MATH 108B - Study Guide Solutions

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MATH 8 Review

Fill blanks with logical symbols

1. If $P \to Q$, then the converse is QP.
2. If $P \to Q$, then the contrapositive is $\neg Q \to \neg P$.
3. The converse of $P \to Q$ is true if the following is true: $P \iff Q$. (iff; if and only if)

5. Eigenvalues, Eigenvectors, and Invariant Subspaces

Fill in the Blank

1. $$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   This is a diagonal (also upper triangular) matrix with eigenvalues 4, 2, and 0.
2. We say a subspace U is **invariant** under some linear operator T, if T maps U back to itself. More formally, for a $T \in \mathcal{L}$, U is invariant under T if for any $u \in U$, $Tu \in U$.
3. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
4. Suppose we have an operator $T \in \mathcal{L}(V)$ and $v_1, ..., v_n$ is a basis for V. Now, suppose

   $$\begin{aligned} Tv_1 &\in \text{span}\,(v_1) \\ Tv_2 &\in \text{span}\,(v_1, v_2) \\ &... \\ Tv_n &\in \text{span}\,(v_1, v_2, ..., v_n) \end{aligned}$$

   Thus, the matrix of T with respect to $v_1, ..., v_n$ is upper triangular (Theorem 5.26).
5. Suppose V is finite-dimensional. Then $T \in \mathcal{L}(V)$ has at most dim V eigenvalues.
6. Let $T \in \mathcal{L}(V)$, then $T^0 = I$.
7. Let $p(z)$ be a polynomial over the complex numbers. If $p(z) = a_0 + a_1 z + a_2 z^3$ then,

   $$p(T) = I + a_1 T + a_2 T^3$$

   (5B - Definition 5.17)

True or False

1. Some operator $T \in \mathcal{L}(V)$ is invertible if and only if some matrix of T has distinct values on its diagonal. **False. You were asked to provide a counterexample of this on the homework on 5B Exercise 14. Also see Theorem 5.30.**

2. Suppose $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some basis of V. Then, T is diagonalizable if and only all the entries on the diagonal are nonzero. **True. See Theorem 5.44.**

3. Let $T \in \mathcal{L}(V)$ with dim V distinct eigenvalues. Then,

$$E(\lambda_1, T) \cap ... \cap E(\lambda_n, T) = 0$$

True. V **is a direct sum of its eigenspaces (Theorem 5.41).**

6. Inner Product Spaces

Fill in the Blank

1. Suppose $\langle v, v \rangle = 0$. Then, $v = \underline{0}$.

2. Suppose $U \subset V$. The **orthogonal complement** of U, denoted $U^{\perp}$ is the set

$$\{v \in V | \langle v, u \rangle = 0 \forall u \in U\}$$

3. Let $v_1, ..., v_n$ be a linearly independent list of vectors where each $\|v_i\| = 1$ for $i = 1, ...n$. Then, $v_1, ..., v_n$ is a(n) **orthonormal list**. (We don't know if it's an orthonormal basis because I didn't state the dimension of V).

6A. Inner Products and Norms

Inner Products An inner product on V is a function which maps **an ordered pair (u, v) of elements in V** to a number in $\mathbb{F}$.

Cauchy-Schwarz Inequality Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

where

$$|\langle u, v \rangle| = \|u\| \, \|v\|$$

if and only if u is a scalar multiple of v.

6B. Orthonormal Bases

Orthonormal A **list of vectors** is orthonormal if each vector in the list has length 1 (normal) and is orthogonal to every other vector in the list.

Example Consider the standard basis of $\mathbb{R}^3$ $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Turn this basis into an orthonormal basis. **This is already an orthonormal basis.**

Proof - An orthonormal list is linearly independent From Theorem 6.25, we know that

$$\|a_1e_1 + ... + a_me_m\|^2 = |a_1|^2 + ... + |a_m|^2$$

Use this to prove that orthonormal lists are linearly independent.

Proof. Let $e_1, ..., e_m$ be any orthonormal list. Now, consider the equation

$$\begin{aligned} a_1e_1 + ... + a_me_m &= 0 \\ \|a_1e_1 + ... + a_me_m\|^2 &= \|0\|^2 \\ |a_1|^2 + ... + |a_n|^2 &= 0 \end{aligned}$$

which implies that $a_1 = ... = a_n = 0$. In other words, $e_1, ..., e_m$ is linearly independent as desired. $\square$

(See the beginning of 6B in LADR for a similar proof.)

Theorem 6.30 – Writing a vector as linear combination of orthonormal basis Suppose $e_1, ..., e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1\rangle e_1 + ... + \langle v, e_n\rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e\rangle|^2$$

This is a rewrite of a proof in the book

Proof. Let $e_1, ..., e_n$ be an orthonormal basis of V and $v \in V$ be arbitrary. First, because $e_1, ..., e_n$ is a basis for V there are some scalars $a_1, ..., a_n \in \mathbb{F}$ such that

$$v = a_1e_1 + ... + a_ne_n$$

Now, notice for the inner product $\langle v, e_i\rangle$ for $i \in 1, ...n$

$$\begin{aligned} \langle v, e_i\rangle &= \langle a_1e_1 + ... + a_ie_i + ... + a_ne_n, e_i\rangle \\ &= \langle a_1e_1, e_i\rangle + ... + \langle a_ie_i, e_i\rangle + ... + \langle a_ne_n, e_i\rangle && \text{By additivity} \\ &= a_1\langle e_1, e_i\rangle + ... + a_i\langle e_i, e_i\rangle + ... + a_n\langle e_n, e_i\rangle && \text{By homogeneity in the first slot} \\ &= 0 + ... + a_i\langle e_i, e_i\rangle + ... + 0 && \text{By orthogonality} \\ &= a_i && \text{Because } e_i \text{ is normal} \end{aligned}$$

Therefore, it is true that $a_ie_i = \langle v, e_i\rangle e_i$ for $i \in 1, ..., n$ and we can write

$$v = \langle v, e_1\rangle e_1 + ... + \langle v, e_n\rangle e_n$$

as desired. Furthermore, we can get the second equation by applying Theorem 6.25 to the previous equation. $\square$

6C. Orthogonal Complements

Prove or give a counterexample Suppose $U \subset V$. Then, $U^\perp$ is a subset of V.

Fill in the Blank Suppose U is a finite-dimensional subspace of V, and $v \in V$. Suppose $w \in U$ is such that

$$\|v - w\| \leq \|v - u\|$$

for any $u \in U$. Then, it must be that $w = P_U v$ (the orthogonal projection).

7. Operators on Inner Product Spaces

7A. Self-Adjoint and Normal Operators

Adjoint Given $T \in \mathcal{L}(V, W)$, the adjoint is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

True or False Suppose $T \in \mathcal{L}(V, W)$. Then, $T^*(0) = 0$. **True. The adjoint is a linear map and all linear maps send 0 to 0.**

Matrix of the Adjoint Suppose

$$\mathcal{M}(T) = \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

then,

$$\mathcal{M}(T^*) = \begin{bmatrix} -i & 2+3i \\ 1+i & 4 \end{bmatrix}$$

Fill in the Blank

1. An operator $T \in \mathcal{L}(V, W)$ is called **self-adjoint** if $T = T^*$.
2. Suppose $T \in \mathcal{L}(V)$ is self-adjoint with some eigenvalue λ. The following equation

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2$$

   shows that the eigenvalues of T **must be real** (Theorem 7.13).
3. Whenever we are discussing the adjoint of $T \in \mathcal{L}(V, W)$, V, W are presumed to be **finite-dimensional** (See Notation 7.1 at the beginning of chapter 7).

True or False

1. All self-adjoint operators are normal. **True.** An operator is normal if it commutes with its adjoint, i.e.

$$TT^* = T^*T$$

   If an operator is self-adjoint then the above equality trivially becomes

$$TT = TT$$

2. An operator can be normal but not self-adjoint. **True. See pg. 212 for an example.**

7B. The Spectral Theorem

The Real Spectral Theorem Suppose V is a vector space over the reals and $T \in \mathcal{L}(V)$. Then, TFAE:

a. T is self-adjoint

b. V has an orthonormal basis consisting of eigenvectors of T

c. T has a diagonal matrix with respect to some orthonormal basis of V

The following is a proof of $(a) \implies (b)$

Proof. We will prove this using induction. For the base case, let $n = 1$. Clearly, if T is an operator on a one-dimensional subspace V, then it maps vectors to scalar multiples of themselves, i.e. V has an orthonormal basis of eigenvectors of T as desired.

Now, for the inductive step assume that $T \in \mathcal{L}(V)$ is self-adjoint and that (a) implies (b) for all $n < \dim V$. By Theorem 7.27, we know that there is some $u \in U$ such that u is an eigenvector of T. Specifically, choose u such that $\|u\| = 1$. As a result, $T|_U$ is a one-dimensional invariant subspace of V. Furthermore, this implies that $T|_U{}^{\perp} \in \mathcal{L}(U^{\perp})$ is self-adjoint. Furthermore, by the inductive hypothesis, $U^p erp$ **has an orthonormal basis of eigenvectors of T**. Recalling that Theorem 6.47 states that $V = U \oplus U^{\perp}$, this implies that adding u to the **orthonormal basis** of $U^{\perp}$ gives an orthonormal basis for V as desired. □

8. Operators on Complex Vector Spaces

8A. Generalized Eigenvectors and Nilpotent Operators

Why do we care about generalized eigenspaces? As you may recall, we can already decompose some $T \in \mathcal{L}(V)$ into the direct sum of one-dimensional invariant subspaces, i.e.

$$V = U_1 \oplus ... \oplus U_n$$

Specifically, each of these U_i is an eigenspace of V. So why do we care about generalized eigenspaces, and how do they extend this idea?

Not all operators have enough eigenvalues for an eigenspace decomposition. However, all operators over a complex vector space have a generalized eigenspace decomposition (Theorem 8.21).

True or False

1. If v is an eigenvector of T, then it is also an generalized eigenvector of T. **True. Recall that** $E(\lambda, T) \subset G(\lambda, T)$.

2. It's **true** that the differentiation operator is nilpotent because **every polynomial of degree** m **has 0 as its** $m + 1^{th}$ **derivative**.

8B. Decomposition of an Operator

True or False Each generalized eigenspace $(T - \lambda_j I)|_{G(\lambda_J, T)}$ is nilpotent. **True, by definition.**

8D. Jordan Form

True or False A matrix is said to be in Jordan Form if it is zero everywhere for square matrices along its diagonal, where all of these square matrices are of the same dimension. **False, see textbook for counterexamples.**