Today

- Union-Find Datastructure to implement Kruskal
- Path Compression

Disjoint Set Data Structure

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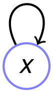

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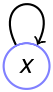

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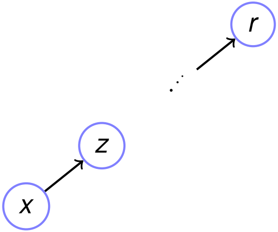
union(x,y)
    $\pi(\text{find}(x)) = \text{find}(y)$



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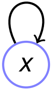

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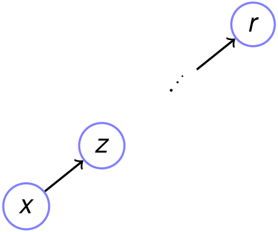

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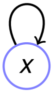

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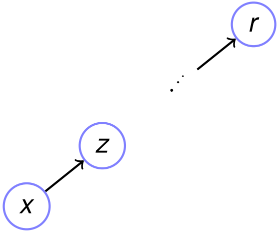

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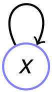

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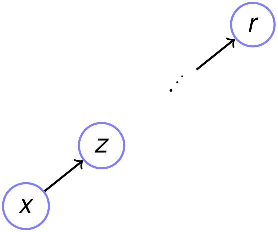

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Want depth to be small!

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Make a bit less deep: union-by-rank.

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union(x,y)
Use roots of x and y.
Which points to which?
"smaller" to "larger" in terms of the height (or what we will call rank)

Union by rank.

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Property of rank

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Test your understanding: Can the rank of a node that is not a root change?

Big rank corresponds to the bigger tree!

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Check your understanding?

Exactly 2^k nodes in tree of rank k? Yes or No?

Check your understanding?

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No.

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Gains nodes without gaining rank!

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Kruskal: Sort edges, $O(n)$ union, $O(m)$ finds.

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Can we do better?

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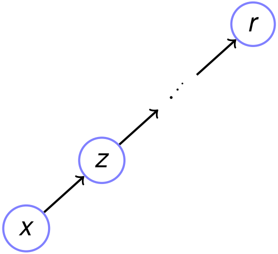

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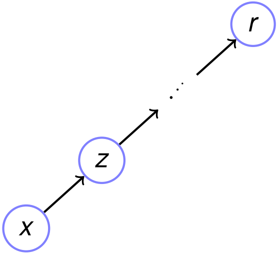
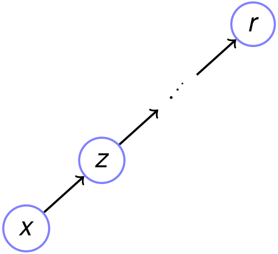


What happens if we find(x) again? We go up the tree again?

Path Compression

find(x)
  if $\pi(x) == x$
    return x
  else
    find($\pi(x)$)



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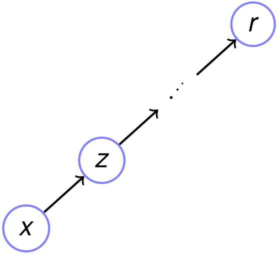
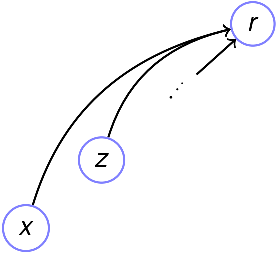


What happens if we find(x) again? We go up the tree again?

Can we avoid this work the next time?

find(x)
  if $\pi(x) == x$
    return x
  else
    $\pi(x) = \text{find}(\pi(x))$
    return $\pi(x)$



Path Compression Analysis

Union is same.

Path Compression Analysis

Union is same. Only affects root nodes.

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Rank properties still hold:

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Every find is asymptotically faster?

(A) Yes

(B) No

No.

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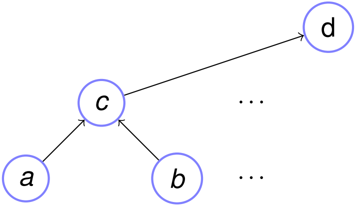

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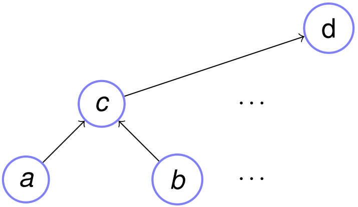

union(a,c)

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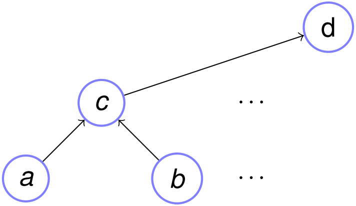

union(a,c) union(b,c)

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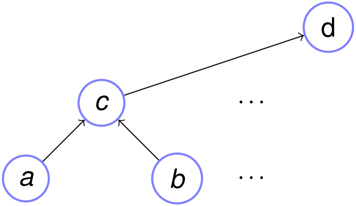

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...

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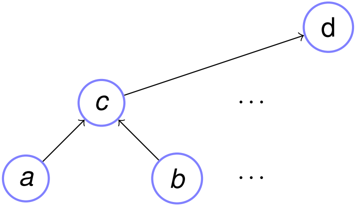

union(a,c) union(b,c)
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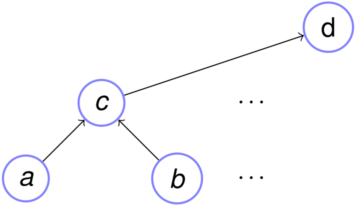

union(a,c) union(b,c)
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union roots to build complete binary tree

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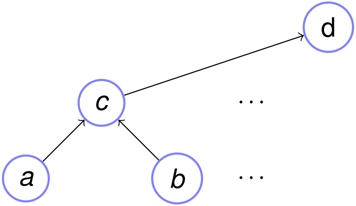

union(a,c) union(b,c)
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union roots to build complete binary tree
find(a)

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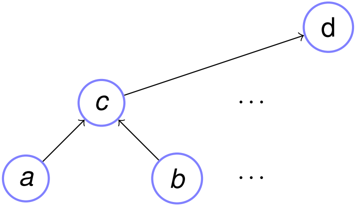

union(a,c) union(b,c)
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union roots to build complete binary tree
find(a)
$\Theta(\log n)$ time for this find.

Amortized Analysis.

Show that m **finds** take $O(m \log^* n)$ time in total.

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Show that m **finds** take $O(m \log^* n)$ time in total.

$O(\log^* n)$ time on average!

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Amortize cost = average over many operations.

What is log star?

$\log^* n$ is number of times one take log to get to 1.

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$\log^*(16)$?

(A) 4

(B) 2

(C) 3

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C. $\log 16 = 4$, $\log 4 = 2$, $\log 2 = 1$. 3 times.

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Also $2^{2^2} = 16$.

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$\log 1,000,000$ versus $\log^* 1,000,000$?

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20 versus 5.

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$\log 1,000,000^{1,000,000}$ versus $\log^* 1,000,000^{1,000,000}$?

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$20,000,000$ versus 6.

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$\log 1,000,000$ versus $\log^* 1,000,000$?

20 versus 5.

$\log 1,000,000^{1,000,000}$ versus $\log^* 1,000,000^{1,000,000}$?

$20,000,000$ versus 6.

Grows very slowly.

Amortized Analysis.

Show that m **finds** take $O(m \log^* n)$ time in total.

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How to do amortized analysis?

Hand out some money

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How to do amortized analysis?

Hand out some money
..... use it to pay for each pointer change.

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$O(\log^* n)$ time on average!

Amortize cost = average over many operations.

How to do amortized analysis?

Hand out some money
..... use it to pay for each pointer change.

Only hand out $O(m \log^* n)$ dollars.

Handing out dollars.

Will hand out money to internal nodes
.....since they change pointers in find.

Handing out dollars.

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Notice: When a node stops being a root

Handing out dollars.

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Notice: When a node stops being a root
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Divide non-zero ranks into levels.

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$\{1\},\{2,3,4\},\{5,\ldots,16\}\cdots\{k+1,\ldots 2^k\}\cdots$

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How many groups of ranks?

(A) $\Theta(\log n)$

(B) $\Theta(\log^* n)$

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How many internal nodes ever get rank r?

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How many internal nodes ever get rank r?

No node contained in more one rank r internal node.

Handing out dollars.

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How many groups of ranks?

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(B) $\Theta(\log^* n)$

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How many internal nodes ever get rank r?

No node contained in more one rank r internal node.
n nodes in total.

Handing out dollars.

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Each rank r node contains 2^r nodes.

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How many internal nodes ever get rank r?

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n nodes in total.
Each rank r node contains 2^r nodes.
$\implies < \frac{n}{2^r}$ rank r nodes

Handing out money!

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If in set of ranks $\{k+1,\ldots,2^k\}$ give node 2^k dollars.

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If in set of ranks $\{k+1, \ldots, 2^k\}$ give node 2^k dollars.

$O(n/2^r)$ internal nodes of rank r.

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Total Doled out:

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Total Doled out:
In a group:

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Total Doled out:
In a group: $2^k(n/2^{k+1} + n/2^{k+2} \cdots) = O(n)$.

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Total Doled out:
In a group: $2^k(n/2^{k+1} + n/2^{k+2}\cdots) = O(n)$.
$O(\log^* n)$ groups. Total money: $O(n\log^* n)$.

Bounding find cost.

Bound cost of find operation.

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Bound cost of find operation.

$O(1)$ plus

Bounding find cost.

Bound cost of find operation.

$O(1)$ plus
    cost of changing pointers to point to higher ranked nodes

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Total cost: $O(m \log^* n)$.

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Bound cost of find operation.

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$O(\log^* n)$ pointers that point to a node to a higher group.

  Total cost: $O(m \log^* n)$.

Node pays for changing a pointer within group.

Bounding find cost.

Bound cost of find operation.

$O(1)$ plus
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$O(\log^* n)$ pointers that point to a node to a higher group.

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Node pays for changing a pointer within group.
Recall group: $\{k+1, \ldots, 2^k\}$

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Enough money?

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Recall group: $\{k+1, \ldots, 2^k\}$

    Enough money?
        fewer than 2^k ranks in group

Bounding find cost.

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Recall group: $\{k+1, \ldots, 2^k\}$

  Enough money?
    fewer than 2^k ranks in group
    each node in group has 2^k dollars.

Bounding find cost.

Bound cost of find operation.

$O(1)$ plus
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Total money: $O(n \log^* n)$.

Total cost of finds:

Bounding find cost.

Bound cost of find operation.

$O(1)$ plus
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$O(\log^* n)$ pointers that point to a node to a higher group.

  Total cost: $O(m \log^* n)$.

Node pays for changing a pointer within group.
Recall group: $\{k+1, \ldots, 2^k\}$

  Enough money?
    fewer than 2^k ranks in group
    each node in group has 2^k dollars.   Enough money!

Total money: $O(n \log^* n)$.

Total cost of finds: $O((m+n)\log^* n)$!

Instant Replay

Intuition:

Instant Replay

Intuition:

Some operations may be expensive.

Instant Replay

Intuition:

Some operations may be expensive.
...but modify data structure so they won't be in future.

Instant Replay

Intuition:

Some operations may be expensive.
...but modify data structure so they won't be in future.

Place credits in data structure to pay for some modifications.

Instant Replay

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