Linear Programming Continued!

Algorithms..

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..multiplication, sorts, depth first search, breadth first search, greedy, dynamic programs.

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Another method.

Make a graph.

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  Make a graph.
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Shortest path.

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Dynamic programming and DAG!

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Different from subroutine call.

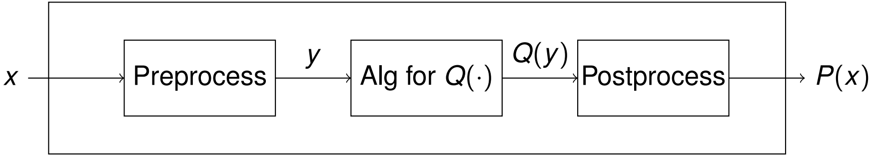
Pre/post process is “easy”.

Reduction.

Algorithm for P.

Reduction.

Algorithm for P. ...from algorithm for Q.



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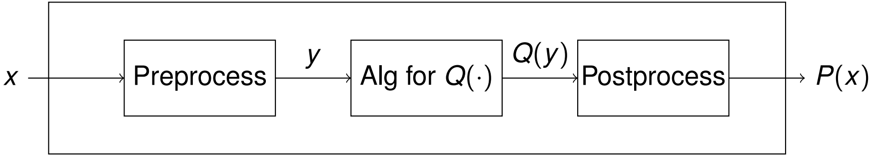

Peas and Carrots, Production. Instances.

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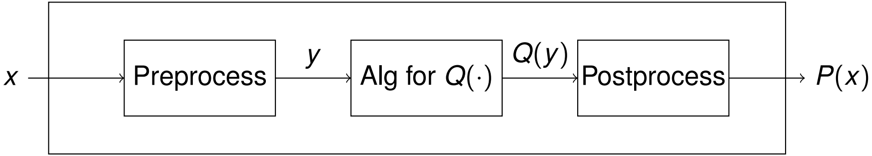

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General problem reduction to linear program.

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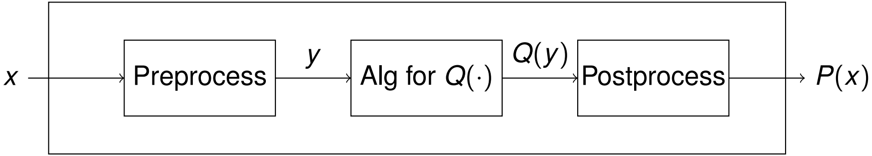

Peas and Carrots, Production. Instances.
General problem reduction to linear program.

New problem: Maximum Flow.

Max-Flow Problem.

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1. Capacity Constraints: $0 \leq f_e \leq c_e$.

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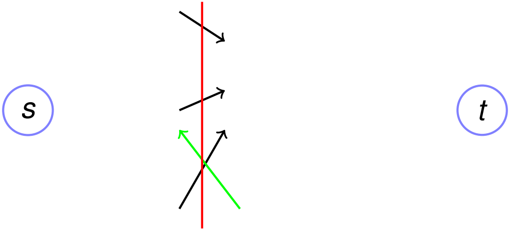
Optimality: upper bound.

s-t Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.



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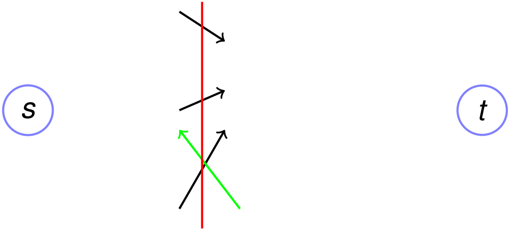

Lemma: Capacity of any $s - t$ cut is an upper bound on the flow.

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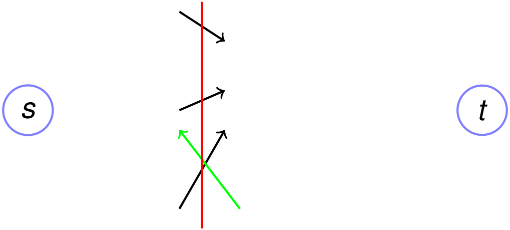

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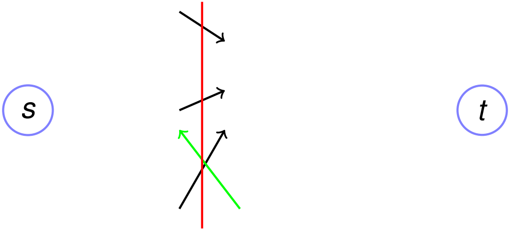

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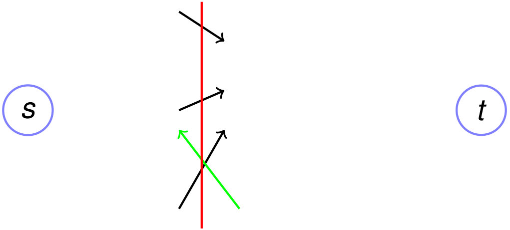

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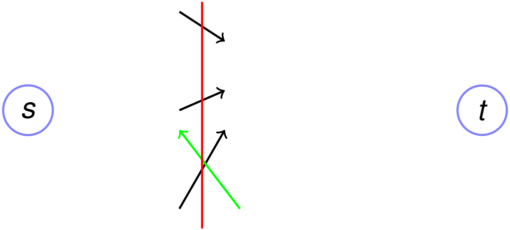

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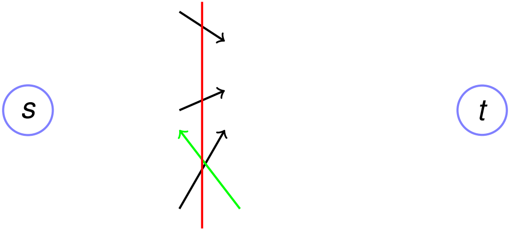

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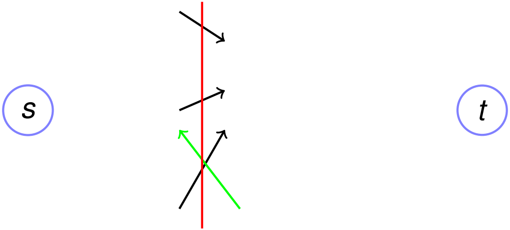

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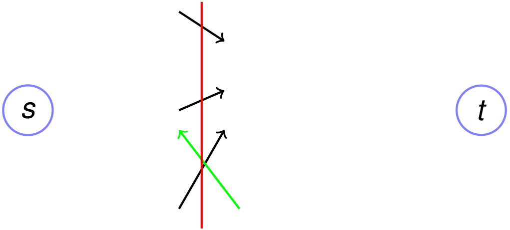

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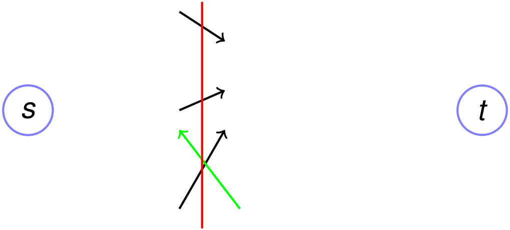

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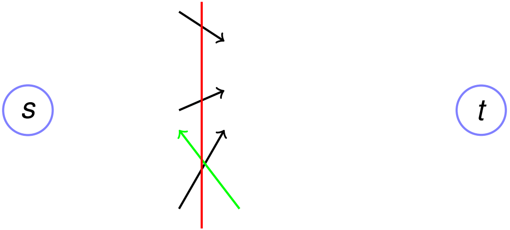

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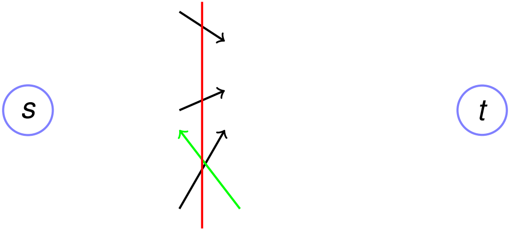

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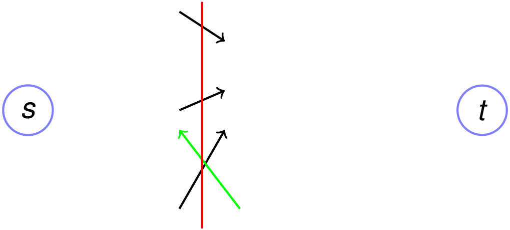

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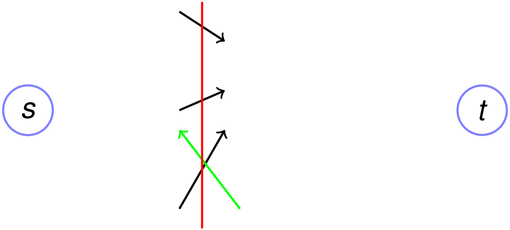

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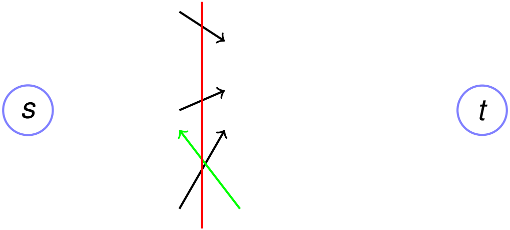

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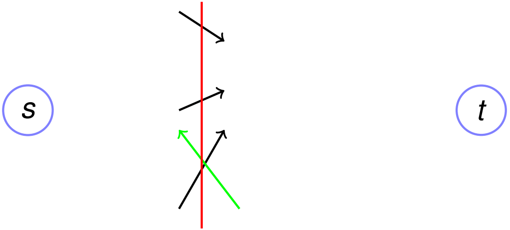

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Optimality: max flow = min cut.

At termination of augmenting path algorithm.

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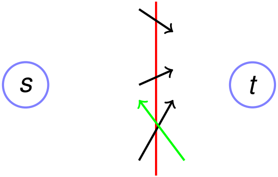

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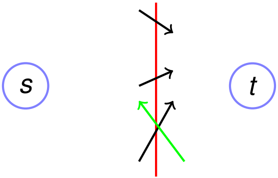

S be reachable nodes.

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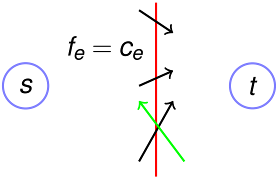

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$\Longrightarrow$ All arcs leaving S are full.

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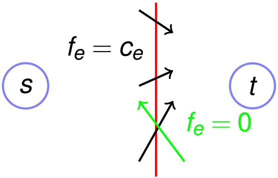

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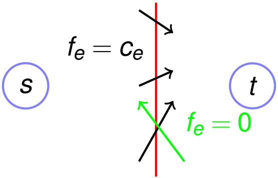

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Total flow leaving S is $C(S,T)$.

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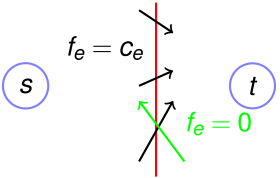

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Total flow leaving S is $C(S, T)$.

Valid flow $\implies$ all that flow from source.

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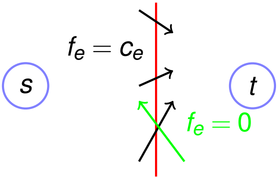

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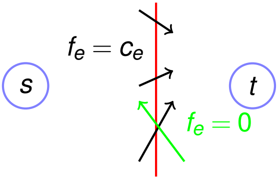

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At termination of augmenting path algorithm.
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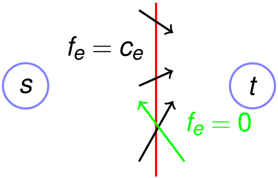

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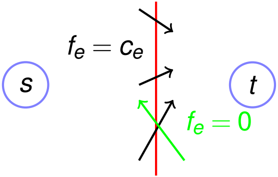

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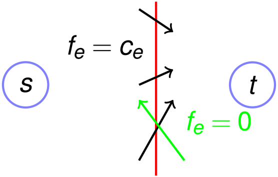

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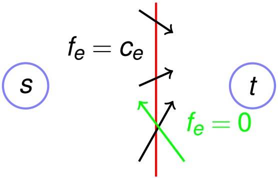

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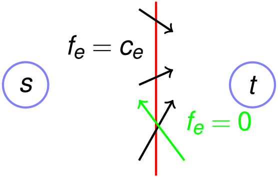

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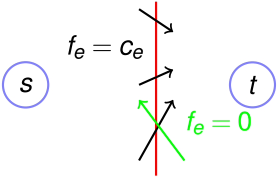

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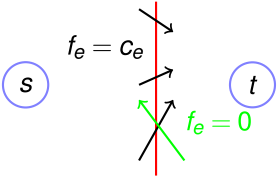

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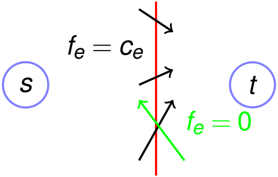

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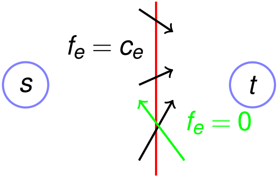

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$\rightarrow$ Flow is maximum!!

Cut is minimum $s - t$ cut too!
"any flow" $\leq$ "any cut" and this flow $=$ this cut.
$\rightarrow$ Maximum flow and minimum $s - t$ cut!

Celebrated max flow -minimum cut theorem.

Theorem: In any flow network, the maximum s-t flow is equal to the minimum cut.

Bipartite Matching

Given a bipartite graph: $B = (L, R, E)$ where $E \subseteq L \times R$.

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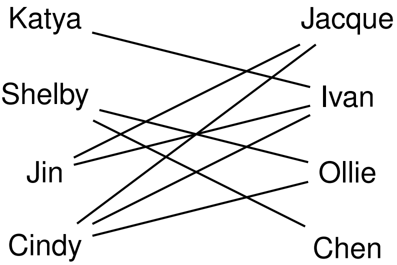

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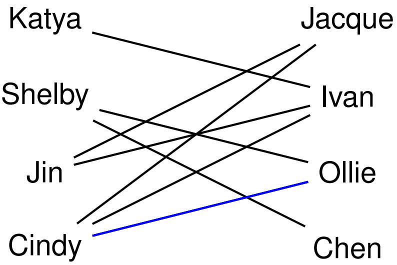

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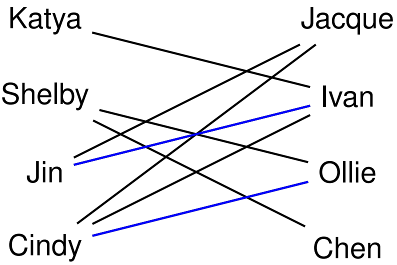

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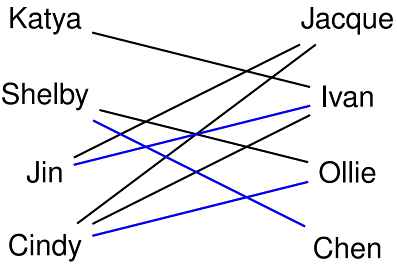

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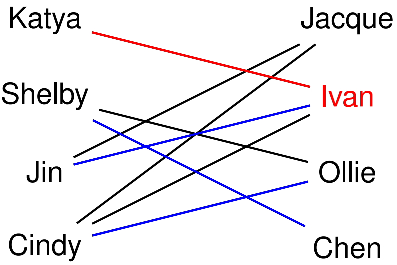

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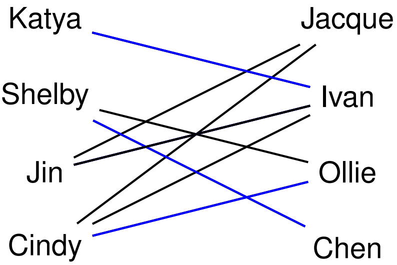

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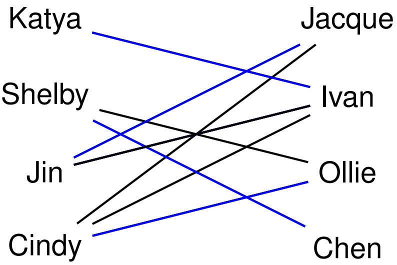

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Algorithm by “Reduction.”:

From matching problem produce flow problem.

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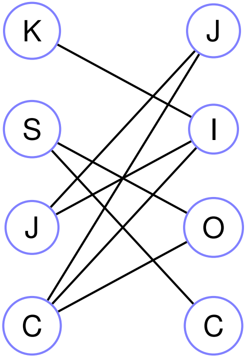

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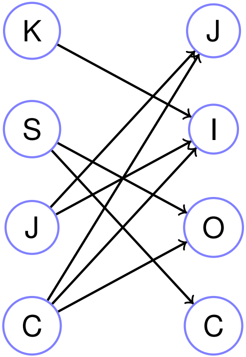

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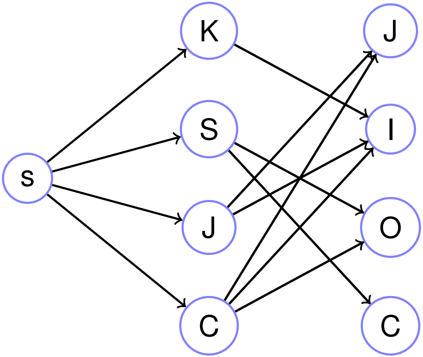

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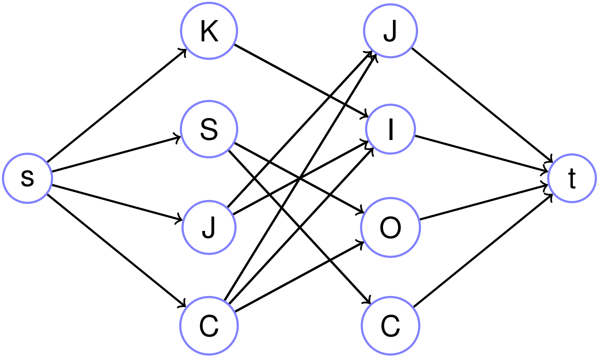

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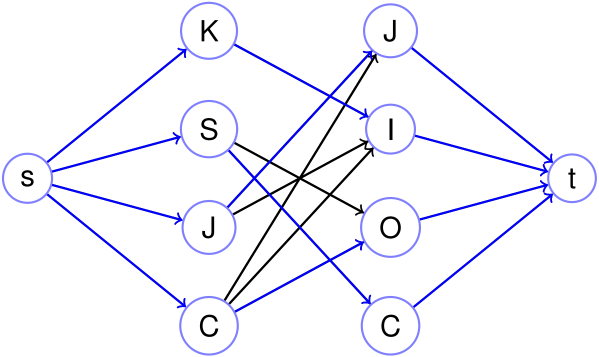

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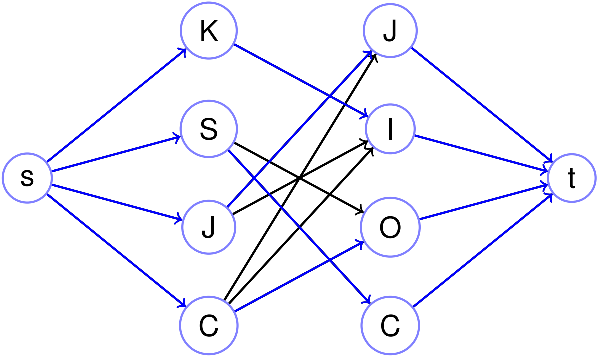

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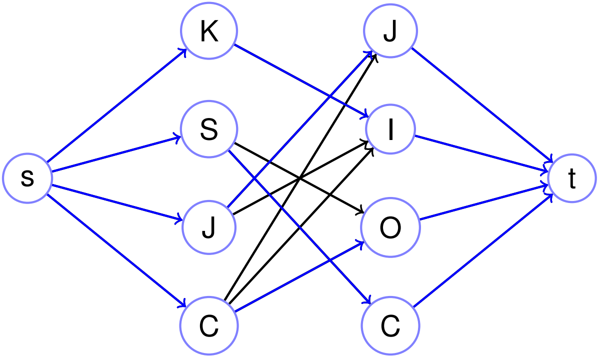

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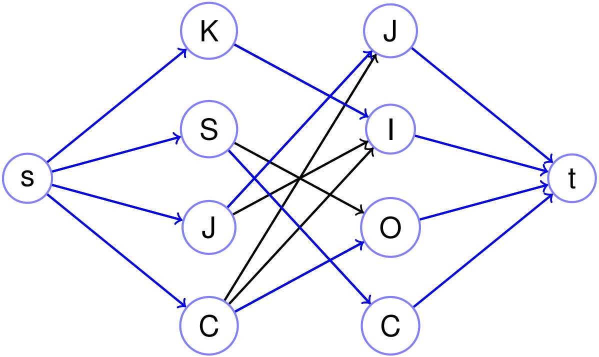

Max flow = Max Matching Size.

Flow is not integer necessarily....

Augmenting path algorithm gives integer flow.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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For any solution.
$x_1 \le 4$ and $x_2 \le 3$ ..

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For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
yields bound on objective..

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$x_1 + 8x_2 \le 4 + 24 = 28$.

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Sure: 6 times equation 2 and 1 times equation 3.

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$x_1 + 8x_2 \le 6(3) + 7 = 25$.

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Thus, the value is at most 25.

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The upper bound is same as solution!

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Proof of optimality!

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Duality:example

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Will this always work?

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
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Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

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The left hand side should "dominate" optimization function:

Duality: computing upper bound.

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The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$

Duality: computing upper bound.

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The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

Duality: computing upper bound.

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y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

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Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

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$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

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A linear program.

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A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

The dual, the dual, the dual.

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Value of both is 25!

The dual, the dual, the dual.

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$$
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y_1 + y_3 \geq 1 \\
y_2 + 2y_3 \geq 8 \\
y_1, y_2, y_3 \geq 0
\end{aligned}
$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal

The dual, the dual, the dual.

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A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

The dual.

In general.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

The dual.

In general.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
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Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

The dual.

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Simplex Algorithm.

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Start at a vertex.

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Move to better neighboring vertex.

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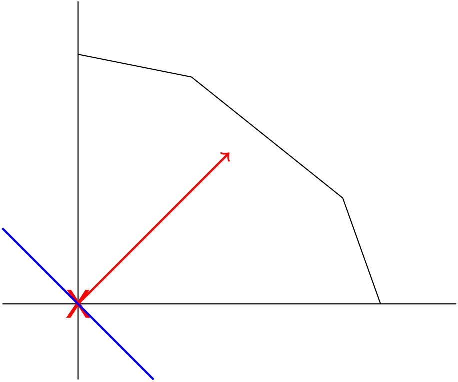
Until no better neighboring vertex.



$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 20$$
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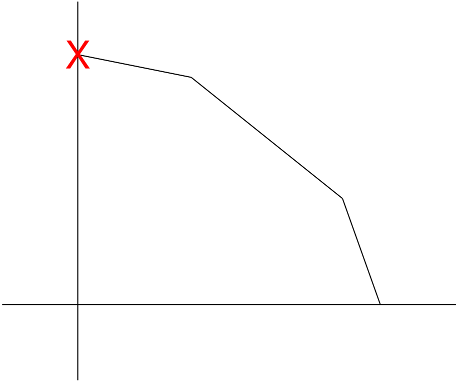

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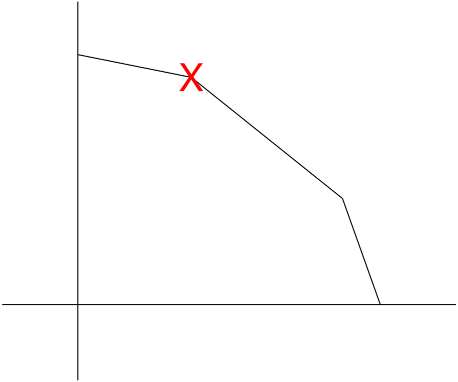

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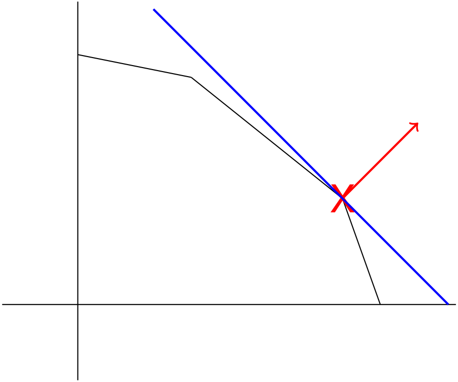

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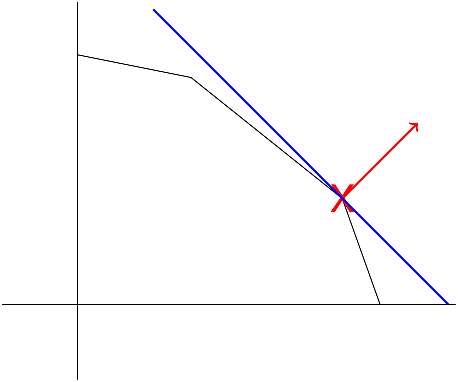

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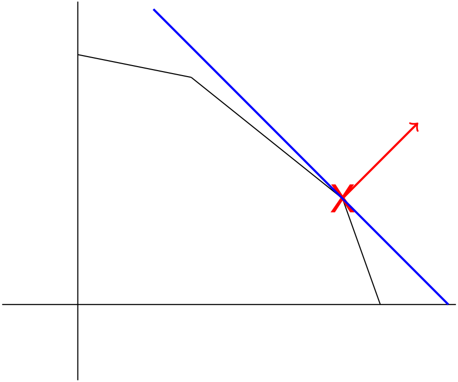

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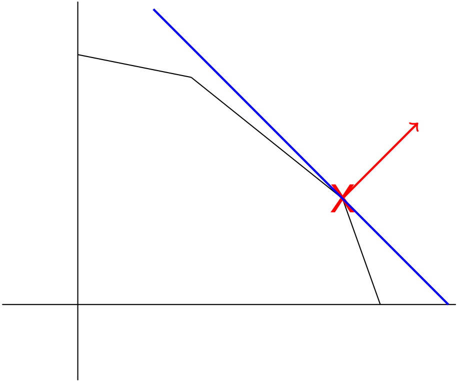

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Why optimal? Draw line corresponding to $cx =$ current value.
Entire feasible region on "wrong" side.

Example: review.

$$\max x_1 + 8x_2$$
$$x_1 \le 4$$
$$x_2 \le 3$$
$$x_1 + 2x_2 \le 7$$
$$y_1, y_2, y_3 \ge 0$$

$$\min 4y_1 + 3y_2 + 7y_3$$
$$y_1 + y_3 \ge 1$$
$$y_2 + 2y_3 \ge 8$$
$$x_1, x_2 \ge 0$$

"Matrix form"

$$\max[1,8] \cdot [x_1, x_2]$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$
$$[x_1, x_2] \ge 0$$

$$\min[4,3,7] \cdot [y_1, y_2, y_3]$$
$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \ge \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$
$$[y_1, y_2, y_3] \ge 0$$

Matrix equations.

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\le\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2]\ge 0$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\ge\begin{bmatrix}1\\ 8\end{bmatrix}$$
$$[y_1,y_2,y_3]\ge 0$$

We can rewrite the above in matrix form.

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\qquad c=[1,8]\quad b=[4,3,7]$$

The primal is $Ax\le b, \max c\cdot x, x\ge 0$.
The dual is $y^TA\ge c, \min b\cdot y, y\ge 0$.

Generality of Linear Programming.

Linear program solves many problems.

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Linear program solves many problems.

How applicable is it?

Circuit Evaluation.

Circuit Evaluation:

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Given: DAG of boolean gates:

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Given: DAG of boolean gates:
  two input **AND**/**OR**.

Circuit Evaluation.

Circuit Evaluation:
Given: DAG of boolean gates:
  two input **AND/OR**.
  One input **NOT**.

Circuit Evaluation.

Circuit Evaluation:
Given: DAG of boolean gates:
  two input **AND**/**OR**.
  One input **NOT**.
TRUE/**FALSE** inputs.

Circuit Evaluation.

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Given: DAG of boolean gates:
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Problem: What is the output of a specified Output Gate?

Circuit Evaluation.

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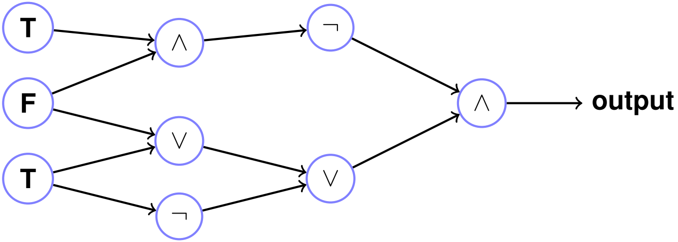

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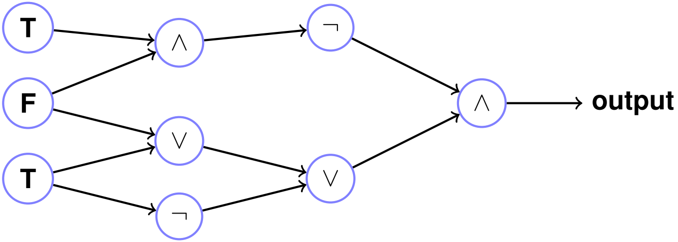

What is the value of the output?

Translation to linear program.

Variable for gate g: x_g.

Translation to linear program.

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Constraints:

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$0 \leq x_g \leq 1$

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Gate g is true gate: $x_g = 1$.

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Variable for gate g: x_g.

Constraints:
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Gate g is true gate: $x_g = 1$.
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Translation to linear program.

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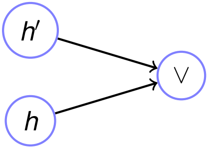

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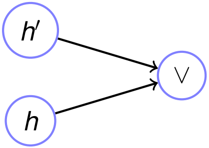

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Constraints:
$0 \leq x_g \leq 1$
Gate g is true gate: $x_g = 1$.
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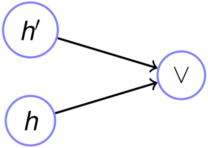
$x_g \geq x_h$



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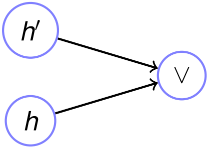

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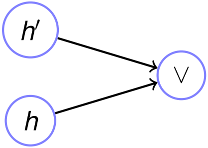
$$x_g \leq x_h + x_{h'}$$



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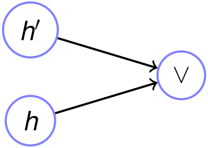

For $\wedge$ gate:
$x_g \leq x_h, \quad x_g \leq x_h'$

Translation to linear program.

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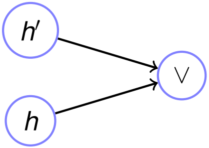

For $\wedge$ gate:
$x_g \leq x_h, \quad x_g \leq x'_h \qquad\qquad x_g \geq x_h + x_{h'} - 1$

Translation to linear program.

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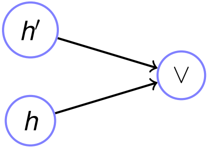

For $\wedge$ gate:
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For $\neg$ gate: $x_g = 1 - x_h$.

Translation to linear program.

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$0 \leq x_g \leq 1$
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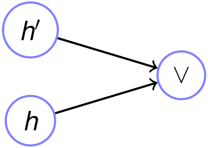

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x_o is 1 if and only if the circuit evaluates to true.

What does this mean?

The circuit value problem is completely general!

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$\Longrightarrow$ circuit value problems model computation.

$\Longrightarrow$ linear programs can model any polynomial time problem!

Warning: existence proof, not generally efficient.