Problem Set 11

Exercises for the lecture Fundamentals of Simulation Methods, WS 2020

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Hand in until Wednesday, 10.02, 23:59
Tutorials on Friday, 12.02
(Group 1, Giovanni Leidi 09:15)
(Group 2, Benedikt Rennekamp 09:15)
(Group 3, Siddhant Deshmukh 11:15)
(Group 4, Fabian Kutzki 14:15)

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1. Isothermal 1D hydrodynamics solver: sound pulse [7 pt.]

1. Use the 1D finite-volume solver that you already developed for the last problem set to solve the following 1D isothermal hydrodynamics problem: the x-grid goes from $x = -1$ to $x = 1$ and it is divided into $N_x = 100$ cells, the boundary conditions are periodic and the isothermal sound speed is $c_s = 1$. The initial conditions are

$$\rho(x, t = 0) = 1 + \epsilon \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad (1)$$

$$u(x, t = 0) = 0 \quad (2)$$

   where $\epsilon = 10^{-4}$ and $\sigma = 0.2$. Use CFL=0.4.

2. How do you expect the system to physically evolve? Why?

3. What is the value of the sound crossing time scale t_{c_s} for this setup?

4. Overplot $\rho(x, 0)$ and $\rho(x, t^*)$ for $t^* = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \times t_{c_s}$. You will need these overplots to make a qualitative comparison, so don't worry if you didn't store the output exactly at t^*, also the previous or the following time step will be fine.

5. Plot the time evolution of the total kinetic energy until $t_{max} = 10 \times t_{c_s}$.

6. Re-do the calculations and the plots using $N_x = 1000$. Explain the results that you get and the differences between the two resolution runs.

2. 1D Euler Riemann problem [13 pt.]

In order to simulate the Riemann problem for the 1D Euler system, you have to implement a few modifications to your previous finite-volume solver:

- the set of conservative variables now is:

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \tag{3}$$

where ρE is the *total energy density*:

$$\rho E \equiv e_{int} + \frac{\rho u^2}{2} \tag{4}$$

and e_{int} is the *internal energy density*.

- the physical fluxes are:

$$\mathbf{q} = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ u(\rho E + P) \end{bmatrix} \tag{5}$$

- The equation of state is now adiabatic. You can recover the value of the pressure from the total energy density:

$$P = (\gamma - 1)(\rho E - \frac{\rho u^2}{2}) \tag{6}$$

where γ is the *adiabatic index*. For this simulation you can set $\gamma = 1.4$.

- In order to calculate the time step you can use the formula given in the last exercise sheet. The only difference is that the sound speed c_s in not constant anymore:

$$c_s \equiv \sqrt{\frac{\gamma P}{\rho}} \tag{7}$$

- Equations 6 and 7 apply both at cell centers and face centers.

- For setting *Dirichlet* boundary conditions, you can simply fill the first and second ghost cells with the provided boundary values. In this case there is no need to update the ghost cells at every time step, just remember to fill them before entering the time loop.

You can now setup the following 1D Riemann problem: the x-grid goes from $x = 0$ to $x = 1$ and it is divided into $N_x = 100$ cells. At $t = 0$ the system is divided into *left* and *right* states:

- Left state ($x \leq 0.5$): $\rho = 1$, $p = 1$, $u = 0$.

- Right state ($x > 0.5$): $\rho = 0.125$, $p = 0.1$, $u = 0$.

and $\gamma = 1.4$. Dirichlet boundary conditions apply, and the boundary values are given by the initial L/R states[1].

[1]Note that: $\rho E_{L/R}(t = 0) = \dfrac{p_{L/R}(t = 0)}{\gamma - 1}$

1. Solve this problem and plot the results for $\rho(x)$, $u(x)$, and $p(x)$ at the final time ($t = 0.2$).

2. Plot the time evolution of these quantities in a $x - t$ *diagram*. (For instance, you can use *pyplot.imshow*)

3. Redo the problem using $N_x = 1000$.

4. Explain the shape of the solution: where is the contact discontinuity, where is the rarefaction wave, and where is the shock wave?

5. Describe the differences that you observe when increasing the resolution, especially with respect to numerical diffusivity across the rarefaction wave, the shock wave and the contact discontinuity.

6. Experiment with the setup values of the Riemann problem and try to find initial conditions such that the outer (non-linear) sonic waves are both shock waves travelling outwards. Plot the results for $\rho(x)$, $u(x)$, and $p(x)$ at what you think it may be a good final time.