Homework 2

ORF 309, Fall 2024

Problem 1. (Rubik's cube) Suppose you have a blank 3x3x3 Rubik's cube, i.e., all faces are white on all sides, and you decide to dip it in green paint. You then break it apart into subcubes and place all the smaller cubes into a bag.

1. You randomly pick a small cube from the bag and see only one face, which is green. What is the probability that the cube was an edge cube?

2. You randomly pick a small cube from the bag and put it on a table. You see that the 5 visible faces are white. What is the probability that the bottom face is painted green?

Problem 2. A *p-coin*, $p \in [0, 1]$, is a coin that lands heads p of the time and tails the remaining $1 - p$ of the time. Suppose $p_1, \dots p_k \in [0, 1]$ satisfy $\sum_{i=1}^{k} p_i = 1$. You have a bag containing a p_1-coin, p_2-coin, ..., and a p_k-coin that are otherwise indistinguishable. Suppose you draw a coin from the bag at random and flip it and it comes up heads. If you flip the same coin it again, what is the probability it comes up heads?

Problem 3 (Gambler's Ruin)**.** Fix $a, b, x \in \mathbb{Z}$ with $a \leq x \leq b$. Suppose you start with $x \in \mathbb{Z}$ dollars of capital (negative means you are borrowing from a bank). You repeatedly play a game that pays a dollar if you flip heads but lose a dollar if you flip tails. You plan to stop once you reach b dollars, but also will stop once you suffer enough losses that you reach a capital of a dollars.

(a) What is the probability you are *ruined*, i.e., you hit a dollars before hitting b dollars?
Hint: Let R_i be the event you are ruined starting with i dollars, $a \leq i \leq b$. Let $r_i := \mathbb{P}(R_i)$ be the probability of ruin starting from a capital of i dollars. The goal is computing r_x. Achieve this by using the law of total probability to deduce a recurrence relation for r_i.

(b) What is the probability you play forever without reaching either boundary?
Hint: Consider probability w_x of reaching b dollars before reading a dollars having started from x (i.e., winning to the desired capital before ruin).

Problem 4. You break a stick at a random point. Then you do the same with each of the resulting two sticks. What is the probability that the resulting line segments are the sides of a quadrilateral?