Today

Linear Programming

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas get 3 sq. yards/bushel of sunny land.
Carrots get 3 sq. yards/bushel of shady land.

Garden has 60 sq. yards of sunny land and 75 sq. yards of shady land.

To pea or not to pea, that is the question!

To pea or not to pea.

4$ for peas. 2$ bushel of carrots. x_1- to pea! x_2 carrots

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Peas take 3 unit of water/bushel.

Carrot take 2 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 3 yards/bushel of sunny land. Have 60 sq. yards.

$3x_1 \leq 60$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative! $x_1, x_2 \geq 0$.

A linear program.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 3x_1 \leq 60 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 3x_1 \leq 60 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Optimal point?

Try every point if we only had time!

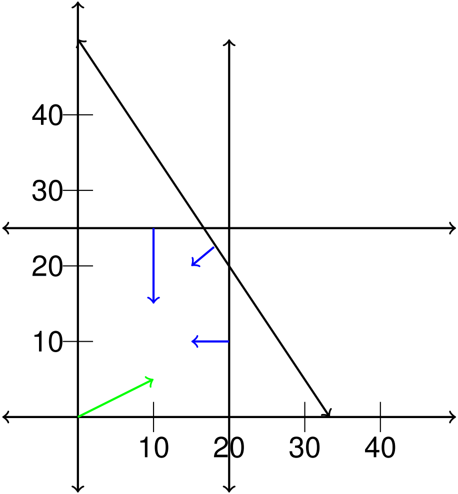
How many points?

Real numbers?

Infinite. Uncountably infinite!

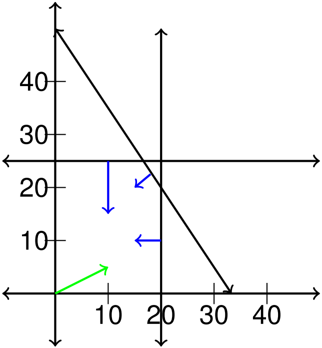
A linear program.

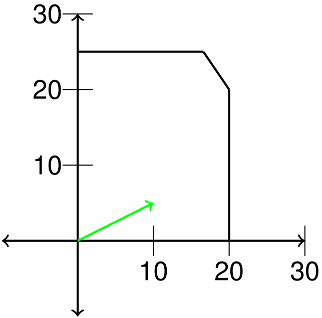
$$\max 4x_1 + 2x_2$$
$$3x_1 \leq 60$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$



Optimal point?

Feasible Region.





Convex.

Any two points in region connected by a line in region.

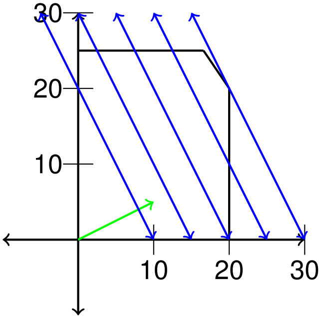
Algebraically:

If x and x' satisfy an constraint, so does $x'' = \alpha x + (1-\alpha)x'$

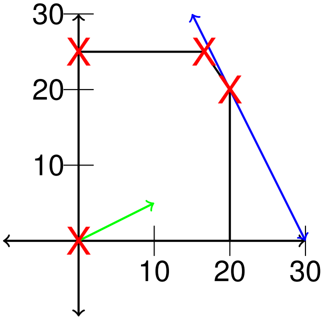
E.g. $3x \leq 60$ and $3x' \leq 60$

$\rightarrow 3\alpha x \leq \alpha(60)$ and $3(1-\alpha)x' \leq (1-\alpha)60$

$\rightarrow 3(\alpha(x) + (1-\alpha)x') \leq (\alpha + (1-\alpha))60 = 60$



Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints! Which are lines.
Try every vertex! Choose best among the ones in the region.
$O(m^2)$ if m constraints and 2 variables.
For n variables, m constraints, how many?
nm? $\binom{m}{n}$? $n+m$?
$\binom{m}{n}$
Finite!!!!!! But exponential in the number of variables.



$$\max 4x_1 + 2x_2$$
$$3x_1 \le 60$$
$$3x_2 \le 75$$
$$3x_1 + 2x_2 \le 100$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Simplex: Start at vertex. Move to better neighboring vertex.
Until you stop. This example.
(0,0) objective 0. $\rightarrow (0,25)$ objective 50.
$\rightarrow (16\frac{2}{3}, 25)$ objective $116\frac{2}{3}$ $\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus third.
Get $4x_1 + 2x_2 \le 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? Yes? No? Maybe? No!
Dual problem: add equations to get best upper bound.

More variables.

More vegetables. How about some Kale!
3$ per bushel.
2 sq. yards/bushel per sunny or 3 sq. yards/bushel for shady land.
2 units of water.
x_3 - sunny kale x_4 - shady kale.

$$\begin{aligned} \max 4x_1 + 2x_2 + 3x_3 + 3x_4 \\ 3x_1 + 2x_3 \leq 60 \\ 3x_2 + 3x_4 \leq 75 \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 \leq 100 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Carpet production planning.

Demands: $d_1, d_2, \ldots, d_{12}$, range: $440 - 920$
30 employees. 20 carpets/month. 2000/month.
Overtime: \$180 extra per carpet. Also at most 30% for one employee.
Hiring/firing: 320/400.
Storage: 8/carpet and no storage at the end of year.

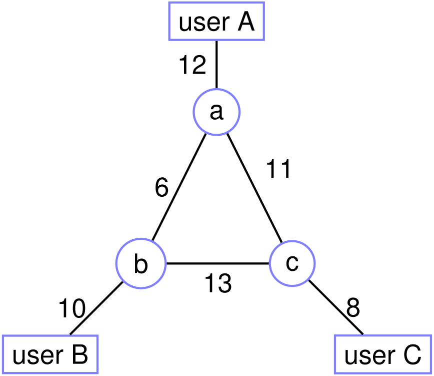
Variables.

w_i - number of workers in month i ; $w_0 = 30$
x_i - carpets made in month i
o_i - overtime carpets in month i
h_i, f_i - hired/fired in month i
s_i - number stored at end of month i ; $s_{12} = 0$

Nonnegative: $w_i, x_i, o_i, h_i, f_i, s_i \geq 0$
Production: $x_i = 20w_i + o_i$
Employment: $w_i = w_{i-1} + h_i - f_i$
Inventory: $s_i = s_{i-1} + x_i - d_i$
Regulations: $o_i \leq 6w_i$
Objective:

$$\min \quad 2000\sum_i w_i + 320\sum_i h_i + 400\sum f_i + 8\sum s_i + 180\sum_i o_i.$$

Bandwidth.



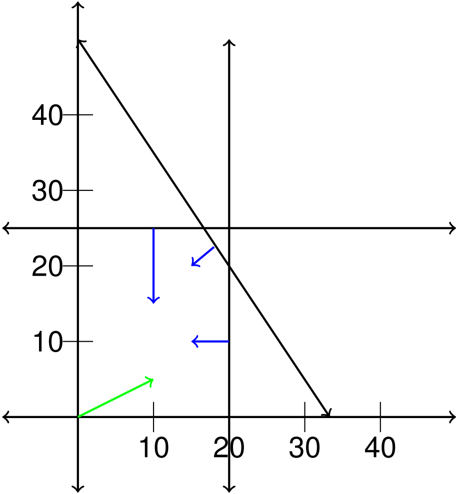
$A - B$ pays 3\$ per unit,
$A - C$ pays 2\$ per unit,
$B - C$ pays 4\$ per unit.
Every pair gets 2 units.
X_{AB} - flow along $A - a - b - B$.
X'_{AB} is flow along path $A - a - c - b - B$
Capacity constraint on edge (a, b):
$X_{AB} + X'_{BC} + X'_{AC} \leq 6$
Bandwidth constraint:
$X_{AB} + X'_{AB} \geq 2$

How many edge constraints? 6.
How many bandwidth constraints? 3.
Objective function?

$$3(X_{AB} + X'_{AB}) + 4(X_{BC} + X'_{BC}) + 2(X_{AC} + X'_{AC})$$

A linear program.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 60 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$



Optimal point?

Again with carpets!

Production: $x_i = 20w_i + o_i$
Employment: $w_i = w_{i-1} + h_i - f_i$
Inventory: $s_i = s_{i-1} + x_i - d_i$
Regulations: $o_i \leq 6w_i$

$$\min 2000\sum_i w_i + 320\sum_i h_i + 400\sum f_i + 8\sum s_i + 180\sum_i o_i.$$

Different form!

Variants of linear programs.

1. Maximization or minimization.
2. Equations or inequalities.
3. Non-negative variables or unrestricted variables.

Reductions.

1. Maximization to minimization?
   Multiply objective function by -1.
2. Less than inequalities into greater than?
   Multiply both sides by (-1) again!
   Example: $4 \geq 3$ to $(-1)4 \leq (-1)3$.
3. Inequalities and equalities.
   (a) $\sum_i a_i x_i \leq b$ into equality?
   $\sum_i a_i x_i + s = b$ and $s \geq 0$.
   (b) $\sum_i a_i x_i = b$ into inequalities?
   $\sum_i a_i x_i \leq b$ and $\sum_i a_i x_i \geq b$
4. Simulate unrestricted variable x with positive variables.
   - Introduce x_+, and x_-.
   - Replace x by $(x_+ - x_-)$.

   $(x_+ - x_-)$ could be any real number!

Standard Form.

Standard form.
Minimization, positive variables, and “greater than” inequalities.

Peas and carrots.

Standard Form.

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \le 60 \\
3x_2 \le 75 \\
3x_1 + 2x_2 \le 100 \\
x_1, x_2 \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\min -4x_1 - 2x_2 \\
-2x_1 \ge -60 \\
-3x_2 \ge -75 \\
-3x_1 - 2x_2 \ge -100 \\
x_1, x_2 \ge 0
\end{aligned}
$$

Matrix Form.

Recall Linear equations: $Ax = b$?

Can do that here, too!

$$\begin{aligned} \min -4x_1 - 2x_2 \\ -2x_1 \geq -60 \\ -3x_2 \geq -75 \\ -3x_1 - 2x_2 \geq -100 \\ x_1, x_2 \geq 0 \end{aligned}$$

$$\min[-4,-2] \cdot [x_1, x_2]$$
$$\begin{pmatrix} -2 & 0 \\ 0 & -3 \\ -3 & -2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} -60 \\ -75 \\ -100 \end{bmatrix}$$
$$[x_1, x_2] \geq 0$$

Inputs:

$m \times n$ matrix A; m length vector b; n length vector c.

Output: n length vector x.

$$\min cx$$
$$Ax \geq b$$

Linear Program Problem

Inputs:
$m \times n$ matrix A; m length vector b; n length vector c.
Output: n length vector x.

$$\min cx$$
$$Ax \geq b$$

Oh yes, some complexities here.

1. Program has constraints $x_1 \leq 1$ and $x_1 \geq 3$?
   Has no feasible solution!
   Infeasible.
2. Program $x_1 \geq 0$, $\max x_1$.
   Optimum?
   100 ,200 ,300 ... no limit!
   Unbounded.