Q1: Out of $2^5 = 32$ configurations, 1 w/ $n=0$, 5 w/ $n=1$, 10 w/ $n=2$

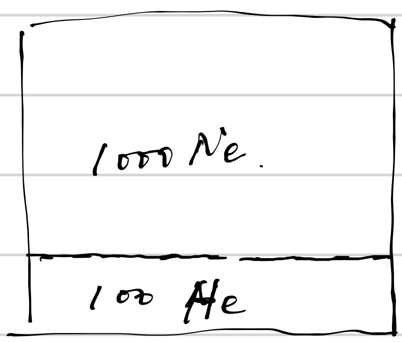
In addition, $P(n=0) = P(n=5)$, $P(n=1) = P(n=4)$, $P(n=2) = P(n=3)$

$$\therefore \; P(n=0) = P(n=5) = \frac{1}{32}$$

$$P(n=1) = P(n=4) = \frac{5}{32}$$

$$P(n=2) = P(n=3) = \frac{10}{32} = \frac{5}{16}$$

Q2.



(a). At equilibrium, $\langle Ne \rangle_L = n_{Ne} P(\text{at Large box}) = 1000 \times \frac{V_L}{V_{tot}} = 750$

using similar logic, $\langle Ne \rangle_S = 1000 \times \frac{1}{4} = 250$

$$\langle He \rangle_L = 100 \times \frac{3}{4} = 75$$

$$\langle He \rangle_S = 100 \times \frac{1}{4} = 25$$

(b). $\left(\frac{3}{4}\right)^{1000} \left(\frac{1}{4}\right)^{100}$

Q3 (a). $\boxed{6^n}$

(b). possible sum : 2 3 4 5 6 $\boxed{7}$ 8 9 10 11 12

combinations : 1 2 3 4 5 6 5 4 3 2 1

(c). $p = \frac{6}{36} = \boxed{\frac{1}{6}}$

(d). (i) $\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \boxed{\frac{5}{72}}$

(ii). $\binom{3}{2}\left(\frac{1}{6}\right)^2 = \boxed{\frac{1}{12}}$

Q4: (a). $E_{total} = E_1 + E_2 = (n_1 + n_2 + 1)\hbar\omega = n'\hbar\omega$

$\therefore n_1 + n_2 + 1 = n'$

Since oscillator 1 and 2 are distinguishable,

n_1 can be any integer from 0 to $n' - 1$

Thus $\boxed{n'}$ microstates available.

$\therefore$ Total entropy $\boxed{S = k_B \ln n'}$

(b). $E_{total} = (n_1 + n_2 + 1)\hbar(2\omega) = n''\hbar\omega$

$\therefore n_1 + n_2 + 1 = n''/2$

Similarly, n_1 can range from 0 to $\frac{n''}{2} - 1$

Thus $n''/2$ microstates available,

$\therefore$ Total entropy $S = k_B \ln \frac{n''}{2}$

(c). Entropy is an additive quantity if two subsystems are completely separated

$\therefore S_{total} = k_B \ln\left(\frac{n'n''}{2}\right) = k_B \ln\left(\frac{E'}{\hbar\omega} \cdot \frac{E''}{\hbar\omega} \cdot \frac{1}{2}\right) = k_B \ln\left(\frac{E'E''}{2\hbar^2\omega^2}\right)$

Q5. Similar to Q4, $E = (n_1 + \frac{1}{2})\hbar\omega + (n_2 + \frac{1}{2})2\hbar\omega = (n + \frac{1}{2})\hbar\omega$

$\Rightarrow n_1 + 2n_2 + 1 = n$

Similar to Q4, n_1 can be any even integer between 0 and $n-1$ (inclusive)

(total of $\frac{n+1}{2}$ choices, $\Omega = \frac{n+1}{2}$)

$\frac{n}{2} + \frac{1}{2} = \frac{E}{2\hbar\omega} + \frac{1}{4}$

$\therefore \boxed{S = k_B \ln \frac{n+1}{2}} = \boxed{k_B \ln\left(\frac{E}{2\hbar\omega} + \frac{1}{4}\right)}$

Contimue to another two non-interacting subsystems similar to 4(c), $S_{total} = k_B \ln \frac{n_1+1}{2} + k_B \ln \frac{n_2+1}{2} = \boxed{k_B \ln\left(\frac{E_1}{2\hbar\omega}+\frac{1}{4}\right)\left(\frac{E_2}{2\hbar\omega}+\frac{1}{4}\right)}$

Q6: $dS = \frac{dQ}{T} \Rightarrow \Delta S = \int \frac{1}{T}\, dQ = mC_V \int_{293}^{373} \frac{dT}{T} = mC_V (\ln T)\Big|_{293}^{373}$

$= 1\,kg \cdot 4.16\,kJ/(kg\cdot K) \cdot 0.241$

$= 1002.6\ J/K$

$\Delta S = k_B \ln(\Omega_f/\Omega_i)$ as well

$\therefore \frac{\Omega_f}{\Omega_i} = \exp\left(\frac{\Delta S}{k_B}\right) = \exp\left(\frac{1002.6\,J/K}{1.381\times10^{-23}\,J/K}\right) = \exp(7.26\times10^{25})$

Q7: $L = K - V = \frac{1}{2}mv^2 - V,\ F = -\nabla V$

Then $\frac{\partial L}{\partial x} = -\nabla V = F$ if we assume v is not explicitly depending on x

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} mv = ma$, if we assume V is not explicitly depending on v

$\therefore F - ma = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

Q8. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial \dot{x}}d\dot{x} + \frac{\partial f}{\partial t}dt$

$\therefore \frac{\partial f}{\partial t} - \frac{d}{dt}\left(f - \dot{x}\frac{\partial f}{\partial \dot{x}}\right) = \cancel{\frac{\partial f}{\partial t}} - \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \cancel{\frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial t}} + \cancel{\frac{\partial f}{\partial t}} - \cancel{\frac{\partial \dot{x}}{\partial t}\frac{\partial f}{\partial \dot{x}}} - \dot{x}\frac{d}{dt}\frac{df}{d\dot{x}}\right)$

$= \dot{x}\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$ $(*)$

$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}$, $(*) = \dot{x}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = 0$

$\therefore$ Two eq.'s are equivalent.

From Euler's eq,

Q9: $\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial v} = \frac{d}{dt}\left(-mc^2 \frac{1}{2\sqrt{1-v^2/c^2}}\frac{-2v}{c^2}\right) = \frac{d}{dt}\left(mv\frac{1}{\sqrt{1-v^2/c^2}}\right)$

Notice $\vec{F} = \frac{\partial L}{\partial \vec{r}}$, each component F_i should satisfy

$$F_i = \frac{d}{dt}\left(mv_i \frac{1}{\sqrt{1-v^2/c^2}}\right)$$

Q10 (i). $g = y(t)$, by definition,

$$\delta J = \int_{t_1}^{t_2} [g(y+\delta y, \dot{y}+\delta\dot{y}, t) - g(y,\dot{y},t)]\,dt = \int_{t_1}^{t_2} \left(\frac{\partial g}{\partial y}\delta y + \frac{\partial g}{\partial \dot{y}}\delta\dot{y}\right)dt$$

and $\delta\dot{y} = \delta\frac{dy}{dt} = \frac{d}{dt}\delta y$, plug into the eq. above,

$$\delta J = \int_{t_1}^{t_2}\left(\frac{\partial g}{\partial y}\delta y - \frac{d}{dt}\frac{\partial g}{\partial \dot{y}}\delta y\right)dt + \cancel{\frac{\partial g}{\partial \dot{y}}\delta y\Big|_{t_1}^{t_2}} \to 0$$

$$= \int_{t_1}^{t_2}\left(\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial g}{\partial \dot{y}}\right)\delta y\,dt$$

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial g}{\partial \dot{y}} = 1 - \frac{d}{dt}\frac{\partial g}{\partial y/\partial t} = 1 - \cancel{\frac{d}{dt}}dt \to 0 = 1 \neq 0,$$ Euler's eq. is not satisfied.

Given $t_1 \neq t_2$, J has no extreme value.

(ii). Using the conclusion we have in (i),

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial f}{\partial y}\delta y - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\delta y\right)dt,$$ extremize I leads to

$$\frac{\partial f}{\partial y}\delta g - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\delta y = 0 \Rightarrow \frac{\partial f_1}{\partial y} - \frac{\partial}{\partial t}\frac{\partial(\dot{y}f_2)}{\partial \dot{y}} = \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial t} = 0$$

$\therefore$ Q.E.D.