PKF on the 2D advection

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Abstract

This notebook illustrates the use of sympkf to build and handle the PKF dynamics associated with the advection in 2D

$$\partial_t c + \mathbf{u}\nabla c = 0,$$

where c is a function t, x, y and $\mathbf{u} = (u(x, y), v(x, y))$ is the stationnary velocity field.

For this dynamics, the resulting PKF system is closed and reads as (in aspect tensor form)

$$\left\{\begin{array}{l}\partial_t c + \mathbf{u}\nabla c = 0,\\ \partial_t V_c + \mathbf{u}\nabla V_c = 0,\\ \partial_t \mathbf{s}_c + \mathbf{u}\nabla \mathbf{s}_c = (\nabla\mathbf{u})\,\mathbf{s}_c + \mathbf{s}_c\,(\nabla\mathbf{u})^T\,.\end{array}\right.$$

1 Definition of the 2D advection equation

[1]:
```
import sympy
sympy.init_printing()
```

Definition of the dynamics from sympy tools

[2]:
```
import sympy
from sympy import init_printing
init_printing()
```

[3]:
```
from sympy import Function, Derivative, Eq, symbols
from sympkf import t
x, y = symbols('x y')
c = Function('c')(t,x,y)
u = Function('u')(x,y)
v = Function('v')(x,y)
dynamics = [
    Eq(Derivative(c,t), u*Derivative(c,x)+v*Derivative(c,y)),
```

```
]
display(dynamics)
```

$$\left[\frac{\partial}{\partial t}c(t,x,y) = u(x,y)\frac{\partial}{\partial x}c(t,x,y) + v(x,y)\frac{\partial}{\partial y}c(t,x,y)\right]$$

[4]:
```
from sympkf import PDESystem
dynamics = PDESystem(dynamics)
```

2 Computation of the PKF dynamics by using SymPKF

[5]:
```
from sympkf.symbolic import SymbolicPKF
pkf_advection = SymbolicPKF(dynamics)
```

[6]:
```
for equation in pkf_advection.in_metric:    display(equation)
```

$$\frac{\partial}{\partial t}c(t,x,y) = u(x,y)\frac{\partial}{\partial x}c(t,x,y) + v(x,y)\frac{\partial}{\partial y}c(t,x,y)$$

$$\frac{\partial}{\partial t}\operatorname{V_{c}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{V_{c}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{V_{c}}(t,x,y)$$

$$\frac{\partial}{\partial t}\operatorname{g_{c,xx}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{g_{c,xx}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{g_{c,xx}}(t,x,y) + 2\operatorname{g_{c,xx}}(t,x,y)\frac{\partial}{\partial x}u(x,y) + 2\operatorname{g_{c,xy}}(t,x,y)\frac{\partial}{\partial x}v(x,y)$$

$$\frac{\partial}{\partial t}\operatorname{g_{c,xy}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{g_{c,xy}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{g_{c,xy}}(t,x,y) + \operatorname{g_{c,xx}}(t,x,y)\frac{\partial}{\partial y}u(x,y) + \operatorname{g_{c,xy}}(t,x,y)\frac{\partial}{\partial x}u(x,y) + \operatorname{g_{c,xy}}(t,x,y)\frac{\partial}{\partial y}v(x,y) + \operatorname{g_{c,yy}}(t,x,y)\frac{\partial}{\partial x}v(x,y)$$

$$\frac{\partial}{\partial t}\operatorname{g_{c,yy}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{g_{c,yy}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{g_{c,yy}}(t,x,y) + 2\operatorname{g_{c,xy}}(t,x,y)\frac{\partial}{\partial y}u(x,y) + 2\operatorname{g_{c,yy}}(t,x,y)\frac{\partial}{\partial y}v(x,y)$$

[7]:
```
for equation in pkf_advection.in_aspect:    display(equation)
```

$$\frac{\partial}{\partial t}c(t,x,y) = u(x,y)\frac{\partial}{\partial x}c(t,x,y) + v(x,y)\frac{\partial}{\partial y}c(t,x,y)$$

$$\frac{\partial}{\partial t}\operatorname{V_{c}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{V_{c}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{V_{c}}(t,x,y)$$

$$\frac{\partial}{\partial t}\operatorname{s_{c,xx}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{s_{c,xx}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{s_{c,xx}}(t,x,y) - 2\operatorname{s_{c,xx}}(t,x,y)\frac{\partial}{\partial x}u(x,y) - 2\operatorname{s_{c,xy}}(t,x,y)\frac{\partial}{\partial y}u(x,y)$$

$$\frac{\partial}{\partial t}\operatorname{s_{c,xy}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\operatorname{s_{c,xy}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\operatorname{s_{c,xy}}(t,x,y) - \operatorname{s_{c,xx}}(t,x,y)\frac{\partial}{\partial x}v(x,y) -$$

$$\mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial x}u(x,y) - \mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial y}v(x,y) - \mathrm{s_{c,yy}}(t,x,y)\frac{\partial}{\partial y}u(x,y)$$

$$\frac{\partial}{\partial t}\mathrm{s_{c,yy}}(t,x,y) = u(x,y)\frac{\partial}{\partial x}\mathrm{s_{c,yy}}(t,x,y) + v(x,y)\frac{\partial}{\partial y}\mathrm{s_{c,yy}}(t,x,y) - 2\,\mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial x}v(x,y) - 2\,\mathrm{s_{c,yy}}(t,x,y)\frac{\partial}{\partial y}v(x,y)$$

2.1 Conclusion

We found that the PKF dynamics for advection dynamics is the closed system given by

$$\begin{cases} \partial_t c + \mathbf{u}\nabla c = 0, \\ \partial_t V_c + \mathbf{u}\nabla V_c = 0, \\ \partial_t \mathbf{s}_c + \mathbf{u}\nabla \mathbf{s}_c = (\nabla \mathbf{u})\,\mathbf{s}_c + \mathbf{s}_c\,(\nabla \mathbf{u})^T. \end{cases}$$

Which also reads as

$$\begin{cases} \frac{\partial}{\partial t}c(t,x,y) = -u(x,y)\frac{\partial}{\partial x}c(t,x,y) - v(x,y)\frac{\partial}{\partial y}c(t,x,y), \\ \frac{\partial}{\partial t}\mathrm{V_c}(t,x,y) = -u(x,y)\frac{\partial}{\partial x}\mathrm{V_c}(t,x,y) - v(x,y)\frac{\partial}{\partial y}\mathrm{V_c}(t,x,y), \\ \frac{\partial}{\partial t}\mathrm{s_{c,xx}}(t,x,y) = -u(x,y)\frac{\partial}{\partial x}\mathrm{s_{c,xx}}(t,x,y) - v(x,y)\frac{\partial}{\partial y}\mathrm{s_{c,xx}}(t,x,y) + 2\,\mathrm{s_{c,xx}}(t,x,y)\frac{\partial}{\partial x}u(x,y) + 2\,\mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial y} \\ \frac{\partial}{\partial t}\mathrm{s_{c,xy}}(t,x,y) = -u(x,y)\frac{\partial}{\partial x}\mathrm{s_{c,xy}}(t,x,y) - v(x,y)\frac{\partial}{\partial y}\mathrm{s_{c,xy}}(t,x,y) + \mathrm{s_{c,xx}}(t,x,y)\frac{\partial}{\partial x}v(x,y) + \mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial x}u(\\ \frac{\partial}{\partial t}\mathrm{s_{c,yy}}(t,x,y) = -u(x,y)\frac{\partial}{\partial x}\mathrm{s_{c,yy}}(t,x,y) - v(x,y)\frac{\partial}{\partial y}\mathrm{s_{c,yy}}(t,x,y) + 2\,\mathrm{s_{c,xy}}(t,x,y)\frac{\partial}{\partial x}v(x,y) + 2\,\mathrm{s_{c,yy}}(t,x,y)\frac{\partial}{\partial y} \end{cases}$$