MATH 108B - Study Guide

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MATH 8 Review

Fill blanks with logical symbols

1. If $P \to Q$, then the converse is ____________________.
2. If $P \to Q$, then the contrapositive is ____________________.
3. The converse of $P \to Q$ is true if P________Q.

5. Eigenvalues, Eigenvectors, and Invariant Subspaces

Fill in the Blank

1.
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   This is a ________________ matrix with eigenvalues ________, ________, and ________.

2. We say a subspace U is **invariant** under some linear operator T, if T maps U back to itself. More formally, for a $T \in \mathcal{L}$, U is invariant under T if for any $u \in U$, $Tu \in$ ________.
3. Eigenvectors corresponding to distinct eigenvalues are ____________.
4. Suppose we have an operator $T \in \mathcal{L}(V)$ and $v_1, ..., v_n$ is a basis for V. Now, suppose

$$\begin{aligned} &Tv_1 \in \operatorname{span}(v_1) \\ &Tv_2 \in \operatorname{span}(v_1, v_2) \\ &\quad ... \\ &Tv_n \in \operatorname{span}(v_1, v_2, ..., v_n) \end{aligned}$$

   Thus, the matrix of T with respect to $v_1, ..., v_n$ is ____________________.

5. Suppose V is finite-dimensional. Then $T \in \mathcal{L}(V)$ has at most ____________ eigenvalues.
6. Let $T \in \mathcal{L}(V)$, then $T^0 =$ ____________.
7. Let $p(z)$ be a polynomial over the complex numbers. If $p(z) = a_0 + a_1 z + a_2 z^3$ then,

$$p(T) = \underline{\hspace{8cm}}$$

True or False

1. ________ Some operator $T \in \mathcal{L}(V)$ is invertible if and only if some matrix of T has distinct values on its diagonal.

2. ________ Suppose $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some basis of V. Then, T is diagonalizable if and only all the entries on the diagonal are nonzero.

3. ________ Let $T \in \mathcal{L}(V)$ with V finite-dimensional. Then,

$$E(\lambda_1, T) \cap ... \cap E(\lambda_n, T) = 0$$

6. Inner Product Spaces

Fill in the Blank

1. Suppose $\langle v, v \rangle = 0$. Then, $v = \underline{0}$.

2. Suppose $U \subset V$. The ______________ of U, denoted $U^\perp$ is the set

$$\{v \in V | \langle v, u \rangle = 0 \forall u \in U\}$$

3. Let $v_1, ..., v_n$ be a linearly independent list of vectors where each $\|v_i\| = 1$ for $i = 1, ...n$. Then, $v_1, ..., v_n$ is a(n) ____________________________.

6A. Inner Products and Norms

Inner Products An inner product on V is a function which maps ____________________________ to a number in $\mathbb{F}$.

Cauchy-Schwarz Inequality Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

where

$$|\langle u, v \rangle| = \|u\| \, \|v\|$$

if and only if u is a scalar multiple of v.

6B. Orthonormal Bases

Orthonormal A **list of vectors** is orthonormal if each vector in the list has length 1 (normal) and is orthogonal to every other vector in the list.

Example Consider the standard basis of $\mathbb{R}^3$ $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Turn this basis into an orthonormal basis.

Proof - An orthonormal list is linearly independent From Theorem 6.25, we know that

$$\|a_1 e_1 + ... + a_m e_m\|^2 = |a_1|^2 + ... + |a_m|^2$$

Use this to prove that orthonormal lists are linearly independent.

Theorem 6.30 – Writing a vector as linear combination of orthonormal basis Suppose $e_1, ..., e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + ... + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + ... + |\langle v, e \rangle|^2$$

This is a rewrite of a proof in the book

Proof. Let $e_1, ..., e_n$ be an orthonormal basis of V and $v \in V$ be arbitrary. First, because $e_1, ..., e_n$ is a basis for V there are some scalars $a_1, ..., a_n \in \mathbb{F}$ such that

$$v = a_1 e_1 + ... + a_n e_n$$

Now, notice for the inner product $\langle v, e_i \rangle$ for $i \in 1, ...n$

$$\begin{aligned}
\langle v, e_i \rangle &= \langle a_1 e_1 + ... + a_i e_i + ... + a_n e_n, e_i \rangle \\
&= \langle a_1 e_1, e_i \rangle + ... + \langle a_i e_i, e_i \rangle + ... + \langle a_n e_n, e_i \rangle && \text{By additivity} \\
&= a_1 \langle e_1, e_i \rangle + ... + a_i \langle e_i, e_i \rangle + ... + a_n \langle e_n, e_i \rangle && \text{By homogeneity in the first slot} \\
&= 0 + ... + a_i \langle e_i, e_i \rangle + ... + 0 && \text{By orthogonality} \\
&= a_i && \text{Because } e_i \text{ is normal}
\end{aligned}$$

Therefore, it is true that $a_i e_i = \langle v, e_i \rangle e_i$ for $i \in 1, ..., n$ and we can write

$$v = \langle v, e_1 \rangle e_1 + ... + \langle v, e_n \rangle e_n$$

as desired. Furthermore, we can get the second equation by applying Theorem 6.25 to the previous equation. □

6C. Orthogonal Complements

Prove or give a counterexample Suppose $U \subset V$. Then, $U^\perp$ is a subset of V.

Fill in the Blank Suppose U is a finite-dimensional subspace of V, and $v \in V$. Suppose $w \in U$ is such that

$$\|v - w\| \leq \|v - u\|$$

for any $u \in U$. Then, it must be that $w =$ ______________________________.

7. Operators on Inner Product Spaces

7A. Self-Adjoint and Normal Operators

Adjoint Given $T \in \mathcal{L}(V, W)$, the adjoint is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

True or False ________ Suppose $T \in \mathcal{L}(V, W)$. Then, $T^*(0) = 0$.

Matrix of the Adjoint Suppose

$$\mathcal{M}(T) = \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

then,

$$\mathcal{M}(T^*) = \begin{bmatrix} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{bmatrix}$$

Fill in the Blank

1. An operator $T \in \mathcal{L}(V, W)$ is called ______________________________ if $T = T^*$.
2. Suppose $T \in \mathcal{L}(V)$ is self-adjoint with some eigenvalue λ. The following equation

   $$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2$$

   shows that the eigenvalues of T ____.
3. Whenever we are discussing the adjoint of $T \in \mathcal{L}(V, W)$, V, W are presumed to be ______________________________.

True or False

1. ________ All self-adjoint operators are normal.
2. ________ An operator can be normal but not self-adjoint.

7B. The Spectral Theorem

The Real Spectral Theorem Suppose V is a vector space over the reals and $T \in \mathcal{L}(V)$. Then, TFAE:

a. T is self-adjoint

b. V has an orthonormal basis consisting of eigenvectors of T

c. T has a diagonal matrix with respect to some orthonormal basis of V

The following is a proof of $(a) \implies (b)$

Proof. We will prove this using induction. For the base case, let $n = 1$. Clearly, if T is an operator on a one-dimensional subspace V, then it maps vectors to scalar multiples of themselves, i.e. V has an orthonormal basis of eigenvectors of T as desired.

Now, for the inductive step assume that $T \in \mathcal{L}(V)$ is self-adjoint and that (a) implies (b) for all $n < \dim V$. By Theorem 7.27, we know that there is some $u \in U$ such that u is an eigenvector of T. Specifically, choose u such that $\|u\| = 1$. As a result, $T|_U$ is a one-dimensional invariant subspace of V. Furthermore, this implies that $T|_U{}^{\perp} \in \mathcal{L}(U^{\perp})$ is self-adjoint. Furthermore, by the inductive hypothesis,

____.

Recalling that Theorem 6.47 states that $V = U \oplus U^{\perp}$, this implies that adding u to the ______________________________ of $U^{\perp}$ gives an orthonormal basis for V as desired. $\square$

8. Operators on Complex Vector Spaces

8A. Generalized Eigenvectors and Nilpotent Operators

Why do we care about generalized eigenspaces? As you may recall, we can already decompose some $T \in \mathcal{L}(V)$ into the direct sum of one-dimensional invariant subspaces, i.e.

$$V = U_1 \oplus ... \oplus U_n$$

Specifically, each of these U_i is an eigenspace of V. So why do we care about generalized eigenspaces, and how do they extend this idea?

____

True or False

1. If v is an eigenvector of T, then it is also an generalized eigenvector of T. ____________
2. It's ______ that the differentiation operator is nilpotent because ____________

8B. Decomposition of an Operator

True or False Each generalized eigenspace $(T - \lambda_j I)|_{G(\lambda_J, T)}$ is nilpotent. ____________

8D. Jordan Form

True or False A matrix is said to be in Jordan Form if it is zero everywhere for square matrices along its diagonal, where all of these square matrices are of the same dimension. __________________