Dynamic Programming Recipe

- Define a set of problems, such that
  - base case - easy to solve
  - final case - matches (closely) the final problem we want to solve.
- Write it as a recursion: Solve bigger problem in terms of the smaller problems. (Should be a DAG on the problem instances!)
- Compute the problems on the DAG in the linearized order!

Reliable Shortest Paths.

Shortest reliable path.

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Given G, k. Find shortest path that uses at most k edges.

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More edges, higher chance of a problem.

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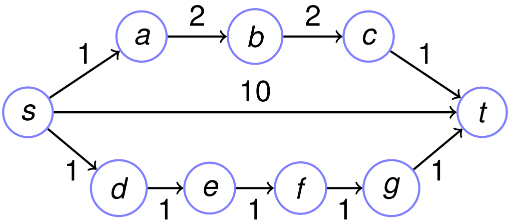

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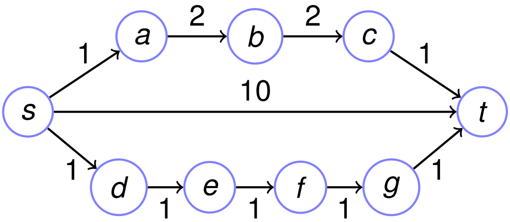

Shortest path? $s \to d \to e \to f \to g \to t$

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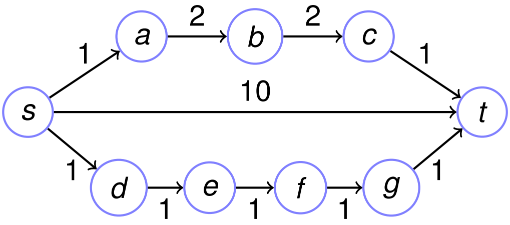

Shortest path? $s \to d \to e \to f \to g \to t$ Cost: 5

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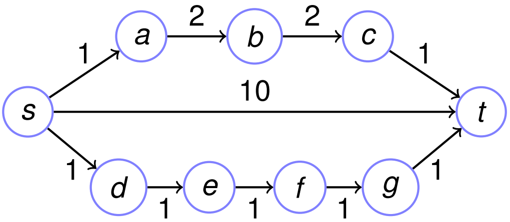

Shortest path? $s \to d \to e \to f \to g \to t$ Cost: 5

Shortest path that uses at most 4 edges?

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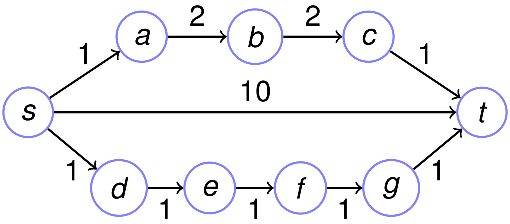

Shortest path? $s \to d \to e \to f \to g \to t$ Cost: 5

Shortest path that uses at most 4 edges? $s \to a \to b \to c \to t$

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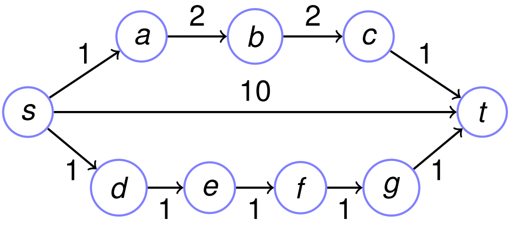

Shortest path? $s \to d \to e \to f \to g \to t$ Cost: 5

Shortest path that uses at most 4 edges? $s \to a \to b \to c \to t$ Cost: 6

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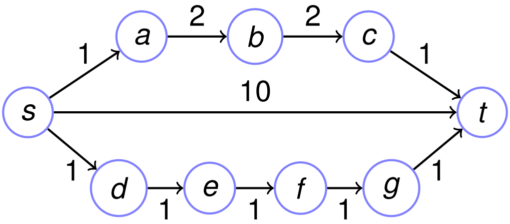

Shortest path? $s \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow t$ Cost: 5

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Shortest path that uses at most 3 edges?

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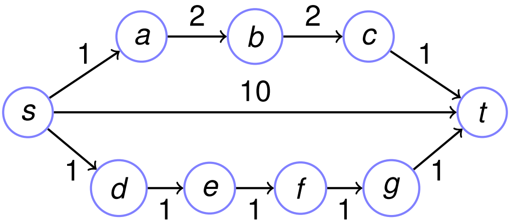

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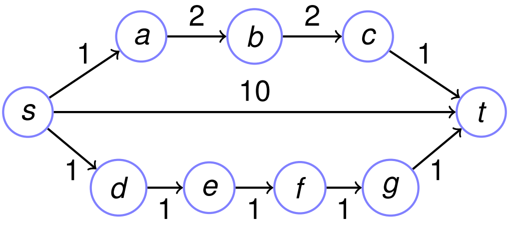

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Shortest path that uses at most 3 edges? $s \rightarrow t$ Cost: 10.

Algorithm

Dijkstra's?

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Dynamic Program.

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Subproblems: shortest path to node using few edges.

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$O(nk)$ table entries.

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Bellman Ford...

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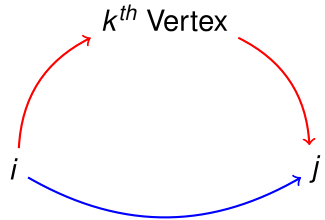

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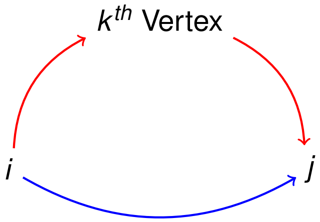

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Runtime? $(n = |V|)$

For each edge $(i,j) \in E$, $d(i,j,0) = l(i,j)$.

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For each edge $(i,j) \in E$, $d(i,j,0) = l(i,j)$.

Initialization time.

(A) $O(n^2)$

(B) $O(|E|)$

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$O(n^3)$ time. (versus $O(n^2|E|)$ for n Bellman-Fords).

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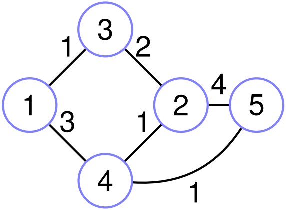
Given distances between n cities, find cycle that visits each city once of minimum cost.



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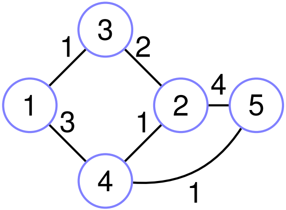

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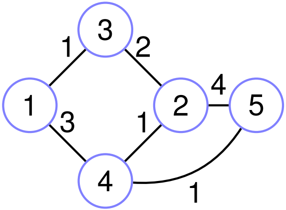

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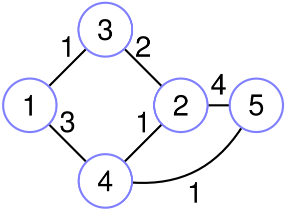
Given distances between n cities, find cycle that visits each city once of minimum cost.



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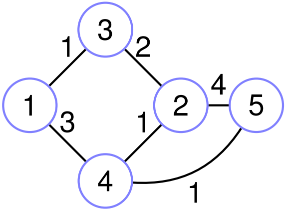

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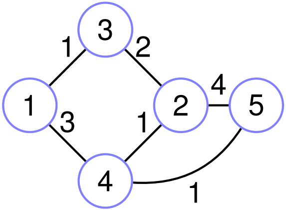

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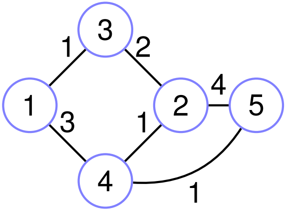

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Much better, but still not polynomial!

TSP Dynamic Program

Subproblem: best tour that visits the first i cities.

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Fill in each entry: $O(n)$ time.
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Answer? (A) $\min_{S,j}\{C(S,j) + d_{j1}\}$ (B) $\min_j\{C(V,j) + d_{j1}\}$, (C) $\min_j\{C(V,j)\}$

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Given a graph $G = (V, E)$, find the smallest subset, S, where $\forall v \in V, v \in S,$ or $,(u, v) \in E$ and $u \in S$.

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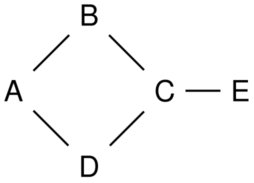

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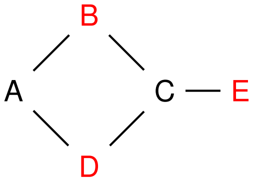

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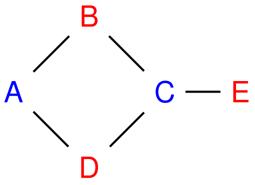

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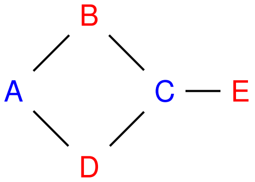

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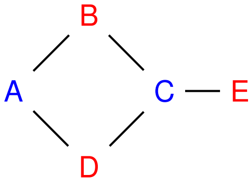

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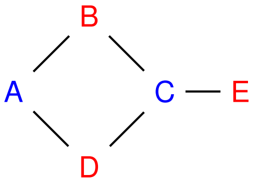

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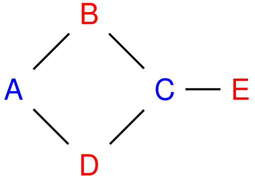

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What could be more important than ice-cream!

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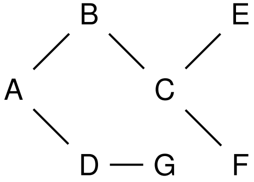

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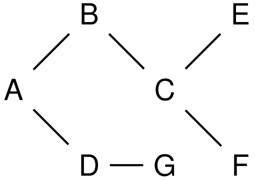

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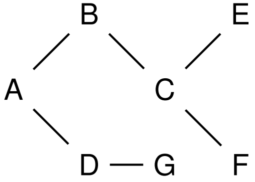

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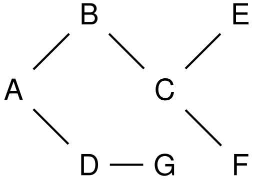

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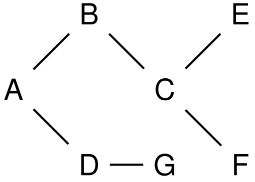

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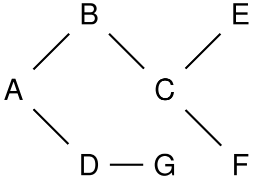

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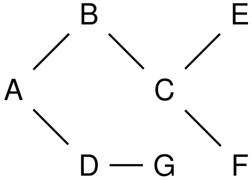

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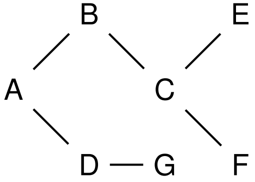

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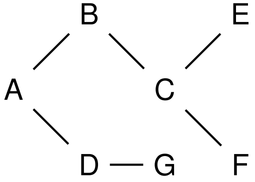

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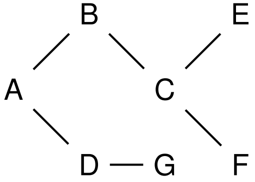

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For leaf u. DS(u,false,false)?

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$= 1 + \Sigma_{\text{subtree}_v} \min\{DS(v, false, false), DS(v, false, true), DS(v, true, true)\}.$

DS(u,false,true) $=$
$\min_{\text{subtree}_x} DS(x, true, true) +$
$\Sigma_{\text{subtree } v \neq x} \min\{DS(v, false, true), DS(v, true, true)\}.$

DS(u,false,false) $= \Sigma_{\text{subtree}_v} DS(v, false, true)$

For leaf u. DS(u,false,false)? 0

Subproblem: DS(u,in_cover,covered).
vertex u, booleans in_cover,covered.
Best solution where u is in cover or not
and covered or not in subtree.

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$O(V)$ table entries.

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$O(V)$ table entries.
Fill in entry for degree d node, in time $O(d^2)$.

Subproblem: DS(u,in_cover,covered).
vertex u, booleans in_cover,covered.
Best solution where u is in cover or not
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$O(V)$ table entries.
Fill in entry for degree d node, in time $O(d^2)$.
$O(d)$ with a bit of care.

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$O(V)$ table entries.
Fill in entry for degree d node, in time $O(d^2)$.
$O(d)$ with a bit of care.
$O(|E|)$ time.