CLASSICAL AND QUANTUM WAVES BRIEF

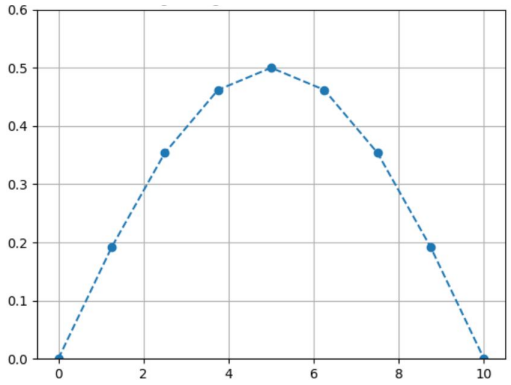
Vincent Cai
February 4, 2024

1. Loaded String System

Typically a system of coupled harmonic oscillators can be solved by finding the normal modes of the system, which are solutions where all components of the system oscillate at the same frequency and in the same phase as each other. Then, the complete solution of the system can be found by considering the space of all linear combinations of the normal modes.

Mathematically, a normal mode is a set of equations, each describing a classical object of the system in queston where each equation is of the form $Ae^{i(\omega t+\delta)}$, with $A, \delta, \omega \in \mathbb{R}$, for a fixed ω and δ, as these represent the frequency and phase of the normal mode, respectively (Note that we also restrict A to be real since a complex amplitude will introduce a phase). Thus, in a system of coupled harmonic oscillators, where the equations of motion of a system can typically all be written in the form $a\ddot{x}_i + b\dot{x}_j = cx_k$, for $a, b, c \in \mathbb{C}$ (usually real system don't have complex coefficients in the equations of motion, but mathematically it does generalize), where x_i is the position element of the ith classical object in the system (this is the defining assumption that characterizes a "couple oscillator system" for us here, with a potential for damping expressed by the first derivative term of position), we can always reduce it down to a linear system of equations by substituting $x_i = A_i e^{i\omega t+\delta}$, and obtaining $-\omega^2 aA_i + i\omega bA_j = cA_k$. All we don't know now is ω.

Now we argue that a loaded string, under certain approximations, can be mathematically represented by this type of system. Consider a system of a series of n masses of mass m, each equidistant from each other by distance l, and each connected by a string with constant tension T, where the two masses at the end are attached to $y = 0$. The setup is shown in the figure below:



Date: February 4, 2024.

Figure 1

We exclusively consider the y-coordinate of each mass, using the approximation that the x-position of each mass stays roughly constant, as held together by opposing forces from the tension in the string.

Next, we make the assumption that the y-dispaclement is much larger than the separation between any two masses. This allows us to use the small angle approximation and say that $tan(\theta) \approxeq sin(\theta)$.

Thus, for some mass i, the equations of motion are:

$$\begin{aligned} F &= ma \\ T_{left} + T_{right} &= m\ddot{y}_i \\ -Tsin(\theta_{left}) + Tsin(\theta_{right}) &= m\ddot{y}_i \\ -Ttan(\theta_{left}) + Ttan(\theta_{right}) &= m\ddot{y}_i \\ T\frac{y_{i-1} - y_i}{l} + T\frac{y_{i+1} - y_i}{l} &= m\ddot{y}_i \\ y_{i-1} - 2y_i + y_{i+1} &= \frac{ml}{T}\ddot{y}_i \end{aligned}$$

On the boundary, at $i = 1$ and $i = n$, we take $y_0 = y_{n+1} = 0$, since the strings at the end attach to $y = 0$.

Now this is a linear system of 2nd order differential equations precisely in the form that we use to solve coupled oscillator systems. Therefore, we can solve for normal modes, where $y_i(t) = A_i e^{i\omega t + \delta}$, so our differential equations become algebraic equations in the form $y_{i-1} - 2y_i + y_{i+1} = -\omega^2 \frac{ml}{T}\ddot{y}_i$.

This system can written as the following matrix equation:

$$\begin{bmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & \dots & \\ 0 & -1 & 2 & -1 & 0 & \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \omega^2 \frac{ml}{T} \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

This is an eigenvalue problem for ω. We can see from this that since the left tridiagonal matrix is it's own conjugate transpose, it is Hermitian (or self-adjoint), which implies there exists an orthonormal eigenbasis of this operator, which therefore implies that there are n distinct 1-dimensional eigenspaces of this operator. Therefore, **a loaded string of n masses will have n distinct normal modes**, since these eigenspaces will span an n dimensional vector space.

2. Numerical Solutions to Schrodinger Equation

We consider the one-dimensional time-independent Schrodinger Equation for a potential $V(x)$. Our goal is to generate numerical solution to potentials that are difficult to analytically solve for. To do this, we need to discretize our Hilbert Space, since we do not have infinite computing power. We approach this by "sampling" values of our wavefunction in discrete intervals, forming a discrete basis $\{x_0, \ldots, x_n\}$ for a new vector space that we can construct our eigenvalue problem for.

We then need to construct our Hamiltonian in this discrete basis. For $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial^2 x} + V(x)$, the potential term is simple, for it is simply a diagonal matrix with the ith diagonal entry as $V(x_i)$ (represents multiplying each position value by $V(x)$). The second derivative at x_i is approximately $\frac{x_{i-1}-2x_i+x_{i+1}}{2dx}$, so our Hamiltonian in this new finite-dimensional sampled vector space is:

$$\bar{H} = \frac{\hbar^2}{2m \times dx}\begin{bmatrix} 2 & -1 & 0 & \ldots & & 0 \\ -1 & 2 & -1 & 0 & \ldots & \\ 0 & -1 & 2 & -1 & 0 & \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \end{bmatrix} + \begin{bmatrix} V(x_0) & \ldots & & & 0 \\ \vdots & V(x_1) & \ldots & & \\ & \vdots & \ddots & & \\ 0 & & & 0 & \end{bmatrix}$$

For this Hamiltonian, we are assuming that $x_0 = x_n = 0$, which is necessary to get a square matrix to represent an operator on a vector space, but introduces boundary conditions that an infinite potential well at both endpoints would introduce. Thus, all solutions using this numerical method model those in an infinite potential well.

Now we can solve the Hamiltonian eigenvalue problem in this discrete position-basis vector space for a vector of position values $\{x_0, \ldots, x_n\}$, which can give us a sample of our wavefunction solutions. Additionally, when this is implemented in code, $\hbar$ is set to a constant, usually 1, so all units are in terms of $\hbar$.