Linear Programming Continued!

# Algorithms..

Designed..

..multiplication, sorts, depth first search, breadth first search, greedy, dynamic programs.

Used previous algorithms as subroutines.
E.g. sort for greedy mst.

Another method.

Make a graph.
Run algorithm.
Pull out answer.

Examples:
Shortest path.
Dynamic programming and DAG!

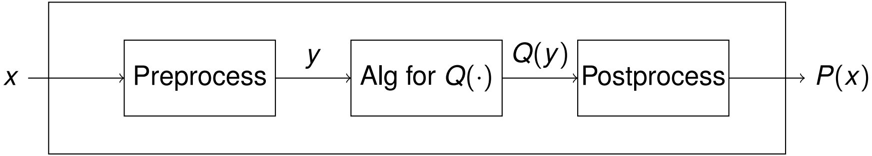
Peas and carrots. Express as a linear program!
Production. Express as a linear program!

Different from subroutine call.

Pre/post process is "easy".

# Reduction.

Algorithm for $P$. ...from algorithm for $Q$.




Peas and Carrots, Production. Instances.
General problem reduction to linear program.

New problem: Maximum Flow.

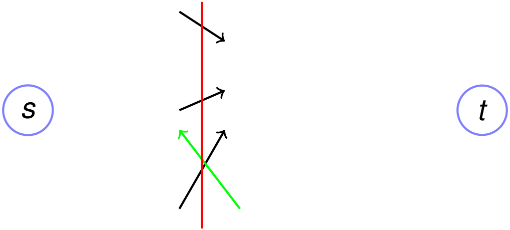
# Max-Flow Problem.

1. Capacity Constraints: $0 \leq f_e \leq c_e$.

2. Conservation Constraints:
"flow into $v$" = "flow out of $v$" (if not $s$ or $t$.)
Algorithm adds flow, say $f$, to path from $s$ to $t$.

# Optimality: upper bound.

$s$-$t$ Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.




Lemma: Capacity of any $s - t$ cut is an upper bound on the flow.

$C(S,T)$ - sum of capacities of all arcs from $S$ to $T$
$C(S,T) = \sum_{e=(u,v):u \in S, V \in T} c_e$

For valid flow:
Flow out of $(S)$ = Flow out of $s$.
Flow into $(T)$ = Flow into $t$.

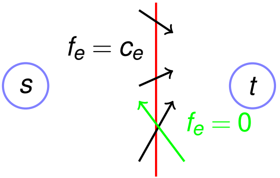
For any valid flow, $f : E \rightarrow Z+$, the flow out of $S$ (into $T$)
$\sum_{e \in S \times T} f_e - \sum_{e \in T \times S} f_e \leq \sum_{e \in S \times T} c_e - \sum_{e \in T \times S} 0 = C(S,T)$.

$\rightarrow$ The value of any valid flow is at most $C(S,T)$! $\square$

# Optimality: max flow = min cut.

At termination of augmenting path algorithm.
 No path with residual capacity!

Depth first search only starting at $s$ does not reach $t$.




$S$ be reachable nodes.

No arc with positive residual capacity leaving $S$

$\implies$ All arcs leaving $S$ are full.

$\implies$ No arcs into $S$ have flow.

Total flow leaving $S$ is $C(S, T)$.

Valid flow $\implies$ all that flow from source.

Value of flow equals value of $C(S, T)$. and Optimal is $\leq C(S, T)$.
$\rightarrow$ Flow is maximum!!

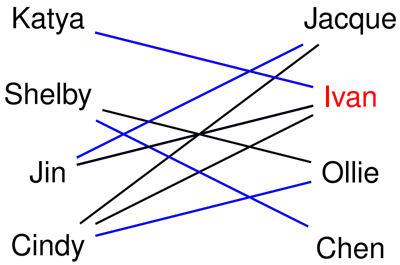
Cut is minimum $s - t$ cut too!
"any flow" $\leq$ "any cut" and this flow $=$ this cut.
$\rightarrow$ Maximum flow and minimum $s - t$ cut!

# Celebrated max flow -minimum cut theorem.

Theorem: In any flow network, the maximum $s$-$t$ flow is equal to the minimum cut.

# Bipartite Matching

Given a bipartite graph: $B = (L, R, E)$ where $E \subseteq L \times R$.



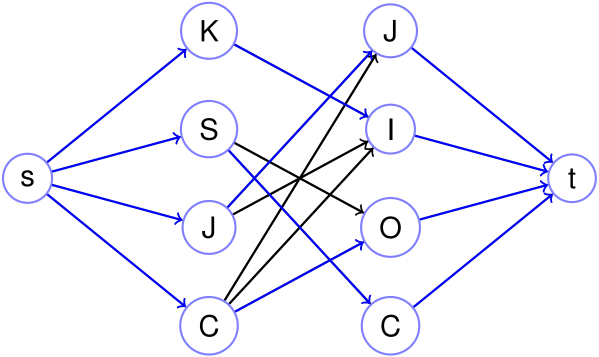

Find largest subset of edges ("matches") which are one to one.

# Bipartite Matching

Algorithm by “Reduction.”:

From matching problem produce flow problem.

From flow solution produce matching solution.




Max flow = Max Matching Size.

Flow is not integer necessarily....

Augmenting path algorithm gives integer flow.

# Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
 yields bound on objective..
Better solution?
Better upper bound?

# Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\le 4 \\ x_2 &\le 3 \\ x_1 + 2x_2 &\le 7 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \le 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
  $x_1 + 8x_2 \le 6(3) + 7 = 25$.

Thus, the value is at most 25.

The upper bound is same as solution!

Proof of optimality!

# Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

How to find best upper bound?

# Duality: computing upper bound.

Best Upper Bound.

| Multiplier | Inequality |
|---|---|
| $y_1$ | $x_1 \leq 4$ |
| $y_2$ | $x_2 \leq 3$ |
| $y_3$ | $x_1 + 2x_2 \leq 7$ |

Adding equations thusly...

The left hand side should "dominate" optimization function:

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Find best $y_i$'s to minimize upper bound?

# The dual, the dual, the dual.

Find best $y_i$'s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

# The dual.

In general.

| Primal LP | Dual LP |
|---|---|
| $\max c \cdot x$ | $\min y^T b$ |
| $Ax \leq b$ | $y^T A \geq c$ |
| $x \geq 0$ | $y \geq 0$ |

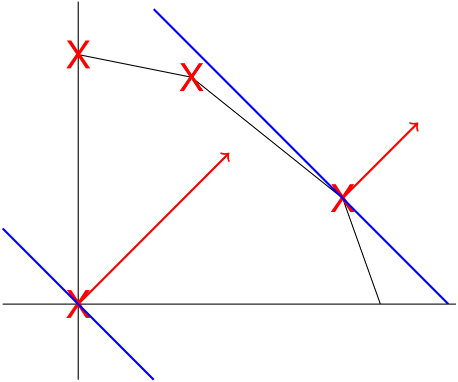
**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

# Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 20$$
$$4x_1 + 5x_2 \leq 21$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Why optimal? Draw line corresponding to $cx =$ current value.
Entire feasible region on "wrong" side.

# Example: review.

$$\begin{aligned} &\max x_1 + 8x_2 \\ &x_1 \leq 4 \\ &x_2 \leq 3 \\ &x_1 + 2x_2 \leq 7 \\ &y_1, y_2, y_3 \geq 0 \end{aligned} \qquad \begin{aligned} &\min 4y_1 + 3y_2 + 7y_3 \\ &y_1 + y_3 \geq 1 \\ &y_2 + 2y_3 \geq 8 \\ &x_1, x_2 \geq 0 \\ & \end{aligned}$$

"Matrix form"

$$\max[1,8]\cdot[x_1,x_2]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$[x_1, x_2] \geq 0$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$

$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$[y_1, y_2, y_3] \geq 0$$

# Matrix equations.

$$\max[1,8]\cdot[x_1,x_2] \qquad \min[4,3,7]\cdot[y_1,y_2,y_3]$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \le \begin{bmatrix}4\\ 3\\ 7\end{bmatrix} \qquad [y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix} \ge \begin{bmatrix}1\\ 8\end{bmatrix}$$

$$[x_1,x_2] \ge 0 \qquad [y_1,y_2,y_3] \ge 0$$

We can rewrite the above in matrix form.

$$A = \begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix} \qquad c = [1,8] \quad b = [4,3,7]$$

The primal is $Ax \le b, \max c\cdot x, x \ge 0$.
The dual is $y^T A \ge c, \min b\cdot y, y \ge 0$.

# Generality of Linear Programming.

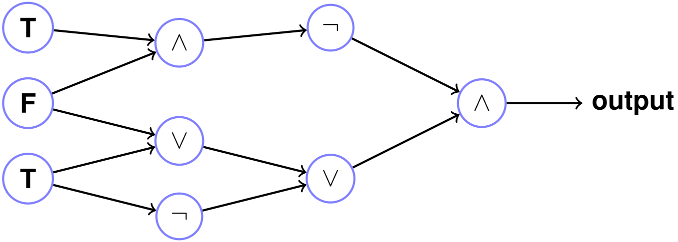
Linear program solves many problems.

How applicable is it?

# Circuit Evaluation.

Circuit Evaluation:
Given: DAG of boolean gates:
two input **AND**/**OR**.
One input **NOT**.
**TRUE**/**FALSE** inputs.

Problem: What is the output of a specified Output Gate?



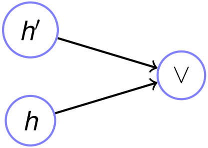

What is the value of the output?

# Translation to linear program.

Variable for gate $g$: $x_g$.

Constraints:
$0 \leq x_g \leq 1$
Gate $g$ is true gate: $x_g = 1$.
Gate $g$ is false gate: $x_g = 0$.

$$x_g \geq x_h$$
$$x_g \geq x_{h'}$$
$$x_g \leq x_h + x_{h'}$$




For $\wedge$ gate:
$x_g \leq x_h, \quad x_g \leq x'_h \qquad\qquad x_g \geq x_h + x_{h'} - 1$

For $\neg$ gate: $x_g = 1 - x_h$.

$x_o$ is 1 if and only if the circuit evaluates to true.

# What does this mean?

The circuit value problem is completely general!

A computer program can be unfolded into a circuit.

Each level is the circuit for a computer.

The number of levels is the number of steps.

$\Longrightarrow$ circuit value problems model computation.

$\Longrightarrow$ linear programs can model any polynomial time problem!

Warning: existence proof, not generally efficient.