CS170 Discussion Section 10: *4/5*

Duality

Consider the following linear program:

$$\begin{aligned} \max\ 4x_1 + 7x_2 \\ x_1 + 2x_2 \leq 10 \\ 3x_1 + x_2 \leq 14 \\ 2x_1 + 3x_2 \leq 11 \\ x_1, x_2 \geq 0 \end{aligned}$$

Construct the dual of the above linear program.

Solution: If we scale the first constraint by $y_1 \geq 0$, the second by $y_2 \geq 0$, the third by $y_3 \geq 0$, and we add them up, we get an upperbound of $(y_1+3y_2+2y_3)x_1+(2y_1+y_2+3y_3)x_2 \leq (10y_1 + 14y_2 + 11y_3)$. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned} \min\ 10y_1 + 14y_2 + 11y_3 \\ y_1 + 3y_2 + 2y_3 \geq 4 \\ 2y_1 + y_2 + 3y_3 \geq 7 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

Remark: It would be instructive to check how the generic duality for linear programme is reflected in this example:

$$\max \begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \min \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 14 \\ 11 \end{bmatrix} \qquad \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} \geq \begin{bmatrix} 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

The above can be recast as

$$\begin{aligned} \max\ \mathbf{c}^T\mathbf{x} \qquad & \min\ \mathbf{y}^T\mathbf{b} \\ \mathbf{A}\mathbf{x} \leq \mathbf{b} \qquad & \mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T \\ \mathbf{x} \geq \mathbf{0} \qquad & \mathbf{y}^T \geq \mathbf{0}^T. \end{aligned}$$

In this concise form, weak duality (that every feasible primal solution has value bounded by every feasible dual solution) is easy to show:

$$\mathbf{c}^T\mathbf{x} \leq (\mathbf{y}^T\mathbf{A})\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^T\mathbf{b}.$$

Note that the first inequality follows from $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$, and the second inequality follows from $\mathbf{y}^T \geq \mathbf{0}^T$ and $\mathbf{A}\mathbf{x} \leq \mathbf{b}$.

Zero-sum Games

Consider a two-player, zero-sum game with the following pay-off matrix (by the column player to the row player):

	A	B
a	4	-2
b	-3	1

1. Assume that you are the row player, and play strategies a and b with probabilities x_1 and x_2 respectively, where x_1 and x_2 are known to the column player. What is your optimal return? Formulate this as a linear program.

   Solution: For given x_1 and x_2, the column player chooses $\min\{4x_1 - 3x_2, -2x_1 + x_2\}$, which the row player wants to maximize over feasible x_1 and x_2. (Note how the first three lines below are equivalent to $\min\{4x_1 - 3x_2, -2x_1 + x_2\}$.)

$$\begin{aligned} \max\ & w \\ & w \leq 4x_1 - 3x_2 \\ & w \leq -2x_1 + x_2 \\ x_1 + x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. Assume that you are the column player, and play strategies A and B with probabilities y_1 and y_2 respectively, where y_1 and y_2 are known to the row player. What is your optimal return? Formulate this as a linear program.

   Solution: For given y_1 and y_2, the row player chooses $\max\{4y_1 - 2y_2, -3y_1 + y_2\}$, which the column player wants to minimize over feasible y_1 and y_2. (Note how the first three lines below are equivalent to $\max\{4y_1 - 2y_2, -3y_1 + y_2\}$.)

$$\begin{aligned} \min\ & z \\ & z \geq 4y_1 - 2y_2 \\ & z \geq -3y_1 + y_2 \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Max of min

Argue that these two problems have the same optimal value:

$$\begin{aligned} \max \quad & \min\{a+b, b+c\} \\ \text{s.t.} \quad & a + 10b + 5c \leq 100 \\ & a \geq 0 \quad b \geq 0 \quad c \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & s \\ \text{s.t.} \quad & s \leq a+b \quad s \leq b+d \\ & a + 10b + 5c \leq 100 \\ & a \geq 0 \quad b \geq 0 \quad c \geq 0 \end{aligned}$$

Solution:

Call the first problem P_0 and the second P_1, and call their optimal values V_0 and V_1.

The basic idea is that in P_1, the optimal solution will always set $s = \min\{a+b, b+c\}$. We will prove $V_0 = V_1$ by showing $V_0 \leq V_1$ and $V_0 \geq V_1$.

Any feasible solution to P_0 can be converted to a feasible solution to P_1 by keeping the values for a, b and c the same, and setting $s = \min\{a+b, b+c\}$. Therefore the optimal solution to P_1 will be at least as good as V_0: $V_1 \geq V_0$.

Conversely, consider any optimal solution to P_1. It must be the case that in this optimal solution, $s = \min\{a+b, b+c\}$ — the constraints enforce that s is no larger than this, and if s is smaller, the solution can be improved by increasing s. Therefore if we keep the values of a, b and c the same, we will get a solution to P_0 with the same value. So $V_0 \geq V_1$.

Bipartite Vertex Cover

A vertex cover of an undirected graph $G = (V, E)$ is a subset of the vertices which touches every edge. In other words, a subset $S \subset V$ such that for each edge $\{u, v\} \in E$, one or both of u, v are in S.

Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. Prove that your reduction is correct.

Solution: Let $G' = (s \cup t \cup V_1 \cup V_2, E')$ be the bipartition of the graph G. Construct a network by adding a dummy source node s, with edges going out to every vertex of V_1, and a dummy target node t, with edges coming in from every vertex of V_2. Direct the remaining original edges so that they go from V_1 to V_2. Let the edges adjacent to s or t have capacity 1 and the original edges have infinite capacity.

Consider now a (s,t)-cut $(S, \bar{S})$ in this network, such that S only cuts edges adjacent to s or t. We claim that the set C_S of original vertices at the end of the edges cut by S must constitute a vertex cover of G. Suppose this is not the case and there exists an edge (u, v) between V_1 and V_2 not covered by C_S. Then $s - u - v - t$ is a path from s to t which is not cut by S. This is a contradiction. Hence, every cut S produces a vertex cover C_S of cardinality $|E(S, \bar{S})|$. Hence, the mincut must yield the minimum vertex cover.

More detailed proof:
We want proof the following:

1. For every (s, t)-cut $(S, \bar{S}) < \infty$ in G', let

$$X = \{v_1 \in V_1 : v_1 \in \bar{S}\} \cup \{v_2 \in V_2 : v_2 \in S\}$$

   X is a vertex cover of G and $|X| = \mathit{capacity}(S, \bar{S})$.

   Proof.
   Because the $(S, \bar{S})$ has finite capacity, there is no edge from V_1 to V_2 that crosses the cut. Thus for all edges e from V_1 to V_2, they either start from V_1 part of $\bar{S}$, end in the

V_2 part of S or both. Thus all edges in G have either have an endpoint in V_1 part of X or V_2 part of X. Hence $|X| = \textit{capacity}(S, \bar{S})$.

2. For every vertex cover X of G, let the cut $(S, \bar{S})$ be

$$S = \{v_1 \in V_1 : v_1 \notin X\} \cup \{v_2 \in V_2 : v_2 \in X\} \cup \{s\}$$
$$\bar{S} = \{v_1 \in V_1 : v_1 \in X\} \cup \{v_2 \in V_2 : v_2 \notin X\} \cup \{t\}$$

$(S, \bar{S})$ has finite capacity and $\textit{capacity}(S, \bar{S}) = |X|$.

Proof.
With a vertex cover X, the $s - t$ cut $(S, \bar{S})$ will not contain an edge of infinite capacity. Assume for the sake of contradiction that we have an edge $e = (u, w)$ going from V_1 to V_2 in the cut. Thus u will be in the V_1 side of S and w will be in the V_2 side of $\bar{S}$. This implies that neither u nor w are in X. Since there is an edge between u and w, X cannot be a vertex cover and we have a contradiction. Thus all edges in $(S, \bar{S})$ has finite capacities. In this case, only edges with capacity of 1 will be in the cut and the capacity of the cut is $|X|$.

Now we show that X is the minimum vertex cover. Since $|X| = \textit{capacity}(S, \bar{S})$ and $\textit{capacity}(S, \bar{S})$ is the minimum $s - t$ cut from max flow, $|X|$ is the smallest as it is the smallest cut.