CHEM 371. PROBLEM SET 2.

(1) A mass m is suspended from the end of a spring of length L and spring constant k. If y is the position of the mass as measured from the top of the spring, write down the expression for the Lagrangian of the system, assuming that the mass can only move up or down in the vertical direction. Derive the associated Euler-Lagrange equation and find its solution.

(2) Find the functions $f[x]$ that extremize

$$(i) \qquad G[f] = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{\partial f(x)}{\partial x}\right)^2}$$

$$(ii) \qquad J[f] = 2\pi \int_{x_1}^{x_2} dx f(x) \sqrt{1 + \left(\frac{\partial f(x)}{\partial x}\right)^2}$$

$$(iii) \qquad H[f] = \int_{x_1}^{x_2} dx \frac{1}{f(x)} \sqrt{1 + \left(\frac{\partial f(x)}{\partial x}\right)^2}$$

(*Hint*: In (ii) and (iii), it helps to multiply the Euler-Lagrange equations you derive by $\partial f / \partial x$.)

(3) Show that the form of Lagrange's equations is preserved under a coordinate transformation. Specifically, show that if

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

for the variables $x_1, x_2, \ldots, x_N$, then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

under the variable change $q_j = q_j(x_1, x_2, \ldots, x_N, t)$.

(4) If the classical Hamiltonian H of a system is not explicitly a function of time, show that $dH / dt = 0$. What does this mean physically? Does this remain true if H *does* depend explicitly on time?

(5) (i) Derive Lagrange's equations for a particle moving in two dimensions under a central potential $u(r)$. Which of these equations illustrates the law of conservation of angular momentum? Is angular momentum conserved if the potential also depends on θ?

(ii) What are the corresponding Lagrangian equations of motion of the particle in three dimensions (in spherical coordinates r, θ, ϕ) under the spherically symmetric potential $U = U(r)$. Show that the Hamiltonian of the system, H, is given by $H = K + V$, where K is the kinetic energy and V the potential energy.

(6) Solve the equation of motion of two masses m_1 and m_2 connected by a harmonic spring with a force constant k.

(7) Consider the rotation of a diatomic molecule with a fixed internuclear separation R and masses m_1 and m_2. By employing center of mass and relative coordinates, show that the rotational kinetic energy can be written in spherical coordinates as

$$\frac{1}{2} I(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

and from this derive the rotational Hamiltonian

$$H_{rot} = \frac{1}{2I}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)$$

where $I = \mu R^2$ is the moment of inertia, with μ the reduced mass.