# Today

- Zero-Sum Games

# Strategic Games.

$N$ players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

| | **C** | **D** |
|---|---|---|
| **C** | (3,3) | (0,5) |
| **D** | (5,0) | (1,1) |

# Famous because?

| | **C** | **D** |
|---|---|---|
| **C** | (3,3) | (0,5) |
| **D** | (5,0) | (.1,.1) |

What is the best thing for the players to do?

Both cooperate. Payoff $(3, 3)$.

If player 1 wants to do better, what does he do?

Defects! Payoff $(5, 0)$

What does player 2 do now?

Defects! Payoff $(.1, .1)$.

Stable now!

Nash Equilibrium: neither player has incentive to change strategy.

# Two Person Zero Sum Games

2 players.

Each player has strategy set:
$m$ strategies for player 1 $n$ strategies for player 2

Payoff function: $u(i,j) = (-a, a)$ (or just $a$).
"Player 1 pays $a$ to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by $m$ by $n$ matrix: $G$.
Row player maximizes, column player minimizes.

Roshambo: rock,paper, scissors.

| | R | P | S |
|---|---|---|---|
| R | 0 | -1 | 1 |
| P | 1 | 0 | -1 |
| S | -1 | 1 | 0 |

Any Nash Equilibrium?

$(R,R)$? no. $(P,R)$? no. $(S,R)$? no.

# Mixed Strategies.

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | -1 | 1 |
| P | $.3\overline{3}$ | 1 | 0 | -1 |
| S | $.3\overline{3}$ | -1 | 1 | 0 |

How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

**Mixed strategies:** Each players plays distribution over strategies.

**Pure strategies:** Each player plays single strategy.

# Payoffs: Equilibrium.

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | -1 | 1 |
| P | $.3\overline{3}$ | 1 | 0 | 1 |
| S | $.3\overline{3}$ | -1 | 1 | 0 |

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable $X$ (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = 0.^{1}$$

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[1] Remember zero sum games have one payoff.

# Equilibrium

| | | R | P | S |
|---|---|---|---|---|
| | | $.3\overline{3}$ | $.3\overline{3}$ | $.3\overline{3}$ |
| R | $.3\overline{3}$ | 0 | 1 | -1 |
| P | $.3\overline{3}$ | -1 | 0 | 1 |
| S | $.3\overline{3}$ | 1 | -1 | 0 |

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times -1 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)}(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium!

# Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

| | R | P | S | E |
|---|---|---|---|---|
| R | 0 | -1 | 1 | -1 |
| P | 1 | 0 | -1 | -1 |
| S | -1 | 1 | 0 | -1 |
| E | 1 | 1 | 1 | 0 |

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$G = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

# Playing the boss...

Row has extra strategy:Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$G = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: column knows row cheats. Why play?
Row is column's boss.

# Equilibrium: play the boss...

$$G = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}$
Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}$

Payoff is $0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}$

Column player: every column payoff is $-\frac{1}{6}$.

Both only play optimal strategies!

Why not play just one? Changes payoff for other guy!

# Two person zero sum games.

$m \times n$ payoff matrix $G$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^t G y$$

That is,

$$\sum_i x_i \left( \sum_j G[i,j] y_j \right) = \sum_j \left( \sum_i x_i G[i,j] \right) y_j.$$

Recall row maximizes, column minimizes.

# Best Response/Defense.

**Row goes first:**
Find $x$, where best column is not too low..

$$R = \min_y \max_x (x^t G y).$$

Note: $y$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?

| | | R | P | S |
|---|---|---|---|---|
| $x_1$ | R | 0 | -1 | 1 |
| $x_2$ | P | 1 | 0 | -1 |
| $x_3$ | S | -1 | 1 | 0 |

Pick $x_1$,$x_2$,$x_3$ to maximize

$$\min\{x_2 - x_3, -x_1 + x_3, x_1 - x_2\}$$

# Best Defense: row

Pick $x_1$,$x_2$,$x_3$ to maximize

$$\min\{x_2 - x_3, -x_1 + x_3, x_1 - x_2\}$$

Linear program: $z = \min\{x_2 - x_3, -x_1 + x_3, x_1 - x_2\}$

$$\begin{aligned} \max z \\ z \leq x_2 - x_3 \\ z \leq -x_1 + x_3 \\ z \leq x_1 - x_2 \\ x_1 + x_2 = 1 \end{aligned}$$

or in standard form...

$$\begin{aligned} \max z \\ x_2 - x_3 - z \geq 0 \\ -x_1 + x_3 - z \geq 0 \\ x_1 - x_2 - z \geq 0 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{aligned}$$

# Computing best defense: column.

**Column goes first:**
Find $y$, where best row is not high.

$$C = \max_x \min_y (x^t G y).$$

Again: $x$ of form $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of $C$? 0.

| | $y_1$ R | $y_2$ P | $y_3$ S |
|---|---|---|---|
| R | 0 | -1 | 1 |
| P | 1 | 0 | -1 |
| S | -1 | 1 | 0 |

Find $y$ to minimize $\max\{y_2 - y_3, -y_1 + y_3, y_1 - y_2\}$.

# Column Best Defense: LP.

Find $y$ to minimize $\max\{y_2 - y_3, -y_1 + y_3, y_1 - y_2\}$.

$$\begin{aligned} \min w \\ y_2 - y_3 \geq w \\ -y_1 + y_3 \geq w \\ y_1 - y_2 \geq w \\ y_1 + y_2 = 1 \end{aligned}$$

..in standard form..

$$\begin{aligned} \min w \\ y_2 - y_3 - w \geq 0 \\ -y_1 + y_3 - w \geq 0 \\ y_1 - y_2 - w \geq 0 \\ y_1 + y_2 = 1 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

# Both.

$$
\begin{array}{ll}
\max z & \min w \\
x_2 - x_3 - z \geq 0 & -y_2 + y_3 - w \geq 0 \\
-x_1 + x_3 - z \geq 0 & y_1 - y_3 - w \geq 0 \\
x_1 - x_2 - z \geq 0 & -y_1 + y_2 - w \geq 0 \\
x_1 + x_2 = 1 & y_1 + y_2 = 1 \\
x_1, x_2 \geq 0 & y_1, y_2, y_3 \geq 0
\end{array}
$$

Note: "column" defense lp is upper bound on "row" defense lp.

Linear programming dual? ...Yes!!!

$\implies z = w$!

$\max_x \min_y x^t G y = \min_y \max_x x^t G y$.

Wow! ! ! ! ! ! ! ! (von Neumann's minimax theorem.)

# Von Neumann's Minimax theorem.

$$\max_x \min_y x^t G y = \min_y \max_x x^t G y.$$

Column goes first, row gets to respond.

same as

Row goes first, column gets to respond.

Test!

Play rock first?

Mixed strategy. Play rock, paper,scissors uniformly.

# Search Problems

Shortest paths, minimum msts, maximum matchings, maximum increasing subsequences, maximum flows.

Efficient algorithms: polynomial.

Try everything: $n$! matchings! $n^{n-2}$ spanning trees...

$2^n$ or worse.

Moore: Computers doubling in speed every 18 month!
We have time !?

Sissa. Not so much!

Are there always *efficient* algorithms for optimization problems?

# Perhaps not.

**Satisfiability** or **SAT**.

An Instance.

$$\phi = (x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x}) \vee (\overline{x} \vee \overline{y} \vee \overline{z})$$

Is there any way to satisfy the formula above?

Search problem: Given instance $I$ find a solution $S$.

$S$ is short and easy to check.

A *search problem* has efficient checking algorithm $\mathscr{C}$:
$S$ is solution for $S$ if and only if $\mathscr{C}(I, S) = true$.

For SAT, what is $S$? assignment. What is $\mathscr{C}$?
Checks if assignment satisfies formula.

Recall: Horn or 2-SAT. Efficient algorithms!
Greedy and strongly connected components.

Efficient algorithm for 3-SAT? Don't think so! More later.