

PKF on 2D multivariate oscillator

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Abstract

This notebook illustrates the use of sympkf to build and handle the PKF dynamics associated with the harmonic oscillator as 2D fields given by

$$\begin{cases} \partial_t u = v \\ \partial_t v = -u \end{cases}$$

where u and v are functions of t, x, y . For this dynamics, the resulting PKF system is not closed because of the cross-correlation.

1 Definition of the 2D multivariate dynamics

```
[1]: import sympy
sympy.init_printing()
```

Definition of the dynamics from sympy tools

```
[2]: from sympy import Function, Derivative, Eq, symbols
from sympkf import SymbolicPKF, t
```

```
[3]: x, y = symbols('x y')
u = Function('u')(t,x,y)
v = Function('v')(t,x,y)
dynamics = [Eq(Derivative(u,t), v), Eq(Derivative(v,t), -u)]
dynamics
```

```
[3]:  $\left[ \frac{\partial}{\partial t} u(t, x, y) = v(t, x, y), \frac{\partial}{\partial t} v(t, x, y) = -u(t, x, y) \right]$ 
```

2 Computation of the PKF dynamics by using SymPKF

```
[4]: pkf_dynamics = SymbolicPKF(dynamics)
```

```
[5]: for equation in pkf_dynamics.in_metric:    display(equation)
```

$$\frac{\partial}{\partial t} u(t, x, y) = v(t, x, y)$$

$$\frac{\partial}{\partial t} v(t, x, y) = -u(t, x, y)$$

$$\frac{\partial}{\partial t} V_u(t, x, y) = 2 V_{uv}(t, x, y)$$

$$\frac{\partial}{\partial t} V_v(t, x, y) = -2 V_{uv}(t, x, y)$$

$$\frac{\partial}{\partial t} V_{uv}(t, x, y) = -V_u(t, x, y) + V_v(t, x, y)$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{u,xx}(t, x, y) = & -\frac{2 V_{uv}(t, x, y) g_{u,xx}(t, x, y)}{V_u(t, x, y)} + \frac{2 \sqrt{V_v(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} + \\ & \frac{\mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_u(t, x, y)}{V_u^{\frac{3}{2}}(t, x, y)} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{u,xy}(t, x, y) = & -\frac{2 V_{uv}(t, x, y) g_{u,xy}(t, x, y)}{V_u(t, x, y)} + \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} + \\ & \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} + \frac{\mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{2 \sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} + \\ & \frac{\mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{2 \sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_u(t, x, y)}{2 V_u^{\frac{3}{2}}(t, x, y)} - \\ & \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_u(t, x, y)}{2 V_u^{\frac{3}{2}}(t, x, y)} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{u,yy}(t, x, y) = & -\frac{2 V_{uv}(t, x, y) g_{u,yy}(t, x, y)}{V_u(t, x, y)} + \frac{2 \sqrt{V_v(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} + \\ & \frac{\mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \frac{\sqrt{V_v(t, x, y)} \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_u(t, x, y)}{V_u^{\frac{3}{2}}(t, x, y)} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{v,xx}(t, x, y) = & \frac{2 V_{uv}(t, x, y) g_{v,xx}(t, x, y)}{V_v(t, x, y)} - \frac{2 \sqrt{V_u(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_v(t, x, y)}} + \\ & \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{V_v^{\frac{3}{2}}(t, x, y)} - \frac{\mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_u(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} \end{aligned}$$

$$\frac{\partial}{\partial t} g_{v,xy}(t, x, y) = \frac{2 V_{uv}(t, x, y) g_{v,xy}(t, x, y)}{V_v(t, x, y)} - \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_v(t, x, y)}} -$$

$$\begin{aligned}
& \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_v(t, x, y)}} + \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{2 V_v^{\frac{3}{2}}(t, x, y)} + \\
& \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{2 V_v^{\frac{3}{2}}(t, x, y)} - \frac{\mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_u(t, x, y)}{2 \sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \\
& \frac{\mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_u(t, x, y)}{2 \sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} \\
& \frac{\partial}{\partial t} g_{v,yy}(t, x, y) = \frac{2 V_{uv}(t, x, y) g_{v,yy}(t, x, y)}{V_v(t, x, y)} - \frac{2 \sqrt{V_u(t, x, y)} \mathbb{E} \left(\frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_v(t, x, y)}} + \\
& \frac{\sqrt{V_u(t, x, y)} \mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{V_v^{\frac{3}{2}}(t, x, y)} - \frac{\mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_u(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}}
\end{aligned}$$

[6]: `for equation in pkf_dynamics.in_aspect: display(equation)`

$$\frac{\partial}{\partial t} u(t, x, y) = v(t, x, y)$$

$$\frac{\partial}{\partial t} v(t, x, y) = -u(t, x, y)$$

$$\frac{\partial}{\partial t} V_u(t, x, y) = 2 V_{uv}(t, x, y)$$

$$\frac{\partial}{\partial t} V_v(t, x, y) = -2 V_{uv}(t, x, y)$$

$$\frac{\partial}{\partial t} V_{uv}(t, x, y) = -V_u(t, x, y) + V_v(t, x, y)$$

$$\begin{aligned}
& \frac{\partial}{\partial t} s_{u,xx}(t, x, y) = \frac{2 V_{uv}(t, x, y) s_{u,xx}(t, x, y)}{V_u(t, x, y)} - \frac{2 \sqrt{V_v(t, x, y)} s_{u,xx}^2(t, x, y) \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} - \\
& \frac{2 \sqrt{V_v(t, x, y)} s_{u,xx}(t, x, y) s_{u,xy}(t, x, y) \mathbb{E} \left(\frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} - \\
& \frac{2 \sqrt{V_v(t, x, y)} s_{u,xx}(t, x, y) s_{u,xy}(t, x, y) \mathbb{E} \left(\frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} - \\
& \frac{2 \sqrt{V_v(t, x, y)} s_{u,xy}^2(t, x, y) \mathbb{E} \left(\frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right)}{\sqrt{V_u(t, x, y)}} - \frac{s_{u,xx}^2(t, x, y) \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \\
& \frac{s_{u,xx}(t, x, y) s_{u,xy}(t, x, y) \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \\
& \frac{s_{u,xx}(t, x, y) s_{u,xy}(t, x, y) \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial x} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} - \\
& \frac{s_{u,xy}^2(t, x, y) \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_u(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_v(t, x, y)}{\sqrt{V_u(t, x, y)} \sqrt{V_v(t, x, y)}} + \frac{\sqrt{V_v(t, x, y)} s_{u,xx}^2(t, x, y) \mathbb{E} \left(\varepsilon_v(t, x, y, \omega) \frac{\partial}{\partial x} \varepsilon_u(t, x, y, \omega) \right)}{V_u^{\frac{3}{2}}(t, x, y)}
\end{aligned}$$

[illegible]

5

[illegible]

[illegible]

[illegible]

[illegible]

$$\frac{V_v^3(t, x, y) s_{v,xy}^2(t, x, y) s_{v,yy}^2(t, x, y) \mathbb{E} \left(\varepsilon_u(t, x, y, \omega) \frac{\partial}{\partial y} \varepsilon_v(t, x, y, \omega) \right) \frac{\partial}{\partial y} V_u(t, x, y)}{\sqrt{V_u(t, x, y)} V_v^{\frac{7}{2}}(t, x, y) s_{v,xx}(t, x, y) s_{v,yy}(t, x, y) - \sqrt{V_u(t, x, y)} V_v^{\frac{7}{2}}(t, x, y) s_{v,xy}^2(t, x, y)}$$