Today

Max Flow

Maximum flow

Flow network $G = (V, E)$, source s, sink $t \in V$, capacities $c_e > 0$.

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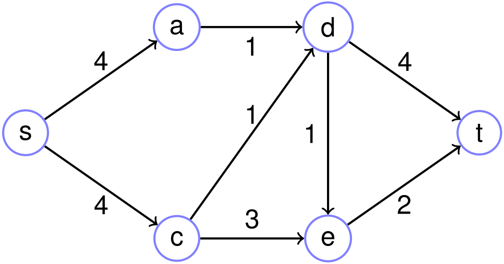

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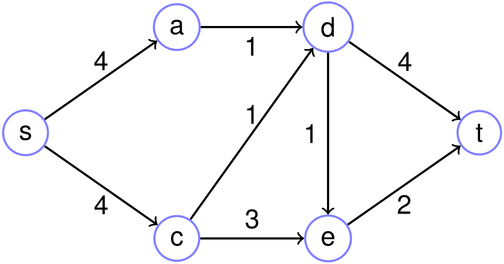

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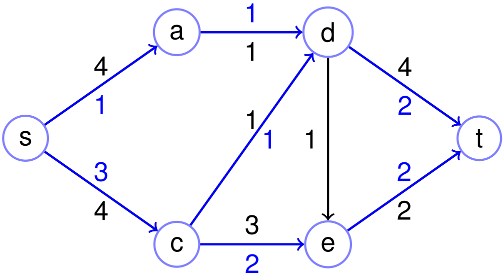

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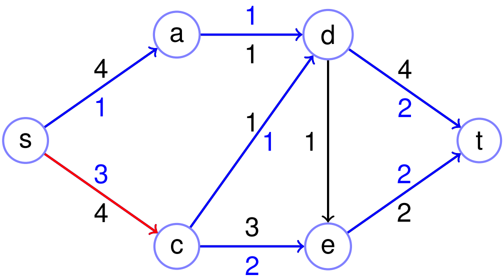

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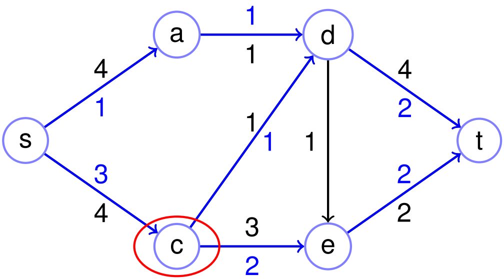

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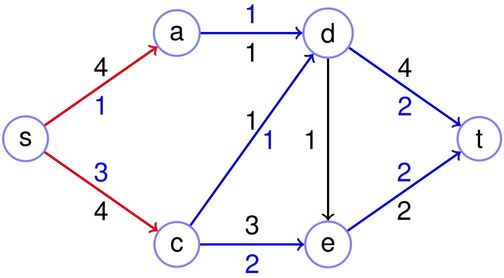

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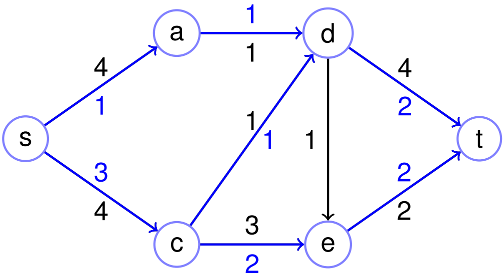

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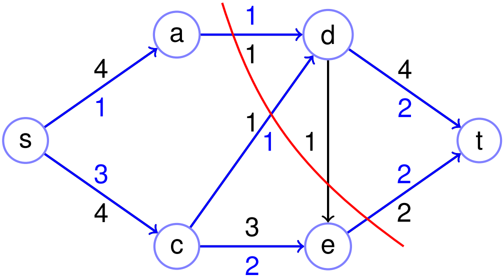

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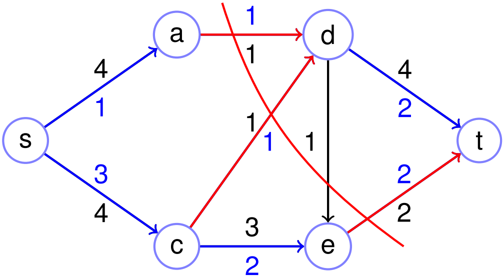

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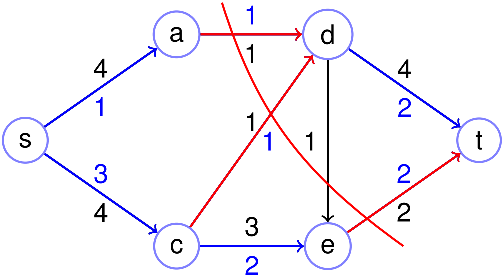

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Optimal? $c_{ad} + c_{cd} + c_{et} = 1 + 1 + 2 = 4.$

Any cut gives an upper bound.

Algorithms.

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Linear program!

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Variables f_e, linear constraints, linear optimization function.

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Cool!

S-T cut.

An $s - t$ cut is a partition of V into S and T where $s \in S$ and $t \in T$. Its capacity is the total capacity of edges from S to T.

Do you know the definition?

Find Flow: f_e

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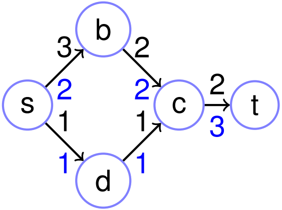

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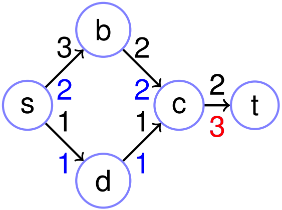

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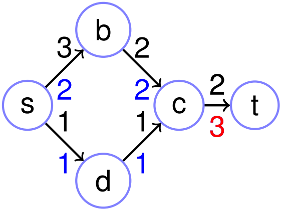

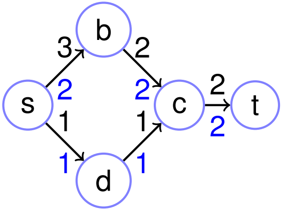

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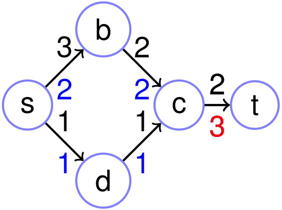

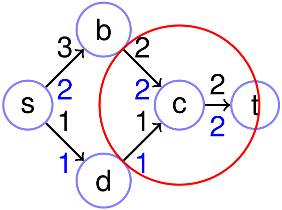

$2 + 1 \neq 2$

Algorithms.

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Vertex solution to linear program (for this problem) must be integral!

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"Simplex" method.

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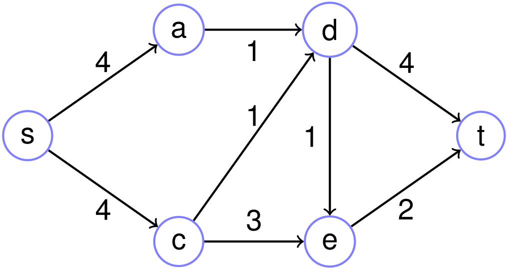

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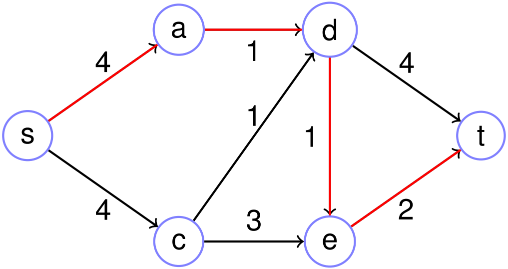

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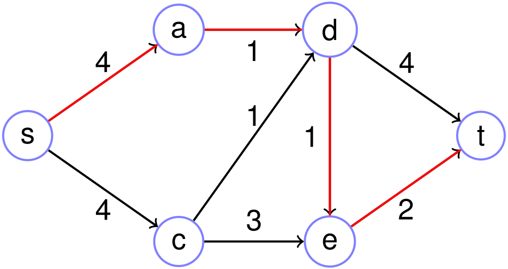

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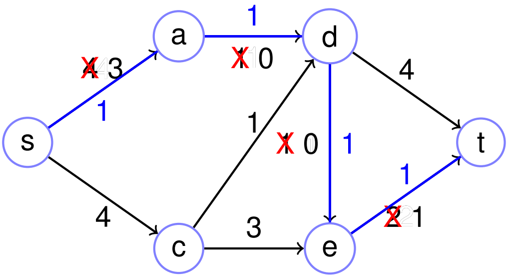

Residual Capacity.

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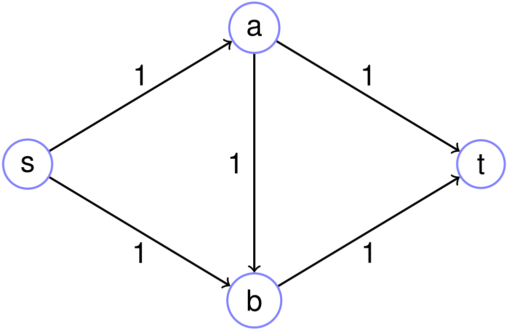
Repeat.



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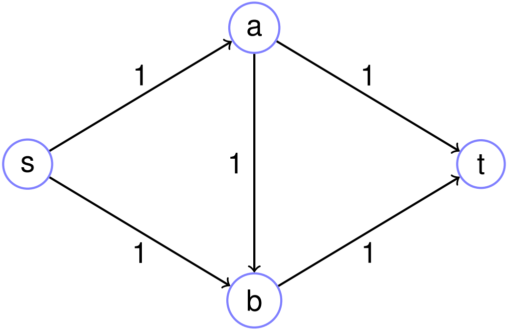

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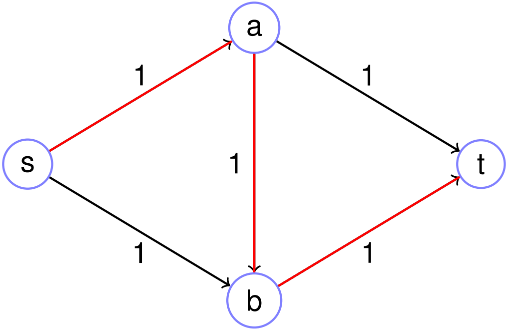

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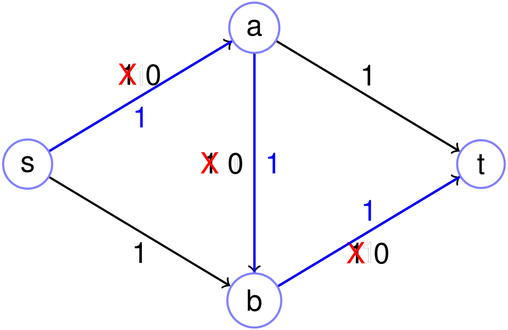

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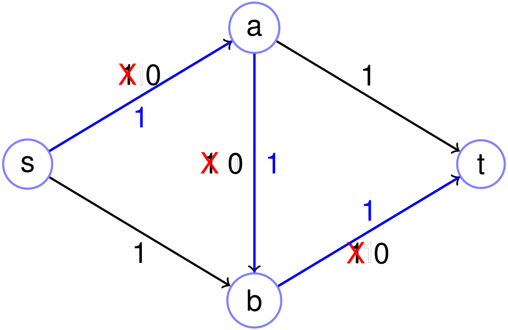

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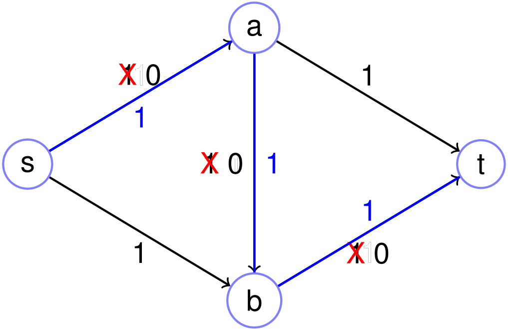

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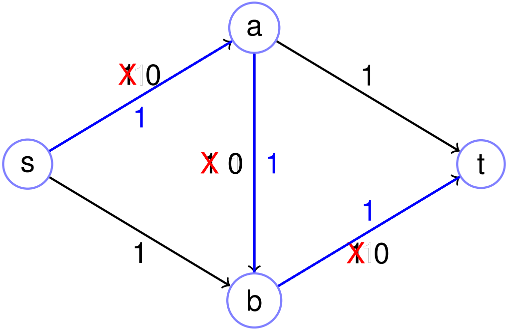

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Residual Capacity.

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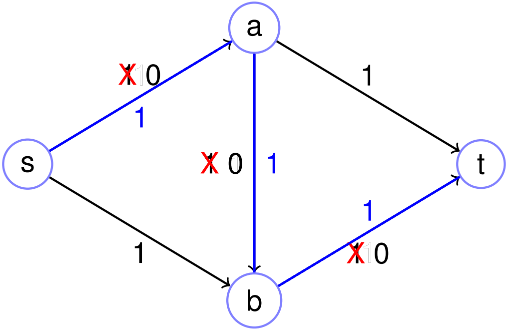
Add to flow along path. Or reduce flow on reverse edge.
Update remaining capacity.

Repeat.



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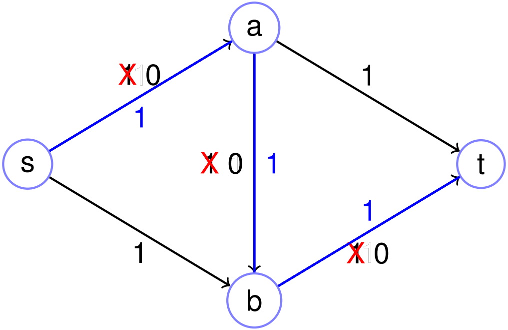
Update remaining capacity.
Reduce $r_e = c_e - f_e$

Repeat.



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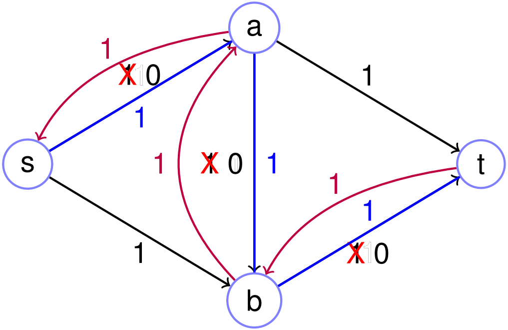

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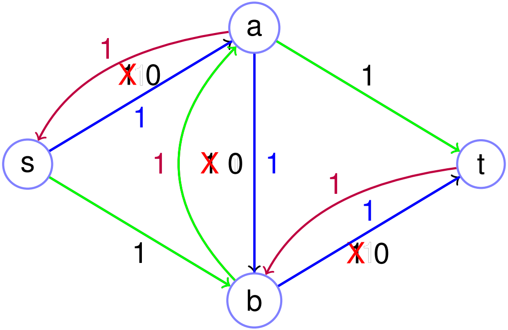

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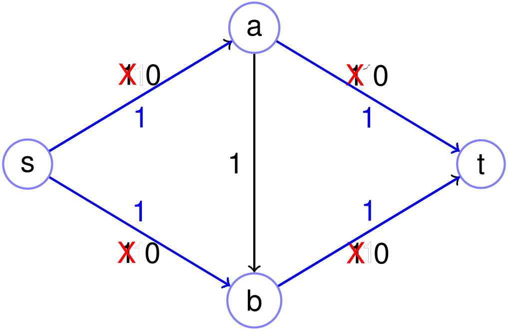
Repeat.



No remaining path. Uh oh! Optimal is 2! (At most 2 due to cut.)
Add reverse arcs to indicate “reverse” capacity.

Bigger Example.

Bigger Example.

Find s to t path with remaining capacity.

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Find s to t path with remaining capacity.

Add to flow along path.

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Update residual capacities: : $r_e = c_e - f_e; r_{uv} = f_{vu}$.

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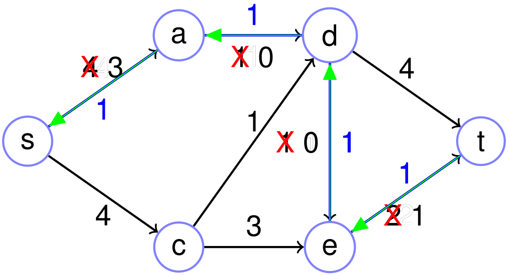

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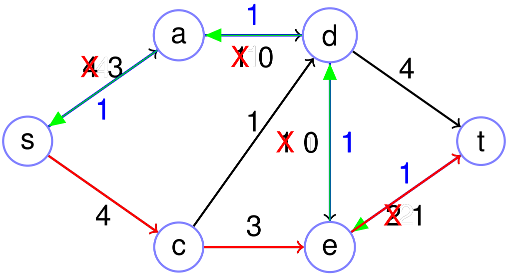

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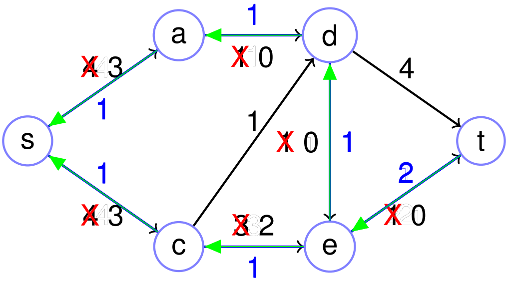

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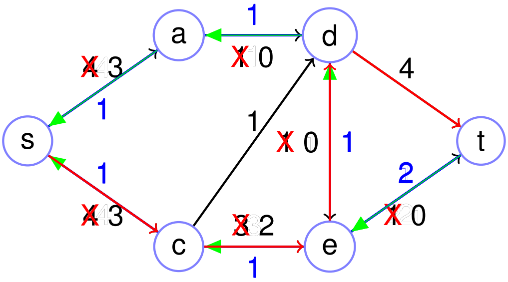

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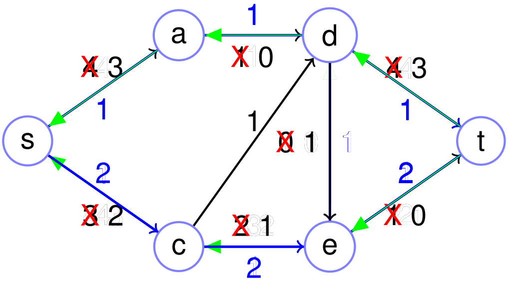

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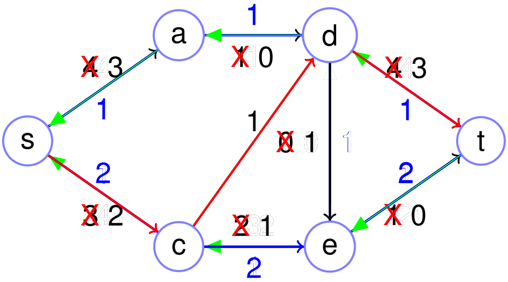

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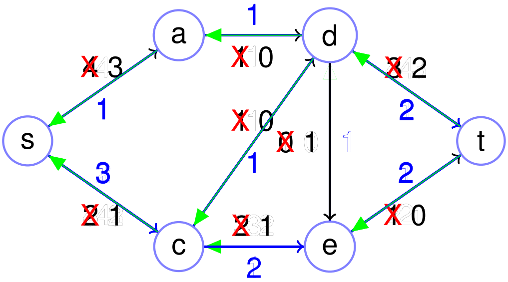

Check Result...

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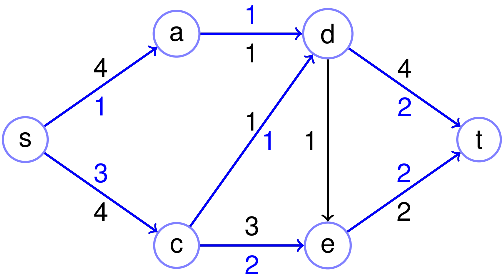

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2. If u is not s or t
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Maximize: size$(f) = \sum_{(s,u)\in E} f_{su}$.

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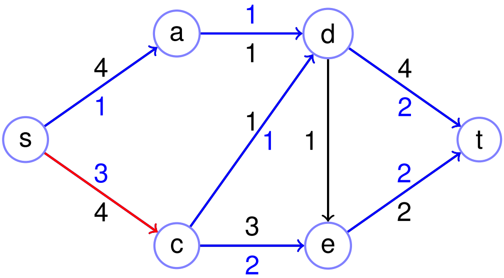

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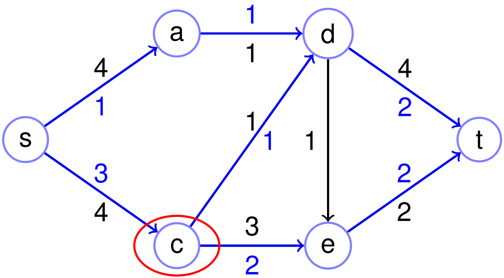

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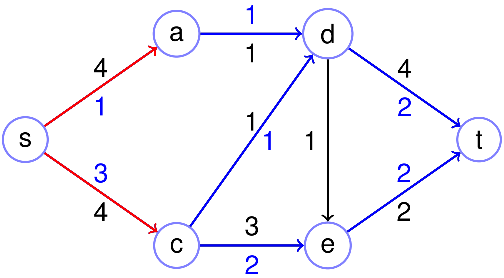

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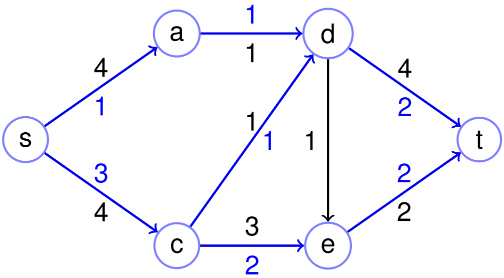

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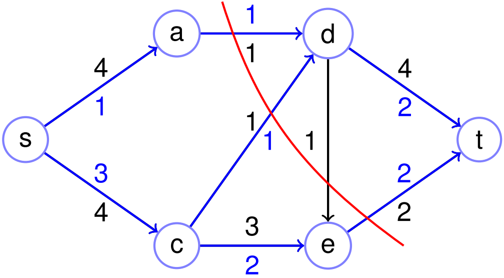

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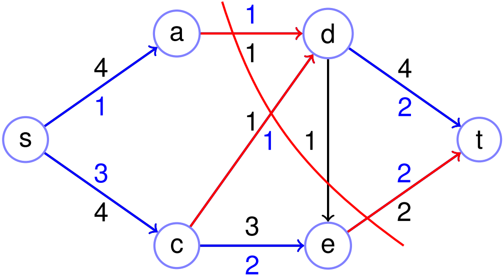

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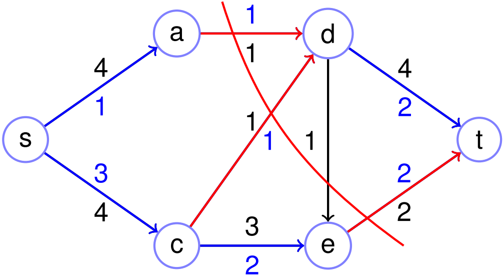

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Any $s - t$ cut gives an upper bound.

Correctness.

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Algorithm adds flow, say f, to path from s to t.

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Flow values to be between 0 and c_e.

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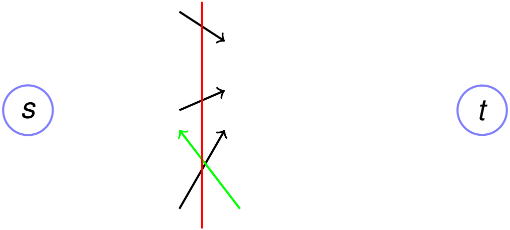
"flow into v" = "flow out of v" (if not s or t.)
Algorithm adds flow, say f, to path from s to t.
Each internal node has f in, and f out.

Optimality: upper bound.

s-t Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.



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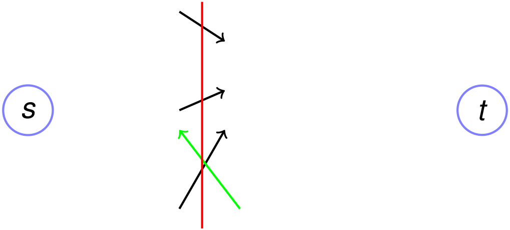

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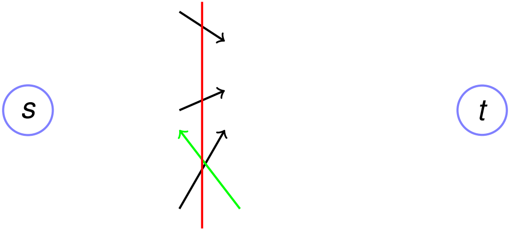

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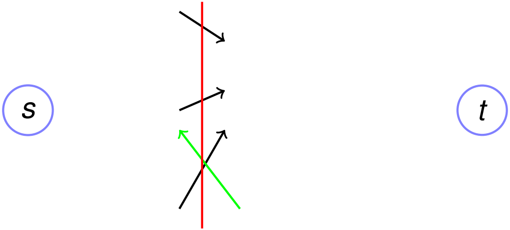

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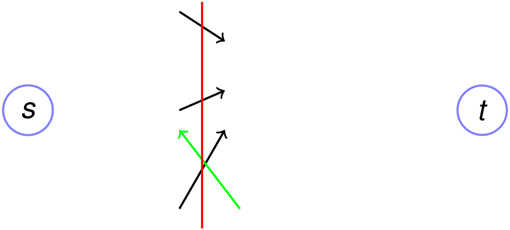

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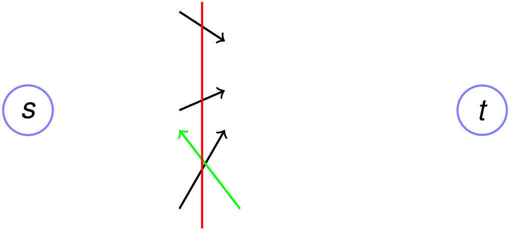

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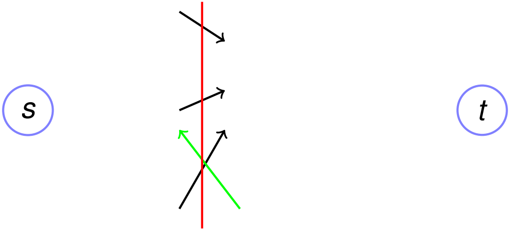

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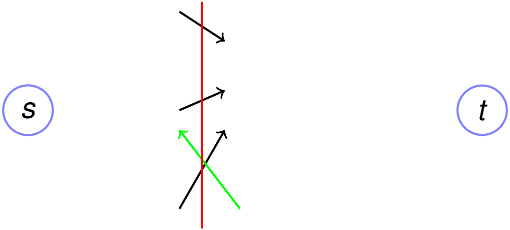

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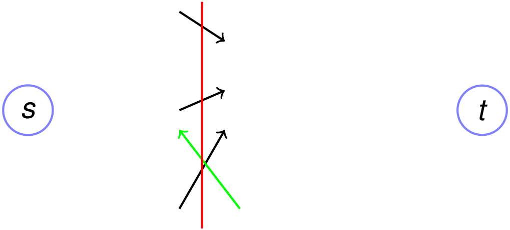

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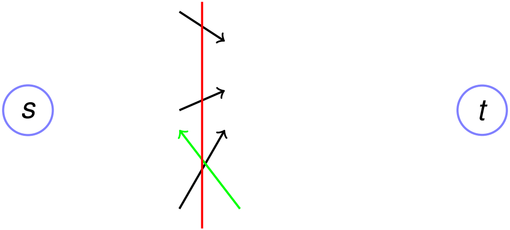

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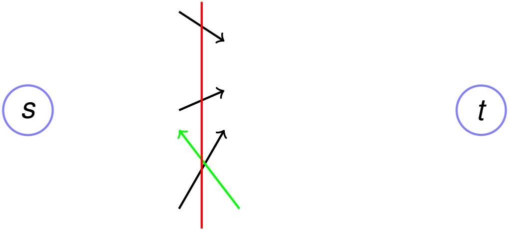

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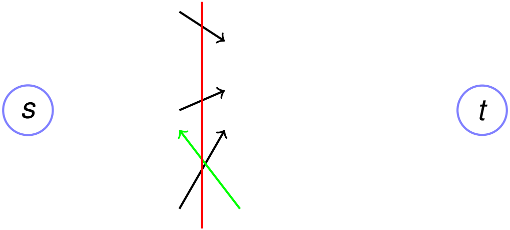

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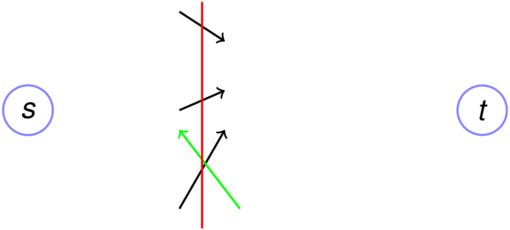

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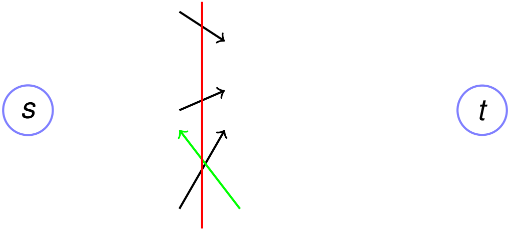

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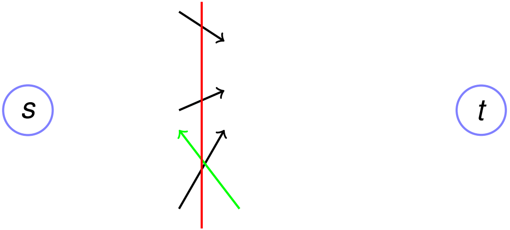

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$\rightarrow$ The value of any valid flow is at most $C(S,T)$!

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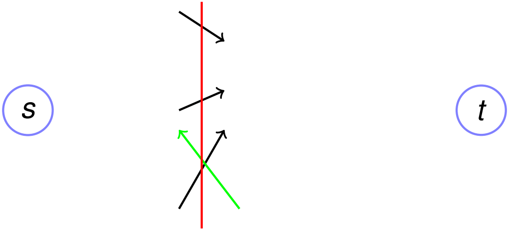

Lemma: Capacity of any $s - t$ cut is an upper bound on the flow.

$C(S,T)$ - sum of capacities of all arcs from S to T
$C(S,T) = \sum_{e=(u,v):u \in S, V \in T} c_e$

For valid flow:
Flow out of (S) = Flow out of s.
Flow into (T) = Flow into t.

For any valid flow, $f : E \to Z+$, the flow out of S (into T)
$\sum_{e \in S \times T} f_e - \sum_{e \in T \times S} f_e \leq \sum_{e \in S \times T} c_e - \sum_{e \in T \times S} 0 = C(S,T)$.

$\to$ The value of any valid flow is at most $C(S,T)$! $\square$

Optimality: max flow = min cut.

At termination of augmenting path algorithm.

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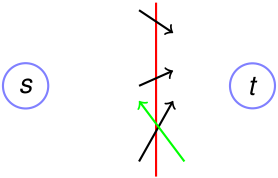

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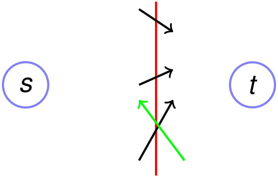

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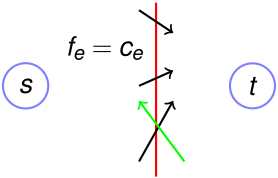

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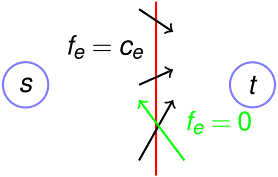

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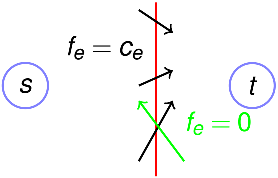

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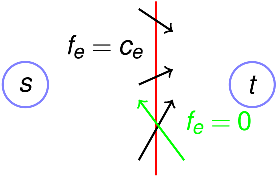

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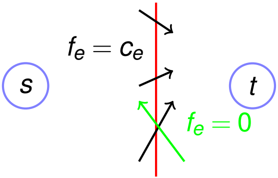

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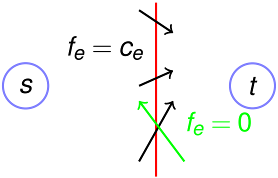

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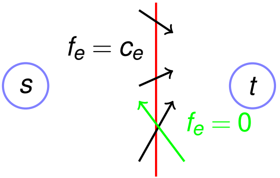

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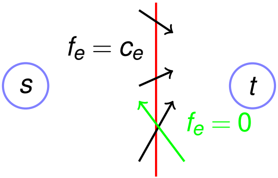

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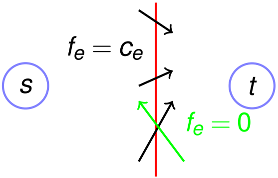

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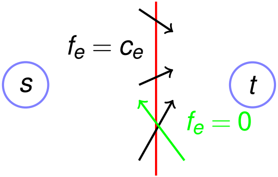

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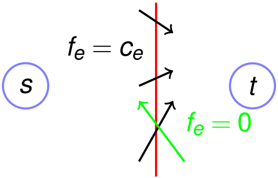

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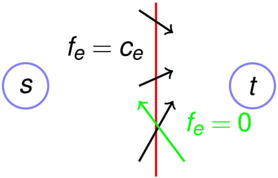

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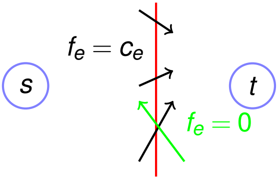

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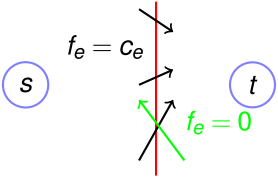

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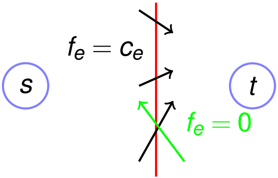

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$\rightarrow$ Flow is maximum!!

Cut is minimum $s - t$ cut too!
"any flow" $\leq$ "any cut" and this flow $=$ this cut.
$\rightarrow$ Maximum flow and minimum $s - t$ cut!

Celebrated max flow -minimum cut theorem.

Theorem: In any flow network, the maximum s-t flow is equal to the minimum cut.

Back to business: Algorithm Terminates?

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It will!!

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Flow keeps increasing.

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How long?

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One more unit every step!

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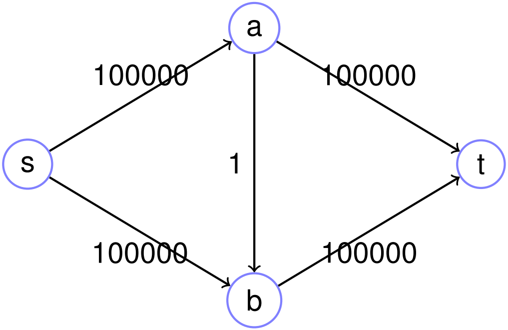

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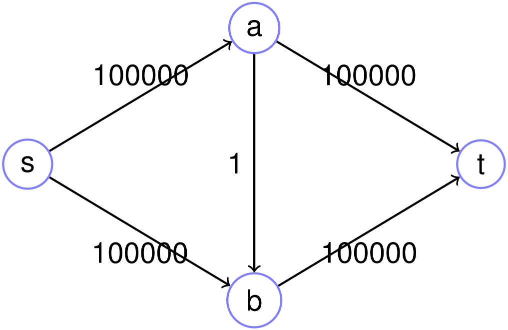

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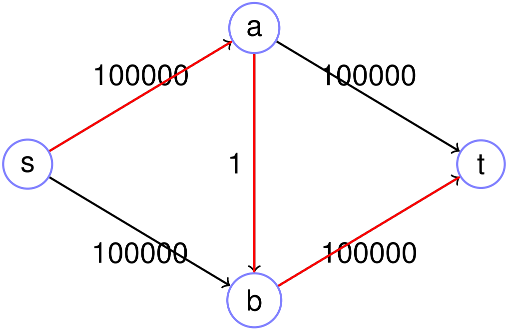

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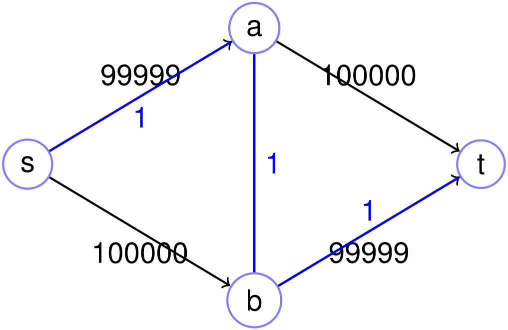

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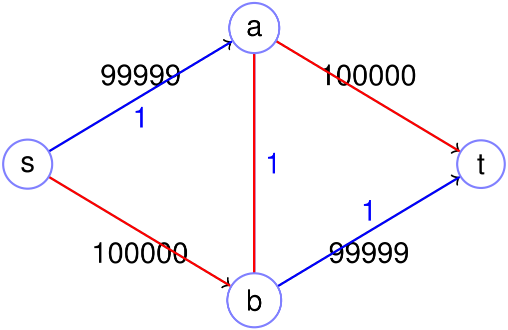

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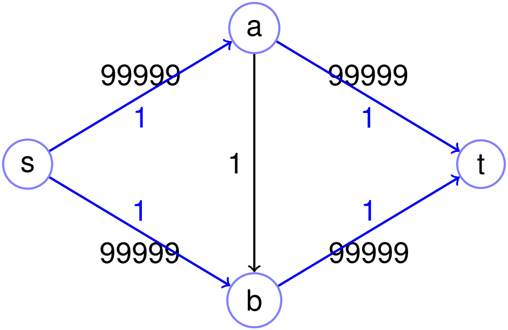

Edmonds-Karp

Augment along shortest path.

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Breadth first search!

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$O(|V||E|)$ augmentations.

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Analysis idea.

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$d(v)$ is distance to sink.

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$d(v)$ is distance to sink.

Only route flow on (u, v) if $d(u) \geq d(v)$.

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$O(|V||E|^2)$ time.