H^{-2}}^2 \leq ||u||_{L^2}^2 ||v||_{L^2}^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{-2}. \tag{2.51}$$

Which concludes the proof. □

Now, as $\mathcal{F}$ is quadratic, the previous lemma ensures us that $\mathcal{F}$ satisfies Assumption 1. Note that for all $\lambda > 0$, we have $m(\lambda) = 3$. Hence, applying Theorem 13 leads to the following immediate corollary.

Corollary 4. *Let $\lambda \in \mathbb{R}_{>0}$. Suppose that $(f_1, f_2, f_3) \in (H^{-2}(\mathbb{T}))^3$ is an F_λ-admissible control operator. Then there exist $K_1, K_2, K_3 \in \mathcal{L}(L^2(\mathbb{T}), \mathbb{C})$ and $\delta > 0$ such that for every $u_0 \in L^2(\mathbb{T})$ with $\|u_0\|_{L^2} \leq \delta$, there exists a unique maximal solution $u(\cdot) \in C^0([0, +\infty); L^2(\mathbb{T}))$ to*

$$\begin{cases} \partial_t u + \Delta^2 u + \Delta u + \frac{1}{2}\partial_x(u^2) + f_1 K_1 u + f_2 K_2 u + f_3 K_3 u = 0, \\ u(0) = u_0. \end{cases} \tag{2.52}$$

Furthermore, there exist $C_\lambda > 0$ such that

$$\forall t \geq 0, \quad ||u(t)||_{L^2} \leq C_\lambda e^{-\lambda t} ||u_0||_{L^2}. \tag{2.53}$$

We now provide a concrete application of the above corollary to demonstrate that the feedback derived from the parabolic F-equivalence is easily constructible.

Suppose we want to stabilize the system at a rate $\lambda = 20$, then $N(\lambda) = 5$. For this example, we define

$$\forall x \in \mathbb{T}, \quad f_1(x) = \frac{1}{\sqrt{2\pi}}, \quad f_2(x) = \frac{1}{\sqrt{2\pi}}(e^{-ix} + e^{-2ix}), \quad f_3(x) = \frac{1}{\sqrt{2\pi}}(e^{ix} + e^{2ix}). \tag{2.54}$$

Thus, we have $f_1 = \tilde{e}_1$, $f_2 = \tilde{e}_2 + \tilde{e}_4$, $f_3 = \tilde{e}_3 + \tilde{e}_5 \in L_\lambda$, and $B = (f_1, f_2, f_3)$ is clearly λ-admissible.

To find our feedbacks, we only need to solve a finite-dimensional F-equivalence problem. Identifying $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_4, \tilde{e}_3, \tilde{e}_5)$ with the canonical basis of $\mathbb{C}^5$, we have

$$A_L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{2.55}$$

Let $\mu \geq \lambda$. If we denote by $(\tilde{T}, \tilde{K})$ the solution of

$$\begin{cases} \tilde{T}(A_L + B\tilde{K}) = (A_L - \mu)\tilde{T}, \\ TB = B, \end{cases} \tag{2.56}$$

then since $TB = B$ is equivalent to $Tf_1 = f_1$, $Tf_2 = f_2$, and $Tf_3 = f_3$, we can decompose the problem into three subproblems. It is straightforward to solve these either manually or numerically, and we find that

$$\begin{aligned} \tilde{T} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{12}+1 & -\frac{\mu}{12} & 0 & 0 \\ 0 & \frac{\mu}{12} & 1-\frac{\mu}{12} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{12}+1 & -\frac{\mu}{12} \\ 0 & 0 & 0 & \frac{\mu}{12} & 1-\frac{\mu}{12} \end{pmatrix}, \\ \tilde{K} &= \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu(\mu+12)}{12} & \frac{\mu(\mu-12)}{12} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mu(\mu+12)}{12} & \frac{\mu(\mu-12)}{12} \end{pmatrix}. \end{aligned} \tag{2.57}$$

By Theorem 9, we know that for almost every $\mu \geq \lambda$, we can use $\tilde{K}$ to define our feedbacks in Corollary 4. Then for all $f \in L^2(\mathbb{T})$, the above corollary applies with

$$\begin{aligned} K_1 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} -\mu f(x)\, dx, \\ K_2 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \left(-\frac{\mu(\mu+12)}{12} e^{ix} + \frac{\mu(\mu-12)}{12} e^{2ix} \right) dx, \\ K_3 f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \left(-\frac{\mu(\mu+12)}{12} e^{-ix} + \frac{\mu(\mu-12)}{12} e^{-2ix} \right) dx. \end{aligned} \tag{2.58}$$

As we will now briefly show, our result can also be applied to the Kuramoto-Sivashinsky system studied in [CL15], as it was one of the first examples of Fredholm backstepping, that is

$$\begin{cases} \partial_t u + \Delta^2 u + \nu \Delta u + \frac{1}{2}\, \partial_x(u^2) = 0 & \text{in } (0,1) \times (0,+\infty), \\ u(t,0) = u(t,1) = 0 & \text{for all } t > 0, \\ \Delta u(t,0) = w(t), \quad \Delta u(t,1) = 0 & \text{for all } t > 0, \\ u(0,\cdot) = u_0 & \text{in } L^2(0,1). \end{cases} \tag{2.59}$$

with $\nu > 0$. Thus, we work in $H = L^2(0,1)$, with $A = -\Delta^2 - \nu\Delta$ and

$$D(A) = \{u \in H^4(0,1) \mid u(0) = u(1) = \Delta u(0) = \Delta u(1) = 0\}.$$

Setting $e_n(x) = \sqrt{2}\sin(\pi n x)$ for all $x \in (0,1)$ and for $n \geq 1$, we observe that $(e_n)_{n\geq 1}$ forms an orthonormal basis of H consisting of eigenvectors of A, with eigenvalues

$$\forall n \geq 1, \quad \lambda_n = -\pi^4 n^4 + \nu \pi^2 n^2. \tag{2.60}$$

Thus, A is a self-adjoint diagonal parabolic operator on H. Multiplying (2.59) by a smooth function in $D(A)$ and integrating by parts, we obtain

$$\forall u \in D(A), \quad B^* u = -\partial_x u(0). \tag{2.61}$$

which defines, by duality, $B \in \mathcal{L}(\mathbb{R}, D(A)')$. Moreover, we have

$$\forall n \geq 1, \quad \langle B \,, e_n \rangle_{D(A)', D(A)} = -\pi n. \tag{2.62}$$

Following [CL15], we assume that

$$\nu \notin \{n^2\pi^2 + k^2\pi^2 \mid n, k \geq 1,\, n \neq k\}, \tag{2.63}$$

which ensures that A has only simple eigenvalues (thus, (A, B) is approximately controllable by Lemma 11). Consequently, (2.62) implies that B is F_λ-admissible for all $\lambda > 0$. The nonlinearity $\mathcal{F}$ can be handled similarly to the previous example. Therefore, applying Theorem 13, we obtain the following corollary.

Corollary 5. *Let $\lambda \in \mathbb{R}_{>0}$. There exists $K \in \mathcal{L}(L^2(0,1), \mathbb{R})$ and $\delta > 0$ such that for every $u_0 \in L^2(0,1)$ with $\|u_0\|_{L^2} \leq \delta$, there exists a unique maximal solution $u \in C^0([0,+\infty); L^2(0,1))$ to*

$$\begin{cases} \partial_t u + \Delta^2 u + \nu\Delta u + \frac{1}{2}\,\partial_x(u^2) = 0 & \text{in } (0,1) \times (0,+\infty), \\ u(t,0) = u(t,1) = 0 & \forall t > 0, \\ \Delta u(t,0) = K u(t,\cdot), \quad \Delta u(t,1) = 0 & \forall t > 0, \\ u(0,\cdot) = u_0 & \text{in } L^2(0,1). \end{cases} \tag{2.64}$$

Furthermore, there exist constants $C_\lambda > 0$ such that

$$\forall t \geq 0, \quad ||u(t,\cdot)||_{L^2} \leq C_\lambda e^{-\lambda t} ||u_0||_{L^2}. \tag{2.65}$$

Remark 17. *As before, the feedback K is constructed by solving a finite-dimensional linear system (see Remark C2). Compared to [CL15], this approach is significantly simpler. One could similarly use our approach for the Kuramoto-Shivashinski system found in [LK01].*

2.4.3 Navier-Stokes equations

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. In this subsection we consider the scalar controlled Navier-Stokes equations which are given by

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = \sum_{i=1}^{m} w_i(t) f_i & \text{in } \Omega \times (0,+\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{2.66}$$

We first need to express (2.66) in our framework. In order to do this we will reuse classical results from the literature about Navier-Stokes systems, such as the ones found in [Ray07a]. The literature on the stabilization of Navier-Stokes equations is extensive, see, for example [Bad+20; BT11; BT14; BT04; Ray06; Ray07b; BLT06; Bad09; BKP19; RT10; AT14; Mit19; BT23; Xia23b] among many others.

We introduce the following functional spaces

$$L^2_\sigma(\Omega) = \overline{\{u \in C_c^\infty(\Omega;\mathbb{R}^2) \mid \nabla\cdot u = 0 \text{ in } \Omega\}}^{L^2}, \tag{2.67}$$

$$V_0^s(\Omega) = \{u \in H^s(\Omega;\mathbb{R}^2) \mid \nabla\cdot u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\},\ s > \frac{1}{2}. \tag{2.68}$$

Recall the Leray-Helmholtz decomposition $L^2(\Omega;\mathbb{R}^2) = L^2_\sigma(\Omega) \oplus \nabla H^1(\Omega)$. We denote by $\mathbb{P}$ the orthogonal projection onto $L^2_\sigma(\Omega)$, which is often called the Leray projection. Our state space will be $H = L^2_\sigma(\Omega)$, and our operator will be the Stokes operator defined by

$$A = \mathbb{P}\Delta \text{ with } D(A) = V_0^2(\Omega). \tag{2.69}$$

It is well-known that A is a self-adjoint diagonal parabolic operator on H with negative eigenvalues, see [Fur01, Lemma 3.1].

Here and in the following, every element of $H^{-1}(\Omega;\mathbb{R}^2)$ is seen as an element in $D_{-1/2}(A)$ by the continuous extension of the Leray projector from $H^{-1}(\Omega;\mathbb{R}^2)$ to $D_{-1/2}(A)$. Now we consider the nonlinearity given by

$$\mathcal{F}(u) = (u\cdot\nabla)u. \tag{2.70}$$

From [BT11, Section 5] the system (2.66) is equivalent to

$$\begin{cases} \partial_t u - Au + \mathcal{F}(u) = 0, \\ u(0) = u_0 \in H. \end{cases} \tag{2.71}$$

Also from [BT11, Section 5], $\mathcal{F}$ goes from $D_{1/2}(A) = V_0^1(\Omega)$ to $H^{-1}(\Omega;\mathbb{R}^2)$, and

$$\forall u, v \in V_0^1(\Omega), \quad \|(u\cdot\nabla)v\|_{H^{-1}} \le C\|u\|_{L^2}\|v\|_{H^1}. \tag{2.72}$$

Here $B = (f_1, \ldots, f_m)$, hence from Remark 12, we deduce that for any control operator such that $B \in D_{-1/2}(A)^m$ (for instance if $B \in H^{-1}(\Omega;\mathbb{R}^2)^m$), $\mathcal{F}$ satisfies Assumption 1.

Now for the sake of doing a simple illustration of Theorem 13, we will assume that Ω is such that the Stokes operator has a simple spectrum (from [OZ01], we know it is generically the case). Hence we only need one forcing term as a control operator, which we denote by $f \in H^{-1}(\Omega;\mathbb{R}^2)$. We will assume that (A, f) is approximately controllable, and thanks to Lemma 11, if we denote by $(e_n)_{n\ge1}$ the eigenfunctions of A, this is equivalent to the following condition

$$\forall n \ge 1, \quad \langle f, e_n\rangle_{H^{-1},H^1} \ne 0. \tag{2.73}$$

The next proposition summarizes the stabilization result in the described setting.

Proposition 5. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$ such that the Stokes operator has a simple spectrum. Let $f \in H^{-1}(\Omega;\mathbb{R}^2)$ such that (A, f) is approximately controllable. (i.e. (2.73) holds) Then for all $\lambda > 0$, there exists a feedback operator $K \in \mathcal{L}(L^2_\sigma(\Omega), \mathbb{R})$ and $\delta > 0$, such that for every $u_0 \in L^2_\sigma(\Omega)$ with $\|u_0\|_{L^2} \leq \delta$, there exists a unique solution $u \in C^0_b([0,+\infty); L^2_\sigma(\Omega)) \cap L^2((0,+\infty); V^1_0(\Omega))$ to*

$$\begin{cases} \partial_t u - \mathbb{P}\Delta u + (u\cdot\nabla)u = (Ku)f, \\ u(0) = u_0. \end{cases} \tag{2.74}$$

Moreover, there exists a constant $C > 0$ (independent of u_0) such that

$$\forall t \geq 0, \quad \|u(t)\|_{L^2} \leq Ce^{-\lambda t}\|u_0\|_{L^2}. \tag{2.75}$$

Remark 18. *Note that again, the strength of Assumption 1 described in Remark 14 allows us to have irregular forcing terms, without any additional work.*

2.4.4 Quasilinear heat equation

Let Ω be a smooth connected bounded open subset of $\mathbb{R}^d$, with $d = 2$.[9] In this subsection we consider a general form of quasilinear heat equation given by

$$\begin{cases} \partial_t u - \operatorname{div}(D(u)\nabla u) = f(u) + w(t)b & \text{in } \Omega\times(0,+\infty), \\ D(0)\nabla u\cdot n = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{2.76}$$

We consider here a classical quasilinear diffusion model. While the literature on feedback stabilization for semilinear parabolic PDEs is extensive, results for quasilinear dynamics are comparatively scarce, see, for instance, the recent 1D stabilization result for a quasilinear heat equation [Bel+25]. In order to keep the presentation simple and the feedback fully explicit (proportional to the spatial average), we restrict ourselves to a single scalar control. This already provides exponential stabilization under the assumptions below. Of course, if one needs rapid stabilization, the approach readily extends by adding more controls, which allows one to accelerate the decay rate at will.

We set $\widetilde{D} = D - D(0)$ and $\widetilde{f}(u) = f(u) - f'(0)u$. Let $\varepsilon \in (0, 1/2)$ and set $s = d/2 + \varepsilon$, we make the following assumptions on D and f:

(D1) We have $D \in C^{2,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{d\times d})$ with $D(0)$ symmetric positive definite.

(D2) We have $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ with $f(0) = 0$.

(D3) We have $b \in H^\varepsilon(\Omega)$ and $\langle b\,, 1\rangle_{L^2} \neq 0$.

[9] We choose $d = 2$ to have sharp assumptions on f and D.

Here $C^{1,1}_{\text{loc}}$ means continuously differentiable with locally Lipschitz derivative and $C^{2,1}_{\text{loc}}$ is defined similarly. Now we express (2.76) in our framework. First, to define A, we define the intermediate operator $A_0 = \operatorname{div}(D(0)\nabla(\cdot))$ on $L^2(\Omega)$ with the usual Neumann boundary condition domain. It is well known (see [Bre10, Chapter 9]) that A_0 is self-adjoint and diagonal parabolic on $L^2(\Omega)$. We denote by $(\lambda'_n)_{n\geq 0}$ its eigenvalues and $(e'_n)_{n\geq 0}$ an associated orthonormal basis of eigenvectors, notice that $\lambda'_0 = 0$ and $e'_0 = 1/\sqrt{|\Omega|}$. We will work with the state space $H = D_s(A_0) = H^s(\Omega)$ (note that, as $\varepsilon < 1/2$, $s < \frac{3}{2}$ and there is no Neumann boundary condition to add). We define A, as the operator with domain $D(A) = D_{s+1}(A_0)$ that act as follows

$$\forall u \in D(A), \quad Au = A_0 u + f'(0)u. \tag{2.77}$$

Hence A is self-adjoint and diagonal parabolic on H, and its eigenvalues are $\lambda_n = \lambda'_n + f'(0)$, associated to the eigenvectors $e_n = e'_n$, for $n \geq 0$. We assume that $f'(0) \geq 0$, such that the uncontrolled system is not exponentially stable.

Now, here $B = b$ and by (D3) we have $B \in H^{\varepsilon}(\Omega) = D_{-1/2}(A)$. Here $\mathcal{F}$ is defined on $D_{1/2}(A) = \{u \in H^{s+1}(\Omega) \mid D(0)\nabla u \cdot n = 0 \text{ in } \partial\Omega\}$, by

$$\mathcal{F}(u) = \operatorname{div}(\widetilde{D}(u)\nabla u) + \widetilde{f}(u). \tag{2.78}$$

We have the following Lemma, shown in Section 2.7:

Lemma 8. *$\mathcal{F}$ satisfies Assumption 1.*

Now we define the following feedback for $\mu > 0$

$$\forall u \in H, \quad Ku = -\frac{\mu}{\langle b\,,1\rangle_{L^2}}\int_{\Omega} u. \tag{2.79}$$

We are now able to state the main result of this subsection.

Proposition 6. *Assume that $0 \leq f'(0) < |\lambda_1|$ and let $\lambda \in (0, |\lambda_1| - f'(0))$. Then, there exists $\delta > 0$ such that for every $u_0 \in H^s(\Omega)$ with $\|u_0\|_{H^s} \leq \delta$, there exists a unique solution $u \in C^0_b([0,+\infty); H^s(\Omega)) \cap L^2((0,+\infty); H^{s+1}(\Omega))$ to*

$$\begin{cases} \partial_t u - \operatorname{div}(D(u)\nabla u) = f(u) + K(u)b & \text{in } \Omega \times (0,+\infty), \\ D(0)\nabla u \cdot n = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega, \end{cases} \tag{2.80}$$

where K is defined in (2.79) with a $\mu \geq \lambda + f'(0)$. Moreover, it is exponentially stable, meaning that there exists a constant $C > 0$ such that

$$\forall t \geq 0, \quad \|u(t)\|_{H^s} \leq Ce^{-\lambda t}\|u_0\|_{H^s}. \tag{2.81}$$

Proof. Here we have $L_\lambda = \operatorname{span}\{e_0\}$ and $m(\lambda) = 1$ as λ_0 is simple. Hence (D3) ensures that B is F_λ-admissible (note that $\langle B, 1\rangle_{H^s}$ is directly deduced from $\langle B, 1\rangle_{L^2}$). Hence, from Lemma

8, there exists a F-equivalence by Theorem 9, and projecting $T(A+BK) = DT$ on e_0, gives

$$K(u)\langle B\,, e_0\rangle_{H^s} = -\mu\langle u\,, e_0\rangle_{H^s}. \tag{2.82}$$

Hence the F-equivalence feedback is given by K defined in (2.79) with some $\mu \geq \lambda + f'(0)$. Then it suffices to apply Theorem 13 to ensures local well-posedness and exponential stability. □

2.5 Main proofs

2.5.1 Existence of parabolic F-equivalence

In this subsection we prove Theorem 9.

Proof strategy

Let A, λ, D, and B be as in Theorem 9. In particular, our frequency decomposition of spaces are always with respect to λ. Below, we briefly outline the proof steps for Theorem 9:

1. We begin by establishing necessary conditions on the form of (T, K) for it to be a parabolic F-equivalence of (A, B, D), as detailed in Proposition 8.

2. In Subsection 2.5.1, we first apply Lemma 6 using the partition induced by the F_λ-admissibility of B. Hence, for each $j \in \{1, \ldots, m(\lambda)\}$, we have A_j, a diagonal parabolic operator on $\mathcal{H}^j$, with $B_j \in D(A_j)'$. Exploiting the fact that A_j has only simple eigenvalues (see, for instance, [CL15, (2.10)]) in L^j_λ, we show in Proposition 9 that for almost every $\mu \geq \lambda + c_A$ and for all $j \in \{1, \ldots, m(\lambda)\}$, there exists a parabolic F-equivalence (T_j, K_j) between (A_j, B_j) and D_j, where ($P_{j\,L}$ and $P_{j\,H}$ are the orthogonal projections on L^j_λ and $\mathcal{H}^j_\lambda$ in $\mathcal{H}^j$)

$$D_j = (A_{j\,L} - \mu)P_{j\,L} + A_{j\,H}P_{j\,H}. \tag{2.83}$$

   The core of the proof lies in this step, with the main technical challenge being to establish that T_j is an isomorphism. To achieve this, we make essential use of the polynomial properties of finite-dimensional F-equivalence feedback, see Theorem 25. Note that D_j explicitly depends on μ, but for notational convenience, we do not indicate this dependence explicitly.

3. Finally, in Subsection 2.5.1, we prove Theorem 9. To this end we set

$$T = T_1 + \cdots + T_{m(\lambda)}, \quad K = (K_1, \ldots, K_{m(\lambda)}). \tag{2.84}$$

   Then we demonstrate that (T, K) indeed forms a parabolic F-equivalence between (A, B) and D.

Necessary conditions on (T, K)

The goal of this subsection is to show some conditions that (T, K) should satisfy to be a parabolic F-equivalence between (A, B) and D (see Proposition 8). This will greatly help us understand the form of (T, K) in Subsection 2.5.1, and we will also reuse it in Section 2.6.

First, let us introduce the following notation: we denote by P_L and P_H the orthogonal projections on L_λ and H_λ in H, and by a slight abuse of notation, we use the same symbols for the orthogonal projections on L_λ and $D(A)'_\lambda$ in $D(A)'$.

Then for every $x \in D(A)'$, we have $x = x_L + x_H$ with $x_L = P_L x$ and $x_H = P_H x$. Now, for every normed vector space E, the previous decompositions give us the following decomposition on the space of bounded operators

$$\mathcal{L}(H, E) = \mathcal{L}(L_\lambda, E) \oplus \mathcal{L}(H_\lambda, E).$$

This is also true when replacing H with $D(A)'$. Notice that $A(L_\lambda) \subset L_\lambda$ and that $A(H_\lambda \cap D(A)) \subset H_\lambda$, so similarly we can define $A_L = AP_L$ and $A_H = AP_H$, and we have $A = A_L + A_H$.

We now decompose $\mathcal{L}(H, D(A)')$ in terms of frequency. Let $M \in \mathcal{L}(H, D(A)')$. We can write

$$M = P_L M P_L + P_L M P_H + P_H M P_L + P_H M P_H, \tag{2.85}$$

which allows us to define the direct sum corresponding to the above decomposition

$$\mathcal{L}(H, D(A)') = LL_\lambda(H, D(A)') \oplus HL_\lambda(H, D(A)') \oplus LH_\lambda(H, D(A)') \oplus HH_\lambda(H, D(A)'), \tag{2.86}$$

where, for instance,

$$HL_\lambda(H, D(A)') = \{M \in \mathcal{L}(H, D(A)') \mid M = P_L M P_H\}, \tag{2.87}$$

and $LL_\lambda(H, D(A)')$, $LH_\lambda(H, D(A)')$ and $HH_\lambda(H, D(A)')$ are defined accordingly. We could apply the same approach to $\mathcal{L}(H)$ or $\mathcal{L}(D(A)')$ and the subspaces defined earlier. In these cases, this decomposition allows us to use a matrix formalism. For example, if $M \in \mathcal{L}(H)$, we write

$$M = \begin{pmatrix} M_{LL} & M_{HL} \\ M_{LH} & M_{HH} \end{pmatrix}, \tag{2.88}$$

with

$$M_{LL} = P_L M P_L, \quad M_{HL} = P_L M P_H, \quad M_{LH} = P_H M P_L, \quad M_{HH} = P_H M P_H,$$

and we verify that the usual matrix multiplication rules apply. For instance, for $x \in H$, we write $x = \begin{pmatrix} x_L \\ x_H \end{pmatrix}$, and we have

$$Mx = M_{LL}x_L + M_{HL}x_H + M_{LH}x_L + M_{HL}x_H = \begin{pmatrix} M_{LL} & M_{HL} \\ M_{LH} & M_{HH} \end{pmatrix} \begin{pmatrix} x_L \\ x_H \end{pmatrix}.$$

Notice that we can define $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ as in Proposition 1. The algebra defined in the proposition below arises naturally in parabolic F-equivalence, as we will see in Sections 2.5 and 2.6.

Proposition 7. *We define the commutant algebra of A_H as*

$$C(A_H) = \{M \in \mathcal{L}_{H_\lambda}(D(A)'_\lambda) \mid A_H M = M A_H \text{ in } \mathcal{L}(H_\lambda, D(A)'_\lambda)\}. \tag{2.89}$$

It is a sub-Banach algebra of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$.

Proof. It's clear that $C(A_H)$ is a subalgebra of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$. We just need to show that it is closed. Let $(M_n)_{n\geq 1}$ be a sequence in $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ such that $M_n \to M$ in $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$. Recall that the norm of $\mathcal{L}_{H_\lambda}(D(A)'_\lambda)$ is equivalent to the following

$$\forall M \in \mathcal{L}_{H_\lambda}(D(A)'_\lambda),\ |||M||| := \|M\|_{\mathcal{L}(D(A)'_\lambda)} + \|M_{|H_\lambda}\|_{\mathcal{L}(H_\lambda)}. \tag{2.90}$$

We have

$$\|MA_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|MA_H - M_n A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} + \|M_n A_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)},$$

for the first them notice that

$$\|MA_H - M_n A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \|M - M_n\|_{\mathcal{L}(D(A)'_\lambda)}.$$

Now using that $M_n A_H = A_H M_n$, we get

$$\|M_n A_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \leq \|A_H\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} \|M - M_n\|_{\mathcal{L}(H_\lambda)}.$$

Then passing to the limit, we finally get $\|MA_H - A_H M\|_{\mathcal{L}(H_\lambda, D(A)'_\lambda)} = 0$, and hence $M \in C(A_H)$.

□

We can now show the following necessary conditions on (T, K):

Proposition 8. *Let $\mu \in \mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h\geq l\geq 1}$, we set*

$$D_\mu = \begin{pmatrix} A_L - \mu & 0 \\ 0 & A_H \end{pmatrix}.$$

Let (T, K) be a parabolic F-equivalence between (A, B) and D (hence $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$). Then, if we denote by $(\tilde{T}, \tilde{K})$ the unique finite-dimensional F-equivalence between (A_L, B_L)

and $A_L - \mu$ given by Theorem 25,[10] *we have*

$$T = \begin{pmatrix} \tilde{T} & 0 \\ \tau & C \end{pmatrix}, \quad K = \tilde{K}, \tag{2.91}$$

with $C \in C(A_H)$ and $\tau \in LH_\lambda(D(A)')$ and is defined as

$$\forall n \in \{1, \ldots, N(\lambda)\},\ \tau(e_n) = \sum_{k\geq 1} \frac{\langle B_H K e_n \,, \sqrt{1+|\lambda_k|^2} e_k \rangle_{D(A)'}}{\lambda_k - \lambda_n} \sqrt{1+|\lambda_k|^2} e_k. \tag{2.92}$$

Furthermore, we have $\tau(L_\lambda) \subset H_\lambda$.

Proof. Let $n > N(\lambda)$. Then $Ke_n = 0$, and by F-equivalence, we have

$$T(A + BK)e_n = \lambda_n T e_n = DTe_n. \tag{2.93}$$

Hence, Te_n is an eigenvector of D associated to the eigenvalue λ_n. Note that $(e_j)_{j\geq 1}$ form a basis of eigenvectors of D, from its definition. Now, because $\mu \notin \{\lambda_l - \lambda_h\}_{h\geq l\geq 1}$, we have $Te_n \in \operatorname{span}(e_k)_{k>N(\lambda)}$. Otherwise, it should exists $l \leq N(\lambda)$ such that $\langle Te_n \,, e_l\rangle_H \neq 0$, but by (2.93) we would have

$$\lambda_n \langle Te_n \,, e_l \rangle_H = (\lambda_l - \mu)\langle Te_n \,, e_l\rangle_H,$$

which would be a contradiction. Thus, we have

$$\forall v \in H_\lambda,\ TA_H v = A_H T v. \tag{2.94}$$

Therefore, $T_{HL} = 0$ and $T_{HH} \in C(A_H)$.

Now let $n \leq N(\lambda)$. Applying this to the F-equivalence equation and using $TB = B$ gives us

$$\lambda_n T e_n + B_L K e_n + B_H K e_n = DTe_n. \tag{2.95}$$

Projecting this on L_λ and $D(A)'_\lambda$ gives us

$$\begin{cases} \lambda_n (Te_n)_L + B_L K e_n = (A_L - \mu)(Te_n)_L, \\ \lambda_n (Te_n)_H + B_H K e_n = A_H (Te_n)_H. \end{cases} \tag{2.96}$$

Now, as $T_{HL} = 0$, we have $T_{LL} B_L = B_L$ and hence the first equation above is equivalent to

$$T_{LL}(A_L + B_L K) = (A_L - \mu) T_{LL}. \tag{2.97}$$

Then, identifying L_λ with $\mathbb{C}^{N(\lambda)}$ using the isomorphism that sends the canonical basis to $(e_n)_{1\leq n\leq N(\lambda)}$, we have, from Theorem 25, $T_{LL} = \tilde{T}$ and $K = \tilde{K}$. Now let's use the second equality in (2.96). We set $h_n := (Te_n)_H$. Note that, given the choice of n, $h_n = T_{LH} e_n$. For

[10] We can apply this theorem using the isomorphism between $\mathbb{C}^{N(\lambda)}$ and L_λ, which sends the canonical basis to $(e_n)_{1\leq n\leq N(\lambda)}$.

$k > N(\lambda)$, we have

$$\lambda_n \langle h_n \, , e_k \rangle_{D(A)'} + \langle B_H K e_n \, , e_k \rangle_{D(A)'} = \langle A_H h_n \, , e_k \rangle_{D(A)'}. \tag{2.98}$$

Using $A^* e_k = \overline{\lambda_k} e_k$, we get

$$\langle h_n \, , e_k \rangle_{D(A)'} = \frac{\langle B_H K e_n \, , e_k \rangle_{D(A)'}}{\lambda_k - \lambda_n}. \tag{2.99}$$

Notice that by definition of $N(\lambda)$, we have $\mathrm{Re}(\lambda_{N(\lambda)+1} - \lambda_{N(\lambda)}) \neq 0$, hence the above expression is well-defined (recall that $k > N(\lambda) \geq n$). Recall that $(\sqrt{1 + |\lambda_m|^2} e_m)_{m \geq 1}$ is an orthonormal basis of $D(A)'$, and so

$$\tau(e_n) := h_n = \sum_{k \geq 1} \frac{\langle B_H K e_n \, , \sqrt{1 + |\lambda_k|^2} e_k \rangle_{D(A)'}}{\lambda_k - \lambda_n} \sqrt{1 + |\lambda_k|^2} e_k.$$

Now, to prove that $\tau(L_\lambda) \subset H$, it suffices to show that $\tau(e_n) \in H$ for each $n \in \{1, \ldots, N(\lambda)\}$.

Let $k \geq 1$. We set $I_k := \langle B_H K e_n \, , \sqrt{1 + |\lambda_k|^2} e_k \rangle_{D(A)'}$. This forms an ℓ^2 sequence since $B_H \in D(A)'$, and we have

$$\frac{|I_k|}{|\lambda_k - \lambda_n|} \sqrt{1 + |\lambda_k|^2} \underset{k \to +\infty}{\sim} |I_k|. \tag{2.100}$$

Thus $\tau(e_n) \in H$.

$\square$

Remark 19. *Here we emphasize that $T_{HL} = 0$ reflects the internal structure of the equation*

$$\partial_t u = Au + BKu, \tag{2.101}$$

with $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$. Notice that because $Kx = Kx_L$, this equation can be decoupled as follows:

$$\partial_t u = Au + BKu \iff \begin{cases} \partial_t u_L = A_L u_L + B_L K u_L, \\ \partial_t u_H = A_H u_H + B_H K u_L. \end{cases} \tag{2.102}$$

The first evolution equation on u_L in the system is independent of u_H and can be solved on its own. This fact is then reflected by $T_{HL} = 0$ in the F-equivalence. Similarly, one can observe that $T_{LH} \neq 0$ if $B_H \neq 0$, as the low-frequency part u_L influences the evolution of u_H.

Remark 20. *Notice that K is entirely determined as soon as A, B and D are given. T is not, however, and the only free component of T is C. As we will discuss in Section 2.6, this is what prevents achieving uniqueness.*

Simple multiplicity case

In this section, we aim to establish the following proposition.

Proposition 9. *For almost every $\mu \geq \lambda + c_A$ and for all $j \in \{1, \ldots, m(\lambda)\}$, there exists a parabolic F-equivalence (T_j, K_j) between (A_j, B_j) and the λ-target D_j.*

To clarify, here we work with the Gelfand triple $(D(A_j), \mathcal{H}^j, D(A_j)')$, and we redefine accordingly all the necessary operator spaces. Then $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$, $K_j \in \mathcal{L}(L^j_\lambda, \mathbb{C})$, and equation 2.17 becomes

$$\begin{cases} T_j(A_j + B_j K_j) = D_j T_j \text{ in } \mathcal{L}(\mathcal{H}^j, D(A_j)'), \\ T_j B_j = B_j \text{ in } D(A_j)'. \end{cases} \tag{2.103}$$

Notice that Proposition 8 also applies here for (A_j, B_j, D_j), and we denote by $\tilde{T}_j$, τ_j, and K_j the operators in (2.91). For the proof of Proposition 9, we will need the following lemma.

Lemma 9. *Let $\mu \in \mathbb{R}_{>0} \backslash \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$. There exists a parabolic F-equivalence of (A_j, B_j, D_j) if and only if there exists $C_j \in C({A_j}_H) \cap \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$ such that*

$$\tau_j {B_j}_L + C_j {B_j}_H = {B_j}_H. \tag{2.104}$$

Proof. Let (T_j, K_j) be a parabolic F-equivalence between (A_j, B_j, D_j). By definition, we have $T_j B_j = B_j$. Using Proposition 8 for (A_j, B_j, D_j), we obtain:

$$T_j B_j = B_j \iff \begin{pmatrix} \tilde{T}_j {B_j}_L = {B_j}_L \\ \tau_j {B_j}_L + C_j {B_j}_H = {B_j}_H \end{pmatrix}. \tag{2.105}$$

Proposition 8 ensures that $C_j \in C({A_j}_H)$, and we have (recall that $D(A_j)'_\lambda$ correspond to the projection on the high frequencies):

$$\forall v \in D(A_j)'_\lambda, \quad T_j v = C_j v. \tag{2.106}$$

Since $T_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$, the equality above implies $C_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$.

Conversely, suppose that there exists $C_j \in C({A_j}_H) \cap \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$ such that $\tau_j {B_j}_L + C_j {B_j}_H = {B_j}_H$ holds. Then, we can define T_j and K_j as in Proposition 8. By Theorem 25, we have that $\tilde{T}_j {B_j}_L = {B_j}_L$, so by Equation (2.105), we have $T_j B_j = B_j$.

Next, let's verify that $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$. Define

$$R_j = \begin{pmatrix} \tilde{T}_j^{-1} & 0 \\ -C_j^{-1} \tau_j \tilde{T}_j^{-1} & C_j^{-1} \end{pmatrix}. \tag{2.107}$$

Noticing that $-C_j^{-1} \tau_j \tilde{T}_j^{-1} \in \mathcal{L}(L^j_\lambda, H_\lambda)$, and using the fact that $C_j \in \mathcal{GL}_{\mathcal{H}^j_\lambda}(D(A_j)'_\lambda)$, we have $R_j \in \mathcal{L}_{\mathcal{H}^j}(D(A_j)')$. Then $T_j \in \mathcal{GL}_{\mathcal{H}^j}(D(A_j)')$ immediately follows from the relation

$$T_j R_j = R_j T_j = \mathrm{Id}_{D(A_j)'}. \tag{2.108}$$

We now need to demonstrate that the first equation in (2.103) holds. Let $n > N_j(\lambda)$, then we have

$$\begin{aligned} T_j(A_j + B_jK_j)e_n^j &= T_jA_je_n^j \\ &= C_jA_je_n^j \\ &= A_jC_je_n^j \\ &= D_jT_je_n^j. \end{aligned} \tag{2.109}$$

Next, consider the case where $n \leq N_j(\lambda)$. Using $T_jB_j = B_j$, we have

$$\begin{aligned} T_j(A_j + B_jK_j)e_n^j &= \lambda_n^j T_je_n^j + B_{j_L}K_je_n^j + B_{j_H}K_je_n^j \\ &= \underbrace{(\tilde{T}_jA_{j_L} + B_{j_L}K_j)e_n^j}_{\in L_\lambda^j} + \underbrace{(\lambda_n^j\tau_j + B_{j_H}K_j)e_n^j}_{\in D(A_j)'_\lambda}. \end{aligned} \tag{2.110}$$

By Theorem 25, we have

$$(\tilde{T}_jA_{j_L} + B_{j_L}K_j)e_n^j = (A_{j_L} - \lambda_n^j)\tilde{T}_je_n^j. \tag{2.111}$$

Then, by the definition of τ_j, and setting $f_k^j := \sqrt{1 + |\lambda_k^j|^2}e_k^j$, we have

$$(\lambda_n^j\tau_j + B_{j_H}K_j)e_n^j = \sum_{k\geq 1} \lambda_k^j \frac{\langle B_{j_H}K_je_n^j, f_k^j\rangle_{D(A_j)'}}{\lambda_k^j - \lambda_n^j} f_k^j = A_{j_H}\tau_je_n^j. \tag{2.112}$$

Thus

$$T_j(A_j + B_jK_j)e_n^j = (A_{j_L} - \lambda_n^j)\tilde{T}_je_n^j + A_{j_H}\tau_je_n^j = D_jT_je_n^j. \tag{2.113}$$

Finally, by continuity and linearity, we have

$$\forall x \in \mathcal{H}^j, \quad T_j(A_j + B_jK_j)x = D_jT_jx. \tag{2.114}$$

□

The preceding lemma is very useful for constructing parabolic F-equivalences, as it shows that we only need to construct C_j. Notice that

$$\forall M \in \mathcal{L}_{\mathcal{H}_\lambda^j}(D(A_j)'_\lambda), \quad M \in C(A_{j_H}) \iff \forall n > N_j(\lambda),\ C_je_n^j \in \ker(A_{j_H} - \lambda_n^j). \tag{2.115}$$

Hence, it is natural to try C_j as a diagonal operator, which means

$$\forall n > N_j(\lambda),\ C_je_n^j = c_n^je_n^j. \tag{2.116}$$

Then the condition $C_j \in \mathcal{GL}_{\mathcal{H}_\lambda^j}(D(A_j)'_\lambda)$ simply becomes

$$\exists c_1^j, c_2^j > 0, \text{ such that } c_1^j \leq |c_n^j| \leq c_2^j. \tag{2.117}$$

Note, however, that it is not clear yet that there would exists a parabolic F-equivalence of (A_j, B_j, D_j) with such a diagonal C_j since it also have to satisfy (2.104). In fact, we are going to show that not only it is possible to have a parabolic F-equivalence of (A_j, B_j, D_j) with C_j, but in addition it is nearly always possible even without additional assumption of B_H. More precisely, we define Λ_j to be the set of all $\mu > 0$ for which there exists a parabolic F-equivalence (T_j, K_j) of (A_j, B_j, D_j)[11] with C_j being diagonal. The following proposition describes important topological properties of Λ_j.

Proposition 10. *Λ_j is open and dense in $\mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$, and $\mathbb{R}_{>0} \setminus \Lambda_j$ is negligible.*

Proof. First, we begin by demonstrating the openness of Λ_j. To avoid confusion, we will explicitly indicate every dependency on μ in our notation throughout this proof. Let $\mu \in \Lambda_j$, then there exists a parabolic F-equivalence (T_j^μ, K_j^μ) of (A_j, B_j, D_j^μ). Let $k > N_j(\lambda)$, and we set

$$f_k^j = \sqrt{1 + |\lambda_k^j|^2} e_k^j, \quad b_k^j := \langle B_j \,, f_k^j \rangle_{D(A_j)'}, \quad K_n^j := K_j f_n^j. \tag{2.118}$$

Then, projecting (2.104) onto f_k^j, we obtain

$$(1 - c_k^{\mu,j}) b_k^j = \sum_{n=1}^{N_j(\lambda)} b_n^j \langle \tau_j^\mu(f_n^j) \,, f_k^j \rangle_{D(A_j)'} = b_k^j \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu,j}}{\lambda_k^j - \lambda_n^j}. \tag{2.119}$$

Hence, if $b_k^j \neq 0$, this imposes

$$c_k^{\mu,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu,j}}{\lambda_k^j - \lambda_n^j}. \tag{2.120}$$

Now, let $\delta \in \mathbb{R}$ such that $\mu + \delta \in \mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$. We will show that if $|\delta|$ is small enough, then $\mu + \delta \in \Lambda_j$, thereby proving that Λ_j is an open set of $\mathbb{R}_{>0} \setminus \{\lambda_l - \lambda_h\}_{h \geq l \geq 1}$.

Let $k > N_j(\lambda)$, and define $C_j^{\mu+\delta}$ as follows:

– If $b_k^j = 0$, then set $c_k^{\mu+\delta,j} = 1$.

– Otherwise, set $c_k^{\mu+\delta,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu+\delta,j}}{\lambda_k^j - \lambda_n^j}$.

Now, by Lemma 9, we only need to check that (2.117) holds for $(c_k^{\mu+\delta,j})_{k > N(\lambda)}$. Notice that the above expressions imply, under Hypothesis (A2), that (note that the number of terms in the sum is finite and does not depend on k)

$$\lim_{k \to +\infty} c_k^{\mu+\delta,j} = 1. \tag{2.121}$$

Hence, the sequence is bounded from above. Since $\mu \in \Lambda_j$, then (2.117) holds for $(c_k^{\mu,j})_{k \geq N(\lambda)}$. Now observe that, from Theorem 25, the $K_n^{\mu+\delta,j}$ are continuous in δ. Also notice that the number of terms in the sum defining $c_k^{\mu+\delta,j}$ is finite and independent of k, and that we have (as $(\mathrm{Re}(\lambda_m^j))_{m \geq 1}$ is non-increasing)

$$\forall k > N_j(\lambda), \ \forall n \leq N_j(\lambda), \ |\lambda_k^j - \lambda_n^j| \geq \mathrm{Re}(\lambda_{N_j(\lambda)+1}^j - \lambda_{N_j(\lambda)}^j) > 0. \tag{2.122}$$

[11] Recall that D_j depends on μ.

Thus, we deduce that the $c_k^{\mu+\delta,j}$ are continuous in δ, uniformly in k. Therefore, using (2.117) with $(c_k^{\mu,j})_{k\geq N(\lambda)}$ if $|\delta|$ is small enough, we obtain that there exists $c_j > 0$ such that

$$|c_k^{\mu+\delta,j}| > c_j, \quad \forall k > N(\lambda). \tag{2.123}$$

which shows that $\mu+\delta \in \Lambda_j$.

Now we prove that $\mathbb{R}_{>0} \setminus \Lambda_j$ is discrete in $\mathbb{R}_{>0}$, hence countable, and this will demonstrate the last two assertions of Proposition 10. We proceed by contradiction, suppose there exists $\mu_\infty \in \mathbb{R}_{>0} \setminus \Lambda_j$ such that there exists a injective sequence $(\mu_m)_{m\geq 1}$ with $\mu_m \in \mathbb{R}_{>0} \setminus \Lambda_j$ and $\mu_m \to \mu_\infty$ as $m \to \infty$. By the previous discussion and Lemma 9, for each $m \geq 1$, there exists $k_m > N(\lambda)$ such that

$$c_{k_m}^{\mu_m,j} = 1 - \sum_{n=1}^{N_j(\lambda)} \frac{b_n^j K_n^{\mu_m,j}}{\lambda_{k_m}^j - \lambda_n^j} = 0. \tag{2.124}$$

Recall that, from (2.121), there exists k_0 large enough such that $c_k^{\mu_\infty,j} \geq 1/2$ for all $k > k_0$, and by the previous discussion, $c_k^{\mu_m,j}$ converge to $c_k^{\mu_\infty,j}$ uniformly in k, when $m \to +\infty$. Therefore, there exists m_1 such that

$$\forall m \geq m_1, \forall k > k_0, c_k^{\mu_m,j} \neq 0. \tag{2.125}$$

Therefore for $m \geq m_1$ there can only be a finite number of k where (2.124) holds. Hence we can find $k_0 > N(\lambda)$ and extract a subsequence ψ such that $k_{\psi_m} = k_0$, namely

$$\forall m \geq 1, \ c_{k_0}^{\mu_{\psi(m)},j} = 0. \tag{2.126}$$

However, by Theorem 25, we know that $K_n^{\mu,j}$, is a polynomial in μ without constant term. Hence $c_{k_0}^{\mu,j}$ must be a non zero polynomial in μ, but the isolated zero theorem and (2.126) imply

$$\forall \mu \in \mathbb{R}, \ c_{k_0}^{\mu,j} = 0, \tag{2.127}$$

which is a contradiction.

□

Finally, we can prove Proposition 9.

Proof of Proposition 9. We set $\Lambda = \bigcap_{j\in\{1,\ldots,m(\lambda)\}} \Lambda_j$. Proposition 10 ensures us that Λ is full measure and dense in $\mathbb{R}_{>0}$, so let $\mu > \lambda + c_A$ within Λ, then for all $j \in \{1, \ldots, m(\lambda)\}$ there exists a parabolic F-equivalence of (A_j, B_j, D_j). Now as $\mu \geq \lambda + c_A$, we have

$$\forall \nu \in \sigma(D_j), \ \nu \leq -\lambda.$$

This implies that D_j is a λ-target. This concludes the proof of Proposition 9. □

Last step

We now finalize the proof of our main result. By Proposition 9, for almost every $\mu \geq \lambda + c_A$ and for all $j \in \{1, \ldots, m(\lambda)\}$, there exists a parabolic F-equivalence (T_j, K_j) between (A_j, B_j) and D_j. Hence, by Lemma 16, if we set

$$T = T_1 + \cdots + T_{m(\lambda)}, \quad K = (K_1, \ldots, K_{m(\lambda)}), \tag{2.128}$$

we have $T \in \mathcal{GL}_H(D(A)')$ and $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$.

Again, by Lemma 16, notice that

$$D = D_1 + \cdots + D_{m(\lambda)}, \tag{2.129}$$

and because $\mu \geq \lambda + c_A$, we know that D is a λ-target. Finally, let $x \in H$. We have $x = x_1 + \cdots + x_{m(\lambda)}$ with $x_j \in \mathcal{H}^j$. By the definition of (T_j, K_j), we have $TB = B$ in $D(A)'$, and

$$T(A + BK)x = (TA + BK)x = \sum_{j=1}^{m(\lambda)} (T_j A_j + B_j K_j) x_j = \sum_{j=1}^{m(\lambda)} D_j T_j x_j = DTx. \tag{2.130}$$

This concludes the proof of Theorem 9.

2.5.2 Stabilization of nonlinear systems

In this subsection we prove Theorem 13. Let A, λ, B as in Theorem 13, and let (T, K) be an F-equivalence of (A, B, D) given by Theorem 9. As B is F_λ-admissible, without loss of generality we can suppose that $m = m(\lambda)$.

We start by showing the following essential lemma.

Lemma 10. *Let $s \in [0,1]$ be such that $B \in (D_{-s}(A))^{m(\lambda)}$, then $T \in \mathcal{GL}(D_r(A))$ for all $r \in [-1, 1-s]$.*

Proof. By Proposition 8, we have, keeping the same notation

$$T = \begin{pmatrix} \tilde{T} & 0 \\ \tau & C \end{pmatrix}. \tag{2.131}$$

Thus as $T \in \mathcal{GL}_H(D(A)')$ by hypothesis, using expression (2.107) to construct an inverse, we only need to show that $\tau(L_\lambda) \subset D_{1-s}(A)$ to conclude. Using that $(\sqrt{1 + |\lambda_k|^2}^r e_k)_{k \geq 1}$ is a Hilbert basis of $D_{-r}(A)$ for all $r \in \mathbb{R}$, and Proposition 8, we have for all $n \leq N(\lambda)$

$$\tau(e_n) = \sum_{k \geq 1} \frac{\langle B_H K e_n \, , \sqrt{1 + |\lambda_k|^2}^s \rangle_{D_{-s}(A)}}{\lambda_k - \lambda_n} \sqrt{1 + |\lambda_k|^2}^s e_k. \tag{2.132}$$

As in the proof of Proposition 8, we set $I_k = \langle B_H K e_n \, , \sqrt{1 + |\lambda_k|^2}^s \rangle_{D_{-s}(A)}$ which forms a ℓ^2

sequence as $B_H K e_n \in D_{-s}(A)$, then we have

$$||\tau(e_n)||^2_{D_{1-s}(A)} = \sum_{k\geq 1} \frac{|I_k|^2}{|\lambda_k - \lambda_n|^2}(1+|\lambda_k|^2) \lesssim \sum_{k\geq 1} |I_k|^2 < +\infty. \tag{2.133}$$

□

Now we set $\gamma = \min(1-s, \frac{1}{2})$ with $s \in [0,1]$ such that $B \in (D_{-s}(A))^{m(\lambda)}$. By F-equivalence, $(T^{-1}(\frac{e_k}{\sqrt{1+|\lambda_k|^2}^\gamma}))_{k\geq 1}$ is a Riesz basis of $D_\gamma(A+BK)$, and by Lemma 10 it is also a Riesz Basis of $D_\gamma(A)$. Hence $D_\gamma(A+BK) = D_\gamma(A)$ and the two norms are equivalent, thus we have the following continuous inclusion

$$D_{1/2}(A+BK) \hookrightarrow D_\gamma(A). \tag{2.134}$$

Now notice that by F-equivalence, in the Hilbert space H endowed with the norm $||T^{-1}\cdot||_H$, $A+BK$ satisfies (H1) in Appendix 2.12 (it suffices to take $\mu > \lambda + c_A$ in Theorem 9), and with this operator, $\mathcal{F}$ satisfies $(H2)$ thanks to (2.134). Hence applying Proposition 15, we get the result of Theorem 13 but for the norm $||T^{-1}\cdot||_H$ and with $C=1$ in (2.29), then to conclude it suffices to notice that

$$\forall x \in H, \quad ||T||^{-1}_{\mathcal{L}(H)}||x||_H \leq ||T^{-1}x||_H \leq ||T^{-1}||_{\mathcal{L}(H)}||x||_H. \tag{2.135}$$

2.6 Approximate controllability and uniqueness

The problem of finding an F-equivalence between a control system (A,B) and a target D is challenging, especially when the problem is ill-posed, meaning there may be multiple pairs (T,K) that satisfy the conditions. In this section, we investigate this issue of uniqueness.

First, in subsection 2.6.1, we introduce an algebraic characterization for the lack of uniqueness on T. This allows us to introduce the weak F-equivalence formalism and to recover a well-posed problem. Then in subsection 2.6.2, we show that our algebraic characterization can be linked to the approximate controllability of (A,B). This allows us to prove Theorem 22, which implies that the parabolic F-equivalence problem is well-posed if and only if (A,B) is approximately controllable.

2.6.1 Weak F-equivalence

Let us fix A, λ, B, μ, and D as in Theorem 9.[12] Let $(T,K),(T',K') \in \mathcal{GL}_H(D(A)') \times \mathcal{L}(H,\mathbb{C}^{m(\lambda)})$ be parabolic F-equivalences of (A,B,D). Proposition 2.91 ensures that $K=K'$, and

$$T_{LL} = T'_{LL},\ T_{LH} = T'_{LH},\ T_{HL} = T'_{HL},\ T_{HH}, T'_{HH} \in C(A_H). \tag{2.136}$$

[12] Hence, with the notations of Section 2.5, $\mu \in \Lambda$.

We adopt the same notation as in the proposition, hence we set $C := T_{HH}$ and $C' := T'_{HH}$. Now, notice that by the definition of F-equivalence, we have

$$(T - T')B = B - B = 0. \tag{2.137}$$

And by Lemma 9, this gives us

$$(C - C')B_H = 0. \tag{2.138}$$

This leads us to define the following closed subspace of $C(A_H)$:

$$N_{B_H} = \{M \in C(A_H) \,|\, MB_H = 0\}. \tag{2.139}$$

Then, we endow $\mathcal{L}_H(D(A)')/N_{B_H}$ with the quotient norm, and because N_{B_H} is closed, the quotient is a Banach space. We denote by $\pi : \mathcal{L}_H(D(A)') \to \mathcal{L}_H(D(A)')/N_{B_H}$ the quotient map. Now, we have found all the possible solutions, as illustrated by the following proposition.

Proposition 11. *Let $\mathcal{S} \subset \mathcal{GL}_H(D(A)') \times \mathcal{L}(H, \mathbb{C}^{m(\lambda)})$ be the set of all parabolic F-equivalences of (A, B, D). We denote by (T_*, K_*) the solution given by Theorem 9. Then we have*

$$\mathcal{S} = (\pi(T_*) \cap \mathcal{GL}_H(D(A)')) \times \{K_*\}. \tag{2.140}$$

Proof. The above discussion shows that $\mathcal{S} \subset (\pi(T_*) \cap \mathcal{GL}_H(D(A)')) \times \{K_*\}$ (as $\pi(T_*) = T_* + N_{B_H}$). Now let $T \in \mathcal{GL}_H(D(A)')$ such that there exists $N \in N_{B_H}$ with $T = T_* + N$. Hence, we have $TB = B$, and if we set $F(T) = T(A + BK) - DT$, using the same notation as in Proposition 2.91, we have

$$F(T) = \begin{pmatrix} \tilde{T}(A_L + B_L K) - (A_L - \mu)\tilde{T} & 0 \\ \tau(A_L + B_L K) + (C + N)B_H K - A_H \tau & (C + N)A_H - A_H(C + N) \end{pmatrix}. \tag{2.141}$$

By definition, $NB_H = 0$, hence we have $F(T) = F(T_*) = 0$, and this concludes the proof. $\square$

We can deduce from this a simple formalism that allows us to restate the parabolic F-equivalence problem so that it becomes well-posed. To this end let fix $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$, if we set $\mathcal{T} := \pi(T)$, we first need to make sense of the following equation

$$\mathcal{T}(A + BK) = D\mathcal{T}. \tag{2.142}$$

We define $F_K : \mathcal{L}_H(D(A)') \to \mathcal{L}(H, D(A)')$ as in the previous proof, which is a bounded linear operator

$$\forall T \in \mathcal{L}_H(D(A)'),\ F_K(T) = T(A + BK) - DT. \tag{2.143}$$

Now for every $T \in \mathcal{L}_H(D(A)')$, we have

$$F_K(T) = \begin{pmatrix} T_{LL}(A_L + B_L K) - (A_L - \mu)T_{LL} + T_{HL}B_H K & T_{HL}A_H - (A_L - \mu)T_{HL} \\ T_{LH}(A_L + B_L K) + T_{HH}B_H K - A_H T_{LH} & T_{HH}A_H - A_H T_{HH} \end{pmatrix}. \tag{2.144}$$

Now let $N \in N_{B_H}$. We have

$$F_K(N) = \begin{pmatrix} 0 & 0 \\ NB_HK & NA_H - A_HN \end{pmatrix} = 0. \tag{2.145}$$

Hence, F_K continuously factors through π, which means that there exists a unique bounded operator $\mathcal{F}_K$ from $\mathcal{L}_H(D(A)')/N_{B_H}$ to $\mathcal{L}(H, D(A)')$ such that

$$\forall T \in \mathcal{L}_H(D(A)'),\ \mathcal{F}_K(\pi(T)) = F(T). \tag{2.146}$$

Now this operator allows us to make sense of (2.142), we simply say that $\mathcal{T}(A+BK) = D\mathcal{T}$ if $\mathcal{F}_K\mathcal{T} = 0$. Finally to define weak F-equivalence, we need to make sense of $\mathcal{T}B = B$, for this we define the following affine subspace

$$\mathcal{F}_B = \{T \in \mathcal{L}_H(D(A)') \,|\, TB = B \,\mathrm{in}\, D(A)'\}. \tag{2.147}$$

Then we define $\mathcal{T}B = B$ as $\mathcal{T} \in \pi(\mathcal{F}_B)$.

Definition 7 (Weak F-equivalence)**.** *Let $(\mathcal{T}, K) \in \mathcal{L}_H(D(A)')/N_{B_H} \times \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$. We say that $(\mathcal{T}, K)$ is a weak F-equivalence of (A, B, D), or that it is a weak F-equivalence between (A, B) and D, if*

$$\mathcal{T} \in \pi(\mathcal{F}_B) \cap \ker \mathcal{F}_K. \tag{2.148}$$

The above condition can also be written with the previous notations as

$$\begin{cases} \mathcal{T}(A + BK) = D\mathcal{T}, \\ \mathcal{T}B = B. \end{cases} \tag{2.149}$$

Equation (2.149) clearly explain why the previous definition is called a weak F-equivalence. Finally the next proposition show that finding a weak F-equivalence is a well-posed problem and the solution is linked to parabolic F-equivalence.

Proposition 12 (Uniqueness of Weak F-Equivalence)**.** *Let (T_*, K_*) be a parabolic F-equivalence of (A, B, D). Then $(\pi(T_*), K_*)$ is the unique weak F-equivalence of (A, B, D).*

Proof. First, note that $(\pi(T_*), K_*)$ is indeed a weak F-equivalence. Now let $K \in \mathcal{L}(L_\lambda, \mathbb{C}^{m(\lambda)})$, $T \in \mathcal{L}_H(D(A)')$ such that $TB = B$ and

$$\pi(T)(A + BK) = D\pi(T). \tag{2.150}$$

This implies that $\mathcal{F}_K(\pi(T)) = F_K(T) = 0$. The same reasoning as in the proof of Proposition 2.91 shows that $K = K_*$ and $T - T_* \in C(A_H)$. Since $TB = B$ and $T_*B = B$, we have $(T - T_*)B = (T - T_*)_{HH}B = 0$, hence $T - T_* \in N_{B_H}$, which gives $\pi(T) = \pi(T_*)$. $\square$

2.6.2 Approximate controllability

One might ask when finding a parabolic F-equivalence becomes a well-posed problem on its own. As we have shown in Proposition 11, the issue of uniqueness is entirely due to the size of N_{B_H}, and having a unique solution is equivalent to $N_{B_H} = \{0\}$. Therefore, in this subsection, we fix A, λ, μ, and D as before, but we let B free.

Definition 8. *Let $B \in (D(A)')^{m(\lambda)}$ and $\tau > 0$, we say that (A, B) is approximately controllable (in time τ) if for all $u_0, u_1 \in H$ and any $\varepsilon > 0$, there exists $w \in L^2(0, \tau; \mathbb{C}^{m(\lambda)})$ such that the solution of the following system*

$$\begin{cases} \partial_t u(t) = Au(t) + Bw(t), \ \forall t \in (0, \tau), \\ u(0) = u_0, \end{cases} \tag{2.151}$$

satisfies $\|u(\tau) - u_1\| \leq \varepsilon$.

Remark 21. *By Lemma 11, as for finite-dimensional systems, the approximate controllability of (A, B) is in fact, independent of τ. Hence, we will simply refer to the approximate controllability of (A, B) without specifying a final time.*

Here our goal is to relate the size of N_{B_H} to the approximate controllability of (A, B) by proving the following theorem.

Theorem 22. *Let $B \in (D(A)')^{m(\lambda)}$ be F_λ-admissible. Then $N_{B_H} = \{0\}$ if and only if (A, B) is approximately controllable.*

The above theorem immediately answers our question and provides a new characterization of approximate controllability for parabolic systems.

Corollary 6. *Let $B \in (D(A)')^{m(\lambda)}$ be F_λ-admissible. Then (A, B) is approximately controllable if and only if there exists a unique parabolic F-equivalence of (A, B, D).*

To prove Theorem 22, we will use the generalized Fattorini criterion introduced by Badra and Takahashi in [BT14]. Let $B = (B_1, \ldots, B_m) \in (D(A)')^m$, since A is normal, we have $\ker(A^* - \overline{\lambda_n}) = \ker(A - \lambda_n)$, hence $D(A)' = \bigoplus_{n \geq 1} \ker(A^* - \overline{\lambda_n})$. We set $l_n := \dim \ker(A - \lambda_n)$, which allows us to find a partition of $(e_n)_{n \geq 1}$ as follows.[13] For each $n \geq 1$, we denote by $(\varepsilon_k^n)_{1 \leq k \leq l_n}$ a basis of $\ker(A - \lambda_n)$ formed by the elements of $(e_k)_{k \geq 1}$, hence (ε_k^n) is a reordering of (e_n). Now, for $j \in \{1, \ldots, m\}$, we can write in $D(A)'$

$$B_j = \sum_{n \geq 1} \sum_{k=1}^{l_n} b_k^{j,n} \varepsilon_k^n. \tag{2.152}$$

With this, we can now state the criterion for approximate controllability.

[13] Note that we will work with $(e_n)_{n \geq 1}$, which is orthogonal but not orthonormal in $D(A)'$.

Lemma 11 (Fattorini-Badra-Takahashi Criterion). *Let $B = (B_1, \ldots, B_{m(\lambda)}) \in (D(A)')^{m(\lambda)}$. The pair (A, B) is approximately controllable (for any time $\tau > 0$) if and only if for every $n \geq 1$, $\operatorname{rank}(\mathcal{B}_n) = l_n$, where $\mathcal{B}_n$ is given by*

$$\mathcal{B}_n = \begin{pmatrix} b_1^{1,n} & b_2^{1,n} & \ldots & b_{l_n}^{1,n} \\ \vdots & \vdots & & \vdots \\ b_1^{m(\lambda),n} & b_2^{m(\lambda),n} & \ldots & b_{l_n}^{m(\lambda),n} \end{pmatrix}. \tag{2.153}$$

Proof. This is a direct application of Theorem 1.3 (using Remark 2.1 which allows us to take $\gamma = 1$) in [BT14]. □

With this criterion we are now able to prove our theorem.

Proof of Theorem 22. Let $C \in C(A_H)$. First we give a characterization of $CB_H = 0$ using a collection of infinite scalar linear systems.

Notice that $CB_H = 0$ is equivalent to $CB_{j_H} = 0$ for all $j \in \{1, \ldots, m(\lambda)\}$. Now, for $n > N(\lambda)$, if we denote by P_n the orthogonal projection onto $\ker(A_n - \lambda_n)$, we have $CP_nB = P_nCB$ because $CA_H = A_HC$. Hence, we have the following characterization of $CB_H = 0$

$$\forall n > N(\lambda), \forall j \in \{1, \ldots, m(\lambda)\},\ CP_nB_{j_H} = 0. \tag{2.154}$$

Thus, for each $n > N(\lambda)$, we have a finite-dimensional linear system. For $n \geq 0$ and for every $k \in \{1, \ldots, l_n\}$, we set $c_k^n := C\varepsilon_k^n \in \ker(A - \lambda_n)$. Then, using (2.154), we have

$$CB_H = 0 \iff \forall n > N(\lambda), \quad \begin{cases} b_1^{1,n} c_1^n + \cdots + b_{l_n}^{1,n} c_{l_n}^n = 0, \\ b_1^{2,n} c_1^n + \cdots + b_{l_n}^{2,n} c_{l_n}^n = 0, \\ \vdots \\ b_1^{m(\lambda),n} c_1^n + \cdots + b_{l_n}^{m(\lambda),n} c_{l_n}^n = 0. \end{cases} \tag{2.155}$$

Now, to obtain scalar linear equations, we fix $n > N(\lambda)$. Then for each $k \in \{1, \ldots, l_n\}$, we set $x_k := (\langle c_1^n, \varepsilon_k^n \rangle,\ \ldots\ , \langle c_{l_n}^n, \varepsilon_k^n \rangle)^T \in \mathbb{C}^{l_n}$, and thus we have

$$\begin{cases} b_1^{1,n} c_1^n + \cdots + b_{l_n}^{1,n} c_{l_n}^n = 0, \\ b_1^{2,n} c_1^n + \cdots + b_{l_n}^{2,n} c_{l_n}^n = 0, \\ \vdots \\ b_1^{m(\lambda),n} c_1^n + \cdots + b_{l_n}^{m(\lambda),n} c_{l_n}^n = 0. \end{cases} \iff \forall k \in \{1, \ldots, l_n\},\ \mathcal{B}_n x_k = 0. \tag{2.156}$$

Now, we show that $N_{B_H} = \{0\}$ is equivalent to the approximate controllability of (A, B). First, suppose that (A, B) is approximately controllable. Then, Lemma 11 ensures that for all $n > N(\lambda)$, we have $\ker \mathcal{B}_n = \{0\}$, and by the above discussion, this implies that $N_{B_H} = \{0\}$. Conversely, suppose that $N_{B_H} = \{0\}$. First, we establish the controllability of (A_L, B_L). Using the decomposition from Lemma 6, it suffices to show that (A_{j_L}, B_{j_L}) is controllable for all $j \in \{1, \ldots, m(\lambda)\}$. This follows from (H_B), thanks to the Kalman criterion. Thus, the pair (A_L, B_L) is controllable. Moreover, the Fattorini criterion applies to finite-dimensional systems as well, where approximate controllability coincides with exact controllability, hence we have

$$\forall n \in \{1, \ldots, N(\lambda)\},\ \operatorname{rank}(\mathcal{B}_n) = l_n. \tag{2.157}$$

Now, suppose by contradiction that there exists $n > N(\lambda)$ such that $\operatorname{rg}(\mathcal{B}_n) < l_n$. Hence, there exists $z \in \mathbb{C}^{l_n} \setminus \{0\}$ such that $\mathcal{B}_n z = 0$. With this, we can construct $C \in C(A_H)$ such that it is zero everywhere except on $\ker(A_n - \lambda_n)$, where we have

$$\forall k \in \{1, \ldots, l_n\},\ C\varepsilon_k^n = z_k \varepsilon_1^n. \tag{2.158}$$

Then, using (2.156) and the above discussion, we would have $CB_H = 0$, which is a contradiction. Hence, by Lemma 11, (A, B) is approximately controllable.

□

Remark 23. *Working with an operator A such that $m(\lambda) \underset{\lambda \to +\infty}{\to} +\infty$, implies that (A, B) is not approximately controllable for any $B \in (D(A)')^{m(\lambda)}$. Indeed, if (A, B) is approximately controllable, then Lemma 11 implies $\sup_{n \geq 1} l_n < +\infty$. Hence, in this case, we know that the problem of finding a parabolic F-equivalence for (A, B, D) is ill-posed.*

2.7 Proof of Lemma 8

In this section, we show Lemma 8. We see from (2.78) that we can decompose $\mathcal{F}$ in two parts, first the bilinear one derived from $B(u, v) = \operatorname{div}(\widetilde{D}(u)\nabla v)$, and the remaining part $\widetilde{f}(u)$. Hence it suffices to check that both parts satisfy Assumption 1 to conclude that $\mathcal{F}$ satisfies it too. In order to do this, we need the following technical lemmas.

Lemma 12. *Let $u, v \in H^s(\Omega)$ and let $g \in C^{1,1}_{loc}(\mathbb{R})$. Then $g(u), g(v) \in H^{s-\varepsilon/2}(\Omega)$ and there exists $C > 0$ such that*

$$\|g(u) - g(v)\|_{H^{s-\varepsilon/2}(\Omega)} \leq C\Phi(\|u\|_{H^s} + \|v\|_{H^s})\|u - v\|_{H^s(\Omega)}, \tag{2.159}$$

where $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as follows

$$\Phi(r) = C_s \sup_{|x| \leq C_s r} |g'(x)| + C_s r \sup_{|x|,|y| \leq C_s r, x \neq y} \frac{|g'(x) - g'(y)|}{|x - y|}, \tag{2.160}$$

where $C_s > 0$ is linked to the Sobolev constant of the embedding $H^s(\Omega) \hookrightarrow C^{0,\varepsilon}(\overline{\Omega})$.

Proof. We show inequality (2.159). Let $u, v \in H^s(\Omega)$, we will use the following equivalent norm for $w \in H^{1+\sigma}(\Omega)$ (where $\sigma \in (0, 1)$)

$$\|w\|^2_{H^{1+\sigma}(\Omega)} = \|w\|^2_{L^2} + [\nabla w]^2_{H^\sigma}, \tag{2.161}$$

where $[\cdot]^2_{H^\sigma}$ is the Gagliardo seminorm defined as

$$[f]^2_{H^\sigma} = \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\sigma}}\,dx\,dy. \tag{2.162}$$

Recall that $H^s(\Omega)$ is an algebra here, and it is continuously embedded in $C^{0,\varepsilon}(\overline{\Omega})$. Now we set $M = \|u\|_{L^\infty} + \|v\|_{L^\infty}$, the L^2 part of the norm is easy to estimate, and gives

$$\|g(u)-g(v)\|_{L^2} \leq \sup_{|x|\leq M} |g'(x)|\,\|u-v\|_{L^2}. \tag{2.163}$$

Then we set $w = u - v$, hence we can write

$$\nabla g(u) - \nabla g(v) = g'(u)\nabla w + (g'(u)-g'(v))\nabla v. \tag{2.164}$$

Let $h \in C^{0,\varepsilon}(\overline{\Omega})$ and $z \in H^{\varepsilon/2}(\Omega)$, recall the classical estimate estimate

$$[hz]^2_{H^{\varepsilon/2}} \leq C_\varepsilon \|h\|^2_{L^\infty}\|z\|^2_{H^{\varepsilon/2}} + C_\varepsilon [h]^2_{C^{0,\varepsilon}}\|z\|^2_{L^2}, \tag{2.165}$$

where $[\cdot]_{C^{0,\varepsilon}}$ is the Hölder seminorm. The proof of (2.165) goes as follows. First, we write for all $x, y \in \Omega$

$$|h(x)z(x)-h(y)z(y)|^2 \leq 2|h(x)(z(x)-z(y))|^2 + 2|z(y)(h(x)-h(y))|^2. \tag{2.166}$$

Then, the first part of the previous inequality gives the bound $\|h\|^2_{L^\infty}\|z\|^2_{H^{\varepsilon/2}}$. For the second part, using the fact that $h \in C^{0,\varepsilon}(\overline{\Omega})$ and $z \in H^{\varepsilon/2}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|z(y)(h(x)-h(y))|^2}{|x-y|^{d+\varepsilon}}\,dx\,dy &\leq [h]^2_{C^{0,\varepsilon}} \int_\Omega |z(y)|^2 \int_{\Omega+\Omega} \frac{1}{|h|^{d-\varepsilon}}\,dh\,dy \\ &\leq C_\varepsilon\,[h]^2_{C^{0,\varepsilon}}\|z\|^2_{L^2}.\end{aligned} \tag{2.167}$$

Now applying this estimate to each term of the right hand side of (2.164) and using the continuous embedding $H^s(\Omega) \hookrightarrow C^{0,\varepsilon}(\overline{\Omega})$, allow us to conclude thanks to direct computations. □

Remark 24. *Notice that as $g \in C^{1,1}_{loc}(\mathbb{R})$, Φ is non-decreasing continuous at 0, and if $g'(0) = 0$, then $\Phi(0) = 0$. However, Φ might not be locally Lipschitz continuous, and this is the reason we needed Assumption 1 to be stated this way (see [BT14] for comparison).*

Lemma 13. *Let $u \in H^s(\Omega)$ and let $h \in C^{2,1}_{loc}(\mathbb{R})$ with $h(0) = 0$. Then $h(u) \in H^s(\Omega)$ and there exists $C > 0$ such that*

$$\|h(u)\|_{H^s(\Omega)} \leq C\chi(\|u\|_{H^s})\|u\|_{H^s(\Omega)}. \tag{2.168}$$

Where $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a non-decreasing function that does not depend on u.

Proof. We use the same notation as in the previous lemma. Let $u \in H^s(\Omega)$, as previously the L^2 part of the norm is easy to estimate given that $h(0) = 0$, hence we focus on the Gagliardo seminorm of $\nabla h(u) = h'(u)\nabla u$. As $h' \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$, Lemma 12 implies that $h'(u) \in H^{s-\varepsilon/2}(\Omega)$. Recall that $s = d/2 + \varepsilon$ so that $s - \varepsilon/2 > d/2$, Sobolev algebra properties gives us that there exists $C_{\varepsilon,s} > 0$ such that

$$\|h'(u)\nabla u\|_{H^{\varepsilon}(\Omega)} \leq C_{\varepsilon,s}\|h'(u)\|_{H^{s-\varepsilon/2}(\Omega)}\|\nabla u\|_{H^{\varepsilon}(\Omega)}. \tag{2.169}$$

Then we conclude using (2.159) and direct computations. □

Now we compile everything in order to show that $\mathcal{F}$ satisfies Assumption 1. For the bilinear part, using the continuity of div from $H^s(\Omega)$ to $H^{s-1}(\Omega)$, and the Sobolev algebra properties, we have, for all $u, v \in H^{s+1}(\Omega)$,

$$\|\operatorname{div}(\widetilde{D}(u)\nabla v)\|_{H^{s-1}} \leq C\|\widetilde{D}(u)\nabla v\|_{H^s} \leq C'\|\widetilde{D}(u)\|_{H^s}\|\nabla v\|_{H^s}, \tag{2.170}$$

where we used again that $s = d/2 + \varepsilon > d/2$. Then using Lemma 13, we obtain

$$\|\operatorname{div}(\widetilde{D}(u)\nabla v)\|_{H^{s-1}} \leq C''\chi(\|u\|_{H^s})\|u\|_{H^s}\|v\|_{H^{s+1}}. \tag{2.171}$$

Hence as χ is non-decreasing, thanks to Remark 12, we see that the bilinear part of $\mathcal{F}$ satisfies Assumption 1 with $\Phi = Id_{\mathbb{R}}$. For the remaining part, applying Lemma 12 to $\widetilde{f}$ and with Remark 24, we see that it also satisfies Assumption 1 with Φ defined in Lemma 12. Hence $\mathcal{F}$ satisfies Assumption 1.

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2.9 Series of functions in a Hilbert space

Let $(f_n)_{n\geq 1}$ be a sequence of $C^0(I, \mathbb{C})$ where I is a real, non-trivial interval. We say that $(f_n)_{n\geq 1}$ is locally uniformly square summable if for all $t_0 \in I$ there exists a neighborhood $J \subset I$ of t_0 and a non-negative sequence $(c_n)_{n\geq 1} \in \ell^2$ such that

$$\forall n \geq 1, \forall t \in J, \ |f_n(t)| \leq c_n.$$

Lemma 14. *Let $(H, \langle\cdot\,,\cdot\rangle)$ be a Hilbert space with an orthonormal basis $(e_n)_{n\geq 1}$. Let $(f_n)_{n\geq 1}$ be locally uniformly square summable sequence of continuous functions. Then the function $F : I \to H$ defined as*

$$\forall t \in I, \ F(t) = \sum_{n\geq 1} f_n(t)e_n,$$

is continuous.

Proof. For all $t \in I$, there exists a neighborhood $J \subset I$ of t such that $(f_n : J \to \mathbb{C})_{n\geq 1}$ is uniformly continuous, hence F is continuous on J and thus continuous at t.

□

2.10 Properties of the operator A

2.10.1 A is normal

We have the following proposition.

Proposition 13. *Let A be a diagonal parabolic operator on H. Then A is normal, and we have*

$$\forall x \in D(A),\ A^*x = \sum_{n\geq 1} \overline{\lambda_n} x_n e_n.$$

Furthermore, it is the infinitesimal generator of an analytic semigroup.

Proof. For all $n \geq 1$, we have $e_n \in D(A)$ by (A1), so $\overline{D(A)} = H$. Notice also that $e_n \in D(A^*)$ for all $n \geq 1$, and we have

$$\forall k, n \geq 1,\ \langle A^* e_n\,, e_k \rangle_H = \langle e_n\,, A e_k \rangle_H = \overline{\lambda_k} \delta_{kn}.$$

Hence, $A^* e_n = \overline{\lambda_n} e_n$, which means that $D(A) = D(A^*)$ and for all $x \in D(A)$, we have

$$A^* x = \sum_{n\geq 1} \overline{\lambda_n} x_n e_n.$$

Thus, $||A^*x||_H = ||Ax||_H$. To conclude that A is normal, we now just need to show that A is closed, but then it is sufficient to prove that A generates a strongly continuous semigroup, see [Paz83, Sec. 1.2].

Let $t > 0$. We define $S(t) : H \to H$ as $S(t)x = \sum_{n\geq 1} x_n e^{\lambda_n t} e_n$ for every $x \in H$. Again by (A1), we have that $S(t)$ is bounded. It is clear that for all $s, t > 0$ we have $S(t+s) = S(t)S(s)$ and that $S(t)x \to x$ as $t \to 0$ by Lemma 14 using Hypothesis (A2). We then see that

$$\frac{S(t)x - x}{t} = \sum_{n\geq 1} x_n \frac{e^{\lambda_n t} - 1}{t} e_n.$$

Setting $f_n(t) = x_n \frac{e^{\lambda_n t} - 1}{t}$ for $t > 0$ and $f_n(0) = x_n \lambda_n$, the inequality of the mean value theorem gives us $|f_n(t)| \leq |x_n \lambda_n|$ for n large enough via (A2). Then, noticing that

$$D(A) = \{x \in H \mid \sum_{n\geq 1} |x_n \lambda_n|^2 < +\infty\},$$

and using Lemma 14, this implies that the limit as $t \to 0$ exists if and only if $x \in D(A)$. Consequently, we have

$$\forall x \in D(A),\ \frac{S(t)x - x}{t} \to Ax.$$

Finally, we show that A generates an analytic semigroup using the resolvent characterization, see [Paz83, Sec. 2.5] First, we set $\omega = \mathrm{Re}(\lambda_1)$. By Hypothesis (A2), the half-plane $\mathrm{Re}(\lambda) > \omega$ is contained in the resolvent set of A.

Then, let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) > \omega$. Using Hypothesis (A3), we can see geometrically that there exists a constant $c > 0$ such that

$$\forall n \geq 1,\ c|\lambda - \lambda_n| \geq |\lambda - \omega|. \tag{2.172}$$

Hence, we have

$$\forall x \in H,\ \|(A-\lambda)^{-1}x\|_H^2 = \sum_{n\geq 1} |\lambda - \lambda_n|^{-2}|x_n|^2 \leq \frac{c^2}{|\lambda - \omega|^2}\|x\|_H^2, \tag{2.173}$$

which gives us the desired estimate on the resolvent:

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \leq \frac{c}{|\lambda - \omega|}. \tag{2.174}$$

□

2.10.2 Extension for normal operators

Proposition 14. *Let N be an unbounded normal operator on H. Then there exists a unique $\tilde{N} \in \mathcal{L}(H, D(N)')$ such that $\tilde{N}_{|D(N)} = N$.*

Proof. This is a straightforward consequence from the normality of the operator N: as N is normal, it is densely defined, so we again have a Gelfand triple $(D(N), H, D(N)')$ as before. The uniqueness follows from the density. Now let $x \in H$, we define $\tilde{N}x$ as

$$\forall y \in D(N),\ (\tilde{N}x)(y) := \langle N^* y\,, x\rangle_H.$$

Thus, for all $y \in D(N)$, since N is normal we have

$$|\langle N^* y\,, x\rangle_H| \leq ||N^* y||_H ||x||_H = ||Ny||_H ||x||_H \leq ||y||_{D(N)} ||x||_H.$$

This shows that $\tilde{N} \in \mathcal{L}(H, D(N)')$. If $x \in D(N)$, then

$$(\tilde{N}x)(y) = \langle y\,, Nx\rangle_H = \langle y\,, Nx\rangle_{D(N),D(N)'}.$$

Which implies that $\tilde{N}x = Nx$ in the sense of the inclusion given by the Gelfand triple. □

2.11 Finite dimensional F-equivalence

We consider a finite dimensional control system

$$\dot{x} = Ax + Bu, \tag{2.175}$$

with $x \in \mathbb{C}^n$, $u \in \mathbb{C}$, $A \in M_n(\mathbb{C})$ and $B \in \mathbb{C}^n$. Here is the main result of this section.

Theorem 25. *Let $\lambda \in \mathbb{R}$ and (A, B) be controllable, then there exists one and only one $(T, K) \in \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{1\times n}$ such that*

$$\begin{cases} T(A + BK) = (A - \lambda)T, \\ TB = B. \end{cases} \tag{2.176}$$

Furthermore, $K_1, \ldots, K_n$ are polynomials in λ, and if A is diagonalizable, they have no constant term.

Proof. First, Theorem 4.1 in [Cor15] gives the existence and uniqueness of (T, K) satisfying (2.176), and its proof ensures that $T_{ij} := \langle Te_j\,, e_i\rangle$ and K_i are polynomials in λ (Note that, in [Cor15], A is assumed to be nilpotent, but the proof can be easily generalized using similar arguments).

Now, assume that A is diagonalizable. Without loss of generality, we can assume that A is diagonal and $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ (note that the controllability is conserved by a change of variable and that A does not depend on λ). Let $j \in \{1, \ldots, n\}$, by (2.176), we have

$$\lambda_j Te_j + K_j B = ATe_j - \lambda Te_j. \tag{2.177}$$

Next, for any $i \in \{1, \ldots, n\}$, projecting the previous equation onto e_i gives (as $\langle ATe_j\,, e_i\rangle = \langle Te_j\,, \bar{A}e_i\rangle$)

$$\lambda_j T_{ij} + K_j B_i = (\lambda_i - \lambda)T_{ij}. \tag{2.178}$$

Setting $i = j$ in this equation yields

$$\forall i \in \{1, \ldots, n\},\ K_i = -\lambda \frac{T_{ii}}{B_i}. \tag{2.179}$$

Thus, we conclude that K_i has no constant term as a polynomial in λ. Note that $B_i \neq 0$ is guaranteed by the fact that (A, B) is controllable. $\square$

Remark 26. *In the diagonal case, using the same notation as in the proof,* (2.176) *is equivalent to*

$$\begin{cases} (\lambda - \lambda_i + \lambda_j)T_{ij} + K_j B_i = 0, \quad \forall i, j \in \{1, \ldots, n\}, \\ \sum_{j=1}^n T_{ij} B_j = B_i, \quad \forall i \in \{1, \ldots, n\}. \end{cases} \tag{2.180}$$

This system consists of $n^2 + n$ linear scalar equations, which allow us to numerically compute (T, K) in practice.

2.12 Nonlinear parabolic systems

In this Appendix, we show well-posedness and stability results for nonlinear parabolic equations, with the following settings:

(H1) Let A be a diagonal parabolic operator, i.e. it satisfies (A1) (and we suppose that $(e_n)_{n\geq 1}$ is a Hilbert basis), (A2), (A3), and $\lambda > 0$ such that

$$\forall n \geq 1, \quad \mathrm{Re}(\lambda_n) < -\lambda.$$

(H2) Let $\mathcal{F}$ be a map from $D_{1/2}(A)$ to $D_{-1/2}(A)$, such that there exists $\eta, K > 0$ and $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ non-decreasing continuous at 0 with $\Phi(0) = 0$, which satisfies the following conditions: for all $u, v \in D_\gamma(A)$ with $\|u\|_H, \|v\|_H \leq \eta$, we have

$$\|\mathcal{F}(u)\|_{D_{-1/2}(A)} \leq \Phi(\|u\|_H)\,\|u\|_{D_{1/2}(A)}, \tag{H2.1}$$

$$\begin{aligned}\|\mathcal{F}(u) - \mathcal{F}(v)\|_{D_{-1/2}(A)} \leq{}& K(\|u\|_{D_{1/2}(A)} + \|v\|_{D_{1/2}(A)})\,\|u-v\|_H \\ &+ K\Phi(\|u\|_H + \|v\|_H)\,\|u-v\|_{D_{1/2}(A)}.\end{aligned} \tag{H2.2}$$

Then we have the following proposition.

Proposition 15. *There exists $\delta > 0$ such that for every $u_0 \in H$, if $\|u_0\|_H \leq \delta$, there exists a unique solution $u \in C_b^0([0,+\infty); H) \cap L^2((0,+\infty); D_{1/2}(A))$ to the following system*

$$\begin{cases} \partial_t u = Au + \mathcal{F}(u), \\ u(0) = u_0. \end{cases} \tag{2.181}$$

Moreover, the previous system is exponentially stable, more precisely

$$\forall t \geq 0, \quad ||u(t)||_H \leq e^{-\lambda t}||u_0||_H. \tag{2.182}$$

The proof is based on a fixed-point argument, in order to do it, we need to study the following inhomogeneous system

$$\begin{cases} \partial_t u = Au + g(t), \\ u(0) = u_0, \end{cases} \tag{2.183}$$

where $g \in L^2((0,+\infty); D_{-1/2}(A))$.

Lemma 15. *Let $g \in L^2((0,+\infty); D_{-1/2}(A))$, for every $u_0 \in H$, there exists a unique solution $u \in C_b^0([0,+\infty); H) \cap L^2((0,+\infty); D_{1/2}(A))$ to* (2.183)*. Moreover, there exists $C \geq 1$ independent of u_0, u and g such that*

$$||u||_{C^0([0,+\infty);H)} + ||u||_{L^2((0,+\infty);D_{1/2}(A))} \leq C(||u_0||_H + ||g||_{L^2((0,+\infty);D_{-1/2}(A))}). \tag{2.184}$$

Remark 27. *In the special case where A is a self-adjoint operator, this result can be found in [Bre10, Chapter 10] and is attributed to Lions.*

Proof. We set $X_{1/2} = C_b^0([0,+\infty);H) \cap L^2((0,+\infty);D_{1/2}(A))$. We want to define u as

$$\forall t \geq 0, \quad u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)\,ds. \tag{2.185}$$

Therefore, we need to check that $e^{(\cdot)A}u_0 \in L^2((0,+\infty),D_{1/2}(A))$, that $e^{(\cdot)A}u_0$ is bounded (semigroup properties ensures that $e^{(\cdot)D}v_0 \in C^0([0,+\infty);H)$) and that the integral part is in $X_{1/2}$. At the same time, we show the different estimates.

For all $n \geq 1$ we have $\mathrm{Re}(\lambda_n) \leq -\lambda$, hence $||e^{tA}u_0||_H \leq ||u_0||_H$ for all $t \geq 0$. We set $u_n := \langle u_0\,, e_n\rangle$, then we have

$$\int_0^{+\infty} ||e^{tA}u_0||^2_{D_{1/2}(A)}\,dt = \sum_{n\geq 1}\int_0^{+\infty} e^{-2t|\operatorname{Re}\lambda_n|}|u_n|^2|\lambda_n|\,dt. \tag{2.186}$$

Now by hypothesis (A3) there exists $c > 0$ such that $c|\lambda_n| \leq |\operatorname{Re}\lambda_n|$ holds for all $n \geq 1$. Thus $e^{-2t|\operatorname{Re}\lambda_n|} \leq e^{-2tc|\lambda_n|}$ and after integration we get

$$\int_0^{+\infty} ||e^{tA}u_0||^2_{D_{1/2}(A)}\,dt \leq \frac{1}{2c}\sum_{n\geq 1}(1-e^{-2\tau c|\lambda_n|})|u_n|^2\,dt \leq \frac{1}{2c}||u_0||^2_H. \tag{2.187}$$

Now we show that $I : t \mapsto \int_0^t e^{(t-s)A}g(s)\,ds$ is in $C_b^0([0,+\infty);H)$. Let $t \geq 0$, we write $g(t) = \sum_{n\geq 1} g_n(t)e_n$, and the hypothesis on g ensures that

$$||g||^2_{L^2((0,+\infty);D_{-1/2}(A))} = \sum_{n\geq 1}\int_0^{+\infty} |\lambda_n|^{-1}|g_n(t)|^2\,dt < \infty. \tag{2.188}$$

On the other hand, we have

$$I(t) = \sum_{n\geq 1}\int_0^t e^{(t-s)\lambda_n}g_n(s)\,ds\,e_n. \tag{2.189}$$

By Cauchy-Schwarz inequality we get

$$\left|\int_0^t e^{(t-s)\lambda_n}g_n(s)\,ds\right|^2 \leq \frac{1}{2c|\lambda_n|}(1-e^{-2c|\lambda_n|t})\int_0^{+\infty}|g_n(s)|^2\,ds. \tag{2.190}$$

Hence we get the continuity using Lemma 14, and we have by the previous inequality

$$||I(t)||^2_H \leq \frac{1}{2c}||g||^2_{L^2((0,+\infty);D_{-1/2}(A))}. \tag{2.191}$$

Now using (2.189) we have

$$||I||^2_{L^2((0,+\infty);D_{1/2}(A))} = \sum_{n\geq 1}\int_0^{+\infty}|d_n|\left|\int_0^t e^{(t-s)d_n}g_n(s)\,ds\right|^2 dt \tag{2.192}$$

$$\leq \sum_{n\geq 1}|d_n|\,||\mathbf{1}_{\mathbb{R}_+}e^{(\cdot)d_n} * \mathbf{1}_{\mathbb{R}_+}g_n(\cdot)||^2_{L^2(\mathbb{R})}. \tag{2.193}$$

Hence applying Young's convolution inequality gives

$$||I||^2_{L^2((0,+\infty);D_{1/2}(A))} \leq \sum_{n\geq 1} \frac{1}{c^2|d_n|} \int_0^{+\infty} |g_n(s)|^2\, ds \tag{2.194}$$

$$\leq \frac{1}{c^2}||g||^2_{L^2((0,+\infty);D_{-1/2}(A))}, \tag{2.195}$$

Which concludes the proof. □

Now, we can use a fixed point argument to prove the first part of Proposition 15.

Proof. First, we define $X = C^0_b([0,+\infty); H) \cap L^2((0,+\infty); D_{1/2}(A))$, it defines a Banach space with the following norm

$$\forall u \in X, \quad ||u||_X := ||u||_{C^0([0,+\infty);H)} + ||u||_{L^2((0,+\infty);D_{1/2}(A))}. \tag{2.196}$$

Let $u_0 \in H$ be such that $||u_0||_H \leq \delta$ and $\delta \in (0, \frac{\eta}{2C})$ to be fixed later on, where C is the constant of Lemma 15, we set $\kappa = 2C\delta$ and we define the following closed subset

$$B(\kappa) = \{u \in X \mid ||u||_X \leq \kappa\}. \tag{2.197}$$

Let us define the mapping M which, to each $u \in B(\kappa)$, associates w, the solution to

$$\begin{cases} \partial_t w = Aw + \mathcal{F}(u(t)), \\ w(0) = u_0. \end{cases} \tag{2.198}$$

Let $u \in B(\kappa)$, hence for all $t \geq 0$ we have $\|u(t)\|_H \leq \kappa$, thus by (H2.1) we have

$$||\mathcal{F}(u)||^2_{L^2((0,+\infty);D_{-1/2}(A))} \leq \Phi(\kappa)^2 \int_0^{+\infty} ||u(s)||^2_{D_{1/2}(A)}\, ds \leq \kappa^2\Phi(\kappa)^2. \tag{2.199}$$

Hence by Lemma 15 we get

$$||M(u)||_X \leq C(||u_0||_H + ||\mathcal{F}(u)||_{L^2((0,+\infty);D_{-1/2}(A))}) \leq C\kappa(\frac{1}{2C} + \Phi(\kappa)). \tag{2.200}$$

As Φ is continuous at 0, we can choose δ small enough so that $\Phi(\kappa) \leq \frac{1}{2C}$ holds, this ensures that M is a mapping from $B(\kappa)$ to itself. Moreover, let $u_1, u_2 \in B(\kappa)$. By first applying Lemma 15 and then using (H2.2), we have

$$||M(u_1) - M(u_2)||^2_X \leq 2C^2K^2(||u_1||^2_X + ||u_2||^2_X + \frac{1}{2}\Phi(2\kappa)^2)||u_1 - u_2||^2_X \tag{2.201}$$

$$\leq C^2K^2(4\kappa^2 + \Phi(2\kappa)^2)||u_1 - u_2||^2_X. \tag{2.202}$$

This shows that taking δ small enough ensures that M is a contraction on $B(\kappa)$, allowing us to apply the Picard fixed-point theorem to conclude that (2.181) is well-posed in X for small initial data. □

Now, let $u_0 \in H$, as u is a mild solution to (2.181) its weak derivative is $Du + \mathcal{F}(u)$, and as $u \in X$ and by the hypothesis on $\mathcal{F}$, we have $\partial_t u \in L^2((0,+\infty); D_{-1/2}(A))$. This implies by classical approximation argument that $||u(\cdot)||_H^2 \in W^{1,1}_{loc}(0,+\infty)$ and

$$\frac{d}{dt}||v(t)||_H^2 = 2\,\mathrm{Re}(\langle \partial_t v\,, v\rangle_{D_{-1/2}(A),D_{1/2}(A)}) \text{ a.e on } (0,+\infty). \tag{2.203}$$

We now proceed to demonstrate the stability part of Proposition 15, which will end the proof.

Proof. We keep the same notation as in the previous fixed-point argument. Let $u_0 \in H$ such that $||u_0||_H \leq \delta$, we denote by u the solution to (2.181) with $u(0) = u_0$, hence by the previous proof we know that

$$\forall t \geq 0, \quad ||u(t)||_H \leq \kappa. \tag{2.204}$$

Thus, for almost every $t > 0$, we have thanks to (2.203) and (H2.1)

$$\frac{1}{2}\frac{d}{dt}||u||_H^2 = \mathrm{Re}(\langle \partial_t u\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) \tag{2.205}$$

$$= \mathrm{Re}(\langle Au + \mathcal{F}(u)\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) \tag{2.206}$$

$$\leq \mathrm{Re}(\langle Au\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) + \Phi(\kappa)||u||^2_{D_{1/2}(A)}. \tag{2.207}$$

By definition, we have for almost every $t > 0$

$$\mathrm{Re}(\langle Au\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) = -\sum_{n\geq 1} |\,\mathrm{Re}\,\lambda_n||u_n(t)|^2. \tag{2.208}$$

Now by Hypothesis (H1), there exists $\varepsilon > 0$ such that $|\,\mathrm{Re}\,\lambda_n| \geq \lambda + \varepsilon$ for every $n \geq 1$, thus we have

$$\mathrm{Re}(\langle Au\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) \leq -(\lambda+\varepsilon)||u||_H^2. \tag{2.209}$$

On the other hand, by hypothesis (A3) there exists $c > 0$ such that $c|\lambda_n| \leq |\,\mathrm{Re}\,\lambda_n|$ holds for all $n \geq 1$. Thus we have, using again (2.208),

$$\mathrm{Re}(\langle Au\,, u\rangle_{D_{-1/2}(A),D_{1/2}(A)}) \leq -c||u||^2_{D_{1/2}(A)}. \tag{2.210}$$

Now we set $\alpha = \frac{\varepsilon}{\lambda+\varepsilon}$, then multiplying the two previous inequality by respectively α and $1-\alpha$, we get that for almost every $t > 0$

$$\frac{1}{2}\frac{d}{dt}||u||_H^2 \leq -\alpha c||u||^2_{D_{1/2}(A)} - \lambda||u||_H^2 + \Phi(\kappa)||u||^2_{D_{1/2}(A)}. \tag{2.211}$$

Hence, if we again reduce δ such that $\Phi(\kappa) \leq \alpha c$ holds, we can apply a classical variant of Gronwall lemma, and as $u \in C^0([0,+\infty); H)$, we have

$$\forall t \geq 0, \quad ||u(t)||_H^2 \leq e^{-2\lambda t}||u_0||_H^2, \tag{2.212}$$

which concludes the proof. $\square$

2.13 Proof of Lemma 6

Let us first show that such a partition exists. Using Appendix G of [Gag+24], we can construct $(e^1_n)_{1\leq n\leq N_1(\lambda)},\dots,(e^{m(\lambda)}_n)_{1\leq n\leq N_{m(\lambda)}(\lambda)}$, which forms a partition of $(e_n)_{1\leq n\leq N(\lambda)}$, such that the restriction of A to $\operatorname{span}((e^j_n)_{1\leq n\leq N_j(\lambda)})$ has only simple eigenvalues for all $j\in\{1,\dots,m(\lambda)\}$. Now, we extend these families as follows

$$\forall j\in\{1,\dots,m(\lambda)\},\forall n>N_j(\lambda),\quad e^j_n=e_{m(\lambda)(n-N_j(\lambda))-(j-1)+N(\lambda)}. \tag{2.213}$$

Thus, $(e^1_n)_{n\geq 1},\dots,(e^{m(\lambda)}_n)_{n\geq 1}$ form a partition of $(e_n)_{n\geq 1}$. Note that this partition is far from being unique, and, in particular, any re-arrangement of $(e^j_n)_{j\in\{1,\dots,m(\lambda)\},n>N_j(\lambda)}$ is suitable. Then, Lemma 6 is a consequence of the following Lemma:

Lemma 16. *Let $d\geq 1$ and let $(e^1_n)_{n\geq 1},\dots,(e^d_n)_{n\geq 1}$ be a partition of $(e_n)_{n\geq 1}$. Define $\mathcal{H}^j=\overline{\operatorname{span}((e^j_n)_{n\geq 1})}^H$ thus $H=\bigoplus_{j=1}^d\mathcal{H}^j$. Then for each $j\in\{1,\dots,d\}$, A induces a diagonal parabolic operator on $\mathcal{H}^j$ such that*

$$A=A_1+\cdots+A_d,\quad D(A)=\bigoplus_{j=1}^d D(A_j),\quad D(A)'=\bigoplus_{j=1}^d D(A_j)'.$$

Proof. Let $j\in\{1,\dots,d\}$. For all $n\geq 1$, define λ^j_n as the eigenvalues associated with e^j_n. Then we set $D(A_j)=\{x\in\mathcal{H}^j\mid\sum_{n\geq 1}|x^j_n|^2|\lambda^j_n|^2<+\infty\}$, which allows us to define A_j as

$$\forall x\in D(A_j),\ A_jx=\sum_{n\geq 1}x^j_n\lambda^j_n e^j_n.$$

Then A_j satisfies (A1). Notice that since $(e^j_n)_{n\geq 1}$ is a subsequence of $(e_n)_{n\geq 1}$, this implies that A also satisfies (A2). Therefore, we have $A=A_1+\cdots+A_d$ and $D(A)=\bigoplus_{j=1}^d D(A_j)$. Taking the dual of the orthogonal sum finally gives the last equality. $\square$

2.14 Proof of Proposition 2

The proof of Proposition 2 is relatively straightforward and we give it here for completeness: since $A\in\mathcal{L}(H,D(A)')$, $B\in(D(A)')^{m(\lambda)}$ (hence it can be seen as belonging to $\mathcal{L}(\mathbb{C}^{m(\lambda)},D(A)')$, see (2.16)) and $K\in\mathcal{L}(H,\mathbb{C}^{m(\lambda)})$, we have $A+BK\in\mathcal{L}(H,D(A)')$.

Now we view $A+BK$ as an unbounded operator on H and define $D(A+BK)=\{x\in H\mid(A+BK)x\in H\}$. Let's show that $D(A+BK)=T^{-1}(D(A))$.

Let $x\in D(A+BK)$. Then $(A+BK)x\in H$, and so equality (2.17) implies $DTx\in H$, hence $x\in T^{-1}(D(A))$. Conversely, let $x\in T^{-1}(D(A))$. Then again by (2.17), we get $T(A+BK)x\in H$, but now because $T\in\mathcal{GL}(D(A)')$ and $T_{|H}\in\mathcal{GL}(H)$, we have $(A+BK)x\in H$. We know that $D(A)$ is dense, then since T is an isomorphism on H, this shows the density of $D(A+BK)$.

Now we set $S(0)=\mathrm{Id}_H$ and for $t>0$, we define $S(t):H\to H$ as

$$\forall x\in H,\ S(t)x=T^{-1}e^{tD}Tx. \tag{2.214}$$

Hence, $S := (S(t))_{t\in\mathbb{R}_{\geq 0}}$ is a differentiable semigroup with growth rate at most $-\lambda$ as $(e^{tD})_{t\in\mathbb{R}_{\geq 0}}$ is a differentiable semigroup with the same growth rate. Now we have to check that $A+BK$ is the infinitesimal generator of S. Let $x \in H$ such that the limit in H as $t \to 0$ of $\frac{S(t)x-x}{t}$ exists. We have

$$\frac{S(t)x-x}{t} = T^{-1}\frac{e^{tD}Tx-Tx}{t}, \tag{2.215}$$

then the existence of the limit is equivalent to $Tx \in D(A)$ and hence $x \in D(A+BK)$ by what we have shown before. The last part of the proposition is an immediate consequence of semigroup theory for evolution equations, see for instance [Paz83, Sec. 4.1]

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Chapter 3

Conclusion

In this last part, we briefly discuss extensions of the results presented in this report that could be considered in future works.

For the first part of this document, as it was mentioned in 1.3, we deeply believe that all of our results can be extended to the case of sub-Riemannian Laplace operators. We first recall the sub-Riemannian setting and we give some references on the observability of sub-Riemannian heat equations. Let $(\mathcal{M}, D, \mu)$ be a smooth connected compact d-dimensional weighted sub-Riemannian manifold, hence μ is a smooth positive measure on $\mathcal{M}$, and $D = \mathrm{Span}(X_1, \dots, X_m)$ with $X_1, \dots, X_m$ smooth vector fields on $\mathcal{M}$ satisfying the Hörmander condition

$$\mathrm{Lie}(D) = \mathrm{Lie}(X_1, \dots, X_m) = T\mathcal{M}. \tag{3.1}$$

We set $L^2(\mathcal{M}) := L^2(\mathcal{M}, \mu)$, and we denote by Δ the nonnegative sub-Laplacian operator on $L^2(\mathcal{M})$, that acts on smooth functions as follows

$$\Delta = \sum_{i=1}^{m} X_i^* X_i, \tag{3.2}$$

where X_i is the $L^2(\mathcal{M})$-adjoint of X_i. Then it is straightforward to check that Δ is a nonnegative symmetric operator on $L^2(\mathcal{M})$. Furthermore condition (3.1) ensures that Δ is subelliptic and essentially self-adjoint as $\mathcal{M}$ is compact, see [Hör67]. To have more details on sub-Riemannian geometry, we refer to [ABB19; CHT21].

In this framework, even for the two-dimensional Grushin operator, originally introduced in [Gru70; Gru71], defined on $\mathbb{R}^2$ by

$$G_\gamma := -\partial_x^2 - |x|^{2\gamma}\partial_y^2, \qquad \gamma > 0, \tag{3.3}$$

deriving an observability inequality is difficult, and genuinely new phenomena arise. On rectangles with control on a vertical strip, [BCG14] established the first positive result: null controllability holds in the weakly degenerate regime $0 < \gamma < 1$, fails in the *strongly degenerate* regime $\gamma > 1$, and a *positive minimal time* appears at the *critical* threshold $\gamma = 1$. This minimal-time barrier was then sharply characterized in [BMM15], including the description of null-controllable data at the critical time. Later, [BDE20] obtained optimal observability estimates and clarified the geometric dependence on the observation set. For further developments on the exact observability of Grushin-type equations, see [DK20; DKR24; Tam22; Van25; Koe17; BC17; Lis25].

As few results are known for the observability of sub-Riemannian heat equations, we believe that our strategy could be of great interest to have a better understanding of observability inequalities for such equations.

Now, for the second part of this document, we think that an important extension of F-equivalence would be to consider operator A with continuous spectrum, as it is the case for many differential operators on unbounded domains. To our knowledge, every result on F-equivalence is stated for operators with compact resolvent. Hence to tackle this issue, it would be a great first step to consider the case of the Laplace operator on the real line.

The main difficulty if A has continuous spectrum, is that there is no Riesz basis of eigenvectors to construct (T, K). Recall that, to go from the functional equality

$$T(A + BK) = DT, \tag{3.4}$$

where B is our control operator and D is our target operator, to a family of tractable scalar equations, we use a Riesz basis of eigenvectors, see Chapter 2 and [HL24; Cor+22; Gag+24; Gag+22; CL15].

References — Conclusion

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