MATH 108A Review Sheet Solutions

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Solutions to Ch 1

1.A $\mathbb{R}^n$ and $\mathbb{C}^n$

1. A **complex number** is an ordered pair (a, b) of real numbers $a, b \in \mathbb{R}$ that we write $a + bi$.
2. Suppose we have $\alpha, \beta \in \mathbb{C}$ such that
$$\alpha := a + bi$$
$$\beta := c + di$$
Then $\alpha \cdot \beta = (ac - bd) + (ad + bc)i$
3. If $a \in \mathbb{R}$ then if $b \in \mathbb{R}$ is the multiplicative inverse of a, $a \cdot b = 1$
4. Two lists of vectors are considered the same if they have the same length and have the same entries/elements in the same order.

1.B Definition of Vector Space

1. A **field** is a set with two operations **vector addition** and **scalar multiplication** such that the operations are commutative, associative, and distributive and both have identity and inverse elements.
2. A **vector space** over a field $\mathbb{F}$ is a set V with two operations
   - Vector addition $+$
   - Scalar multiplication $\cdot$
3. In general, is there a such thing as multiplication between vectors? No.
4. An element of a vector space is called a **vector**.
5. The **trivial vector space** is the set containing the zero vector $\{0\}$.
6. The set F^∞ is the set of all sequences.
7. The set $\mathbb{R}^\mathbb{R}$ is the set of continuous real valued functions.
8. What are the sets $\mathbb{R}^n, \mathbb{C}^n$, and $\mathbb{F}^n$?
   - All lists of real numbers of length n
   - All lists of complex numbers of length n
   - All lists of numbers of length n

   (See pre-midterm review lecture on February 7)

1.C Subspace

1. Subspace Test
   (a) A set U is a subspace of some vector space V if it meets the following three conditions of the Subspace Test
      i. The zero vector is in U.
      ii. U is closed under vector addition.
      iii. U is closed under scalar multiplication.
   (b) Write out the Subspace Test using logical symbols.
      i. $\vec{0} \in U$
      ii. $\forall u_1, u_2 \in U,\ u_1 + u_2 \in U$
      iii. $\forall u \in U,\ \forall \lambda \in \mathbb{F},\ \lambda \cdot u \in U$.

2. Suppose that $U_1, U_2, ...U_m$ are subspaces of V such that

$$0 = u_1 + u_2 + ... + u_m$$

   where $u_1 \in U_1, ..., u_m \in U_m$, has a unique solution.

   Then we can say that $u_1+u_2+...+u_m$ is a **direct sum**, where $u_i = 0$ for $i = 1, 2, ...m$.

   (See textbook).

3. If $U + W$ are subspaces of V, then $U + W$ is a direct sum if and only if there is a unique way to write zero. (Equivalently, $U + W$ is a direct sum if $U \cap W = \{0\}$.)

4. Let $v_1, v_2, ..., v_m$ be a list of vectors in V. Then, $\text{span}(v_1, v_2, ..., v_m)$ is a **subspace** of V.

5. Let U_1, U_2, U_3 be subspaces of V such that $U_1 \cap U_2 \cap U_3 = \{0\}$ and that $U_1+U_2+U_3 = V$. Can we conclude that $U_1 \oplus U_2 \oplus U_3 = V$?

   No. See midterm/lecture notes.

Solutions to Ch 2

2.A Span and Linear Independence

1. The set $\mathscr{P}(\mathbb{F})$ is infinite-dimensional while $\mathscr{P}_m(\mathbb{F})$ is finite-dimensional.

2. Two functions p, q are the **same** if for all $z \in \mathbb{F}$ $p(z) = q(z)$ (i.e. they have the same output).

3. Suppose for there exists a $v_j \in \text{span}(v_1, v_2, ..., v_n)$. Then, we know that $v_1, v_2, ..., v_j, ..., v_n$ is **linearly dependent**.

4. True or False. If $a_1v_1 + a_2v_2 + ... + a_mv_m = 0$, given that $a_1 = a_2 = ... = a_m = 0$, then $v_1, v_2, ..., v_m$ are linearly independent.

   False. This is trivially true for all vectors. We need $a_1 = a_2 = ... = a_m = 0$ to be **the only** solution to this equation to know that $v_1, v_2, ..., v_m$ are linearly independent. (See after midterm lecture on February 14).

2.B Bases

1. If a list $v_1, v_2, ..., v_n \in V$ is both linearly independent and spanning, then it is a **basis** for V.
2. If a list is spanning but not a basis, then it is **linearly dependent** (and we can drop some vectors to make it a basis). (See Linear Dependence Lemma).
3. The standard basis for $\mathbb{F}^n$ is the list
   - $(1, 0, 0, ..., 0)$
   - $(0, 1, 0, ..., 0)$
   - ...
   - $(0, 0, 0, ..., 1)$
4. If a list $v_1, v_2, ..., v_n$ (of length n) is a basis for V, then dim $V = n$.
5. If $v_1, v_2, ..., v_j$ is linearly independent in V, and $v_1, v_2, ..., v_k$ spans V, then j $\leq$ k. (This is based on the fact that all linear independent lists are either spanning or shorter than a spanning list.)

2.C

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Solutions to Ch 3

3.A The Vector Space of Linear Maps

1. A linear map (linear transformation) from V to W is a function $T : V \to W$ such that
   - $T(u + v) = Tu + Tv$ for all $u, v \in V$
   - $\lambda T(u) = T(\lambda u)$ for all $\lambda \in \mathbb{F}$ and $u \in V$

3.B Null Space and Range

1. Suppose $v_1, v_2, ..., v_n$ is linearly independent in V. Do we know that $v_1, v_2, ..., v_n$ can be extended to be a basis for V?

   No. This is true if V was finite-dimensional, but we don't know that. (See midterm/beginning of February 14 lecture).

2. Let $T \in \mathscr{L}(V, W)$. Then,

   $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. (This is the **Fundamental Theorem of Linear Maps**).

3. Let $T : L \to W$, be a linear map. Then the **null space** of T is the set

$$\{v \in V | Tv = 0\}$$

4. Does the null space contain the zero vector?

   Yes. It's also true that the null space is a subspace.

5. A linear map $T : V \to W$ if injective if whenever $u \neq w$, then $Tu \neq Tw$. (From lecture notes. This is the contrapositive of the normal definition of injectivity.)

6. Prove that T is injective, if and only if null $T = \{\vec{0}\}$.

   (See pg. 61 of LADR or February 14 lecture for another version of this proof)

   Proof. For one direction, suppose that null $T = \{0\}$ and show that T is injective. This implies we want to show that if $Tu = Tv$ then $u = v$.

   So suppose that $Tu = Tv$, implying

$$\begin{aligned} Tu &= Tv \\ Tu - Tv &= 0 \\ T(u - v) &= 0 && \text{Additivity of linear maps} \\ u - v &= 0 && \text{Because only } T0 = 0 \text{ as implied by null } T = \{0\} \\ u &= v \end{aligned}$$

   as we set out to prove.

   Now, for the other direction, suppose that T is injective and show that null $T = \{0\}$.

   Suppose for a contradiction that null T is bigger than $\{0\}$. That implies there is some v such that $Tv = 0$, but $v \neq 0$. However, because $T0 = Tv$ but $0 \neq v$, T is not injective. So it must be that null $T = \{0\}$.

   This completes the proof. □

7. If a linear map $T : V \to W$ is such that range $T = W$ then we say T is **surjective** (or **onto**).

3.C Matrices

1. $\mathbb{F}^{m,n}$ is the set of all **m × n matrices**.

2. Suppose $T \in \mathscr{L}(V)$ is such that with respect to the basis v_1, v_2, v_3

$$\mathscr{M}(T) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

   What are Tv_1, Tv_2, Tv_3 equal to?

$$Tv_1 = 8v_1 + 0v_2 + 0v_3$$
$$Tv_2 = 0v_1 + 5v_2 + 0v_3$$
$$Tv_3 = 0v_1 + 0v_2 + 5v_3$$

   Source: Last lecture

3.D Invertibility and Isomorphic Vector Spaces

1. Suppose $T \in \mathscr{L}(V, W)$ is such that null $T = \{\vec{0}\}$ and range $T = W$. Then, we know that T is **invertible**.

   Explanation null $T = \{0\}$ implies T is injective, and range $T = W$ implies T is surjective. Now recall that T is invertible if and only if it it is injective and surjective.

2. Let $T \in L(V, W)$ be such there are $R, S \in L(W, V)$ such that R, S are inverses of T. Is $R = S$?

   Yes. $R = S$ because inverses are unique.

3. Suppose $T \in L(V, W)$ is invertible. Then, we can say V and W are **isomorphic** and T is **an isomorphism**.

4. Let V, W be finite-dimensional and isomorphic. Then,

$$\dim V = \dim W$$

5. Suppose T is a linear function from $\mathbb{R}^3$ to $\mathbb{R}^3$. Then we can say, T is a(n) **linear operator** on $\mathbb{R}^3$.

Solutions to Ch 4

1. Let $p(z) = 0$ for all $z \in \mathbb{F}$, i.e. the zero polynomial. Then, $\deg(p) = -\infty$.

2. Let $p(z) = a_0 + a_1z + a_2z^2 + ... + a_mz^m$ for all $z \in \mathbb{F}$. Then, $\deg(p) = m$.

Solutions to Ch 5

5.A Invariant Subspaces

1. Suppose U is a subspace of V and $T \in \mathscr{L}(V)$. If for any $u \in U$, Tu is also in U, we say U is **invariant under** $\boldsymbol{T}$**.** (Def 5.2: invariant subspace)

2. λv.

3. By definition, for any $v \in \text{null } T$, $Tv = 0$. Because $0 \in \text{null } T$, the null space is invariant under T.

4. Because T maps any vector into its range, range T is invariant under T.

5. Suppose that U is a one-dimensional subspace invariant under T and v is a basis for U. Then v is a(n) eigenvector for T and there exists a scalar λ such that $Tv = \lambda v$. (See March 7 lecture notes)

6. No, T is not invertible.

   Theorem 5.30: Determination of invertibility from upper-triangular matrix
   Suppose $T \in \mathscr{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then T is invertible if and only if all the entries of the diagonal of that upper-triangular matrix are nonzero.

7. Linearly independent.